# DESIGN OF SEARCH BY COMMITTEES

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Abstract. I apply a mechanism design approach to committee search problems, such as hiring by a department or a couple's search for a house. A special class of simple dynamic decisions rules have agents submit in each period one of three votes: veto, approve, or recommend; the current option is adopted whenever no agent vetoes and at least one agent recommends. I show that every implementable payoff can be attained by randomizing among these simple rules. This result dramatically simplifies the design problem.

## 1. INTRODUCTION

Consider the example of a married couple who keeping looking for houses until they collectively decide on purchasing one. Assume the housing market is competitive and a house is gone before a new one becomes available, then the couple faces a committee search problem.

For every house indexed by $t$ that is available, the husband and the wife have two separate ratings $v_{1,t}$ and $v_{2,t}$, respectively. Examples of decision rules they can use include,

(1) dictator rule: buy the house whenever $v_{2,t} \geqslant \theta$, or
(2) unanimity rule: buy the house if $v_{1,t} > \theta_1$ and $v_{2,t} > \theta_2$, or
(3) ternary rule: buy the house if $v_{i,t} > \theta_1$ for both $i = 1, 2$ and $v_{i,t} > \theta_2 > \theta_1$ for some $i = 1, 2$, or
(4) sum rule: buy the house if $v_{i,t} > \theta_1$ for both $i = 1, 2$ and $v_{1,t} + v_{2,t} > \theta_2$, or
(5) log rule: buy the house if $\log(v_{1,t}) + \log(v_{2,t}) > \theta$.

In this paper, I show that decision rules such as (5) are not implementable because they do not induce truth-telling, and rules like (4) is not optimal for the couple. In the case where previous ratings cannot affect the current decision, only the dictator rule (1) and the

*Date*: October 26, 2017.

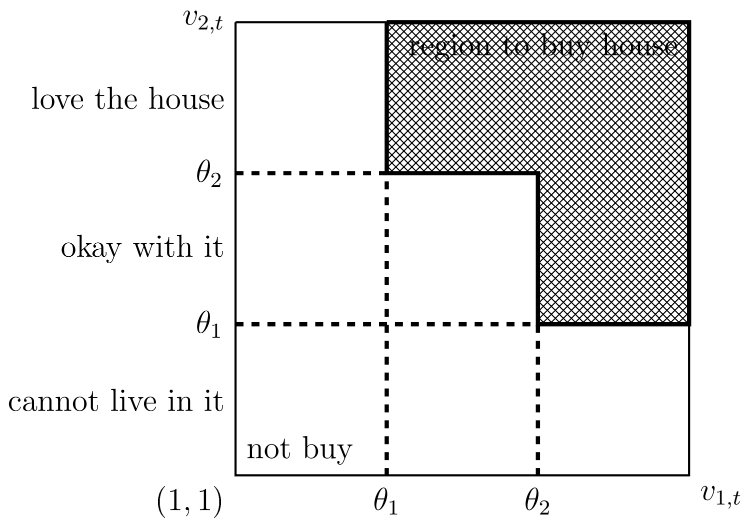


FIGURE 1. Decision rule (3)

unanimity rule (2) are implementable and optimal, and if I allow decisions to be linked over time, then all implementable rules are equivalent to randomizations among the rules of the form described in (3), and in particular, all the optimal decision rules are rules like (3). In general, (3) represents a class of simple voting rules with two thresholds that divide the domain of ratings into three regions: love the house (recommend), okay with it (approve), absolutely cannot live in it (veto), and the house is purchased as long as both the husband and the wife is at least okay with it and one of them loves it. I call this class of decision rules *ternary* mechanisms because each member has *three* choices. Note that (1) and (2) are special cases of ternary rules. (3) is a typical ternary rule and I use a diagram like Figure 1 to represent such decision rule.

Ternary rules can be naturally extend to problems with more than two agents. Economic examples include the following.

(1) A hiring committee with possibly different preferences observes one candidate at a time. Every rejected candidate cannot be called back, and the committee keeps searching until one candidate is hired.
(2) An oppotunity of the costly provision of a public good appears once in a while, and a group of voters who may value the good differently make irreversible decisions to spend tax money on one of the alternatives.

(3) A team observes project proposals that arrives sequentially. Members with differing opinions on each proposal need to collectively decided whether to select a particular one. Once rejected, the project might be adopted by another team.

In all of the above examples, multiple agents have to make a collective and irreversible decision on an option to adopt when facing options that are presented to them sequentially. I analyze the mechanism design for these *committee search problems with private value, public allocation, and without transfers.* The contribution of this paper is showing that every implementable decision rules is equivalent to some randomization among ternary rules, and as a result, the designer can restrict attention to using only these simple voting rules.

In specific, I consider a model where two agents observe private values of, for example hiring a candidate, every period. The values are independent over time but the values different agents observe in the same period can be correlated. In each period, the candidate is either hired, and in that case, every agent gets her private value, or the decision is delayed to the next period. The mechanism incentivizes the agents to report their values truthfully without using transfers. It uses differing future decision rules to link incentives over time. For a ternary rule, a design specifies a lower threshold of value below which an agent vetoes and an upper threshold above which an agent recommends. If the decision is delayed due to an agent exercising her veto power, the mechanism updates the lower and upper thresholds for the future periods to give her a lower expected payoff as punishment to ensure that veto power will be exercised with discretion.

Given a decision rule in some period, there are three regions of values for each agent,

(1) the region where the candidate is not hired no matter what the other agents' values are, called the veto region,
(2) the region where the candidate is hired as long as no other agent vetoes, called the recommendation region, and,
(3) the remaining region, called the approval region.

There are many rules that incentivize truth-telling, where the candidate is sometimes hired and sometimes not if every agent has value in her approval region. The mechanism that always reject the candidate if every agent has value in her approval region is a ternary rule. This ternary rule results in higher payoffs for every agent since, intuitively, if everyone approves a candidate, but no one recommends him, then continue searching will likely provide a better payoff for at least one agent and on average will not decrease the payoff for everyone else. In addition, there are mechanisms such as the dictator rules that give one or more agents the worst possible payoffs and they are special cases of ternary rules. Therefore, I conclude that every implementable rule, which results in payoffs between the best and the worst, is payoff-equivalent to some randomization among these ternary rules.

Technically, the model involves a principal trying to implement an incentive compatible mechanism. Dynamic ex post incentive compatibility is used as the definition of implementability so that the solution is robust to private communication between agents and robust to correlation between values and beliefs of the agents. I also restrict attention to mechanisms that are deterministic in every period, where either the candidate is hired for certain, or the decision is delayed. These assumptions are made so that the model is tractable. I also make other assumptions for the purpose of presentation, such as there are only two agents and finite number of periods, and the private values are drawn from distributions with full support on some compact set and are independent over time. Those assumptions can be relaxed without significantly changing the main results.

This paper is closely related to the literature on dynamic mechanism design without transfers. Guo and Horner (2015) study the problem without transfers where goods can be allocated the agent in multiple periods. They characterize the optimal mechanism and provide a simple implementation where agents have virtual budgets. Lipnouski and Ramos (2016) have a similar model with similar results where the principal has less commitment power. In this model, the principal can decrease the veto threshold in the next period if she exercises her veto power. It is similar to the budget mechanism in Guo and Horner (2015) in the sense that the agent have a virtual budget of veto power, and if she vetoes excessively, she will gradually lose her power to veto.

Kovac, Krahmer and Tatur (2013) study the stopping problem where only a single good is allocated. In their model, the optimal mechanism is one where the principal chooses different probabilities for assigning the good at different times to incentivize the agent to use the stopping rule the principal prefers. I focus on allocation rules that are deterministic in each period. The good is either allocated for certain or the decision is delayed to the next period. Instead, the principal changes the amount of control an agent has over the allocation in every period to incentivize an agent. The size of the region where an agent have more control over the allocation, for example, to veto or recommend it, is different every period, and this is used in place of the allocation probabilities in the deadline mechanism in Kovac, Krahmer and Tatur (2013). Both Guo and Horner (2015) and Kovac, Krahmer and Tatur (2013) focus on problems with one agent, where as this paper focus on problems with multiple agents.

Johnson (2014) considers the problem with multiple agents, but the good is allocated to only one of the agents. They use the promise utility approach where the principal promises future allocation rules that correspond to different expect payoffs to incentivize truth telling. I use the same approach, but I consider the problem with a public good. In their model, the agents trade favors in a virtual market for decision rights and as a result, they take turns to obtain their favorite private allocation. In this model, the agents also trade decision rights, but the goal is wait for the public allocation that is preferable to every agent.

Moldovanu and Shi (2013) study the committee search similar to the model in this paper. They solve the stationary voting rules with a single threshold wheve each agent votes for the current option if its value is above the threshold and votes against it otherwise. Compte and Jehiel (2010) and Albrecht, Anderson and Vroman (2010) also have a similar model and focus on majority voting rules. This paper considers all direct revelation mechanisms, and one result states that if decisions cannot be linked over time, then all implementable decision rules are the voting rules with a single threshold as in Moldovanu and Shi (2013). Additional decision rules that are not voting rules can be implementable in this environment through the linking of decisions over time, but I show that every implementable rule is payoff-equivalent to some randomization among ternary voting rules and this justifies the restriction to only using simple voting rules when solving committee search problems.

Section 2 introduces the model. Section 3 characterizes implementability. Section 4 states and explains the main result that every implementable payoff can be attained by randomizing among ternary rules. Section 5 discusses examples and concludes.

To do list:

Diagrams need axes.

Side-by-side diagrams need common captions.

Indentation.

Citation.

## 2. MODEL

In this section, I describe the agent's stopping problem and the principal's design problem. In particular, I define implementability as dynamic ex post incentive compatibility, and explain why it is appropriate for this model.

2.1. **Valuations.** In this subsection, I describe the payoffs and the class of mechanisms.

I describe the model with two agents and discuss possible extensions to $N > 2$ agents in section 5. A principal hires a new employee through a committee. In each period, a new candidate appears and each agent in the committee observes the value of hiring the candidate. The number of periods $T$ is finite and there is no discounting. I assume the value in period $t, v_t = (v_{1,t}, v_{2,t}) \in \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ are independently distributed over time with continuous density $f_t$, full support $\mathcal{V}$ and the means are normalized to 0 in each coordinate. $\mathcal{V}$ is compact, and without loss of generality, I assume $\mathcal{V}_1 = \mathcal{V}_2 = [\underline{v}, \bar{v}]$.

I use $v^t$ to denote a history of values, or reports, from period 1 to period $t$,

$$\begin{aligned} v^t &= (v_1, v_2, ..., v_t) \\ &= ((v_{1,1}, v_{2,1}), (v_{2,1}, v_{2,2}), ..., (v_{t,1}, v_{t,2})) . \end{aligned}$$

I assume independence over time but not between agents. The value of hiring a candidate to each agent may be correlated. In the public good example, a new potential provision appears each period independently and the value to each voter could be different. Similarly, in the project selection example, an independent proposal is drafted every period but the values to the members can be correlated.

I need to find out which revelation principle is used.

Due to the revelation principle, I assume the principal designs a direct mechanism, where the agents only report their own valuation every period, that specifies whether to hire the candidate in each period. I define the mapping $q : \mathcal{V} \to [0,1]$ as a stage mechanism after some history $v^{t-1} = (v_1, v_2, ..., v_{t-1})$ of reports until period $t-1$. Here, $q\left(v_t | v^{t-1}\right)$ is the conditional probability that the candidate is hired in period $t$ if the agents report $v_t$ after the history $v^{t-1}$ condition on the candidate not hired in periods before $t$. When the context is clear, I omit the history after which the stage mechanism is used and use the expression $q\left(v_t\right)$. The grand mechanism, $Q$, is the collection of stage mechanisms following every possible history,

$$Q : \bigcup_{t \leqslant T} \mathcal{V}^t \to [0,1] .$$

The values of hiring a candidate are independent over time, but the mechanism can depend on historical reports. As a result, the principal can choose different allocations in the future as reward or punishment to incentivize truthful reports.

