SCHOOL ALLOCATION PROBLEM WITH OBSERVABLE CHARACTERISTICS

YOUNG WU

1. Model

1.1. **Allocations.** There are finite number of schools $l \in L = \left\{1, 2, ..., \bar{L}\right\}$ and finite number of groups of students divided by characteristics $k \in K = \left\{1, 2, ..., \bar{K}\right\}$. Let c_l be the measure of seats in school l and μ_k be the measure of of students in group k.

Definition 1. A profile of orderings is a full support distribution $\mu \in \Delta\left(K \times \mathcal{P}\left(L\right)\right)$, where $\mathcal{P}\left(L\right)$ is the set of all permutations (strict orderings) of L, and μ satisfies,

$$\sum_{p \in \mathcal{P}(L)} \mu\left(k, p\right) = \mu_{k.}$$

We implicitly assume that all preferences are strict, and use the notation $\mu\left(k, p\right)$ to denote the fraction of students in group k who have preference ordering p. We call these students' type $\left(k, p\right)$.

We also assume full support

$$\mu\left(k, p\right) > 0 \;\forall\; \left(k, p\right) \in \left(K, \mathcal{P}\left(L\right)\right).$$

Definition 2. Let the set of students be $\mathcal{I}$. An allocation is a function $q : \mathcal{I} \to \Delta\left(L\right)$, where $q\left(l; i\right)$ is the probability that a student $i \in \mathcal{I}$ is assigned to school l.

In the case where every student with the same type gets the same allocation, we will use the notation $q\left(l; k, p\right)$ to denote the probability that a student with type $\left(k, p\right)$ is assigned to school l. We call this condition group symmetric. Then the amount of students with type

Date: January 24, 2017.

(k,p) who are assigned to school l satisfies

$$\mu(k,p) \cdot q(l;k,p) = \int_{i \in \mathcal{I}\colon \text{type } (i)=(k,p)} q(l;i)\, d\mu,$$

where $\mu(k,p)$ is the mass of students with type (k,p) and

$$\mu(k,p) = \int_{i \in \mathcal{I}\colon \text{type } (i)=(k,p)} d\mu.$$

We also impose feasibility assumption that no student will be left unassigned

$$\begin{aligned}\sum_{l \in L} c_l &\geq \sum_{k \in K} \sum_{p \in \mathcal{P}(L)} \mu(k,p) \\ &= \sum_{k \in K} \mu_k \\ &= 1.\end{aligned}$$

Definition 3. An allocation q is feasible given a profile μ if

$$\int_{i \in \mathcal{I}} q(l;i)\, d\mu \leq c_l.$$

In the group symmetric case, the condition becomes

$$\sum_{k \in K} \sum_{p \in \mathcal{P}(L)} \mu(k,p)\, q(l;k,p) \leq c_l.$$

Definition 4. A mechanism is function $q : \mathcal{I} \to Q$, where Q is the set of all feasible allocations.

1.2. **Ordinal Efficiency.** We first define ordinal efficiency using first order stochastic domination of the allocation distribution:

Definition 5. An allocation q is dominated by q' for a student with type (k,p) where $p = l_1 \succ l_2 \succ l_3 ... \succ l_L$ if

$$\sum_{s=1}^{t} q(l_s;k,p) \leq \sum_{s=1}^{t} q'(l_s;k,p) \;\; \forall\, t \in L.$$

The main desirable properties of a ordinal mechanism in this model are:

Definition 6. Group Symmetry: $q(i) = q(j)$ for any $i, j \in \mathcal{I}$ such that $\text{ type } (i) = \text{ type } (j)$.

Definition 7. Envy-free: $q(i)$ is not dominated by any $q(j)$ for any $i, j \in \mathcal{I}$ such that $\text{ group } (i) = \text{ group } (j)$.

Definition 8. Efficiency: $q(i)$ is not dominated by any $q'(i)$ for any $i \in \mathcal{I}$.

Group symmetry states that students with the same characteristics are assigned the same probabilistic allocation. Given this assumption, we can use the notation $q(l; k, p)$ to denote the probability of a student with type (k, p) getting allocated the school l. Envy-free states that any a student (k, p) will not prefer the allocation of another student (k, p') for any $p' \neq p$. It is weaker condition of strategy-proof, but it guarantees that no student has incentive to misreport her preference p. We will show that a mechanism satisfying all three properties Definition 6, Definition 7, Definition 8 must be the modified Probabilistic Serial mechanism (PS) defined in the following section.

