Iterative sorting algorithms

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Insertion sort

Sorting

- Input
  - Symbols $<a_1, a_2, \ldots, a_n>$ belonging to a set having an order relation $\leq$
- Output
  - Permutation $<a'_1, a'_2, \ldots, a'_n>$ of the input for which the order relation holds $a'_1 \leq a'_2 \leq \ldots \leq a'_n$

Insertion sort

Order relation $\leq$

Binary relation between elements of a set A satisfying the following properties

- reflexivity $\forall\ x \in A\ x \leq x$
- antisymmetry $\forall\ x, y \in A\ \ x \leq y \wedge y \leq x \Rightarrow x = y$
- transitivity $\forall\ x, y, z \in A\ \ x \leq y \wedge y \leq z \Rightarrow x \leq z$

A is a partially ordered set (poset)

If relation $\leq$ holds $\forall\ x, y \in A$, A is totally ordered set

Insertion sort

Examples of order relations $\leq$

- (total) relation $\leq$ on natural, relative, rational and real numbers (sets N, Z, Q, R)
- (partial) relation: divisibility on natural numbers, excluding 0

Approach

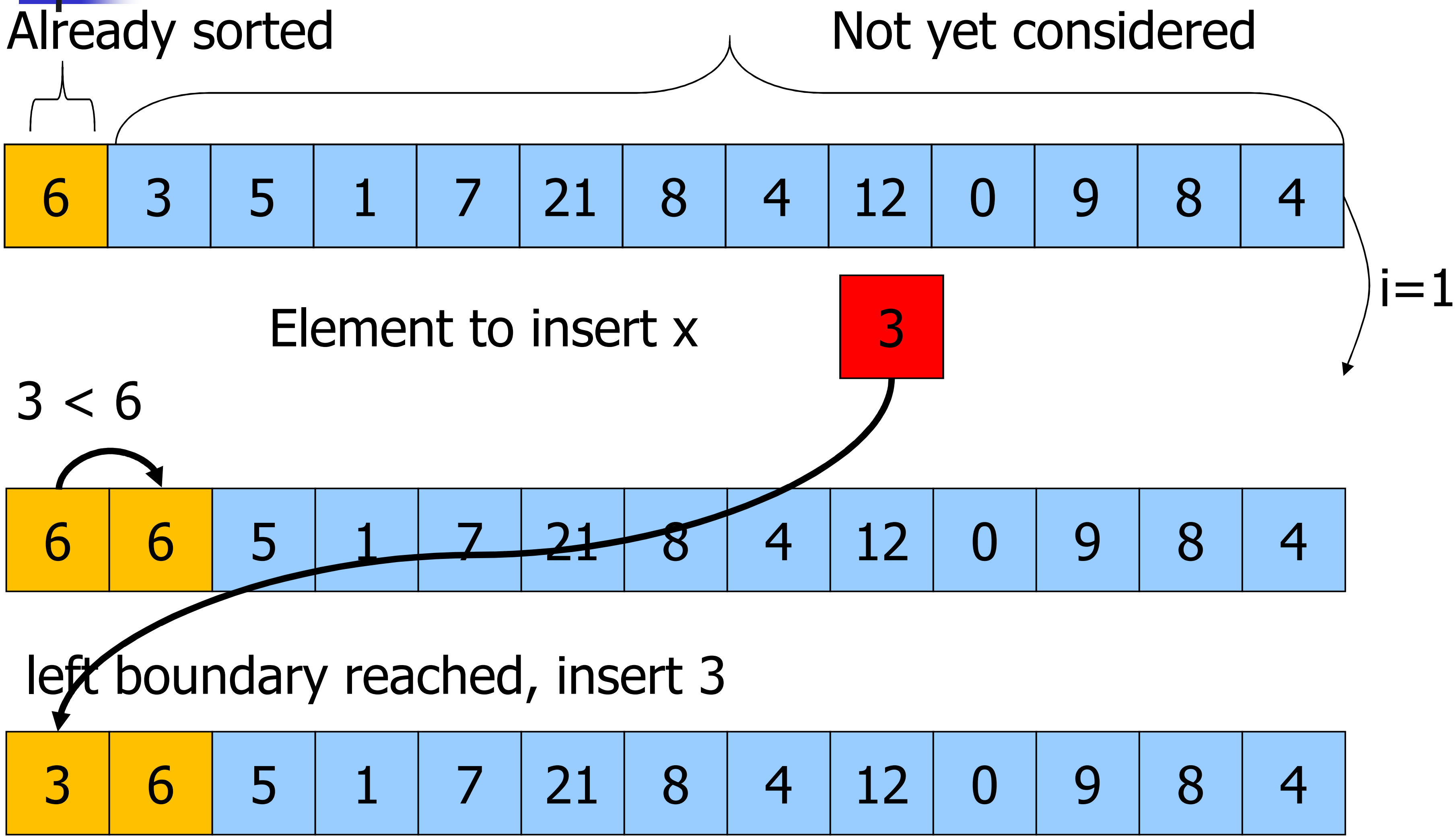
- Data: integers in array A
- Array partitioned in 2 sub-arrays
  - Left: sorted
  - Right: unsorted
- An array of just one element is sorted
- Incremental approach: at each step we expand the sorted sub-array by inserting one more element (invariance of the sorting property)
- Termination: all elements inserted in proper order

