Recursive sorting algorithms

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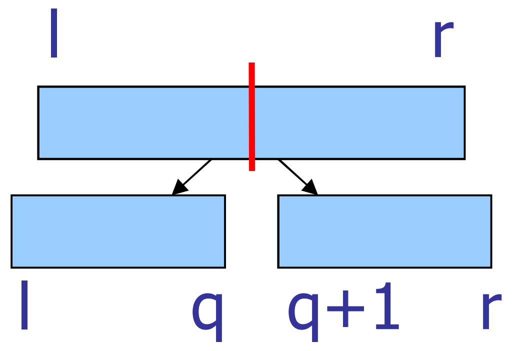
Merge Sort

- Division
  - Partition the array into 2 subarrays L and R with respect to the array's middle element



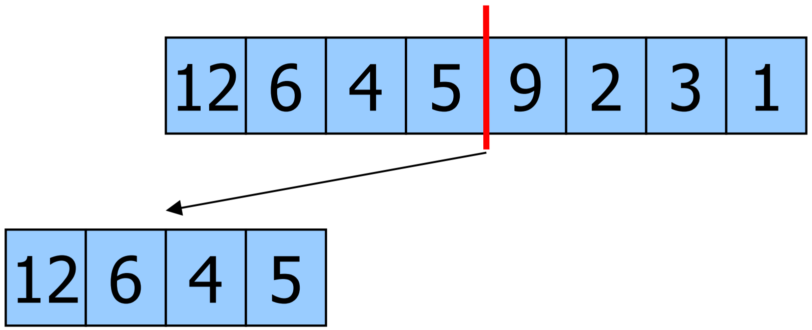
Merge Sort

- Recursion
  - Merge sort on subarray L
  - Merge sort on subarray R
  - Termination condition: with 1 (l=r) or 0 (l>r) elements the array is sorted
- Ricombination
  - **Merge** 2 sorted subarrays into one sorted array

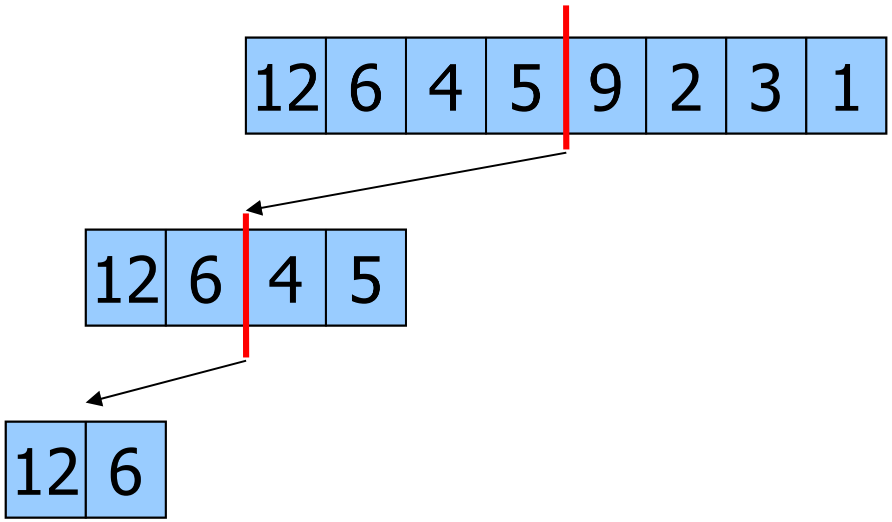
Example

12	6	4	5	9	2	3	1

Example



Example



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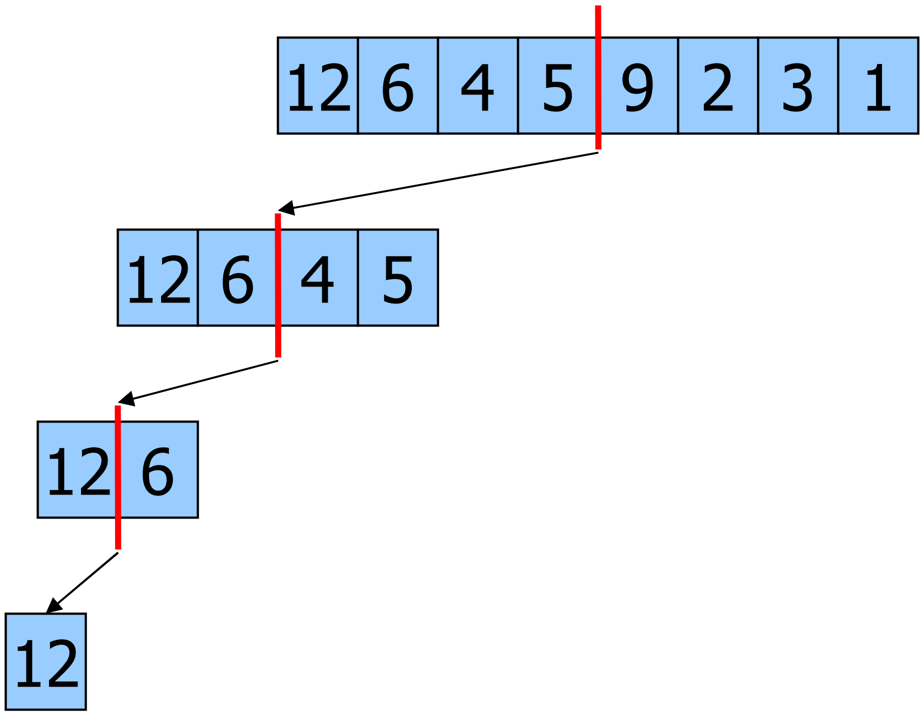
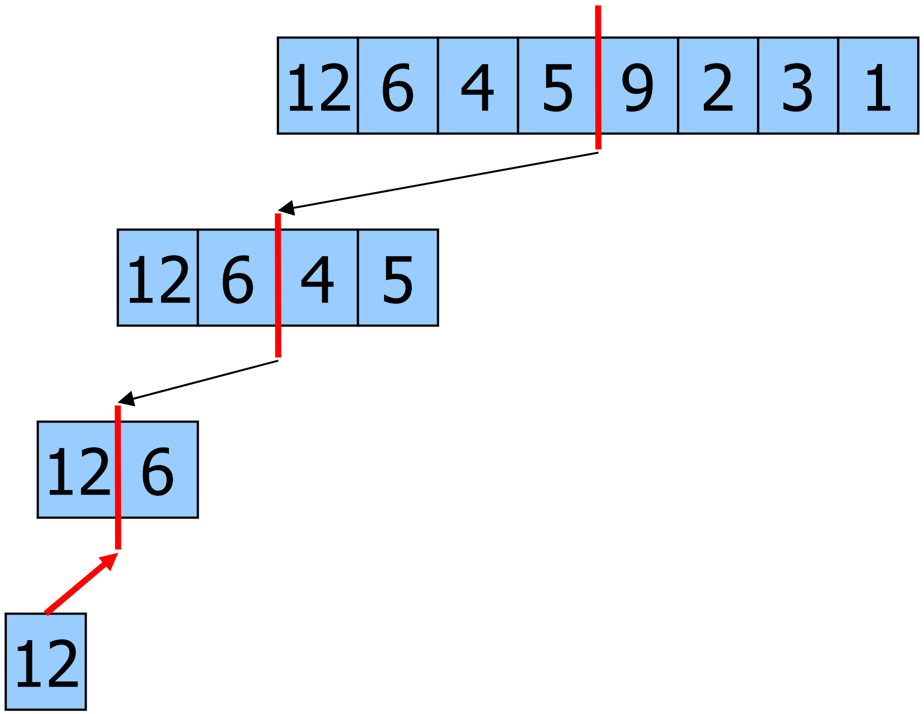


Example



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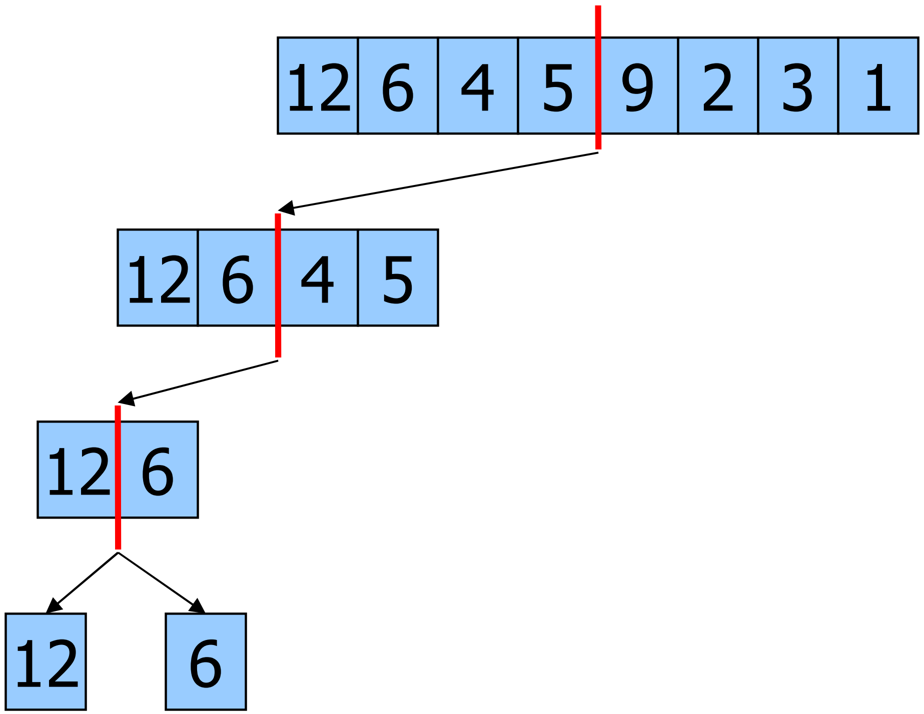

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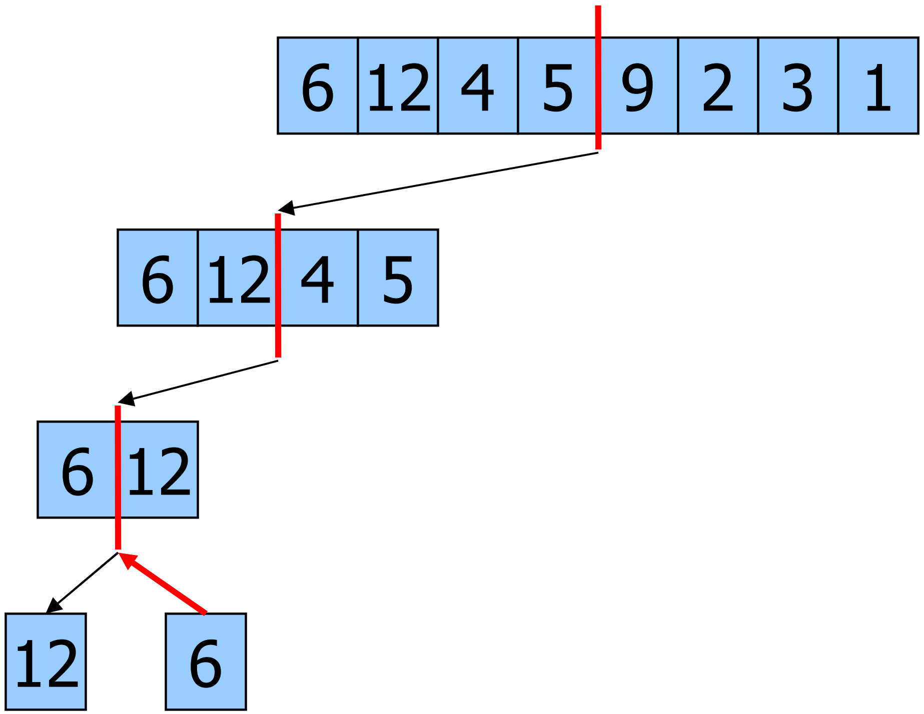

