1 TA Allocation

Students: $s \in S$ with assigned hours h_s

Courses: $c \in C$ with requested hours h_c

Students have incomplete preferences:

$$c_s^{(1)} \succ_s c_s^{(2)} \succ_s ... \succ_s c_s^{(\underline{c})} \succ_s \emptyset \succ_s c_S^{(\bar{c})} \succ_s ... \succ_s c_s^{(C)}$$

Courses have incomplete preferences:

$$s_c^{(1)} \succ_c s_c^{(2)} \succ_c ... \succ_c s_c^{(\underline{s})} \succ_c \emptyset \succ_c c_S^{(\bar{s})} \succ_c ... \succ_c s_c^{(S)}$$

Algorithm:

1. Each student s proposes to $c_s^{(1)}$ with $h_{c^{(1)}}$ hours, $c_s^{(2)}$ with $h_{c^{(2)}}$ hours ... until she uses up all the assigned hours $h_{s.}$

2. Each course c rejects only dominated proposals. The courses can temporarily accept proposals with sum of hours larger than $h_{c.}$

   A proposal (s, h) is dominated by (s', h') if $s' \succ_c s$ and $h' \geq h$.

3. Repeat (1) and (2) until no more rejection.

4. Let excess proposal e_c be the sum of number of hours course c does not reject. For the course with the largest e_c, accept the best combination among the ones with minimum number of proposals, reject all other proposals. Repeat $(1), (2), (3)$.

5. Repeat (4) until $\max e_c = 0$.

2 Room Allocation

CDF Characterization does not work in two dimensions:

$$1 - F(x) < 1 - G(x) \;\; \forall\, x \text{ iff } \exists \phi \text{ incr } \text{ such that } F(x) = G(\phi(x))$$

Example: F has masses at $(0,1), (1,0)$, G has masses at $(1,1), (0,0)$

F$(0,1) = 0 < G(0,1) = \dfrac{1}{2}$

$F(1,0) = 0 < G(1,0) = \frac{1}{2}$

No matching works.

Alternative CDF Characterization:

Courses: $c \sim F \in C \subseteq \mathbb{R}^L$

Rooms: $r \sim G \in R \subseteq \mathbb{R}^L$, with $|R| \geq |C|$

Sufficient Condition (C1): if for each $c \in C, \exists$ a dimension $i^\star \in L$, such that $\mathbb{P}\{C >_i c\} < \mathbb{P}\{R >> c\}$, then there is a matching $\phi : C \to R$ such that $\phi(c) = r$ iff $r >> c$.

Algorithm: take dimension $i^\star$, start with the c with largest $c_{i^\star}$, call it $c^\star$, match it with one of the rooms $r^\star >> c^\star$ such that remaining set $(C \setminus c^\star, R \setminus r^\star)$ still satisfies C1. Repeat until all courses are matched.

Base case (need strict inequality):

$$\mathbb{P}\{C >_i c\} = 0$$
$$\text{need } \mathbb{P}\{R >> c\} > 0$$

Induction:

For each c with $c_{i^\star} < c^\star_{i^\star}$, C1 is satisfied.

After removing $c^\star, \mathbb{P}\{C >_i c\}$ is reduced by 1 unit, and $\mathbb{P}\{R >> c\}$ is reduced by 0 or 1 unit. Therefore, C1 will still be satisfied on $(C \setminus c^\star, R \setminus r^\star)$ for each remaining c.

3 Room Allocation 2

3.1 Generating Preferences

Courses: $n \in \{1, 2, 3, ..., N\}$

Rooms: $m \in \{1, 2, 3, ..., M\}$

Characteristics: $k \in \{1, 2, 3, ..., K\}$

Weights of characteristics: $w_1, w_2, ...w_K$

$w_k = \infty$ if k characteristics constraint must be satisfied: class size, projector etc.

other weights $0 < w_k \leq 1$ could be estimated: tables, location etc.

