Iterative sorting algorithms

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Insertion sort

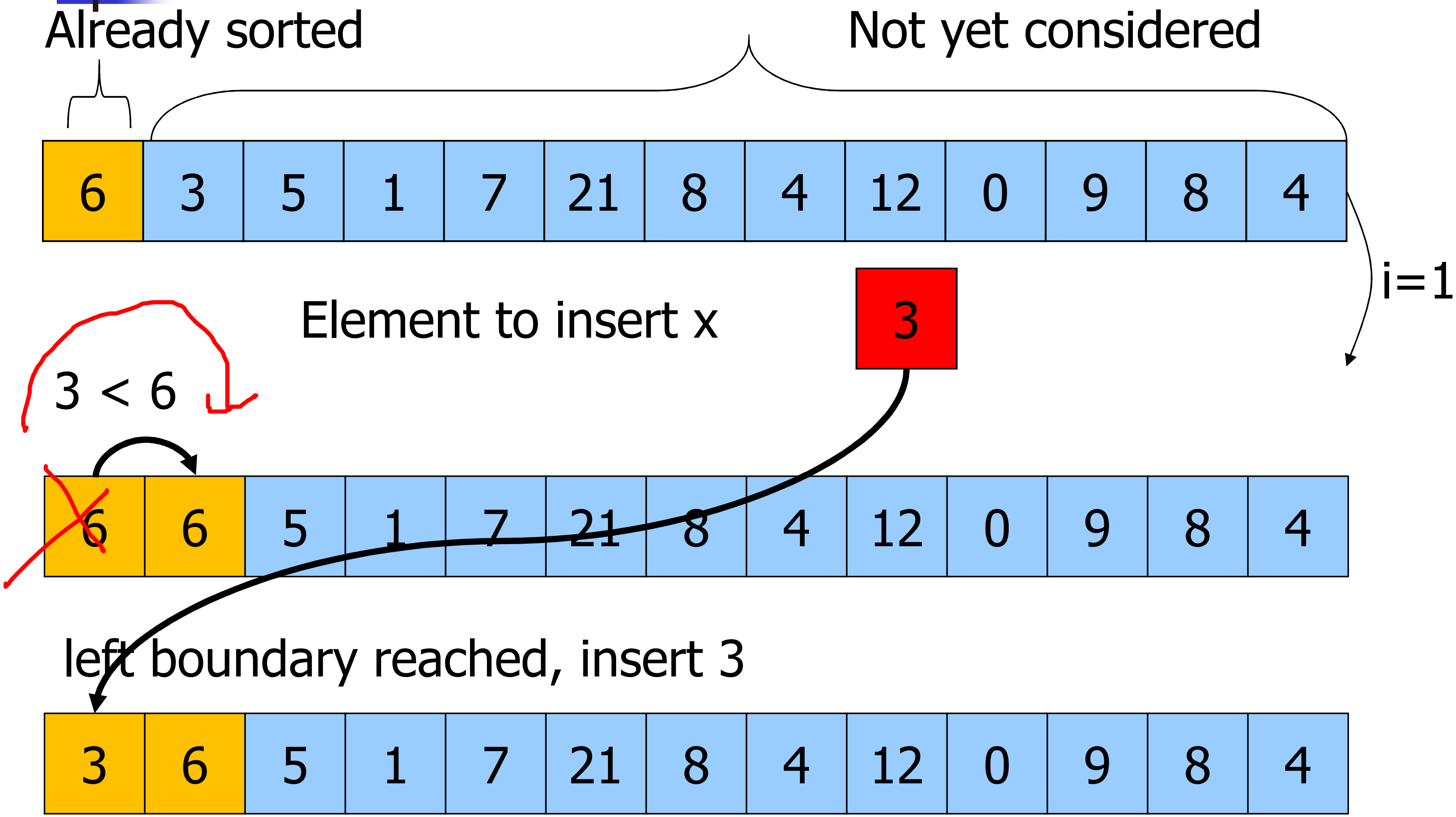
- Data: integers in array A
- Array partitioned in 2 sub-arrays
  - Left: sorted
  - Right: unsorted
- An array of just one element is sorted
- Incremental approach
  - At each step we expand the sorted sub-array by inserting one more element (invariance of the sorting property)
- Termination
  - All elements inserted in proper order

i-th step: sorted insertion

I-th step: put in the proper position $x = A_i$

- Scan the sorted subarray (from A_{i-1} to A_0) until we find $A_k > A_i$
- Right shift by one position of the elements from A_k to A_{i-1}
- Insert A_i in k-th position

