Goldbach's Conjecture II: Chasing Goldbach on the Unit Circle

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Introduction

Some problems in mathematics are deceptively simple yet remarkably enduring, and Goldbach's conjecture is one of the most well-known. It states that **every even number no less than 6 can be written as the sum of two prime numbers.** It is also proposed that **Every odd number no less than 9 can be written as the sum of three prime numbers**, which is now know as Goldbach's weak conjecture. Despite centuries of effort and many numerical evidence, a complete proof remains out of reach[1]. In my article from the previous issue of this magazine [1], we explored the conjecture's historical development and highlighted some of the key breakthroughs along the way. These seemingly irregular primes consistently sum to even numbers calls for deeper analytical insight. To that end, we now turn to one of the most influential analytic technique developed in the 20th century.

Idea behind the circle method

In 1920, Hardy and Littlewood published a series of papers under the general title Some Problems of "Partitio Numerorum." In these papers, they systematically created and developed an analytic method in additive number theory. Among them, the third paper published in 1923 were specifically devoted to discussing Goldbach's conjecture [4]. The core ideas behind this new method had already appeared in a 1918 paper by Hardy and Ramanujan, which were later referred to as the Hardy–Littlewood–Ramanujan *circle method* [3]. Building on this foundation, Soviet mathematician Ivan Vinogradov made significant improvements to the method leading to major breakthroughs in expressing odd numbers as sums of primes.

Generally speaking, the circle method turns the conjecture into studying an integral along the unit circle. Consider the equations

$$N = p_1 + p_2, \quad p_1, p_2 \text{ are even primes.} \tag{1}$$

$$N = p_1 + p_2 + p_3, \quad p_1, p_2, p_3 \text{ are even primes.} \tag{2}$$

Given N, we can solve these equations for p_1 and p_2. Then to prove Goldbach's conjecture, it is to prove that equation (1) has solutions for all even $N \geq 6$. To prove Goldbach's weak conjecture, it is to prove that equation (2) has solutions for all odd $N \geq 9$. It turns out that the number of solutions can be written in an interesting form.

Theorem 1. *The number of solutions to equation (1),*

$$D(N) = \int_0^1 S^2(\alpha, N) e(-N\alpha)\ \mathrm{d}\alpha, \tag{3}$$

[1]Peruvian mathematician Harald Helfgott released a series of papers in 2013 and the following years, which are now widely accepted as a proof of Goldbach's weak conjecture. [6][7][8]

and the number of solutions to equation (2)

$$T(N) = \int_0^1 S^3(\alpha, N) e(-N\alpha) \, \mathrm{d}\alpha \tag{4}$$

where $e(x) = e^{2\pi i x}$*,* $S(\alpha, N) = \sum_{2<p\le N} e(\alpha p)$*,* p *is a prime.*

Proof. By direct expansion we have:

$$\begin{aligned}
D(N) &= \int_0^1 S^2(\alpha, N) e(-N\alpha) \, \mathrm{d}\alpha \\
&= \int_0^1 \left(\sum_{2<p\le N} e(\alpha p) \right)^2 e(-N\alpha) \, \mathrm{d}\alpha \\
&= \int_0^1 \left(\sum_{2<p_1,p_2\le N} e(\alpha(p_1+p_2)) \right) e(-N\alpha) \, \mathrm{d}\alpha \\
&= \int_0^1 \left(\sum_{2<p_1,p_2\le N} e(\alpha(p_1+p_2-N)) \right) \mathrm{d}\alpha \\
&= \sum_{2<p_1,p_2\le N} \int_0^1 e(\alpha(p_1+p_2-N)) \, \mathrm{d}\alpha \qquad (*)
\end{aligned}$$

Recall that for any $n \in \mathbb{Z}$ we have the orthogonality relation

$$\int_0^1 e(nx) \, \mathrm{d}x = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

Every integral in $(*)$ evaluates to 0 if $N \neq p_1 + p_2$, and to 1 if $N = p_1 + p_2$, so the integral in (3) gives the number of pairs of primes $\{p_1, p_2\}$ that solves (1). Similarly, we can show that the integral in (4) gives the number of solutions to equation (2). □

We can restate the conjectures as follows.

Definition 1 (Goldbach's conjecture)**.** *For all even* $N \geq 6$*,* $D(N) > 0$*.*

Definition 2 (Goldbach's weak conjecture)**.** *For all odd* $N \geq 9$*,* $T(N) > 0$*.*

Hence, the problem of both conjectures is reduced to the analysis of the integrals in (3) and (4). Of course, that would require us to study the trigonometric sum (over prime numbers) $S(\alpha, N)$.

