Algorithms

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Algorithm

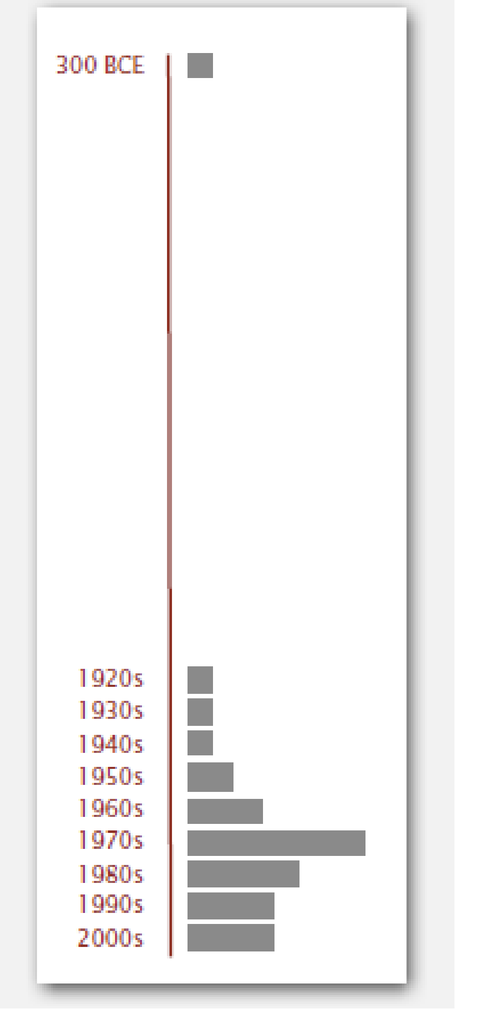
Finite sequence of instructions that

- Solve a problem
- Satisfy the following criteria
  - They receive input values
  - They produce output values
  - They are clear, unambiguous and executable
  - They terminate after a finite number of steps
- They work on data structures

Algorithm: from al-Huarizmi, Persian IX cent. DC mathematician

Why study algorithms?

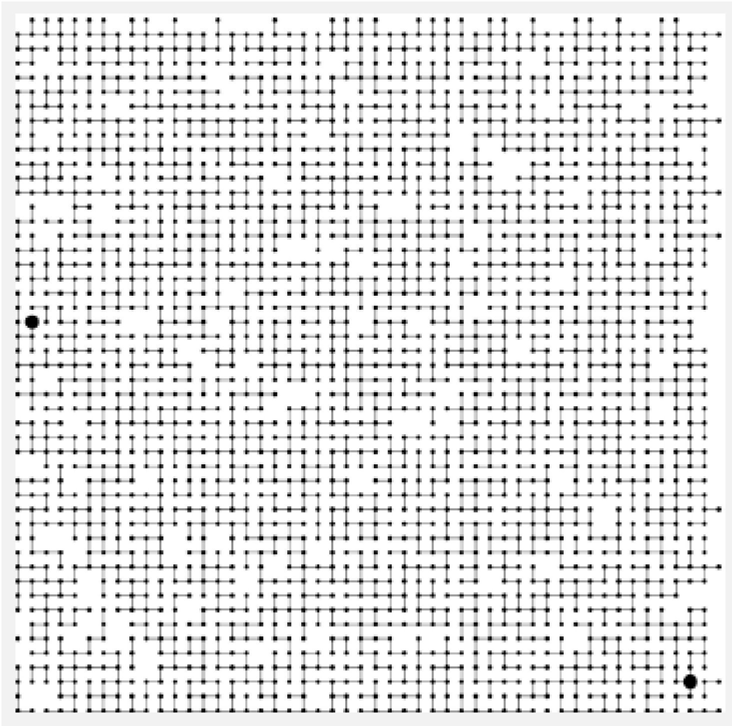
- Algorithms have ancient roots
  - Euclid (IV cent, B.C.)
  - Formalization by Church and Turing (XX cent., ’30s)
  - Recent developments



Why study algorithms?

- To do something otherwise impossible
  - Are the two dark dots connected in this network (network connectivity)?

Netword connectivity



Why study algorithms?

