ECO2401 Term Project

Test of Equilibrium Belief in Coordination Games

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March 28, 2012

1 Introduction

The objective of this project is to examine a test of equilibrium beliefs in a coordination game proposed in the paper "Estimation of Dynamic Discrete Games When Players' Beliefs are Not in Equilibrium" by Victor Aguirregabiria and Arvind Magesan (2012, later abbreviated as AM). By changing the key parameters in a simple model of coordination games, Monte Carlo simulation is performed to find the power of the test of whether the players' beliefs of other players' actions are in equilibrium, and express the power as function of these parameters. The exercise is interesting for several reasons: 1) coordination games often appear in the literature of industrial organization to model interaction between firms, for example when they make decisions about store location or capital investment; therefore, the test of whether the firms' beliefs coincide with the behavior of their competitors is important; 2) the estimation of the beliefs in a coordination game depends on the whether the beliefs are in equilibrium; thus, it is useful to know the power of such test, which represents the probability of correct rejection of the hypothesis of unbiased beliefs; 3) it is a good computational exercise to implement Monte Carlo simulation of coordination games, the estimation of the parameters in the model, as well as the test of equilibrium beliefs.

2 Model

Consider a coordination game with two players 1 and 2, and two actions 0 and 1. Each player i chooses $Y_i = \{0, 1\}$ to maximize expected utility in the form of:

$$\begin{cases} u_i(Y_i = 1) = & u_0 \cdot (1 - Y_{-i}) + u_1 \cdot Y_{-i} + \theta \cdot Z_i - \varepsilon_i \\ u_i(Y_i = 0) = & 0 \end{cases},$$

where Y_{-i} is the strategy of the other player, Z_i is the common knowledge state variable that only enters the utility function of player i and ε_i is the private knowledge state variable only known to player i given their action Y_i.

Informally, in strategic form, the game could be represented by:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & (0, 0) & (0, u_0 + \theta\cdot Z_2 - \varepsilon_2) \\ 1 & (u_0 + \theta\cdot Z_1 - \varepsilon_1, 0) & (u_1 + \theta\cdot Z_1 - \varepsilon_1, u_1 + \theta\cdot Z_2 - \varepsilon_2) \end{array}$$

Assuming, as in AM that:

1) (Payoff relevant state variables) The strategy function $\sigma_i(Z, \varepsilon_i)$ and $B_i(Z, \varepsilon_i)$ are independent of any variables other than Z, ε_i.

2) (Maximization of expected payoffs) Each player i choose $Y_i = \underset{\{Y_i\}}{\operatorname{argmax}}\, u_i(Y_i)$.

3) (State space) $Z \in \mathcal{Z}$ with discrete and finite space $\mathcal{Z}$.

The equilibrium belief assumption is dropped: a player's belief about the behavior of other players do not necessarily represent the actual behavior of the other players.

Let $B_i(Y_{-i} \mid Z)$ be the belief as the probability of the opponent choosing Y_{-i} given the public information state variable Z, then the expected utility given the beliefs are computed as:

$$u_i^B(Y_i, Z) = u_i(Y_i, Y_{-i} = 0, Z)\cdot B_i(Y_{-i} = 0 \mid Z) + u_i(Y_i, Y_{-i} = 1, Z)\cdot B_i(Y_{-i} = 1 \mid Z),$$

The conditional choice probability defined as the probability of choosing Y_i given the state Z is:

$$P_i(Y_i \mid Z) = \int \mathbb{1}_{\{\sigma_i(Z, \varepsilon_i) = Y_i\}}\, dF_i(\varepsilon_i)$$