2. Modified Probabilistic Serial

2.1. **Algorithm.** We define the probabilistic serial mechanism as a simultaneous eating mechanism with equal constant eating speed over all groups.

Definition 9. subcapacity c_l^k of school l for students in group k, are functions $\mu \to [0, c_l]$ satisfying

$$\sum_{k \in K} c_l^k(\mu) \leq c_l \; \forall \, l \in L.$$

Algorithm 1. *A simultaneous eating mechanism (PSKT) with eating speed* $\omega = \{\omega_1, \omega_2, ..., \omega_{\bar{K}}\}$, *where* $\omega_k : [0,1] \to \mathbb{R}_+$ *satisfying* $\int_0^1 \omega_k(t)\, dt = 1$, *and subcapacities* $\left\{c_l^k\right\}_{l \in L, k \in K}$, *is given by*

Initialize: $L_k^0 = L, y_k^0 = 0$ *for each* $k \in K$ *and* $q^0(l; k, p) = 0$ *for each* $l \in L, k \in K, p \in \mathcal{P}(L)$, *Iteration: assume $L_k^{s-1}, y_k^{s-1}, q_k^{s-1}$ are defined for each k, set:*

$$M\left(l,A\right)=\left\{p\in\mathcal{P}\left(L\right):l\succ l'\;\forall\,l'\in A\setminus\left\{l\right\}\right\},$$

$$y_k^s\left(l\right)=\arg\min_y\left\{\sum_{p\in M\left(l,L_k^{s-1}\right)}\int_{y_k^{s-1}}^{y}\mu\left(k,p\right)\omega_k\left(t\right)dt+\sum_{p\in\mathcal{P}\left(L\right)}\mu\left(k,p\right)q^{s-1}\left(l;k,p\right)=c_l^k\left(\mu\right)\right\},$$

$$F_k^s=\arg\min_l y_k^s\left(l\right),$$

$$y_k^s=\min_l y_k^s\left(l\right),$$

$$L_k^s=L_k^{s-1}\setminus F_k^s,$$

$$t_l^k=y_{s_k},\;\mathit{for}\;l\in F_k^s,$$

$$q^s\left(l;k,p\right)=q^{s-1}\left(l;k,p\right)+\mathbb{I}_{p\in M\left(l,L_k^{s-1}\right)}\int_{y^{s-1}}^{y^s}\omega_k\left(t\right)dt.$$

Here, in step s,

$M\left(l,A\right)$ is the set of students where will be consuming school l if the remaining set of available schools is A,

$y_k^s\left(l\right)$ is the smallest amount of time needed for students in group k to finish consuming school l,

y_k^s is the smallest amount of time needed for students in group k to finish consuming any school,

F_k^s is the set of schools that are completely consumed by students in group k,

t_l^k is the time school l is completely consumed by students in group k,

L_k^s is the remaining set of available schools for students in group k,

$q^s\left(l;k,p\right)$ is the temporary allocation of school l for students with type $\left(k,p\right)$.

Algorithm 2. *A probabilistic serial mechanism (PS) is a simultaneous eating algorithm with* $\omega_k\left(t\right)=\omega=1\;\forall\;k\in K,t\in\left[0,1\right]$.

2.2. **Ordinal Properties.** For a full support profile, Bogomolnaia and Moulin showed that an allocation is efficient if and only if it is generated by a simultaneous eating algorithm, and Liu and Pycia showed that an allocation is efficient and envy-free if and only if it is generated by probabilistic serial. The result can be extended to the problem with multiple groups with a similar proof to Theorem 1 in Liu and Pycia.

Proposition 1. *An allocation q is group symmetric, envy-free and efficient for full support profile μ if and only if it is generated by Algorithm 2 (PS) with subcapacities,*

$$c_l^k(\mu) = \sum_{p \in \mathcal{P}(L)} \mu(k,p)\, q(l;k,p).$$

2.3. **Cardinal Efficiency.** We also define cardinal utility functions that induce the preference relations in the following way.

Definition 10. A utility distribution that is consistent with a preference profile μ is one density function $f_k : \mathbb{R}^L \to \mathbb{R}$ for each k such that:

$$\int_{p(u)=p} f_k(u)\, du = \frac{\mu(k,p)}{\mu_k}\ \forall\, p \in \mathcal{P}(L).$$

where $p(u)$ is the preference relation induced by the utility ranking u.