- To solve problems in many domains
  - Internet: Web search, packet routing, distributed file sharing
  - Biology: human genome
  - Computers: CAD tools, file systems, compilers
  - Graphics: virtual reality, videographics
  - Multimedia: MP3, JPG, DivX, HDTV
  - Social Networks: recommendations, news feed, advertisement
  - Security: e-commerce, cell phones
  - Physics: particle collision simulation …

Why study algorithms?

- For intellectual stimulation
- To become a proficient programmer

" I will, in fact, claim that the difference between a bad programmer and a good one is whether he considers his code or his data structures more important. Bad programmers worry about the code. Good programmers worry about data structures and their relationships. "

— Linus Torvalds (creator of Linux)

Why study algorithms?

- To unlock the secrets of life and of the universe by creating models
  - In many sciences computational models are replacing mathematical ones

$$E = mc^2$$

$$F = ma$$

$$F = \frac{G m_1 m_2}{r^2}$$

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \Psi(r) = E \, \Psi(r)$$

```
for (double t = 0.0; true; t = t + dt)
   for (int i = 0; i < N; i++)
   {
      bodies[i].resetForce();
      for (int j = 0; j < N; j++)
         if (i != j)
            bodies[i].addForce(bodies[j]);
   }
```

Mathematical formulae

Computational model

Why study algorithms?

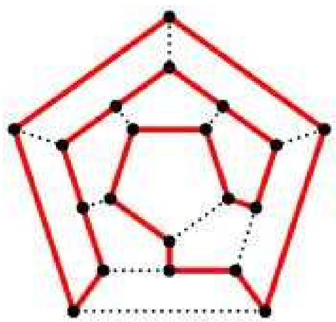
- For fun
- For money
- To increase speed
- To process more data
- To satisfy intellectual curiosity

Types of problems

- Decision problems
  - Problems with a yes/no answer
  - Examples
    - Given 2 integers x and y, does x exactly divide y?
    - Given a positive integer x, is it prime?
    - Given a positive integer n, do 2 positive and > 1 integers p and q exist such that n = pq?
    - Determine whether a number is prime

Types of problems

- Search problem
  - Does a valid solution exist and which one is it?
  - Examples
    - Hamiltonian cycle: given an undirected graph, does a simple cycle spanning all vertices exist? Which one is it?

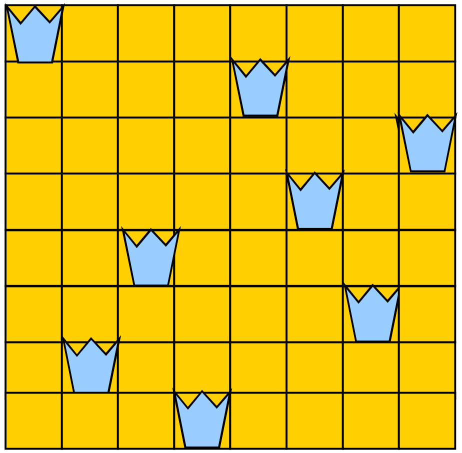


    - Which one is the k-th prime number
    - Given an array of integers, sort it in ascending order

Types of problems

- Verification problems
  - Given a solution (certificate), make sure that it is really one
  - Examples
    - Sudoku
    - The eight queen problem

8 queens



Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Types of problems

- Optimization problems
  - If a solution exists, which one is the best one?
  - Examples
    - Given a weighted directed graph, which is the shortest simple path, if it exists, between nodes i and j?

Decision Problems

- Decision problems may be
  - Decidable
    - There exists an algorithm that solves them
  - Undecidable
    - There is no algorithm that solves them
- Decidable decision problems may be
  - Tractable, i.e., solvable in “reasonable” time
  - Intractable, i.e. non solvable in “reasonable” time

The P Class

Decidable and tractable decision problems

$\Leftrightarrow$

$\exists$ a polynomial algorithm that solves them (Edmonds-Cook-Karp thesis, '70)

An algorithm is polynomial iff, working on n data, given a constant c>0, it terminates in a finite number of stemps upper-bounded by n^c