Assuming the private state variables follow logit distribution (whose distribution function is given by $F_i(\varepsilon_i) = \exp(\exp(-\varepsilon_i))$, and let $P_i(Z)$ represent for the probability $P_i(Y_i = 1 \mid Z)$. Similarly let $u_i^B(Z)$ $\equiv u_i^B(Y_i = 1, Z)$ and $u_i^B(Y_i = 0, Z) = 0$, then:

$$P_i(Z) \equiv P_i(Y_i = 1 \mid Z) = \frac{\exp\left(u_i^B(Y_i = 1, Z) - u_i^B(Y_i = 0, Z)\right)}{\exp\left(u_i^B(Y_i = 1, Z) - u_i^B(Y_i = 0, Z)\right) + 1} = \frac{\exp\left(u_i^B(Z)\right)}{\exp\left(u_i^B(Z)\right) + 1} \text{ and}$$

$$P_i(Y_i = 0 \mid Z) = 1 - P_i(Z) = \frac{1}{\exp\left(u_i^B(Y_i = 1, Z) - u_i^B(Y_i = 0, Z)\right) + 1} = \frac{1}{\exp\left(u_i^B(Z)\right) + 1}$$

For the logit model, also define $q_i(Z) = \log(P_i(Z)) - \log(1 - P_i(Z))$.

Note that if the beliefs are in equilibrium, the CCP should solve the fixed point equation $P_i(Z) = B_{-i}(Y_i = 1 \mid Z)$ for i = 1, 2. The conditional choice probability should be consistent with the belief of the other player.

3 Estimation of Parameters

Estimation of the parameters $\{u_0, u_1, \theta\}$ is not relevant to the test of unbiased beliefs, but are still performed in order to compare the results with AM and verify that the algorithm is correct. The output in the appendix is very similar to the results given in AM.

However, estimation of the beliefs are relevant to the test of equilibrium beliefs, and a consistent estimator for CCP is given by the nonparametric estimator:

$$\widehat{P}_i(Z) = \frac{\sum_{m=1}^{M} Y_{im} \cdot \mathbb{1}_{\{Z_m = Z\}}}{\sum_{m=1}^{M} \mathbb{1}_{\{Z_m = Z\}}}$$ where M is the number of data.

Consequently, $\widehat{q}_i(Z)$ could be obtained using the equation $\widehat{q}_i(Z) = \log\left(\widehat{P}_i(Z)\right) - \log\left(1 - \widehat{P}_i(Z)\right)$.

4 Test of Equilibrium Beliefs

Fix arbitrary and distinct points in the state space $s^0, s^{(a)}, s^{(b)} \in \mathcal{Z}$ and define

$$\delta_i = \frac{q_i(s^{(a)}) - q_i(s^0)}{q_i(s^{(b)}) - q_i(s^0)} - \frac{P_{-i}(s^{(a)}) - P_{-i}(s^0)}{P_{-i}(s^{(b)}) - P_{-i}(s^0)}.$$

Consider the relationship:

$$q_i(Z) = u_i^B(Z) = u_i(Y_{-i} = 0, Z) \cdot B_i(Y_{-i} = 0 \mid Z) + u_i(Y_{-i} = 1, Z) \cdot B_i(Y_{-i} = 1 \mid Z)$$

In the logit case, note that $q_i(Z) = \log\left(P_i(Z)\right) - \log\left(1 - P_i(Z)\right)$

$$= \log\left(\frac{\exp\left(u_i^B(Z)\right)}{\exp\left(u_i^B(Z)\right) + 1}\right) - \log\left(\frac{1}{\exp\left(u_i^B(Z)\right) + 1}\right) = u_i^B(Z) - 0 = u_i^B(Z).$$

Note that here, without loss of generality, assume Z enters only player 2's utility function, then for i = 1, $u_i(Z) = u_i$ is a constant.

Then, $q_i(s^{(a)}) - q_i(s^0) = u_i \cdot \left(B_i\left(Y_{-i} = 1 \mid s^{(a)}\right) - B_i\left(Y_{-i} = 1 \mid s^0\right)\right)$

Similarly, $q_i(s^{(b)}) - q_i(s^0) = u_i \cdot \left(B_i\left(Y_{-i} = 1 \mid s^{(b)}\right) - B_i\left(Y_{-i} = 1 \mid s^0\right)\right)$

Thus, $\dfrac{q_i(s^{(a)}) - q_i(s^0)}{B_i\left(Y_{-i} = 1 \mid s^{(a)}\right) - B_i\left(Y_{-i} = 1 \mid s^0\right)} = \dfrac{q_i(s^{(b)}) - q_i(s^0)}{B_i\left(Y_{-i} = 1 \mid s^{(b)}\right) - B_i\left(Y_{-i} = 1 \mid s^0\right)} = u_i.$

Also, under the null hypothesis of equilibrium beliefs: $B_i\left(Y_{-i} = 1 \mid Z\right) = P_{-i}(Z)$.