Requirements for Courses: $x_{n,k} \in [-1, 1]$

Characteristics of Rooms: $y_{m,k} \in [-1, 1]$

Utilities: $u_n(m) = -\sum_k w_k \cdot \max\{0, x_{n,k} - y_{m,k}\}$

Use the convention that $\infty \cdot 0 = 0$

Preferences can be given by $m \succ_n m'$ iff $u_n(m) \geq u_n(m')$

Let $m = \emptyset$ represent unassigned courses with $u_n(\emptyset) = -2$ K

3.2 Generating Allocations

Given a priority list $\pi(N)$, a permutation of $\{1, 2, 3, ..., N\}$

Use serial dictatorship to get allocation $d(\pi)$

Randomize over the allocations with the smallest number of unassigned courses

4 School Alloaction

4.1 Cardinal $Mo\delta$

Finite number of schools (districts): i, scs $\in \{1, 2, 3, ..., S\}$

Utility for students living near school i going to school s is:

$$u(i, s) = -\delta_{is} + \varepsilon_{is}, \varepsilon_{is} \sim N(0, 1)$$

where δ_{is} = distance between student i and school s.

$\delta_{ii} = 0$ and $\delta_{ij} = \delta_{ji} > 0$.

Designer wants to maximize total utility of all students.

1. If no report of preferences:

$$\max_{\mu} \mathbb{E}\left[\sum_i u(i, \mu(i))\right]$$
$$\Rightarrow \min_{\mu} \sum_i \delta_{i,\mu(i)}$$

Algorithm:

Probablistic serial with preference ranking over $s = -\delta_{is}$

$\Leftrightarrow$ DA with preferences -δ_{is} on both schools and students (indifferences broken by randomization)

2. If truthful report of ordinal preferences:

$$\max_{\mu} \mathbb{E}\left[\sum_i u(i,\mu(i)) | \hat{\succ}_i\right]$$
$$\Rightarrow \min_{\mu} \sum_i \mathbb{E}\left[\delta_{i,\mu(i)} - \varepsilon_{i,\mu(i)} | \hat{\succ}_i\right]$$

In 2 school case with $\delta_{12} = 1$:

$$\begin{aligned}
&\mathbb{E}\left[u(1,1) | 1 \succ 2\right] \\
&= \mathbb{E}\left[-\delta_{11} + \varepsilon_{11} | -\delta_{11} + \varepsilon_{11} > -\delta_{12} + \varepsilon_{12}\right] \\
&= \mathbb{E}\left[\varepsilon_{11} | \varepsilon_{11} > -1 + \varepsilon_{12}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\varepsilon_{11} | \varepsilon_{11} > -1 + \varepsilon_{12}, \varepsilon_{12}\right]\right] \\
&= \int_{-\infty}^{\infty} \int_{-1+\varepsilon_{12}}^{\infty} \varepsilon_{11} dF(\varepsilon_{11}) \, dF(\varepsilon_{12}) \\
&= \frac{1}{2\sqrt{\pi}} e^{\frac{-1}{4}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[u(1,2) | 1 \succ 2\right] \\
&= \mathbb{E}\left[-\delta_{12} + \varepsilon_{12} | -\delta_{11} + \varepsilon_{11} > -\delta_{12} + \varepsilon_{12}\right] \\
&= \mathbb{E}\left[-1 + \varepsilon_{12} | \varepsilon_{11} > -1 + \varepsilon_{12}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[-1 + \varepsilon_{12} | \varepsilon_{11} > -1 + \varepsilon_{12}, \varepsilon_{11}\right]\right] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\varepsilon_{11}+1} -1 + \varepsilon_{12} dF(\varepsilon_{12}) \, dF(\varepsilon_{11}) \\
&< \int_{-\infty}^{\infty} \int_{-\infty}^{\varepsilon_{11}+1} \varepsilon_{12} dF(\varepsilon_{12}) \, dF(\varepsilon_{11}) \\
&= -\frac{1}{2\sqrt{\pi}} e^{\frac{-1}{4}}
\end{aligned}$$

In 3 school case with $\delta_{12} = 1, \delta_{13} = 2, \delta_{23} = 3$:

$$\mathbb{E}\left[u(1,1) | 1 \succ 2 \succ 3\right]$$

Algorithm:

Random priority with priority list ranking = lexicographical ordering on difference between $\hat{\succ}_i$ and -δ_{is}.