If no candidate is hired by the end of the last period, the agents get outside option $v_{T+1} = \left(v_1^\star, v_2^\star\right)$ in the interior of $\mathcal{V}$ in period $T+1$. For each terminal history $v^{T+1} \in \mathcal{V}^{T+1}$, the total probability of hiring the candidate the should add up to 1,

$$\sum_{t=1}^{T+1} q\left(v_t\right) \prod_{s=t}^{t-1} \left(1 - q\left(v_s\right)\right) = 1.$$

Given a grand mechanism $Q$, let $w_{i,t}\left(Q; v^{t-1}\right)$ denote the ex ante continuation value of agent $i$ in period $t$ after some history $v^{t-1} \in \mathcal{V}^{t-1}$, before $v_t, v_{t+1}, ..., v_T$ are realized, assuming both agents report truthfully.

$$w_{i,t}\left(Q;v^{t-1}\right)=\mathbb{E}\left[\sum_{s=t}^{T+1}q\left(v_s\right)\prod_{s'=1}^{s-1}\left(1-q\left(v_{s'}\right)\right)v_{i,s}\right].$$

For a mechanism, $Q$, the *continuation value after history* $v^{t-1}$ is a pair,

$$w_t\left(Q;v^{t-1}\right)=\left(w_{1,t}\left(q;v^{t-1}\right),w_{2,t}\left(q;v^{t-1}\right)\right).$$

In particular, the total payoff an agent, $i$, gets from a mechanism, $Q$, is, $w_{i,1}(Q)$.

I also define two grand mechanisms, $Q_1, Q_2$, to be *payoff-equivalent* after history $v^{t-1}$ if,

$$w_t\left(Q_1;v^{t-1}\right)=w_t\left(Q_2;v^{t-1}\right).$$

2.2. **Implementability.** In this subsection, I define dynamic ex post implementability and explain the reasons it is used for this problem. Also, I define and restrict attention to a subset of random mechanisms I call quasi-deterministic mechanisms where the mechanism is determinsitic in each stage but between period randomization is allowed.

**Definition 1.** A mechanism, $Q$, is (dynamic ex post) incentive compatible, if in each period $t \in \{1, 2, ..., T\}$, after each history $v^{t-1} \in \mathcal{V}^{t-1}$ and every $v_{-i,t} \in \mathcal{V}_{-i}$,

$$v_{i,t}\in\arg\max_{\hat{v}_{i,t}\in\mathcal{V}_i}v_{i,t}q\left(\hat{v}_{i,t},v_{-i,t}\right)+\left(1-q\left(\hat{v}_{i,t},v_{-i,t}\right)\right)w_{i,t+1}\left(Q;v^{t-1},\hat{v}_{i,t},v_{-i,t}\right).$$

A mechanism is dynamic ex post incentive compatible (or the allocation is dynamic ex post implementable) if, in every period, it is optimal to report the true valuation an agent observes given the other agents' true valuations.

This definition is the same as the one in Noda (2017), but without transfers and discounting. He calls this type of implementability *within-period ex-post incentive compatibility* , or wp-EPIC. Bergemann and Valimaki (2010), Cavallo, Parkes, and Singh (2010), Athey and Segal (2013) also use a more complicated version of ex-post implementability since valuations (or types) can be correlated over time.

Ex post implementability is used because it is robust to private communication between agents, within period correlation between valuations, and variation in agents' beliefs about each other's types. This definition of incentive compatibility also makes the model tractable.

Need to add more explanation why ex-post ic is used. Need to read Bergemann and Morris (2005).

One-shot deviation principle is built into the definition of implementability. I require the agents to report truthfully after all histories, not only on equilibrium paths. Therefore, I omit the possibility that an agent misreports in multiple periods.

I omit individual rationality constraints and agents are forced to participate.

I focus on mechanisms that are deterministic in every stage where the candidate is either hired for sure, or the the decision is delayed to the next period.

Try to prove payoff-equivalence between quasi-determinstic mechanisms and general random mechanisms to justify the use of this.

**Definition 2.** A stage mechanism, $q$, is *deterministic* if $q(\cdot) \in \{0, 1\}$. A grand mechanism, $Q$, is *quasi-deterministic* if after every history $v^{t-1} \in \mathcal{V}^{t-1}$, only randomization among multiple deterministic stage mechanisms are used.

For example, after $(v_{1,t}, v_{2,t})$ are reported, the principal must either hire the candidate with probability 1 or delay hiring to the next period, but in case the principal chooses to delay, he can randomize among multiple stage mechanisms in period $t + 1$.

I removed the $\tilde{\Delta}$ Q definition of quasi-deterministic mechanisms. I will add it back if I am able to prove some kind of decomposibility result.

## 3. IMPLEMENTABILITY

In this section, I characterize the conditions for implementability. It is convenient to divide the analysis into few steps. In the first step, I characterize implementable stage mechanisms in the last period. Next, I explain that the implementability in the dynamic setting can be reduced to a version of the static problem with appropriately chosen continuation values. Finally, I explain why the dynamic problem is substantially richer than a sequence of static ones.

3.1. **Static Implementation.** In this subsection, I consider the problem when $T = 1$. In this case, the continuation value is fixed at the outside option $(v_1^\star, v_2^\star)$. I describe a class of mechanisms called binary stage mechanisms that characterizes implementability. The same characterization applies to the last period when $T > 1$.

**Definition 3.** A stage mechanism, $q$, is binary, with outside option $(a_1, a_2)$, if for each $i \in \{1, 2\}$, for every $v_{-i,t} \in \mathcal{V}_{-i}$, either $q\left(v_{i,t}, v_{-i,t}\right)$ is constant (0 or 1) for every $v_{i,t} \in \mathcal{V}_i$, or,

$$q\left(v_{i,t}, v_{-i,t}\right) = \begin{cases} 0 & \text{if } v_{i,t} < a_i \\ 1 & \text{if } v_{i,t} > a_i \end{cases}.$$

There are only six mechanisms that are binary as shown in Figure 2. The shaded regions are where the candidate is hired, or the acceptance region $\{v_t : q\left(v_t\right) = 1\}$, and the remaining region is where the candidate is not hired and the outside option $(a_1, a_2)$ is given to the agents.

**Lemma 1.** *For $T = 1$, if a quasi-deterministic mechanism, $Q$, is incentive compatible, then the (only) stage mechanism $q\left(\cdot|\varnothing\right)$ must be binary with outside option $(v_1^\star, v_2^\star)$.*

The intuition for Lemma 1 is as follows. Note that in each of the six binary stage mechanisms, whenever there is a threshold above which the candidate is hired and below which the candidate is not, the threshold value must be $v_1^\star$ for agent 1 and $v_2^\star$ for agent 2. This is because, in the last period, if the candidate is still not hired, the agents gets $(v_1^\star, v_2^\star)$ in period $T + 1$. For a particular agent and a fixed report from the other agent, if the mechanism always hires the candidate or never hires the candidate, this agent will be indifferent between reporting any value, and as a result, she will not misreport. Otherwise, if the mechanism hires the candidate when she reports a value less than the outside option, she will misreport a higher value and obtain the outside option instead; and if the mechanism does not hire

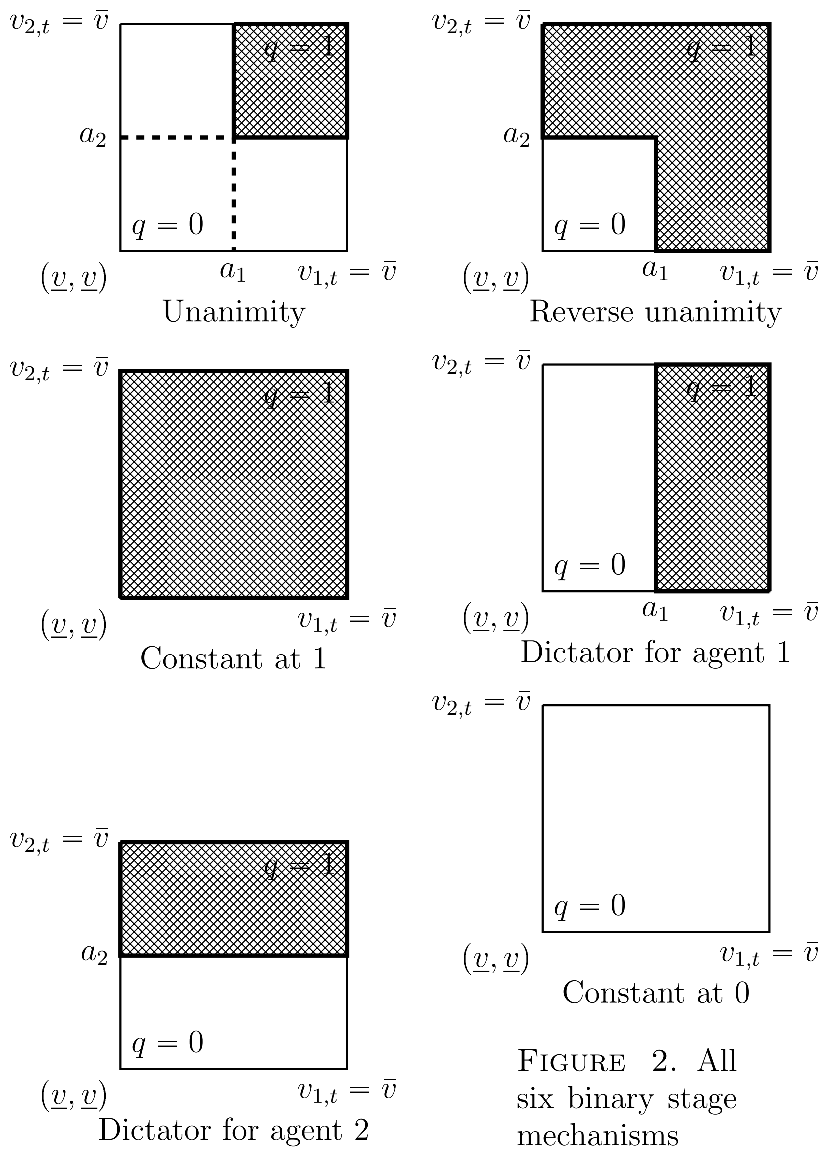


FIGURE 2. All six binary stage mechanisms

the candidate when she reports a value more than the outside option, she will misreport a lower value to get the candidate hired. Therefore, the only incentive compatible mechanism hires the candidate if and only if an agent reports a value larger than the outside option.

3.2. **Dynamic Implementation.** In this subsection, I show how the dynamic problem can be reduced to a static one but with more than one possible outside options. Then I state a monotonicity condition as a characterization of all quasi-deterministic incentive compatible mechanisms in this environment.

An incentive compatible grand mechanism consists of a collection of incentive compatible stage mechanisms, one after each history $v^{t-1}$, with continuation value pairs for all reports in the rejection region $\{v_t : q(v_t) = 0\}$, where each continuation value pair corresponds to some sequence of incentive compatible stage mechanisms in periods $t+1, t+2, ..., T$.

An incentive compatible mechanism must be monotonic in the sense that the candidate is more likely to be hired if the agents observe higher valuations. Monotonicity is necessary, but not sufficient, for implementability.

Move this lemma to the appendix.

**Lemma 2.** *If a grand mechanism, $Q$, is incentive compatible, then after every history $v^{t-1}, q(v_t|v^{t-1})$ is weakly increasing in $v_{i,t}$ for each agent $i$.*

Given a stage mechanism $q(\cdot|v^{t-1})$ and the report of every agent other than $i$, define the following threshold function,

$$R_i(v_{-i,t}) = \inf\{v_{i,t} : q(v_t) = 1\}, \tag{1}$$

with the convention that $\inf \varnothing = \infty$.

For deterministic stage mechanisms, Lemma 2 means there are thresholds $(R_1, R_2)$ such that the candidate is hired if and only if an agent reports a value above the threshold. Also, by incentive compatibility, the continuation value function must be constant whenever the value observed is less than the threshold, since if not, an agent can misreport and obtain a different, and possibly highly, continuation value. This observation is stated as the following lemma.