The relation is true for all states, thus:

$$\frac{P_{-i}(s^{(a)}) - P_{-i}(s^0)}{B_i\left(Y_{-i} = 1 \mid s^{(a)}\right) - B_i\left(Y_{-i} = 1 \mid s^0\right)} = \frac{P_{-i}(s^{(b)}) - P_{-i}(s^0)}{B_i\left(Y_{-i} = 1 \mid s^{(b)}\right) - B_i\left(Y_{-i} = 1 \mid s^0\right)}.$$

Combining the two equation gives $\delta_i = \dfrac{q_i(s^{(a)}) - q_i(s^0)}{q_i(s^{(b)}) - q_i(s^0)} - \dfrac{P_{-i}(s^{(a)}) - P_{-i}(s^0)}{P_{-i}(s^{(b)}) - P_{-i}(s^0)} = 0$ under null hypothesis.

Let $\widehat{\delta}_i = \dfrac{\widehat{q}_i(s^{(a)}) - \widehat{q}_i(s^0)}{\widehat{q}_i(s^{(b)}) - \widehat{q}_i(s^0)} - \dfrac{\widehat{P}_{-i}(s^{(a)}) - \widehat{P}_{-i}(s^0)}{\widehat{P}_{-i}(s^{(b)}) - \widehat{P}_{-i}(s^0)}$ be a consistent non-parametric estimator of δ_i, and $\widehat{\mathrm{Var}}\left[\widehat{\delta}_i\right]$ be the bootstrap variance of the estimator $\widehat{\delta}_i$. Then the statistic $\widehat{D} = \dfrac{\widehat{\delta}_i^2}{\widehat{\mathrm{Var}}\left[\widehat{\delta}_i\right]} \sim X^2(df = 1)$ under the null hypothesis of equilibrium beliefs could be used as a test.

5 Monte Carlo Simulation

The game in strategic form with the notation is AM is given by:

$$\begin{bmatrix} & 0 & 1 \\ 0 & (0, 0) & \left(0, u_0 + \theta \cdot Z - \varepsilon_2\right) \\ 1 & \left(u_0 - \varepsilon_1, 0\right) & \left(u_0 + \tilde{u} - \varepsilon_1, u_0 + \tilde{u} + \theta \cdot Z - \varepsilon_2\right) \end{bmatrix}, \text{ where } \tilde{u} = u_1 - u_0 \text{ and } Z_1 = 0$$

Also suppose biased beliefs comes in the linear form $B_i(Z) = \lambda_i \cdot P_i(Z)$, with $\mathcal{Z} = \{-2, -1, 0, 1, 2\}$ and

$$\lambda_i = \begin{cases} 1 & \textbf{if } Z \in \{-2, 2\} \\ 0.7 & \textbf{if } Z \in \{-1, 0, 1\} \end{cases}.$$

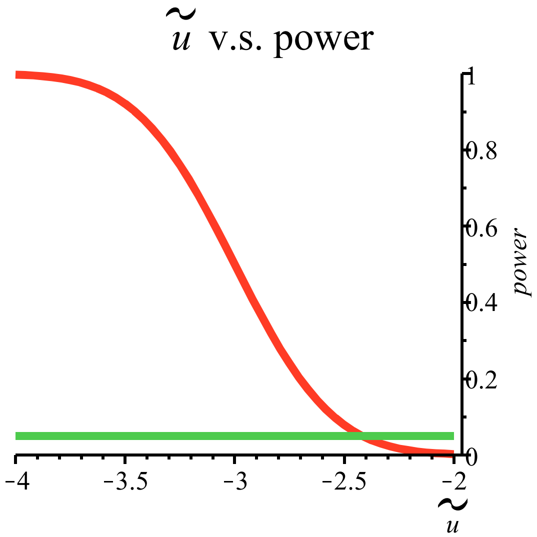
The data generated will be the sets $\left\{\{Y_{imt}\}_{i=1}^{2}\right\}_{m=1}^{M}$ and $\{Z_m\}_{m=1}^{M}$ and these could be used to compute the test statistics, and thus determine the power of the tests.