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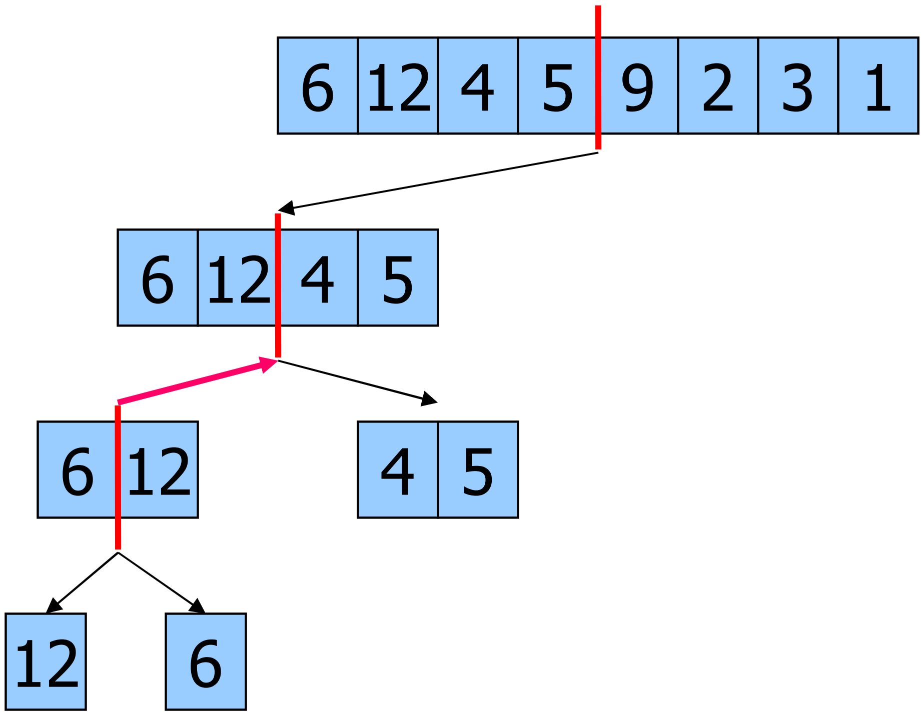

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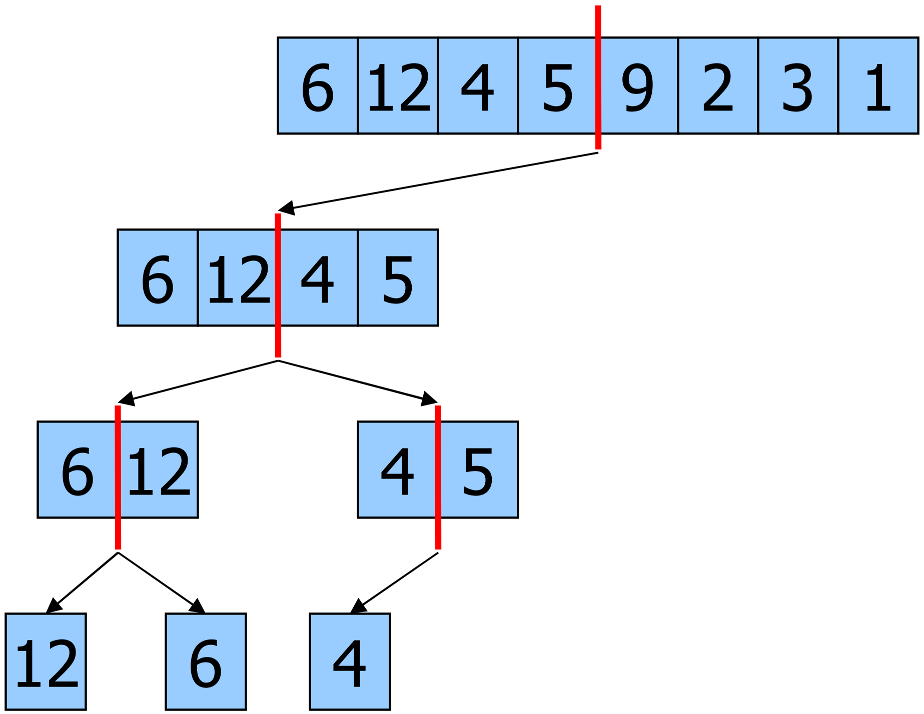

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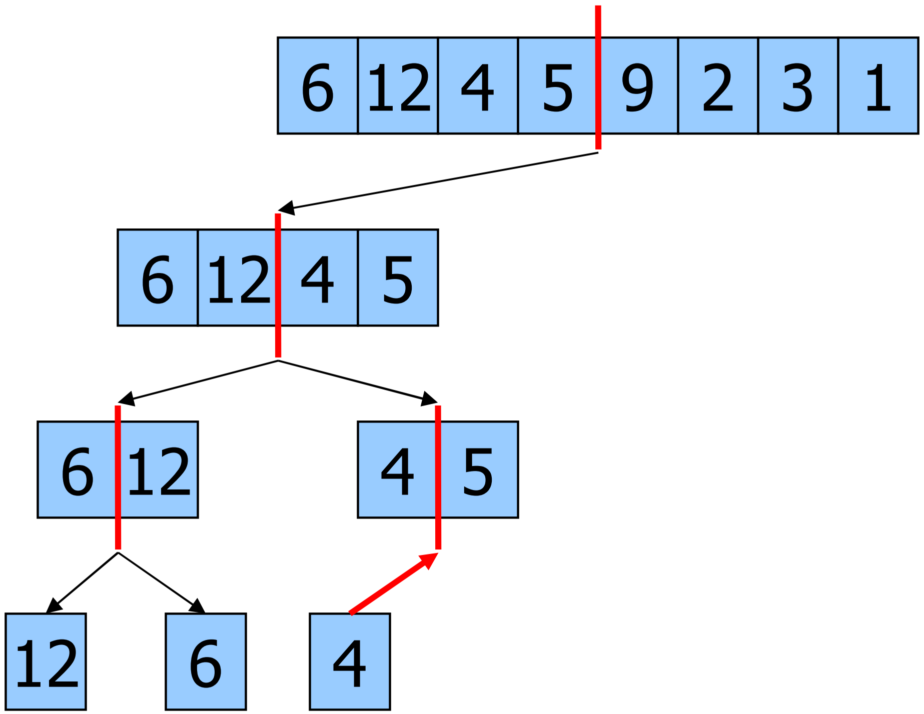

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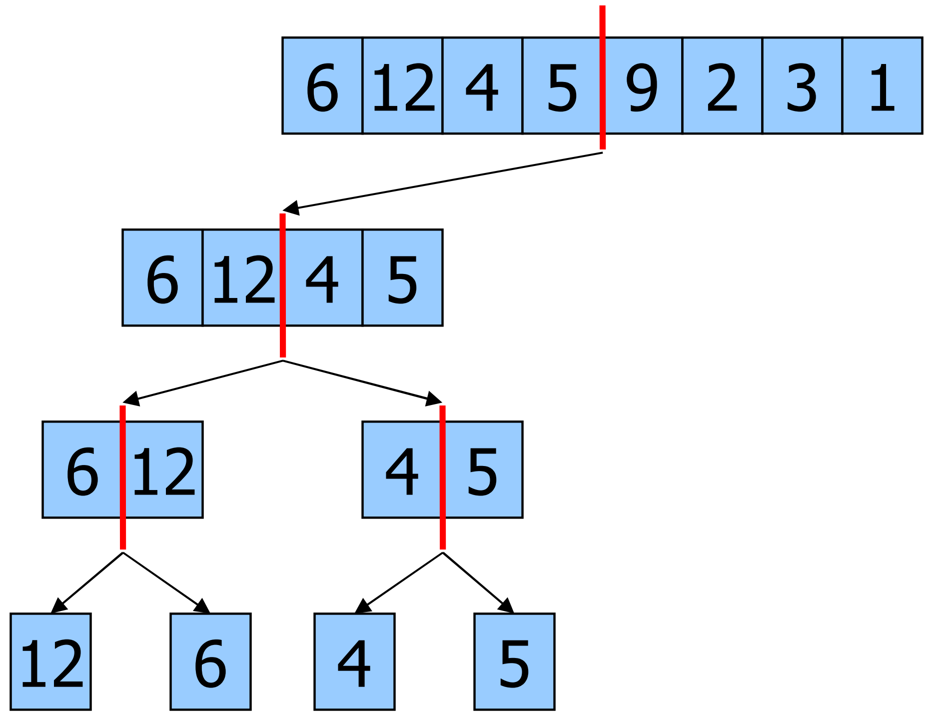

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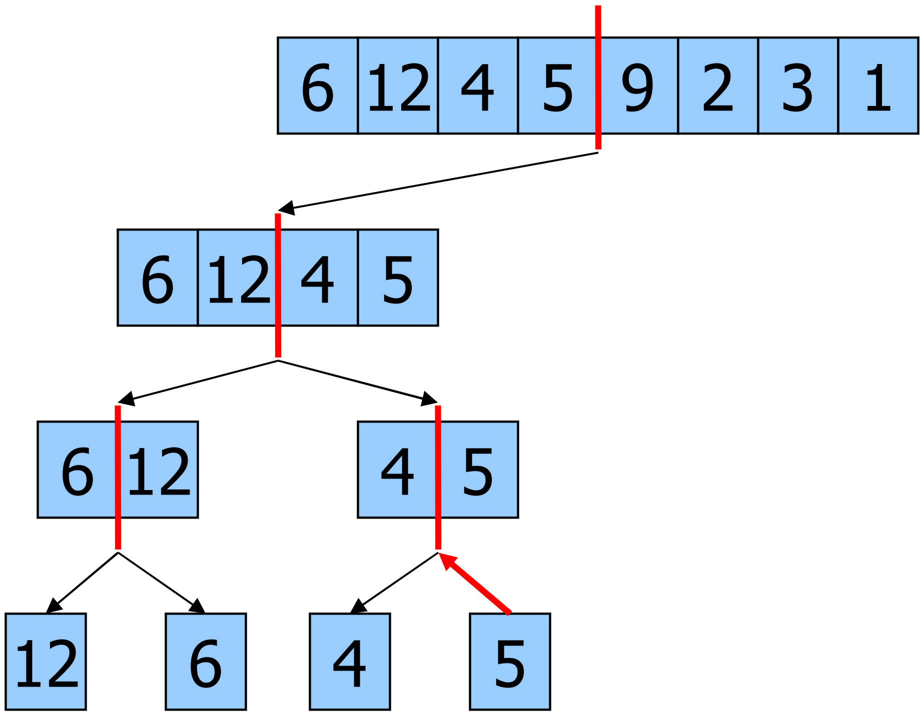

