Introduction to Finite Geometry

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Project Paper for MATD02H3:
Classical Plane Geometries and their Transformations

Computer and Mathematical Sciences
University of Toronto at Scarborough
Nov 21, 2024

1 Abstract

Finite geometries are axiomatic systems, like Euclidean geometry, but with a key distinction: they contain a finite number of points. Rather than referring to a single structure, finite geometry encompasses various systems, each with its own set of axioms. For instance, a 4-point geometry differs from a 5-point geometry, though both are examples of finite geometries. In these systems, familiar geometric concepts such as distance and parallel lines, differ significantly from their Euclidean counterparts due to the finite nature of the points.

This paper focuses on spaces (mostly planes) within finite geometry. Although projective geometry is another topic choice for the project, finite projective geometry still falls under finite geometry. We will explore key examples, examine the underlying axioms and related theorems of two types of planes: affine and projective planes. Finally, we study two theorems concerning projective planes.

2 Planes in Finite Geometry

Finite plane geometries can be broadly classified into two categories: projective plane geometries and affine plane geometries. In projective planes, there are no parallel lines—all lines intersect, at "points at infinity". In contrast, affine planes allow the concept of parallel lines, which is more aligned with our intuitive understanding of Euclidean geometry.

Before delving into the specifics, it is important to emphasize that, as with Euclidean geometry, the meaning of points and lines in finite geometries depends on the underlying model or interpretation. So, we begin by defining some basic notions that are essential to describe objects in these finite planes.

Definition 2.1 (Intersect/incident)**.** Two lines are said to incident with (intersect at) a point p if they both contain p. These two lines are called intersecting lines

Definition 2.2 (Parallel)**.** Two lines that do not intersect are called parallel lines.

Definition 2.3 (Concurrent)**.** Three or more lines that intersect at the same point are called concurrent.

Definition 2.4 (Collinear)**.** Points that lie in the same line are collinear.

Now we are ready to study affine planes.

2.1 Finite Affine Planes

A finite affine plane is an affine plane with finitely many lines and points. The Euclidean spaces are examples of affine spaces (planes)[6], but they are not finite. Despite this, finite affine planes share many similarities with the Euclidean geometry. The following is the axioms of finite affine planes. Of course, given any specific affine plane, additional axioms are introduced in addition to the following.
Axioms of finite affine planes:

1. Every two points are incident with a unique line.
2. Given a point p and a line ℓ such that $p \in \ell$ then there exists a unique line m such that $p \in m$ and $m \cap \ell = \emptyset$. This is saying for any line through a point, there is another line that is parallel to it.
3. There are three points that are not collinear.

Proposition 2.1.1 (Properties of affine planes)**.**

1. *Parallelism is an equivalence relation in affine spaces.*
2. *In an affine plane, every line is incident with a constant n points and every point is incident with $n+1$ lines. We define this constant n to be the **order** of an affine plane.*
3. *An affine plane of order n has n^2 points and n^2+n lines.*

Remark: Proof of 1 is straight-forward, if we adopt the convention that a line is parallel to itself; Proving 2 and 3 require lots of counting and induction, but it can be done in an much simpler way with some new notions. See section 3.

Now let's look at some examples to solidify our understanding.

2.1.1 4-point Geometry

In addition to the axioms for finite affine planes, we introduce the following axioms for 4-point geometry. **Axioms of 4-point geometry**:

1. There exist exactly four points.
2. Any two distinct points have exactly one line on (or contain) both of them.
3. Each line is on exactly two points.

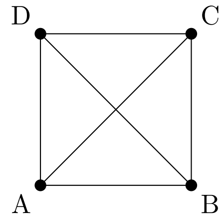

Figure 1: 4-point geometry

This is a geometry with only 4 points. In fact, it is the simplest (non-trivial) affine plane.

Note that lines AC and BD seem to intersect, but they actually do not because if they did intersect, that would give us 5 points instead of 4 for the geometry. Therefore the "seemingly" intersection in the middle is not labeled with a black dot.

We can easily verify from the picture that the 4-point geometry satisfies the axioms of finite geometry.