For the results in the next section, the specific parameters used in the simulation is M = 2000, number of replication = 1000, number of re-sampling for the bootstrap variance estimation = 100, and $u_0 = 2.4$, $\tilde{u} = -3.0$, $\theta = -0.5$ initially, and 100 evenly distributed points are chosen in the intervals $u_0 \in (1.5, 3.5]$, $\tilde{u} \in (-4, -2]$, $\theta \in (-1, 1]$.

6 Results

The power of the test is the probability that the test will reject the equilibrium belief when there the beliefs are indeed not in equilibrium. Thus, a Monte Carlo estimation of power would be, fixing the size of the test at 5% as shown in the green lines in the plots, the proportion of the rejections given the beliefs are not in equilibrium, which is represented by the red lines in the plots.

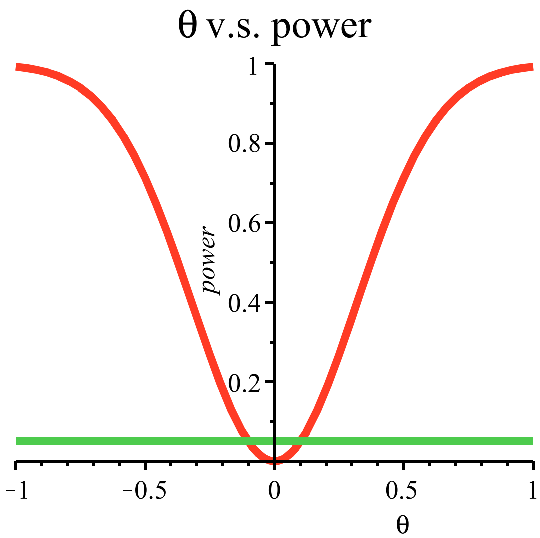
Note that the diagrams below are very crude approximation of the shape of the functions, and are used for illustration only: the actual counts and a polynomial approximations are given the appendix.



From the above diagram for the power of the test against the value of the parameter $\tilde{u}$, it is possible to conclude that the power is a decreasing function of the $\tilde{u}$ assuming the other parameters are fixed. Note that $\tilde{u}$ represents the difference between the utilities given different actions of the other player, and $\tilde{u}$ is assumed to be always less than 0 such that the game is a coordination game which does not have a dominant strategy in expectation for any of the players. Therefore, if the difference is close to 0, the game becomes:

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (0, 0) & \left(0, u_0 + \theta \cdot Z - \varepsilon_2\right) \\ \left(u_0 - \varepsilon_1, 0\right) & \left(u_0 - \varepsilon_1, u_0 + \theta \cdot Z - \varepsilon_2\right) \end{bmatrix} \end{array}$$

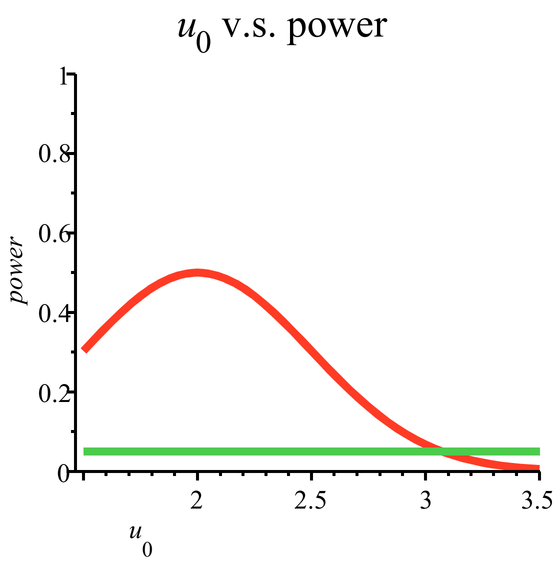
Here, the decisions of both of the players are independent of the choice of the other player. This means that the belief of the action of other player is no longer relevant, thus resulting in a low power of the test of equilibrium beliefs.