i-th step: sorted insertion

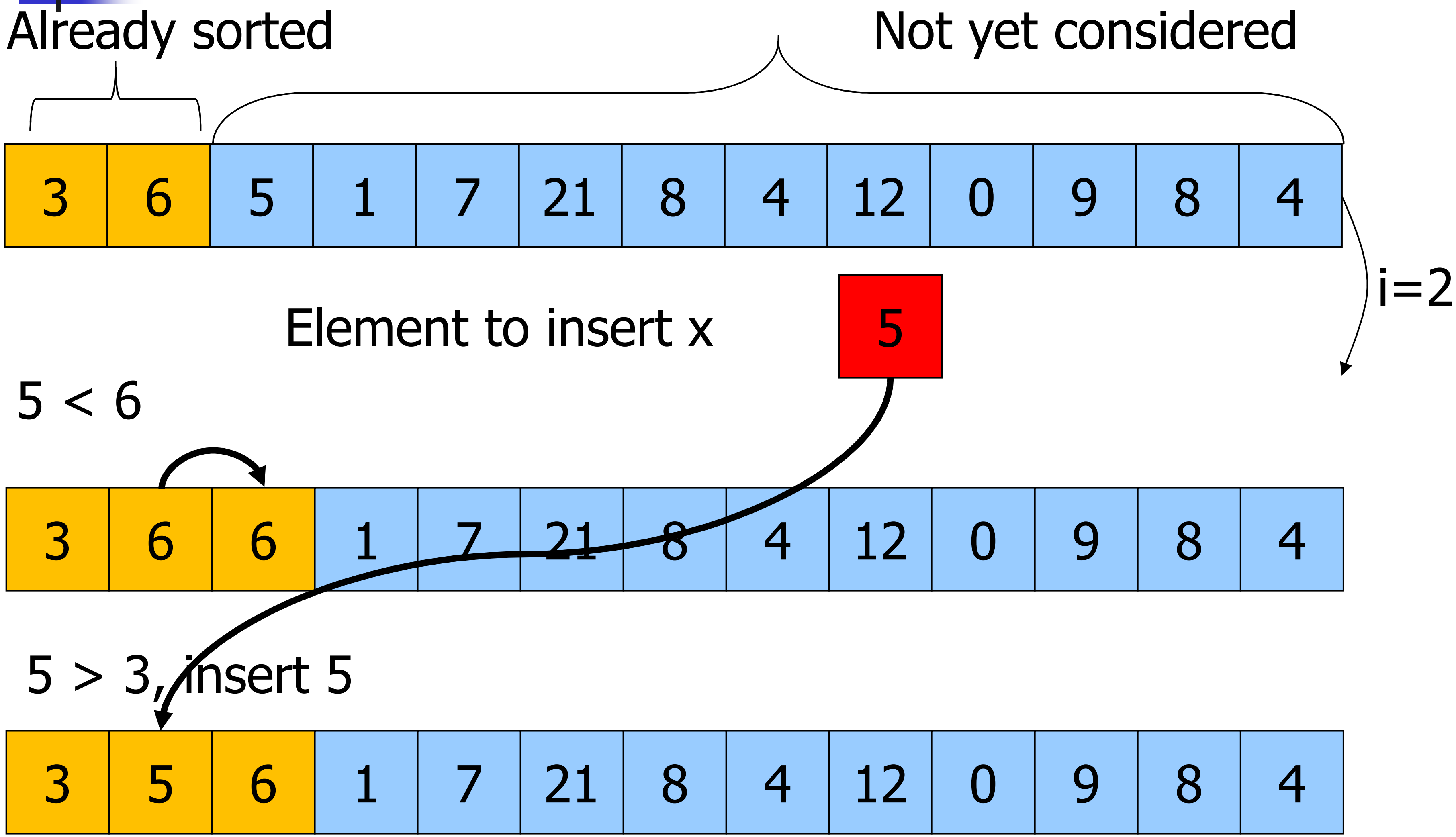
I-th step: put in the proper position $x = A_i$

- Scan the sorted subarray (from A_{i-1} to A_0) until we find $A_k > A_i$
- Right shift by one position of the elements from A_k to A_{i-1}
- Insert A_i in k-th position

Example


Already sorted
Not yet considered
6 3 5 1 7 21 8 4 12 0 9 8 4
i=1
Element to insert x
3
3 < 6
6 6 5 1 7 21 8 4 12 0 9 8 4
left boundary reached, insert 3
3 6 5 1 7 21 8 4 12 0 9 8 4

Example


Already sorted
Not yet considered
3 6 5 1 7 21 8 4 12 0 9 8 4
i=2
Element to insert x
5
5 < 6
3 6 6 1 7 21 8 4 12 0 9 8 4
5 > 3, insert 5
3 5 6 1 7 21 8 4 12 0 9 8 4