**Lemma 3.** *A quasi-deterministic mechanism, $Q$, is incentive compatible if and only if after every history $v^{t-1}$, for each $i \in \{1,2\}$, and every $v_{-i,t} \in \mathcal{V}_{-i}$,*

$$q(v_t) = \begin{cases} 0 & \textit{if } v_{i,t} < R_i(v_{-i,t}) \\ 1 & \textit{if } v_{i,t} > R_i(v_{-i,t}) \end{cases},$$

*and for every* $v_t = (v_{i,t}, v_{-i,t})$ *such that* $q(v_t) = 0$,

$$w_{i,t+1}\left(Q; v^{t-1}, v_t\right) \text{ is independent of } v_{i,t}, \text{ and,}$$

$$w_{i,t+1}\left(Q; v^{t-1}, v_t\right) = R_i(v_{-i,t}) \text{ if } R_i(v_{-i,t}) \neq \infty.$$

The property is stated recursively. The notation $w_{i,t+1}$ implies that it is the payoff that corresponds to some sequence of stage mechanisms in periods $t+1, t+2, ...T$ that also satisfy the above property.

Next, I describe some implication of Lemma 3. First, I describe two stage mechanisms that cannot be incentive compatible.

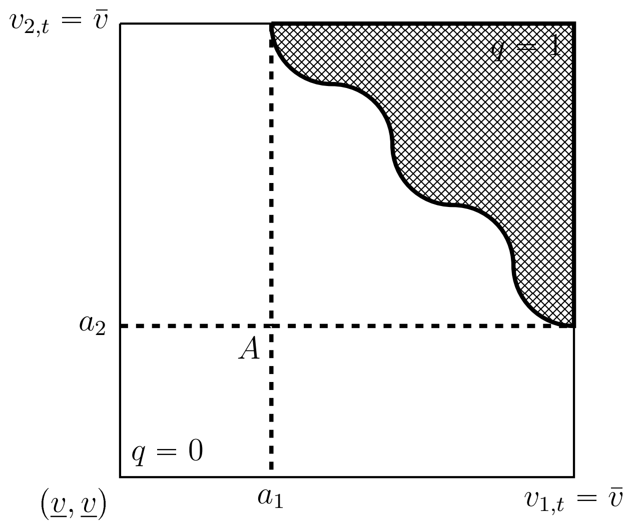


FIGURE 3. Example of a stage mechanism that is not incentive compatible

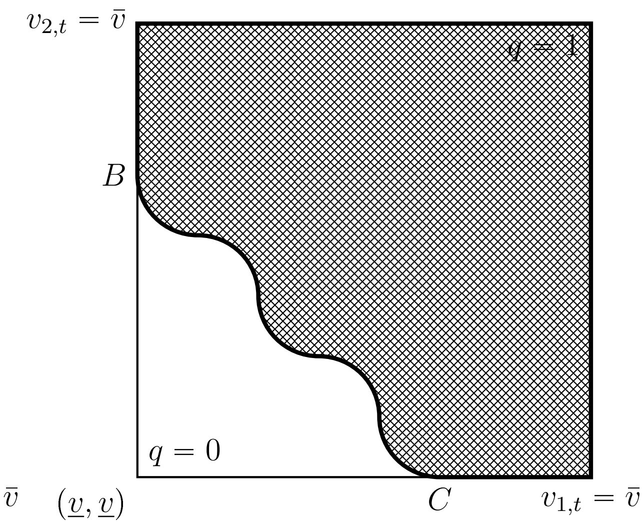


FIGURE 4. Another example of a stage mechanism that is not incentive compatible

Stage mechanisms with acceptance region, $\{v_t : q(v_t) = 1\}$, like the area shaded in Figure 3 cannot be a part of an incentive compatible grand mechanism because points like $A$ require value pair $w_{t+1}(v^{t-1}, A)$ to be equal to the thresholds $(R_1(a_2), R_2(a_1)) = (\bar{v}, \bar{v})$ according Lemma 3. It means some incentive compatible mechanism must generate a payoff of $\bar{v}$ for the agents through a finite sequence of stage mechanisms from period $t+1$ to period $T$. This is not possible unless the outside option $v^\star$ is $(\bar{v}, \bar{v})$, which violates the assumption that $(v_1^\star, v_2^\star)$ is in the interior of $\mathcal{V}$.

Similarly, stage mechanisms with acceptance region like the area shaded in Figure 4 cannot be a part of an incentive compatible grand mechanism because due to the condition in Lemma 3 that requires the continuation value to be equal to the threshold, points like $B = (0, b)$ and $C = (c, 0)$ require $w_{1,t+1}(v^{t-1}, B) = R_1(b) = \underline{v}$ and $w_{2,t+1}(v^{t-1}, C) = R_2(c) = \underline{v}$, which also violates the assumption that $(v_1^\star, v_2^\star)$ is in the interior of $\mathcal{V}$.

This is not related to the lemma, but I need it to define $a_i$ and $r_i$ which will be used later in explaining the main results. No other place to put this.

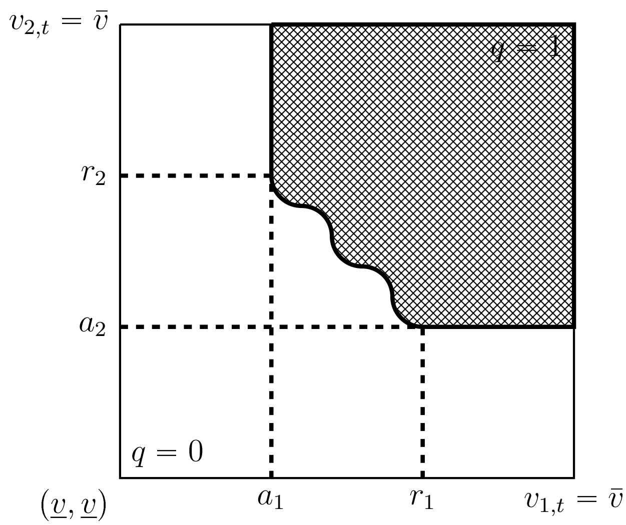


FIGURE 5. Example of a stage mechanism that is possibly incentive compatible

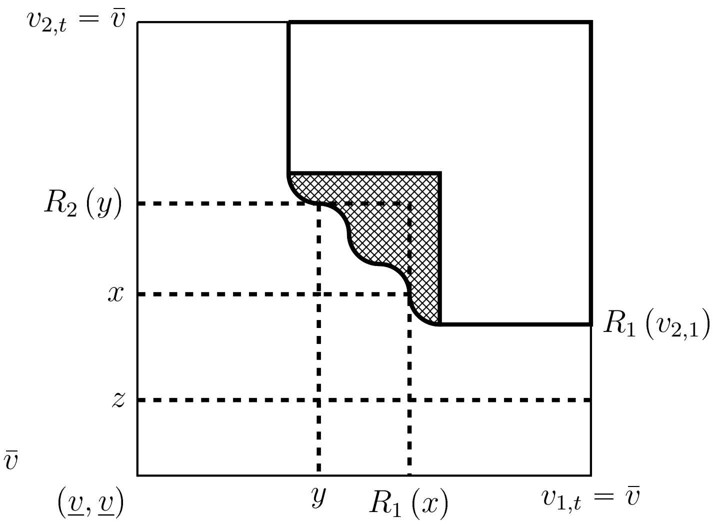


FIGURE 6. Threshold continuation value

The previous observation implies that incentive compatible mechanism must consist of stage mechanisms that are the ones described in Figure 5 where there exist $a_1, a_2 \notin \{\underline{v}, \bar{v}\}$,

$$a_i = \sup_{v_{i,t}} \{v_{i,t} : q(v_{i,t}, v_{-i,t}) = 0 \text{ for all } v_{-i,t} \in \mathcal{V}_{-i}\}, \tag{2}$$

such that if agent $i$ observes a value less than $a_i$, the candidate will not be hired no matter what value the other agent reports, and there exist $r_1, r_2 \notin \{\underline{v}, \bar{v}\}$,

$$r_i = \lim_{\varepsilon \to 0^+} R_i(a_{-i} + \varepsilon), \tag{3}$$

such that if agent $i$ observes a value more than $r_i$, the candidate will be hired as long as the other agent reports a value larger than $a_{-i}$. Recall $R_i$ is the threshold function defined in Equation 1.

In the case where the acceptance region is closed,

$$r_i = R_i\left(a_{-i}\right) < \infty. \tag{4}$$

I call $a_i$ the approval threshold and $r_i$ the recommend threshold for an agent $i$. When agent $i$ observes a value less than $a_i$, the candidate will never be hired for any value the other agent reports, so intuitively, agent $i$ disapproves (or vetoes) hiring. When agent $i$ observes a value larger than $r_i$, the candidate will be hired as long as the other agent does not veto, so intuitively, agent $i$ recommends hiring. When both agents approve but none recommends, the stage mechanism can choose an arbitrary hiring rule of the form in Figure 5.

If this is moved up, then I need to define $a_i$ and $r_i$ without Figure 3 and Figure 4, but then those two examples are not necessary?

An example of a stage mechanism that satisfies the condition of Lemma 3 is depicted in Figure 5. The shaded area corresponds to the acceptance region. According to the first part of Lemma 3, the acceptance region has the monotonicity property: if the candidate is accepted at some values, he is also accepted at the values that dominate the previous ones.

The continuation values for agent 1 along the line $v_{2,t} = z$ in Figure 6 are constant, for example,

$$w_{1,t+1}\left(v^{t-1}, \left(v_{1,t}, z\right)\right) = a_1 \ \forall\ v_{1,t} \in \mathcal{V}_1.$$

If continuation values for agent 1 are not the same along this line, then the type of agent 1 who has a value that leads to a lower continuation value will misreport and get a different and higher continuation value.

The continuation values of agent 1 along $v_{2,t} = x$ in Figure 6 are equal to the lowest valuation for which the candidate is accepted $R_1(x)$, or,

$$w_{1,t+1}\left(v^{t-1},(v_{1,t},x)\right) = R_1(x)\ \forall\ v_{1,t} \text{ such that } q(v_{1,t},x) = 0.$$

Not only must the continuation value be constant for every $v_{1,t}$, it must be equal to $R_1(x)$ because if the continuation value is higher, say $R_1(x)+\delta$, then the type of agent 1 with value $R_1(x)+\frac{\delta}{2}$ will have incentive to misreport a lower value and delay the hiring to get a higher continuation value. Similarly, if the continuation value is lower, say $R_1(x)-\delta$, then an agent 1 observing $R_1(x)-\frac{\delta}{2}$ will have incentive to misreport a higher value to get the candidate hired in the current period.

Similar requirements apply to agent 2 along vertical line segments on the diagram. For example, the continuation value at the point $(x,y)$ is fixed for both players at,

$$w_{t+1}\left(v^{t-1},(x,y)\right) = \left(R_1(y), R_2(x)\right).$$

This leads to the observation that every value pair in the shaded region in Figure 6 must be a continuation value that corresponds to a sequence of stage mechanisms in periods $t+1, t+2, ..., T$. It is difficult to formally state this observation since $r_i$ is not well defined when $R_i(a_{-i}) = \infty$, but this idea of required continuation value region will be formally defined and used in the proofs of the main results in the next section.

This is still unclear. Remove?

3.3. **Linking Decisions.** The principal can link payoffs over time by using history dependent mechanisms. In this subsection, I demonstrate, with two examples, that such mechanisms can lead to Pareto improvements over history independent ones.

As demonstrated in Figure 5, there are many incentive compatible stage mechanisms in periods $1, 2, ..., T-1$ that look different from the ones in the last period. One main reason that more allocation rules are implementable is the linking of decisions: the principal can use different randomizations between stage mechanisms after different reports to punish or reward the agents. I call these mechanisms history dependent mechanisms.