We now derive some properties that from the axioms of 4-point geometry.

Proposition 2.1.2.

1. *In 4-point geometry, if two distinct lines intersect, they have exactly one point in common.*
   proof. *Suppose line ℓ_1 and ℓ_2 intersect at two distinct points A and B, then 2 distinct lines ℓ_1 and ℓ_2 contain points A, B, violating Axiom 2.*
2. *There are exactly six lines in the 4-point geometry.*
   proof. *By Axiom 2 and 3 we know that a pair distinct points determines a line uniquely. Therefore the number of lines is simply the number of pairs of distinct points. Since there are 4 points in total, this is given by $\binom{4}{2} = 6$ lines.*
3. *In the 4-point geometry, each point has exactly three lines on it.*
   proof. *By Axiom 2, given a point A, and three other (distinct) points B, C, D, there is exactly one line through A and each of B, C, D respectively. Therefore there are 3 lines through A by definition.*
4. *In the 4-point geometry, each line has at least one line that is parallel to it.*
   proof. *Let A, B be 2 distinct points in a line ℓ. By Axiom 3 we know that the line through C, D (any two points that are not A and B) do not intersect ℓ, otherwise, we would have a line that is on three (or more) points.*
5. *In the 4-point geometry, there exists a set of two lines that contain all of the points in the geometry.*
   proof. *By 4. we can take lines ℓ and ℓ' that are parallel; By Axiom 2 and 3 we know these two lines determine 4 distinct points, which are all the points we have in the 4-point geometry.*

2.1.2 5-point Geometry

The axioms of the 5-point geometry is the same as what's for the 4-point geometry, except the first axiom will say "There are exactly **5** points". This can be extended to any n-point geometry.[1] Some properties we proved for the 4-point geometry hold any n-point-geometry, of course, the numbers are different, depending on how many points the geometry has.

Let us study some properties of the 5-point geometry to see what carries over from the 4-point geometry.

Proposition 2.1.3 (Properties of 5-point geometry)**.**

1. *In 5-point geometry, if two distinct lines intersect, they have exactly one point in common.*
2. *There are exactly 10 lines in the 5-point geometry.*
   Remark: *In general, we have $\binom{n}{2}$ lines in $n-$point geometry.*
3. *In the 5-point geometry, each point has exactly four lines on it.*
   Remark: *In general, we have $n-1$ lines on a point in $n-$point geometry.*

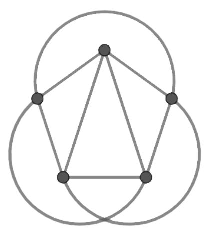

Figure 2: 5-point geometry

4. *In the 5-point geometry, each line has at least one line that is parallel to it.*
   Remark: *Can we do better than just saying at least one line?*

5. *In the 5-point geometry, there exists a set of three lines that contain all of the points in the geometry.*
   Remark: *How can we generalize this theorem to any n-point geometry?*

It is worth noting that, although the 5 point geometry possesses similar properties to that of the 4=point geometry, but it is not an affine plane because uniqueness in axiom 2 is not satisfied (parallel axiom). You can also see this by noting 5 is not a square, but an affine plane of order n has n^2 points.

2.2 Finite Projective Planes

Now we turn our attention to another type of planes in finite geometry: projective planes. The following axioms for projective planes are very similar to those of affine planes, but the second axioms here doesn't guarantee parallel lines (as mentioned earlier, every line intersects in projective plane), but instead addresses unique intersection.

1. Every two points are incident with a unique line.

2. Every two lines are incident with a unique point.

3. There exists a set of four points, no three of which are collinear.

From these axioms, we have some interesting properties.

Proposition 2.2.1. *Let p be a point not incident with a line ℓ. The number of points incident with ℓ is equal to the number of lines incident with ℓ.*

proof. This follows directly from the first and second axiom of projective planes.

Proposition 2.2.2. *Every point in a projective plane is incident with $n+1$ lines, and every line is incident with $n+1$ points, for some constant $n \in \mathbb{N}$.*

proof. By **Proposition 2.2.1**, the first are the second half of **Proposition 2.2.2** are equivalent. So it suffices to the first half.