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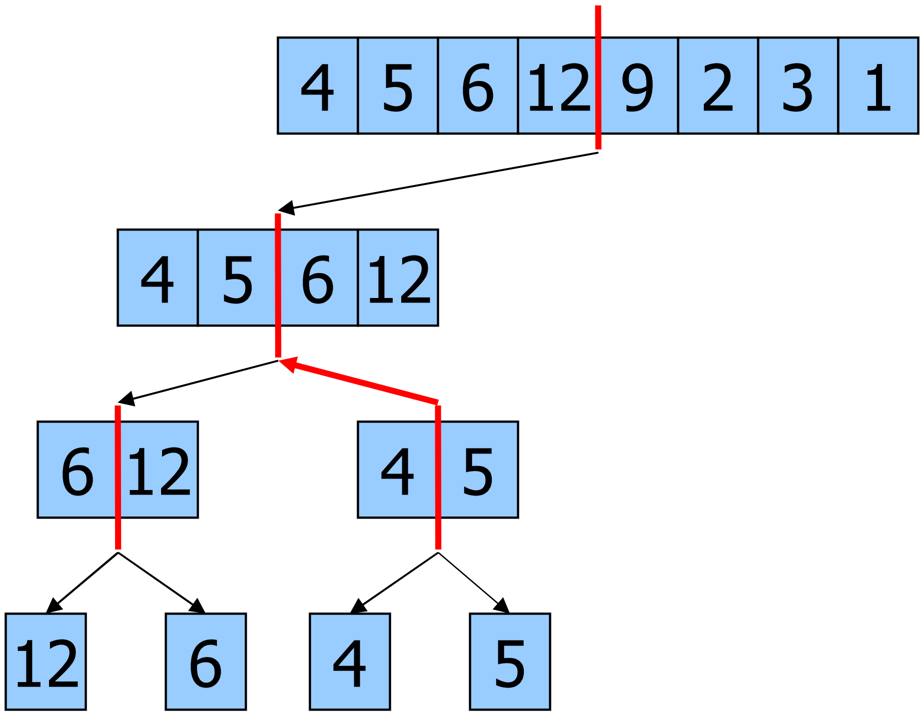

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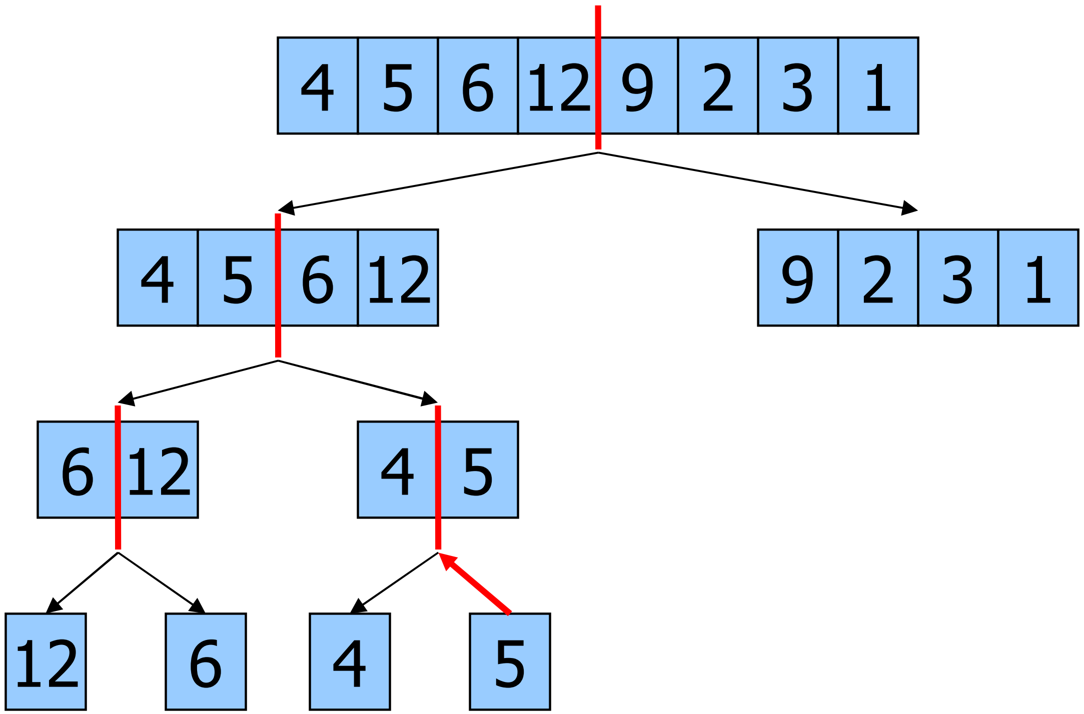

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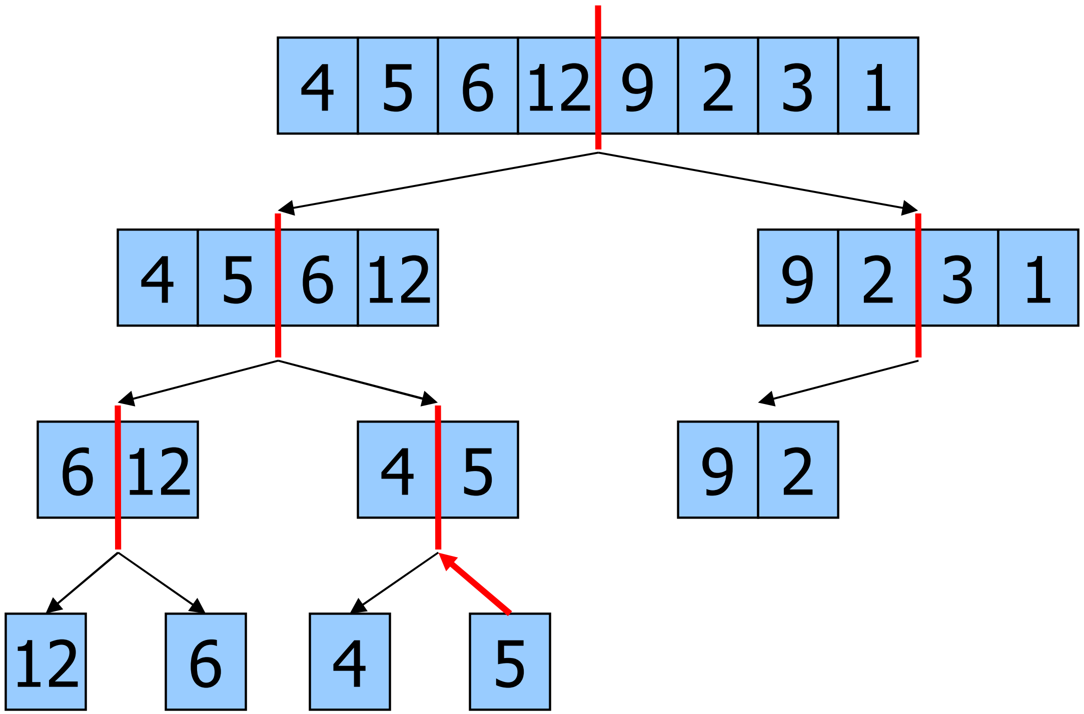

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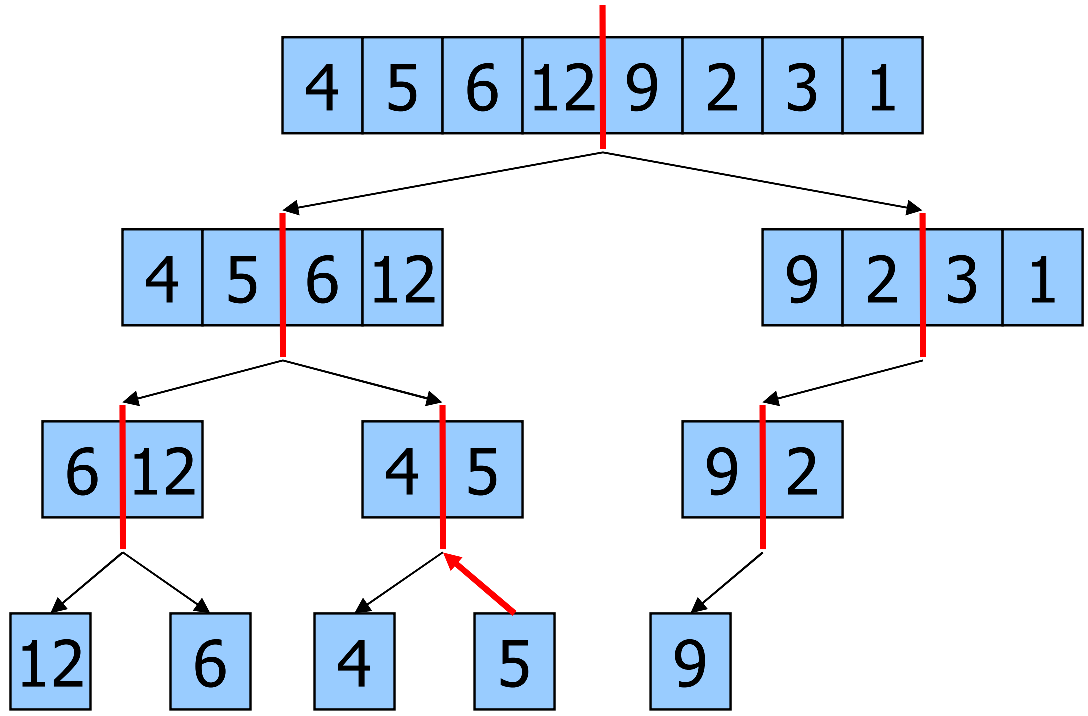