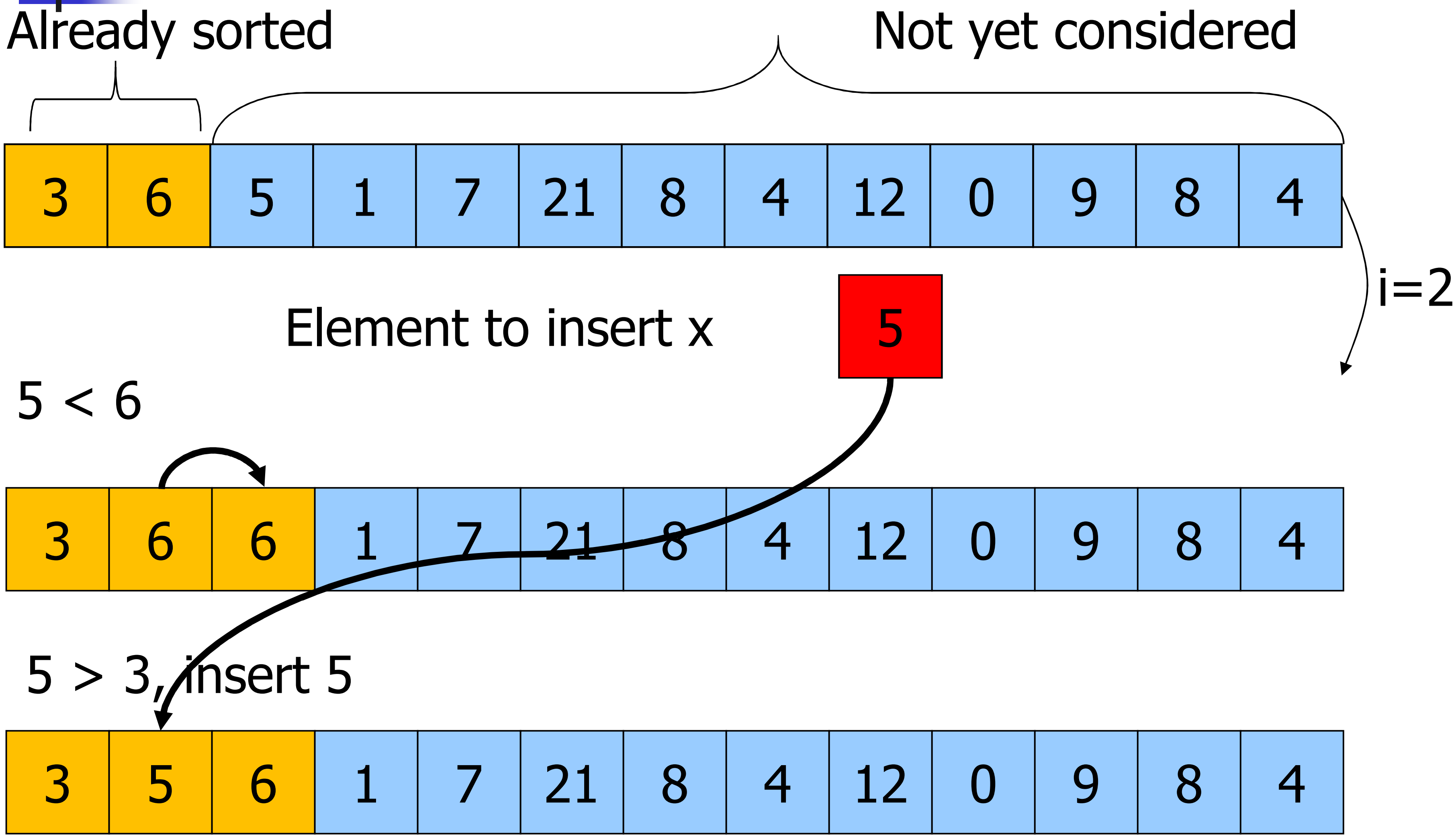
Example


Already sorted
Not yet considered
6 3 5 1 7 21 8 4 12 0 9 8 4
i=1
Element to insert x
3
3 < 6
6 6 5 1 7 21 8 4 12 0 9 8 4
left boundary reached, insert 3
3 6 5 1 7 21 8 4 12 0 9 8 4