Hardy and Littlewood believed that on certain small intervals (centered at irreducible fractions[2]), $S(\alpha, N)$ attains larger values, meaning that most of the contributions to $D(N)$ and $T(N)$ come from these intervals. Outside of them, the integrands are comparatively small and thus the contribution to the integrals is insignificant. Based on this observation, they proposed dividing the interval of integration into what we call the major and minor arcs[2]. To approach the conjecture, we first identify the major and minor arcs, then compute the corresponding integrals over them. The goal is to show that the contribution from the minor arcs is negligible compared to that of the major arcs.

Let us introduce some technical details. Let $Q, r > 0$ with $1 \leq Q \leq r \leq N$. To capture the idea of irreducible fractions, consider the Farey sequence[3] of order Q,

$$F_Q = \left\{ \frac{a}{b} \;\middle|\; 0 \leq a \leq b, \ 1 \leq b \leq Q, \ \gcd(a, b) = 1 \right\}, \tag{5}$$

[2] We will define these terms more precisely later.

[3] A Farey sequence of order $n \in \mathbb{Z}^+$ is the set of irreducible fractions $\frac{a}{b}$ arranged in increasing order, with $0 \leq a \leq b$ and $1 \leq b \leq n$.

and the corresponding set of intervals centered at $\dfrac{a}{b}$,

$$I(a,b) = \left[\frac{a}{b} - \frac{1}{r}, \frac{a}{b} + \frac{1}{r}\right], \quad a < b.^4 \tag{6}$$

These intervals will be used to construct major and minor arcs mentioned earlier. We put a condition on the choice of Q and r to give $I(a,b)$ some nice properties.

Theorem 2. *If $2Q^2 < r$, then every $I(a,b)$ lies strictly inside $\left[-\dfrac{1}{r}, 1 - \dfrac{1}{r}\right]$ and they are pairwise disjoint.*

Proof. If $2Q^2 < r$ then $\dfrac{1}{Q^2} > \dfrac{2}{r}$. For any two distinct terms $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2} \in F_Q$, we have

$$\left|\frac{a_1}{b_1} - \frac{a_2}{b_2}\right| \geq \frac{1}{b_1 b_2} \geq \frac{1}{Q^2} > \frac{2}{r}.^5$$

By definition, the length of each $I(a,b)$ is $\frac{2}{r}$, the above inequality implies that they do not intersect with each other. Now by assumption $Q \geq 1$, so $2Q \leq 2Q^2 < r$. We can simply check that the endpoints of the "rightmost" and "leftmost" interval lies inside $\left[-\frac{1}{r}, 1 - \frac{1}{r}\right]$:

$$-\frac{1}{r} < \frac{1}{Q} - \frac{1}{r} \leq \frac{a}{b} - \frac{1}{r} \quad \text{and} \frac{a}{b} + \frac{1}{r} \leq \frac{Q-1}{Q} - \frac{1}{r} < 1 - \frac{1}{r}.$$

So, every $I(a,b)$ lies strictly inside $\left[-\dfrac{1}{r}, 1 - \dfrac{1}{r}\right]$. ☐

Definition 3 (Major and minor arcs[6])**.** *Define the union of these intervals*

$$E_1 = \bigcup_{1 \leq b \leq N} \bigcup_{\substack{0 \leq a < b \\ \gcd(a,b)=1}} I(a,b) \tag{7}$$

to be ***major arcs****, and define the* ***minor arcs*** *to be*

$$E_2 = \left[-\frac{1}{r}, 1 - \frac{1}{r}\right] \setminus E_1. \tag{8}$$

We have divided the interval $\left[-\frac{1}{r}, 1 - \frac{1}{r}\right]$ into two parts, E_1 and E_2, as proposed in the circle method. Note that $\left[-\frac{1}{r}, 1 - \frac{1}{r}\right]$ differs from the integration range $[0,1]$ of $D(N)$ and $T(N)$. However, this is not an issue: both $S(\alpha, N)$ and the integrands of $D(N)$ and $T(N)$ are periodic functions with period 1, so the integrals over both intervals yield the same value. For convenience, we will work with $\left[-\frac{1}{r}, 1 - \frac{1}{r}\right]$.

Definition 4. *The denominator of a fraction is* ***small*** *if it is no more than Q, and* ***large*** *otherwise. Two points are said to be* ***close*** *if they are no more than r^{-1} apart.*

We previously noted that $S(\alpha, N)$ tends to be larger on certain intervals. This behavior is not coincidental: Hardy and Littlewood predicted that $S(\alpha, N)$ is large when α is close to an irreducible fraction with a small denominator, and small when α is close to one with a large denominator. The following theorem offers an intuitive justification for the way we defined the major and minor arcs.

