15.053 Tutorial 04

LP Transformation Tricks

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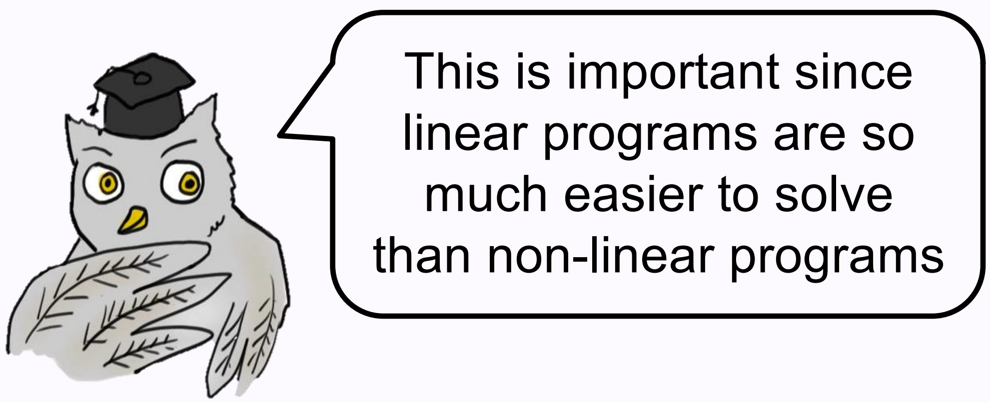
Introduction and Definitions

- The tutorial will show three different types of non-linear constraints that can be transformed into linear constraints:
  - ***Absolute value constraints***, in which the absolute value of a linear combination of decision variables is constrained
  - ***Maximizing minimum constraints*** (and vice versa), in which we find the largest value of a “min” function (taking the minimum value from a set), or find the smallest value of a “max” function (taking the maximum value from a set)
  - ***Ratio constraints***, in which the constraint is non-linear due to fractional coefficients



Example 1

Marketing Problem (1 of 2)

From Tutorial 02: Algebraic Formulations

Problem Statement

- Begin with the same marketing problem as seen in Tutorial 02, plus an absolute value constraint:
  - ***min*** $500\,x_1 + 200\,x_2 + 250\,x_3 + 125\,x_4$

    s.t. $50{,}000\,x_1 + 25{,}000\,x_2 + 20{,}000\,x_3 + 15{,}000\,x_4 \geq 1{,}500{,}000$

    $0 \leq x_1 \leq 20$

    $0 \leq x_2 \leq 15$

    $0 \leq x_3 \leq 10$

    $0 \leq x_4 \leq 15$

    $\mathbf{|x_1 + x_2 - x_3 - x_4| \leq 5}$

	TV	**Radio**	**Mail**	**Newspaper**
Audience Size	50,000	25,000	20,000	15,000
Cost/Impression	\$500	\$200	\$250	\$125
Maximum # Ads	20	15	10	15

Task

- Solve with the added absolute value constraint (total number of ads in electronic media is within 5 of the number in paper-based media), and solve

Example 1

Marketing Problem (2 of 2)

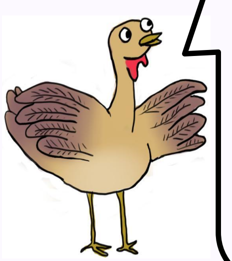
- There is a "trick" we can use to change the constraint $|x_1 + x_2 - x_3 - x_4| \leq 5$ into a linear constraint
- $|x_1 + x_2 - x_3 - x_4| \leq 5$ is equivalent to a combination of the following two constraints:
  - $x_1 + x_2 - x_3 - x_4 \leq 5$ and $-x_1 - x_2 + x_3 + x_4 \leq 5$
- The new problem is linear, and still equivalent to the original non-linear program:
  - ***min*** $500\,x_1 + 200\,x_2 + 250\,x_3 + 125\,x_4$

    s.t. $50{,}000\,x_1 + 25{,}000\,x_2 + 20{,}000\,x_3 + 15{,}000\,x_4 \geq 1{,}500{,}000$