**Definition 4.** A grand mechanism is *history independent* if the stage mechanisms after any two histories with the same length are the same, that is, if

$$q(\cdot|v^t) = q(\cdot|\tilde{v}^t) \text{ for every } v^t, \tilde{v}^t \in \mathcal{V}^t, \text{ and every } t \in \{1, 2, ..., T\}.$$

In general, if the principal is restricted to use history independent mechanisms, then every stage mechanism in every period must be binary. This observation is stated in the following corollary to Lemma 1.

**Corollary 1.** *If a quasi-deterministic incentive compatible mechanism is history independent, then the stage mechanism after every history must be binary.*

To see why Corollary 1 is true, notice that after every history $v^t$, only one continuation value is allowed in a history independent mechanism because there is only one sequence of stage mechanisms in periods $t+1, t+2, ..., T$. Then every period is similar to the last period except that the outside option may differ from $(v_1^\star, v_2^\star)$. The set of stage mechanisms that is incentive compatible with a single outside option is the set of binary stage mechanisms for that outside option.

The example below with $T = 2$ illustrates the difference between a history dependent mechanism a history independent mechanism.

Suppose the value distributions are symmetric, the outside option is $v^\star = (0, 0)$, and the stage mechanism in period 1 involves a symmetric unanimity rule. The two choices for the stage mechanisms in period 2 are:

(1) Constant 0 (or unanimity or reverse unanimity) after all histories where the candidate is not hired in period 1,
(2) Dictatorship by agent 1 after histories where the candidate is not hired because agent 1 reports a value less than $r_1$ but agent 2 reports a value larger than $r_2$; dictatorship by agent 2 after histories where the candidate is not hired because agent 2 reports a value less than $r_2$ but agent 1 reports a value larger than $a_1$; and constant 0 in period

2 after histories where the candidate is not hired because both agents report values less than their respective $r_i$.

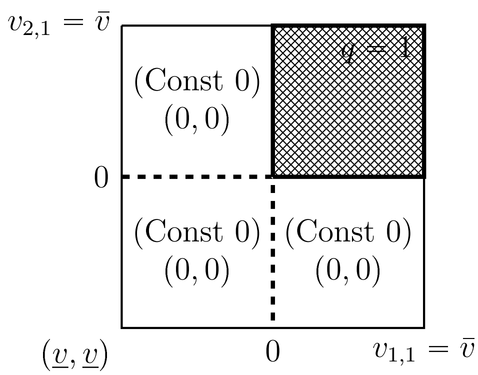


FIGURE 7. History independent unanimity in period 1

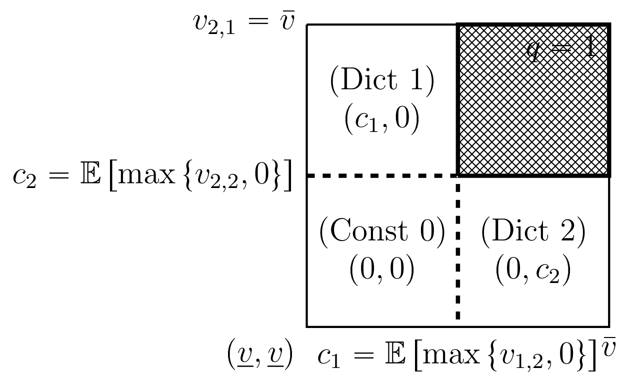


FIGURE 8. History dependent unanimity in period 1

These two mechanisms are described in diagrams Figure 7 and Figure 8 respectively. Mechanism (1) is history independent and mechanism (2) is history dependent. Both diagrams depict the unanimity stage mechanism in period 1, and the name of the stage mechanism and its resulting continuation value pair from period 2 is written in each region in which that continuation mechanism is used.

Note that $r_1 = \mathbb{E}[\max\{v_{1,2}, 0\}]$ and $r_2 = \mathbb{E}[\max\{v_{2,2}, 0\}]$ because the property in Lemma 3 must be satisfied and the $(\mathbb{E}[\max\{v_{1,2}, 0\}], 0)$ and $(0, \mathbb{E}[\max\{v_{2,2}, 0\}])$ are the continuation value pairs generated by the two dictator stage mechanisms in the second period.

Give a numerical example of ternary mechanism here? And move the general definition of ternary mechanisms to next section?

The following definition describes an important class of history dependent mechanisms.

**Definition 5.** A stage mechanism, $q$, is ternary if there are $a_1, r_1, a_2, r_2$, where $\underline{v} \leqslant a_i \leqslant r_i \leqslant \bar{v}$ for each $i \in \{1, 2\}$ such that,

$$q(v_t) = \begin{cases} 0 & \text{if } v_{i,t} < a_i \text{ for some } i \text{ or } a_i \leqslant v_{i,t} < r_i \text{ for all } i \\ 1 & \text{if } v_{i,t} > r_i \text{ for some } i \text{ and } v_{i,t} > a_i \text{ for all } i \end{cases}.$$

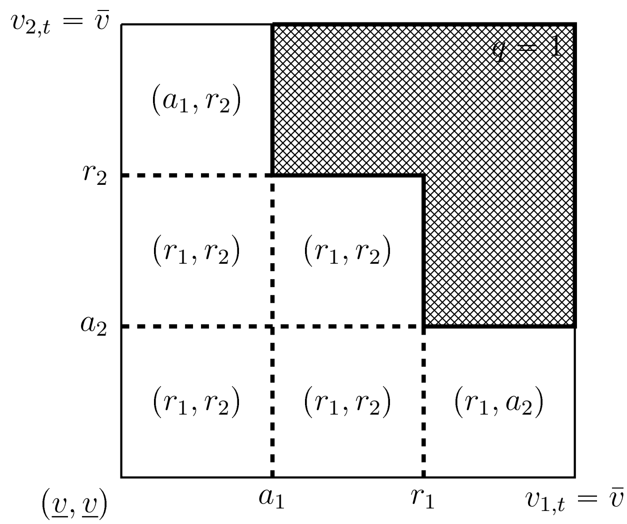


FIGURE 9. An example of a ternary stage mechanism

Figure 9 represents a typical ternary stage mechanism. The continuation values are put in each rectangular region instead of the names of the continuation stage mechanisms from the next period. For general $T > 2$, the complete sequence of stage mechanisms after the current period is irrelavent and difficult to state explicitly, therefore, I write only the pair of continuation values such as $(r_1, r_2)$ in a region to mean that if the agents report $v_t$ is in this region, the candidate will not be hired and a sequence of stage mechanisms that results in payoff $w_t(Q; v^{t-1}) = (r_1, r_2)$ will be used in periods $t+1, t+2, ..., T$. In Figure 9, the continuation values without a star are fixed due to Lemma 3, and the values with a star are just an example of possible continuation values.

Ternary stage mechanisms are voting mechanisms where each agent can cast one of three votes: veto, approve, or recommend. Every agent can veto the candidate and the principal needs at least one person recommending the candidate to hire the candidate. In particular, this voting rule does not hire the candidate if both agents approve and neither recommends. Each agent has three intervals separated by $a_i$ and $r_i$, the smallest (less than $a_i$) where the agent can veto the candidate, the largest (larger than $r_i$) where the agent recommends and the candidate is hired as long as the other agent do not veto, and the one in the middle where the candidate is hired only when the other agent recommends.

# 4. TERNARY MECHANISMS

I start by definition ternary mechanism, an important class of mechanisms with a very simple interpretation.

This section contains the main result of the paper. The main result shows that any quasi-deterministic mechanism can be constructed by randomization among ternary stage mechanisms. The proof is divided into two steps. In the first subsection, I show that every Pareto optimal stage mechanism must be ternary. In the second subsection, I show that the rest of the boundary of the set of payoffs that can be generated by quasi-deterministic incentive compatible mechanisms, that is not Pareto optimal, consists of only randomizations among binary mechanisms. The main result is stated and discussed in the last subsection.

4.1. **Pareto Optimal Mechanisms.** In this subsection, I define Pareto optimal mechanisms as the ones that are restricted Pareto optimal within the set of quasi-deterministic incentive compatible mechanisms. I briefly explain why if a stage mechanism is not ternary after some history, then it is Pareto dominated by one that is ternary after the same history.

**Definition 6.** For incentive compatible mechanisms $Q$ and $\tilde{Q}$, the mechanism $Q$ *Pareto dominates* $\tilde{Q}$ after history $v^{t-1}$ if

$$w_{i,t}\left(Q; v^{t-1}\right) \geqslant w_{i,t}\left(\tilde{Q}; v^{t-1}\right) \text{ for each } i \in \{1, 2\},$$

with strict inequality for at least one agent.

An incentive compatible mechanism, $Q$, is *Pareto optimal after history* $v^{t-1}$, if it is not Pareto dominated by any other incentive compatible mechanism, $\tilde{Q}$, after the same history $v^{t-1}$.

**Lemma 4.** *If a stage mechanism is Pareto optimal after some history, then it is payoff-equivalent to a randomization among ternary stage mechanisms after the same history.*

Define ternary mechanisms as having constant continuation values in the veto region?

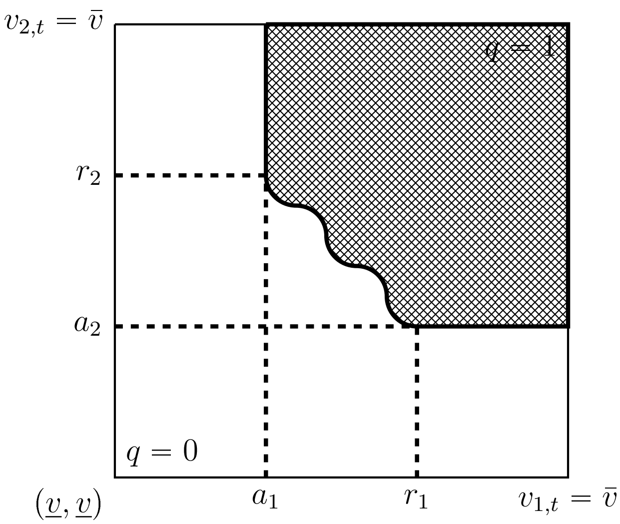


FIGURE 10. An arbitrary stage mechanism

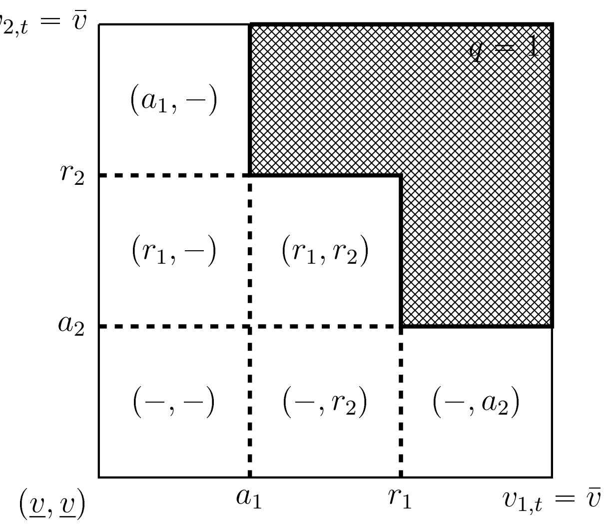


FIGURE 11. Pareto dominating ternary mechanism

For an arbitrary incentive compatible stage mechanism like the one in Figure 10, I define $a_i$ and $r_i$ the same way as in Equation 2 and Equation 4. I compare the original stage mechanism in Figure 10 with another one where the candidate is never hired in the region $[a_1, r_1] \times [a_2, r_2]$ in Figure 11. That other stage mechanism is ternary. In the ternary mechanism when $v_{2,t}$ is between $a_2$ and $r_2$, agent 1 gets constant continuation value $a_1$ if the candidate is not hired, whereas in the original mechanism, agent 1 gets continuation values that are smaller or equal to $a_1$ whenever she has a value less than $a_1$. When $v_{2,t}$ is smaller than $a_2$ or larger than $r_2$, the ternary mechanism yields the same expected payoff for agent 1 by construction.

