DC Motor Position: PID Controller Design

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1 Preliminary

1.1 System Identification

The system is modeled as a plant $P(s)$ connected in series with a PID controller $C(s)$. The plant is a first order linear system with time constant $\tau = 0.085$s and DC gain $K = 4.9\text{rad} \cdot \text{V/s}$. The input of the plant $P(s)$ is the input voltage and the output of the system $Y(s)$ is the angular displacement of the DC motor. A proportional-integral-derivative(PID) controller $C(s)$ is connected to the plant in series to complete a closed loop system.

1.2 Block Diagram

The block diagram of the system is as follows:

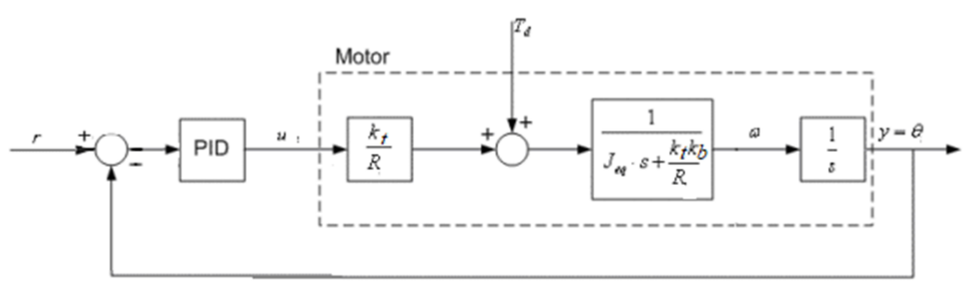

Figure 1: Block diagram

2 Background Questions

2.1 Plant Transfer Function

The plant transfer function for the DC motor position system is as follows:

$$\begin{aligned} P(s) &= \frac{Y(s)}{V(s)} = \frac{1}{s}\frac{1}{J_{eq}s + \frac{k_t k_b}{R}}\frac{k_t}{R} \\ &= \frac{k_t}{J_{eq}Rs + k_t k_b}\frac{1}{s} = \frac{\frac{1}{k_b}}{\frac{J_{eq}R}{k_t k_b}s + 1}\frac{1}{s} \\ &= \left(\frac{K}{\tau s + 1}\right)\frac{1}{s} \end{aligned}$$

where K is a DC gain and τ is a time constant.

2.2 PD controller

PD controller Transfer Function The transfer function of the system with a PD controller can be found by equating $k_i = 0$; i.e.:

$$C(s) = k_p + sk_d$$

Therefore, the closed loop transfer function from the position reference, r, to the angular position output, θ, is:

$$\begin{aligned} G_{yr} &= \frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{(k_p + sk_d)\left(\frac{K}{\tau s+1}\frac{1}{s}\right)}{1 + (k_p + sk_d)\left(\frac{K}{\tau s+1}\frac{1}{s}\right)} = \frac{K(k_p + sk_d)}{s(\tau s + 1) + K(k_p + sk_d)} \\ &= \frac{Kk_d s + Kk_p}{\tau s^2 + (Kk_d + 1)s + Kk_p} \end{aligned}$$

Determination of gains Controller gains k_p and k_d can be determined by equating the denominator of the closed loop transfer function to the canonical form of second order system.

$$s^2 + \frac{Kk_d + 1}{\tau}s + \frac{Kk_p}{\tau} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

which yields the relationship between the gains and the parameters.

$$\begin{aligned} \frac{Kk_p}{\tau} &= \omega_n^2 \Rightarrow k_p = \frac{\tau\omega_n^2}{K} \\ \frac{Kk_d + 1}{\tau} &= 2\zeta\omega_n \Rightarrow k_d = \frac{2\zeta\omega_n\tau - 1}{K} \end{aligned}$$

Steady-State error and DC gain The steady-state error of the system can be calculated by using the final value theorem.

$$\begin{aligned} \lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{sR(s)}{1 + P(s)C(s)} = \lim_{s\to 0}\frac{s \cdot \frac{1}{s}}{1 + \left(\frac{K}{\tau s+1}\frac{1}{s}\right)(k_p + sk_d)} \\ &= \lim_{s\to 0}\frac{s(\tau s + 1)}{s(\tau s + 1) + K(k_p + sk_d)} = 0 \end{aligned}$$

Therefore, the steady-state error is 0.
The DC gain of the system can be found by using the final value theorem along with the input signal

$R(s) = \frac{1}{s}$(step input).

$$
\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\cdot R(s)G_{yr}(s) = \lim_{s\to 0} s\cdot\frac{1}{s}\cdot\frac{Kk_d s + Kk_p}{\tau s^2 + (Kk_d+1)\,s + Kk_p} \\
&= \frac{Kk_p}{Kk_p} = 1
\end{aligned}
$$

Therefore, the DC gain of the system is 1, and the output of the system y becomes identical to the step input $r = 1$ in steady-state.

