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The paradox of the floating candle that continues to burn

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What happens after lighting a paraffin candle that is barely floating in water and kept upright with the aid of an appropriately weighted nail attached to its bottom? Presumably, it should sink because the buoyant force will decrease more than the weight. Surprisingly, the candle will continue to burn, rising slowly above the surface of the water. The reason for this is that the flame forms a well around the wick filled with molten paraffin, while the water keeps the outer walls of the candle cool and unscathed. Thus, the buoyancy hardly changes while the weight is reduced through burning, resulting in a floating candle that will rise above water. We present a quantitative model that describes the formation of the well and verify it experimentally, examining first the case of a candle in the air and then the case of a candle immersed in water.
[http://dx.doi.org/10.1119/1.4726320]

I. INTRODUCTION

Towards the end of the 19th century Arthur Good, under the pseudonym Tom Tit, wrote weekly articles in the French magazine L'Illustration, titled La Science Amusante. In these articles, he presented amusing physical experiments using everyday objects, demonstrations, recreations, scientific games, and paradoxes, all with classic engravings by Poyet. Three hundred of these experiments were published in 1890, 1892, and 1893.[1] These volumes became very popular and have been reprinted many times.[2]

Among these experiments is a particularly interesting one, the *singular candle* (*un singulier bougeoir*), involving an unsinkable burning candle.[3] A nail is inserted in the bottom of a cylindrical candle of appreciable diameter. The weight of the nail is such that the candle can float upright in a glass of water with its wick being the only part of the candle above the surface. The glass of water thus serves as a candlestick. We then light the wick and ask what happens.

In a time interval dt, a volume dV of paraffin will burn. Let us make the elementary assumption that the paraffin burns uniformly across the top of the cylindrical candle, in which case all of dV contributes to the decrease in buoyancy. In this case, the buoyancy is reduced by $\rho_w\,dV$, where ρ_w is the density of the water, while the weight is reduced by $\rho\,dV$, where ρ is the density of the solid paraffin. Consequently, the weight should be reduced less than the buoyancy ($\rho < \rho_w$), resulting in the sinking of the candle and the extinction of the flame.

Actually, the candle will continue to burn and will slowly rise above the water, demonstrating that our assumption that the paraffin is burned uniformly across the entire cylinder is erroneous. As Tom Tit remarked, the cylindrical walls of the candle are kept cold by the water so the flame has no other option but to melt only the interior of the candle, digging a deep cavity filled with molten paraffin.

We can understand this result if we perform a little thought experiment. Let us assume the candle mentioned above has an extremely large diameter. When we light the wick at the middle, the remote cylindrical walls of the candle will not be affected at all. The paraffin will be burned only at the center, resulting in the inevitable formation of a well. For the few moments just after lighting the candle, we expect the buoyancy to be unaffected, while the weight will be slowly reduced. This reduction in weight will result in some lifting of the candle above the surface. Nonetheless, the candle will keep burning.

Let us assume now that the diameter of the candle is intermediate in size so that the well formed occupies most of the cross section of the candle. We will still have a deepening depression around the wick as the candle burns, and again the weight will be reduced (initially) without any appreciable change in the external shape of the candle. As a result, the candle will rise a little above the surface, enabling the flame to melt the paraffin at the rim. Because the rim is above the surface of the water, this burning will reduce the weight without affecting the buoyancy. Thus, the candle will tend to sink back down a little. This process is a continuous one, of course, but the end result is that the burning candle continues to float with the top of the candle remaining at the same position.

The paradox is thus explained, for candles of appreciable diameter, by the formation of a well around the wick at the center of the candle. This situation is illustrated in Fig. 1, using a picture from the book of Tom Tit.[4]

What about a small-diameter candle? If the candle is relatively thin, having a diameter smaller than the size of the well that would naturally develop, it will be impossible for the flame to dig a well at the center. The burning of the candle will create a mass of molten paraffin that cannot find refuge in a well around the flame. The melting paraffin will encounter the cold water soon after its creation, before having time to burn, and thus will solidify on the surface of the water, forming a thin but wide annular collar around the cylindrical candle. This collar will keep the candle afloat due to the surface tension of the water.

Floating candles were not without practical applications. Before gas lamps or electric light was available, people used candles as a source of light. With a candle in an ordinary candlestick, the height of the light source varies significantly as the candle burns. In contrast, a floating candle is a light source that maintains a constant height. Furthermore, the floating candle has become a favorite classroom demonstration and an excellent physics puzzle.[5]

II. THEORETICAL MODEL

Studies on the behavior of combustion have been widely performed from the viewpoint of engineering and are generally aimed at controlling combustion in order to avoid explosions

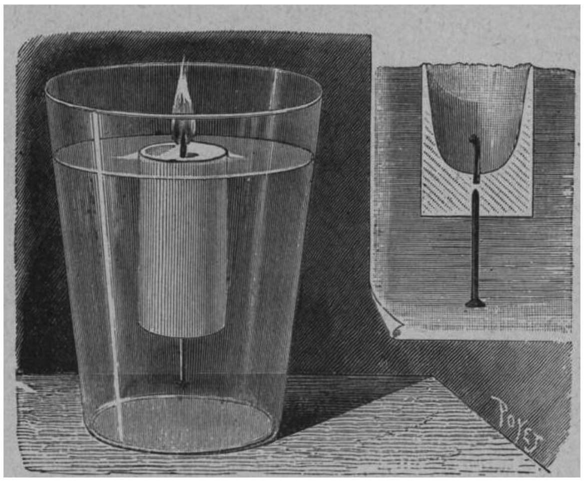

Fig. 1. The well formed around the flame of the candle, as depicted in Ref. 3.