    $0 \leq x_1 \leq 20, \quad 0 \leq x_2 \leq 15, \quad 0 \leq x_3 \leq 10, \quad 0 \leq x_4 \leq 15$

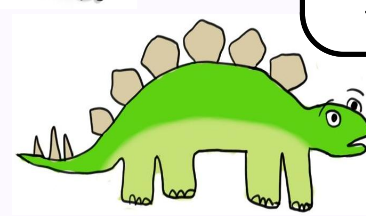
    $x_1 + x_2 - x_3 - x_4 \leq 5$

    $-x_1 - x_2 + x_3 + x_4 \leq 5$



What do you mean by equivalent? The two problems look different to me…

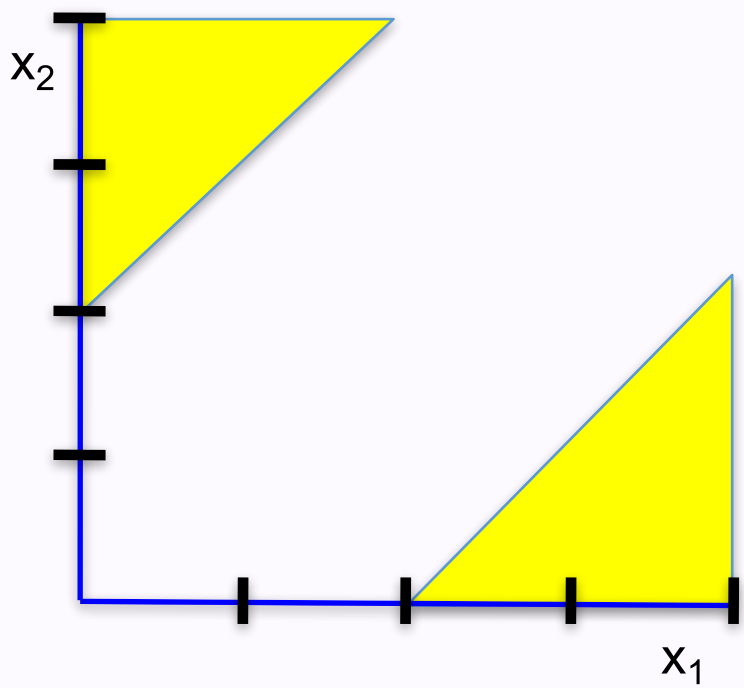
The problems are equivalent in the sense that any **feasible solution** for the non-linear program is feasible for the linear program, and vice versa. That is, **the feasible regions are exactly the same**



I'm extinct… The set of living dinosaurs is equivalent to a linear program with no feasible solutions!

Absolute Value Constraints

- We now know how to transform constraints of the form $|\ ax_1 + bx_2\ | \leq c$
- However, our trick will not work on absolute value constraints of the form $|\ ax_1 + bx_2\ | \geq c$
  - We can still transform this into two equivalent constraints, but the feasible region will no longer be convex, meaning we can't solve the LP by traditional methods
  - For example, try out $|\ x_1 - x_2\ | \geq 2$:
    - This is equivalent to $x_1 - x_2 \geq 2$ OR $-x_1 + x_2 \geq 2$, which can't be made linear:

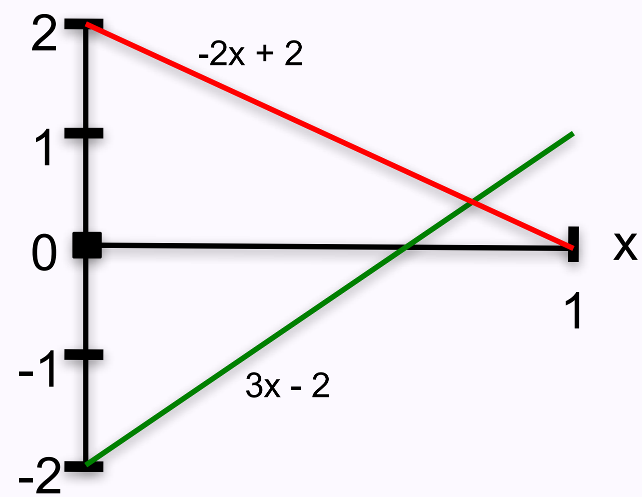


- The feasible region is in yellow, and it's in two separate pieces
- Recall from class that a linear programming feasible region is always connected, and convex In fact, it's always **convex**: if two points are feasible, then so is the line segment joining the two points.