Similarly, for the parameter θ, note that the sign of θ is not relevant because Z is assumed symmetric around 0. Therefore, the symmetry of the power of the test around 0 is not surprising. Note that the power of the test is increasing in the absolute value of θ. Consider when θ is close to 0, then the game is given by:

$$\begin{bmatrix} & 0 & 1 \\ 0 & (0, 0) & \left(0, u_0 - \varepsilon_2\right) \\ 1 & \left(u_0 - \varepsilon_1, 0\right) & \left(u_0 + \tilde{u} - \varepsilon_1, u_0 + \tilde{u} - \varepsilon_2\right) \end{bmatrix}$$

Here $\varepsilon_1 \overset{d}{=} \varepsilon_2$ implies that the game is symmetric and the players know that his belief should be consistent with his action in equilibrium. Thus, again, the belief of the other player's action is no longer relevant, causing the power of the test to be very low.



u_0 is only a unit of measuring utility, so the effect of this parameter on the power of the test is not as meaningful. The relationship is bell-shaped centering at around 2.0 mainly due to the other two parameters being fixed at $\tilde{u} = -3.0$ and $\theta = -0.5$, so the shape does not have any particular meaning and should be changing as the other two parameters change. Writing the power of the test as a function of all three parameters jointly would give a better understanding of the situation but the experiment is not done for this project.

7 Conclusion

From the Monte Carlo experiments, the power of the test of equilibrium beliefs in a coordination game is shown as a function of the three parameters $\{u_0, \tilde{u}, \theta\}$ separately, where u_0 is the utility of choosing action 1 given that the other player chooses 0, $\tilde{u}$ is the difference in utility due to the action of the other player, and θ is the coefficient of a public information variable specific to only one of the players.

The relationships are given by the diagrams in the previous section and in the appendix. Notably, the power of the test is decreasing in $\tilde{u}$ and increasing in the absolute value of θ. A possible explanation is given in the previous section, and the main idea is that as $\tilde{u} \uparrow 0$ or $\theta \to 0$, the belief of the action of the other player becomes irrelevant to decision, thus making the power of the test very close to 0.

Further investigation could be made to 1) write the power of the test as a function of all three parameters at the same time, and 2) dynamic coordination game could be considered where players have beliefs of the other player's actions at each time period, and they are updated according to Bayes rule, and 3) games with three or more players or actions could be considered.

8 References

[1] Aguirregabiria, Victor and Magesan, Arvind, (2012): "Estimation of Dynamic Discrete Games When Players' Beliefs are Not in Equilibrium", Working paper.

[2] Aguirregabiria, Victor, (2009): "Estimation of Dynamic Discrete Games Using the Nested Pseudo Likelihood Algorithm: Code and Application", Working paper.

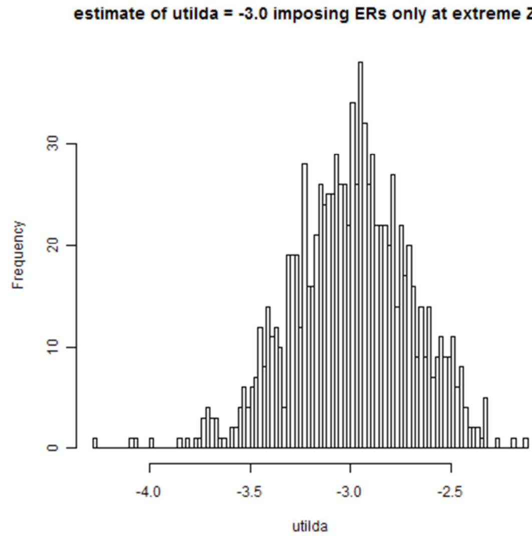
[3] Aguirregabiria, Victor and Mira, Pedro, (2007): "Sequential estimation of dynamic discrete games", Econometrica 75, 1—53.

