DC Motor Speed: PID Controller Design

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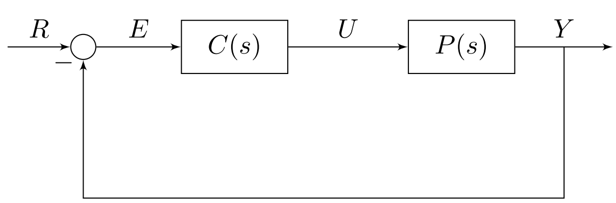
1 Preliminary

1.1 System Identification

The system is modeled as a plant $P(s)$ connected in series with a PID controller $C(s)$. The plant is a first order linear system with time constant $\tau = 0.085\text{s}$ and DC gain $K = 4.9\text{rad}\cdot\text{V/s}$. The input of the plant $P(s)$ is the input voltage and the output of the system $Y(s)$ is the angular velocity of the DC motor. A proportional-integral-derivative controller(PID) controller $C(s)$ is connected to the plant in series to complete a closed loop system.

1.2 Block Diagram

The block diagram of the controlled system is as follows:



where

$$C(s) = k_p + \frac{k_i}{s} + sk_d$$
$$P(s) = \frac{K}{\tau s + 1}$$

2 Background Questions

2.1 PID controller

PID controller Transfer Function The transfer function of the system with a PID controller from the reference speed, R, to the output, Y, the angular motor speed output, is as follows:

$$
\begin{aligned}
G_{yr} &= \frac{P(s)C(s)}{1+P(s)C(s)} \\
&= \frac{\frac{K}{\tau s+1}(k_p + \frac{k_i}{s} + sk_d)}{1+\frac{K}{\tau s+1}(k_p + \frac{k_i}{s} + sk_d)} \\
&= \frac{K(sk_p + k_i + s^2 k_d)}{(\tau s+1)s + K(sk_p + k_i + s^2 k_d)} \\
&= \frac{Kk_d s^2 + Kk_p s + Kk_i}{(Kk_d + \tau)s^2 + (Kk_p + 1)s + Kk_i}
\end{aligned}
$$

Where K is a DC gain, τ is a time constant, k_p is a proportional gain, k_i is an integral gain, and k_d is a derivative gain.

2.2 PI controller

PI controller Transfer Function The transfer function of the system with a PI controller can be found by equating $k_d = 0$:

$$G_{yr} = \frac{Kk_p s + Kk_i}{\tau s^2 + (Kk_p + 1)s + Kk_i}$$

PI controller Closed Loop Poles Poles of the closed loop system can be found by equating the denominator of the closed loop transfer function to zero.

$$\tau s^2 + (Kk_p + 1)s + Kk_i = 0$$

Which is equivalent to

$$s^2 + \frac{Kk_p + 1}{\tau}s + \frac{Kk_i}{\tau} = 0 \tag{1}$$

The poles of the system is therefore

$$s_{1,2} = -\frac{Kk_p + 1}{2\tau} \pm \sqrt{\left(\frac{Kk_p + 1}{2\tau}\right)^2 - \frac{Kk_i}{\tau}}$$

Effect of k_p and k_i Observe that due to the PI controller, the overall system is now a second order system. Comparing equation (1) with the canonical form of the second order system,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

following relationships can be found:

$$
\begin{aligned}
2\zeta\omega_n &= \frac{Kk_p + 1}{\tau} \\
\omega_n^2 &= \frac{Kk_i}{\tau}
\end{aligned}
$$

Solving for k_p ad k_i,

$$
\begin{aligned}
k_p &= \frac{2\tau\zeta\omega_n - 1}{K} \\
k_i &= \frac{\tau\omega_n^2}{K}
\end{aligned}
$$

As the proportional gain k_p increases, the damping effect ζ of the system will increase. Also, as the integral gain k_i increases, the closed loop natural frequency of the system ω_n will increase.