or instabilities.[6] Wax candles, on the other hand, have received relatively little attention in the combustion literature, owing partly to experimental and analytical difficulties.[7] Elaborate theoretical and computational models have been developed to describe the behavior of candle flames,[8] but they are rather complicated and they describe only the flame, not the cavity formed by the flame. In fact, unlike the shape of the flame, the shape of the cavity formed in a candle has not even been examined experimentally.[9]

We present in this paper a simple quantitative model that describes the formation of the well in a burning candle of appreciable diameter (larger than the radius of the well formed). Because the walls of the candle will be cooler when in water than in air, the radius of the well will depend on whether the candle is immersed in water or air. The structure of the model, however, will not depend on whether the candle is in water or not. We will adopt simple, reasonable assumptions and see if the main features of the burning candle can be reproduced. We note that even when the candle is in air a well will be formed, provided the radius of the well is smaller than the radius of the candle. If this were not the case, then the molten paraffin would overflow onto the walls of the candle to form beautiful "stalactites" on solidification.

Our first assumption is that the flame emits a steady axisymmetric lateral radiative heat flux $f(r)$, which is a function of only the radial distance r from the wick. This vertically averaged $f(r)$, which has units of energy per unit area per unit time, is constant as a function of the height above the base of the flame by assumption. In reality, the radiative flux does vary as a function of the height,[10] but our assumption is reasonable because the paraffin of the wall is melted by the lower heat flux and flows down into the pool, leaving the wall vertical in shape and rendering the influence of the upper heat flux irrelevant. We will therefore neglect any heat flux variations in the vertical direction as the flame is digging a well around the wick.

Let us begin by examining a cylindrical paraffin candle surrounded by air, the diameter of which exceeds the radius of the well that will be formed. In this situation, a cylindrical well will be created by the flame. The actual candles we investigated had the following physical properties: a diameter of 7 cm, a wick radius of 1.5 mm, a height of 10.2 cm, and a density of $\rho = 0.9\ \mathrm{g/cm^3}$; the density of molten paraffin is $\rho_m = 0.8159\ \mathrm{gr/cm^3}$. The mean height of the flame was 2.5 cm while its average width was about 7 mm. The base of the flame was about 4 mm above the molten paraffin pool.

Figure 2 shows a schematic drawing of the candle that helps define our variables. We let $R(t)$ be the radius of the well, $y(t)$ is the depth of the well, $u(t)$ is the distance from the top of the molten paraffin pool to the top of the candle, and $w(t)$ is the depth of the molten paraffin pool within the well. These variables are constrained by the relation

$$u(t) = y(t) - w(t). \tag{1}$$

A. Radial energy flux and well radius

The outgoing energy flux will diminish as we move radially away from the flame. In fact, recent experiments show that the energy flux is approximately a linearly decreasing function of the radial distance from the flame centerline.[10] To understand this experimental result, we let $T(r,z)$ be the temperature field with r the radial distance from, and z the distance along, the axis of the flame. We let ℓ be the radial distance beyond which the flame cannot melt the paraffin. Then, for $r < \ell$ the paraffin will melt and for $r > \ell$ we will have solid paraffin at an essentially constant temperature T_0, and consequently $T(\ell, z) = T_0$. Because the temperature reaches its minimum at $r = \ell$, we expect

$$\frac{\partial T}{\partial r}(\ell, z) \approx 0. \tag{2}$$

If we now expand $T(r, z)$ about the boundary $r = \ell$, we find

$$T(r, z) \approx T_0 + \frac{\partial^2 T}{\partial r^2}(\ell, z)\frac{(r-\ell)^2}{2}. \tag{3}$$

However, Fourier's law of thermal conduction indicates that the energy flux is proportional to $-\nabla T$. Making use of Eq. (3), we find that the local radial energy flux is then proportional to

$$\frac{\partial T}{\partial r} \approx \frac{\partial^2 T}{\partial r^2}(\ell, z)(r-\ell). \tag{4}$$

We therefore deduce that the radial energy flux depends on the height z above the base of the flame, a deduction

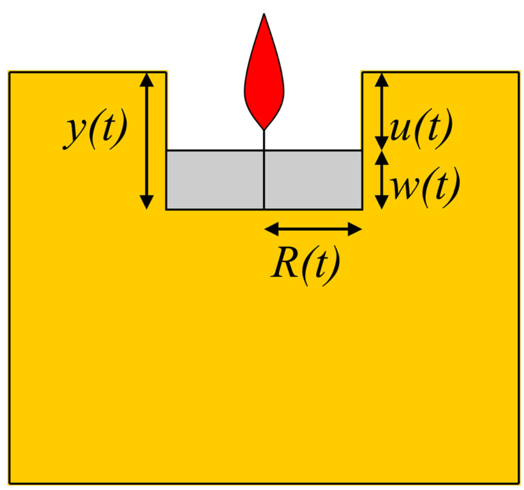

Fig. 2. Schematic diagram of the well carved around the wick by the flame.

confirmed by recent experiments,[10] and it varies like $(r-\ell)$. We then make the simple assumption that the radial energy flux, averaged over the z-axis, diminishes linearly as

$$f(r) = f_0\left(1-\frac{r}{\ell}\right), \tag{5}$$

where f_0 is a measure of the energy flux emanating from the candle flame.