2.3 PID controller

Torque disturbance transfer function The closed-loop torque disturbance to position transfer function, assuming zero reference input, can be found as:

$$
\begin{aligned}
G_{yT} = \frac{Y(s)}{T(s)} &= \frac{\left(\frac{1}{s}\right)\left(\frac{R}{RJ_{eq}s+k_tk_b}\right)}{1+\left(\frac{1}{s}\right)\left(\frac{R}{RJ_{eq}s+k_tk_b}\right)\left(\frac{k_t}{R}\right)\left(k_p+\frac{k_i}{s}+sk_d\right)} \\
&= \frac{\left(\frac{1}{s}\right)\left(\frac{R}{RJ_{eq}s+k_tk_b}\right)}{1+\left(\frac{1}{s}\right)\left(\frac{k_t}{RJ_{eq}s+k_tk_b}\right)\left(k_p+\frac{k_i}{s}+sk_d\right)} = \frac{\left(\frac{1}{s}\right)\left(\frac{\frac{R}{k_tk_b}}{\frac{RJ_{eq}}{k_tk_b}s+1}\right)}{1+\left(\frac{1}{s}\right)\left(\frac{\frac{1}{k_b}}{\frac{RJ_{eq}}{k_tk_b}s+1}\right)\left(k_p+\frac{k_i}{s}+sk_d\right)} \\
&= \frac{\frac{1}{s}\frac{\frac{R}{k_tk_b}}{\tau s+1}}{1+\left(\frac{1}{s}\right)\left(\frac{K}{\tau s+1}\right)\left(k_p+\frac{k_i}{s}+sk_d\right)} = \frac{s\frac{R}{k_tk_b}}{s^2\left(\tau s+1\right)+K\left(s^2k_d+sk_p+k_i\right)} \\
&= \frac{\tau s}{J_{eq}\left[\tau s^3+\left(Kk_d+1\right)s^2+Kk_ps+Kk_i\right]}
\end{aligned}
$$

Steady-state error The steady-state error to a unit step torque disturbance on a PID controller can be found using the final value theorem along with the condition $G_{yT}(s) = Y(s)/T(s) = -E(s)/T(s)$.

$$
\begin{aligned}
\lim_{s\to 0} sE(s) &= \lim_{s\to 0}\left[-sG_{yT}(s)T(s)\right] = \lim_{s\to 0}\left[-s\cdot\frac{1}{s}\cdot G_{yT}(s)\right] \\
&= \lim_{s\to 0}\frac{-\tau s}{J_{eq}\left[\tau s^3+\left(Kk_d+1\right)s^2+Kk_ps+Kk_i\right]} \\
&= 0
\end{aligned}
$$

The steady-state error to a unit step torque disturbance on a PD controller can be found by letting $k_i = 0$.

$$
\begin{aligned}
\lim_{s\to 0} sE(s) &= \lim_{s\to 0}\frac{-\tau s}{J_{eq}\left[\tau s^3+\left(Kk_d+1\right)s^2+Kk_ps\right]} \\
&= \lim_{s\to 0}\frac{-\tau}{J_{eq}\left[3\tau s^2+2\left(Kk_d+1\right)s+Kk_p\right]} \\
&= -\frac{\tau}{J_{eq}Kk_p}
\end{aligned}
$$

Determination of gains Controller gains k_p, k_i, and k_d can be determined by equating the denominator of the closed loop transfer function to the canonical form of third order system.

$$
s^3+\frac{Kk_d+1}{\tau}s^2+\frac{Kk_p}{\tau}s+\frac{Kk_i}{\tau} = s^3+(2\zeta\omega_n+p_0)s^2+(\omega_n^2+2\zeta\omega_np_0)s+\omega_n^2p_0
$$

which yields the relationship between the gains and the parameters.

$$\frac{Kk_p}{\tau} = \omega_n^2 + 2\zeta\omega_n p_0 \Rightarrow k_p = \frac{\tau(\omega_n^2 + 2\zeta\omega_n p_0)}{K}$$
$$\frac{Kk_i}{\tau} = \omega_n^2 p_0 \Rightarrow k_i = \frac{\tau\omega_n^2 p_0}{K}$$
$$\frac{Kk_d + 1}{\tau} = 2\zeta\omega_n + p_0 \Rightarrow k_d = \frac{\tau(2\zeta\omega_n + p_0) - 1}{K}$$

3 Experiment

3.1 Qualitative analysis

Effect of k_p The step responses of the system with varying k_p are plotted below.