Example

Already sorted — Not yet considered

3	5	6	1	7	21	8	4	12	0	9	8	4

i=3

Element to insert x: 1

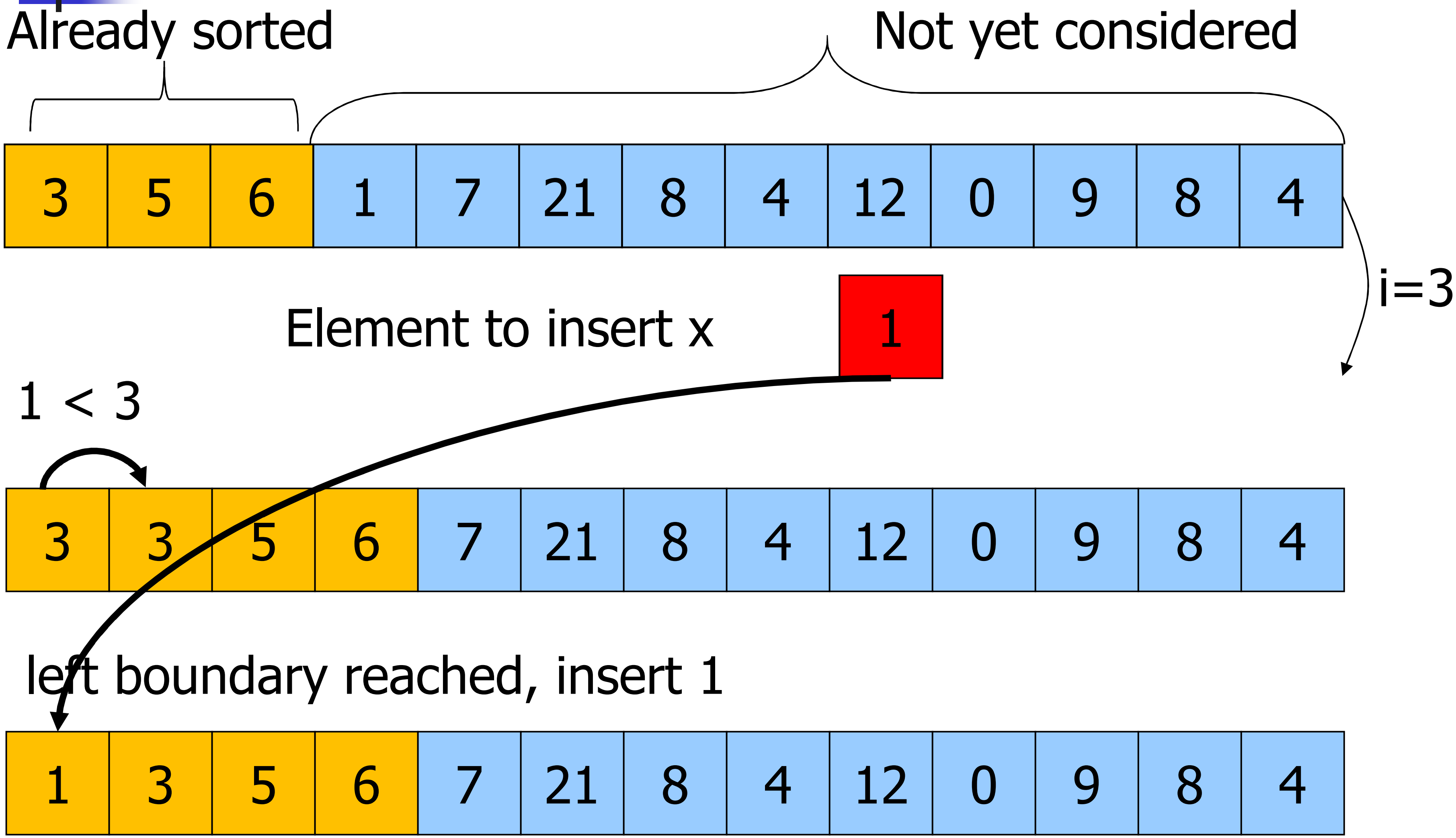
$1 < 6$

3	5	6	6	7	21	8	4	12	0	9	8	4

$1 < 5$

3	5	5	6	7	21	8	4	12	0	9	8	4

Example



Already sorted
Not yet considered
3 5 6 1 7 21 8 4 12 0 9 8 4
i=3
Element to insert x
1
1 < 3
3 3 5 6 7 21 8 4 12 0 9 8 4
left boundary reached, insert 1
1 3 5 6 7 21 8 4 12 0 9 8 4

C Code

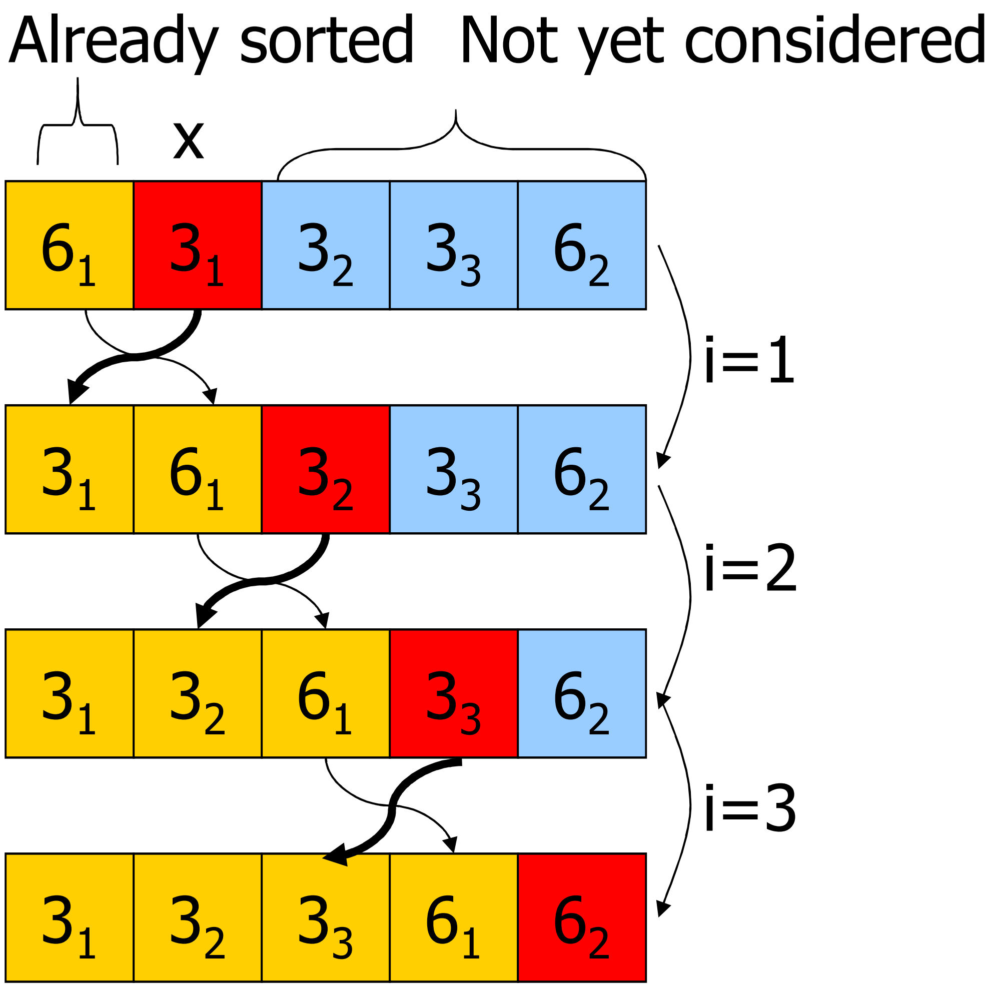
```
void InsertionSort (int A[], int n) {
  int i, j, x;

  for (i=1; i<n; i++) {
    x = A[i];
    j = i - 1;
    while (j >= 0 && x < A[j]) {
      A[j+1] = A[j];
      j--;
    }
    A[j+1] = x;
  }

  return;
}
```