Definition 11. Efficiency: the allocation $q^\star$ is cardinally efficient if it maximizes the following expected welfare:

$$q^\star = \arg\max_q \sum_{k=1}^{K} \int u \cdot q(l;k,p(u)) \cdot f_k(u)\, du.$$

2.4. **Cardinal Extension.** Since PS is ordinally efficient, the cardinally efficient allocation must be obtained by PS for some subcapacities. We will rewrite the welfare maximization problem Definition 11 as a maximization problem by choosing the subcapacities c_l^k.

Definition 12. The welfare function $W_k : \mathbb{R}^k \to \mathbb{R}$ for group k is defined as the following,

$$W_k\left(c^k\right) = \int u \cdot q(l;k,p(u)) \cdot f_k(u)\, du.$$

The welfare maximization problem becomes:

$$\max_{\{c^k\}_{k\in K}} \sum_{k\in K} W_k\left(c^k\right)$$

$$\text{such that } \mu_k = \sum_l c_l^k \; \forall \; k \in K$$

$$\text{and } c_l = \sum_k c_l^k \; \forall \; l \in L$$

Proposition 2. *The function $W_k\left(c^k\right)$ is non-decreasing and concave in c^k.*

Conjecture 1. *The function $W_k\left(c^k\right)$ is piecewise linear in each subcapacity c_l^k.*

The above two properties of the welfare function guarantee that the problem can be solved by linear programming, although formulating the problem into a simple linear program is difficult.

3. Equivalence between PS and DA

3.1. **Two-school example.** Optimal PS and DA are equivalent if there are two characteristics and two schools. Suppose students live in two districts $(k = 1, 2)$ and there is one school in each district $(l = 1, 2)$.

Fix k, let,

$$\mu_1 = \int_{\{u_1 > u_2\}} f(u)\, du$$

$$\mu_2 = \int_{\{u_1 < u_2\}} f(u)\, du$$

Then,

$$\mu = \mu_1 + \mu_2$$

The PS allocations are:

If $c_1 \le c_2 \cdot \dfrac{\mu_1}{\mu_2}$,

$$q(k, 1 \succ 2) = 1^{\frac{c_1}{\mu_1}} 2^{1-\frac{c_1}{\mu_1}}$$

$$q(k, 2 \succ 1) = 1^0 2^1$$

If $c_1 \ge c_2 \cdot \dfrac{\mu_1}{\mu_2}$,

$$q(k, 1 \succ 2) = 1^1 2^0$$

$$q(k, 2 \succ 1) = 1^{1-\frac{c_2}{\mu_2}} 2^{\frac{c_2}{\mu_2}}$$

The value function is:

$$W(c) = \begin{cases} \int_{\{u_1 > u_2\}} \left(\frac{c_1}{\mu_1}\right) \cdot u_1 + \left(1 - \frac{c_1}{\mu_1}\right) \cdot u_2 du + \int_{\{u_1 < u_2\}} 0 \cdot u_1 + 1 \cdot u_2 du & \text{if } c_1 \le c_2 \cdot \frac{\mu_1}{\mu_2}, \\ \int_{\{u_1 > u_2\}} 1 \cdot u_1 + 0 \cdot u_2 du + \int_{\{u_1 < u_2\}} \left(1 - \frac{c_2}{\mu_2}\right) \cdot u_1 + \left(\frac{c_2}{\mu_2}\right) \cdot u_2 du & \text{otherwise .} \end{cases}$$

Or

$$W(c) = \begin{cases} \mu_1 \cdot \left(\frac{c_1}{\mu_1} \cdot \mathbb{E}[u_1 | u_1 > u_2] + \left(1 - \frac{c_1}{\mu_1}\right) \cdot \mathbb{E}[u_2 | u_1 > u_2]\right) + \mu_2 \cdot \mathbb{E}[u_2 | u_1 < u_2] & \text{if } c_1 \le c_2 \cdot \frac{\mu_1}{\mu_2} \\ \mu_1 \cdot \mathbb{E}[u_1 | u_1 > u_2] + \mu_2 \cdot \left(\left(1 - \frac{c_2}{\mu_2}\right) \cdot \mathbb{E}[u_1 | u_1 < u_2] + \left(\frac{c_2}{\mu_2}\right) \cdot \mathbb{E}[u_2 | u_1 < u_2]\right) & \text{otherwise .} \end{cases}$$

The derivative with respect to c_1 for fixed c_2 is

$$\frac{\partial W}{\partial c_1}(c) = \begin{cases} \mathbb{E}[u_1 - u_2 | u_1 > u_2] > 0 & \text{if } c_1 \le c_2 \cdot \frac{\mu_1}{\mu_2} \\ 0 & \text{otherwise ,} \end{cases}$$

and for $c_2 = \mu - c_1$ is

$$\frac{\partial W}{\partial c_1}(c) = \begin{cases} \mathbb{E}[u_1 - u_2 | u_1 > u_2] > 0 & \text{if } c_1 \le \mu_1 \\ \mathbb{E}[u_1 - u_2 | u_1 < u_2] < 0 & \text{otherwise .} \end{cases}$$

Therefore resulting allocation is cardinally efficient and it is identical to the one obtained from the differed acceptance algorithm.