When the wall of the well has a radius $R(t)$ and a depth $y(t)$, the radial energy flux during a time interval dt causes the melting of a ring of thickness dR and height $y(t)$ that contains a mass $2\pi Ry\rho dR$ of paraffin. The amount of energy expended during this time is $2\pi Ryf(R)dt$. If L is the latent heat for melting paraffin, we have

$$2\pi Ryf(R)dt = L2\pi Ry\rho\, dR, \tag{6}$$

which reduces to

$$\frac{dR}{dt} = \frac{(\ell - R)}{\tau}, \tag{7}$$

where $\tau = \rho\ell L/f_0$ is a natural time constant for this problem—the time it takes to melt a thickness ℓ of a unit area of paraffin. Introducing $\kappa = 1/\tau$ for convenience, Eq. (7) can be solved to give

$$R(t) = \ell - (\ell - d)e^{-\kappa t}, \tag{8}$$

where we have assumed an initial well radius $R(0) = d$ is created when the wick is first lit.

B. Vertical energy flux and well depth

We now examine the transformation of an elementary mass of paraffin at the bottom of the well. At time t, the well has a depth $y(t)$ while the molten paraffin pond has a depth $w(t)$. If the flame was extinguished, the pond would solidify and contract in volume. The depth of the (solidified) well would be $y_s = y - \rho_m w/\rho$. At a later time $t+dt$, the well will have depth $y+dy$ and the molten paraffin pond will have depth $w+dw$. If we were to extinguish the flame now, the solidified well would have depth $y_s + dy_s$, where $dy_s = dy - \rho_m dw/\rho$. This increase in the depth of the solidified well would be solely due to the burning of paraffin. If the paraffin combustion rate is j, then the burned mass is $j\,dt$ and the corresponding difference in depth dy_s would be $j\,dt/(\rho\pi R^2)$. Putting this all together gives

$$\frac{jdt}{\rho\pi R^2} = dy_s = dy - \frac{\rho_m}{\rho}dw, \tag{9}$$

or equivalently

$$\frac{dy}{dt} = \frac{j}{\rho\pi R^2} + \frac{\rho_m}{\rho}\frac{dw}{dt}. \tag{10}$$

This differential equation includes both the well depth and the molten paraffin depth and can be integrated to give

$$y(t) = w\frac{\rho_m}{\rho} + \frac{j(R-d)}{\pi\rho R\kappa\ell d} + \frac{j}{\pi\rho\kappa\ell^2}\ln\left(\frac{R}{d}\right) + \frac{jt}{\pi\rho\ell^2}, \tag{11}$$

where we used Eq. (8) to simplify the result.

Of course, the energy flux from the candle is directed vertically as well as laterally. The local vertical energy flux is proportional to the temperature gradient $\partial T/\partial z$ and, according to Eq. (3), is proportional to $(r-\ell)^2$. We assume then that the steady axisymmetric radiative flux that falls vertically on the bottom of the well is given by

$$F(r) = \eta f_0\left(1-\frac{r}{\ell}\right)^2, \tag{12}$$

where η is a dimensionless parameter that governs the fact that the flux emitted vertically is different from the average flux emitted radially. As with the radial situation, the vertical flux becomes ineffective beyond a radius ℓ. Using Eq. (12), we can calculate the total power received by the bottom of the pool as

$$P = \int_0^R 2\pi rF(r)dr = \pi\eta f_0 R^2\left[1 - \frac{4R}{3\ell} + \frac{R^2}{2\ell^2}\right], \tag{13}$$

which, using Eq. (8), can be written as

$$P = \pi\eta L\rho\kappa\left[\frac{(\ell-d)^4}{2\ell}e^{-4\kappa t} - \frac{2}{3}(\ell-d)^3 e^{-3\kappa t} + \frac{\ell^3}{6}\right]. \tag{14}$$

This power is necessary to maintain the molten paraffin of thickness $w(t)$ as well as for melting some additional solid paraffin at the bottom of the well. In a time interval dt, the energy needed to melt a cylindrical slice of solid paraffin of thickness dy is $L\rho\pi R^2 dy$. Meanwhile, to maintain the pool in liquid form requires a power that is proportional to the mass of the already molten paraffin. We take the constant of proportionality to be kL, where k is a constant with units of s^{-1}, so the stabilizing power is $kL\rho_m w\pi R^2$. Putting this all together, we find the energy radiated vertically in time dt is

$$Pdt = L\rho\pi R^2 dy + kL\rho_m w\pi R^2 dt. \tag{15}$$

Combining Eq. (9) with Eqs. (13) and (15), we obtain

$$\frac{dw}{dt} + kw + \frac{j}{\pi R^2\rho_m} = \frac{\eta\kappa\rho}{\rho_m}\left(\ell - \frac{4R}{3} + \frac{R^2}{2\ell}\right), \tag{16}$$

or equivalently

$$\frac{d}{dt}(we^{kt}) = e^{kt}\frac{\eta\kappa\rho}{\rho_m}\left(\ell - \frac{4R}{3} + \frac{R^2}{2\ell}\right) - e^{kt}\frac{j}{\pi\rho_m R^2}. \tag{17}$$