Stability conditions for PI controller The system is stable if all poles reside at left half plane. This happens when all the coefficients of the polynomial are positive.

$$\frac{Kk_p + 1}{\tau} > 0$$
$$\frac{Kk_i}{\tau} > 0$$

Given τ is a positive number, and using $K = 4.9\text{rad/Vs}$ and $\tau = 0.085s$, conditions above are equivalent to:

$$k_p > -0.204\frac{\text{V}\cdot\text{s}}{\text{rad}}$$
$$k_i > 0\frac{\text{V}}{\text{rad}}$$

2.3 P controller

P controller and steady-state error The transfer function of a P controller system can be found by equating $k_i = 0$ and $k_d = 0$:

$$G_{yr} = \frac{Kk_p s}{\tau s^2 + (Kk_p + 1)s}$$

The steady-state transfer function can be found as follows.

$$\begin{aligned}\lim_{s\to 0} G_{yr}(s) &= \lim_{s\to 0}\frac{Kk_p s}{\tau s^2 + (Kk_p + 1)s}\\ &= \lim_{s\to 0}\frac{Kk_p}{2\tau s + Kk_p + 1}\\ &= \frac{Kk_p}{Kk_p + 1}\end{aligned}$$

This implies that, since $Y(s) = G(s)R(s)$, there will be a steady-state error of

$$E(s) = R(s) - Y(s) = (1 - G(s))\,R(s) = \left(1 - \frac{Kk_p}{Kk_p + 1}\right)R(s) = \frac{1}{Kk_p + 1}R(s)$$

which is $\frac{1}{Kk_p+1} \times 100\%$ of the reference point.

3 Experiment

3.1 Qualitative analysis

3.1.1 P Controller

Experimental determination of steady-state error of P controller The response of the P control motor system with gain of $k_p = 0.5\text{V}\cdot\text{s/rad}$ is described in the plot below.

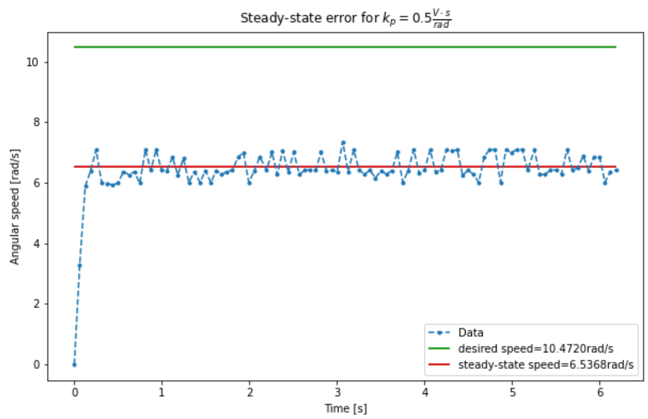


Figure 1: Steady-state error, $k_p = 0.5$, setpoint=10.4720rad/s

The steady-state % error is calculated as follows.

$$\%error = \frac{\omega_{desired} - \omega_{steady}}{\omega_{desired}} \times 100 = \frac{10.4720 - 6.5358}{10.4720} \times 100\% = 37.59\%$$

Comparison between experimental and theoretical steady-state error The theoretical steady-state error is

$$\left(1 - \frac{Kk_p}{Kk_p + 1}\right) \times 100\% = \left(1 - \frac{4.9\frac{\text{rad}}{\text{V}\cdot\text{s}} 0.5 \frac{\text{V}\cdot\text{s}}{\text{rad}}}{4.9\frac{\text{rad}}{\text{V}\cdot\text{s}} 0.5 \frac{\text{V}\cdot\text{s}}{\text{rad}} + 1}\right) \times 100\% = 28.99\%$$

Theoretically, the steady-state % error should not vary as the speed setpoints $R(s)$ vary. However, the steady-state % error tends to decrease as the speed setpoints increased. Following table lists experimentally determined steady-state % errors for different speed setpoints.

Setpoints[rad/s]	10.4720	20.9440	31.4159	41.8879	52.3599
Steady-state speed[rad/s]	6.5368	15.9749	25.0641	33.4821	41.1053
% error	37.58%	23.73%	20.22%	20.07%	21.49%

Steady-state error for different k$_p$ According to the relationship $\%error = \frac{1}{Kk_p+1} \times 100\%$, it is expected that the steady-state error will decrease as k_p increases. Theoretical steady-state error and experimental steady-state error as functions of k_p are plotted in Figure 2.