Let ℓ be any line, and by the third axiom of projective geometry, we choose arbitrary points p and q, $p \neq q$ such that p, q do not incident with ℓ. Then again by **Proposition 2.2.1**, the number of lines incident with q is equal to the number of lines incident with p. This means any two points incident with the same number of lines.

Definition 2.5. The constant n in **Proposition 2.2.2** is called the **order** of a projective plane. In other words, it is the number of points incident with a line minus one.

Proposition 2.2.3. *A projective plane of order n has n^2+n+1 points and n^2+n+1 lines.*

proof. Let p be a point of a projective plane. By **Proposition 2.2.2** there are $n+1$ lines incident with p and each is incident with n other points (these are distinct, since lines are pairs of points uniquely determine each other by axiom 1 and 2). So we found $n(n+1)$ points other than p. Together with the point p, there are $n(n+1)+1 = n^2+n+1$ points in a projective plane of order n.

We can repeat the above argument for lines to conclude that there are $n(n+1)+1 = n^2+n+1$ lines in a projective plane of order n.

Corrollary 2.2.4. *A projective plane has an equal number of points and lines.*

We now turn our head to a simple example of projective plane.

2.2.1 The Fano Plane

In addition to the Axioms of projective plane, the following are the axioms for the Fano plane. **Axioms of Fano Plane:**

1. There exists at least one line.
2. There are exactly three points on each line.
3. Not all points are on the same line.

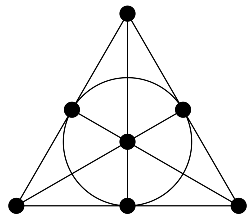

Figure 3: 5-point geometry [**?**]

Axioms 1 and 3 ensure that the Fano plane is not trivial (Contains no or too little points/lines to be interesting), and Axiom 2 defines the number of points (thus, lines) in the Fano plane. Axiom 2 also tells us that the Fano plane is a projective plane of order 2. See 2.3.

This is one of the simplest projective planes one can have. Note that in the picture, every line in the geometry intersect with some other line so there are no parallel lines.

From **Propositions 2.21 - 2.23** and the axioms, we can easily prove that in the Fano Plane, there are exactly seven points and seven lines, and each line is incident with three points.

2.3 Projective Planes as Vector Spaces

Euclidean geometry can have different models. For example, $\mathbb{R}^2$. This model allows us to use other powerful tools like calculus, analysis and topology to study Euclidean geometry.

Similarly, we can introduce a model on finite projective spaces to allow use to use powerful tools. One way to do this is to think of finite projective spaces as (subspaces of) vector spaces over finite field $GF(q^n)$ where q^n is a prime power.

To do so, we will need some definitions and theorems from algebra. The following can be found in Gallion's *Contemporary Abstract Algebra*[3] and Rotman's *Galois Theory*[4], so we will omit the details, especially the proofs.

2.3.1 Preliminary

Theorem 2.3.1. *For each prime p and each positive integer n, there is a unique finite field of order p^n (up to isomorphism). We denote this field as $GF(p^n)$. If $n = 1$, we have $GF(p) \cong \mathbb{Z}_p$. In addition, all finite fields must be of order p^n.*

Theorem 2.3.2. *$GF(p^n)$ is a vector space of dimension n over $GF(p)$.*

Notation: $V(n, q)$ denotes a vector space of dimension n over $GF(q)$.

Definition 2.6. The projective space, denoted as $PG(n, q)$, is the geometry whose points, lines, planes, ... and hyperplanes are the subspaces of $V(n + 1, q)$ of dimension $1, 2, 3, \ldots, n$.

Remark: The Fano plane can be denoted as $PG(2, 2)$.
Remark: The number of points and lines we proved for projective planes translate exactly into the number of corresponding subspaces of $PG(n, q)$. One can prove them by counting linearly independent sets. This left as an exercise for the reader.

