Function Spaces

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MATC27 Introduction to Topology

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Introduction

What is a function space?

- In simple words, a function space is a topological space whose points are functions.
- There are many different types of function spaces, and we can give several different topologies on them.

Introduction

In this presentation, we will study the following topologies on a function space:

- Product topology and box topology
- Uniform topology
- Topology of compact convergence, and compact-open topology.

We might skip some of the last three topologies in exchange for a better understanding of the most basic concepts of function spaces.

Basic Notions

Before we study topologies on function spaces, we need to understand the following concepts:

Definition: Subbasis

A **subbasis** for a topology $\mathcal{T}_x$ on X is a collection of subsets of X whose union equals X.
The topology generated by $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of.

Note that a subbasis need not to be a basis.

Basic Notions

Definition: The product topology

Let $\{X_i\}_{i \in I}$ be an indexed family of topological spaces.
A basis for a topology on the product space $\prod X_i$ is the collection of all sets of the form

$$\prod_{i \in I} U_i$$

where U_i is open in X_i, and $U_i = X_i$ for but finitely many i.
The topology generated by this basis is called the **product topology**.

This isn't the usual definition provided in our text, but we use this definition for better understanding and more convenience in later proofs.

Basic Notions

Definition: The box topology

Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces.
A basis for a topology on the product space $\prod X_i$ is the collection of all sets of the form

$$\prod_{i\in I} U_i, \text{ where } U_i \text{ is open in } X_i.$$

The topology generated by this basis is called the **box topology**.

This seems like an extension to the product topology in the infinite case (note that we've only defined product topology on finite products in lecture), but the product topology for infinite products is different from the box topology.

Basic Notions

We now turn our attention to set of functions.

Notation

Let X and Y be sets, we write Y^X to denote the set of all functions from X to Y.

Why is the set of functions is in the form of some power of Y?

Basic Notions

The idea is that to know everything about a function $f : X \to Y$, we only need to know what f does to every point in its domain X, namely, we want to know what is $f(x)$ for all $x \in X$. So, if we put all such $f(x)$ into a tuple, we have a tuple (sequence) with every entry being an element of Y, thus this tuple belongs to some power of Y.

Example

Any function f from a finite set $S = \{s_i\}_{i=1}^{n}$ to a set Y can be represented as the sequence $(f(s_1), f(s_2), \ldots f(s_n)) \in Y^n$

Basic Notions

The next 2 examples illustrate the case for infinite domains.

Example

Any function f from $\mathbb{N}$ to a set Y can be represented as an infinite sequence $(f(1), f(2), \dots) \in Y^{\mathbb{N}} = \prod_{n \in \mathbb{N}} Y = Y \times Y \times \dots$

Example

The set $\mathbb{R}^{\mathbb{R}} = \prod_{x \in \mathbb{R}} \mathbb{R}$ is the set of all $f : \mathbb{R} \to \mathbb{R}$.

Basic Notions

We know a sequence a_n of elements in a topological space X can converge to a point $x \in X$, but if every point in X is a function, what does it mean for a sequence of functions to converge to a function?

Definition: Pointwise Convergence

Let X be a set, let Y be a topological space, and let $f_n : X \to Y$ be a sequence of functions. Then f_n **converges pointwise** to a function $f : X \to Y$ if for all $x \in X$, the sequence $f_n(x)$ converges to $f(x) \in Y$, namely $\lim_{n\to\infty} f_n(x) = f(x)$

Basic Notions

Example

The sequence of functions $f_n(x) = n\sin(\frac{x}{n})$ converges to $f(x) \in \mathbb{R}$ for all $x \in [0, 2\pi]$

Basic Notions

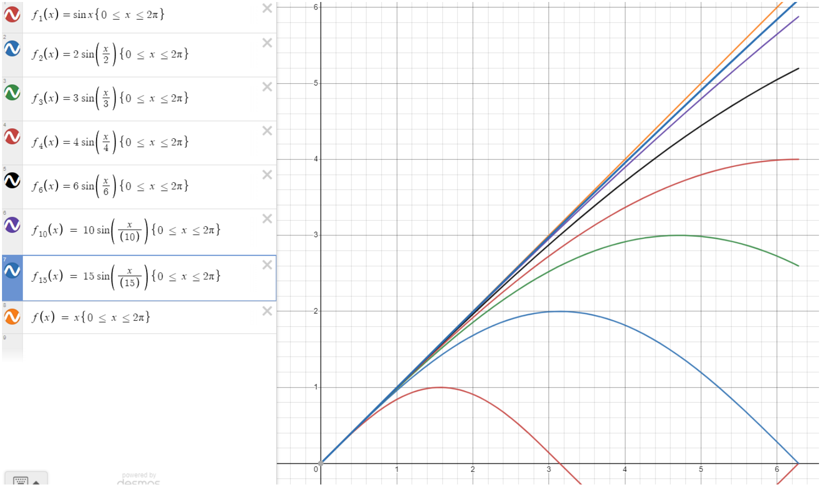

Figure 1: Convergence of $f_n(x) = n\sin(\frac{x}{n})$

The Product Topology

Definition: The Product Topology

Let X be a set, and let Y be a topological space. Given any $x \in X$ and any open set $U \subseteq Y$, we define

$$S(x, U) = \{f \in Y^X | f(x) \in U\}.$$

The collection of all $S(x, U)$ is a subbasis for a topology on Y^X, known as the **product topology**, or the **topology of pointwise convergence**.