We then substitute for R using Eq. (8) and integrate the resulting equation subject to the initial conditions $w(0) = 0$ and $R(0) = d$, to obtain

$$\begin{aligned} w(t)e^{kt} = &\frac{\eta\kappa\rho}{6\rho_m}\left[\frac{\ell}{k}(e^{kt}-1) + \frac{2(\ell-d)}{(k-\kappa)}(e^{(k-\kappa)t}-1)\right.\\ &\left.+\frac{3\ell - 6d + 3d^2/\ell}{k-2\kappa}(e^{(k-2\kappa)t}-1)\right]\\ &-\int_d^R \frac{j\left(\frac{\ell-d}{\ell-r}\right)^{k/\kappa}}{\pi\kappa\rho_m r^2(\ell-r)}dr. \end{aligned} \tag{18}$$

Making the variable substitution $\xi = 1/(\ell - r)$ allows us to perform the above integral, giving

$$w(t) = \frac{\eta\kappa\rho}{6\rho_m}\left[\frac{\ell}{k}(1-e^{-kt}) + \frac{2(\ell-d)}{(k-\kappa)}(e^{-\kappa t}-e^{-kt}) + \frac{3\ell-6d+3d^2/\ell}{k-2\kappa}(e^{-2\kappa t}-e^{-kt})\right] - \frac{j(R(t)/\ell)^{k/\kappa}}{k\pi\rho_m\ell^2}\,{}_2F_1\left(-\frac{k}{\kappa},-1-\frac{k}{\kappa},1-\frac{k}{\kappa},\frac{d-\ell}{R(t)}e^{-\kappa t}\right) + \frac{j(d/\ell)^{k/\kappa}}{k\pi\rho_m\ell^2}e^{-kt}\,{}_2F_1\left(-\frac{k}{\kappa},-1-\frac{k}{\kappa},1-\frac{k}{\kappa},\frac{d-\ell}{d}\right), \tag{19}$$

where ${}_2F_1$ is a hypergeometric function.

Equation (19) provides the final piece of the puzzle. The well radius $R(t)$ is given by Eq. (8), which is used in Eq. (19) to find $w(t)$; finally $y(t)$ is determined via Eq. (11). Using this procedure, we can determine what happens at long times by letting $t \to \infty$. In this case $R \to \ell$,

$$w \to w_\infty \equiv \frac{\eta\kappa\rho\ell}{6k\rho_m} - \frac{j}{k\pi\rho_m\ell^2}, \tag{20}$$

and

$$y \to \frac{\rho_m w_\infty}{\rho} + \frac{j(\ell-d)}{\pi\rho\ell^2\kappa d} + \frac{j}{\pi\rho\kappa\ell^2}\ln\left(\frac{\ell}{d}\right) + \frac{jt}{\pi\rho\ell^2}. \tag{21}$$

Our model thus predicts that the well will eventually acquire a constant radius and a linearly increasing depth, while the pool of molten paraffin attains a fixed thickness. In particular, our model predicts that the radius of the well eventually becomes equal to ℓ, that the thickness of the layer of molten paraffin approaches w_∞, and that $y(t)$ takes the asymptotic form $a + bt$, with

$$a = \frac{\rho_m w_\infty}{\rho} + \frac{j(\ell-d)}{\pi\rho\ell^2\kappa d} + \frac{j}{\pi\rho\kappa\ell^2}\ln\left(\frac{\ell}{d}\right) \tag{22}$$

and

$$b = \frac{j}{\pi\rho\ell^2}. \tag{23}$$

As discussed more fully below, these are precisely the results of our experiments. In fact, we can use the observed values of a, b, and w_∞ to solve for the parameters k, j, and κ

$$j = \pi b\ell^2\rho, \tag{24}$$

$$k = \frac{\eta b\ell\rho^2}{6d\rho_m w_\infty}\frac{\ell-d+d\ln(\ell/d)}{a\rho - w_\infty\rho_m} - \frac{b\rho}{\rho_m w_\infty}, \tag{25}$$

and

$$\kappa = \frac{\rho b}{d}\frac{\ell-d+d\ln(\ell/d)}{a\rho - w_\infty\rho_m}. \tag{26}$$

III. COMPARISON WITH EXPERIMENTS

A. A candle in air

We performed experiments with candles in air and in water and measured $R(t)$, $w(t)$, $y(t)$, and $u(t)$. The radial distance was measured using a tape measure. To measure the depth of the pool, we lowered thin sticks of wood into the molten paraffin until they touched the bottom and then measured the length of the solidified paraffin on the sticks. The same sticks were used to measure the depth of the well.

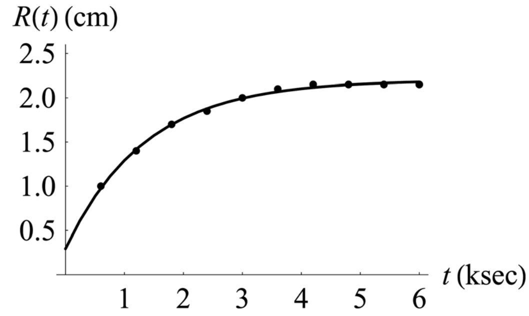

Fig. 3. The radius of the well formed around the wick of a candle surrounded by air as a function of time. The dots are the experimental points while the curve is the prediction of Eq. (8) using $\ell = 2.2$ cm and $d = 0.29$ cm.