2.3.2 The Bruck-Ryser-Chowla theorem

We know that a projective plane of order n has $n^2 + n + 1$ points and $n^2 + n + 1$ lines. If we think of points on a line as 1-dimensional subspaces of this line (as a vector space over $GF(p)$), then with simple counting we get that this line has $\dfrac{p^2 - 1}{p - 1} = p + 1$ subspaces, or points. So if every (projective) line contains $p + 1$ points, then for a projective plane $PF(2, p)$, it must be of order p. p is a prime power, so we have many examples of projective planes of order of prime power. But what about non-prime powers? It was conjectured that

The order of a projective plane is the power of a prime.

This question in finite geometry remains unsolved, but this brings our attention to studying order of a projective plane.

We will present a theorem that implies the non-existence of projective planes of certain orders.

Theorem 2.3.3 (The Bruck-Ryser-Chowla theorem, or BRC theorem)**.** *If there is a projective plane of order n and $n \equiv 1$ or $2 \mod 4$, then n is the sum of two squares.*

Since both 6 and $14 \equiv 2 \mod 4$ but they are not the sum of squares, there doesn't exist projective planes of such orders.

To prove this theorem, we will need some results from number theory, of which the proofs we will omit, as it is not the focus of this paper. They can be found in [5].

Lemma 2.3.4. *We have*

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2) = y_1^2 + y_2^2 + y_3^2 + y_4^2,$$

where

$$\begin{aligned} y_1 &= a_1x_1 - a_2x_2 - a_3x_3 - a_4x_4, \\ y_2 &= a_1x_2 + a_2x_1 + a_3x_4 - a_4x_3, \\ y_3 &= a_1x_3 + a_3x_1 + a_4x_2 - a_2x_4, \\ y_4 &= a_1x_4 + a_4x_1 + a_2x_3 - a_3x_2. \end{aligned}$$

Remark: Exact coefficients $a_i's$ come from direct computation, but is not needed for our purpose. We just need that $y_i's$ are linear combinations of $x_i's$.

Lemma 2.3.5. *Every number is the sum of four squares.*

Lemma 2.3.6. *If $nx^2 = w^2 + y^2$ has integer solutions, then n is the sum of two squares.*

Lemma 2.3.7. *If $n \equiv 1$ or $2 \mod 4$, then $[(n^2 + n + 1) + 1] \mod 4 = 0$*

Let us now prove the theorem.

proof of Theorem 2.3.3. Suppose we have a projective plane PG of order n, $n \equiv 1$ or $2 \mod 4$.

Let $\{P_i\}_{i=1}^N$ be the points in PG and $\{L_i\}_{i=1}^N$ be line in PG, where $N = n^2 + n + 1$.

We begin by defining the following matrix:

$$A = [a_{ij}],\ a_{ij} = \begin{cases} 1, & \text{if } P_i \in L_i \\ 0, & \text{if } P_i \notin L_i \end{cases}$$

Using the 1st and 2nd axiom of projective plane, A is an $N \times N$ matrix where every row and column has exactly $(n + 1)$ 0's. Direct computation gives that

$$A^T A = \mathbf{1} + nI,$$

where $\mathbf{1}$ is the matrix of all $1's$ and I is the identity matrix.

Now let $z = Ax$, where $x = (x_1, \ldots, x_N)$ is an N-dimensional vector, $x_i \in \mathbb{R}$.

It follows that $z^T z = x^T A^T Ax = x^T$. Expanding gives

$$z_1^2 + \cdots + z_N^2 = (x_1 + \cdots + x_n)^2 + n(x_1^2 + \cdots + x_N^2).$$

Adding nx_{N+1}^2 to both sides we have

$$z_1^2 + \cdots + z_N^2 + nx_{N+1}^2 = (x_1 + \cdots + x_n)^2 + n(x_1^2 + \cdots + x_{N+1}^2).$$

By **Lemma 2.3.5** n is a sum of 4 squares. Also, number of squares in $(x_1^2 + \cdots + x_{N+1}^2)$ is a multiple of 4 by **Lemma 2.3.7**. So if we apply **Lemma 2.3.4** $\frac{N+1}{4}$ times, we arrive to

$$z_1^2 + \cdots + z_N^2 + nx_{N+1}^2 = (x_1 + \cdots + x_n)^2 + (y_1^2 + \cdots + y_{N+1}^2), \quad (*)$$

where $y_i's$ are linear combinations of $x_i's$.