Example

selection fast,done all things in insert phrase

contract to selction sort



Example

Already sorted

Not yet considered

3	5	6	1	7	21	8	4	12	0	9	8	4

i=3

Element to insert x

1

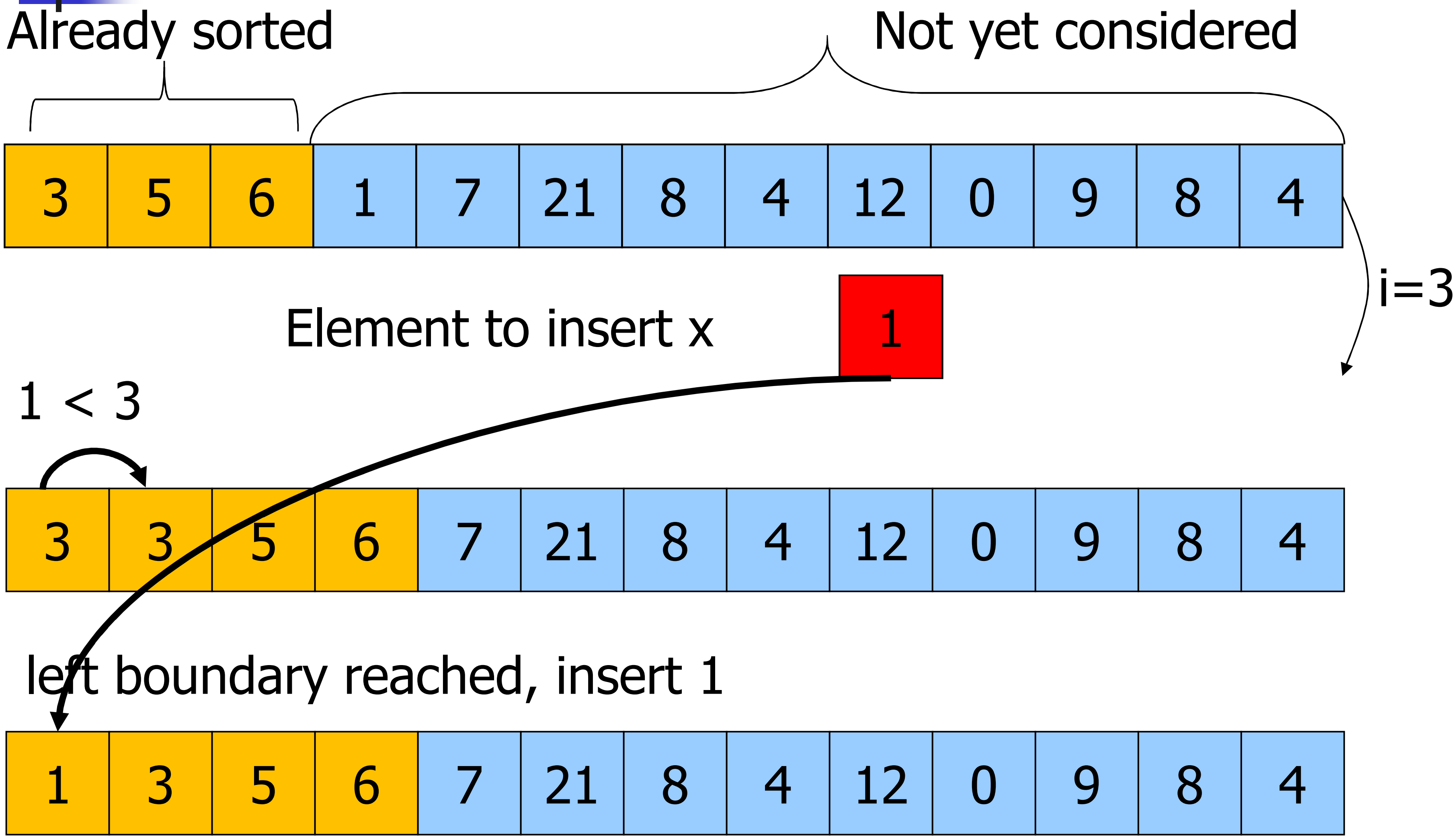
$1 < 6$

3	5	6	6	7	21	8	4	12	0	9	8	4

$1 < 5$

3	5	5	6	7	21	8	4	12	0	9	8	4

Example


Already sorted
Not yet considered
3 5 6 1 7 21 8 4 12 0 9 8 4
i=3
Element to insert x
1
1 < 3
3 3 5 6 7 21 8 4 12 0 9 8 4
left boundary reached, insert 1
1 3 5 6 7 21 8 4 12 0 9 8 4

many movements

C Code

size

```
void InsertionSort (int A[], int n) {
  int i, j, x;
```

i star from 1 bcz o is sorted

```
  for (i=1; i<n; i++) {
    x = A[i];
    j = i - 1;
    while (j>=0 && x<A[j]) {
      A[j+1] = A[j];
      j--;
    }
    A[j+1] = x;
  }

  return;
}
```

tmp

insertion phrase

~~It is also possibile to reason in terms of~~
~~l = index of the leftmost element~~
~~r = index of the rightmost element~~

Complexity Analysis

- Analytic analysis
  - Worst case
  - We assume unit cost for all operations

```
for (i=1; i<n; i++) {
    x = A[i];
    j = i - 1;
    while (j>=0 && x<A[j]){
      A[j+1] = A[j];
      j--;
    }
    A[j+1] = x;
}
```

#checks #iterations

`for (i=1; i<n; i++) {`	1 + (n-1+1) + (n-1)
`x = A[i];`	n - 1
`j = i - 1;`	n - 1
`while (j>=0 && x<A[j]){`	$\sum_{i=1}^{n-1}(i+1)$
`A[j+1] = A[j];`	$\sum_{i=1}^{n-1} i$
`j--;`	$\sum_{i=1}^{n-1} i$
`}`	
`A[j+1] = x;`	n - 1
`}`	

Complexity Analysis

$1 + (n-1+1) + (n-1)$
$n - 1$
$n - 1$
$\sum_{i=1}^{n-1}(i+1)$
$\sum_{i=1}^{n-1} i$
$\sum_{i=1}^{n-1} i$
$n - 1$

Recalling that
$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$
$\sum_{i=1}^{n-1}(i+1) = \frac{n(n+1)}{2} - 1$

$T(n) =$

$= 2n+(n-1)+(n-1)+\sum_{i=1}^{n-1}(i+1)+\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i+(n-1)$

$= 2n + 3(n-1) + \frac{n(n+1)}{2} - 1 + 2\frac{(n-1)n}{2}$

$= 2n + 3n - 3 + \frac{1}{2}n^2 + \frac{1}{2}n - 1 + n^2 - n$

$= \frac{3}{2}n^2 + \frac{9}{2}n - 4 = \Theta(n^2)$

T(n) grows quadratically

Worst case
$O(n^2)$ overall

Complexity Analysis

- More intuitive analysis
  - Two nested cycles
  - Outer loop: n-1 executions
  - Inner loop: the worst-case → i executions at the i-th iteration of the outer loop
- Complexity
  - $T(N) = 1 + 2 + 3 + 4 + \cdots + (n-2) + (n-1)$
  - $T(N) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$

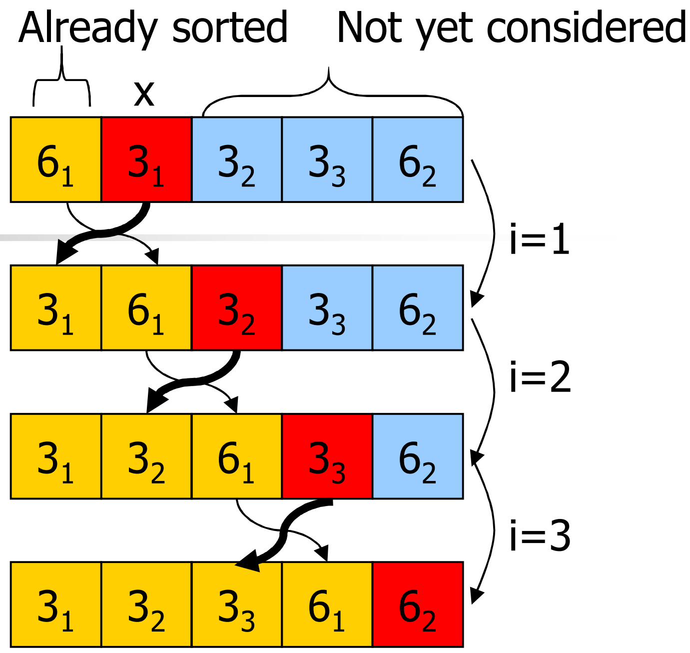
T(n) grows quadratically with n

Complexity Analysis

already sorted array

- Best case scenario
  - Inner loop: 1 execution at the i-the iteration of the outer loop
  - Complexity
  - $T(n) = \underbrace{1 + 1 + 1 + \dots\dots\dots + 1}_{\text{n-1 times}} = n-1$
  - T(n) grows linearly with n