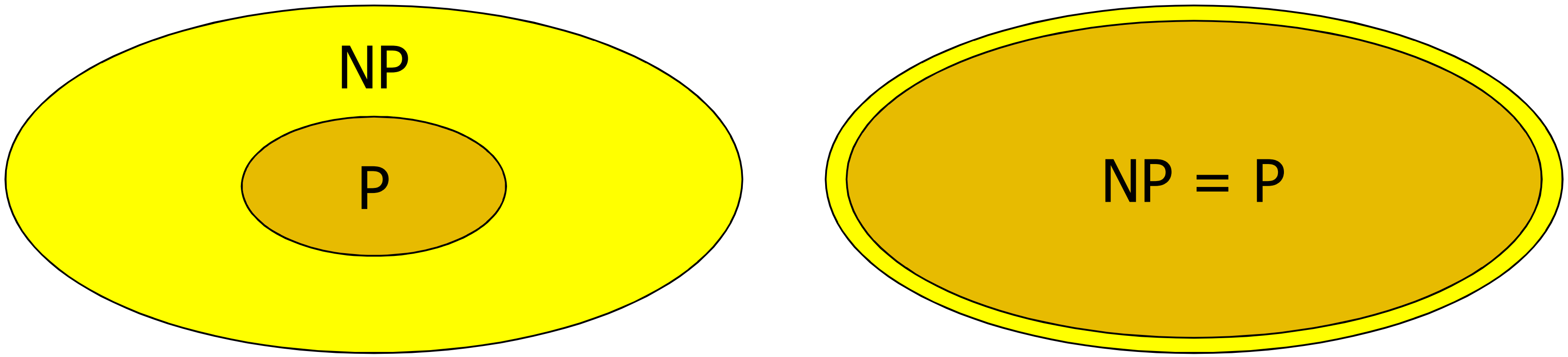
In practice c should not exceed 2

The NP Class

- NP stands for Non-deterministic Polynomial
- There exist decidable problems for which
  - we have exponential algorithms, but we don’t know any polynomial algorithms
  - However we can’t rule out the existence of polynomial algorithms
- We have polynomial **verification** algorithms, to check whether a solution (certificate) is really such
  - Sudoku, satisfyability of a boolean function

P versus NP

- $P \subseteq NP$, but we don't know whether P is a proper subset of NP or it coincides with NP. It is probable that P is a proper subset of NP



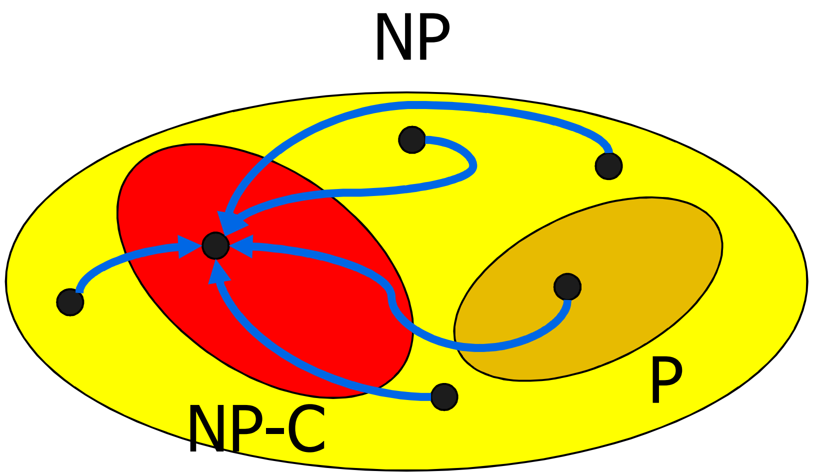
Probable

Improbable

The NP-C Class

A problem is NP-**complete** if

- It is NP
- Any other problem in NP may be reduced to it by means of a polynomial transformation



P versus NP versus NP-C

If we find a polynomial algorithm for any problem in this class, we could find polynomial algorithms for all NP problems, through transformations

HIGHLY IMPROBABLE!