Example

Consider the set of functions $Y^{\{1,2\}}$ (basically Y^2).
Let $\mathcal{S}_1 = \{S(1, U) | U \in \mathcal{T}_y\} = \{U \times Y | U \in \mathcal{T}_y\}$ and $\mathcal{S}_2 = \{S(2, U) | U \in \mathcal{T}_y\} = \{Y \times U | U \in \mathcal{T}_y\}$. Then $\mathcal{S}_1 \cup \mathcal{S}_2$ is a subbasis because $\{s_1 \cap s_2 | s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\} = \{U \times U | U \in \mathcal{T}_y\}$ is a basis for the "usual" product topology.

The Product Topology

Theorem

Let X be a set, let Y be a topological space, and let $\mathcal{B}$ be a basis for the topology on Y . Then $\{S(x, B) | x \in X, B \in \mathcal{B}\}$ is a subbasis for the product topology on Y^X.

The proof is easy, as for any open set $U \subseteq Y$, we can express it as a union of $B_i \in \mathcal{B}$. So $S(x, U)$ open in Y^x, can be written as $\bigcup_{i \in I} s(x, B_i)$, which means $S(x, U)$ lies in the basis generated by $\{S(x, B) | x \in X, B \in \mathcal{B}\}$

Example

The space of all real-valued infinite sequences is $\mathbb{R}^{\mathbb{N}}$. Given a basis element (a, b) from the topology on $\mathbb{R}$, we can get a subbasic element $S(2, (a, b)) = \prod_{n \in \mathbb{N}} \mathbb{R} \times (a, b) \times \mathbb{R} \times \dots$

The Product Topology

We now introduce the following theorem:

Theorem: Convergence in The Product Topology

Let X be a set, let Y be a topological space, let f_n be a sequence in Y^X, and let $f \in Y^X$. Then $f_n \to f$ under the product topology if and only if f_n converge pointwise to f.

The Product Topology

Proof:
Suppose $f_n \to f$ under the product topology, if $f(x) \in U$ is open in Y, then $f_n \in S(x, U)$ for all but finitely many n by definition of the product topology. It follows that $f_n(x) \in U$ for all but finitely many n (sequence convergence doesn't care about finitely many n's), which shows that $f_n(x) \to f(x)$.

Now suppose f_n converges pointwise to f, and let $S(x, U)$ be a neighborhood of f in Y^X. Then U is a neighborhood of $f(x)$ in Y. Since $f_n(x) \to f(x)$, it follows by convergence that $f_n(x) \in U$ for all but finitely many n. Then $f_n \in S(x, U)$ for all but finitely many n, so $f_n \to f$ under the product topology. $\square$

The Product Topology

Let X be a set, Y be a topological space. For $x \in X$, $\pi_x : Y^X \to Y$ be the projection function $\pi_x(f) = f(x)$. Then we have the following important properties of the product topology:

- Every π_x is continuous under the product topology.
- The product topology is the smallest topology on Y^X for which all of the π_x are continuous.
- If A is a topological space and $g : A \to Y^X$ is a function, then g is continuous under the product topology if and only if every function $\pi_x \circ g : A \to Y$ is continuous.

The first 2 points are easily proven by noting that $\pi_x^{-1}(U) = S(x, U)$ for any open set U of Y; For the last point, note that $g^{-1}(S(x, U) = (\pi_x \circ g)^{-1}(U)$ which is open if $\pi_x \circ g$ is continuous, and that the composition of continuous functions is continuous. $\square$

The Uniform Topology

We first introduce the metric that is required to for the uniform topology:

Definition: Uniform Distance

Let X be a set, Y be a metric space with metric d. Let f, g be functions from X to Y.
We define the **uniform distance** $p(f, g)$ as

$$p(f, g) = \sup\{d(f(x), g(x)) | x \in X\}$$

This is kind of saying that the uniform distance is the maximum distance between 2 functions, expect that this distance can be unbounded, for example, consider $f(x) = x$ and $g(x) = -x$.