Features

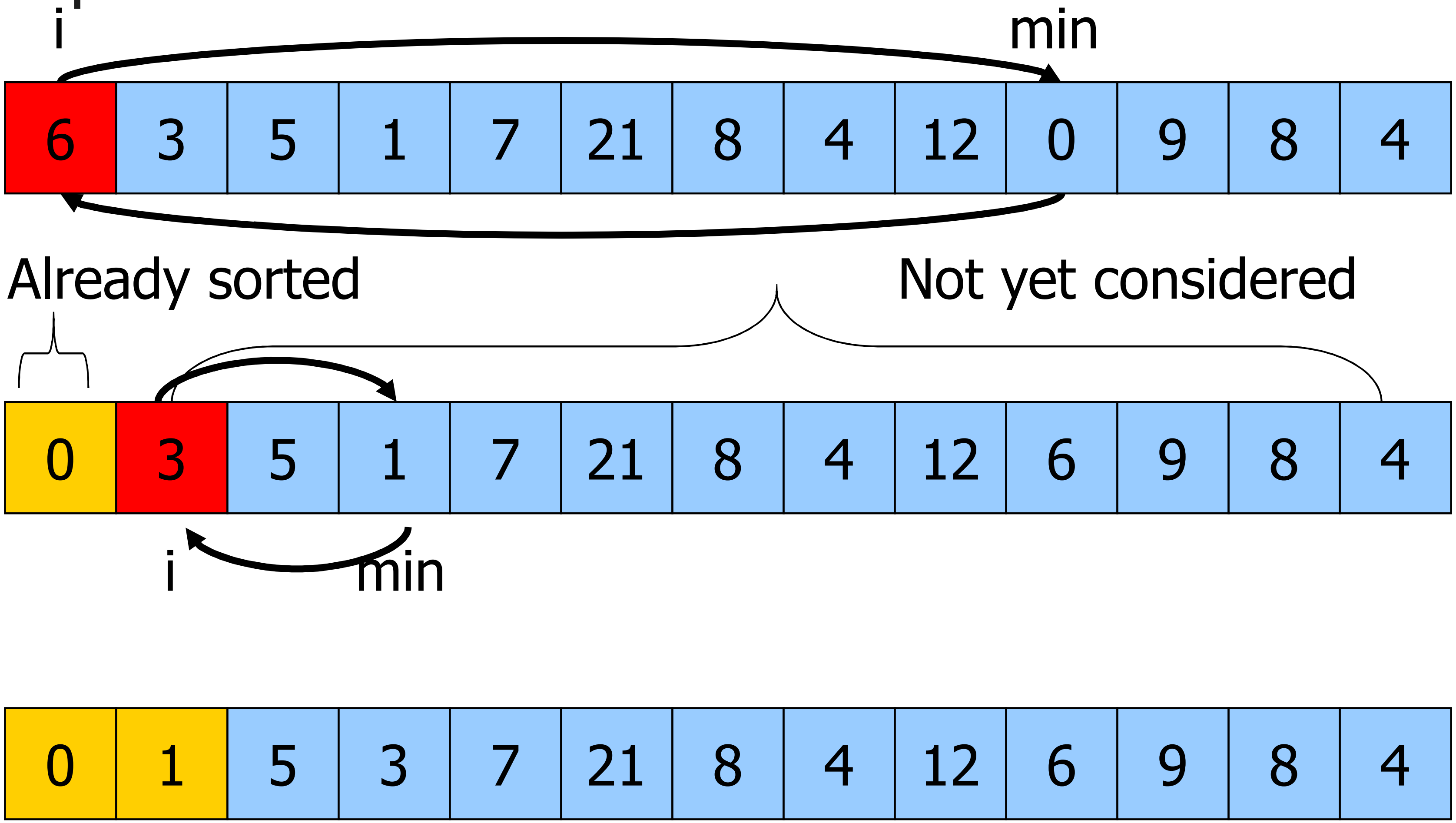


- In place
- Number of exchanges in worst-case
  - $O(n^2)$
- Number of comparisons in worst-case
  - $O(n^2)$
- Stable
  - If the element to insert is a duplicate key, it can't pass over to the left a preceeding occurrence of the same key

Selection sort

- Sort n integers in array A
- Array divided into two sub-arrays
  - Left: sorted, initially empty
  - Right: unsorted, initially it coincides with A
- Incremental approach
  - Iteration i: the minimum of the right sub-array (A_i ... A_r) is assigned to a A[i]; increment i
- Termination: all elements are inserted in the correct location
- Searching for the minimum in the right sub-array entails scanning the sub-array

Example



i
min
6 3 5 1 7 21 8 4 12 0 9 8 4
Already sorted
Not yet considered
0 3 5 1 7 21 8 4 12 6 9 8 4
i
min
0 1 5 3 7 21 8 4 12 6 9 8 4

C Code

```
void SelectionSort (int A[], int n) {
   int i, j, min, temp;
                    n-2
   for (i=0; i<n-1; i++) {
temp  min = i;
      for (j=i+1; j<n; j++) {                 select
         if (A[j] < A[min]) {
            min = j;
         }
      }
      temp = A[i];                       swap
      A[i] = A[min];
      A[min] = temp;
   }

   return;
}
```

Complexity Analysis

- Analytic analysis
  - Worst case
  - We assume unit cost for all operations

```
for (i=0; i<n-1; i++) {
     min = i;
     for (j=i+1; j<n; j++) {
        if (A[j] < A[min]) {
            min = j;
        }
     }
     temp = A[i];
     A[i] = A[min];
     A[min] = temp;
}
```

Number of operations

#checks #iterations

1 + (n-1+1) + (n-1)

n-1

$\sum_{i=0}^{n-2}(1 + (n - (i + 1) + 1) + (n - (i + 1))$

$\sum_{i=0}^{n-2}(n - (i + 1))$

$\sum_{i=0}^{n-2}(n - (i + 1))$

n-1

n-1

n-1

Complexity Analysis

```
1 + (n-1+1) + (n-1)
n-1
```

$\sum_{i=0}^{n-2}(1 + (n - (i + 1) + 1) + (n - (i + 1))$

$\sum_{i=0}^{n-2}(n - (i + 1))$

$\sum_{i=0}^{n-2}(n - (i + 1))$

```
n-1
n-1
n-1
```

Recalling that
$\sum_{i=0}^{n-2} n = \mathrm{n}(\mathrm{n}-1)$
$\sum_{i=0}^{n-2} i = \frac{(n-2)(n-1)}{2}$
$\sum_{i=0}^{n-2} 1$ =n-1