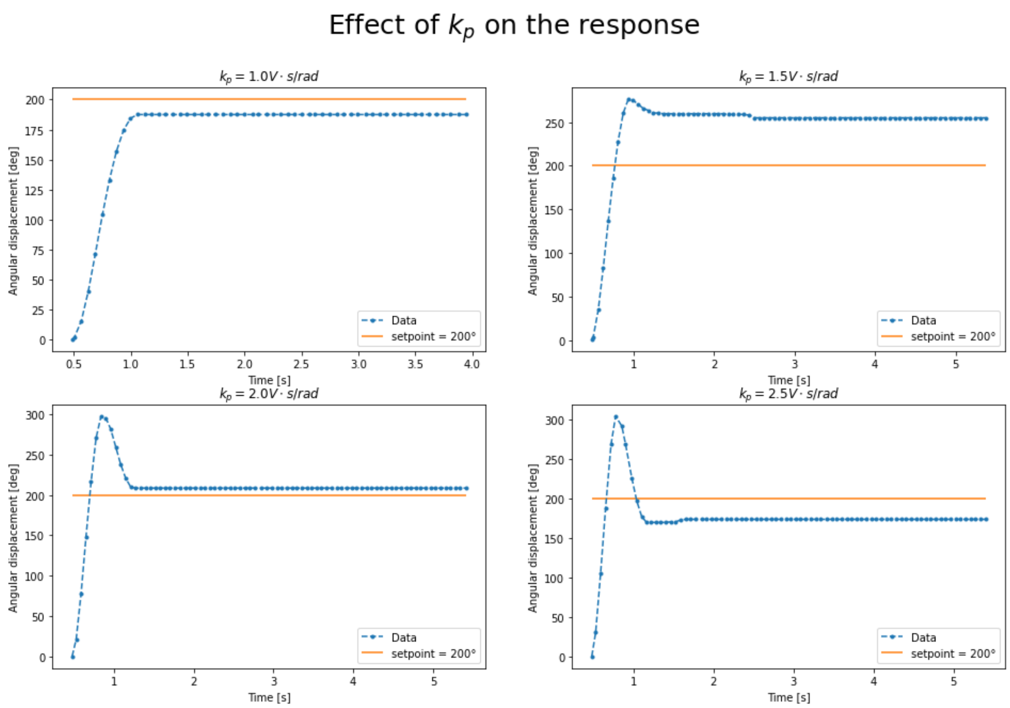

Figure 2: Step response with $k_p = 1.0$, $k_p = 1.5$, $k_p = 2.0$, $k_p = 2.5$

Significant steady state errors are present in all of the responses, regardless of the value of k_p. As k_p increases, an overshoot is observed; yet k_p alone is not enough to provide enough input voltage to eliminate the steady state response.

Effect of k_d Introduction of k_d helps the response to start off quicker. From the plots below, it can be observed that the systems with nonzero k_d exhibit immediate response at the beginning. However too large k_d resulted in poorer performance in long term. The derivative feedback constantly hindered the response from approaching the reference point.

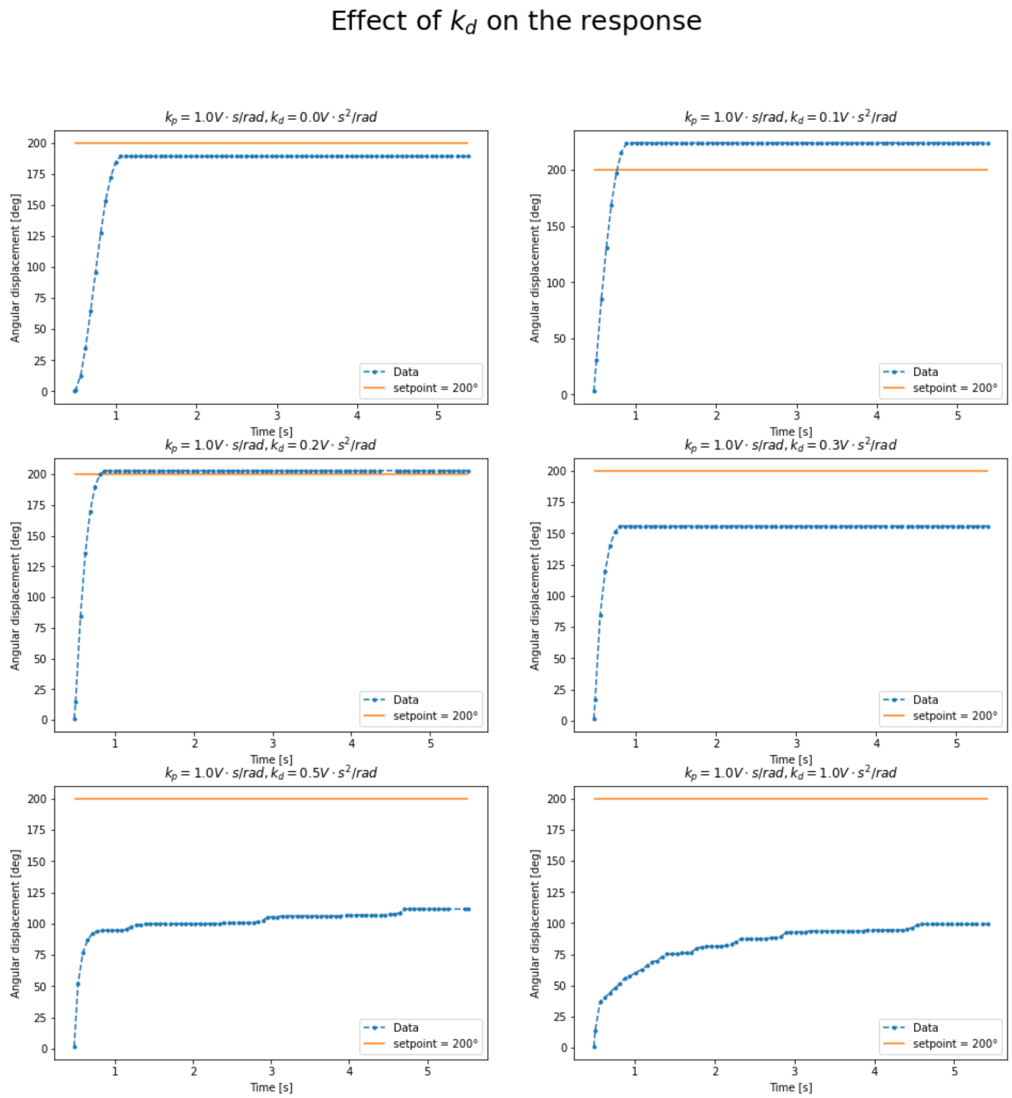

Figure 3: Step response with $k_d = 0.0$, $k_d = 0.1$, $k_d = 0.2$, $k_d = 0.3$, $k_d = 0.5$, $k_d = 1.0$