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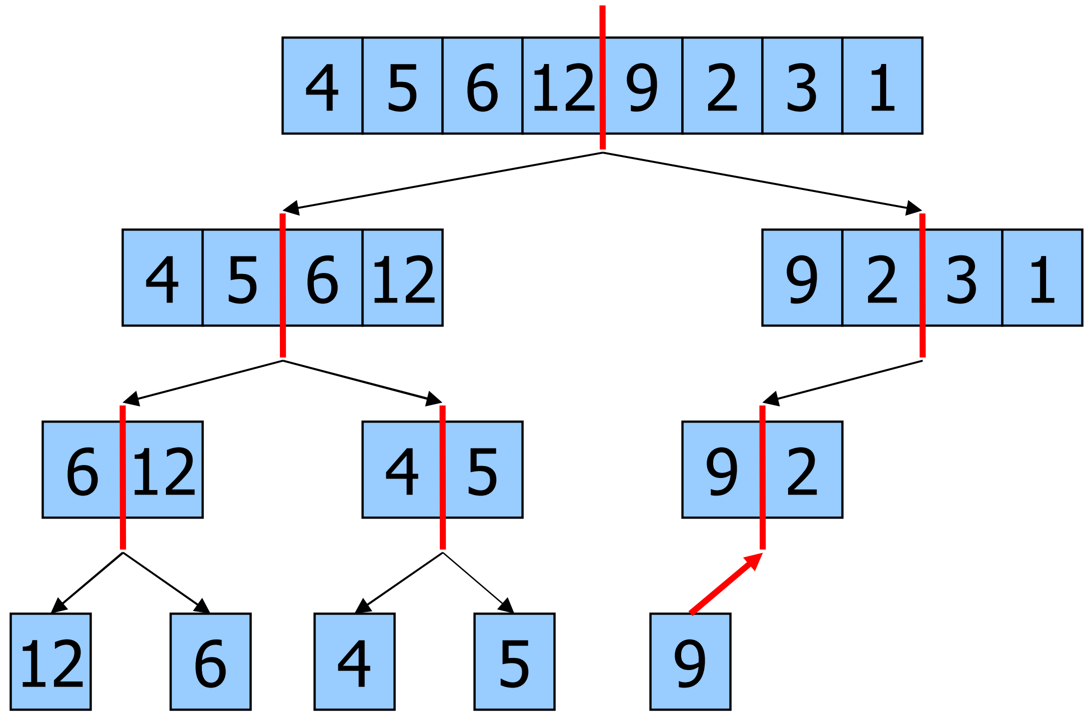

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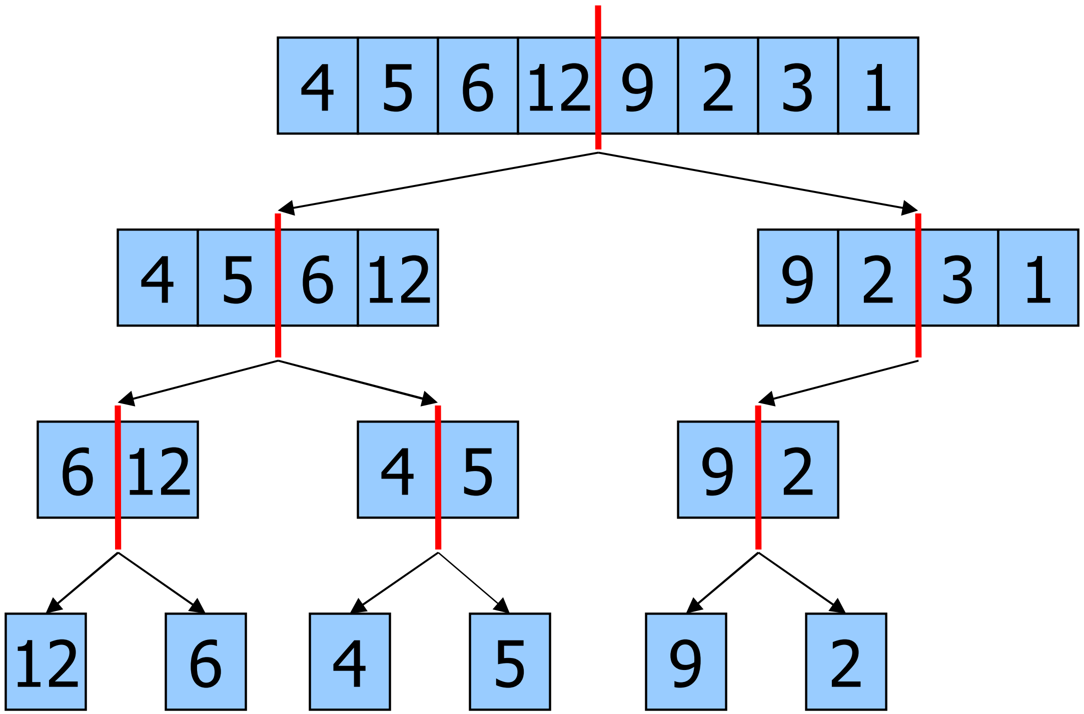

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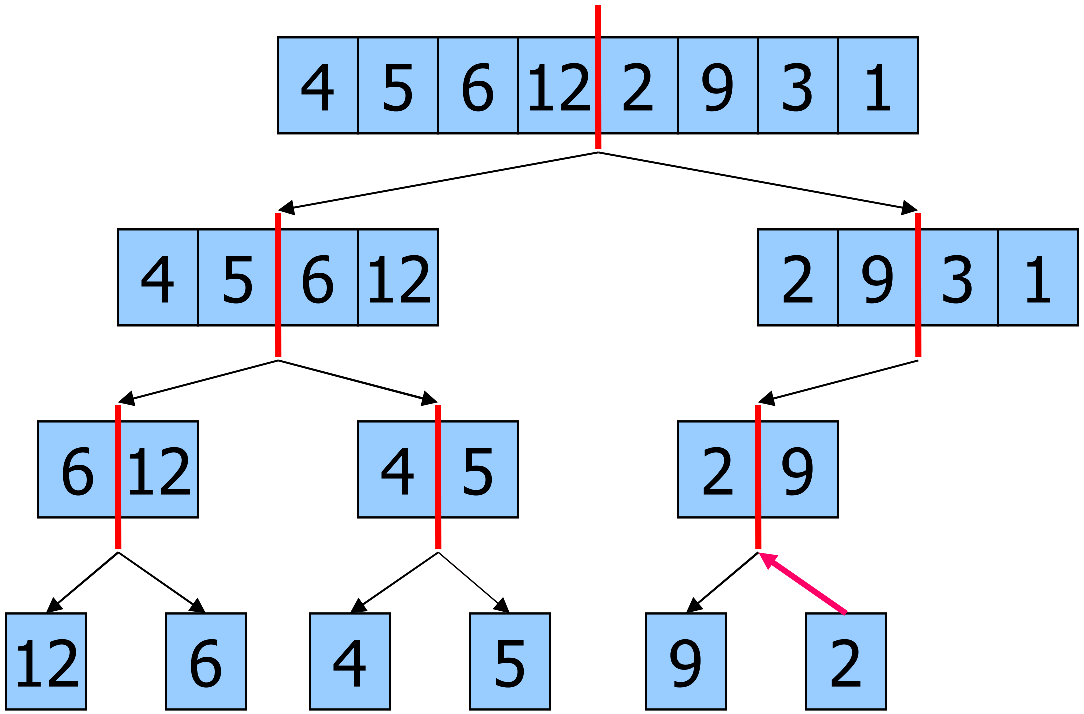

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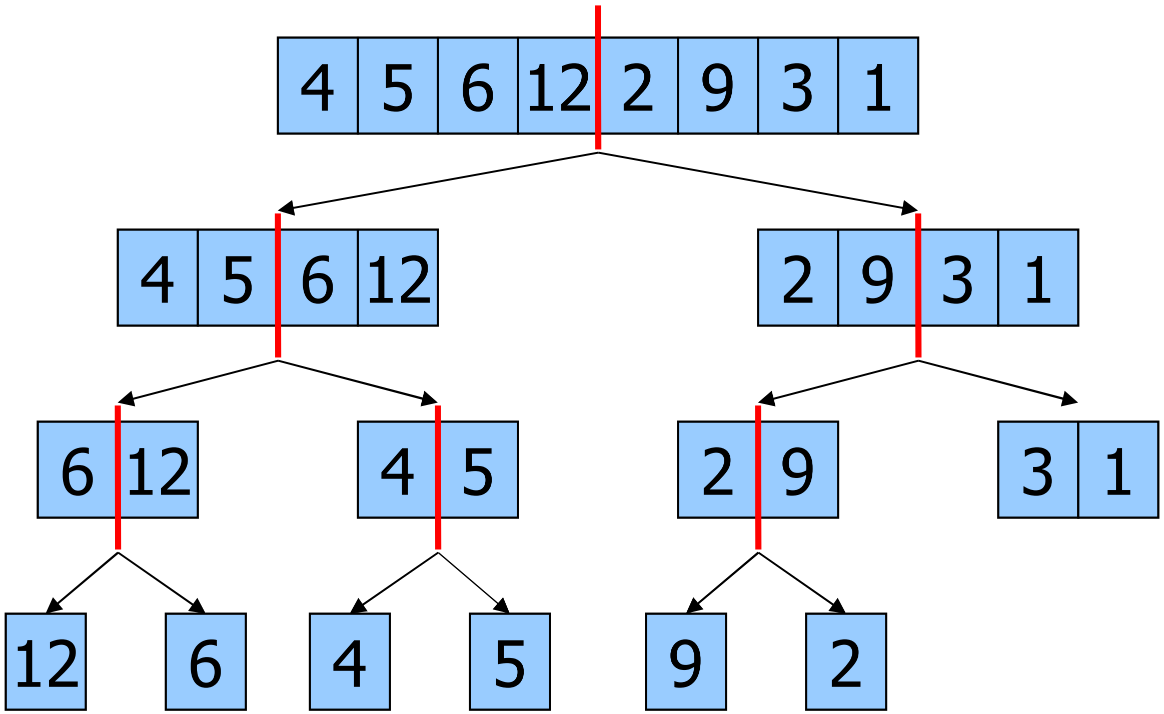