T(n) =

$= 2n + 4(n\text{-}1) + 2\sum_{i=0}^{n-2}(n - i) + 2\sum_{\mathrm{i}=0}^{n-2}(n - i - 1)$

$= 6n - 4 + 4\sum_{i=0}^{n-2} n - 4\sum_{i=0}^{n-2} i - 2\sum_{\mathrm{i}=0}^{n-2} 1$

$= 6n - 4 + 4n(n-1) - 4\frac{(n-2)(n-1)}{2} - 2(n-1)$

$= 6n - 4 + 4n^2 - 4n - 2n^2 + 6n - 4 - 2n + 2$

$= 2n^2 + 6n - 6 = \Theta(n^2)$

T(n) grows quadratically

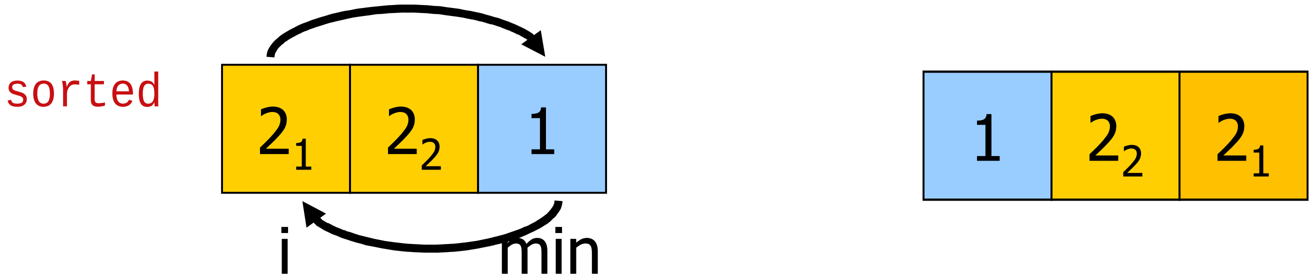
Worst case
$O(n^2)$ overall

Worst-case asymptotic analysis

- More intuitive analysis
  - Two nested loops
  - Outer loop: executed n-1 times
  - Inner loop: at the i-th iteration executed n-i-1 times
    - $T(n) = (n-1) + (n-2) + \ldots 2 + 1 = O(n^2)$
- Number of exchanges in worst-case $O(n)$
- Number of comparisons in worst-case $O(n^2)$

Features

- In place
- Not stable
  - A swap of "far away" elements may result in a duplicate key passing over to the left of a preceding instance of the same key



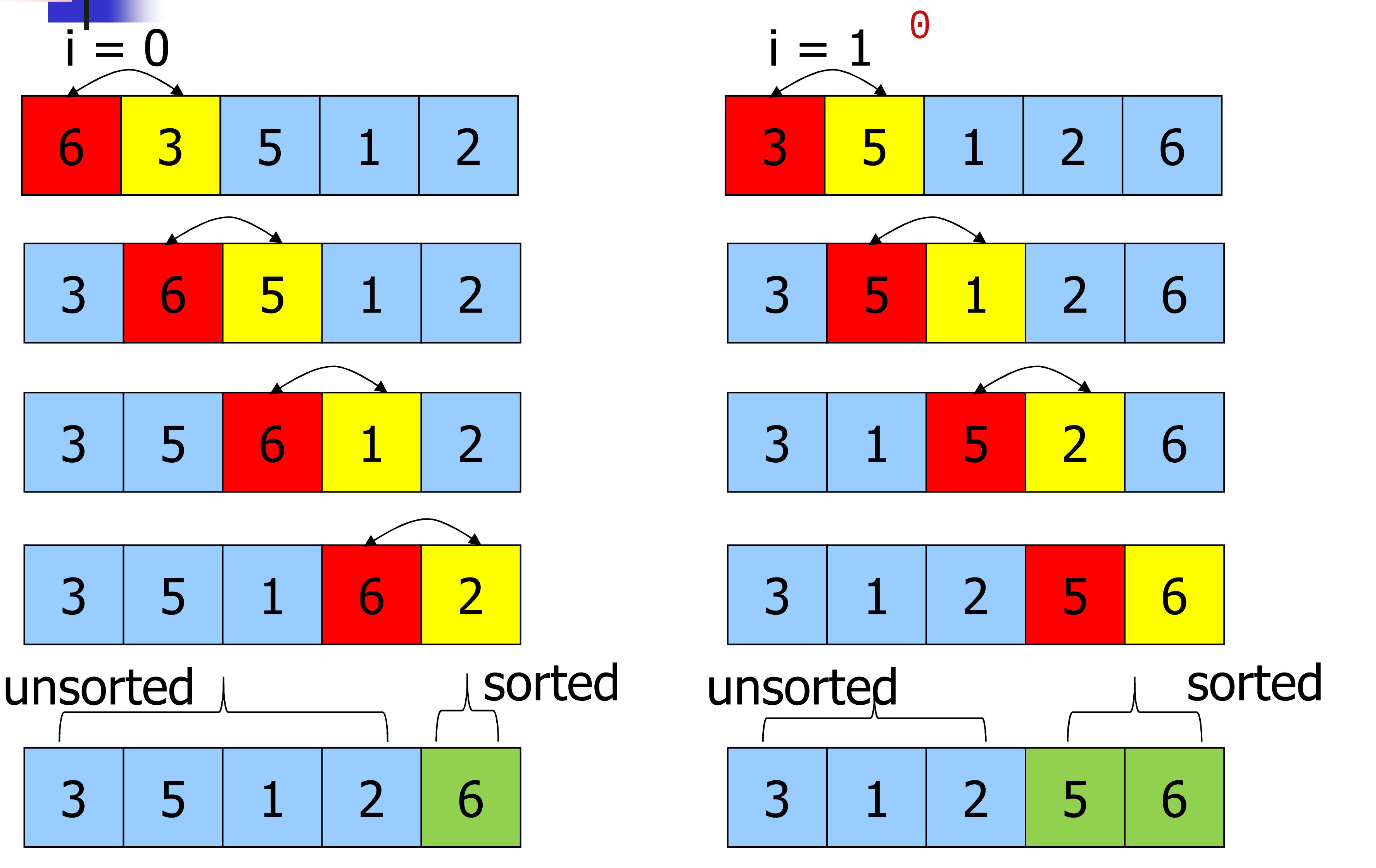
Exchange (Bubble) Sort

- Data: integers in array A delimited by left and right indices l and r
- Array divided in 2 sub-arrays
  - Right : sorted, initially empty
  - Left: unsorted, initially it coincides with A
- Elementary operation
  - Compare successive elements of the array A[j] and A[j+1], swap if A[j] > A[j+1]