Features of Insertion sort

- In place
- Number of exchanges in worst-case
  - $O(n^2)$
- Number of comparisons in worst-case
  - $O(n^2)$
- Stable
  - If the element to insert is a duplicate key, it can't pass over to the left a preceeding occurrence of the same key


Already sorted
Not yet considered
x
6_1 3_1 3_2 3_3 6_2
i=1
3_1 6_1 3_2 3_3 6_2
i=2
3_1 3_2 6_1 3_3 6_2
i=3
3_1 3_2 3_3 6_1 6_2

Insertion Sort: Complexity Analysis

Two nested cycles

- Outer loop: n-1 executions
- Inner loopin the worst-case: i executions at the i-th iteration of the outer loop

Complexity

$$T(N) = 1 + 2 + 3 + 4 + \cdots + (n-2) + (n-1)$$

$$T(N) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

T(n) grows quadratically with n

Finite arithmetic progression with ratio = 1 (Gauss, end of XVII cent.)

Insertion Sort: Complexity Analysis

- Inner loop in the best case: 1 execution at the i-the iteration of the outer loop
- Complexity

$$T(n) = \underbrace{1 + 1 + 1 + \dots\dots\dots + 1}_{\text{n-1 times}} = n-1$$

T(n) grows linearly with n

Insertion Sort: Complexity Analysis

Analytically, in the worst case, assuming unit cost for all operations

```
for(i=1; i<n; i++) {                         n
     x = A[i];                               n - 1
     j = i - 1;                              n - 1
     while (j >= 0 && x < A[j]){             Σ_{k=2}^{n} k
       A[j+1] = A[j];                        Σ_{k=2}^{n} (k-1)
       j--;                                  Σ_{k=2}^{n} (k-1)
     }
     A[j+1] = x;                             n - 1
  }
```

$$T(n) = n+(n-1)+(n-1)+\sum_{k=2}^{n} k+\sum_{k=2}^{n}(k\text{-}1) + \sum_{k=2}^{n}(k\text{-}1)+(n\text{-}1)$$

Number of operations

Insertion Sort: Complexity Analysis

Recalling that

$\sum_{k=2}^{n} k$ = n*(n+1)/2 -1

$\sum_{\mathrm{k}=2}^{n}(k\text{-}1)$ = n*(n-1)/2

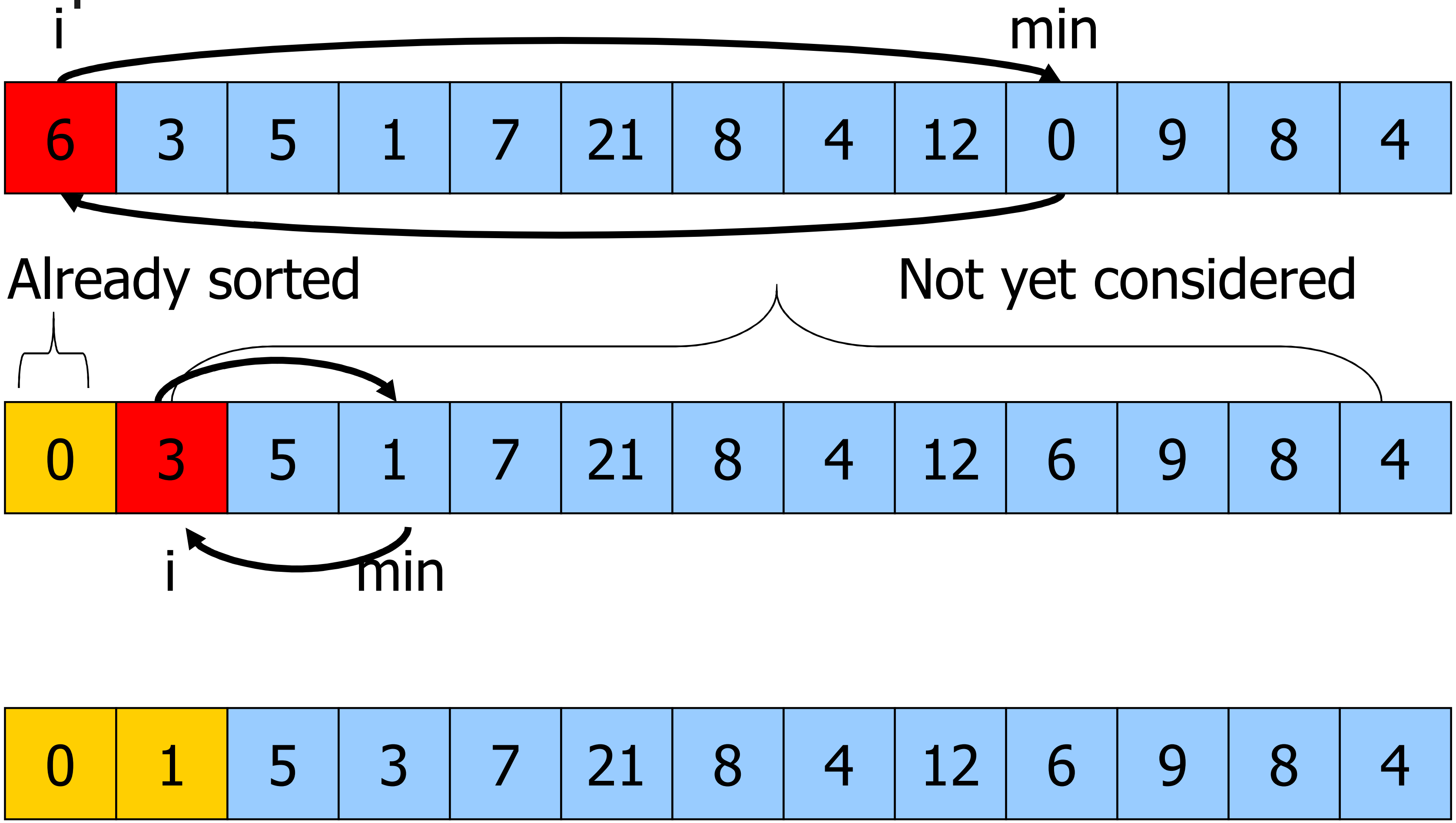
T(n) = n + 3*(n-1) + n*(n+1)/2 -1
+ 2*(n*(n-1)/2)
= $3/2n^2 + 7/2n - 4$