From our experiments with candles in air we find $w_\infty = 0.5$ cm. For long times, we find $y(t) \approx a + bt$, with $a = 1.05$ cm and $b = 5.2 \times 10^{-5}$ cm/s. Furthermore, fitting our experimental results for $R(t)$ to Eq. (8) yields the values $\ell = 2.2$ cm and $d = 0.29$ cm. As shown in Fig. 3, the agreement between theory and experiment is quite good. We note that the initial radius d of the molten paraffin pond that appears immediately after lighting the candle is around 0.3 cm, while the average radius of the flame was observed to be about 0.35 cm. Thus, it is the area exactly beneath the flame that melts instantaneously after lighting the wick.

Using Eqs. (24) and (26), we find $j = j_a = 7.15 \times 10^{-4}$ g/s and $\kappa = 7.4 \times 10^{-4}$ s^{-1}. The parameter j_a is the combustion rate for our paraffin candle when it is surrounded by air and it is approximately 0.043 g/min. The time constant $\tau = 1/\kappa \approx 22.5$ min is a measure of the time needed for the radius to reach its asymptotic value. For example, the time needed for the radius of the well to reach the 90% of its asymptotic value is 48.5 min.

Figure 4 shows a fit of Eq. (19) to our experimental values of $w(t)$, giving $\eta = 0.723$ and consequently $k = 3.2 \times 10^{-4}$ s^{-1}, where Eq. (25) is used to find k. Again, we see reasonably good agreement between theory and experiment. We can infer from Eqs. (5) and (12) that the heat flux incident on the bottom of the well at a distance of 1.1 cm from the axis is a fraction $\eta(1-1.1/\ell)^2/(1-1.1/\ell) = 0.36$ of the horizontal heat flux, a value confirmed by recent experiments.[10] In addition, Figs. 5

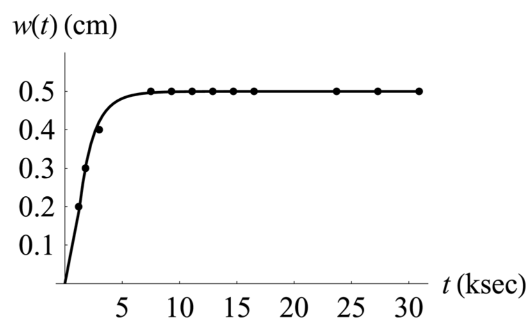

Fig. 4. The depth of the molten paraffin layer as a function of time for a candle surrounded by air. The dots are the experimental points while the curve is the prediction of Eq. (19) using $\ell = 2.2$ cm, $d = 0.29$ cm, and $\eta = 0.723$.

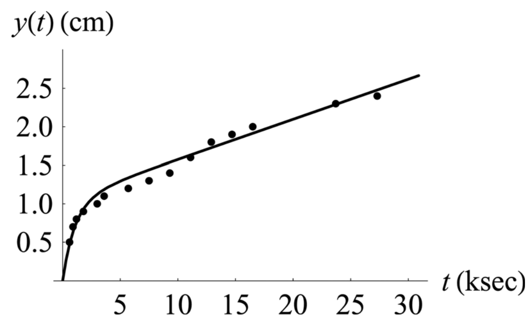

Fig. 5. The depth of the well as a function of time for a candle surrounded by air. The dots are the experimental points, while the curve is the prediction of Eq. (11) using $\ell = 2.2$ cm, $d = 0.29$ cm, and $\eta = 0.723$.

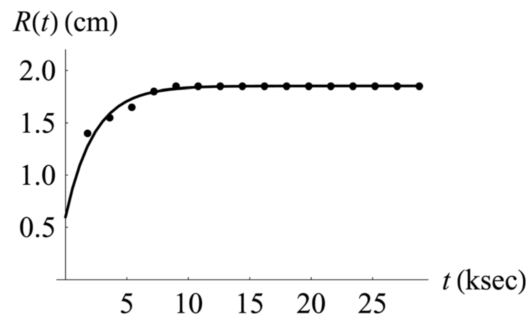

Fig. 7. The radius of the well formed around the wick of a candle immersed in water as a function of time. The dots are the experimental points while the curve is the prediction of Eq. (8) using $\ell = 1.85$ cm and $d = 0.597$ cm.

and 6 show our experimental data for $y(t)$ and $u(t)$ along with the predictions of Eqs. (11) and (1).

B. A candle wholly immersed in water

We now examine the case of a candle immersed in water up to the base of the wick. The theoretical model developed earlier remains valid here as long as ℓ is smaller than the radius of the candle. Otherwise, the paraffin on the rim will melt and spill into the water where it solidifies to form a collar around the floating candle.

For the candle immersed in water, we find that for long times, the thickness $w(t)$ of the molten paraffin pond reaches the asymptotic value $w_\infty = 0.4$ cm, while $y(t) \approx a + bt$ with $a = 1.04$ cm and $b = 9.16 \times 10^{-5}$ cm/s. We observe that the depth of the well increases more quickly than before. This makes sense if we consider that the outside of the candle is cooler than before, which means more energy will be required to melt the walls of the candle so that the well will end up with a smaller radius. A smaller well radius then confines the vertical energy flow into a smaller area, resulting in a depth that increases more quickly with time.