Example 2

Maximin Problem (1 of 2)

- The next constraint we will make linear appears when we maximize the value of a **min function**, which looks like ***min* {3x - 2, -2x + 2}**



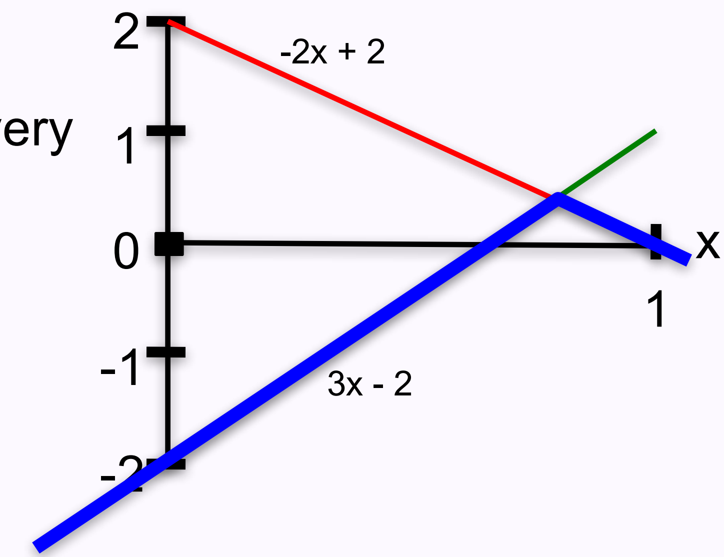
Problem Statement

- maximize $\quad$ *min* {3x - 2, -2x + 2}
  s.t. $\quad 0 \leq x \leq 4$

Task

- Turn this into an LP

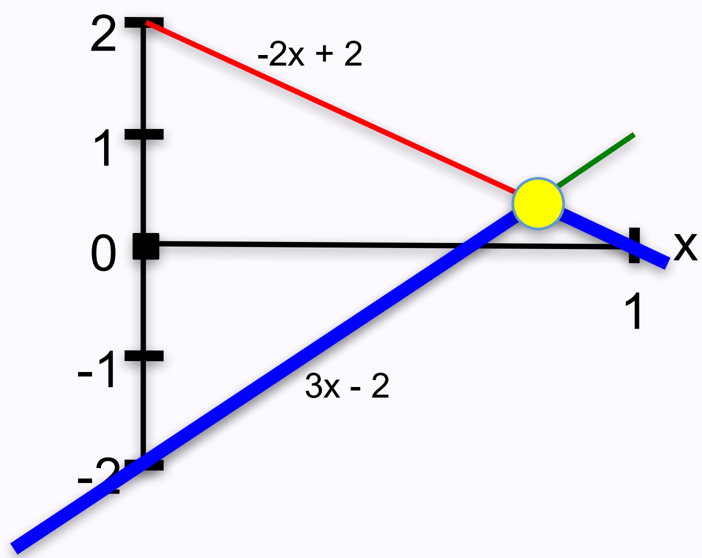
- To start, we can look at the graph of the problem shown above, and identify the minimum value of the two functions for every value of x (shown here in blue):



Example 2

Maximin Problem (2 of 2)

- Looking at these minimum values, we then find the maximum value of the blue line (shown as a yellow dot) to solve the problem
- The max of the min has a value of 2/5 at x = 4/5:



Example 3

Marketing Problem (1 of 2)