Continuation mechanism is not Pareto optimal. Define Pareto optimal after all histories and Pareto optimal in period 1? Also need to define continuation mechanism somewhere earlier.

4.2. **Non-Pareto Optimal Mechanisms.** The boundary of the set of continuation values that correspond to some incentive compatible mechanism consists of Pareto optimal mechanisms and the ones that are the worst for one agent fixing the payoff of the other agent. In this subsection, I explain why the part of the boundary that is not Pareto optimal is made up of ternary mechanisms and as a result conclude that every incentive compatible mechanism is payoff-equivalent to some randomization among ternary mechanisms.

For an incentive compatible mechanism, $Q$, it is *on the non-Pareto optimal boundary* after history $v^{t-1}$, if there does not exist another incentive compatiable mechanism, $\tilde{Q}$ with the property that

$$\begin{aligned} w_{i,t}\left(\tilde{Q}; v^{t-1}\right) &= w_{i,t}\left(Q; v^{t-1}\right) \text{ for some } i \text{ and,} \\ w_{-i,t}\left(\tilde{Q}; v^{t-1}\right) &< w_{-i,t}\left(Q; v^{t-1}\right). \end{aligned}$$

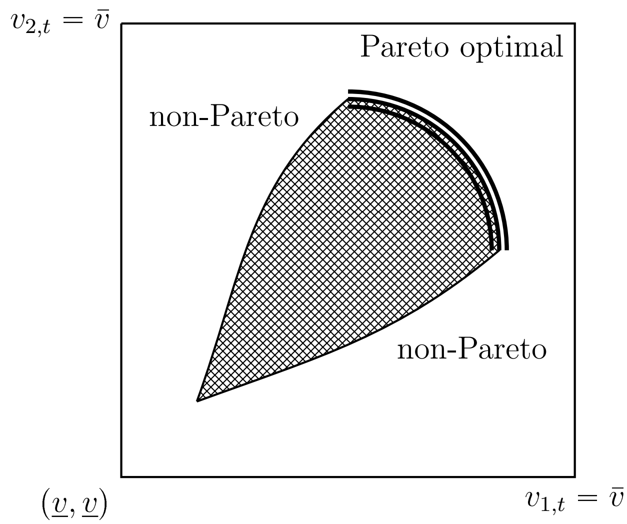


FIGURE 12. Boundary of a set of continuation values that corresponds to some incentive compatible mechanism

The shaded region in Figure 12 depicts a set of continuation values that are achievable by some incentive compatible mechanisms. The Pareto optimal boundary is highlighted and the remaining boundary is the non-Pareto optimal boundary.

**Lemma 5.** *If a stage mechanism is on the non-Pareto optimal boundary after some history, then it is payoff-equivalent to a randomization among binary stage mechanisms after the same history.*

Much better explainations is needed.

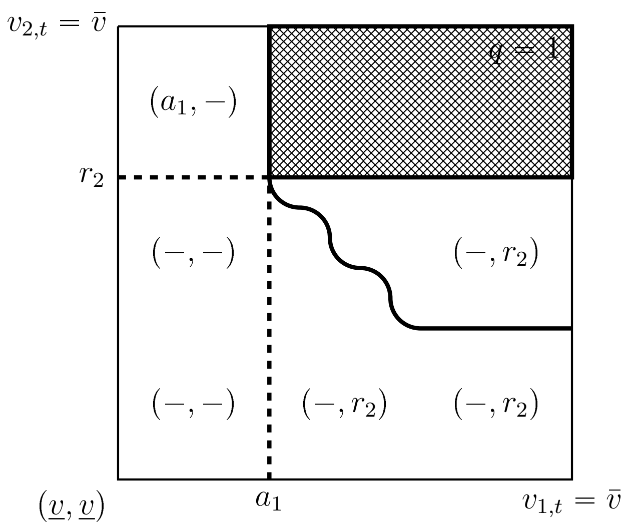


FIGURE 13. Binary mechanism that is worse for agent 1 but better for agent 2

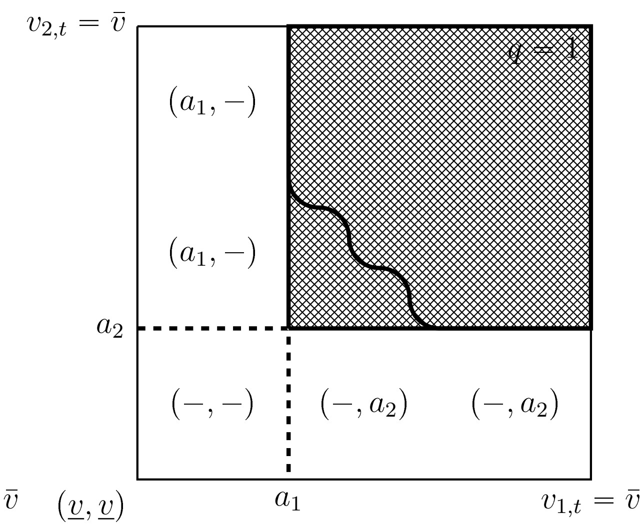


FIGURE 14. Binary mechanism that is worse for both agent 1 and agent 2

For an arbitrary stage mechanism like the one in Figure 10, I define $r_i$ and $a_i$ the same way as in Lemma 4. I show that a binary mechanism like the one in Figure 13, with carefully constructed continuation value pairs in each rectangular region, results in a larger payoff for one agent but smaller for the other agent, and a ternary mechanism like the one in Figure 14 results in a smaller payoff for both agents.

**Theorem 1.** *For every quasi-deterministic incentive compatible mechanism, $Q$, there is a mechanism $\tilde{Q}$ that satisfies,*

(1) *Every stage mechanism of $\tilde{Q}$ is ternary,*
(2) *$Q$ is payoff-equivalent to $\tilde{Q}$.*

The result follows directly from Lemma 4 and Lemma 5 since the best and worst payoff for an agent are randomizations between ternary stage mechanisms and it applies to every agent and the stage mechanism after every history. Theorem 1 implies that a principal can use only ternary mechanisms to obtain the every possible payoff for the agents.

# 5. EXAMPLES

In this section, I discuss an example of a mechanism that is Pareto optimal after every history and an example where the set of continuation values that correspond to some incentive compatible mechanism are triangles.

Not sure of the purpose of this section yet.

**Example 1.** Recall the dictator stage mechanism of agent $i$ is given by,

$$q(v_t) = \begin{cases} 0 & \text{if } v_{i,t} < r_i \\ 1 & \text{if } v_{i,t} > r_i \end{cases}.$$

This stage mechanism yields the maximal continuation value for agent $i$ at the beginning of period $T$ when $r_i = v_i^\star$. The resulting continuation value for player $i$ is $\mathbb{E}[\max\{v_{i,T}, v_i^\star\}]$. Similarly, if the dictator mechanism is used in period $T-1$ with $r_i = \mathbb{E}[\max\{v_{i,T}, v_i^\star\}]$, and after all histories where the candidate is not hired in $T-1$, the dictator mechanism is used in period $T$, then the continuation value for agent $i$ at the beginning of period $T-1$ is,

$$\mathbb{E}[\max\{v_{i,T-1}, \mathbb{E}[\max\{v_{i,T}, v_i^\star\}]\}].$$

I define these maximal continuation values backward inductively for an agent $i$, as

$$v_{i,T+1}^{max} = v_i^\star,$$

$$v_{i,t}^{max} = \mathbb{E}\left[\max\left\{v_{i,t}, v_{i,t+1}^{max}\right\}\right] \text{ for } t \in \{1, 2, ..., T\}.$$

After every history $v^{t-1} \in \mathcal{V}^{t-1}$, the dictator stage mechanism with $r_i = \mathbb{E}\left[\max\left\{v_{i,t}, v_{i,t+1}^{max}\right\}\right]$ is used. Then, the continuation value for agent $i$ in period $t$ is $v_{i,t.}^{max}$

For any problem, this grand mechanism is Pareto optimal after all histories because the continuation value for the dictator cannot be improved.

I define the set of continuation values in period $t$ that can be generated by some incentive compatible mechanism as the *feasible continuation set in period* $t$, or,

$$\Phi_t = \left\{w_t\left(Q; v^{t-1}\right) : Q \text{ is incentive compatible }, v^{t-1} \in \mathcal{V}^{t-1}\right\}.$$

Regular feasible region is not that useful.

**Definition 7.** The feasible continuation set in period $t, \Phi_t$, is regular if,

(1) $\Phi_t \subseteq [0, \bar{v}] \times [0, \bar{v}]$,

(2) If $v_t \in \Phi_t$, and $\tilde{v}_{i,t} \leqslant v_{i,t}$ for every $i$, then $\tilde{v}_t \in \Phi_t$.

0 is special in the definition because of the normalization $\mathbb{E}\left[v_{i,t}\right] = 0$. Regularity requires that only continuation values above the mean, 0, correspond to some incentive compatible mechanism, and regularity also requires that the continuation set can be characterized by a Pareto frontier.

**Lemma 6.** *The continuation set is regular in each period if and only if* $v^\star = 0$.

**Lemma 7.** *The feasible continuation set* $\Phi_t$ *is a triangle with vertices* $(0,0)$, $\left(v_{1,t}^{max}, 0\right)$, $\left(0, v_{2,t}^{max}\right)$ *in every period if,*

$$v^\star = (0,0) \text{ and },$$
$$\frac{\partial f_t}{\partial v_1} \leqslant 0 \text{ and } \frac{\partial f_t}{\partial v_2} \leqslant 0 \text{ for every } t \in \{1, 2, ..., T\}.$$

**Example 2.** If the values of hiring the candidates are independently uniformly distributed over [-1, 1] in every period, then by Lemma 7, the feasible continuation set is a triangle with $(0,0)$, $\left(0, \frac{1}{4}\right)$, $\left(\frac{1}{4}, 0\right)$ in the last period.

To illustrate Lemma 7, it is possible to algebraically solve for all possible continuation value pairs on the Pareto frontiers. For each $(r_1, r_2)$, all potential Pareto optimal mechanisms can be found by changing $(a_1, a_2)$ in a way such that the resulting allocation rule is not Pareto

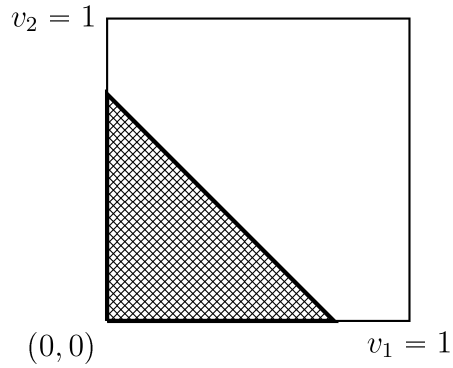


FIGURE 15. Set of continuation values generated by incentive compatible mechanisms

dominated, namely the $(a_1, a_2)$ pairs that satisfy the following relationship,

$$\frac{\frac{\partial w_{1,t}}{\partial a_1}}{\frac{\partial w_{1,t}}{\partial a_2}} = \frac{\frac{\partial w_{2,t}}{\partial a_1}}{\frac{\partial w_{2,t}}{\partial a_2}},$$

which comes from the maximization problem,

$$\max_{a_1,a_2} w_{1,t}\left(Q; v^{t-1}\right) \text{ such that } w_{2,t}\left(Q; v^{t-1}\right) = u.$$

It implies for this uniform[-1, 1] example,

$$a_1^{\star}\left(a_2\right) = \frac{1}{4}\frac{r_1 r_2 + r_1 + r_2 - 4a_2 - 3}{a_2 + 1}.$$

Substituting this into the continuation value functions, I get,

$$\frac{dw_{1,t}}{dw_{2,t}} < 0,$$

$$\frac{d^2 w_{1,t}}{d\left(w_{2,t}\right)^2} > 0,$$

so the Pareto frontier for each $(r_1, r_2)$ pair is a decreasing convex function, meaning the resulting Pareto frontier, which is the maximum over every $(r_1, r_2)$ in the feasible continuation set, will also be decreasing and convex.