3.2 PD control

3.2.1 Step response with $\zeta = 0.8$, $\omega_n = 10rad/s$

With $K = 4.9rad/Vs$, $\tau = 0.085s$, $\zeta = 0.8$, $\omega_n = 10rad/s$, and $p_0 = 0$, the gains of a PD controller are found to be $k_p = 1.73V \cdot s/rad$ and $k_d = 0.073V \cdot s^2/rad$. The expected position response and the empirical response to step input are plotted below.

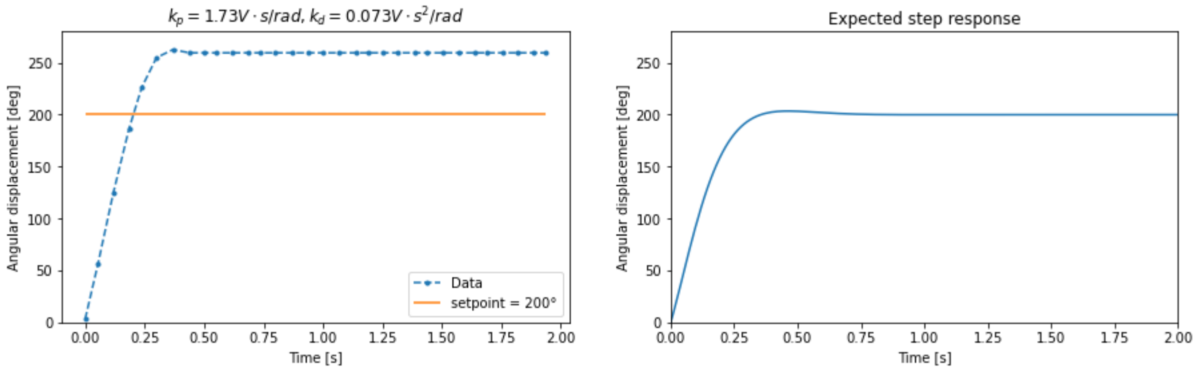

Figure 4: Step response with $k_p = 1.73$ and $k_d = 0.073$

Different from the expected behavior, there exists a steady-state error. This is mainly due to the face that the input voltage was not high enough to overcome the physical resistances that are not included in the model; for example, resistance, inductance, frictions, and internal energy losses. However it is notable that both plots exhibit similar behavior, such as slight overshoot behavior above its steady state equilibrium, and time required to reach the peak output value.

3.2.2 Step response with $\zeta = 0.6$, $\omega_n = 15rad/s$

With $K = 4.9rad/Vs$, $\tau = 0.085s$, $\zeta = 0.6$, $\omega_n = 15rad/s$, and $p_0 = 0$, the gains of a PD controller are found to be $k_p = 3.903V \cdot s/rad$ and $k_d = 0.108V \cdot s^2/rad$. The empirical and the analytically expected step responses are plotted below.

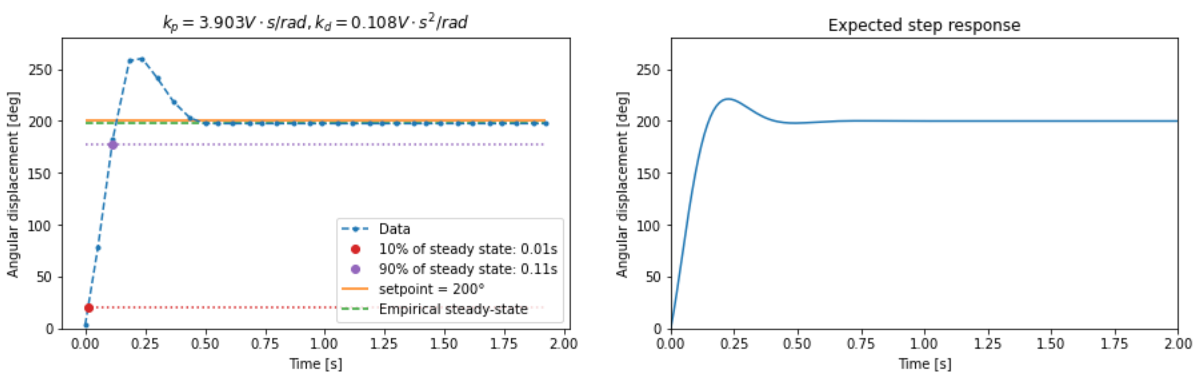

Figure 5: Step response with $k_p = 3.903$ and $k_d = 0.108$

Increasing the natural frequency made the system response quicker in the beginning. Observe that the initial slope of the response is steeper for $\omega_n = 15rad/s$ than $\omega_n = 10rad/s$. Also, decrease in the damping coefficient ζ resulted in a larger overshoot above the reference point. It can be observed that the magnitude of the overshoot for $\zeta = 0.6$ is larger than that for $\zeta = 0.8$.