T(n) grows quadratically

Selection sort

- Sort n integers in array A
- Array divided into two sub-arrays
  - Left: sorted, initially empty
  - Right: unsorted, initially it coincides with A
- Incremental approach: iteration i: the minimum of the right sub-array (A_i ... A_r) is assigned to a A[i]; increment i
- Termination: all elements are inserted in the correct location
- Searching for the minimum in the right sub-array entails scanning the sub-array

Example


i
min
6 3 5 1 7 21 8 4 12 0 9 8 4
Already sorted
Not yet considered
0 3 5 1 7 21 8 4 12 6 9 8 4
i
min
0 1 5 3 7 21 8 4 12 6 9 8 4

C Code

```
void SelectionSort (int A[], int l, int r) {
  int i, j, min, temp;

  for(i=l; i<r; i++) {
    min = i;
    for (j = i+1; j <= r; j++) {
      if (A[j] < A[min]) {
        min = j;
      }
    }
    temp = A[i];
    A[i] = A[min];
    A[min] = temp;
  }

  return;
}
```

Features

- In place
- Not stable
  - A swap of "far away" elements may result in a duplicate key passing over to the left of a preceding instance of the same key

Worst-case asymptotic analysis

- Two nested loops
  - outer loop: executed n-1 times
  - inner loop: at the i-th iteration executed n-i-1 times

  $T(n) = (n-1) + (n-2) + \ldots 2 + 1 = O(n^2)$
- Number of exchanges in worst-case $O(n)$
- Number of comparisons in worst-case $O(n^2)$

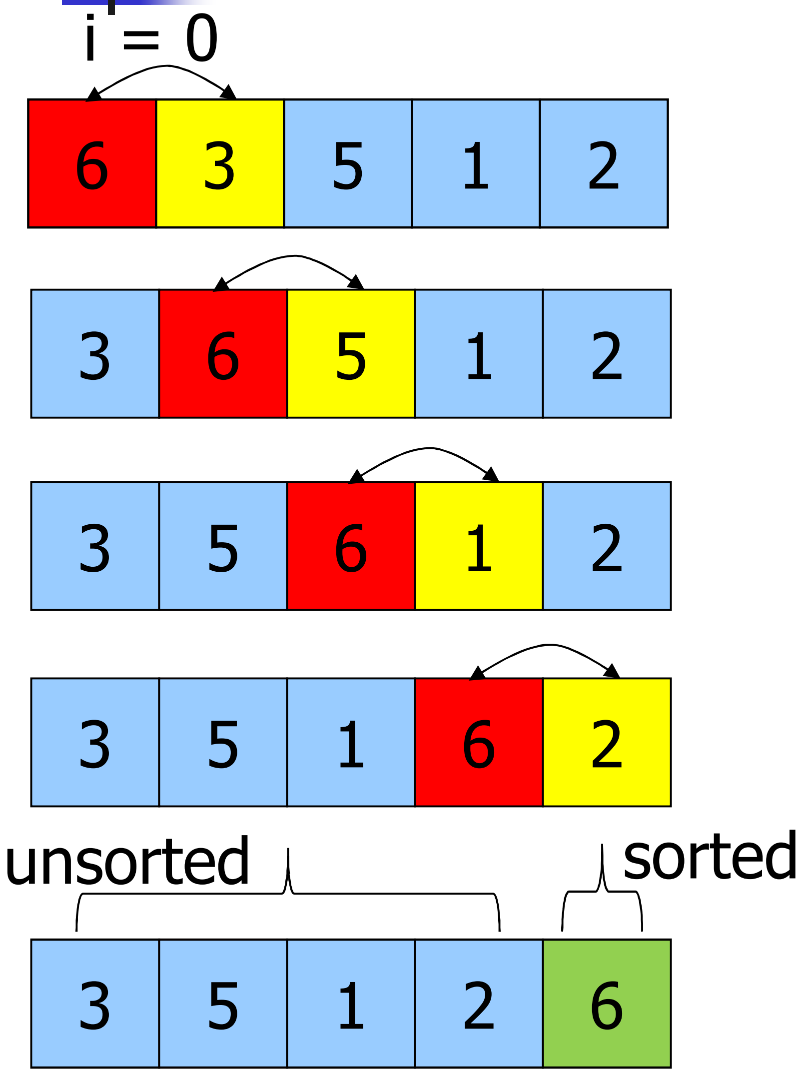
Exchange (Bubble) Sort

- Data: integers in array A delimited by left and right indices l and r
- Array divided in 2 sub-arrays
  - Right : sorted, initially empty
  - Left: unsorted, initially it coincides with A
- Elementary operation
  - Compare successive elements of the array A[j] and A[j+1], swap if A[j] > A[j+1]