We know $Ax = z$, and the first row of A is nonzero, so there is some z_i that is a linear combination where the coefficient of x_1 is nonzero.

Similarly, by definition of $y_i's$, it must also be some y_i that is a linear combination of $x_j's$ where the coefficient of x_1 is nonzero, since the term $n(x_1^2 + \cdots + x_{N+1}^2)$ has a nonzero x_1 term.

Process:

WLOG assume z_1 and y_1 contain x_1. Recall that x_i are arbitrary real numbers of our choice, so there exists x_1 such that $z_1 = y_1$. (If the coefficient of x_1 in both y_1 and z_1 are the same, put $z_1 = y_1$ instead) Solving $z_1 = y_1$ (or $z_1 = -y_1$) for x_1 and substitute into z_1 will allow use to cancel z_1^2 and y_1^2 from both sides.

We repeat this process for all $x_1, \ldots, x_N$.

If at some step k, one or more of y_k, z_k do not contain x_k, then we know when **Lemma 2.3.4** is applied, x_k is one of the four terms $(x_s, x_{s+1}, x_{s+2}, x_{s+3},\ s \leq k \leq s+3)$. We can write $(y_s, y_{s+1}, y_{s+2}, y_{s+3}) = C(x_s, x_{s+1}, x_{s+2}, x_{s+3})$, where C is some coefficient matrix dependent on n. Since we are free to choose $x_i's$, if we set $x_i = 0$ for $i > k$, then $(y_s, y_{s+1}, y_{s+2}, y_{s+3}) = C(x_s, x_{s+1}, x_{s+2}, x_{s+3}) = 0$, so C is a non-invertible matrix. However, if we let $n = \sum_{i=1}^4 a_i^2$ like in **Lemma 2.3.4**, then we can explicitly write out

$$C = \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix}.$$

Direct computation gives $\det(C) = \left(\sum_{i=1}^4 a_i^2\right)^2 = n^2$ which is nonzero for $n \geq 1$. We have a contradiction. Hence, at every step k, we always have some y_k and z_k that contain x_k. So the above process may be repeated to cancel out z_k^2 and y_k^2 from $(*)$ until we reach the last step, which leaves us with

$$nx_{N+1}^2 = (x_1 + \cdots + x_n)^2 + y_{N+1}^2 \quad (\Delta)$$

Notice that each step of cancellation we are only performing addition and multiplication (division) of integers or rationals, so we know $(x_1 + \cdots + x_n)^2$ and y_{N+1} are just rational multiples of X_{N+1}. Again, we are free to choose x_{N+1}, so simply choose one such that (Δ) has integer solutions.
Finally, **Lemma 2.3.6** implies n is a sum of two squares. □

2.4 Desargues' theorem

Last but not least, we will prove Desargues' theorem in a 3-dimentional projective space. Before we do so, we need some useful properties that can be derived from axioms for projective planes (spaces).

Lemma 2.4.1. *Any two (distinct) planes in a projective space are incident with exactly one line; Any two (distinct) lines in a projective space are incident with exactly one plane.*

To prove this, we would first start arguing the intersection of a line and a plane. The proof can be found here [7].

Theorem 2.4.2 (Desargues' theorem)**.** *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles lying in the same or different planes. Let the lines AA', BB', and CC' intersect in the point O. Then*

$$\begin{aligned} &BC \text{ meets } B'C' \text{ in } Y, \\ &CA \text{ meets } C'A' \text{ in } X, \\ &AB \text{ meets } A'B' \text{ in } Z, \end{aligned}$$

where X, Y, Z are collinear.

proof.

- Case 1: $\triangle ABC$ and $\triangle A'B'C'$ lie in different planes.
  Clearly, it follows that BB' and CC' are in the same plane since these two lines from different planes intersect at O. Then by axiom 2, BC and $B'C'$ incident uniquely at Y.

  Analogously, CA and $C'A'$ are incident with X; AB and $A'B'$ incident with Z.