$\Leftrightarrow$ DA with preferences -δ_{is} on schools and $\hat{\succ}$ on students (indifferences broken by randomization)

3. If truthful report of cardinal preferences:

$$\max_{\mu} \sum_i \hat{u}\left(i, \mu\left(i\right)\right)$$

Algorithm:

Serial dictatorship with priority list ranking = $\hat{u}$ restricted on remaining seats.

$\Leftrightarrow$ DA with preferences $\hat{u}\left(\cdot, s\right)$ for schools s and preferences $\hat{u}\left(i, \cdot\right)$ for students.

4.2 Ordinal $Mo\delta$

Given preferences: $\succ_i$

$$u\left(i, s\right) = j \text{ if } s = s^{(j)}$$

where $s^{(j)}$ is the j-th school in $\succ_i$

the $\mathbb{P}\left\{s \succ s'\right\} > \mathbb{P}\left\{s' \succ s\right\}$ if $\delta_{is} < \delta_{is'}$

1. If no report of preferences:

$$\begin{aligned}
&\max_{\mu} \mathbb{E}\left[\sum_i u\left(i, \mu\left(i\right)\right)\right] \\
&\Rightarrow \max_{\mu} \sum_i \sum_j j \cdot \mathbb{P}\left\{\mu\left(i\right) = s^{(j)}\right\} \\
&\Rightarrow \min_{\mu} \sum_i \delta_{i,\mu(i)}
\end{aligned}$$

since $\sum_j j \cdot \mathbb{P}\left\{\mu\left(i\right) = s^{(j)}\right\}$ is increasing in -$\delta_{i,\mu(i)}$

2. If truthful report of ordinal preferences:

Cardinal preferences can be inferred from ordinal preferences.

3. If truthful report of cardinal preferences:

$$\max_{\mu} \sum_{i} \hat{u}\left(i, \mu\left(i\right)\right)$$

4.3 Other

Deferred Acceptance with student proposing, school priority list ordered by δ_{ij}. Indifferences broken by randomization.

(NOT) Random Serial Dictorship with priority list with lexicographical ordering on α_{is} where s is the preference of a student in i on the list of schools. Indifferences broken by randomization.

With 2 schools:

1. D1 with $1 \succ 2$ and D2 with $2 \succ 1$
2. D1 with $2 \succ 1$ and D2 with $1 \succ 2$

   In DA round 1, class (1) are all accepted, class (2) are accepted until the school is full.

   In DA round 2, remaining of class (2) are accepted to second preferred.

   With 3 schools with $\alpha_{13} < \alpha_{23} < \alpha_{12} < 0$:

1. D1 with $1 \succ 2 \succ 3$ and D2 with $2 \succ 1 \succ 3$
2. D2 with $2 \succ 3 \succ 1$ and D3 with $3 \succ 2 \succ 1$
3. D1 with $1 \succ 3 \succ 2$ and D3 with $3 \succ 1 \succ 2$
4. D1 with $2 \succ 1 \succ 3$ and D2 with $1 \succ 2 \succ 3$
5. D2 with $1 \succ 3 \succ 2$
6. D1 with $2 \succ 3 \succ 1$
7. D2 with $3 \succ 2 \succ 1$ and D3 with $2 \succ 3 \succ 1$
8. D2 with $3 \succ 1 \succ 2$
9. D3 with $2 \succ 1 \succ 3$
10. D1 with $3 \succ 1 \succ 2$ and D3 with $1 \succ 3 \succ 2$
11. D1 with $3 \succ 2 \succ 1$

12. D3 with $1 \succ 2 \succ 3$

    In DA round 1, classes (1) and (2) and (3) are all accepted, all other classes are accepted until full.

    In DA round 2, remaining of classes (4) and (5) and (6) are all accepted.