Overshoot and rise time From the above experiment, the overshoot in % and rise time were calculated. For the calculations, an empirical steady-state value was used instead of the reference point.
The overshoot in % is found as follows:

$$overshoot = \frac{overshoot_{max} - \theta_{ss}}{\theta_{ss}} \times 100\% = \frac{260.33 - 197.57}{197.57} \times 100\% = 31.766\%$$

In order to find the rise time, time taken to reach the 10% and 90% of the steady-state value were first calculated by interpolation.

$$t_r = t_{90\%} - t_{10\%} = 0.11s - 0.01s = 0.10s$$

3.2.3 Response to Load Disturbance

The controller with parameters $K = 4.9rad/Vs$ and $\tau = 0.085s$ was designed to be a system with $\zeta = 0.6$, $\omega_n = 15rad/s$, and $p_0 = 0$. Corresponding gains were found to be $k_p = 3.903V \cdot s/rad$ and $k_d = 0.108V \cdot s^2/rad$. The setpoint was set to be 0 to observe the behavior against external load disturbance. Following plot describes how the PD controller counteracts the load disturbance.

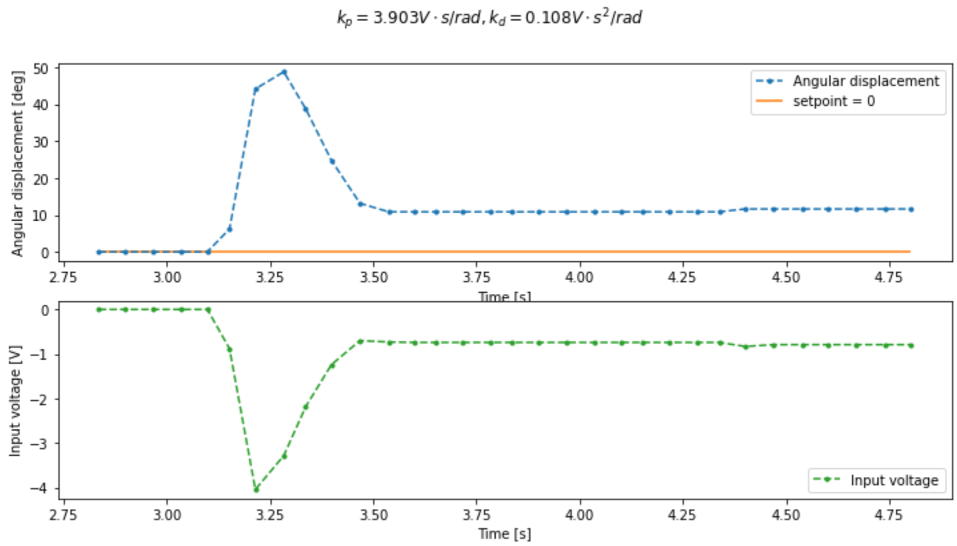

Figure 6: Effect of the disturbance on PD controller

As it can be observed from the plot above, as soon as external disturbance is applied, the input voltage is implemented so as to counteract the effect of the disturbance. As a result, the angular displacement tends to recover its initial value. However a permanent steady state error is still observable.

3.3 PID control

3.3.1 Step response with $\zeta = 0.6$, $\omega_n = 15rad/s$, $p_0 = 1$

The controller was designed for $\zeta = 0.6$, $\omega_n = 15rad/s$, and $p_0 = 1$. Corresponding gains were found to be $k_p = 4.215V \cdot s/rad$, $k_i = 3.903V/rad$, and $k_d = 0.1255V \cdot s^2/rad$. Following plot shows the empirical and expected step responses of the system with the PID controller.

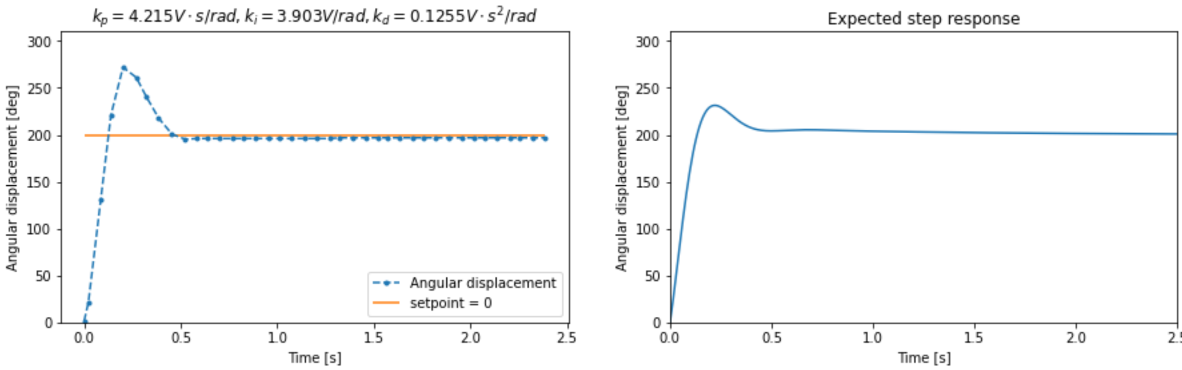

Figure 7: Step response with $k_p = 4.215$ $k_i = 3.903$, and $k_d = 0.1255$

Effect of k_i The most notable effect of adding k_i is that the steady state error is eliminated. The accumulated error results in increasing/decreasing in the input voltage which, when becomes large enough to overcome the resistance, actuates the DC motor to ultimately achieve the desired reference point.