  Now X, Y, Z lie on the lines that lie both in the plane of $\triangle ABC$ and $\triangle A'B'C'$, so the three points lie in the intersection of two planes. By **Lemma 2.4.1**, this intersection is a line, thus X, Y and Z are collinear, as in Figure 4.

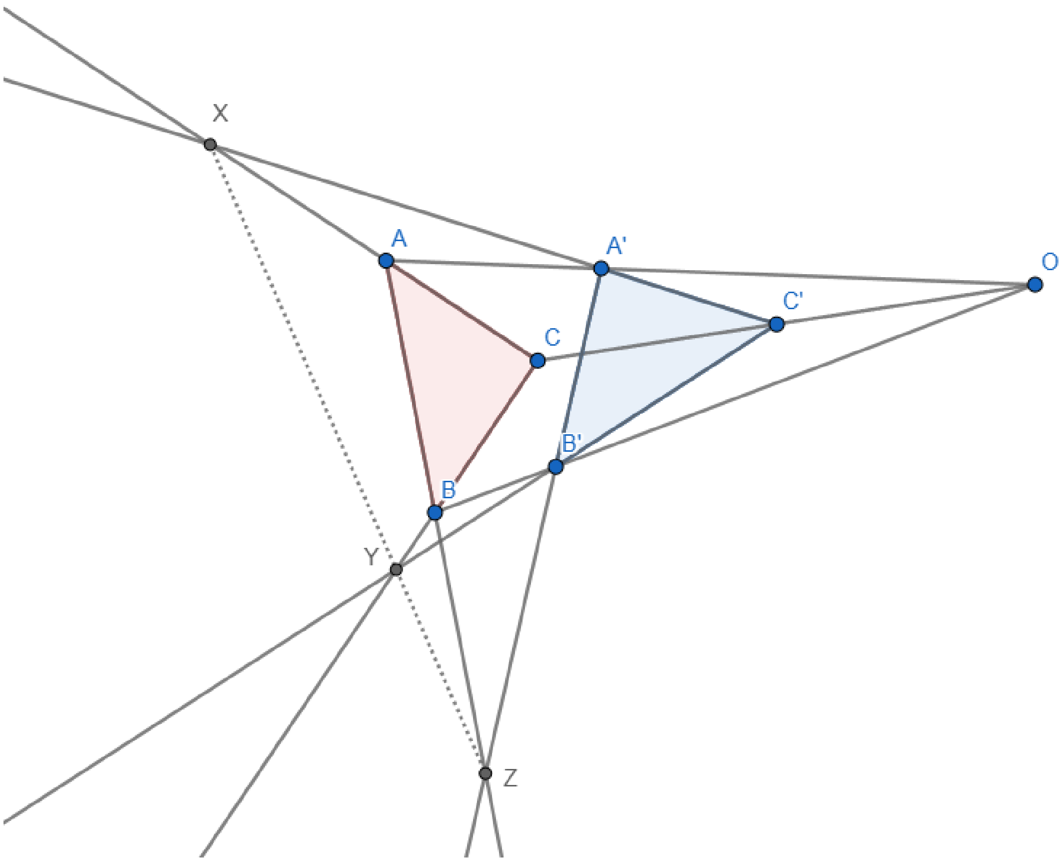

Figure 4: Desargue's theorem case 1

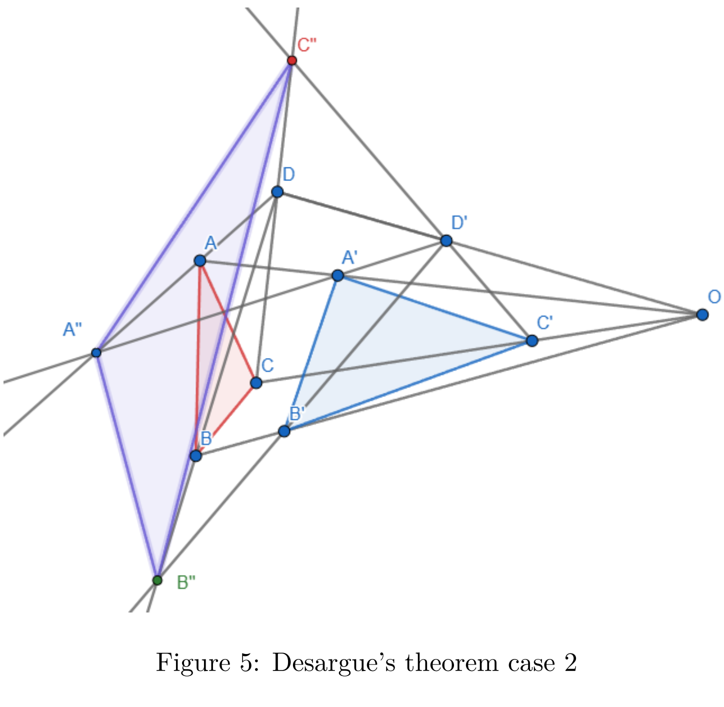

Figure 5: Desargue's theorem case 2

- Case 2: $\triangle ABC$ and $\triangle A'B'C'$ lie in the same plane say P.
  Intuitively, we want to "lift up" $\triangle A'B'C'$ to a new $\triangle A''B''C''$ that lives in a different plane from $\triangle ABC$ and apply case 1.

  WLOG, take two distinct points D, D' that passes through O and are not contained in the plane P.

  By our assumption, DD' and AA' intersect at O thus they lie in the same plane. Now by axiom 1 and 2, DA and $D'A'$ are incident with some A''.

  We see that DB and $D'B'$, DC and $D'C'$ are incident with some B'', C'' respectively by the same argument. This forms $\triangle A''B''C''$, as shown in figure 5.

  Now notice that $BC \in P \cap DBC$, $B'C' \in P \cap DBC$ by construction. Also, since DB and $D'B'$ are incident with B'' implies $B'' \in DBC \cap DB'C'$; and DC and $D'C'$ are incident with C'' implies $C'' \in DBC \cap DB'C'$; By Axiom 2 and **Lemma 2.4.1**, $B''C'' = DBC \cap DB'C'$.

  We know BC and $B'C'$ are incident with $Y \in P$, and $B''C'' \notin P$ by construction, so $B''C''$ must meet BC and $B'C'$ at $Y \in P$, otherwise we have two points of incidents, which forms a new line that incident with $B''C''$ at two points, contradicting Axiom 2.

  We repeat the same argument (which we did above for BC) for AB and CA to get that $C''A''$ must meet CA and $C'A'$ at $X \in P$ and $A''B''$ must meet AB and $A'B'$ at $Z \in P$ By construction, $\triangle A''B''C''$ is not in the same plane as $\triangle ABC$. Since the corresponding sides meet at the same points given in the questin, we repeat Case 1 for but replace $\triangle A'B'C'$ with $\triangle A''B''C''$ and the theorem is proven.