Therefore, with randomization, the continuation set for the previous period will be a triangle. This observation holds for each period.

## 6. MORE THAN TWO AGENTS

This section needs to be rewritten. DO NOT read.

For general $N > 2$, a binary stage mechanism is a voting rule where each agent is given one of two possible votes: recommend and not recommend. The candidate is hired whenever a subset of the agents vote recommend. In particular, unanimity is a binary mechanism where the candidate is hired only when every agent recommends, and dictatorship is the one where the candidate is hired as long as one agent, called the dictator, recommends the hiring. The stage mechanism where the candidate is always hired is the constant at 1 mechanism and the one where the candidate is never hired is the constant at 0 mechanism.

These binary stage mechanisms are the only incentive compatible ones in the static case.

For general $N > 2$, ternary stage mechanisms are voting mechanisms where every agent can cast one of three votes: veto, approve, or recommend. Every agent can veto the candidate and if the principal needs at least one person recommending the candidate to hire the candidate. In particular, this voting rule does not hire the candidate if all agents approve and no one recommends. Each agent has three intervals separated by $a_i$ and $r_i$, the smallest (less than $r_i$) where the agent can veto the candidate, the largest (larger than $a_i$) where the agent recommends and the candidate is hired as long as every other agent does not veto, and the one in the middle where the candidate is hired only when no other agent vetoes and some other agent recommends.

All the lemmas hold for general $N > 2$, but they do not impley Theorem 1 since the non-Pareto boundary in does not cover the whole non-Pareto boundary. The proofs for the lemmas in the first two sections do not change for $N > 2$ and but Lemma 4 and Lemma 5 need significant changes.

## 7. PROOFS (for the Lemmas in Section 3)

This section contains the proof of Lemma 2 and Lemma 3. I use the results to prove Lemma 1.

*Proof of Lemma 2:* Let $v^t$ and $\tilde{v}^t$ be two histories that are the same except for the value for agent $i$ in period $t, v_{i,t}$ and $\tilde{v}_{i,t}$, respectively, such that $v_{i,t} > \tilde{v}_{i,t.}$

Ex post implementability implies, for a stage mechanism $q(\cdot) = q_{\cdot|v^{t-1}}$ of $Q$,

$$q(v_t)\, v_{i,t} + (1 - q(v_t))\, w_{i,t}\left(Q; v^t\right) \geqslant q(\tilde{v}_t)\, v_{i,t} + (1 - q(\tilde{v}_t))\, w_{i,t}\left(Q; \tilde{v}^t\right) \text{ and },$$
$$q(v_t)\, \tilde{v}_{i,t} + (1 - q(v_t))\, w_{i,t}\left(Q; v^t\right) \leqslant q(\tilde{v}_t)\, \tilde{v}_{i,t} + (1 - q(\tilde{v}_t))\, w_{i,t}\left(Q; \tilde{v}^t\right).$$

Taking the difference between the above inequalities gives the result.

$$(q(v_t) - q(\tilde{v}_t)) \cdot (v_{i,t} - \tilde{v}_{i,t}) \geqslant 0$$
$$q(v_t) - q(\tilde{v}_t) \geqslant 0$$

Since $v^t$ and $\tilde{v}^t$ only differ in the component $v_{i,t} > \tilde{v}_{i,t}, q$ is monotonic in the component $v_{i,t}$ for every $i$ and every $t$. □

*Proof of Lemma 3:* Fix a grand mechanism $Q$ and its stage mechanism after the history $v^{t-1}, q(\cdot) = q(\cdot|v^{t-1})$, recall the definition of the threshold function in Equation 1, $R_i(v_{-i,t}) = \inf\{v_{i,t} : q(v_t) = 1\}$, with the convention that $\inf \varnothing = \infty$.

I first prove that the conditions (monotonicity on $q$ and threshold condition on $w_{t+1}$) are necessary. Assume $Q$ is a quasi-deterministic incentive compatible mechanism.

By Lemma 2, since $q$ is monotonic in $v_{i,t}$, and,

$$q(v_{i,t}, v_{-i,t}) = \begin{cases} 0 & \text{if } v_{i,t} < R_i(v_{-i,t}) \\ 1 & \text{if } v_{i,t} > R_i(v_{-i,t}) \end{cases}.$$

The continuation value for agent $i$ for fixed $v_{-i,t}, w_{i,t+1}(Q; v^{t-1}, (v_{i,t}, v_{-i,t}))$, must be constant for all values $v_{i,t}$ in the rejection region $\{v_t : q(v_t) = 0\}$. If not, the agent could always find it optimal to report the valuation with the largest possible continuation value. From now on, fix $v_{-i,t}$ and let $w_{i,t+1}(Q; v^{t-1}, (v_{i,t}, v_{-i,t})) = c$ be the constant continuation value.

For an agent observing $v_{i,t} < R(v_{-i,t})$, incentive compatibility implies reporting $v_{i,t}$ is preferred to reporting another $\hat{v}_{i,t} > R_i(v_{-i,t})$ to get the allocation $q = 1$, meaning,

$$c \geqslant v_{i,t}.$$

This is true for every $v_{i,t} < R(v_{-i,t})$,

$$c \geqslant R_i(v_{-i,t}).$$

For an agent observing $v_{i,t} > R_i(v_{-i,t})$, incentive compatibility implies reporting $v_{i,t}$ is preferred to reporting another $\hat{v}_{i,t} < R_i(v_{-i,t})$ to get the allocation $q = 0$, meaning,

$$c \leqslant v_{i,t}.$$

This is true for every $v_{i,t} > R(v_{-i,t})$,

$$c \leqslant R_i(v_{-i,t}).$$

Therefore,

$$c = R_i(v_{-i,t}),$$

and,

$$w_{i,t+1}\left(Q; v^{t-1}, (v_{i,t}, v_{-i,t})\right) = R_i(v_{-i,t}) \text{ for each } v_{i,t} \text{ such that } q(v_{i,t}, v_{-i,t}) = 0.$$

Now I prove that the monotonicity and threhold conditions are sufficient. Fix $v_{-i,t}$ and assume $q$ is a stage mechanism that satisfy these conditions.

For an agent $i$ with $v_{i,t} < R\left(v_{-i,t}\right)$, reporting $\hat{v}_{i,t}$ results in payoff,

$$w_{i,t}\left(Q;v^{t-1}\right)=\begin{cases} R_i\left(v_{-i,t}\right) & \text{if } \hat{v}_{i,t}=v_{i,t} \\ R_i\left(v_{-i,t}\right) & \text{if } \hat{v}_{i,t}\neq v_{i,t}, \hat{v}_{i,t}\leqslant R\left(v_{-i,t}\right) \\ v_{i,t} & \text{if } \hat{v}_{i,t}> R\left(v_{-i,t}\right) \end{cases}.$$

Since $v_{i,t} < R_i\left(v_{-i,t}\right)$, it is optimal to report truthfully.

For an agent $i$ with $v_{i,t} > R\left(v_{-i,t}\right)$, reporting $\hat{v}_{i,t}$ results in payoff,

$$w_{i,t}\left(Q;v^{t-1}\right)=\begin{cases} v_{i,t} & \text{if } \hat{v}_{i,t}=v_{i,t} \\ v_{i,t} & \text{if } \hat{v}_{i,t}\neq v_{i,t}, \hat{v}_{i,t}> R\left(v_{-i,t}\right) \\ R_i\left(v_{-i,t}\right) & \text{if } \hat{v}_{i,t}\leqslant R\left(v_{-i,t}\right) \end{cases}.$$

Since $v_{i,t} > R_i\left(v_{-i,t}\right)$, it is optimal to report truthfully.

Therefore, truthful reports are optimal, $q$ is incentive compatible. □

*Proof of Lemma 1:* Fix a grand mechanism, $Q$. Since for any history $v^T \in \mathcal{V}^{T,}$

$$w_{i,T+1}\left(Q;v^T\right)=v_i^\star,$$

by Lemma 3, for any $v_{-i,T}$, either $q\left(v_{i,T},v_{-i,T}|v^{T-1}\right)$ is constant in $v_{i,T}$ or,

$$R_i\left(v_{-i,T}\right)=v_i^\star.$$

By Definition 3, these stage mechanism are binary with outside option $v_i^\star$ for agent $i$. □

## 8. PROOFS (for the Lemmas and Propositions in Section 4)

In this section, I start by defining some shorthand notations for the proofs in this section and some preliminary observations that simplify the shapes and continuation values of an incentive compatible mechanism. Then, I prove Lemma 4 and Lemma 5. Theorem 1 follows directly from Lemma 4 and Lemma 5.

From now on, I am going to fix the grand quasi-deterministic incentive compatible mechanism $Q$, period $t$ and history $v^{t-1}$. In order to simplify the subsequent notation, I write,

$$q\left(v_t\right) = q\left(v_t | v^{t-1}\right).$$

I also write,

$$w_{t+1}\left(v_t\right) = w_{t+1}\left(Q; v^{t-1}, v_t\right),$$

and,

$$w_t = w_t\left(Q; v^{t-1}\right).$$

I define the following constants and sets, I assume the acceptance region $\{v_t : q\left(v_t\right) = 1\}$ is closed for these definitions.

(1) Recall from Equation 2 that the *approval threshold* , $a_i$, is the threshold below which the agent $i$ has veto power: if $v_{i,t} < a_i$, the candidate is vetoed by $i$ and will never be hired for any value of $v_{-i,t}$,

$$a_i = \sup_{v_{i,t}} \left\{v_{i,t} : q\left(v_{i,t}, v_{-i,t}\right) = 0 \text{ for all } v_{-i,t} \in \mathcal{V}_{-i}\right\}.$$

(2) The *non-threshold region* for agent $i$, is the set of $v_{-i,t}$ such that the candidate is never hired for any value of $v_{i,t}$,

$$\left\{v_{-i,t} : q\left(\tilde{v}_{i,t}, v_{-i,t}\right) = 0 \text{ for all } \tilde{v}_{i,t} \in \mathcal{V}_i\right\} = \left[\underline{v}, a_{-i}\right).$$

The *average non-threshold continuation value* , $c_i$, is the expected continuation value within the non-threshold region for agent $i$,

$$c_i = \mathbb{E}\left[w_{i,t+1}\left(v_{i,t}, v_{-i,t}\right) | v_{-i,t} < a_i\right].$$

(3) The definition for the *recommendation threshold* , $r_i$, is simplified from Equation 4 due to the assumption that the acceptance region is closed,

$$r_i = R_i\left(a_{-i}\right).$$

One useful relation due to the definition of these constants is,

$$a_i \leqslant R_{v_{-i,t}} \leqslant r_i \text{ for all } v_{-i,t} \geqslant a_{-i}. \tag{5}$$

Then, I state an corollary to Lemma 3 that makes computation easier in all the following proofs.

**Corollary 2.** *Fix a history $v^{t-1}$ and a stage mechanism of a quasi-deterministic incentive compatible $Q$ after this history, $q\left(\cdot\right) = q\left(\cdot | v^{t-1}\right)$,*

$$\mathbb{E}\left[\max\left\{v_{i,t}, R_i\left(v_{-i,t}\right)\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} + c_i \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\}.$$

*Proof of Corollary 2:* It follows directly from Lemma 3. □

In the course of this section, I will construct alternative stage mechanisms with new continuation values. I say *a continuation value is incentive compatible* if they can be implemented in an incentive compatible quasi-deterministic grand mehcanisms.

As an example, I show the following Compactness Lemma that ensures that the acceptance region is closed.

**Lemma 8.** *There exists an incentive compatible stage mechanism $\tilde{q}$, with incentive compatible continuation values, that is payoff equivalent to $q$, such that,*

$$\left\{v_t : \tilde{q}\left(v_t\right) = 1\right\} \textit{ is closed}.$$

Recall that payoff equivalence in this section means $\tilde{w}_t = w_t$ where $\tilde{w}_t$ is the continuation value of mechanism that has $\tilde{q}$ after history $v^{t-1}$ in place of $q$.