We measure $R(t)$ and fit the data to the predictions of Eq. (8) using $\ell = 1.85$ cm and $d = 0.597$ cm. As shown in Fig. 7, the results are quite good. Using Eqs. (24) and (26), we find $j = j_w = 8.88 \times 10^{-4}$ g/s (0.053 g/min) for the combustion rate and $\kappa = 4.3 \times 10^{-4}\,\mathrm{s}^{-1}$. Again, the time constant $\tau = 1/\kappa \approx 38.4$ min is a measure of the time needed for the radius to reach its asymptotic value. For example, the time needed for the radius of the well to reach the 90% of its asymptotic value is 73.4 min when the candle is surrounded by water. We see that the candle burns more intensely when it is surrounded by water, but the flame takes more time to melt the walls of the well because the water does a better job of keeping the candle cool.

We also measured the thickness $w(t)$ of the molten paraffin pond when it is immersed in water and fit the data using Eq. (19). The best fit is obtained for $\eta = 1.46$, in which case $k = 2.9 \times 10^{-4}\,\mathrm{s}^{-1}$. Note that this value of k is essentially the same as the value of k obtained for a candle burning in air; this is to be expected because the power required to maintain a unit mass of paraffin in the pool in molten form does not depend appreciably on the state of the walls of the candle. Figure 8 shows a reasonably good fit between theory and experiment.

We note that there is a hump in both the experimental points and the theoretical curve of $w(t)$ for a candle immersed in water (see Fig. 8). This feature is caused by the fact that paraffin melts both on the wall and at the bottom of the well initially, whereas it melts only at the bottom once the radius of the well is constant. For a candle surrounded by air, the well is wider and thus the amount of paraffin melted off the wall is not as important in this case.

All the unknown parameters have been determined, so we can plot $y(t)$ and $u(t)$ as functions of time. The comparison of the experimental points with the predictions of Eqs. (11) and (1) shows that there is again reasonable agreement, as can be seen in Figs. 9 and 10.

IV. A CANDLE PARTLY IMMERSED IN WATER

We have seen that our model reproduces quite adequately the experimental behavior of the candle that is fully immersed in air or in water. There is, however, one subtle

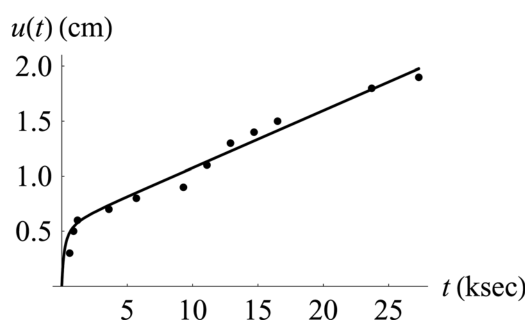

Fig. 6. The distance from the surface of the pond to the top of the candle as a function of time for a candle surrounded by air. The dots are the experimental points while the curve is the prediction of Eq. (1) using $\ell = 2.2$ cm, $d = 0.29$ cm, and $\eta = 0.723$.

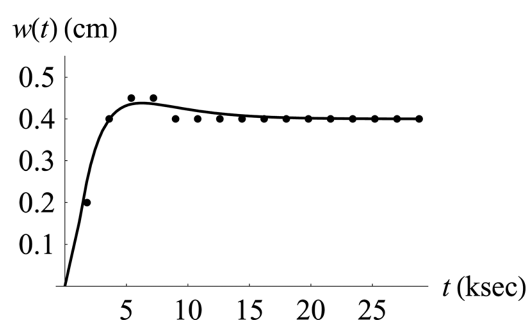

Fig. 8. The thickness of the molten paraffin layer as a function of time for a candle surrounded by water. The dots are the experimental points while the curve is the prediction of Eq. (19) using $\ell = 1.85$ cm, $d = 0.597$ cm, and $\eta = 1.46$.

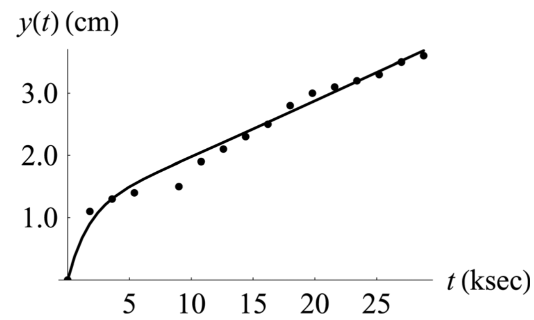

Fig. 9. The depth of the well as a function of time for a candle surrounded by water. The dots are the experimental points while the curve is the prediction of Eq. (11) using $\ell = 1.85\,\text{cm}$, $d = 0.597\,\text{cm}$, and $\eta = 1.46$.

point that we have overlooked. The parameters j and ℓ depend on whether the candle is surrounded by water or air. In our treatment of the candle in water we assumed that it was wholly immersed. In fact, as the candle becomes lighter, it will rise a small amount above the surface of the water. Thus, a large part will be beneath the surface of the water and the rest will be in air. As a result, the well will not be a perfect cylinder. Because the asymptotic radius is 2.2 cm in air but only 1.85 cm in water, the well will be a bit wider at the top than at the bottom.