Include an example of 3 school case when PS and DA are not equivalent?

4. Proofs

Proofs still have a lot of inconsistent notations.

4.1. **Proof of Proposition 1.**

Proof. The equivalence can be obtained from the following three lemmas. $\square$

Lemma 1. *(Modified from Liu and Pycia Theorem 1) If an allocation q is group symmetric, envy-free and efficient for profile $\mu >> 0$, then it is generated by PS with constraints $c_l^k\left(\mu\right) = \sum_p \mu\left(k, p\right) q\left(l; k, p\right)$.*

Proof. Consider any allocation q' and the allocation obtained by PS q^1. Let q^t be the partial allocation at time $t \in [0, 1]$ for PS.

Need to show that for any $\left(k, p\right), l$ and at any time $t \in [0, 1]$:

$$\sum_{l' \succ_{k,p} l} q'\left(l'; k, p\right) \geq \sum_{l' \succ_{k,p} l} q^t\left(l'; k, p\right)$$

Assume for a contradiction, there is $\tau \in [0, 1]$ such that:

$$\tau = \inf \left\{ t : \sum_{l' \succ_{k,p} l} q'\left(l'; k, p\right) < \sum_{l' \succ_{k,p} l} q^t\left(l'; k, p\right) \text{ for some } \left(k, p\right) \in \left(K \times \mathcal{P}\left(L\right) l \in L\right.\right.$$

Note that the original inequality are satisfied for all $t \in [0, \tau]$ and $\left(k, p\right)$ must be eating l at τ, which implies.

$$\sum_{l' \succ_{k,p} l} q'\left(l'; k, p\right) \geq \sum_{l' \succ_{k,p} l} q^{\tau}\left(l'; k, p\right) = \tau$$

By continuity of q^t in t:

$$\sum_{l' \succ_{k,p} l} q'\left(l'; k, p\right) = \sum_{l' \succ_{k,p} l} q^{\tau}\left(l'; k, p\right) = \tau$$

If assumed full support condition, l is favorite object of some agent (k,p'), then:

$$q\left(l;k,p'\right) \geq q^{\tau}\left(l;k,p'\right) = \tau$$

Envy-free assumption implies (k,p) does not prefer the allocation of (k,p'):

$$q\left(l;k,p'\right) \leq \tau$$

Therefore,

$$q\left(l;k,p'\right) = \tau$$

Since l is not exhausted at τ:

$$q^{1}\left(l;k,p'\right) > \tau$$

Therefore, (k,p') gets less l in q than q^1, efficiency assumption implies that there is another student $(k,\hat{p})$ who gets:

$$q\left(l;k,\hat{p}\right) > q^{1}\left(l;k,\hat{p}\right)$$

And there is some $\hat{l} \neq l$ that student $(k,\hat{p})$ prefers just more than l:

$$\sum_{l' \succ_{k,\hat{p}} \hat{l}} q'\left(l';k,\hat{p}\right) \geq \sum_{l' \succ_{k,\hat{p}} \hat{l}} q^{\tau}\left(l';k,\hat{p}\right) = \tau - q^{\tau}\left(l;k,\hat{p}\right) \geq \tau - q^{1}\left(l;k,\hat{p}\right)$$
$$\Rightarrow \sum_{l' \succ_{k,\hat{p}} l} q'\left(l';k,\hat{p}\right) > \tau$$

Comparing students (k,p') and $(k,\hat{p})$, envy-free assumption leads to a contradiction. $\square$

Lemma 2. *(Bogomolnaia and Moulin Proposition* 1*) PS is envy-free.*

Proof. For $y_k^{s-1} \leq t \leq y_k^s$, define:

$$N(l,t) = M\left(l, L^{s-1}\right) \text{ if } l \in L^{s-1} \text{ and } \emptyset \text{ otherwise}$$

$$n(l,t) = \sum_{p,k} \mu(k,p)\, \mathbb{I}_{(k,p)\in M(l,L^{s-1})}$$

$$t(l) = \sup\left\{t | n(l,t) \geq c_l^k(\mu)\right\}$$

Consider a student (k,p), let q be the allocation if she reports p and q' be the allocation if she reports p'.