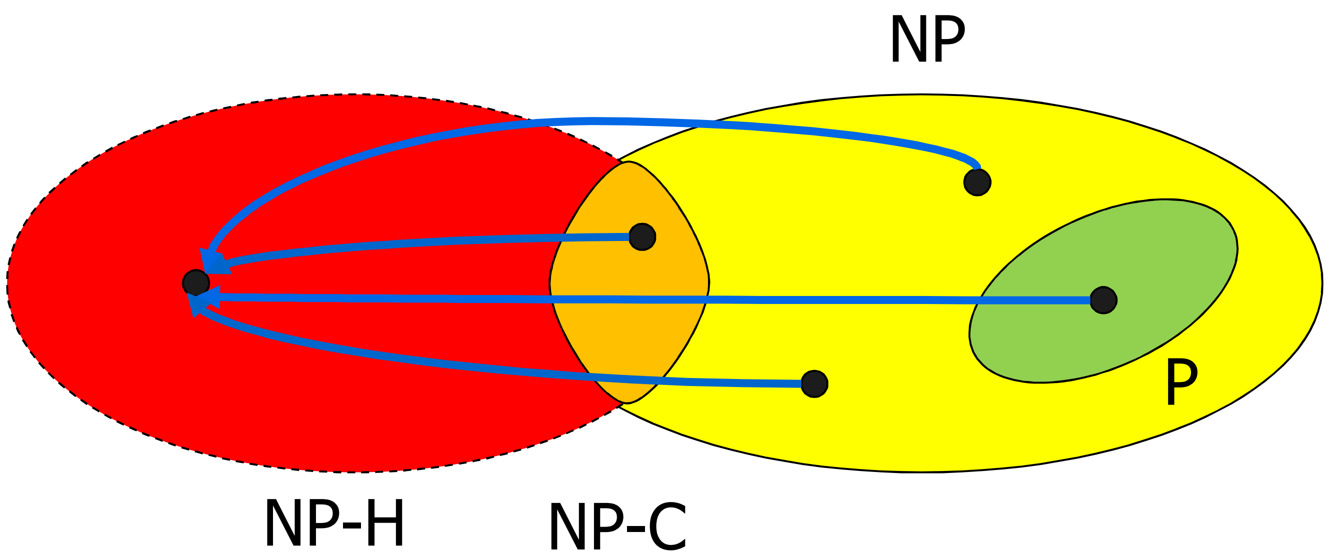
The existence of the NP-C class makes it probable that P $\subset$ NP

Example of NP-C problem: satisfyiability

given a Boolean function, find if there exists an assignment to the input variables such that the function is true.

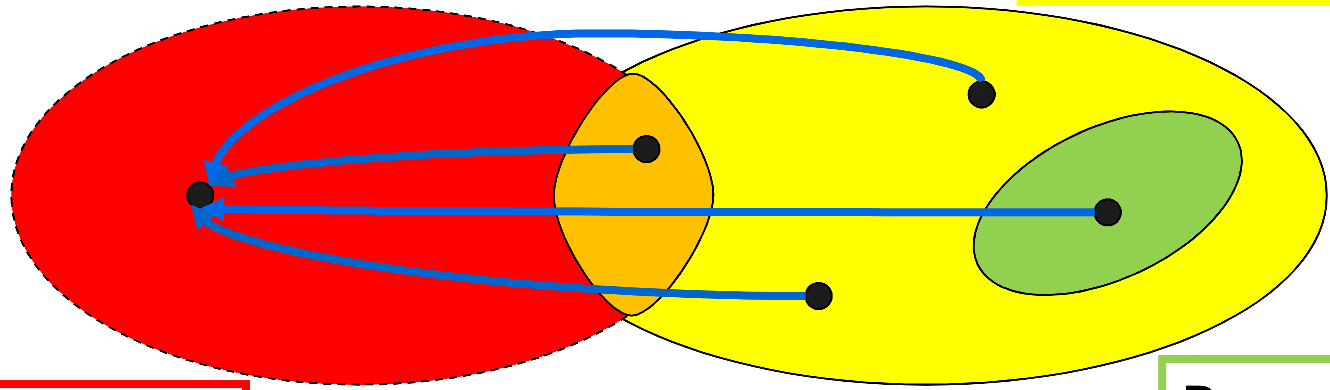
The NP-H Class

- A problem is NP-**hard** if every problem in NP may be reduced to it in polynomial time (even if it doesn't belong to NP)
- Any other problem in NP may be reduced to it by means of a polynomial transformation



The NP-H Class

NP:
- Factorization
- Graph isomorphism



NP-H:
- Permanent of a matrix

NP-C:
- Satisfyiability
- Hamilton Cycle
- Clique

P:
- Graph Connectivity
- Primality
- Determinant