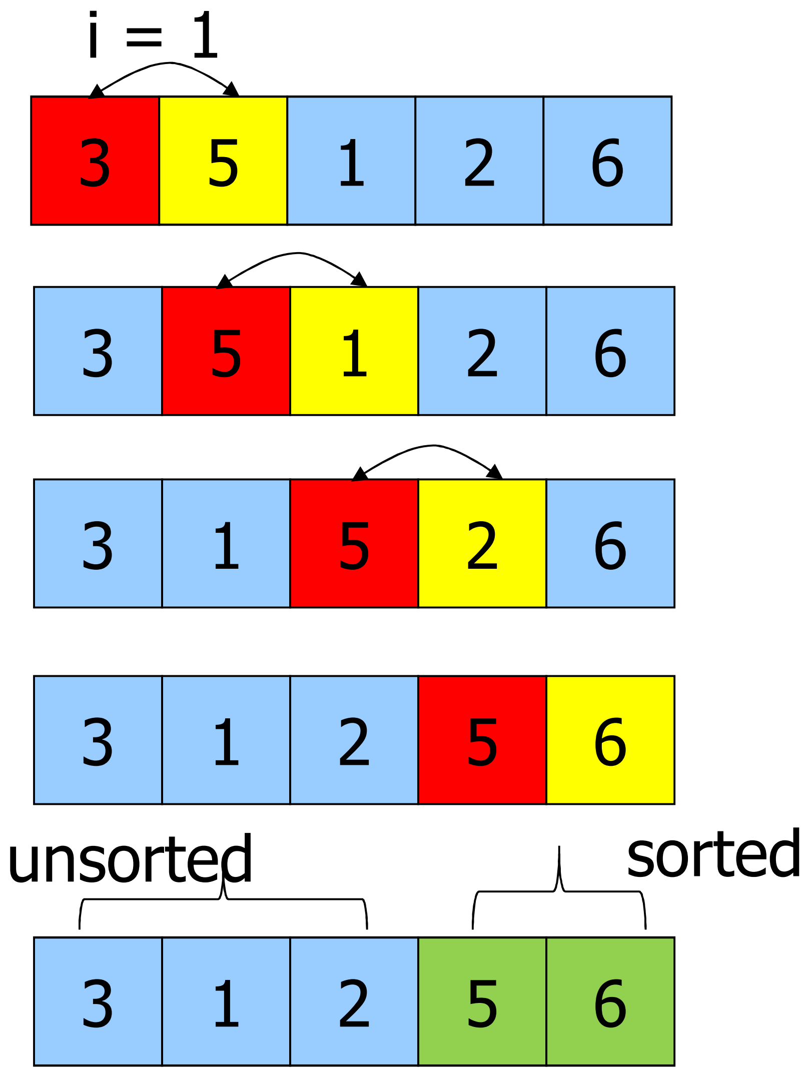
Exchange (Bubble) Sort

- Incremental approach: iteration i: the maximum of the left sub-array SX (A_l ... A_{r-i+l}) is assigned to A[r-i+l]; increment i. The sorted right sub-array increases in size by 1 to the left, dually the left sub-array decreases in size by 1
- Termination: all elements are inserted in the correct location
- Possible optimization: flag to record that there have been swaps, early loop exit.

Example



i = 0
6 3 5 1 2
3 6 5 1 2
3 5 6 1 2
3 5 1 6 2
unsorted
sorted
3 5 1 2 6



i = 1
3 5 1 2 6
3 5 1 2 6
3 1 5 2 6
3 1 2 5 6
unsorted
sorted
3 1 2 5 6

C Code

```
void BubbleSort (int A[], int l, int r){
  int i, j, temp;

  for( i = l; i < r; i++) {
    for (j = l; j < r - i +l; j++) {
      if (A[j] > A[j+1])) {
        temp = A[j];
        A[j] = A[j+1];
        A[j+1] = temp;
      }
    }
  }
  return;
}
```

i-l is the number of already sorted elements

Optimized C Code

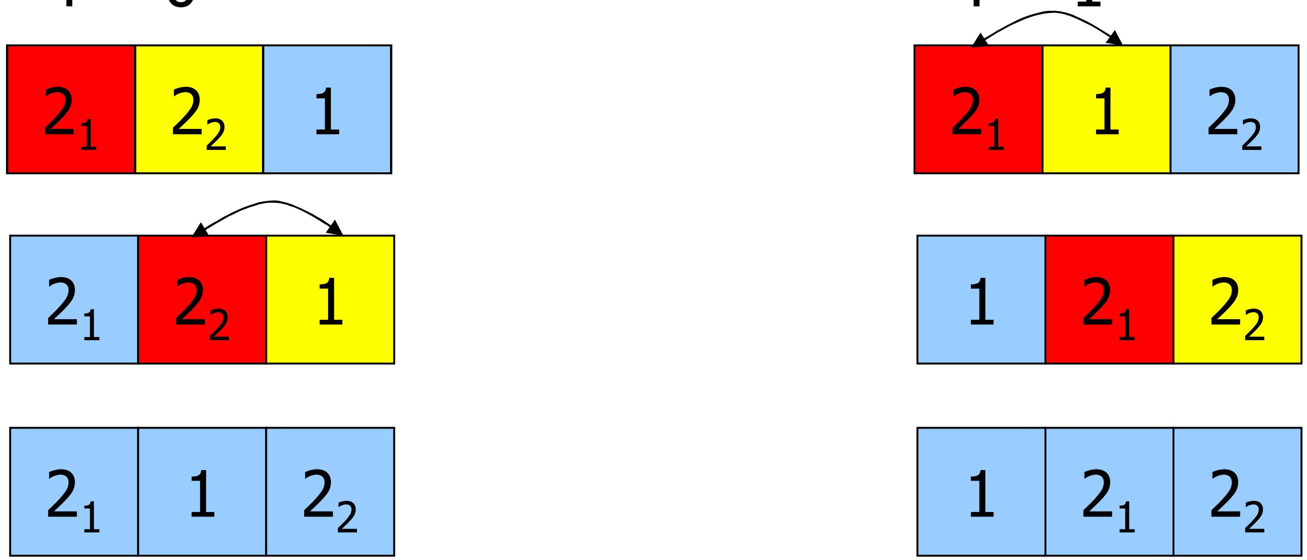
```
void OptBubbleSort (int A[], int l, int r) {
   int i, j, flag, temp;

   flag = 1;
   for(i = l; i < r && flag==1; i++) {
     flag = 0;
     for (j = l; j < r - i + l; j++)
       if (A[j] > A[j+1]) {
         flag = 1;
         temp = A[j];
         A[j] = A[j+1];
         A[j+1] = temp;
     }
   }
   return;
}
```

Features

- In place
- Stable
  - Among several duplicate keys, the rightmost one takes the rightmost position and no other identical key ever moves past it to the right:

i = 0   i = 1



Worst-case asymptotic analysis

- Features: stable, in place
- Two nested loops:
  - outer loop: executed n-1 times
  - Inner loop: at the i-th iteration executed n-1-i times