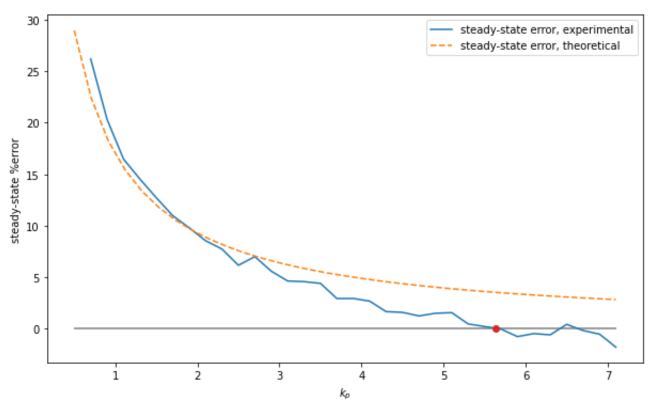


Figure 2: Steady-state error versus k_p

Theoretically, the steady-state error never becomes zero. However, the experimentally determined effective steady-state error becomes zero when k_p is around 5.63. Although it is practically possible to accomplish zero steady-state error using k_p only, the trade-off exists as well; for the input is proportional to the error, there always is an significant overshoot before the system goes to the equilibrium. Figure 3 is an example of an overshoot behavior.

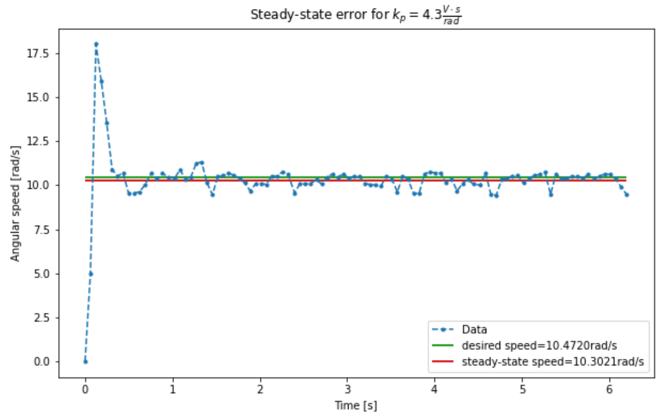


Figure 3: Overshoot behavior, $k_p = 4.3$, setpoint=10.4720rad/s

3.1.2 PI controller

Effect of $\mathbf{k}_i$ Proportional gain, k_p, is fixed at $0.5\text{V}\cdot\text{s/rad}$ and k_p is varied from 0V/rad by setps of 0.2V/rad. For any positive k_i, the steady-state error is essentially eliminated in the end. For larger k_i values, the response of the motor speed was quicker. However, for k_i larger than 4, an overshoot in response occurred, and for k_i larger than 10, oscillatory behavior was significant. Figure 4 displays different response for different k_i gains.

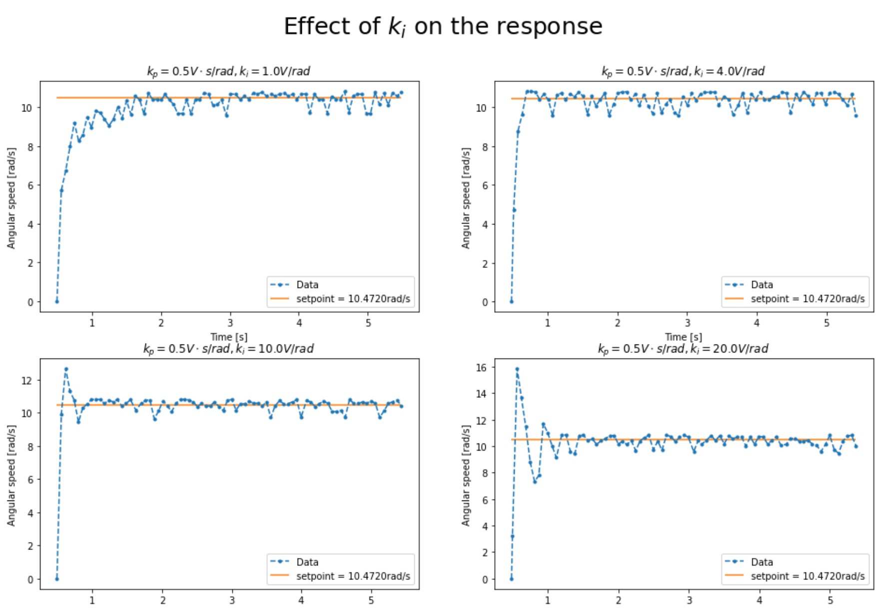


Figure 4: Effect of k_i on the response

In general, when k_i was too low, the system was not responsive enough; it took longer time to achieve the steady-state equilibrium. However, when k_i is too high, oscillatory behavior was observed.