Let p be the preference $l_1 \succ_{k,p} l_2 \succ_{k,p} l_3 \succ_{k,p} ...$

If $q(l_1;p,k) \leq q'(l_1;p,k)$, then $t(l_1) \leq t'(l_1)$.

We want to show that $N(l,t) = N'(l,t) \;\; \forall \; t \in (0, t(l))$, which implies $q(l_1;p,k) = q'(l_1;p,k)$

Repeat this process for l_2, l_3 ... □

Lemma 3. *(Bogomolnaia and Moulin Theorem* 1*) PS is efficient.*

Proof. Suppose, for a contradiction that q is obtained by PS and it is not efficient, and q is dominated by q'.

Define binary relation $\tau : l\tau l' \Leftrightarrow \{\exists (k,p) \in (K \times \mathcal{P}(L)) : l \succ_{k,p} l' \text{ and } q(l;k,p) > 0\}$.

Let (k_1,p_1) be the student such that $q(k_1,p_1) \neq q'(k_1,p_1)$, then there are l_0, l_1 such that

$$l_1 \succ_{p_1} l_0, q(l_1;k_1,p_1) > q'(l_1;k_1,p_1), q(l_0;k_1,p_1) < q'(l_0;k_1,p_1)$$

Then $l_0 \tau l_1$. Similarly, there is $l_1 \tau l_2$ and since L is finite, there $\exists$ a cycle in the relation τ:

$$l_0 \tau l_1, ..., l_R \tau l_0$$

Let (k_r,p_r) be the student such that $l_{r-1} \succ_{p_r} l_r$ and $q(l_r;k_r,p_r) > 0$.

Define $s_r = \min_s \{s : p^s(l_r;k_r,p_r)\}$, and note that $l_{r-1} \notin A^{s_r - 1}$, meaning $s_{r-1} < s_r$.

This implies $s_0 < s_1 < ... < s_{R-1} < s_0$, contradiction. □

4.2. **Proof of.** The concavity of the welfare functions can be obtained using the following lemmas:

Lemma 4. *Convex combination of envy free allocations are envy free.*

Proof. Consider arbitrary pair of students (k,p) and (k,p') under two different allocations q_1 and $q_{2.}$

Let $p = l_1 \succ l_2 \succ l_3 ... \succ l_L$ be the preference ranking of the first student, and define the following,

$$t_1^i = \min_t \left\{ \sum_{s=1}^{t} q_1 (l_s; k, p) < \sum_{s=1}^{t} q_1 (l_s; k, p') \right\}$$
$$t_1^a = \max_t \left\{ \sum_{s=1}^{t} q_1 (l_s; k, p) \leq \sum_{s=1}^{t} q_1 (l_s; k, p') \right\}$$
$$t_2^i = \min_t \left\{ \sum_{s=1}^{t} q_2 (l_s; k, p) < \sum_{s=1}^{t} q_2 (l_s; k, p') \right\}$$
$$t_2^a = \max_t \left\{ \sum_{s=1}^{t} q_2 (l_s; k, p) \leq \sum_{s=1}^{t} q_2 (l_s; k, p') \right\}$$

By envy-freeness, $t_1^i \neq t_1^a$ and $t_2^i \neq t_{2,}^a$

Consider a convex combination $q_0 = (\alpha) q_1 + (1-\alpha) q_2$ for $\alpha \in [0,1]$,

For $t \leq \min \left\{ t_1^i, t_2^i \right\}$,

$$\begin{aligned}
\sum_{s=1}^{t} q_0 (l_s; k, p) &= \sum_{s=1}^{t} (\alpha) q_1 (l_s; k, p) + (1-\alpha) q_2 (l_s; k, p) \\
&< \sum_{s=1}^{t} (\alpha) q_1 (l_s; k, p') + (1-\alpha) q_2 (l_s; k, p') \\
&= \sum_{s=1}^{t} q_0 (l_s; k, p')
\end{aligned}$$

And for $t \geq \max\{t_1^a, t_2^a\}$,

$$\begin{aligned}\sum_{s=1}^{t} q_0\left(l_s; k, p\right) &= \sum_{s=1}^{t} (\alpha)\, q_1\left(l_s; k, p\right) + (1-\alpha)\, q_2\left(l_s; k, p\right) \\ &> \sum_{s=1}^{t} (\alpha)\, q_1\left(l_s; k, p'\right) + (1-\alpha)\, q_2\left(l_s; k, p'\right) \\ &= \sum_{s=1}^{t} q_0\left(l_s; k, p'\right)\end{aligned}$$

Therefore, under q_0, no student strictly prefer the alloaction of another student, q_0 is envy-free. ☐

Lemma 5. *Any inefficient envy-free allocation has an envy-free Pareto improvement.*

Proof. Consider an allocation q and another allocation q' that (Pareto) dominates q.

