Week 4.2

Thursday, February 1, 2024 4:55 PM

ordering $\sqsubseteq$

$\sqsubseteq$ ← some ordering

binary relation

an ordering on set $S \in \mathcal{P}(S \times S)$

ex: $(\{1, 2, 3, 6\}, \sqsubseteq)$ where $\sqsubseteq$ means "divides evenly" then

$\sqsubseteq = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$

instead of $(1,3) \in \sqsubseteq$ we write $1 \sqsubseteq 3$

properties

- reflexive: $\forall a \in S,\ a \sqsubseteq a$
- transitive: $\forall a,b,c \in S,\ a \sqsubseteq b \wedge b \sqsubseteq c \Rightarrow a \sqsubseteq c$
- symmetric: $\forall a,b \in S,\ a \sqsubseteq b \Leftrightarrow b \sqsubseteq a$
- anti-symmetric: $\forall a,b \in S,\ a \sqsubseteq b \wedge b \sqsubseteq a \Rightarrow a = b$

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partial order

- a binary relation that is reflexive, transitive, & anti-symmetric ex

- reachability on DAG
- $\mathbb{N}$ & divisibility : $(\mathbb{N}, |)$
- $\mathcal{P}(S)$ and $\subseteq$

$\{1,2,3\}$
$\{1,3\}$
$\{2,3\}$
$\{1\}$
$\{2\}$
$\{3\}$
$\{\}$
$\{1,2\}$

$\{1,2\} \stackrel{?}{\sqsubseteq} \{1,3\}$

total order

$(\mathbb{Z}, \leq)$

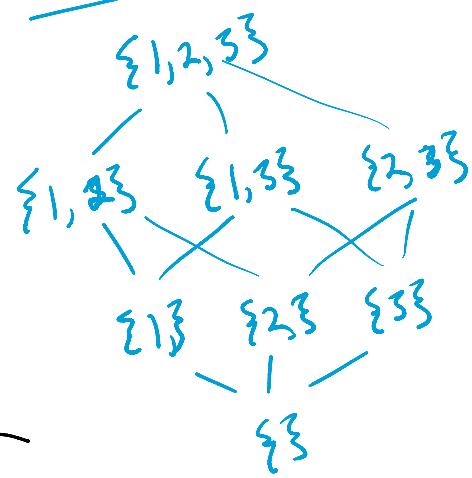
counter-example:

$(\mathbb{Z}, |)$ ← divides evenly

not a partial order

$(2,-2)$ $(-2,2)$

hasse diagram



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least upper bound (aka join) $\sqcup$

- for $D \subseteq S$, the least $x \in S$
  s.t. $\forall d \in D,\ d \sqsubseteq x$

greatest lower bound (aka meet) $\sqcap$

- the dual of $\sqcup$

$\boxed{\text{the } \sqcup \text{ \& } \sqcap \text{ are not guaranteed to exit}}$

$\mathcal{L}$ $(\mathbb{Z}, \leq)$

let D = set of even integers

$\sqcup D = \mathbb{Z}$

chain : a totally ordered subset of a partial order

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lattice

- a partial order s.t. any two elements are guaranteed to have $\sqcup$ & $\sqcap$

ex

- $(\mathbb{N}, |)$

  $\sqcup$ = least common multiple

  $\sqcap$ = greatest common divisor

  $\bot = 1$

  $\top = \emptyset$

- $(\mathcal{P}(S), \subseteq)$

  $\sqcup = \cup$

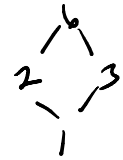
  $\sqcap = \cap$

  $\bot = \{\}$

  $\top = S$

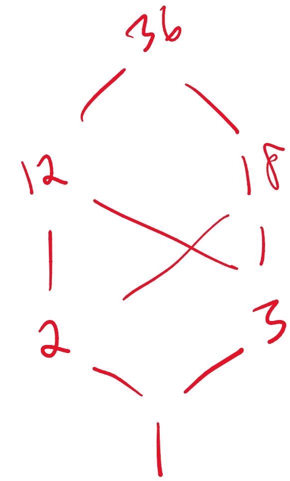
- $(\{1, 2, 3\}, |)$ NO

1. $(\{1,2,3\}, |)$ NO



2. $(\{1,2,3,6\}, |)$ YES

3. $(\{1,2,3,12,18,36\}, |)$ NO



4. $(\mathbb{N}, \leq)$ YES

Noetherian lattice

- a lattice s.t. all ascending chains are finite

ex

- any finite lattice
- $(\mathbb{N}, \geq)$

Complete lattice

L a lattice s.t. every subset has $\sqcap$ & $\sqcup$

$(\mathbb{N}, \geq)$

D = all even integers

MD where D = all even integers

all finite lattices are complete

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binary relation
- partial order (includes total order)
  - lattice
    - complete lattice