3.2 Integrator Anti-Windup

Integrator windup occurs when the desired input is out of the limit that can be provided by the actuator. For example, for $k_p = 1\text{V} \cdot \text{s/rad}$, $k_i = 2\text{V/rad}$, and setpoint of 550rpm, the motor speed does not reach the desired output with the maximum input voltage. This irresolvable error accumulates, and affects the response of the system after the setpoint within the controllable limit is set: the input remains saturated for some time until the negative error compensates for the accumulated error, resulting in longer settling time. The effect of the integrator windup is described in Figure 5. Notice how the response is delayed when the setpoint drops from 550rpm to 200rpm.

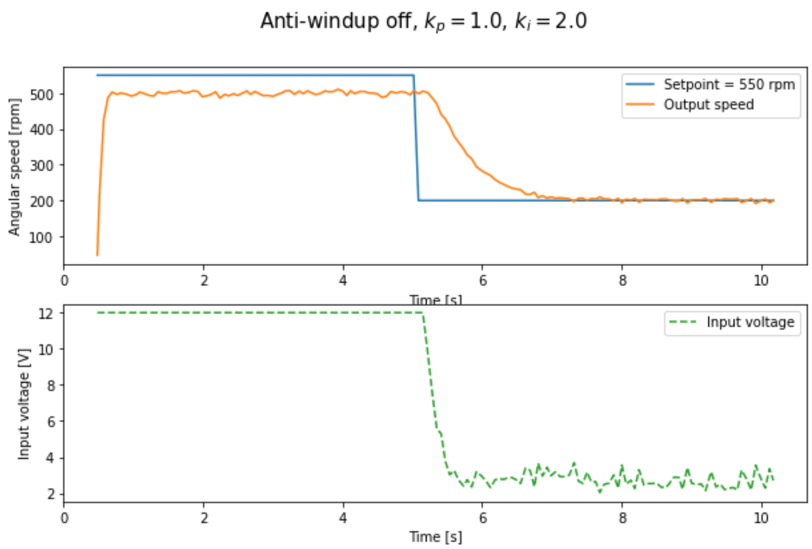


Figure 5: Anti-windup off

When the integrator anti-windup is activated, the input signal is immediately saturated to the lower limit to make the output more responsive. Figure 6 describes the effect of integrator anti-windup.

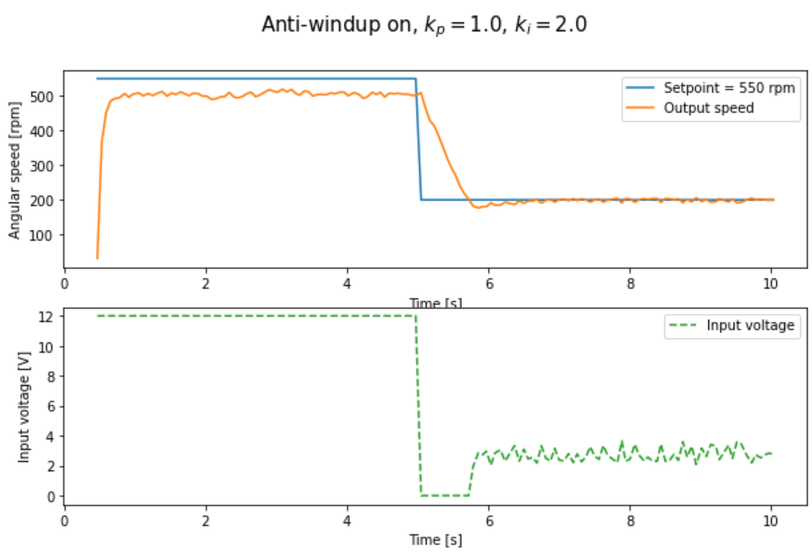


Figure 6: Anti-windup on

3.3 PID controller

Small quantity of derivative gain k_d is added and the response is analyzed. While k_p and k_i are fixed to $0.5\text{V}\cdot\text{s/rad}$ and 4.0V/rad respectively, the system response to the varying k_d of 0, 0.001, 0.1, and 1.0 $\text{V}\cdot\text{s}^2\text{/rad}$ are compared.