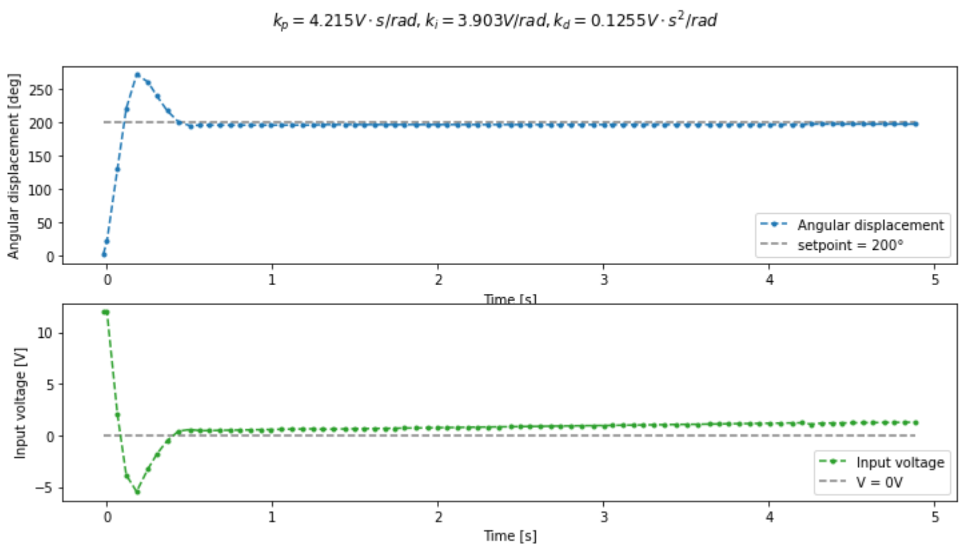

Figure 8: Effect of k_i in the input voltage

The figure above shows how k_i uses the accruing errors to achieve the desired point. Notice that the angular displacement being constantly below the setpoint from $t = 0.5$s resulted in the constant increasing in the input voltage, until it becomes large enough to overcome the physical resistance and actuate the DC motor around $t = 4.3$s.

3.3.2 Step response with $\zeta = 0.6$, $\omega_n = 15rad/s$, $p_0 = 2$

Now the pole is moved from 1 to 2. The gains are calculated to be $k_p = 4.528V \cdot s/rad$, $k_i = 7.806V/rad$, and $k_d = 0.1429V \cdot s^2/rad$. Following figures show the empirical and expected step response of the system.

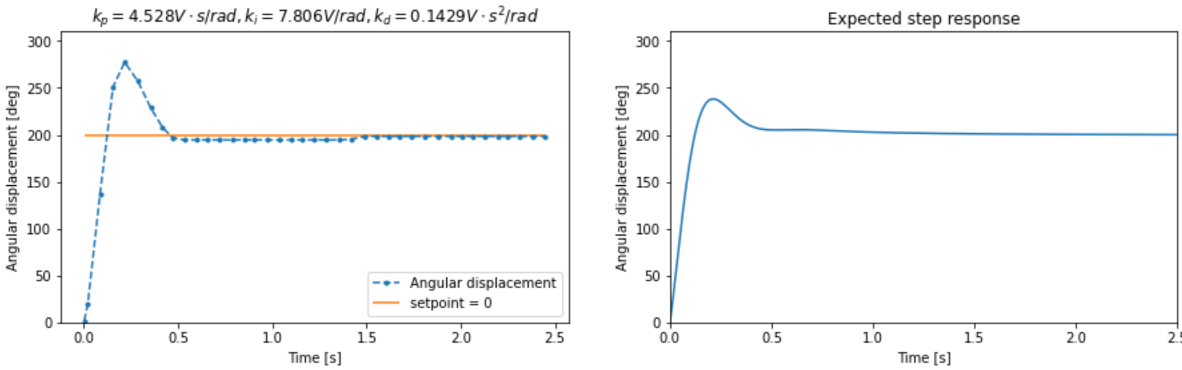

Figure 9: Step response with $k_p = 4.528$ $k_i = 7.806$, and $k_d = 0.1429$

When the poles are placed further into the LHP, the response becomes more rapid. This trend can be observed from the figure below.