[4] Strictly speaking, the definition of the Farey sequence allows $a = b$, but we omit this case for technical convenience, as it has no effect on the analysis of the integral.

[5] A Farey sequence has the property that the distance between any two distinct terms $\left|\frac{a_1}{b_1} - \frac{a_2}{b_2}\right| \geq \frac{1}{b_1 b_2}$.

[6] Major arcs are sometimes called basic intervals, and minor arcs, supplementary intervals.

Lemma 3 (Dirichlet's approximation theorem[7]). *For any $\alpha, M \in \mathbb{R}$, there exists integers a, b with $\gcd(a,b) = 1$, such that $1 \le b \le M$ and $|b\alpha - a| < \dfrac{1}{M}$.*

Theorem 4. *Every point in E_1 is close to an irreducible fraction with small denominator; Every point in E_2 is close to an irreducible fraction with large denominator.*

Proof. Let $e_1 \in E_1$. By definition $e_1 \in I(a,b)$ for some $\dfrac{a}{b} \in F_Q$. Clearly e_1 is close to $\dfrac{a}{b}$, an irreducible fraction with small denominator ($b \le Q$). Now let $e_2 \in E_2$. In Lemma 3 take $\alpha = e_2,\ M = r$, then there exists $c, d \in \mathbb{Z}$ satisfying $\gcd(c,d) = 1,\ 1 \le d \le r$ such that

$$|de_2 - c| < \frac{1}{r}, \quad \text{or equivalently,} \ \left| e_2 - \frac{c}{d} \right| < \frac{1}{dr} \le \frac{1}{r}.$$

Thus, every point in E_2 is close to an irreducible fraction $\dfrac{c}{d}$. Here, if $d \le Q$ then $\dfrac{c}{d} \in F_Q$ and the above inequality would imply $e_2 \in I(c,d) \subseteq E_1$. But since $E_1 \cap E_2 = \emptyset$ by definition, this leads to a contradiction. Therefore, $\dfrac{c}{d}$ must have a large denominator. ☐

Theorem 4 shows that the major arcs E_1 are precisely the intervals where α is close to irreducible fractions with small denominators, and the minor arcs E_2 correspond to those near fractions with large denominators, exactly capturing the structure Hardy and Littlewood had in mind. We can now analyze $D(N)$ and $T(N)$ on E_1 and E_2 separately.

$$D(N) = \int_{-\frac{1}{r}}^{1-\frac{1}{r}} S^2(\alpha, N) e(-N\alpha)\ \mathrm{d}\alpha = D_1(N) + D_2(N), \tag{9}$$

where

$$D_1(N) = \int_{E_1} S^2(\alpha, N) e(-N\alpha)\ \mathrm{d}\alpha, \quad D_2(N) = \int_{E_2} S^2(\alpha, N) e(-N\alpha)\ \mathrm{d}\alpha. \tag{10}$$

Similarly,

$$T(N) = \int_{-\frac{1}{r}}^{1-\frac{1}{r}} S^3(\alpha, N) e(-N\alpha)\ \mathrm{d}\alpha = T_1(N) + T_2(N), \tag{11}$$

where $T_1(N)$, $T_2(N)$ are defined analogously.

References

[1] Y. Li, *Introduction to Goldbach's Conjecture and its History*, $U(t)$-Mathazine. **9**(2024).

[2] W. Yuan, *The Goldbach Conjecture*, World Scientific, 2002.

[3] Pan, Cheng Dong, and Pan Cheng Biao, *Goldbach Conjecture*, Science Press, 2011. [潘承洞，潘承彪，哥德巴赫猜想. 科学出版社, 2011.]

[4] G. H. Hardy, J. E. Littlewood, *Some problems of 'Partitio numerorum; III: On the expression of a number as a sum of primes*, Acta Mathematica, 44(none) 1-70 1923, DOI 10.1007/BF02403921.

[5] S. Daniel, et al. *A Numerical Verification of the Strong Goldbach Conjecture up to* 9×10^{18}, GPH - International Journal of Mathematics, vol. 06, no. 11, Global Publication House, Dec. 2023, pp. 28–37, DOI 10.5281/zenodo.10391440.

[6] H. A. Helfgott, *Minor arcs for Goldbach's problem*, arXiv:1205.5252, 2012.

[7] H. A. Helfgott, *Major arcs for Goldbach's theorem*, arXiv:1305.2897, 2013.

[8] H. A. Helfgott, *The ternary Goldbach problem*, arXiv:1501.05438, 2015.

[7] A full proof can be found on page 90 of [3]. Some versions of the theorem omit the coprime condition, but diving two numbers by their GCD always gives a coprime pair.