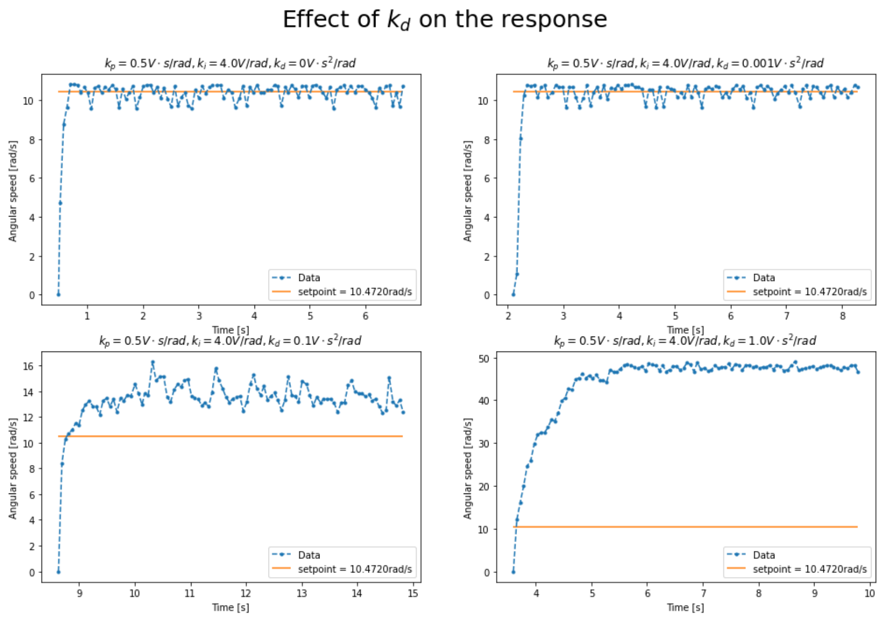


Figure 7: Effect of k_d on the response

From Figure 7, it can be seen that small k_d can help the system achieving the steady state equilibrium slightly quicker, yet larger k_d can result in instable system. Since the advantage of adding k_d is minuscule whereas the potential disadvantage can be detrimental, adding k_d is inappropriate for speed control.

3.4 Design of PI controller

3.4.1 Theoretical approach

PI control according to specifications questions From the transfer function of the plant with PI controller:

$$G_{yr} = \frac{K(k_p s + k_i)}{\tau s^2 + (1 + K k_p)s + K k_i}$$

the poles of the transfer function are the zeros of the denominator

$$\tau s^2 + (1 + K k_p)s + K k_i = 0$$

which is equivalent to

$$s^2 + \frac{1 + K k_p}{\tau} s + \frac{K k_i}{\tau} = 0$$

Observe that due to the PI controller, the overall system is now a second order system. Comparing this equation with the canonical form of the second order system,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{2}$$

following relationships can be found:

$$2\zeta\omega_n = \frac{1 + K k_p}{\tau} \tag{3}$$

$$\omega_n^2 = \frac{Kk_i}{\tau} \tag{4}$$

Combining equation (3) and equation (4) gives

$$k_i = \frac{\tau}{K}\left(\frac{Kk_p + 1}{2\tau\zeta}\right)^2 \tag{5}$$

Using $k_p = 0.5\mathrm{V}\cdot\mathrm{s/rad}$, $K = 4.9\mathrm{rad/V}\cdot\mathrm{s}$, $\tau = 0.085\mathrm{s}$, and $\zeta = 0.7$, it can be found that

$$\begin{aligned} k_i &= \frac{0.085\mathrm{s}}{4.9\mathrm{rad/V}\cdot\mathrm{s}}\left(\frac{(4.9\mathrm{rad/V}\cdot\mathrm{s})(0.5\mathrm{V}\cdot\mathrm{s/rad}) + 1}{2(0.085\mathrm{s})(0.7)}\right)^2 \\ &= 14.58\mathrm{V/rad} \end{aligned}$$

$$\begin{aligned} \omega_n &= \sqrt{\frac{Kk_i}{\tau}} = \sqrt{\frac{(4.9\mathrm{rad/V}\cdot\mathrm{s})(14.58\mathrm{V/rad})}{0.085\mathrm{s}}} \\ &= 28.99\mathrm{Hz} \end{aligned}$$

Effect of ζ What is the effect of ζ values to the system in general? From equation (2), notice that the poles are

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

For $0 < \zeta < 1$, an oscillatory behavior is expected, since the poles are complex; on the other hand, a system with $\zeta > 1$ is not expected to display any oscillatory behavior, for both poles are real.
What exactly is happening for $0 < \zeta < 1$? From equation (5), it can be noticed that increasing ζ decreases k_i. Plotting ζ versus k_i:

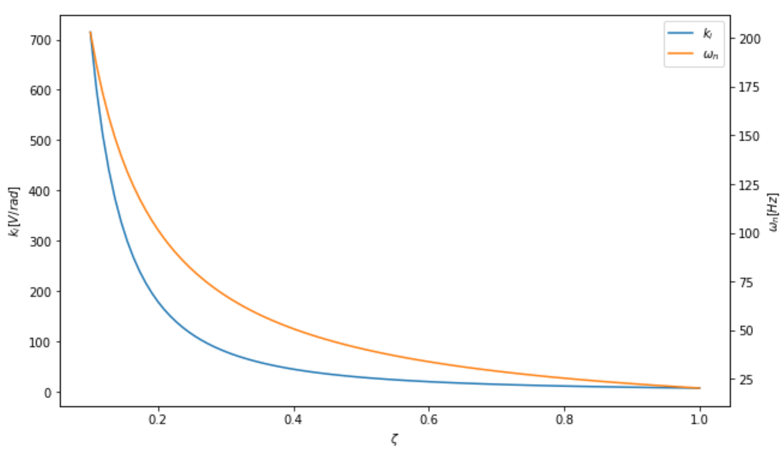


Figure 8: Effect of ζ on k_i and ω_n

As it can be inferred from Figure 8, having small ζ can cause excessive oscillation. On the other hand, having too large ζ can make the system require too much time to achieve the steady-state equilibrium.

3.4.2 Experimental results

Experimental effect of ζ Different ζ values and corresponding theoretical k_i values are tabulated below. Note that k_p was kept at $0.5\mathrm{V}\cdot\mathrm{s/rad}$.

ζ	0.3	0.5	0.7	0.9	3.0	10.0
k_i[V/rad]	79.38	28.58	14.58	8.82	0.79	0.071

The response of the system for each ζ values are obtained. Figure 9 describes how the response changes for different ζ values.

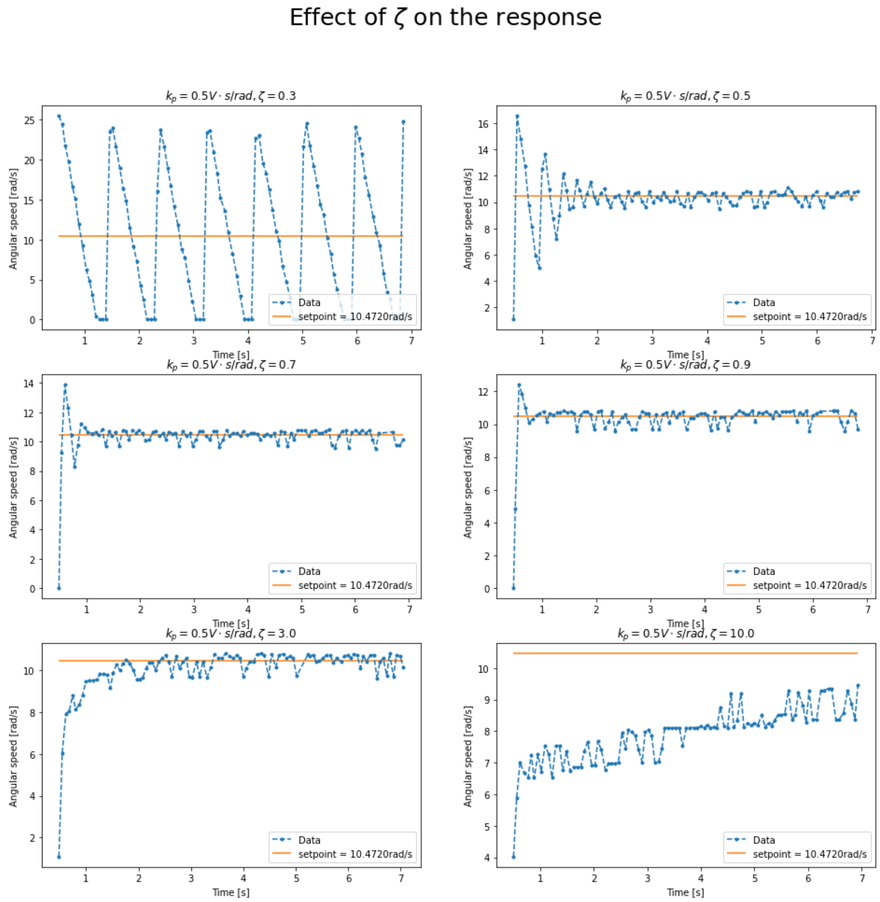


Figure 9: Effect of ζ on the response

As expected, small ζ values causes oscillation, whereas large ζ causes slow response of the system.

Trade-off between k_p and k_i In general, k_p provides steady-state stability whereas k_i provides quicker response and elimination of steady-state error. Having either one of them will result in unstable system or a system with steady-state error. Figure 10 illustrates such behavior.