For each student (k, p) and pair of schools (i, j), define the flow from school i to j by $\Delta(k, p; i, j)$ satisfying:

$$\sum_{j} \Delta\left(k, p; i, j\right) = \max\left\{0, q\left(k, p; i\right) - q'\left(k, p; i\right)\right\};$$
$$\sum_{i} \Delta\left(k, p; i, j\right) = \max\left\{0, q\left(k, p; j\right) - q'\left(k, p; j\right)\right\};$$

$$\Delta\left(k, p; i, j\right) \geq 0$$

Then define another allocation $q^\star$ by:

$$\begin{aligned}q^\star\left(k, p; i\right) = q\left(k, p; i\right) - \sum_{j} \Big(&\mathbb{I}_{\Delta(k,p;i,j)>0 \text{ or } j \succ_p i} \cdot \Delta^\star\left(k, p; i, j\right) \\ &+ \sum_{j} \mathbb{I}_{\Delta(k,p;j,i)>0 \text{ or } j \succ_p i} \cdot \Delta^\star\left(k, p; j, i\right)\end{aligned}$$

where $\Delta^\star$ is defined as:

$$\Delta^\star\left(k, p; i, j\right) = \frac{\sum_{k', p'} \Delta\left(k', p'; j, i\right) \cdot \mu\left(k', p'\right)}{\sum_{k', p' : i \succ_{p'} j \text{ and } \Delta(k', p'; i, j) = 0} \mu\left(k', p'\right) + \sum_{k', p'} \Delta\left(k', p'; j, i\right) \cdot \mu\left(k', p'\right)}$$

Note that the flows from q to q' and the flows from q to $q^\star$ are the same (the previous system for Δ is still satisfied).

Also, $q^\star$ still dominates q since:

$$\begin{cases} \Delta^\star(k,p;i,j) > 0 & \text{if } i \succ_p j \\ \Delta^\star(k,p;i,j) < \Delta(k,p;i,j) & \text{if } j \succ_p i \end{cases}$$

And $q^\star$ is envy-free since:

$$\begin{cases} \Delta^\star(k,p;i,j) \geq \Delta^\star(k,p';i,j)\ \forall\, p' & \text{if } i \succ_p j \\ \Delta^\star(k,p;i,j) \geq 0 & \text{if } j \succ_p i \end{cases}$$

Therefore, $q^\star$ is an envy-free Pareto improvement to q. □

Lemma 6. *The set of envy-free allocations are closed.*

Proof. Consider any sequence of allocations $\{q_i\}_{i=1}^{\infty}$ and the limit $q^\star$.

Fix any two students (k,p) and (k,p'), since q_i are envy-free for each i:

$$\sum_{s=1}^{t} q_i(l_s;k,p') \leq \sum_{s=1}^{t} q_i(l_s;k,p)\ \forall\, t$$

where l_s is the s-th school in the preference ranking of student (k,p).

Then,

$$\lim_{i\to\infty}\sum_{s=1}^{t} q_i(l_s;k,p') \leq \lim_{i\to\infty}\sum_{s=1}^{t} q_i(l_s;k,p)\ \forall\, t$$
$$\sum_{s=1}^{t} q^\star(l_s;k,p') \leq \sum_{s=1}^{t} q^\star(l_s;k,p)\ \forall\, t$$

Therefore, $q^\star$ is envy-free. The set is closed under limits.

Similarly, the set of Pareto improvements of any allocation is closed. □

Proof. Let c, c' be two vector of capacities, and q, q' be the PS allocation with capacities c, c' respectively.

Consider allocation $q_0 = \frac{1}{2}q + \frac{1}{2}q'$ and the welfare of allocation q_0 is $\frac{1}{2}\left(W(c) + W(c')\right)$

If q_0 can be obtained from PS with capacities $\frac{1}{2}(c+c')$, then $\frac{1}{2}(W(c)+W(c')) = W\left(\frac{1}{2}(c+c')\right)$.

Suppose q_0 cannot be obtained from PS, then q_0 is not both envy-free and efficient.

Since q_0 is envy-free from Lemma 6, q_0 is not efficient.