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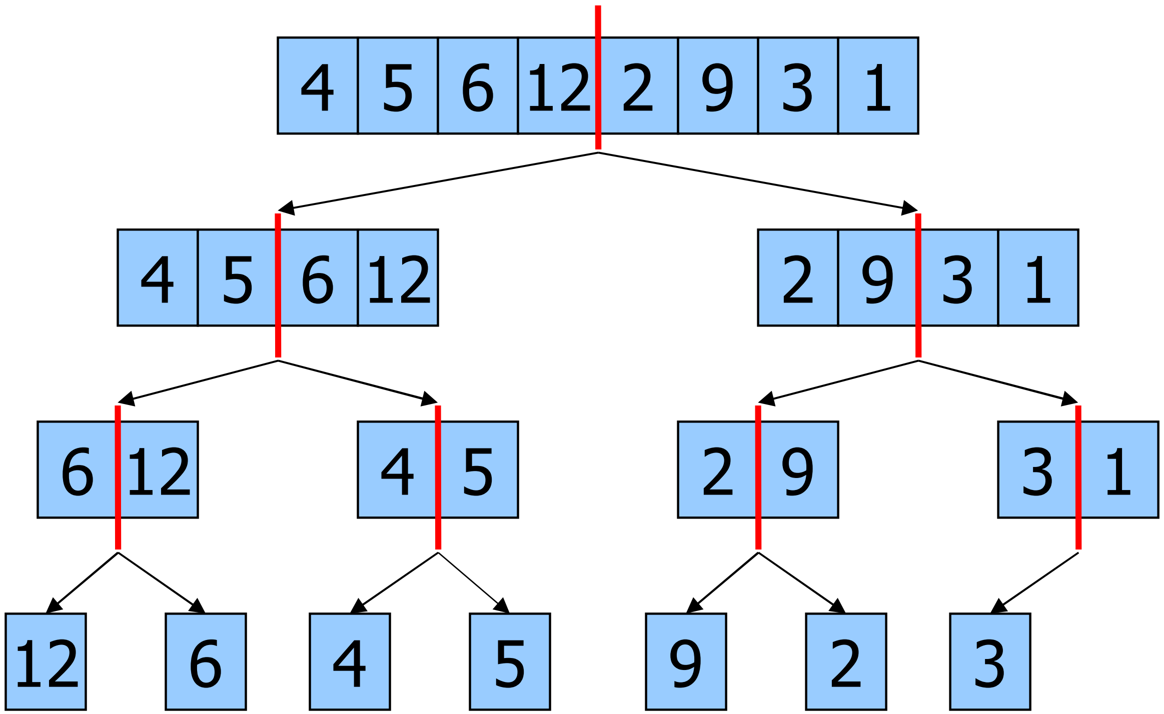

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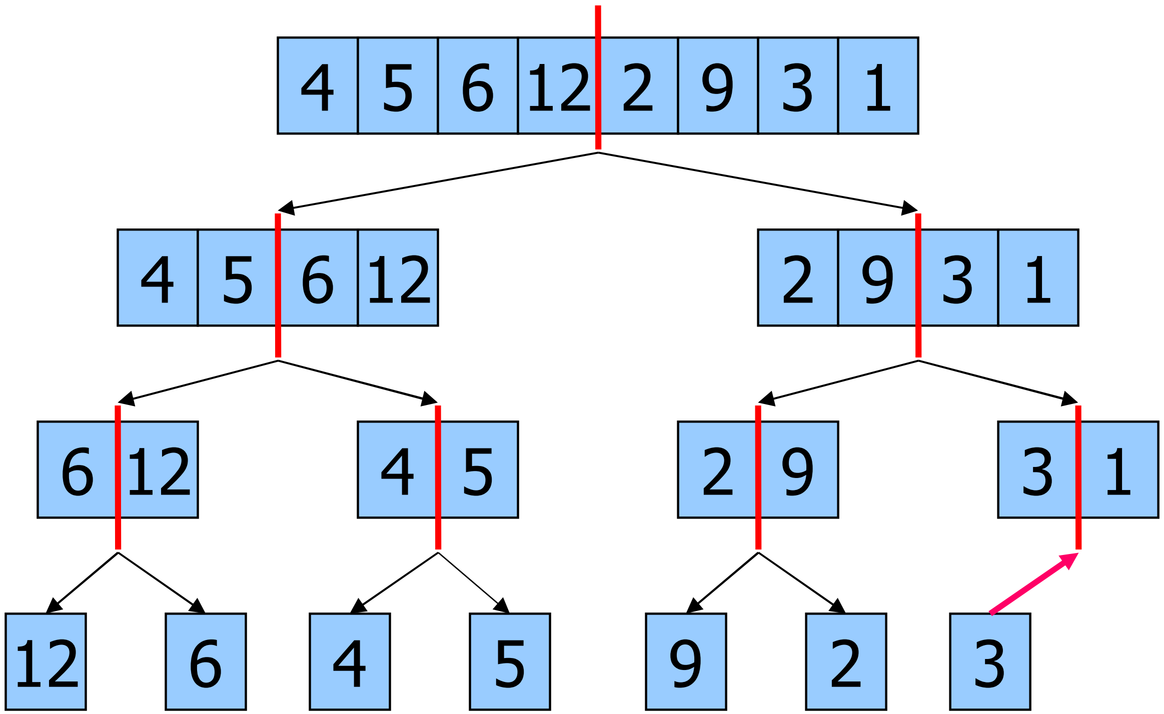

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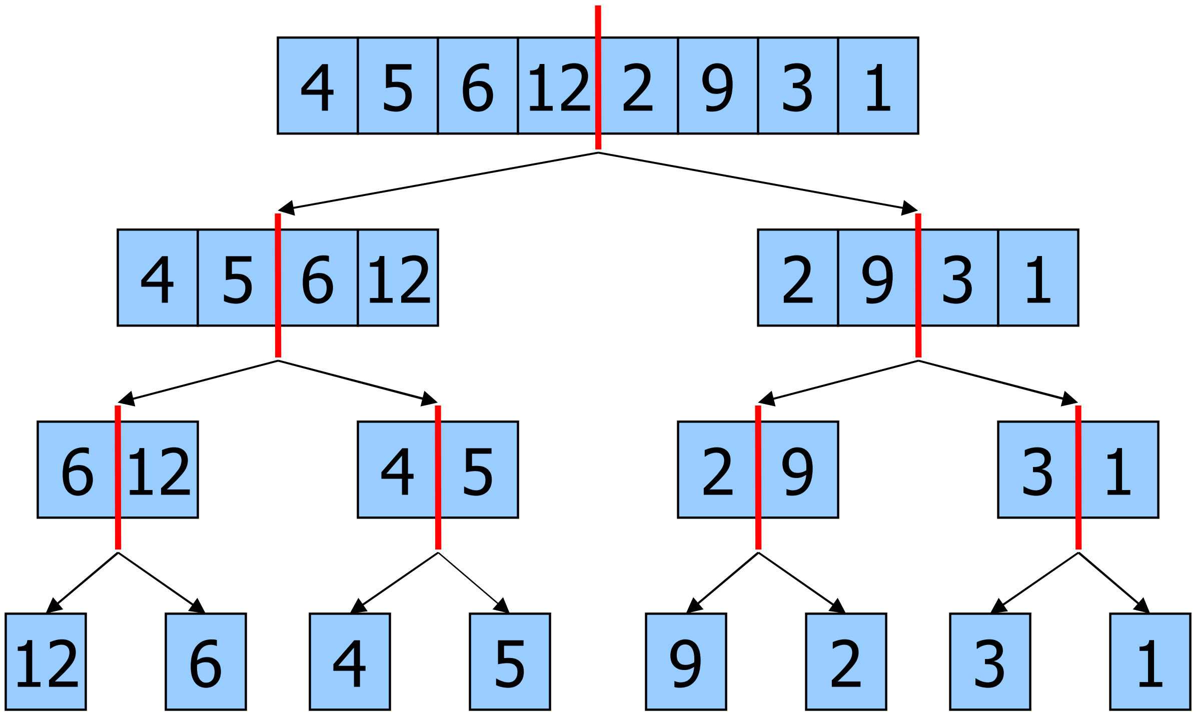

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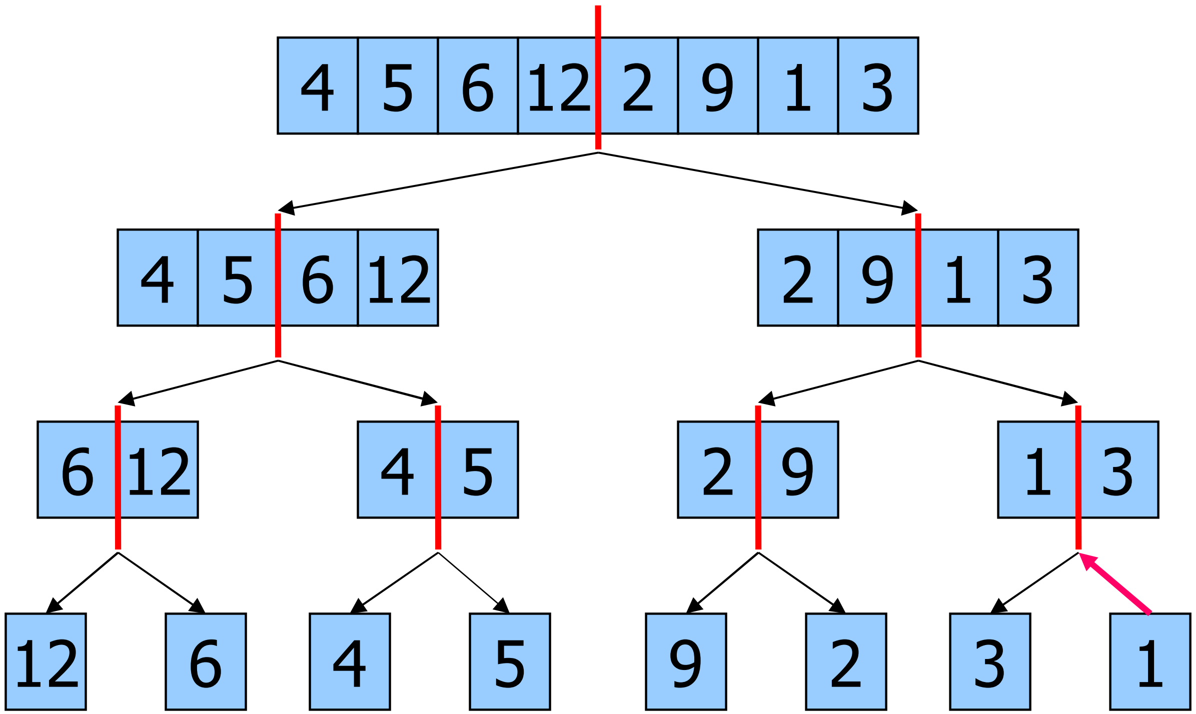