Exchange (Bubble) Sort

- Incremental approach
  - At iteration i the maximum of the left sub-array (A_l ... A_{r-i+l}) is assigned to A[r-i+l]; increment i
  - The sorted right sub-array increases in size by 1 to the left, dually the left sub-array decreases in size by 1
- Termination: all elements are inserted in the correct location
- Possible optimization: flag to record that there have been swaps, early loop exit

Example


i = 0
6 3 5 1 2
3 6 5 1 2
3 5 6 1 2
3 5 1 6 2
unsorted
sorted
3 5 1 2 6
i = 1
0
3 5 1 2 6
3 5 1 2 6
3 1 5 2 6
3 1 2 5 6
unsorted
sorted
3 1 2 5 6

C Code

```
void BubbleSort (int A[], int n){
  int i, j, temp;

  for (i=0; i<n-1; i++) {
    for (j=0; j<n-i-1; j++) {
      if (A[j] > A[j+1])) {
        temp = A[j];
        A[j] = A[j+1];
        A[j+1] = temp;
      }
    }
  }
  return;
}
```

i already sorted element

i-l is the number of already sorted elements

Complexity

- Analytic analysis
  - Worst case
  - We assume unit cost for all operations

opposite order

```
for (i=0; i<n-1; i++) {
    for (j=0; j<n-i-1; j++) {
      if (A[j] > A[j+1])) {
        temp = A[j];
        A[j] = A[j+1];
        A[j+1] = temp;
      }
    }
}
```

1 + (n-1+1) + (n-1)

$\sum_{i=0}^{n-2}(1+(n-i-1+1)+(n-i-1))$

$\sum_{i=0}^{n-2}(n-i-1)$

$\sum_{i=0}^{n-2}(n-i-1)$

$\sum_{i=0}^{n-2}(n-i-1)$

$\sum_{i=0}^{n-2}(n-i-1)$

Complexity

```
1 + (n-1+1) + (n-1)
```

$\sum_{i=0}^{n-2}(1+(n-i-1+1)+(n-i-1))$

$\sum_{i=0}^{n-2}(n-i-1)$

$\sum_{i=0}^{n-2}(n-i-1)$

$\sum_{i=0}^{n-2}(n-i-1)$

$\sum_{i=0}^{n-2}(n-i-1)$

T(n) =

$= 2n + \sum_{i=0}^{n-2} 1 + \sum_{i=0}^{n-2}(n-i) + 5\sum_{i=0}^{n-2}(n-i-1)$

$= 2n + \sum_{i=0}^{n-2} 1 + \sum_{i=0}^{n-2} n - \sum_{i=0}^{n-2} i + 5\sum_{i=0}^{n-2} n - 5\sum_{i=0}^{n-2} i - 5\sum_{i=0}^{n-2} 1$

$= 2n + 6\sum_{i=0}^{n-2} n - 6\sum_{i=0}^{n-2} i - 4\sum_{i=0}^{n-2} 1$

$= 2n + 6n(n-1) - 6\frac{(n-2)(n-1)}{2} - 4(n-1)$

$= 3n^2 + n - 2$

T(n) grows quadratically

Worst case $O(n^2)$ overall

Complexity

- More intuitive analysis
  - Two nested loops
  - Outer loop
    - Executed n-1 times
  - Inner loop
    - At the i-th iteration executed n-1-i times
  - $T(n) = (n-1) + (n-2) + \ldots + 2 + 1$

    $= O(n^2)$

Optimized C Code

```
void OptBubbleSort (int A[], int n) {
  int i, j, flag, temp;

  flag = 1;
  for(i=0; i<n-1 && flag==1; i++) {
    flag = 0;
    for (j=0; j<n-i; j++)
      if (A[j] > A[j+1]) {
        flag = 1;
        temp = A[j];
        A[j] = A[j+1];
        A[j+1] = temp;
    }
  }
  return;
}
```

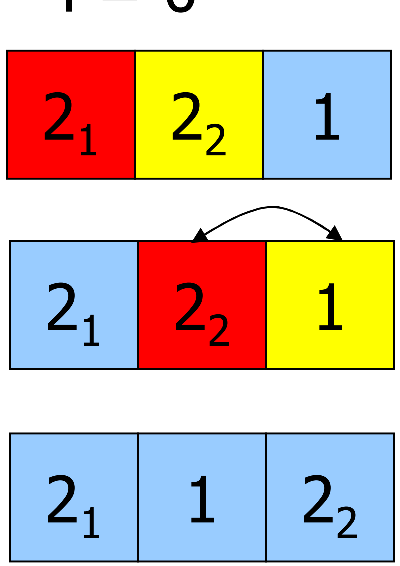
only swap something set flag=1

Features

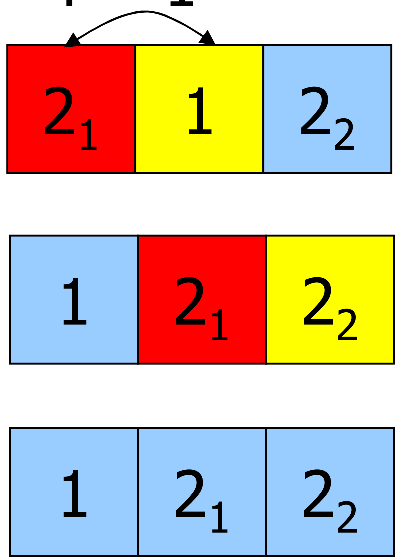
- In place
- Stable
  - Among several duplicate keys, the rightmost one takes the rightmost position and no other identical key ever moves past it to the right

compare with sorted so
stable

i = 0



i = 1



Shellsort (Shell, 1959)