  $T(n) = (n-1) + (n-2) + \ldots 2 + 1 = O(n^2)$

Finite arithmetic progression with ratio 1 (Gauss, end of XVII cent)

Shellsort (Shell, 1959)

Limit of insertion sort: comparison, thus exchange takes place only between adjacent elements

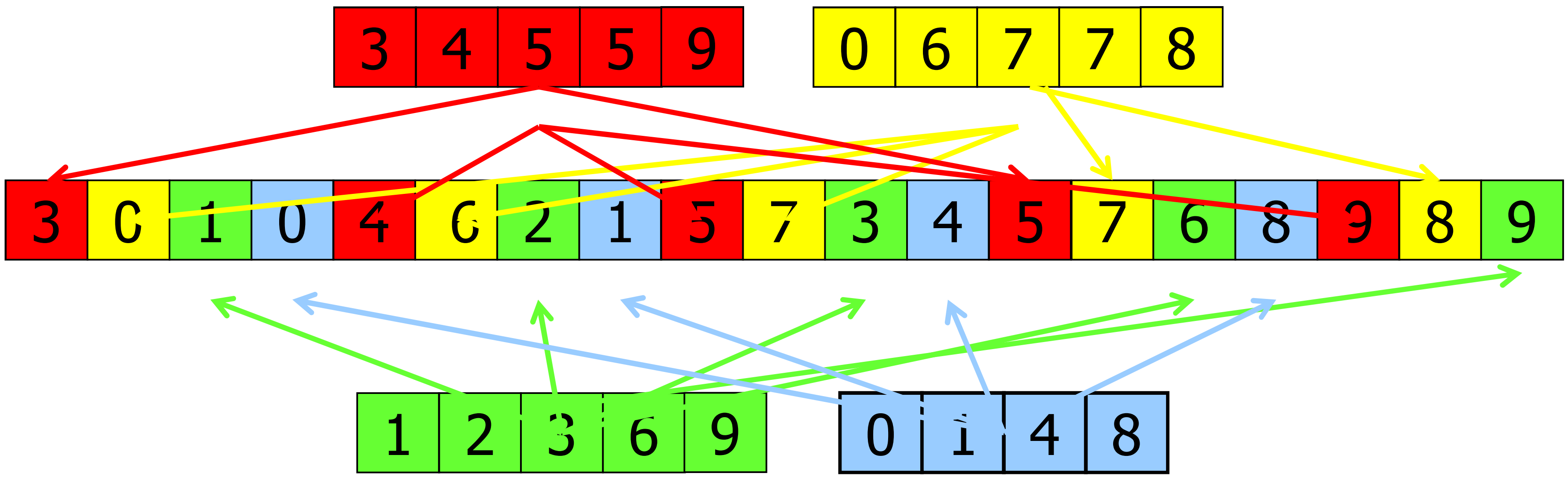
Rationale of Shellsort

- Compare, thus possibly exchange, elements at distance h
- Defining a decreasing sequence of integers ending with 1

Shellsort (Shell, 1959)

An array formed by non contiguous sequences composed by elements whose distance is h is h-sorted

Example with h=4 (sorted non contiguous subsequences)



Shellsort (Shell, 1959)

For each of the subsequences we apply insertion sort. The elements of the subsequence are those at distance h from the current one.

```
for (i = l+h; i <= r; i++) {
  int j = i, v = A[i];
  while (j>= l+h && v < A[j-h])) {
    A[j] = A[j-h];
    j -=h;
  }
  A[j] = v;
}
```

Example

5	8	9	0	4	6	3	1	3	7	6	4	5	7	1	8	9	0	2

sequence h: 13, 4, 1

Step1: h=13

5	1	8	0	0	2	3	1	3	7	6	4	5	7	8	9	9	4	6

Step 2: h=4

0	1	3	0	3	2	6	1	5	4	6	4	5	4	8	9	9	7	8

Step 3: h=1

0	0	1	1	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9

Choosing the sequence

- Has an impact on performance
- Knuth's sequence

  $h = 3 \cdot h + 1 = 1\ 4\ 13\ 40\ 121\ \ldots$
- Sequence

  $h = 1$ then $4^{i+1} + 3 \cdot 2^{i} + 1 = 1\ 8\ 23\ 77\ 281\ 1073\ \ldots$
- Sedgewick's sequence

  $h = 1, 5, 19, 41, 109, 209, 505, 929, 2161, 3905, \ldots$

C Code

```
void ShellSort(int A[], int l, int r) {
  int i, j, temp, h, n;
  h=1; n = r - l +1;
  while (h < n/3)
    h = 3*h+1;
  while (h >= 1) {
    for (i = l + h; i <= r; i++) {
      j = i;
      temp = A[i];
      while (j >= l + h  && temp<A[j-h]) {
        A[j] = A[j-h];
        j -=h;
      }
      A[j] = temp;
    }
    h = h/3;
  }
}
```

Features

- In place
- Not stable
    - An exchange between "far away" elements may result in a duplicate key that passes over to the left a preceding occurrence of the same key

Example

2_1	2_2	2_3	2_4	2_5	0

sequence h: 4, 1

- Step 1: h=4

2_1	2_2	2_3	2_4	2_5	0

2_1	0	2_3	2_4	2_5	2_2

- Step 2: h=1

0	2_1	2_3	2_4	2_5	2_2

Worst-case asymptotic analysis

Shellsort

- With Knuth's sequence: 1 4 13 40 121 …
  - It executes less than $O(n^{3/2})$ comparisons
- With the sequence 1 8 23 77 281 1073 …
  - It executes less than $O(n^{4/3})$ comparisons
- With Shell's original sequence 1 2 4 8 16 …
  - It may degenerate to $O(n^2)$

Counting sort

- Sorting based on computation: find, for each element to sort x, how many elements are less than or equal to x
- x directly assigned to final location
- Features: stable, not in place

Data Structures

- 3 arrays
  - Starting array
    - A[0..n-1] of n integers
  - Resulting array
    - B [0..n-1] of n integers
  - Occurrence array
    - C of k integers if data belong to the range [0..k-1]

Algorithm

- Step 1: simple occurrences
  - C[i] = number of elements of A equal to i
- Step 2: multiple occurrences
  - C[i] = number of elements of A <= i
- Step3: $\forall$ j
  - C[A[j]] = number of elements <= A[j]
- Thus final location of A[j] in B

B[C[A[j]]] = A[j]

(beware of indices in C, see code!)