Let V be the set of envy-free allocations that is more efficient than $q_{0.}$

V is bounded since the set of allocations is bounded and the set of all envy-free allocations and the set of allocations that are Pareto improvements to q_0 are closed by Lemma 8. Then, V is an intersection of two compact sets implying that V is compact.

Therefore, there is an allocation $q^\star \in V$ that maximizes $W\left(\frac{1}{2}(c+c')\right)$.

Note that $q^\star$ must be efficient because if not, by Lemma 7, there is a envy-free Pareto improvement of $q^\star$ in V which contradicts the definition that $q^\star$ maximizes $W\left(\frac{1}{2}(c+c')\right)$.

$q^\star$ is envy-free and efficient, implying that $q^\star$ is the PS allocation with capacity $\frac{1}{2}(c+c')$.

Therefore, $\frac{1}{2}(W(c)+W(c')) \leq W\left(\frac{1}{2}(c+c')\right)$, W is concave in c. □

The function $W_k(c)$ is strictly increasing in c_l^k for $c_l^k \in \left[0, \sum_{p:l \text{ is the most preferred school}} \mu(k,p)\right]$, and non-decreasing for c_l^k in $\left[\sum_{p:l \text{ is the most preferred school}} \mu(k,p), c_l\right]$.

4.3. **Proof of Conjecture 1.**

Proof. Fix ak, define the set of capacities for which no two schools are finished being eaten at the same time, $\mathcal{D}$:

$$\mathcal{D} = \{c : y^s(l) \neq y^s(l') \ \forall\, s\, \forall\, l' \neq l\}$$

Define the set of capacities for which l and l' are finished being eaten at the same time, $E_{l:}$

$$E_{l,l'} = \{c_l : y^s(l) = y^s(l') \text{ for some } s\}$$

We first show that the value function on this set is linear.

Take $\varepsilon < \min_s y^s$, consider the change from c_l to $c_l + \varepsilon$ and $c_{l'}$ to $c_{l'} - \varepsilon$.

Let s be the iteration with $y^s(l) = y^s$, and s' be the iteration with $y^{s'}(l') = y^{s'}$

Then $W(c_l)$ will change by

$$\frac{\varepsilon}{|M(l, L^{s-1})|} \cdot \left(\sum_{p \in M(l, L^{s-1})} u(p, l) \right) - \frac{\varepsilon}{|M(l', L^{s'-1})|} \cdot \left(\sum_{p \in M\left(l, L'^{s'-1}\right)} u(p, l') \right)$$

where the $u(p, l)$ is the expected utility of students with preference p getting into school l.

The students with preferences in $M(l, L^{s-1})$ will spend $\frac{\varepsilon}{|M(l, L^{s-1})|}$ extra time on eating l, and L^s will stay the same since $\varepsilon < y^{s.}$

Similarly, the students with preferences in $M\left(l', L^{s'-1}\right)$ will spend $\frac{\varepsilon}{|M(l', L^{s'-1})|}$ less time on eating l', and $L^{s'}$ will stay the same since $\varepsilon < y^{s'}$.

Then note that $E_{l,l'}$ contains at most one point, since otherwise, $y^s(l) \neq y^s(l')$ in one of the points in $E_{l,l'}$. Therefore, $\mathcal{D}^c$ is a finite union of $E_{l,l'}$, the value function is linear on all but a finite set of points, i.e. piecewise linear. □

Lemma 7. *The value function is piecewise linear in the capacities c_l^k*

Proof. Fix k, re-index the school according the time it is eaten in the PS_k algorithm. From the definition of PS, if school s_1 and $s_2, s_1 < s_2$ are eaten at the same time, $y_{s_2} = 0$, the school with smaller original index is eaten first, then the school with larger index is eaten in 0 units of time.

Then the time school l is eaten is $t_l = \sum_{s=1}^{l} y_{s.}$

Define the change in finish time of l:

$$\Delta t_l = \frac{1}{\sum_{p \in M\left(l, L^{l-1}\right)} \mu(p)}$$

and the change in total value due to change in capacity l:

$$\Delta V_l(\varepsilon) = V(c_l + \varepsilon) - V(c_l)$$

Define expected utility from eating l:

$$\Delta u_l = \sum_{p \in M\left(l, L^{l-1}\right)} \mathbb{E}\left[u\left(l; p\right) \mu\left(p\right)\right]$$