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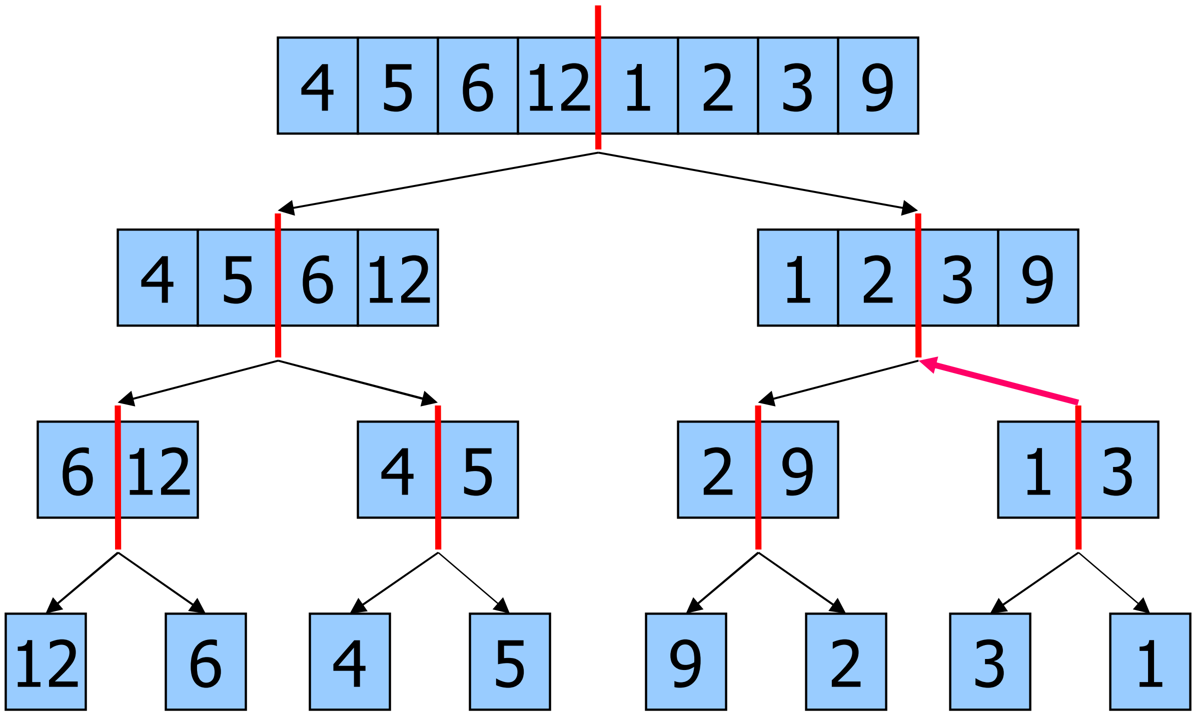

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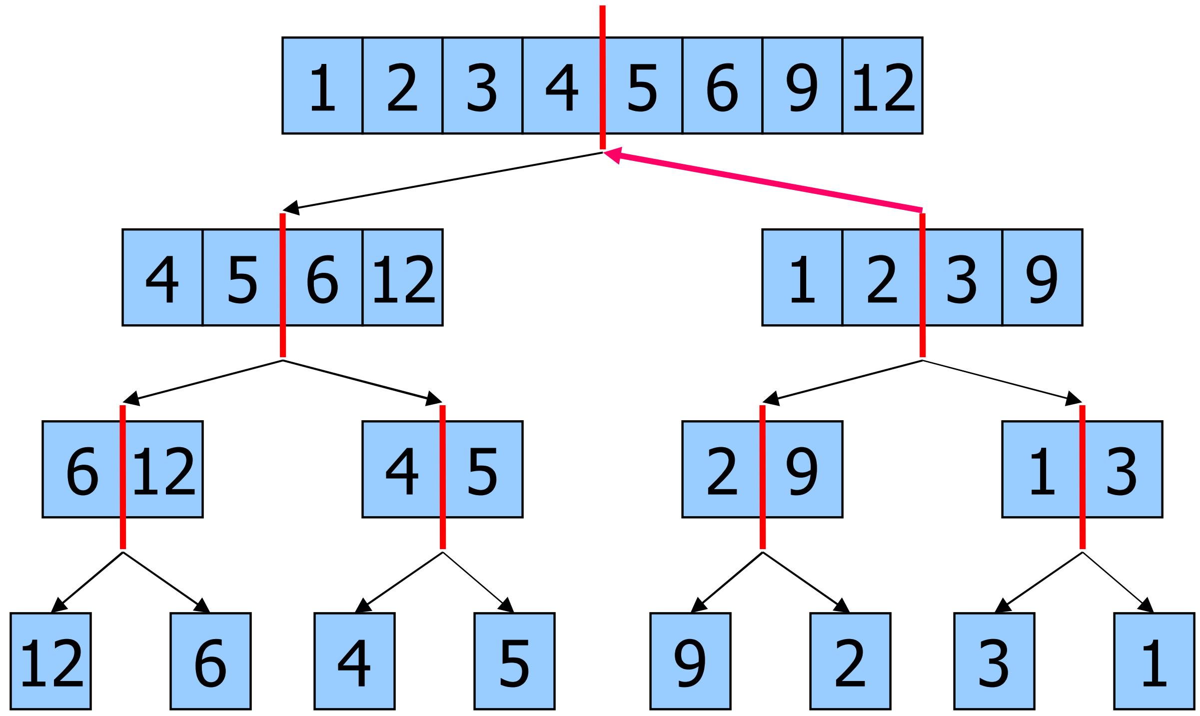


Solution

Wrapper

Auxiliary array (check and free missing)

```
void merge_sort (int *A, int N) {
  int l=0, r=N-1;
  int *B = (int *)malloc(N*sizeof(int));
  merge_sort_r (A, B, l, r);
}
```

Boundaries

Recursive function call

```
void merge_sort_r (int *A, int *B, int l, int r){
  int c;
  if (r <= l)
    return;
  c = (l + r)/2
  merge_sort_r (A, B, l, c);
  merge_sort_r (A, B, c+1, r);
  merge (A, B, l, c, r);
}
```

Division

Termination

Recursive calls

Recombination

Solution

```
void merge (int *A, int *B,  int l, int c, int r) {
  int i, j, k;

  for (i=l, j=c+1, k=l; i<=c && j<=r; )
    if (A[i]<=A[j])
      B[k++] = A[i++];
    else
      B[k++] = A[j++];

  while (i<=c)
    B[k++]=A[i++];
  while (j<=r)
    B[k++]=A[j++];

  for (k=l; k<=r; k++)
    A[k] = B[k];

  return;
}
```

<= for stability

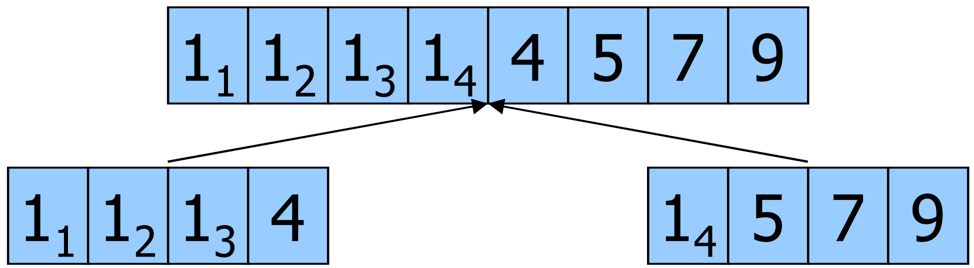
Compare and merge

Copy one of the tail

Copy array back

Features

- Not in place
  - It uses an auxiliary array
- Stable
  - Function merge takes keys from the left subarray in the case of duplicate values



Complexity analysis

- Assumption
  - $n = 2^k$
- Divide array in 2
  - $D(n)=\Theta(1)$
- Solve 2 subproblems of size n/2 each
  - $2T(n/2)$
- Termination
  - Simple test $\Theta(1)$
- Combine
  - Based on merge $C(n) = \Theta(n)$

Complexity analysis

- Recurrence equation
  - $T(n) = D(n) + a\,T(n/b) + C(n)$

$D(n)+C(n) = \Theta(n)$

- That is
  - $T(n) = n + 2T(n/2)$ $\quad n>1$
  - $T(1) = 1$ $\quad n=1$
- Resolution by unfolding
  - $T(n) = n + 2T(n/2)$
  - $T(n/2) = n/2 + 2T(n/4)$
  - $T(n/4) = n/4 + 2T(n/8)$
  - …