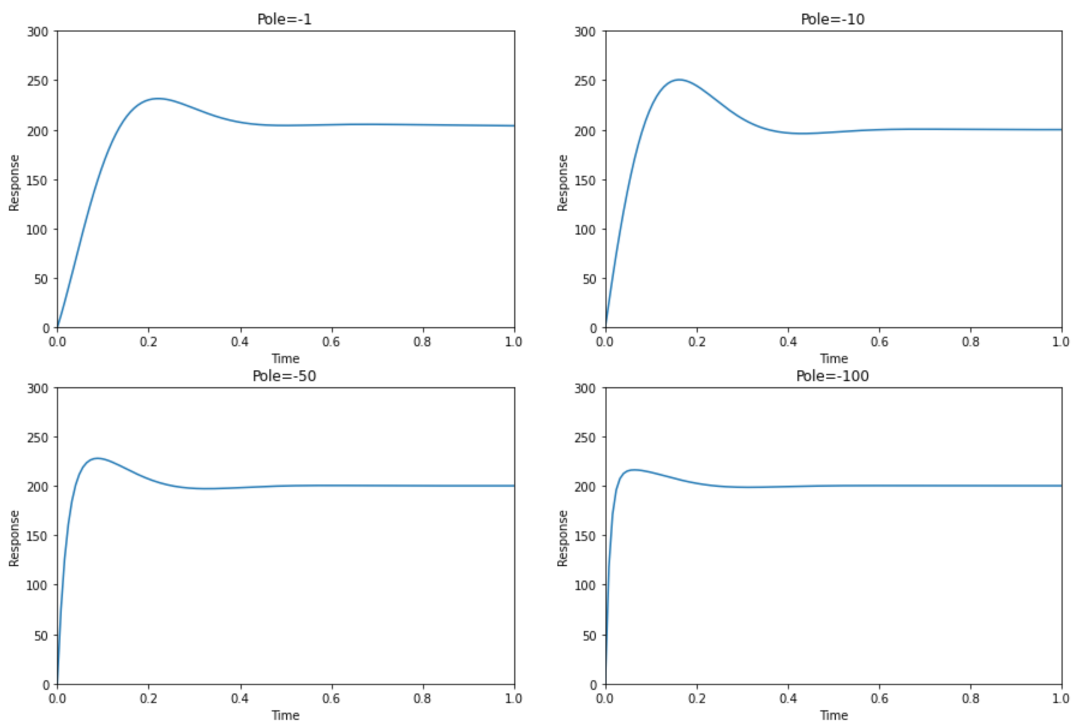

Figure 10: Effect of pole p_0

Notice that, as p_0 is placed further into the LHP, the system responds much quicker and achieves steady state in less time. However, in reality, this extreme case of the pole placement is limited by the input saturation.

3.3.3 Response to Load Disturbance

The controller with parameters $K = 4.9rad/Vs$ and $\tau = 0.085s$ was designed to be the system with $\zeta = 0.6$, $\omega_n = 15rad/s$, and $p_0 = 1$. Corresponding gains were found to be $k_p = 4.215V \cdot s/rad$, $k_i = 3.903V/rad$, and $k_d = 0.1255V \cdot s^2/rad$. The setpoint was set to be 0 to observe the behavior against external load disturbance. Following plot describes how the PID controller counteracts the load disturbance.

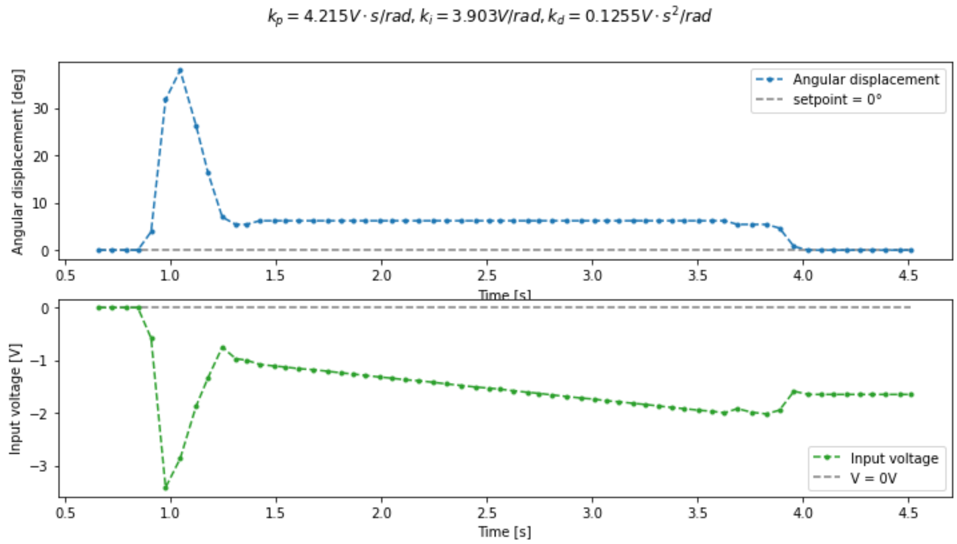

Figure 11: Effect of the disturbance on PID controller

The most significant characteristic of the PID controller compared to the PD controller is that the permanent steady-state error is eliminated by the integral gain k_i. As it can be seen from the plot above, the integral gain k_i accumulates the error between the response and the setpoint, until the input voltage becomes large enough to overcome the physical resistance and actuate the DC motor to achieve the zero steady-state error.

3.4 Tuning and Root Locus

3.4.1 Root Locus of Closed Loop Transfer Function

The system was tuned by using the root locus. The controller transfer function is first rewritten as

$$C(s) = k_p + \frac{k_i}{s} + sk_d = k_p \left(1 + \frac{1}{T_i s} + T_d s\right)$$

where T_i and T_d are integral time constant and derivative time constant respectively. The closed loop poles are written in Evan's form for the root locus investigation.

$$1 + P(s)C(s) = 1 + k_p \left(\frac{1}{T_i s} + T_d s\right)\left(\frac{K}{\tau s + 1} \cdot \frac{1}{s}\right) = 1 + k_p G(s)$$

with

$$G(s) = \left(1 + \frac{1}{T_i s} + T_d s\right)\left(1 + \frac{K}{\tau s + 1} \cdot \frac{1}{s}\right)$$

Pole-zero map of $G(s)$ with different T_i and T_d values were tested to find the optimal distribution of the poles and zeros that can potentially produce the best performance; i.e. $\zeta = 0.7$ and as high ω_n as possible. As a result, the combination of $T_i = 0.31$ and T_d=0.08 were selected.
Next, the root locus of $G(s)$ is plotted, and the best compensator value k_p was determined to be 2.9. Poles of the closed loop transfer function that corresponds to this compensator value are indicated in magenta in the figure below.