The Uniform Topology

Since we require a metric to be finite, we modify the uniform distance to give a metric:

Definition: Bounded Uniform Metric

Let X be a set, Y be a metric space. Let f, g be functions from X to Y, and let p be the uniform distance. Then we define the **bounded uniform metric** $\overline{p}$ as:

$$\overline{p}(f, g) = \min\{p(f, g), 1\}$$

This ensures that $\overline{p}$ is finite.

Definition: The Uniform Topology

The metric topology induced by the **bounded uniform metric** is the **uniform topology**, generated by $\{B_{\overline{p}}(f, \epsilon) | f \in Y^X, \epsilon > 0\}$

The Uniform Topology

Uniform Convergence

Let X be a set, Y be a metric space. Let $\{f_n\}$ be a sequence of function. We say the f_n uniformly converges to $f : X \to Y$, if

$$\forall \epsilon > 0, \exists N > 0 \text{ such that } n > N \implies p(f_n, f) < \epsilon$$

This is basically saying that $p(f_n, f) \to 0$ as $n \to \infty$.

Example

Let $f_n = (\underbrace{0, \dots, 0}_{n-1 \text{ times}}, n, n, \dots)$. Then $p(f_n, f) = n$ for all $n \in \mathbb{N}$, where $f = (0, 0, \dots)$. So f_n doesn't uniformly converge to f, but it does pointwise.

The Uniform Topology

Theorem: Convergence in The Uniform Topology

Let X be a set, let Y be a topological space, let f_n be a sequence in Y^X, and let $f \in Y^X$. Then $f_n \to f$ under the uniform topology if and only if f_n uniformly converges to f.

The proof comes for free, given that convergence is defined using the metric $\overline{p}$.

The Uniform Topology

We now study a nice property of the uniform topology:

Theorem: Set of Continuous Functions is Closed

Let X be a set and let Y be a topological space with uniform topology. Let $\mathcal{C}(x, Y)$ be the set of all continuous functions from X to Y. Then $\mathcal{C}(x, Y)$ is a closed set in Y^X.

Proof: Let $f \in \overline{C(X,Y)}$ ($\overline{A}$ means closure of A) , let d be the metric on Y, and let $x_0 \in X$. We show that f is continuous at x_0. Given $\epsilon > 0$, $f \in \overline{C(X,Y)} \implies \exists g \in \mathcal{C}(x, Y)$ such that $\overline{p}(f, g) < \frac{\epsilon}{3}$. g continuous $\implies \exists U$ open in X such that $g(U) \subseteq B_d(g(x_0), \frac{\epsilon}{3})$. By our free choice of U we have $d(f(x), g(x)) < \frac{\epsilon}{3}$, $d(g(x), g(x_0)) < \frac{\epsilon}{3}$ and $d(f(x_0), g(x_0)) < \frac{\epsilon}{3}$. Then by triangle inequality we have $d(f(x), f(x_0)) < \epsilon$, so $f(U) \subseteq B_d(f(x_0), \epsilon)$, meaning f is continuous at x_0. $\square$

Compact Convergence and Compact Open Topology

There two more topologies that we can put on a function space: the compact convergence topology and the compact open topology. These topologies concern then X in Y^X is compact, which leads to many more nice, but in my opinion, highly nontrivial results. For the time sake, we omit these topologies in this presentation.

The Box Topology

Definition: The box topology

Let X be a set and Y be a topological space. Let $\{U_i\}_{x \in X}$ be an indexed family of open sets of Y.
The product $\prod_{x \in X} U_x = \{f \in Y^X | f(x) \in U_x, \forall x \in X\}$ is an **open box** in Y^X.
The collection of all such open boxes forms a basis, and the topology generated by this basis is called the **box topology**.

Theorem: Subbasis for The Box Topology

Let X be a set, let Y be a topological space, and let $\mathcal{B}$ be a basis for the topology on Y . Then $\{\prod B_x | B_x \in \mathcal{B}\ \forall x \in X\}$ is a subbasis for the product topology on Y^X.

Note that in the product topology version of the above definition/theorem, we do not require arbitrary open sets from Y.

The Box Topology

In the box topology, many sequences do not converge as illustrated by the following example.

Example

$\{f_n = (\frac{1}{n}, \frac{1}{n}, \dots)\}$ converges to $f = (0, 0, \dots)$ under the product topology, but not the box topology, because for an open set $\prod_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$ in the box topology, it contains f, but not f_n for any value of n.

The Box Topology

While the box topology is not useful in functional analysis, the following theorem serves and a way to give counterexamples to general statements about arbitrary topological spaces.

Theorem: Non-metrizable Space

$\mathbb{R}^{\mathbb{N}}$ is not metrizable under the box topology.

Last Thoughts

- Define convergence in box topology?
- What can we say about sizes of $\mathcal{T}_{product}$, $\mathcal{T}_{uniform}$ and $\mathcal{T}_{box}$?

Thank you for listening!