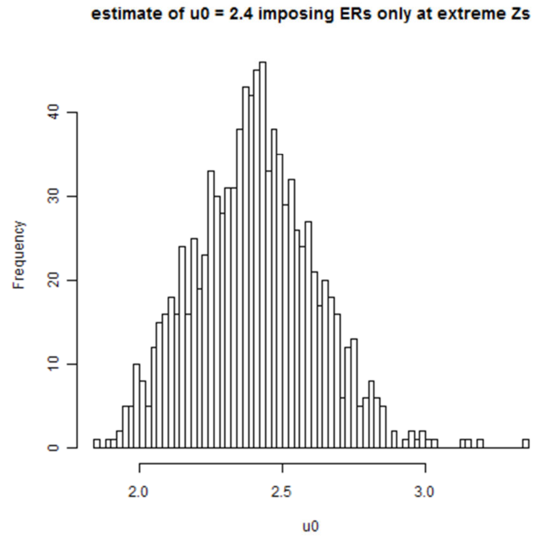
[4] Aguirregabiria, Victor and Mira, Pedro, (2002): "Swapping the nested fixed point algorithm: A class of estimators for discrete Markov decision models", Econometrica 70, 1519-1543.

9 Appendix

The plots for estimation and tests are provided in the next two pages.

The Fortran code, R script and other output files used to generate the plots, and the description of these files could be found at http://individual.utoronto.ca/youngwu/ECO2401Project.htm