Let the distance from the surface of the water to the top of the candle be x. In a time interval dt, the candle will lose an amount of mass given by $j\,dt$. This mass loss will be accompanied by a decrease in buoyancy of $\rho_w \pi D^2 dx$, where $D = 3.5\,\text{cm}$ is the radius of the candle and ρ_w is the density of water. Equating these leads to

$$\frac{dx}{dt} = \frac{j}{\rho_w \pi D^2}. \tag{27}$$

Unfortunately, it is not obvious what value of j should be used. We shall assume that initially j has the value j_0. This value will be of the order of the already measured combustion rates j_a and j_w (around $8 \times 10^{-4}\text{g/s}$). Equation (27) then tells us that the initial value of dx/dt is around $2 \times 10^{-5}\text{cm/s}$. Hence in 90 min the candle will have risen above the water by only 1 mm, which was indeed observed experimentally. In these first 90 min, however, the radius of the well has ample time to reach the maximum size of $\ell_w = 1.85\,\text{cm}$, it can have when surrounded by water. From this point on the well can only become deeper, not wider. And because the wick is at the bottom of the well, the burning takes place below the surface of the water where the burn rate is j_w.

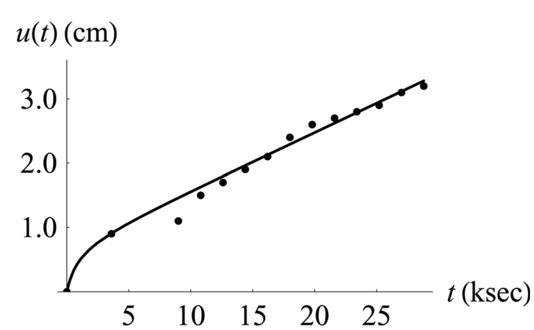

Fig. 10. The distance from the surface of the pond to the top of the candle as a function of time for a candle surrounded by water. The dots are the experimental points while the curve is the prediction of Eq. (1) using $\ell = 1.85\,\text{cm}$, $d = 0.597\,\text{cm}$, and $\eta = 1.46$.

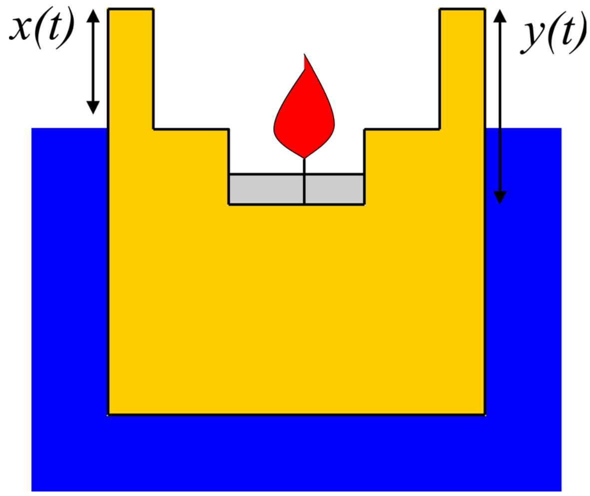

Fig. 11. Once a portion of the candle rises above water, the flame carves out a wider cylinder of radius ℓ_a above the water level, while it carves out a cylinder of radius ℓ_w below the water level. The top of the candle is at a height x above water while the depth of the well is y.

However, as the candle continues to burn, more of it rises above the surface of the water, which results in a larger asymptotic well radius ($\ell_a = 2.2\,\text{cm}$) for the top of the candle. Thus, the candle will adopt a shape as shown in Fig. 11. In fact, this shape of two successive cylindrical cavities appears more clearly when the candle is fixed to the bottom of a jar of water with a substantial portion sticking out of the water.

Let us assume then that the burning of the paraffin takes place initially at a rate j_0 for a time t_0. From then on, the burning takes place at an effective burn rate j_{eff}. This scenario results in a total mass loss in time t of $j_0 t_0 + j_{\text{eff}}(t - t_0)$. This mass loss will be offset by a reduction in buoyancy, hence the candle will rise a height $x(t)$ above water, where $x(t)$ satisfies

$$j_0 t_0 + j_{\text{eff}}(t - t_0) = \rho_w \pi D^2 x(t). \tag{28}$$

Using this model, the portion of the candle above the surface of the water is given by

$$x(t) = \begin{cases} \dfrac{j_0 t}{\rho_w \pi D^2}, & \text{for } t < t_0 \\ \dfrac{j_0 t_0 + j_{\text{eff}}(t - t_0)}{\rho_w \pi D^2}, & \text{for } t > t_0. \end{cases} \tag{29}$$

We have experimentally measured $x(t)$ and fitted Eq. (29) to the experimental points to find $j_0 = 7.127 \times 10^{-4}\text{g/s}$, $t_0 = 6607\,\text{s}$, and $j_{\text{eff}} = 1.12 \times 10^{-3}\text{g/s}$ (see Fig. 12). We note that j_{eff} is larger than j_w. If the candle had been forced to be under the water up to its top, then for long times, when the radius would have stopped increasing, we would have had a combustion rate of j_w, since the only melting and burning would occur at the bottom of the well. However, having a well that consists of two cylindrical cavities raises the combustion rate from j_w to j_{eff} as more paraffin is consumed within a given time interval. The additional paraffin burned comes from the lower horizontal wall of the upper cavity. This is shown quantitatively below. We note that there is always a horizontal wall between the cavity above water and the cavity below water.