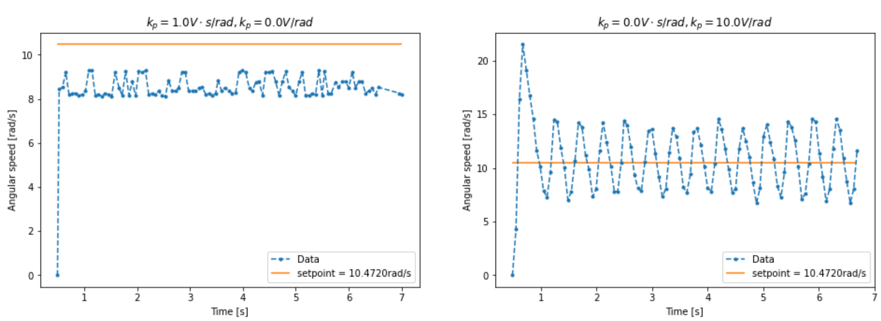


Figure 10: Effect of k_p and k_i

Observe that k_p without k_i results in a system with constant steady-state error, and k_i without k_p results in a permanently oscillating unstable system. Figure 11 describes the different behaviors of the system for different combinations of k_p and k_i.

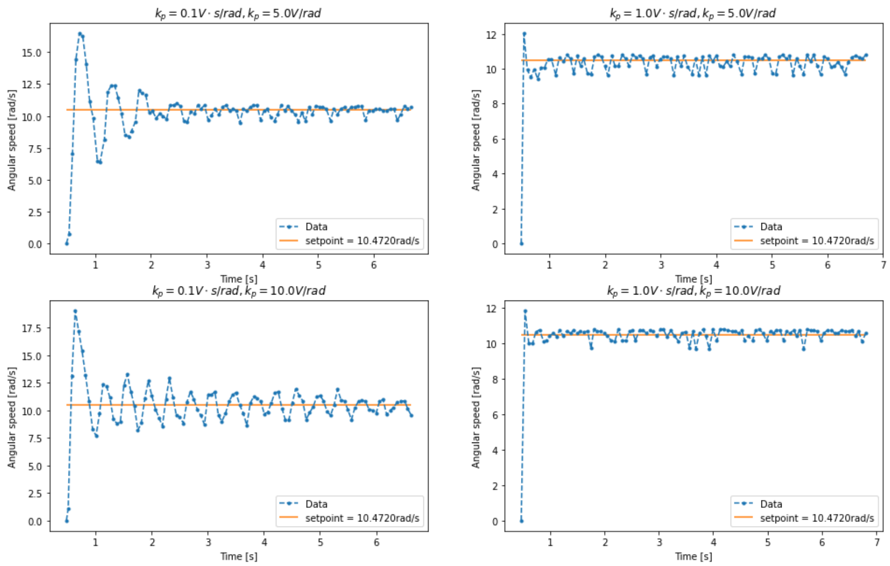


Figure 11: Effect of different combinations of k_p and k_i

Top left plot is both slow and unstable. Increasing k_p provides more stability, resulting in the plot on the top right; and increasing k_i makes the system more responsive, resulting in the plot on the bottom left(take a close look at the tickmarks). Both effect combined together, the bottom right plot produces the best result.

3.4.3 Motor speed control Tuning

We want to find the optimal values for k_p and k_i so that the system with PI controller will achieve the equilibrium state as quick as possible.
First, it should be made sure that the input value that comes out from the controller does not exceed the actual available input. Here, let us assume that the input voltage up to 12V is available; thus $u_{max} = 12$. Also, let us assume that the system was initially at rest; i.e. $y(0) = 0$. Suppose we want to have our motor to be operating at 250rpm = 26.180rad/s at the steady state, which is equivalent to $r = 26.180$. Therefore the maximum error is $e_{max} = r - y(0) = 26.180$.
The heuristic determination of the maximum value for the proportional gain k_p can be performed from the following relationship:

$$\begin{aligned} k_{p,max} &= \frac{u_{max}}{e_{max}} \\ &= \frac{12}{26.180} = 0.458 \end{aligned}$$

Next, in order to find the optimal k_i that will allow the system to achieve the equilibrium state quickly, we use a heuristics for setting the damping coefficient ζ. A damping coefficient that is too small ($\zeta << 1$) will cause an unnecessary oscillation, and damping coefficient that is too large ($\zeta > 1$) will add excessive underdamping effect to the system, resulting in longer time to achieve the steady state equilibrium. Therefore it is optimal to use $\zeta = 0.7$.