estimate of utilda = -3.0 imposing ERs only at extreme Zs
Frequency
utilda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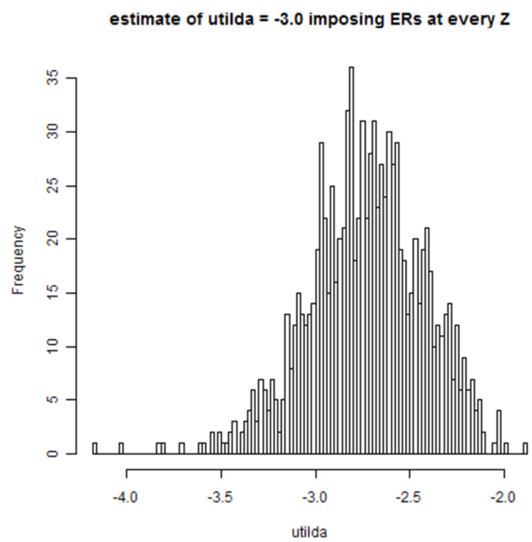
Frequency
u0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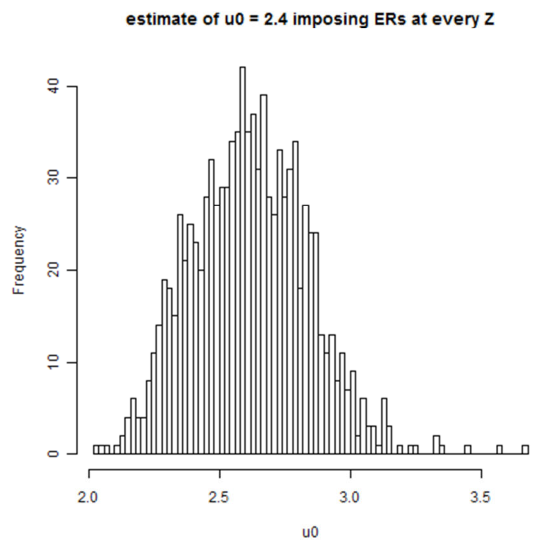
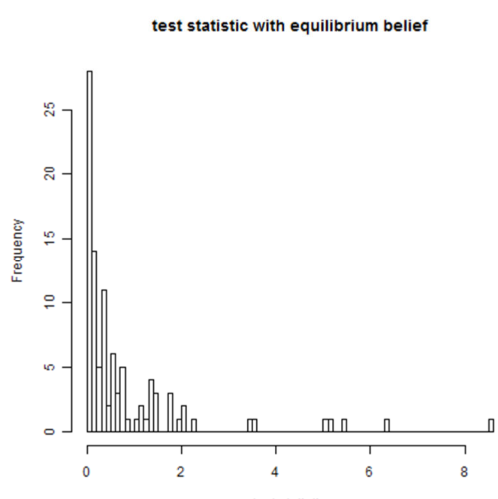
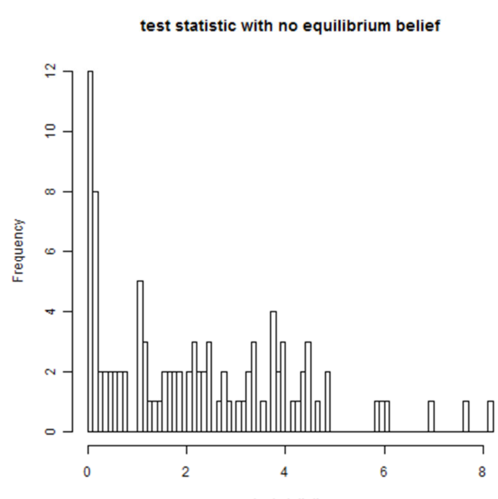
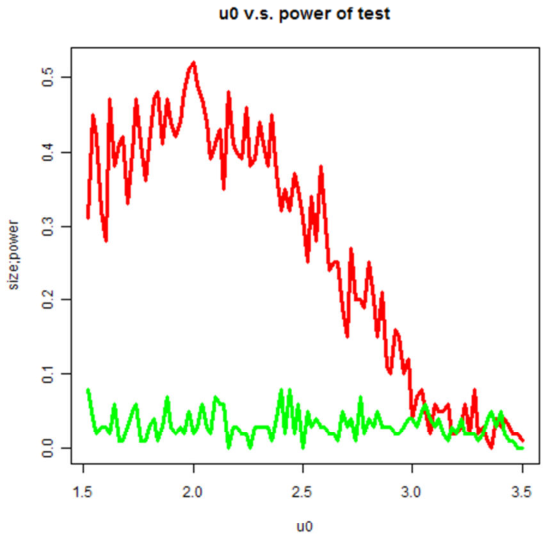
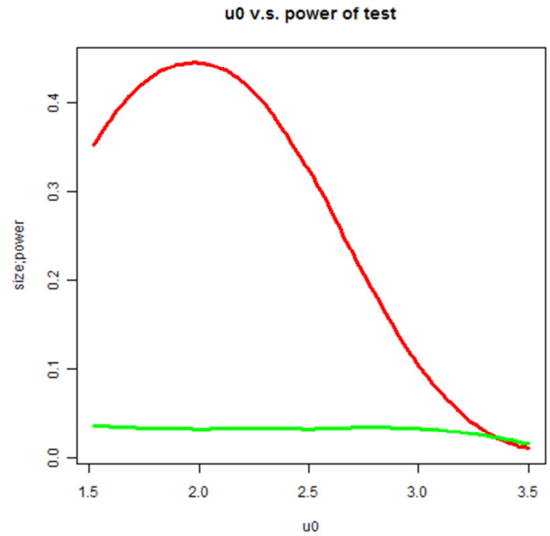
Frequency
utilda