- Limit of insertion sort
  - Comparison, thus exchange takes place only between adjacent elements
- Rationale of Shellsort

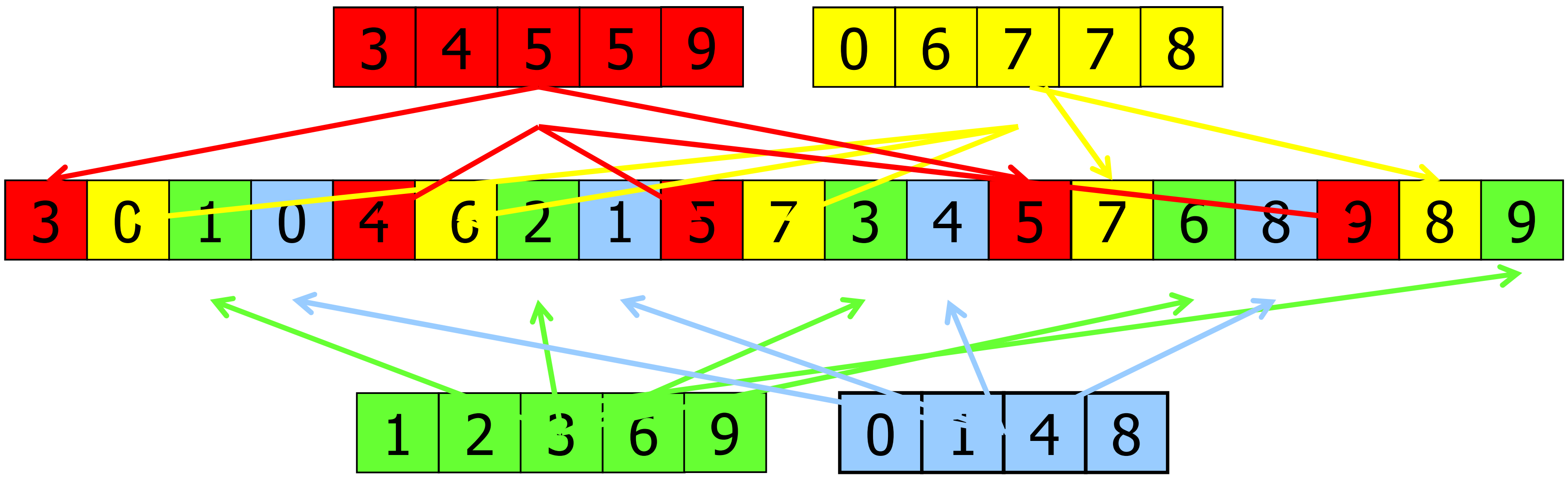
  insertion sort
  - Compare, thus possibly exchange, elements at distance h
  - Defining a decreasing sequence of integers ending with 1

Shellsort (Shell, 1959)

- An array formed by non contiguous sequences composed by elements whose distance is h is h-sorted

move faster

- Example
  - Sorted non contiguous subsequences with h=4



Shellsort (Shell, 1959)

- For each of the subsequences we apply insertion sort
- The elements of the subsequence are those at distance h from the current one

```
for (i=h; i<n; i++) {
  int j = i, v = A[i];
  while (j>=h && v<A[j-h])) {
    A[j] = A[j-h];
    j -=h;
  }
  A[j] = v;
}
```

1 -> h

Example

5	8	9	0	4	6	3	1	3	7	6	4	5	7	1	8	9	0	2

Sequence h: 13, 4, 1

Step1: h=13

5	1	8	0	0	2	3	1	3	7	6	4	5	7	8	9	9	4	6

Step 2: h=4

0	1	3	0	3	2	6	1	5	4	6	4	5	4	8	9	9	7	8

Step 3: h=1

0	0	1	1	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9

Choosing the sequence

h

- Has an impact on performance
- Knuth’s sequence
  - $h = 3 \cdot h + 1$ = 1 4 13 40 121 …
- Sequence
  - h = 1 then $4^{i+1} + 3 \cdot 2^{i} + 1$ = 1 8 23 77 281 1073 …
- Sedgewick’s sequence
  - h = 1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, …

C Code

largest h

insertion sort

```
void ShellSort(int A[], int n) {
  int i, j, temp, h;
  h=1;
  while (h < n/3)
    h = 3*h+1;
  while (h >= 1) {
    for (i=h; i<n; i++) {
      j = i;
      temp = A[i];
      while (j>=h  && temp<A[j-h]) {
        A[j] = A[j-h];
        j -=h;
      }
      A[j] = temp;
    }
    h = h/3;
  }
}
```

last step origin sort

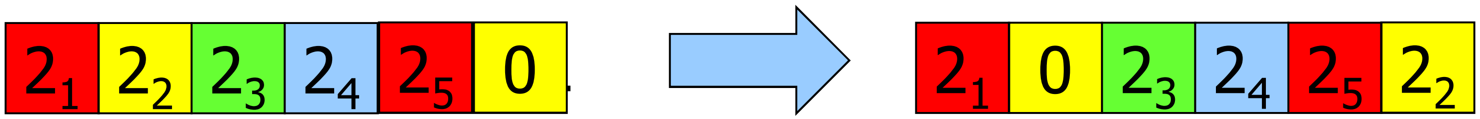
move faster than origin

Features

- In place
- Not stable
  - An exchange between "far away" elements may result in a duplicate key that passes over to the left a preceding occurrence of the same key

Example

2_1	2_2	2_3	2_4	2_5	0



- Step 1: h=4

0	2_1	2_3	2_4	2_5	2_2

- Step 2: h=1

Worst-case asymptotic analysis

Shellsort

- With Knuth's sequence:
  - 1 4 13 40 121 …
  - It executes less than $O(n^{3/2})$ comparisons
- With the sequence
  - 1 8 23 77 281 1073 …
  - It executes less than $O(n^{4/3})$ comparisons
- With Shell's original sequence
  - 1 2 4 8 16 …
  - It may degenerate to $O(n^2)$

Counting sort

- Sorting based on computation (not on comparison)
  - Find, for each element to sort x, how many elements are less than or equal to x
  - Assigne x directly to its final location
- Features
  - Stable
  - Not in place

Data Structures

- 3 arrays
  - Starting array
    - A[0..n-1] of n integers
  - Resulting array
    - B [0..n-1] of n integers
  - Occurrence array
    - C of k integers **if** data belong to the range [0..k-1]

Algorithm

```
c[i]={0};
```

- Step 1: simple occurrences
  - C[i] = number of elements of A equal to i
- Step 2: multiple occurrences
  - C[i] = number of elements of A <= i
- Step3: $\forall$ j
  - C[A[j]] = number of elements <= A[j]
- Thus final location of A[j] in B

B[C[A[j]]] = A[j]

(beware of indices in C, see code!)