may or not be noetherian

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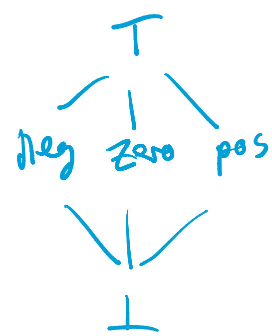
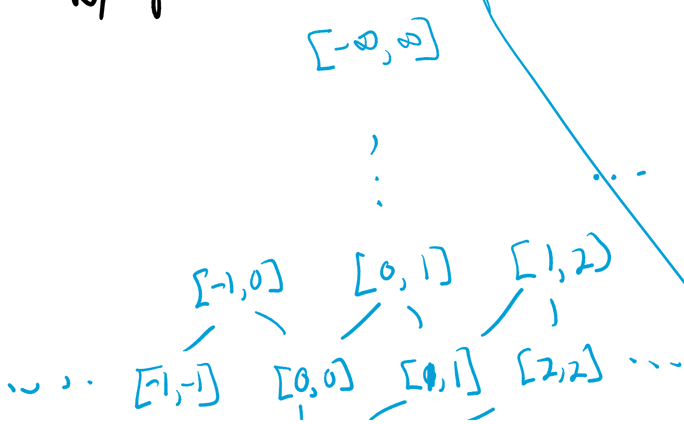
relation to DFA

- require our abstract domains to be complete noetherian lattices w/ partial order is based on precision





... [-1,-1] [0,0] [1,1] [2,2] ...

[ ]

→ **DFA**

most precise = $\top$

least precise = $\bot$

A.I.

most precise = $\bot$

least precise = $\top$

Computability

fixpoints

- $f(x) = x$
- $\mathrm{lfp}_x\, F = u$ s.t. $x \sqsubseteq u$, $F(u) = u$, & $\forall y,\ F(y) = y \Rightarrow u \sqsubseteq y$

least fixpoint

- pre-fixpoint: $x \sqsubseteq f(x)$
- post-fixpoint: $f(x) \sqsubseteq x$

ex

- $f(x) = x + 2$ — no fixpoints
- $f(x) = x^2 - x$ — $\emptyset$ & 2
- $f(x) = x^2/x$ — ∞, no least fixpoint if $x \in \mathbb{Z}$

($\cdot\ f(x) = x^{\cdot}/x$ ←

MFP worklist alg is computing $\mathrm{lfp}(F^{\#})$

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monotonicity

- function f is monotone iff
  $\boxed{x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)}$ ← "preserves order"
  $\boxed{\left(\text{or equiv. } f(x) \sqcup f(y) \sqsubseteq f(x \sqcup y)\right)}$

ex
- $f(x) = x - 1$

counter-example

$$f(x) = \begin{cases} x+2 & \text{if } x \leq \emptyset \\ x & \text{else} \end{cases}$$

~~$f(x) = -x$~~
~~$x \leq y$~~
~~$2 \leq 4$~~
~~$f(2) \overset{?}{\leq} f(4)$~~
~~$-2 \not\leq -4$~~

an extensive function : $x \sqsubseteq f(x)$

$\boxed{\text{fixpoint thms}}$

- Tarski : let L be a complete lattice and $F : L \to L$ be monotone, then

- Tarski : let L be a complete lattice &
  $F : L \to L$ be monotone, then
  the set of fixpoints of F is a
  complete lattice & thus there is
  a least fixpoint

- Kleene : let L be a complete <u>noetherian</u> lattice
  & $F : L \to L$ be monotone & x be a
  pre-fixpoint of F, then $\exists n \geq \emptyset$ s.t.
  $F^n(X)$ is stationary & $\mathrm{lfp}_x F = F^n(x)$

kleene iterations : $\bot \sqsubseteq F(\bot) \sqsubseteq F(F(\bot)) \sqsubseteq \cdots$
$\cdots \sqsubseteq F^n(\bot)$

$\boxed{\text{widening redux}}$

- widening forces convergence

cost : $\mathrm{lfp}_{orig} \sqsubseteq \mathrm{lfp}_{widened}$
& we have to prove each widening
operator works

$\nabla \in D \times D \to D$ over poset D
is a widening operator iff :

∇ over approximates

- $x \sqsubseteq x \nabla y$ & $y \sqsubseteq x \nabla y$

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- for all ascending chains $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \ldots$ then the following stabilizes in finite steps:
  - $acc_0 = x_0$
  - $acc_{n+1} = acc_n \nabla x_{n+1}$

∇ on ∞ ascending chain yields a finite ascending chain