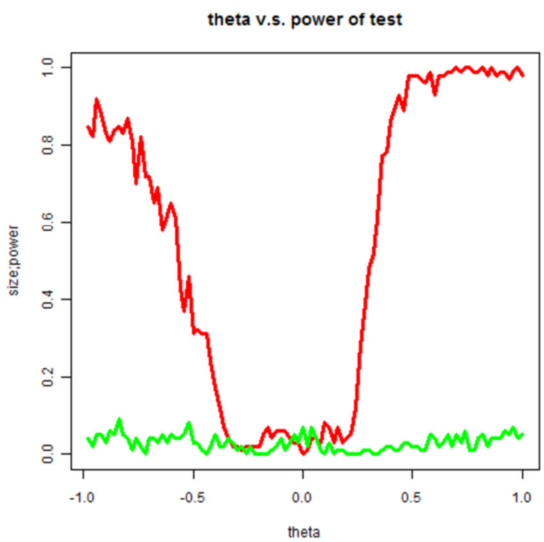

estimate of u0 = 2.4 imposing ERs at every Z
Frequency
u0

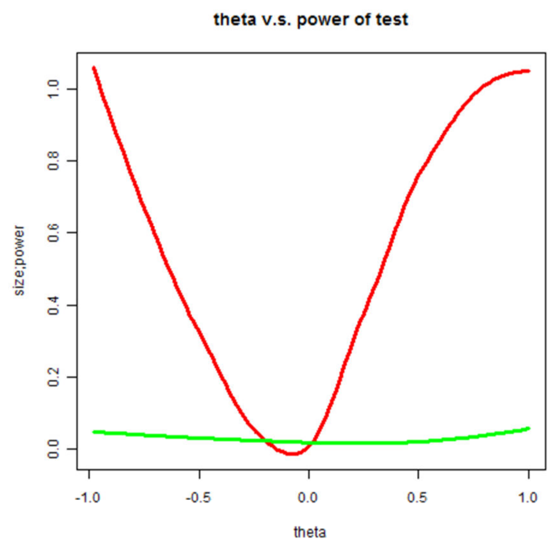

test statistic with equilibrium belief
Frequency
test statistic

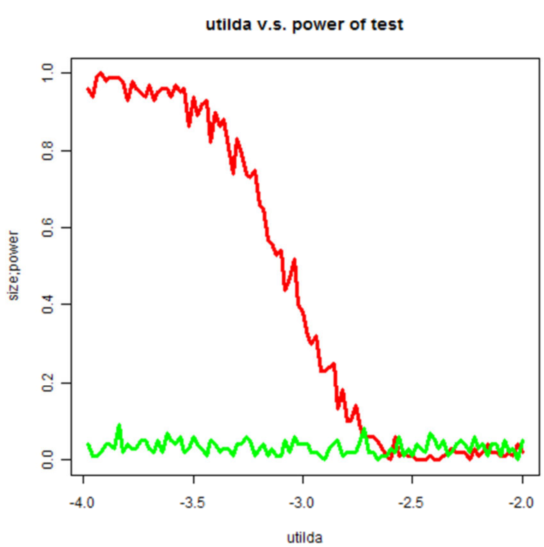

test statistic with no equilibrium belief
Frequency
test statistic

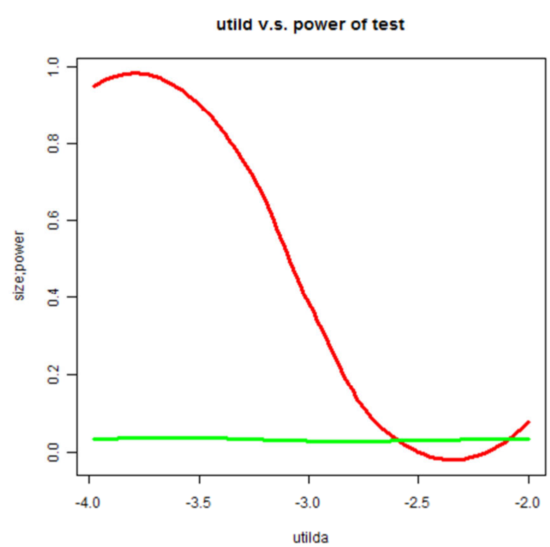

u0 v.s. power of test
size;power
u0


u0 v.s. power of test
size;power
u0


theta v.s. power of test
size;power
theta


theta v.s. power of test
size;power
theta


utilda v.s. power of test
size;power
utilda


utild v.s. power of test
size;power
utilda