3 Extensions and Future work

Introduced a model on finite geometries allows us to use algebra to study it, which leads to a vast different directions along which we can extend our work. Following is just a glimpse of whats awaiting us:

1. A vector space model for affine planes (spaces): Similar to our treatment of projective spaces, affine spaces can be analyzed as cosets of vector spaces over finite fields. This perspective enables us to investigate properties such as the order of planes, the existence of affine spaces of specific sizes, isomorphisms between spaces, and more. Notably, this approach provides a direct way to proving 2 and 3 of **Proposition 2.1.1** by noting removing some line and all the points on it from an affine plane gives a projective plane.

2. Conics, ovals, and further topics: The study of objects in projective spaces extends beyond points and lines to encompass shapes encountered in Euclidean spaces, such as conics and ovals. These objects can be further explored using the vector space model, which connects to foundational results like Segre's theorem. Moreover, familiar concepts from linear algebra, including quadratic forms and unitary/orthogonal spaces, naturally arise within the context of finite geometry, offering new insights.

This paper merely scratches the surface of the vast and intricate world of finite geometry. The topics outlined here represent just a fraction of the potential directions for further study, each with the promise of uncovering deeper connections and insights. While it is impossible to exhaust such a rich field within the scope of a single paper, these ideas serve as a foundation for continued exploration and discovery. Until next time, we leave these topics open for future investigation.

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