Then we have

$$\Delta V_l\left(\varepsilon\right) = \Delta t_l \cdot \varepsilon \cdot \Delta u_l + \sum_{i=1}^{L-l} \Delta V_{l+i}\left(\eta_{l+i}\right) \text{ for some } \eta_s < \varepsilon \,\forall\, s.$$

$$\text{where } \eta_s = -\Delta t_l \cdot \varepsilon \cdot \left(\sum_{p \in M\left(s, L^{l-1}\right)} \mu\left(p\right)\right) \cdot \Delta t_s$$

$$\text{and } \Delta V_L\left(\varepsilon\right) = \Delta t_L \cdot \varepsilon \cdot \Delta u_L$$

$$\begin{aligned}\frac{dV}{dc_l} &= \Delta t_l \cdot \left(\Delta u_l + \sum_{i=1}^{L-l}\left(\sum_{p \in M\left(l+i, L^{l-1}\right)} \mu\left(p\right)\right) \cdot \Delta t_{l+i} \frac{dV}{dc_{l+i}}\right) \\ &= \sum_{i=0}^{L-l} w_i \Delta t_{l+i} \cdot \Delta u_{l+i} \text{ for some weights } w_i \text{ with } w_0 = 1, w_i < 0 \text{ for } i > 0\end{aligned}$$

For $\varepsilon_{l_1}, \varepsilon_{l_2}$ small enough, define the following for $l_1 < l_2$:

$$\Delta V\left(\varepsilon_{l_1}, \varepsilon_{l_2}\right) = V\left(c_{l_1} + \varepsilon_{l_1}, c_{l_2} + \varepsilon_{l_2}\right) - V\left(c_{l_1}, c_{l_2}\right)$$

If $t_{l_1} = t_{l_2}$, meaning l_1 and l_2 are eaten at the same time, then:

(1) If $\Delta t_{l_1} \varepsilon_{l_1} > \Delta t_{l_2} \varepsilon_{l_2}$,

l_2 will be eaten before l_1 after the ε change in capacity:

$$\begin{aligned}\Delta V\left(\varepsilon_{l_1}, \varepsilon_{l_2}\right) = \Delta t_{l_2} \cdot \varepsilon_{l_2} \cdot \Bigg(\Delta u_{l_1} + \Delta u_{l_2} + \sum_{i=1}^{L-l} \Delta V_{l_2+i}\left(\eta_{l+i}\right) \\ + \Delta V_{l_2}\left(\varepsilon_{l_1} - \frac{\Delta t_{l_2} \varepsilon\left(l_2\right)}{\Delta t_{l_1}}\right)\Bigg)\end{aligned}$$

(2) If $\Delta t_{s_1} \varepsilon_1 < \Delta t_{s_2} \varepsilon_2$,

s_1 will be eaten before s_2 after the ε change in capacity:

$$\Delta V\left(\varepsilon_{l_1}, \varepsilon_{l_2}\right) = \Delta t_{l_1} \cdot \varepsilon_{l_1} \cdot \left(\Delta u_{l_1} + \Delta u_{l_2} + \sum_{i=1}^{L-l} \Delta V_{l_2+i}\left(\eta_{l+i}\right)\right.$$
$$\left. + \Delta V_{l_1}\left(\varepsilon_{l_2} - \frac{\Delta t_{l_1} \varepsilon\left(l_1\right)}{\Delta t_{l_2}}\right)\right)$$

(3) If $\Delta t_{s_1}\varepsilon_1 = \Delta t_{s_2}\varepsilon_2$,

s_1 and s_2 will remain getting eaten at the same time. Same as the one-dimension case.

The directional derivative in the direction v is:

$$\nabla_v V = \sum_{l=1}^{L} w_l \Delta t_l \cdot \Delta u_l \text{ for some weights } w_l$$
$$= \sum_{l=1}^{L} w_l \frac{\sum_{p \in M\left(l, L^{l-1}\right)} \mathbb{E}\left[u\left(p, l\right) \cdot \mu\left(p\right)\right]}{\sum_{p \in M\left(l, L^{l-1}\right)} \mu\left(p\right)} \text{ for some weights } w_l$$

Moving in direction v satisfying the following condition will maintain the ordering of the schools:

$$\Delta t_{l_1} v_{l_1} = \Delta t_{l_2} v_{l_2} \ \forall\ l_1, l_2 \text{ such that } t_{l_1} = t_{l_2}$$
$$\frac{v_{l_1}}{\sum_{p \in M\left(l_1, L_1^{l_1-1}\right)} \mu\left(p\right)} = \frac{v_{l_2}}{\sum_{p \in M\left(l_2, L_2^{l_2-1}\right)} \mu\left(p\right)} \ \forall\ l_1, l_2 \text{ such that } t_{l_1} = t_{l_2}$$

This can be extended to cases where more than two schools are eaten at the same time.

$\square$