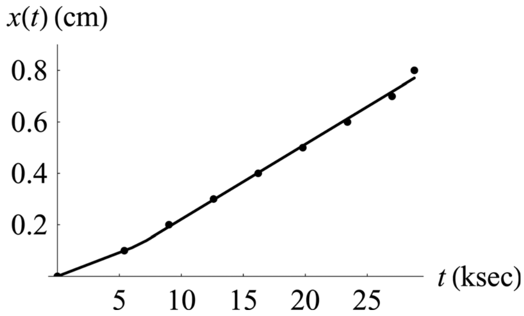

Fig. 12. The portion of the candle above the water level as a function of time. The dots are the experimental points while the curve is the piecewise continuous function of Eq. (29) using $j_0 = 7.127 \times 10^{-4}\,\mathrm{g/s}$, $t_0 = 6607\,\mathrm{s}$, and $j_{\mathrm{eff}} = 1.12 \times 10^{-3}\,\mathrm{g/s}$.

We also note that j_0 is practically equal to j_a, which indicates that in the beginning, the combustion rate is not affected by the water, presumably because the temperature of the paraffin at the surface around the wick is essentially determined by the contact of the entire cross section of the candle with air.

Let us now examine how our other equations are affected. Equation (8) remains valid as long as we use ℓ_a and d_a for the region above the water level and ℓ_w and d_w for the region below. Equation (19) also remains valid because its derivation was based on the action of the heat flux on the bottom of the well. In particular, $w(t)$ will still reach the asymptotic value of 0.4 cm for long times because the parameters involved are all for a candle immersed in water. However, we need to modify Eq. (9). For times longer than t_0 the radius has reached its asymptotic value both above and below the water level. At this stage of the evolution, we consider the melting that takes place during a time dt. This melting occurs only on the horizontal portions of the well, so a mass of $\rho\pi(\ell_a^2 - \ell_w^2)dx + \rho\pi\ell_w^2\,dy$ is melted and partly burned. The burned mass is $j_{\mathrm{eff}}dt$ whereas the molten mass augments the amount of liquid paraffin already present at the bottom of the well by $\rho_m\pi\ell_w^2 dw$. Hence

$$\rho\pi(\ell_a^2 - \ell_w^2)dx + \rho\pi\ell_w^2 dy = j_{\mathrm{eff}}dt + \rho_m\pi\ell_w^2 dw, \tag{30}$$

which leads to

$$\rho\pi(\ell_a^2 - \ell_w^2)\frac{dx}{dt} + \rho\pi\ell_w^2\frac{dy}{dt} = j_{\mathrm{eff}} + \rho_m\pi\ell_w^2\frac{dw}{dt}. \tag{31}$$

As previously discussed, we know that $dw/dt \to 0$ for large times and Eq. (29) yields $dx/dt = j_{\mathrm{eff}}/(\rho_w\pi D^2)$, allowing us to find

$$j_{\mathrm{eff}} = \frac{\rho\pi\ell_w^2(dy/dt)}{1 - \dfrac{\rho}{\rho_w}\dfrac{\ell_a^2 - \ell_w^2}{D^2}}. \tag{32}$$

The rate dy/dt is the one measured in water because the flame is deep inside the body of the immersed candle, below the surface of the water. If the lengths ℓ_a and ℓ_w were equal, j_{eff} would be simply j_w. However, the existence of the denominator renders $j_{\mathrm{eff}} > j_w$, precisely because in the two-cylinder geometry additional paraffin that has melted from the horizontal wall of the upper cavity is being burned.

The experimental data in Fig. 9 have an asymptotic slope of $dy/dt = 9.64 \times 10^{-5}\,\mathrm{cm/s}$, so Eq. (32) yields $j_{\mathrm{eff}} = 1.04 \times 10^{-3}\,\mathrm{g/s}$. This value agrees quite well with our previous estimate of $j_{\mathrm{eff}} = 1.12 \times 10^{-3}\,\mathrm{g/s}$ obtained using the experimental data of Fig. 12.

V. CONCLUSION

In this paper, we discuss the paradox of the floating candle and develop a mathematical model that explains the main features of this system. In particular, we see that the radius of the well reaches an asymptotic value that is greater when the candle is surrounded by air than when immersed in water. We also demonstrate that the depth of the molten paraffin pond within the well reaches a constant value. As the candle gets lighter, it rises above the water level and as a result the well becomes wider at the top. However, once the asymptotic radius is attained, only the depth of the well can change; this leads to a well with two cylindrical chambers. The formation of the well explains Tom Tit's paradox since the candle gets lighter while continuing to float. The rise of the candle above the water level is very slow and gradual, so the candle gives to the spectator the impression of always being at the brink of extinction.

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[2]For the impact of these books, see ⟨http://www.ciencianet.com/scienceamusante.html⟩.

[3]Tom Tit (Arthur Good), *La Science Amusante, 1e série* (Librairie Larousse, Paris 1890), pp. 77–78.

[4]Available at ⟨http://gallica.bnf.fr/ark:/12148/bpt6k2029874/f78.image⟩.

[5]See, for example, Edward Shevick, *Science puzzlers: Solving Science Mysteries* (Teaching and Learning Company, Dayton 1998), p. 54, Goran Grimvall, Brainteaser Physics: Challenging Physics Puzzlers (The Johns Hopkins U.P., Baltimore, 2007), p. 2 and 8, H. J. Schlichting, "Physikalische Anmerkungen zur schwimmenden Kerze," Prax. Naturwiss., Phys. Sch. **43/4**, 15 (1994).

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