Using $k_p = 0.458$, $\tau = 0.085$s, $K = 4.9\text{rad} \cdot \text{V/s}$, and $\zeta = 0.7$ to the equation (7),

$$
\begin{aligned}
k_i &= \frac{\tau}{K}\left(\frac{Kk_p + 1}{2\tau\zeta}\right)^2 \\
&= \frac{0.085}{4.9}\left(\frac{4.9 \cdot 0.458 + 1}{2 \cdot 0.085 \cdot 0.7}\right)^2 \\
&= 12.907
\end{aligned}
$$

Below is the response of the system that is tuned with $k_p = 0.4584$ and $k_i = 12.907$. Note that the response of the system is similar regardless of the setpoint.

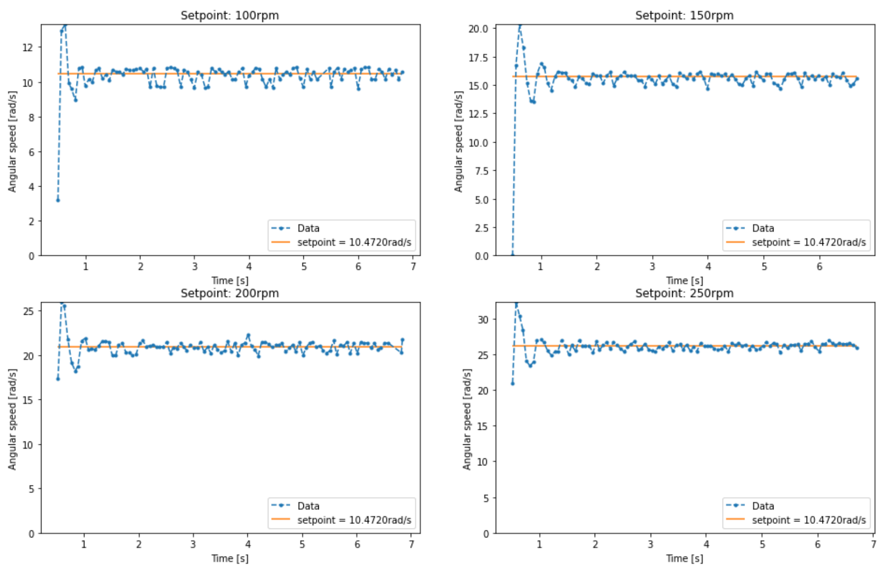


Figure 12: Response of the tuned system with different setpoints

There is a slight overshoot and a small oscillation at the beginning. This is due to the assumptions made when modeling a DC motor. Although the system does not perfectly exhibit the expected response, the selected pair of the gains is a good place to start tuning.

3.4.4 Root Locus

The effect of varying k_i can be analyzed with the root locus. In order to draw the root locus of k_i, the transfer function of the system is first considered.
We already know that the transfer function of the system with PI controller is

$$G_{yr} = \frac{P(s)C(s)}{1 + P(s)C(s)}$$

with

$$
\begin{aligned}
P(s) &= \frac{K}{\tau s + 1} \\
C(s) &= k_p + \frac{k_i}{s}
\end{aligned}
$$

The closed loop poles are

$$
\begin{aligned}
1 + P(s)C(s) = 0 \\
1 + \frac{K}{\tau s + 1}\left(k_p + \frac{k_i}{s}\right) = 0
\end{aligned}
$$

Simple arithmetic yields

$$1 + k_i G(s) = 0$$

with

$$G(s) = \frac{K}{s\,(\tau s + K k_p + 1)}$$

This $G(s)$ is what we want to plot as a root locus. The root locus plot is plotted below, as well as the closed loop poles for $k_i = 12.907$.

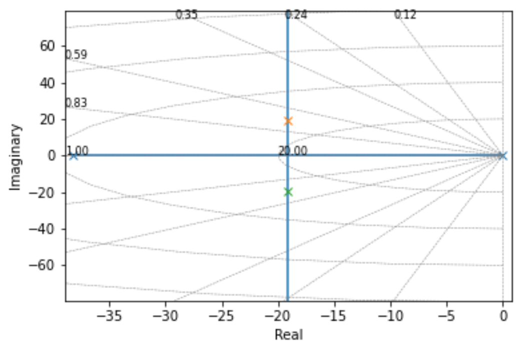


Figure 13: Root locus