Need to write this proof up. Prove continuity.

*Proof.* The only issue is how to define the continuation payoffs for player $i$ when the other player says $a_i$. But use the continuity and just take the limit of continuation values when $v_{-i} > a_{i.}$ □

Also, I show the followng Flattening Lemma that ensures that continuation values in the non-threshold region is constant.

**Lemma 9.** *There exists an incentive compatible stage mechanism $\tilde{q}$, with incentive compatible continuation values, that is payoff equivalent to $q$, such that,*

$$\tilde{w}_{t+1}(v_{i,t}, v_{-i,t}) = c_i \textit{ for all } v_{-i,t} < a_{-i}.$$

*Proof of Lemma 9:* Let $\tilde{q}$ be the stage mechanism from replacing the continuation value of agent $i$ in $q$ in the non-threshold region $v_{-i,t} < a_{-i}$ by $c_{i.}$

I divide the proof into three main parts.

(1) $\tilde{q}$ is incentive compatible.

(2) $\tilde{q}$ payoff equivalent to $q$.

*Part (1)* There are two types of new continuation values that is different from the ones used in $q$.

(1) $(c_i, R_{-i}(v_{i,t}))$ from the region $v_{i,t} \geqslant a_i$ and $v_{-i,t} < a_{-i}$,

(2) $(c_i, c_{-i})$ from the region $v_{i,t} < a_i$ and $v_{-i,t} < a_{-i}$.

Fix $\tilde{v}_{i,t} \geqslant a_i$, $R_{-i}(\tilde{v}_{i,t})$ is finite due to the definition of $a_{i.}$

$$\begin{aligned}
(c_i, R_{-i}(\tilde{v}_{i,t})) &= \left(\mathbb{E}\left[w_{i,t+1}(v_{i,t}, v_{-i,t})\,|v_{-i,t} < a_{-i}\right], R_{-i}(\tilde{v}_{i,t})\right) \\
&= \left(\mathbb{E}\left[\mathbb{E}\left[w_{i,t+1}(v_{i,t}, v_{-i,t})\,|v_{-i,t} < a_{-i}\right], w_{-i,t+1}(\tilde{v}_{i,t}, v_{-i,t})\right)\right. \\
&= \mathbb{E}\left[w_{t+1}(\tilde{v}_{i,t}, v_{-i,t})\,|v_{-i,t} < a_{-i}\right].
\end{aligned}$$

The second equality is due to Lemma 3 which states that $w_{i,t+1}(v_{i,t}, v_{-i,t})$ is constant in $v_{i,t}$ in the non-threshold region $[\underline{v}, a_{-i})$.

Similarly,

$$\begin{aligned}(c_i, c_{-i}) &= \left(\mathbb{E}\left[w_{i,t+1}\left(v_{i,t}, v_{-i,t}\right)|v_{-i,t} < a_{-i}\right], \mathbb{E}\left[w_{-i,t+1}\left(v_{i,t}, v_{-i,t}\right)|v_{i,t} < a_{-i}\right]\right) \\ &= \mathbb{E}\left[w_{t+1}\left(v_{i,t}, v_{-i,t}\right)|v_{i,t} < a_{-i}, v_{-i,t} < a_{-i}\right].\end{aligned}$$

The second equality is due to Lemma 3 which states that $w_{i,t+1}\left(v_{i,t}, v_{-i,t}\right)$ is constant in $v_{i,t}$ in the non-threshold region $[\underline{v}, a_{-i})$.

Therefore, $\left(c_i, R_{-i}\left(v_{i,t}\right)\right)$ and $(c_i, c_{-i})$ are the expected value other continuation value pairs so they are incentive compatible too. Since $\tilde{q}$ is the same as $q$, so incentive compatible of $\tilde{q}$ follows from the incentive compatibility of $q$.

*Part (2)* Note that,

$$\begin{aligned}\tilde{c}_i &= \mathbb{E}\left[\tilde{w}_{i,t+1}\left(v_{i,t}, v_{-i,t}\right)|v_{-i,t} < a_{-i}\right] \\ &= \mathbb{E}\left[c_i | v_{-i,t} < a_{-i}\right] \\ &= c_i.\end{aligned}$$

I check $\tilde{q}$ is payoff equivalent to $q$ using Corollary 2,

$$\begin{aligned}\tilde{w}_{i,t} &= \mathbb{E}\left[\max\left\{v_{i,t}, \tilde{R}_i\left(v_{-i,t}\right)\right\}|v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} + \tilde{c}_i \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\} \\ &= \mathbb{E}\left[\max\left\{v_{i,t}, R_i\left(v_{-i,t}\right)\right\}|v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} v_{-i,t} \geqslant a_{-i}\right\} + c_i \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\} \\ &= w_{i,t.}\end{aligned}$$

□

**Lemma 10.** *Ternary stage mechanisms with continuation values described in Figure 17 are incentive compatible as long as the continuation values are incentive compatible.*

*Proof.* It follows directly from Lemma 3. □

In the following proofs, by Lemma 8 and Lemma 9, without loss of generality, assume the acceptance region of $q$ is closed and continuation value for each agent $i$ in its non-threshold

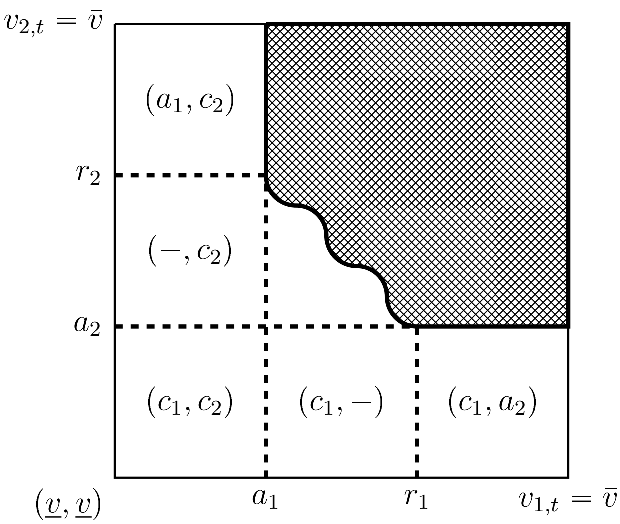


FIGURE 16. Mechanism $q$

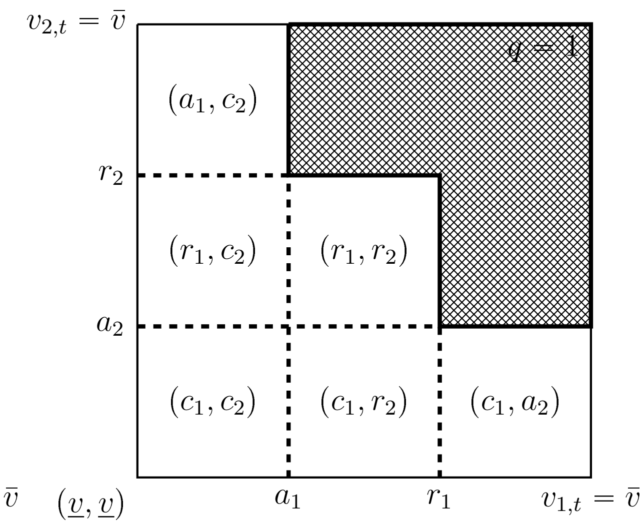


FIGURE 17. Mechanism $\tilde{q}$

region $v_{-i,t} \in [\underline{v}, a_{-i})$ is constant at $c_i$. Let the continuation values and the shape of the acceptance region of $q$ be ones as in Figure 16.

*Proof of Lemma 4:* Consider a ternary stage mechanism, $\tilde{q}$, with the thresholds $a_i, r_i$ with continuation values specified in Figure 17.

I divide the proof into three main parts.

(1) $\tilde{q}$ is incentive compatible.

(2) $\tilde{q}$ Pareto dominates $q$.

*Part (1)* I only need to show that the continuation values are incentive compatible, the rest follows from Lemma 10. There are three types of new continuation values that is different from the ones used in $q$.

(1) $(r_i, r_{-i})$ in the region $a_j \leqslant v_{j,t} < r_j$ for both $j \in \{1, 2\}$,

(2) $(r_i, c_{-i})$ in the region $v_{i,t} < a_i$ and $a_{-i} \leqslant v_{-i,t} < r_{-i}$.

$$
\begin{aligned}
(r_i, r_{-i}) &= \left(R_i\left(a_{-i}\right), R_{-i}\left(a_i\right)\right) \\
&= w_{t+1}\left(a_i, a_{-i}\right),
\end{aligned}
$$

and fix $a_i \leqslant v_{i,t} < r_i$,

$$\begin{aligned}(r_i, c_{-i}) &= (R_i(a_{-i}), c_{-i})\\ &= w_{t+1}(v_{i,t}, a_{-i}),\end{aligned}$$

are both incentive compatible continuation values due to implementability of $q$.

*Part (2)* I use Corollary 2 to compare $\tilde{q}$ and $q$.

$$\begin{aligned}&\tilde{w}_{i,t} - w_{i,t}\\ &= \mathbb{E}\left[\max\left\{v_{i,t}, \tilde{R}_{i,t}(v_{-i,t})\right\} - \max\left\{v_{i,t}, R_{i,t}(v_{-i,t})\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\}\\ &\quad + (\tilde{c}_i - c_i) \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\}\\ &= \mathbb{E}\left[\max\left\{v_{i,t}, r_i\right\} - \max\left\{v_{i,t}, R_{i,t}(v_{-i,t})\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\}\\ &\geqslant \mathbb{E}\left[\max\left\{v_{i,t}, r_i\right\} - \max\left\{v_{i,t}, r_i\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\}\\ &= 0.\end{aligned}$$

The inequality is due to the observation Equation 5.

Therefore, $q$ is Pareto dominated by $\tilde{q}$. □

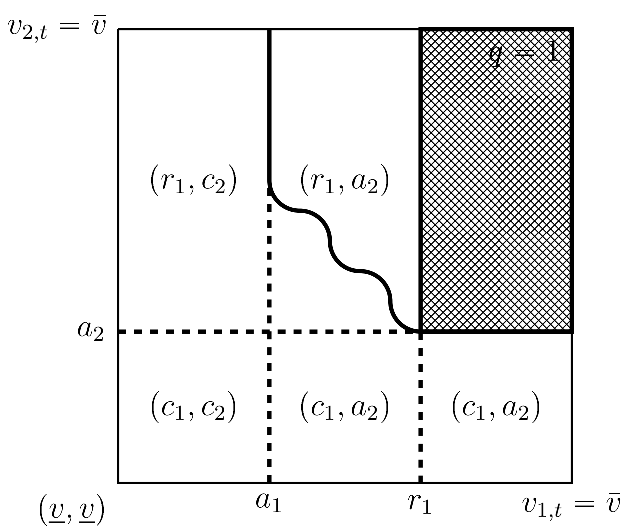


FIGURE 18. Mechanism $q^{max}$

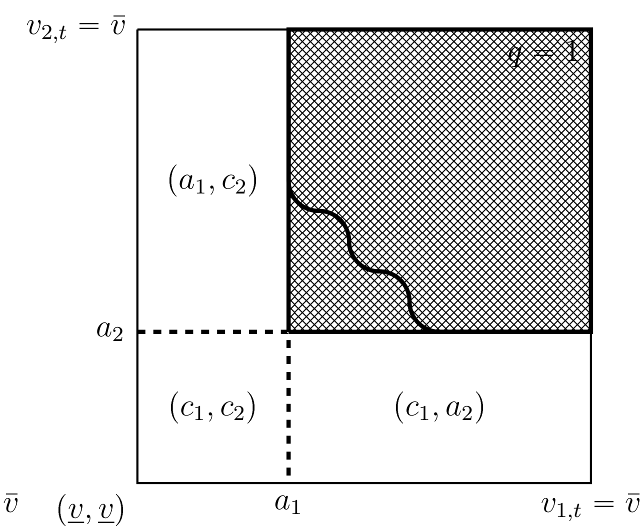


FIGURE 19. Mechanism $q^{min}$

*Proof of Lemma 5:* Consider another two stage mechanism $q^{min}$ and $q^{max}$ in place of $q$ after history $v^{t-1}$, where $q^{max}$ is the binary mechanism with thresholds $(a_i, r_{-i})$ and $q^{min}$ is the binary mechanism with thresholds $(a_i, a_{-i})$, and with continuation values specified in Figure 18 and Figure 19. I add superscript *min* and *max* to denote the modified recommendation thresholds,

$$r_i^{min} = a_i,$$

$$r_{-i}^{min} = r_{-i},$$

and,

$$r_i^{max} = a_i,$$

$$r_{-i}^{max} = a_{-i}.$$

with continuation values specified in Figure 17.