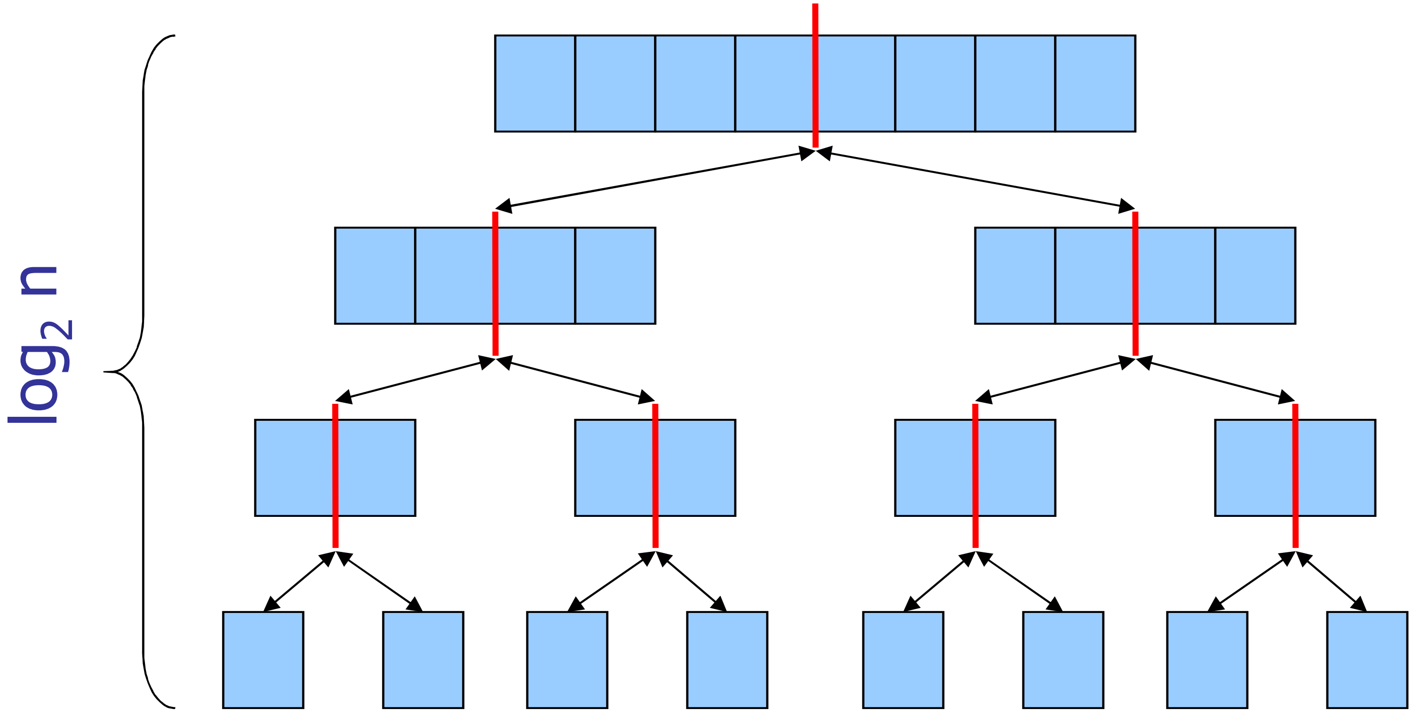
Complexity analysis

Termination condition
$n/2^i = 1$
$i = \log_2 n$

- Replacing in T(n)
  - $T(n) = n + n + n + n + \ldots$

    $= n \sum_{i=1}^{log_2 n} 1 = n \log n$
- Thus
  - $T(n) = O(n \log n)$

Intuitive analysis



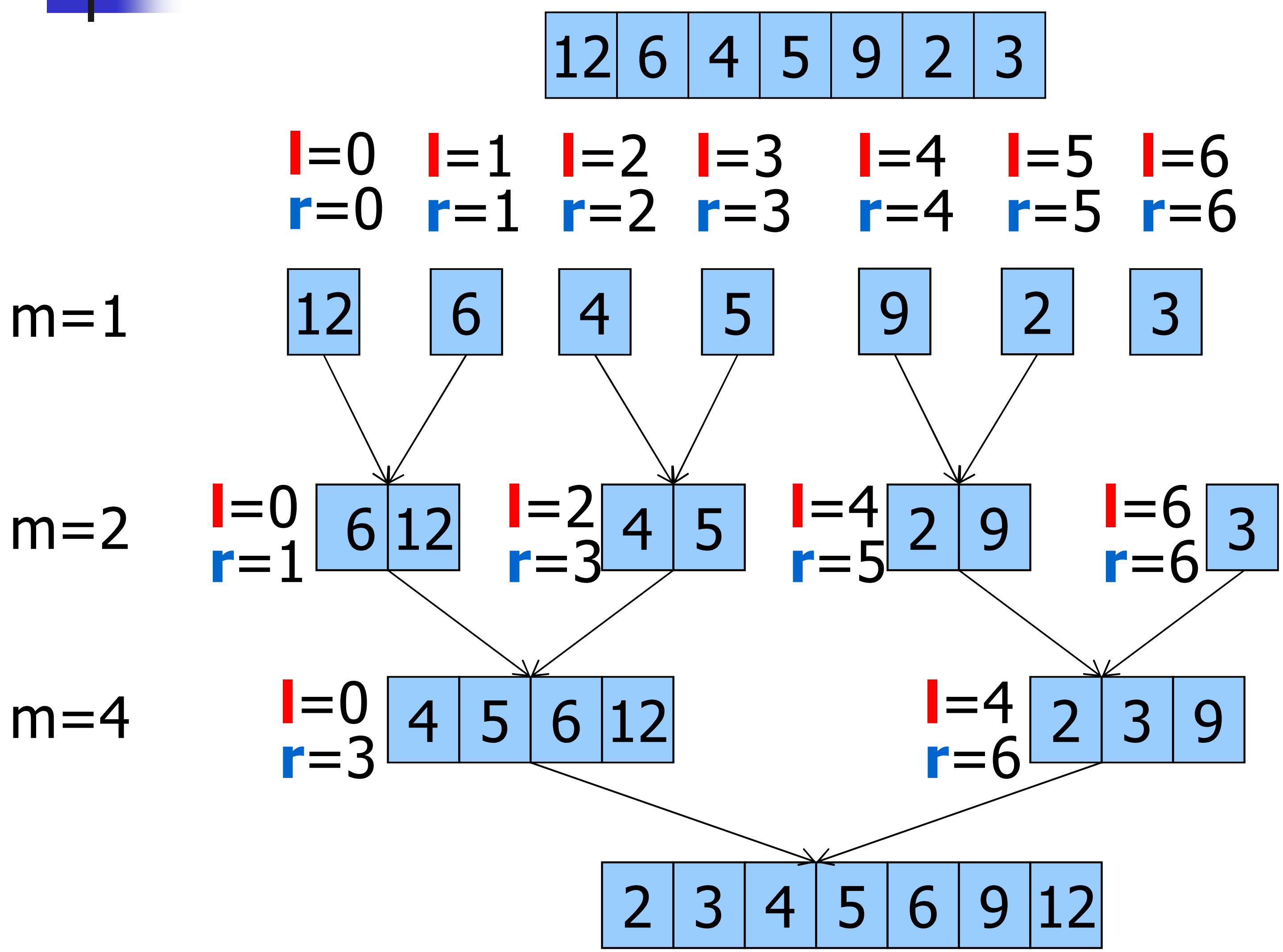
Recursion levels: $\log_2 n$ Operations at each level: n

Total operations: $n \log_2 n$

Bottom-up merge sort

- Non recursive version
- Starting from subarrays of length `1` (thus sorted), apply `Merge` to obtain at each step sorted arrays whose length is twice as big
- Termination
  - The length of the sorted array equals the length of the initial array

Example



12 6 4 5 9 2 3
l=0 r=0
l=1 r=1
l=2 r=2
l=3 r=3
l=4 r=4
l=5 r=5
l=6 r=6
m=1
12
6
4
5
9
2
3
m=2
l=0 r=1
6 12
l=2 r=3
4 5
l=4 r=5
2 9
l=6 r=6
3
m=4
l=0 r=3
4 5 6 12
l=4 r=6
2 3 9
2 3 4 5 6 9 12

Solution

```
int min(int i, int j) {
   if (i < j)
      return i;
   else
      return j;
}
void bottom_up_merge_sort (int *A, int l, int r){
  int i, m, l=0, r=N-1;
  int *B = (int *)malloc(N*sizeof(int));
  for (m = 1; m <= r-l; m = m + m)
    for (i = l; i <= r-m; i += m + m)
      merge (A, B, i, i+m-1, min(i+m+m-1,r));
}
```

B: Auxiliary array (check and free missing)

Boundaries

Consider the pairs of sorted subarrays of size m

Merge them

Quicksort (Hoare, 1961)

- In the division step, the array A[l..r] is partitioned in 2 subarrays L and R
  - Given a pivot element x
  - L, i.e., A[l..q-1], contains all elements $< x$
  - R, i.e., A[q+1..r], contains all elements $> x$
  - x is in the right place
  - Division doesn't necessarily halve the array

Quicksort (Hoare, 1961)

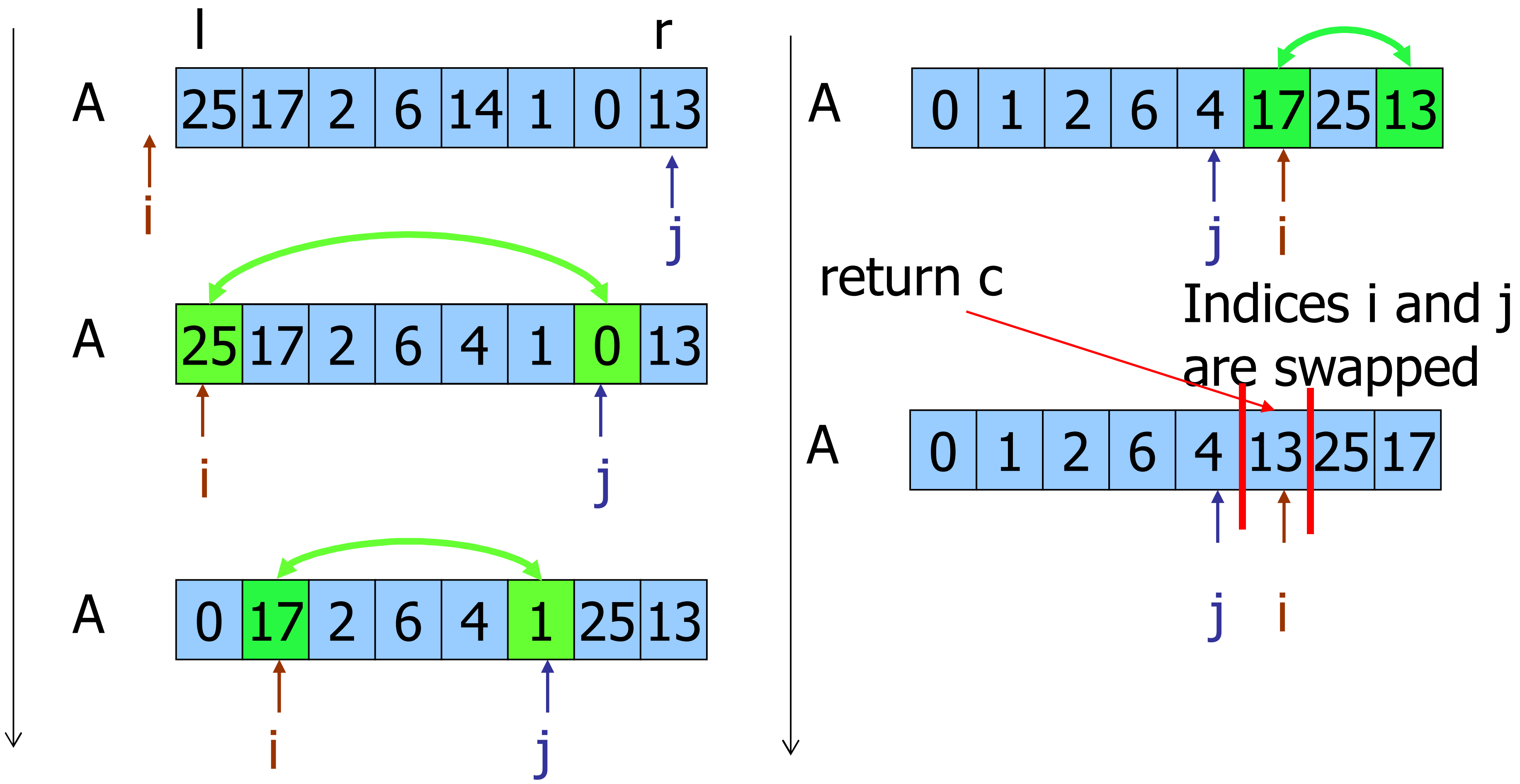
- Recursion
  - Quicksort on subarray L, i.e., A[l..q-1]
  - Quicksort on subarray R, i.e., A[q+1..r]
  - Termination condition: if the array has 1 element it is sorted
- Ricombination
  - None