Example (n=8, k=6)

Array to sort

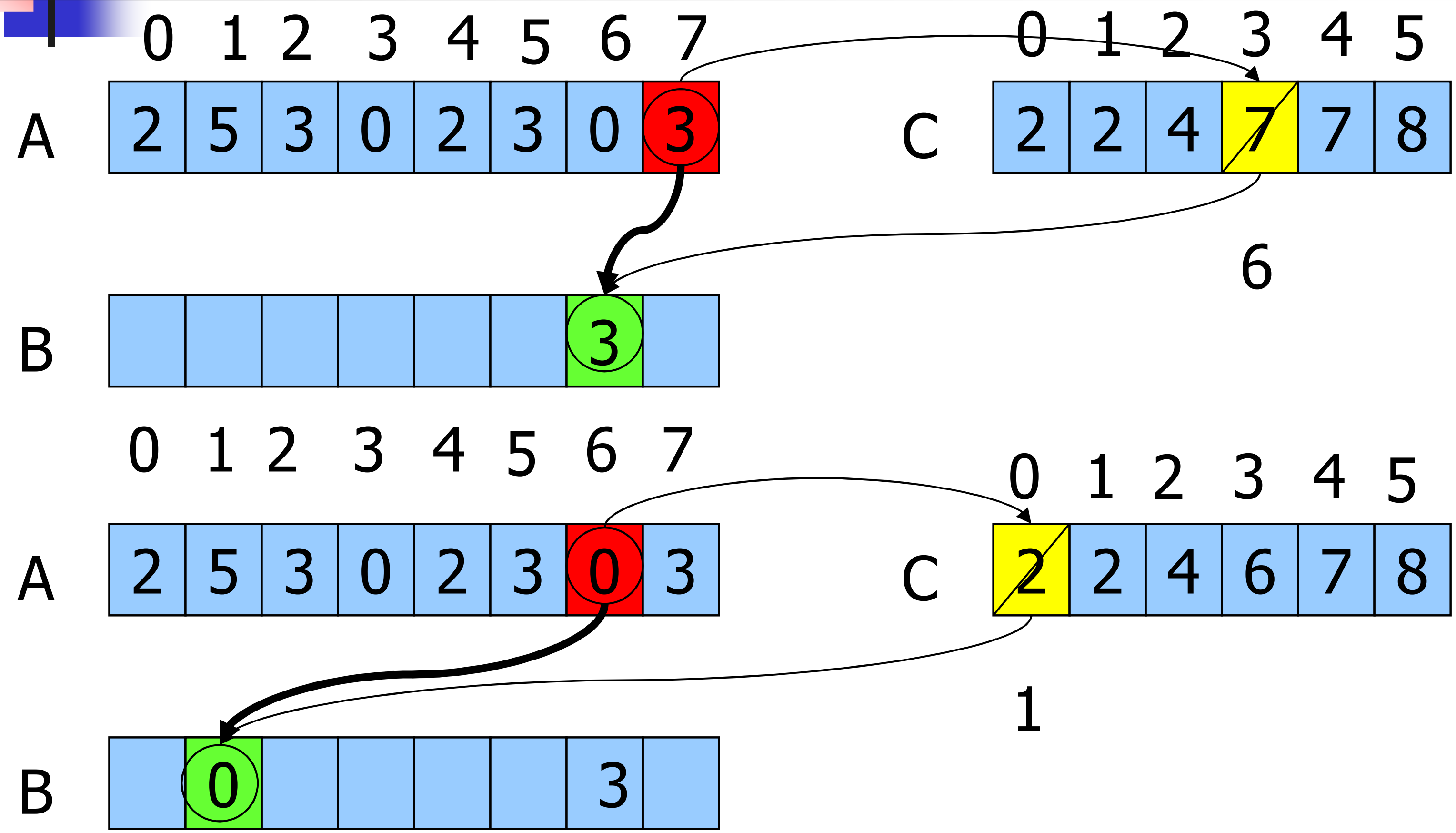
	0	1	2	3	4	5	6	7
A	2	5	3	0	2	3	0	3

Simple occurrences

	0	1	2	3	4	5
C	2	0	2	3	0	1

Multiple occurrences

	0	1	2	3	4	5
C	2	2	4	7	7	8


0 1 2 3 4 5 6 7
A 2 5 3 0 2 3 0 3
C 2 2 4 7 7 8
0 1 2 3 4 5
6
B 3
0 1 2 3 4 5 6 7
A 2 5 3 0 2 3 0 3
C 2 2 4 6 7 8
0 1 2 3 4 5
1
B 0 3

```
#define MAX 100
```

```
void CountingSort(int A[], int l, int r, int k) {
  int i, n, C[MAX], B[MAX];
  n = r - l + 1;
  for (i = 0; i < k; i++)
    C[i] = 0;
  for (i = l; i <= r; i++)
    C[A[i]]++;
  for (i = 1; i < k; i++)
    C[i] += C[i-1];
  for (i = r; i >= l; i--) {
    B[C[A[i]]-1] = A[i];
    C[A[i]]--;
  }
  for (i = l; i <= r; i++)
    A[i] = B[i];
}
```

Worst-case asymptotic analysis

- Initialization loop for C: O(k)
- Loop to compute simple occurrences: O(n)
- Loop to compute multiple occurrences: O(k)
- Loop to copy result in B: O(n)
- Loop to copy in A: O(n)

$$T(n) = O(n+k)$$

Applicability: $k=O(n)$, thus $T(n) = O(n)$