I divide the proof into two parts.

(1) $q^{min}$ and $q^{max}$ are incentive compatible.

(2) A randomization between $Q^{min}$ and $Q^{max}$ results in the same continuation value for $i$ and is a smaller continuation value for $-i$.

*Part (1)* The continuation values are incentive compatible for similar reasons as the continuation values of $\tilde{q}$ are incentive compatible in the proof of Lemma 4. The rest follows from Lemma 10.

*Part (2)* There are three things to show.

(1) $q^{min}$ is worse than $q$ for every agent,

(2) $q^{max}$ is better than $q$ for $i$,

(3) $q^{max}$ is worse then $q$ for $-i$.

I use Corollary 2 for all three comparisons.

*Comparison (1)* : For every agent $j \in \{1, 2\}$,

$$
\begin{aligned}
& w_{j,t} - w_{j,t}^{min} \\
& = \mathbb{E}\left[\max\left\{v_{j,t}, R_{j,t}\left(v_{-j,t}\right)\right\} - \max\left\{v_{j,t}, R_{j,t}^{min}\left(v_{-j,t}\right)\right\} | v_{-j,t} \geqslant a_{-j}\right] \cdot \mathbb{P}\left\{v_{-j,t} \geqslant a_{-j}\right\} \\
& \quad + \left(c_j^{min} - c_j\right) \cdot \mathbb{P}\left\{v_{-j,t} < a_{-j}\right\} \\
& = \mathbb{E}\left[\max\left\{v_{j,t}, R_{j,t}\left(v_{-j,t}\right)\right\} - \max\left\{v_{j,t}, a_j\right\} | v_{-j,t} \geqslant a_{-j}\right] \cdot \mathbb{P}\left\{v_{-j,t} \geqslant a_{-j}\right\} \\
& \geqslant \mathbb{E}\left[\max\left\{v_{j,t}, a_j\right\} - \max\left\{v_{j,t}, a_j\right\} v_{-j,t} \geqslant a_{-j}\right] \cdot \mathbb{P}\left\{v_{-j,t} \geqslant a_{-j}\right\} \\
& = 0.
\end{aligned}
$$

The inequality is due to the observation Equation 5.

*Comparison (2)* : For agent $i$,

$$
\begin{aligned}
& w_{i,t}^{max} - w_{i,t} \\
& = \mathbb{E}\left[\max\left\{v_{i,t}, R_{i,t}^{max}\left(v_{-i,t}\right)\right\} - \max\left\{v_{i,t}, R_{i,t}\left(v_{-i,t}\right)\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} \\
& \quad + \left(c_i^{max} - c_i\right) \cdot \mathbb{P}\left\{v_{-i,t} < a_{-i}\right\} \\
& = \mathbb{E}\left[\max\left\{v_{i,t}, r_i\right\} - \max\left\{v_{i,t}, R_{i,t}\left(v_{-i,t}\right)\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} \\
& \geqslant \mathbb{E}\left[\max\left\{v_{i,t}, r_i\right\} - \max\left\{v_{i,t}, r_i\right\} | v_{-i,t} \geqslant a_{-i}\right] \cdot \mathbb{P}\left\{v_{-i,t} \geqslant a_{-i}\right\} \\
& = 0.
\end{aligned}
$$

The inequality is due to the observation Equation 5.

*Comparison (3)* : For agent $-i$,

$$
\begin{aligned}
& w_{-i,t} - w^{max_{-i,t}} \\
& = \mathbb{E}\left[\max\left\{v_{-i,t}, R_{-i,t}\left(v_{i,t}\right)\right\} - \max\left\{v_{-i,t}, R^{max_{-i,t}}\left(v_{i,t}\right)\right\} | v_{i,t} \geqslant a_i\right] \cdot \mathbb{P}\left\{v_{i,t} \geqslant a_i\right\} \\
& \quad + \left(c_j^{max} - c_j\right) \cdot \mathbb{P}\left\{v_{i,t} < a_i\right\} \\
& = \mathbb{E}\left[\max\left\{v_{j,t}, R_{j,t}\left(v_{-j,t}\right)\right\} - a_j | a_i \leqslant v_{i,t} < r_i\right] \cdot \mathbb{P}\left\{a_i \leqslant v_{i,t} < r_i\right\} \\
& \geqslant \mathbb{E}\left[\max\left\{v_{j,t}, a_j\right\} - a_j | a_i \leqslant v_{i,t} < r_i\right] \cdot \mathbb{P}\left\{a_i \leqslant v_{i,t} < r_i\right\} \\
& \geqslant 0.
\end{aligned}
$$

The first inequality is due to the observation Equation 5. Note that this abuses Corollary 2 to since the non-threshold regions are not the same.

Therefore, there is a randomization between $q^{min}$ and $q^{max}$ such that the continuation value for $i$ is the same and lower for agent $-i$. □

*Proof of Theorem 1:* Since binary mechanisms in Lemma 5 are ternary and the set $-i$ is a singleton, the result follows from Lemma 5 and Lemma 4. □

## 9. PROOFS (for the Lemmas in Section 5)

NEED TO FIX notations incorrect. Do NOT read.

*Proof of Lemma 6:* I first show that $v_i^\star = 0$ results in a regular allocation.

$\Phi_{T+1} = \{v^\star\}$ and regularity conditions (1) and (2) are both satisfied. I proceed by induction on $t$ to show that $\Phi_t \subseteq [0, \infty)^{N.}$

From the proof of Lemma 5, the dictator mechanisms result in the maximal continuation value for agent $i, v_{i,t}^{max}$ and 0 for all other agents, so regularity condition (2) is satisfied.

Condition (1) is satisfied because any continuation value less than 0 is not possible. For any $v_{-i,t}$, the payoff for $i$ is at least,

$$\min_{v_{i,t+1} \in \Phi_{i,t+1}} \left\{\mathbb{E}\left[v_{i,t} | v_{i,t} > v_{i,t+1}\right], v_{i,t+1}\right\} \geqslant 0,$$

as $w_{i,t+1} \geqslant 0$ by induction assumption.

Now I show that $v_i^\star \neq 0$ results in continuation sets that are not regular.

If $v_i^\star < 0$ for any $i$, then the continuation set does not satisfy condition (1) of regularity in period $T+1$.

If $v_i^\star > 0$ for every $i$, then the maximal dictator mechanism result a continuation value of every non-dictator that is strictly larger than 0, so the continuation set does not satisfy condition (2) of regularity in period $T+1$. □

*Proof of Lemma 7:* The following proof only works for $N=2$, the proof for general $N$ should be similar but with very complicated notations.

Fix some history $v^{t-1}$ and the Pareto optimal mechanism $q$, which must be ternary by , in period $t$ after this history,

$$\begin{aligned}
w_{i,t}\left(q;v^{t-1}\right) = & \int_{c_{-i}}^{\bar{v}}\int_{\underline{v}}^{r_i} r_i f\left(v_i,v_{-i}\right)dv_{-i}dv_i + \int_{r_{-i}}^{c_{-i}}\int_{\underline{v}}^{a_i} a_i f\left(v_i,v_{-i}\right)dv_{-i}dv_i + \int_{\underline{v}}^{r_{-i}}\int_{0}^{1} a_i f\left(v_i,v_{-i}\right)dv_{-i}dv_i \\
& + \int_{c_{-i}}^{\bar{v}}\int_{r_i}^{\bar{v}} v_i f\left(v_i,v_{-i}\right)dv_{-i}dv_i + \int_{r_{-i}}^{c_{-i}}\int_{a_i}^{\bar{v}} v_i f\left(v_i,v_{-i}\right)dv_{-i}dv_i.
\end{aligned}$$

I show that $w_{i,t}\left(q;v^{t-1}\right)$ is increasing $a_i, c_{-i}, r_i$ and decreasing in $r_{-i}$ by taking the partial derivatives,

$$\begin{aligned}
\frac{\partial w_{i,t}\left(q;v^{t-1}\right)}{\partial a_i} &= \int_{r_{-i}}^{c_{-i}}\int_{\underline{v}}^{a_i} f\left(v_i,v_{-i}\right)dv_i dv_{-i} \\
&\geqslant 0, \\
\frac{\partial w_{i,t}\left(q;v^{t-1}\right)}{\partial c_{-i}} &= \int_{\underline{v}}^{a_i} a_i f\left(v_i,c_{-i}\right)dv_i - \int_{\underline{v}}^{r_{-i}} r_{-i} f\left(v_i,c_{-i}\right)dv_i - \int_{r_{-i}}^{a_i} v_i f\left(v_i,c_{-i}\right)dv_i \\
&\geqslant \int_{\underline{v}}^{a_i} a_i f\left(v_i,c_{-i}\right)dv_i - \int_{\underline{v}}^{r_{-i}} c_{-i} f\left(v_i,c_{-i}\right)dv_i - \int_{r_{-i}}^{a_i} a_i f\left(v_i,c_{-i}\right)dv_i \\
&= 0,
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\partial w_{i,t}\left(q;v^{t-1}\right)}{\partial r_i} &= \int_{c_{-i}}^{\bar{v}}\int_{\underline{v}}^{r_i} f\left(v_i,v_{-i}\right)dv_i dv_{-i} \\
&\geqslant 0, \\
\frac{\partial w_{i,t}\left(q;v^{t-1}\right)}{\partial r_{-i}} &= \int_{a_i}^{\bar{v}} a_i f\left(v_i,r_{-i}\right)dv_i - \int_{a_i}^{\bar{v}} v_i f\left(v_i,r_{-i}\right)dv_i \\
&\leqslant \int_{a_i}^{\bar{v}} a_i f\left(v_i,r_{-i}\right)dv_i - \int_{a_i}^{\bar{v}} a_i f\left(v_i,r_{-i}\right)dv_i \\
&= 0.
\end{aligned}$$

I show that without randomizing, the Pareto frontier is concave.

For each fixed $(a_i,c_{-i})$, consider $(r_i,r_{-i})$ and $\left(r_i',r_{-i}'\right)$.

Let the mechanism corresponding to $\alpha\left(r_i,r_{-i}\right)+(1-\alpha)\left(r_i',r_{-i}'\right)$ be $q_\alpha$. I show that

$$w_{i,t}\left(q_\alpha;v^{t-1}\right)\leqslant \alpha w_{i,t}\left(q_1;v^{t-1}\right)+(1-\alpha)\,w_i\left(q_0;v^{t-1}\right)$$

Equality holds with $\alpha=0$ and $\alpha=1$, and

$$\begin{aligned}
V_i''\left(q_\alpha;v^{t-1}\right) &= \left(r_i-r_i'\right)^2\int_{c_{-i}}^{\bar{v}} f\left(\alpha,v_{-i}\right)dv_{-i} - \left(r_{-i}-r_{-i}'\right)^2\int_{a_i}^{\bar{v}}\left(v_i-a_i\right)\frac{\partial f}{\partial v_{-i}}\left(v_i,\alpha\right)dv_i \\
&\geqslant 0.
\end{aligned}$$

Therefore, the Pareto frontier without randomization is decreasing and concave. The continuation set will be generated by convexifying the region with the three points,

$$\left(v_1^{max},0\right),(0,0),\left(0,v_2^{max}\right),$$

due to Lemma 5. □