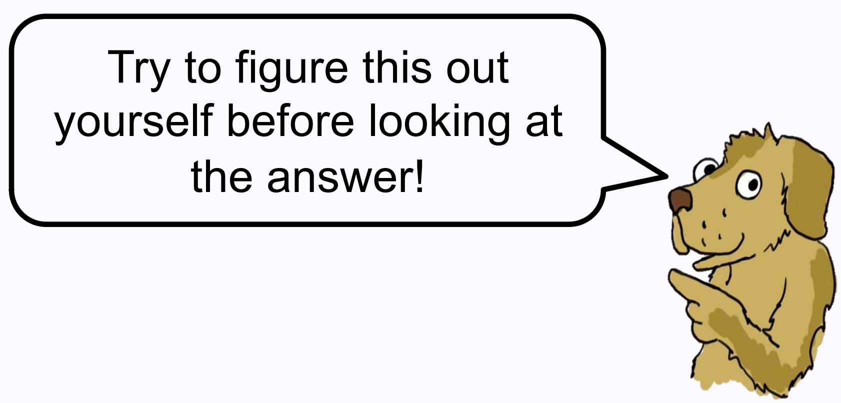
Problem Statement

- Begin with the same marketing problem as seen in Tutorial 02, but with a new objective function:
  - ***max*** *min* $\{50x_1, 25x_2, 20x_3, 15x_4\}$
    s.t. $50{,}000\, x_1 + 25{,}000\, x_2 + 20{,}000\, x_3 + 15{,}000\, x_4 \geq 1{,}500{,}000$
    $0 \leq x_1 \leq 20$
    $0 \leq x_2 \leq 15$
    $0 \leq x_3 \leq 10$
    $0 \leq x_4 \leq 15$

- The minimum of $\{50x_1, 25x_2, 50x_3, 15x_4\}$ is the smallest number of persons reached by one of the four media (in 1000's)

Task

- Formulate this as an LP whose solution will find the maximum value of z such that each medium reaches at least 1000z people



Example 3

Marketing Problem (2 of 2)

- Now, we want to maximize z subject to the constraint that z is at most the number of ads seen for each media
- z is upper bounded by each expression in the min function, which we can represent with the constraints:
  - $z \le 50x_1$
    $z \le 25x_2$
    $z \le 20x_3$
    $z \le 15x_4$
- Then, we can maximize z to get our final LP:
  - ***max*** z
    s.t. $50x_1 + 25x_2 + 20x_3 + 15x_4 \ge 1{,}500$
    $0 \le x_1 \le 20, \quad 0 \le x_2 \le 15, \quad 0 \le x_3 \le 10, \quad 0 \le x_4 \le 15$
    $z \le 50x_1, \quad z \le 25x_2, \quad z \le 20x_3, \quad z \le 15x_4$

- *Note:* This technique works whenever you need to maximize the minimum ("**maximin**") of linear functions. A similar trick works whenever you want to minimize the maximum ("**minimax**") of linear functions

Example 4

Ratio Constraints

Problem Statement

- Consider the Marketing Problem again
- Suppose that we wanted to add the **ratio constraint** that at least 20% of all ads had to be by mail:
  - $x_1 / (x_1 + x_2 + x_3 + x_4) \geq .2$

Task

- Formulate this as an LP

- The "trick" here is to multiply by the denominator so that the constraint becomes linear:
  - $x_1 \geq .2\,(x_1 + x_2 + x_3 + x_4)$
  - $\mathbf{.8x_1 - .2\,x_2 - .2\,x_3 - .2\,x_4 \geq 0}$
- *Note*: You can only do this if you know that the denominator's value is positive for all possible x
  - If you multiply both sides of an inequality by a negative number, the direction of the inequality reverses
  - The new constraint is valid if x = 0, so we don't need to consider that case

Summary on Non-linear Constraints

- Most of the time, if there is a constraint or objective that isn't linear, it *cannot* be transformed into a linear constraint or objective

- In the 3 cases presented above, transformations into LPs *can* be done:
  - Certain absolute value constraints
    - Remember to check if the transformation works by graphing the feasible region!
  - Maximizing the minimum or mimizing the maximum
  - Ratio constraints