Example (n=8, k=6)

A — Array to sort

0	1	2	3	4	5	6	7
2	5	3	0	2	3	0	3

C — Simple occurrences

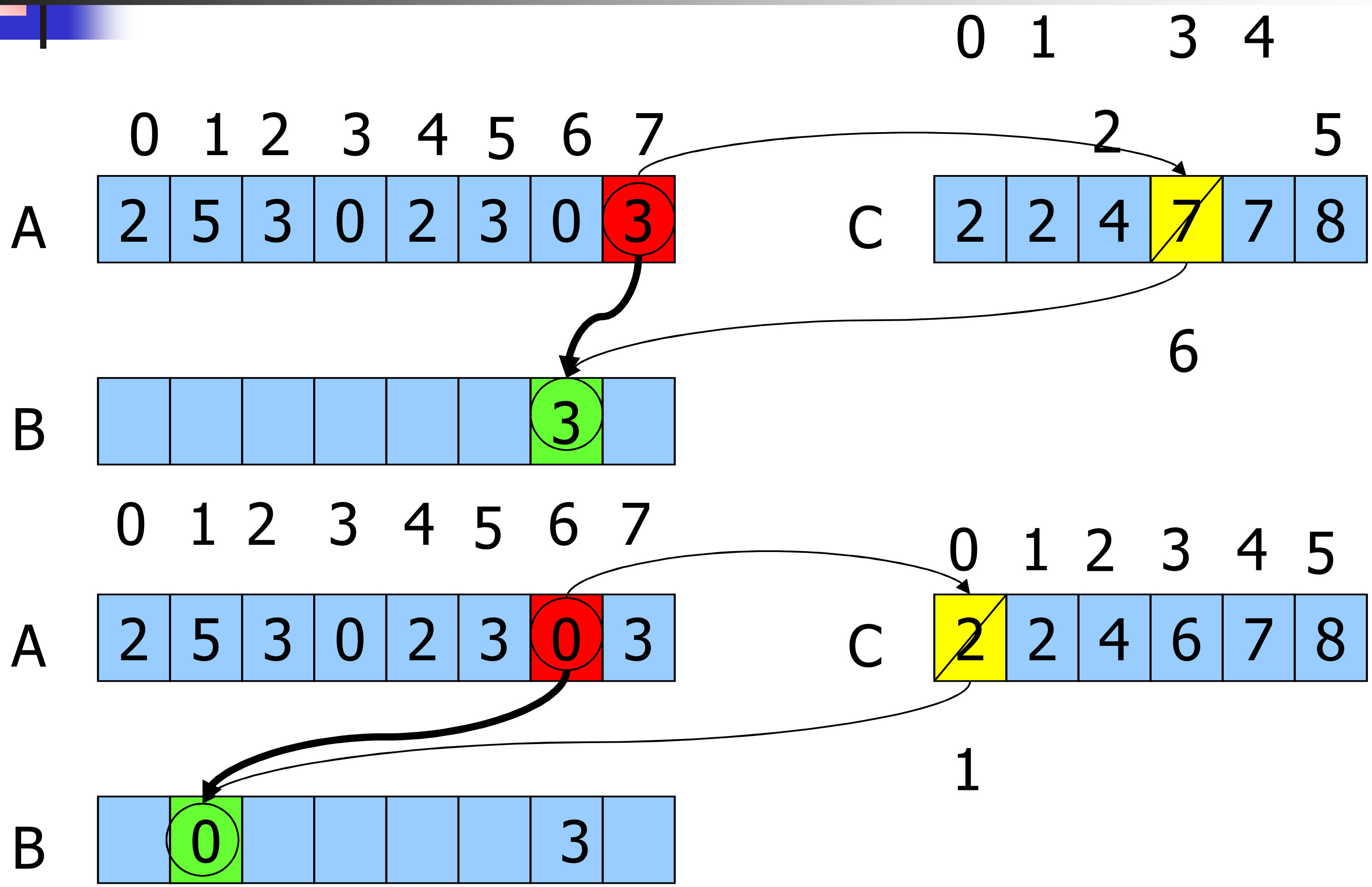
0	1	2	3	4	5
2	0	2	3	0	1

C — Multiple occurrences

0	1	2	3	4	5
2	2	4	7	7	8

`sum(A[i+1])`

Example (n=8, k=6)



Example (n=8, k=6)

```
#define MAX 100
```

```
void CountingSort(int A[], int n, int k) {
  int i, C[MAX], B[MAX];

  for (i=0; i<k; i++)
    C[i] = 0;
  for (i=0; i<n; i++)
    C[A[i]]++;
  for (i=1; i<k; i++)
    C[i] += C[i-1];
  for (i=n-1; i>=0; i--) {
    B[C[A[i]]-1] = A[i];
    C[A[i]]--;
  }
  for (i=0; i<n; i++)
    A[i] = B[i];
}
```

Worst-case asymptotic analysis

- Initialization loop for C: O(k)
- Loop to compute simple occurrences: O(n)
- Loop to compute multiple occurrences: O(k)
- Loop to copy result in B: O(n)
- Loop to copy in A: O(n)

$$T(n) = O(n+k)$$

Applicability

waste memory

- k=O(n), thus T(n) = O(n)