Partition

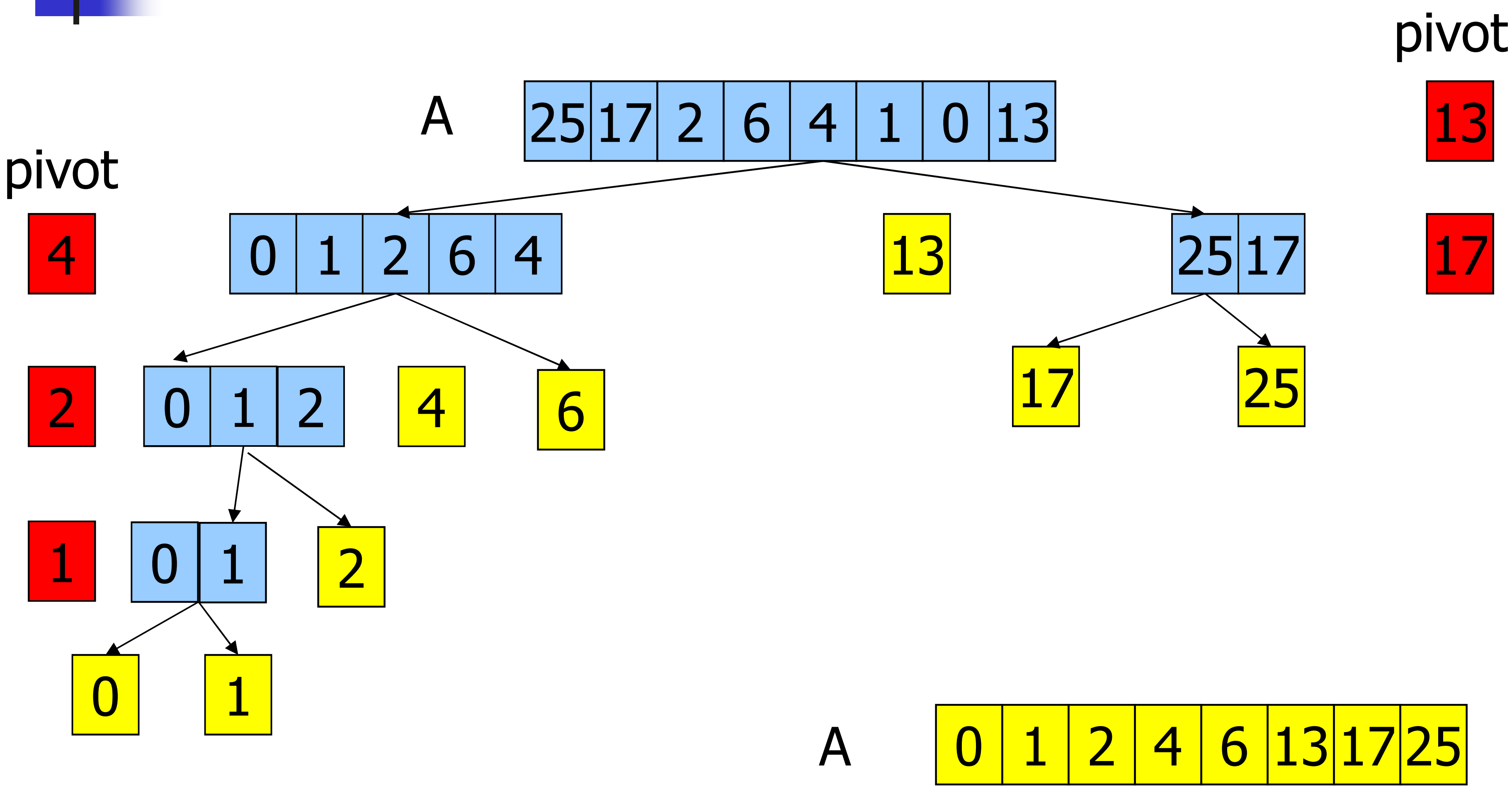
- Pivot pivot = A[r]
- Find A[i] and A[j], i.e., 2 elements not in the proper partition
  - Ascending loop on i until we find an element greater than the pivot x
  - Descending loop on j until we find an element less than the pivot x
- Swap A[i] and A[j]
- Repeat until i < j
- At the end swap A[i] and pivot x
- Return q = i

Partition example

pivot 13



Quicksort example



Solution

Wrapper

```
void quick_sort(int *A, int N) {
  int l, r;
  l = 0;
  r = N-1;
  quick_sort_r (A, l, r);
}
```

Boundaries

Recursive function call

```
void quick_sort_r (int *A, int l, int r){
  int c;
  if (r <= l)
    return;
  c = partition (A, l, r);
  quick_sort_r (A, l, c-1);
  quick_sort_r (A, c+1, r);
  return;
}
```

Termination

Division

Recursive calls

Element c is not moved any more

Solution

```
int partition (int *A, int l, int r ){
  int i, j, pivot;

  i = l-1;
  j = r;
  pivot = A[r];
  while (i<j) {
    while(A[++i]<pivot);
    while(j>l && A[--j]>=pivot);
    if (i < j)
      swap(A, i, j);
  }

  swap (A, i, r);
  return i;
}
```

Pivot values are moved in the right sub-array; worst case: stop on pivot

Pivot values stay in the right sub-array; worst case: stop on element l

Solution

```
void swap (int *v, int n1, int n2) {
   int temp;

   temp  = v[n1];
   v[n1] = v[n2];
   v[n2] = temp;

   return;
}
```

Example: Inverse ordered array

Initial array	:	9 8 7 6 5 4 3 2 1 0
pivot: 0 -	array:	0 8 7 6 5 4 3 2 1 9
pivot: 9 -	array:	0 8 7 6 5 4 3 2 1 9
pivot: 1 -	array:	0 1 7 6 5 4 3 2 8 9
pivot: 8 -	array:	0 1 7 6 5 4 3 2 8 9
pivot: 2 -	array:	0 1 2 6 5 4 3 7 8 9
pivot: 7 -	array:	0 1 2 6 5 4 3 7 8 9
pivot: 3 -	array:	0 1 2 3 5 4 6 7 8 9
pivot: 6 -	array:	0 1 2 3 5 4 6 7 8 9
pivot: 4 -	array:	0 1 2 3 4 5 6 7 8 9

Example: Ordered array

Initial array		:	0	1	2	3	4	5	6	7	8	9
pivot:	9	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	8	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	7	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	6	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	5	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	4	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	3	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	2	- array:	0	1	2	3	4	5	6	7	8	9
pivot:	1	- array:	0	1	2	3	4	5	6	7	8	9

Example: Scrambled array

Initial array :	1	8	0	2	3	9	4	6	5	7
pivot: 7 - array:	1	5	0	2	3	6	4	7	8	9
pivot: 4 - array:	1	3	0	2	4	6	5	7	8	9
pivot: 2 - array:	1	0	2	3	4	6	5	7	8	9
pivot: 0 - array:	0	1	2	3	4	6	5	7	8	9
pivot: 5 - array:	0	1	2	3	4	5	6	7	8	9
pivot: 9 - array:	0	1	2	3	4	5	6	7	8	9

Features

- In place
- Not stable
  - Partition may swap "far away" elements
  - Then occurrence of a duplicate key moves to the left of a previous occurrence of the same key

Analysis

- Efficiency depends on the partition balance
- At each step **partition** returns
  - In the worst case a subarray with n-1 elements and a subarray with 1 element
  - In the best case 2 subarrays with n/2 elements
  - In the average case 2 subarrays of different sizes
- Balancing depending on the choice of the pivot

Worst case

- Worst case
  - Pivot = minimum or maximum (array already sorted)
- Recusion equation
  - $T(n) = T(n-1) + n \qquad n>=2$
  - $T(1) = 1 \qquad n = 1$
- Solution
  - $T(n) = O(n^2)$

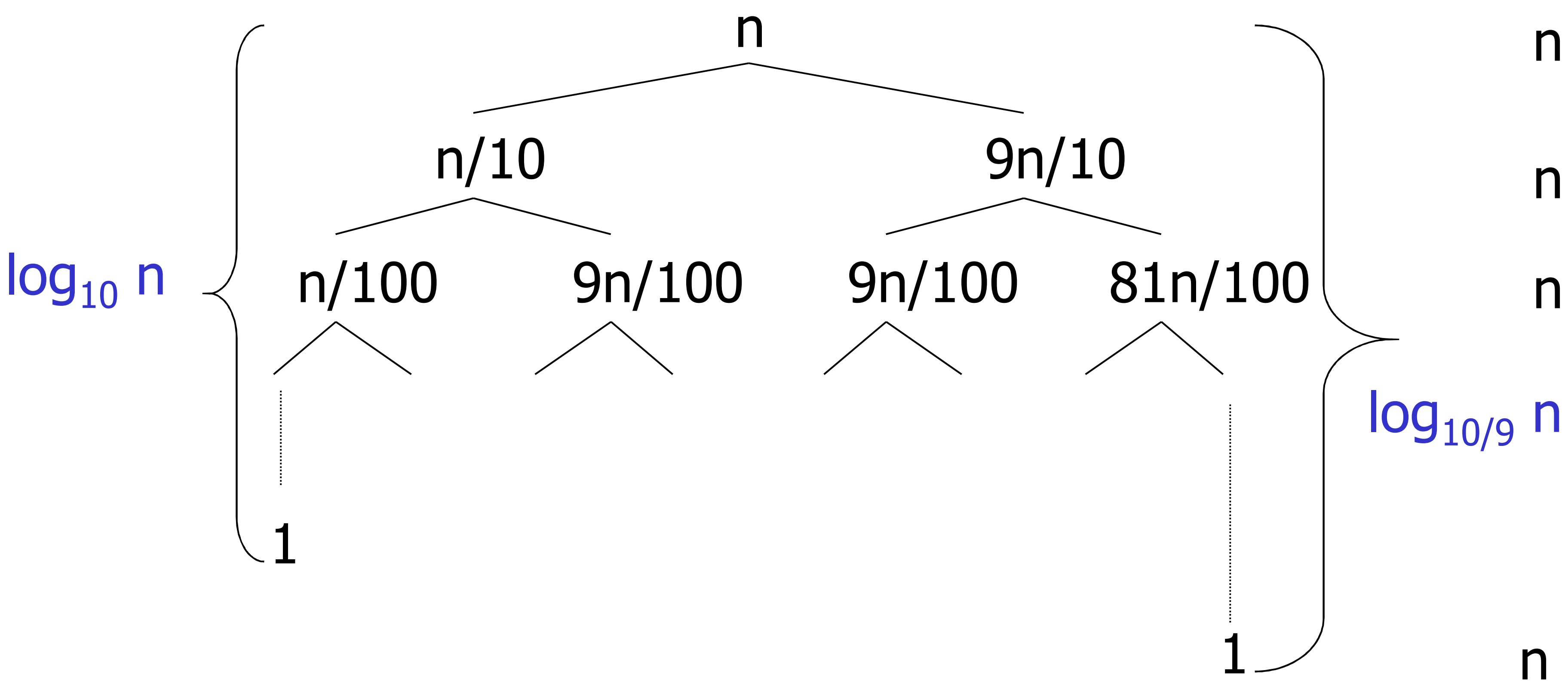
Best case

- Recursion equation
  - $T(n) = 2T(n/2) + n$ $\quad n>=2$
  - $T(1) = 1$ $\quad n = 1$
- Solution
  - $T(n) = O(n \lg n)$

Average case

- Provided we are not in the worst case, though partitions may be strongly unbalanced
  - Average case = best case
- Example
  - At each step **partition** generates 2 partitions
  - Let us suppose the first one has $(9/10 \cdot n)$ elements and the second one $(n/10)$ elements

Average case



$T(n) = O(n \lg n)$

Pivot selection

- Random element
  - Generate a random number i with $p \le i \le r$, then swap A[r] and A[i], use A[r] as pivot
- Middle element
  - $x \leftarrow A[(p+r)/2]$
- Select average between min and max
- Select median of 3 elements chosen randomly in array
- …

Summary on sorting algorithm

Algorithm	In place	Stable	Worst-Case
Bubble sort	Yes	Yes	$O(n^2)$
Selection sort	Yes	No	$O(n^2)$
Insertion sort	Yes	Yes	$O(n^2)$
Shellsort	Yes	No	depends
Mergesort	No	Yes	$O(n \cdot \log n)$
Quicksort	Yes	No	$O(n^2)$
Counting sort	No	Yes	$O(n)$