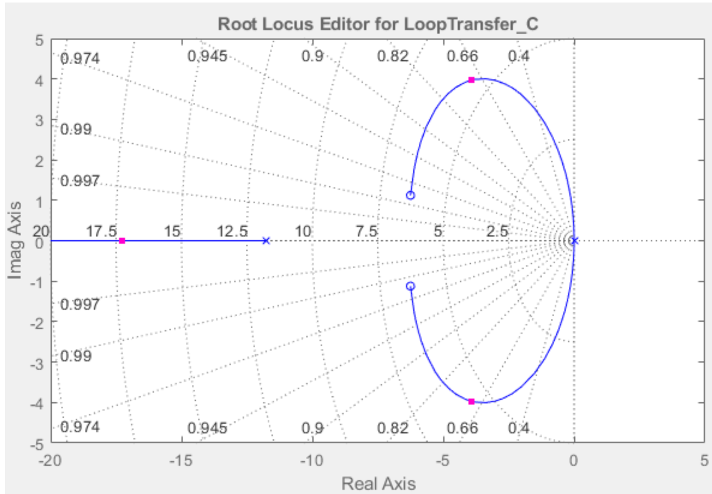

Figure 12: Root locus of the closed loop transfer function G_{yr}

Corresponding k_i and k_d were 9.355 and 0.232, respectively. Following figure shows the pole zero map of the closed loop transfer function G_{yr}. It can be observed that, as expected, a pair of complex conjugate poles are situated above and below the real axis, creating angles of 45° with the imaginary axis.

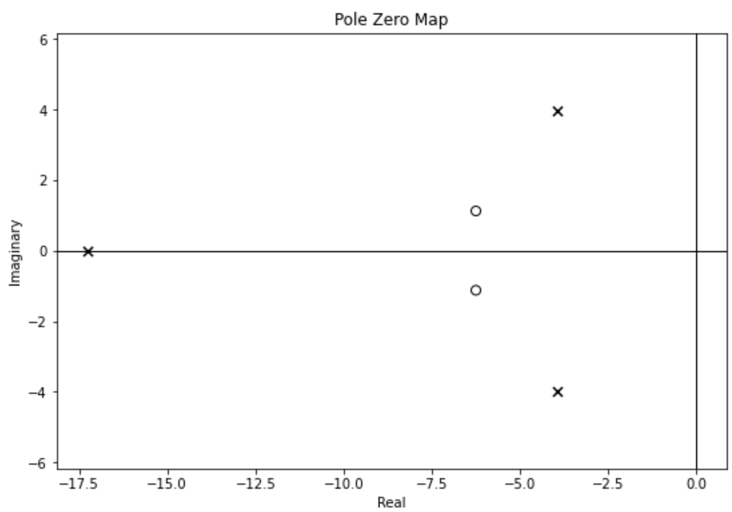

Figure 13: Pole-zero map of the closed loop transfer function G_{yr}

3.4.2 Experimental result

These gain values, $k_p = 2.9$, $k_i = 0.355$, and $k_d = 0.232$ were implemented to obtain the experimental response. Below is a figure in which the empirical and analytically expected step responses are plotted.

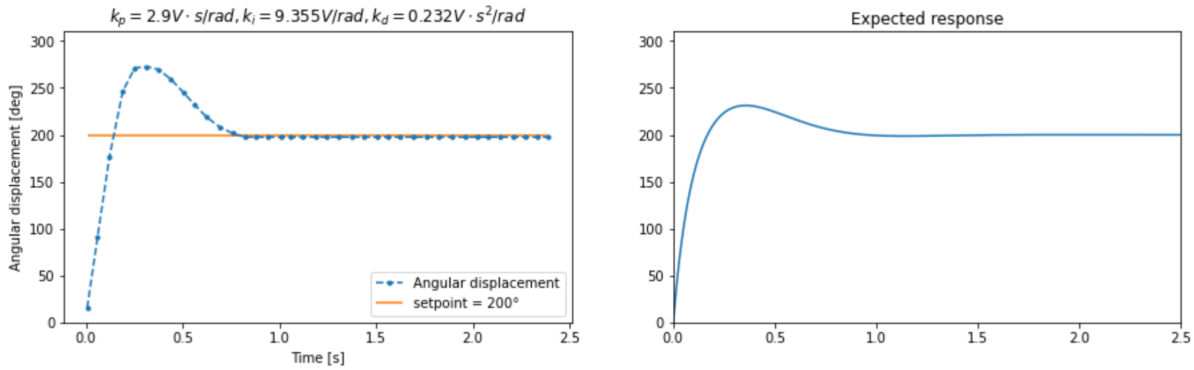

Figure 14: Step response of the system tuned with root locus

The overall behavior of the system is as expected; however, the magnitude of the overshoot is larger than expected. The system can be further fine tuned by increasing the damping to reduce the overshoot.