Discrete Mathematics

Chapter 2, Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Part 2

Reference book: Discrete Mathematics and Its Applications 8^{th} ed. (Rosen, 2019)

Sequences and Summations

Section 2.4

Sequences[1]

Definition: A *sequence(수열)* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.

The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of $f(n)$ where f is a function from {0,1,2,…..} to S. We call a_n a *term* of the sequence.

Sequences 2

Example: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1,\ a_2,\ a_3 \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar^2, \ldots, ar^n, \ldots$

where the *initial term* ***a*** and the *common ratio* ***r*** are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$

2. Let $a = 2$ and $r = 5$. Then:

$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$

3. Let $a = 6$ and $r = 1/3$. Then:

$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \left\{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\right\}$

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form: $a,\ a+d,\ a+2d, \ldots, \ a+nd, \ldots$

where the *initial term* ***a*** and the *common difference* ***d*** are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:
$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15\ldots\}$

2. Let $a = 7$ and $d = -3$:
$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7,4,1,-2,-5, \ldots\}$

3. Let $a = 1$ and $d = 2$:
$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1,3,5,7,9, \ldots\}$

Strings 1

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

Sequences of characters or bits are important in computer science.

The *empty string* is represented by λ.

The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations [1]

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots$. and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations [2]

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6.

Answer:

$f_2 = f_1 + f_0 = 1 + 0 = 1,$

$f_3 = f_2 + f_1 = 1 + 1 = 2,$

$f_4 = f_3 + f_2 = 2 + 1 = 3,$

$f_5 = f_4 + f_3 = 3 + 2 = 5,$

$f_6 = f_5 + f_4 = 5 + 3 = 8.$

Solving Recurrence Relations

Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

Such a formula is called a *closed formula*.

Iterative Solution Example[1]

Method 1: Working upward, forward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example 2

Method 2: Working downward, backward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3\,) + 3 \cdot 2 \;= a_{n-3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n-2) \;= (a_1 + 3) + 3(n-2) \;= 2 + 3(n-1)
\end{aligned}
$$

Special Integer Sequences (*opt*)

Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.

Some questions to ask?

- Are there repeated terms of the same value?
- Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
- Can you obtain a term by combining the previous terms in some way?
- Are they cycles among the terms?
- Do the terms match those of a well known sequence?

Questions on Special Integer Sequences (*opt*) 1

Example 1: Find formulae for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16

Solution: Note that the denominators are powers of 2. The sequence with $a_n = 1/2^n$ is a possible match. This is a geometric progression with $a = 1$ and $r = ½$.

Example 2: Consider 1,3,5,7,9

Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is $a_n = 2n + 1$. This is an arithmetic progression with $a = 1$ and $d = 2$.

Example 3: 1, -1, 1, -1,1

Solution: The terms alternate between 1 and -1. A possible sequence is $a_n = (-1)^n$. This is a geometric progression with $a = 1$ and $r = -1$.

Questions on Special Integer Sequences (*opt*) 2

TABLE 1 Some Useful Sequences.

nth Term	***First 10 Terms***
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,…
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,…
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,…
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,…
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,…
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,…
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,…

Summations 1

Sum of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \qquad \sum\nolimits_{j=m}^{n} a_j \qquad \sum\nolimits_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

Summations 2

More generally for a set S:

$$\sum_{j \in s} a_j$$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

If $S = \{2, 5, 7, 10\}$ then $\sum_{j \in s} a_j = a_2 + a_5 + a_7 + a_{10}$

Product Notation (*optional*)

Product of the terms $a_m, a_{m+1}, \ldots, a_n$
from the sequence $\{a_n\}$

The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod_{j=m}^{n} a_j \qquad \prod_{m \leq j \leq n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	***Closed From***
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1},\ r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Cardinality of Sets

Section 2.5

Cardinality 1

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted $|A| = |B|$,

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality $_2$

Definition: A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called *countable*. A set that is not countable is *uncountable*.

The set of real numbers **R** is an uncountable set.

When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet).

We write $|S| = \aleph_0$ and say that *S* has cardinality "aleph null."

Showing that a Set is Countable

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Showing that a Set is Countable 1

Example 1: Show that the set of positive even integers *E* is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 & 12 \end{array}$$

Then *f* is a bijection from **N** to *E* since *f* is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that *t* is an even positive integer. Then $t = 2k$ for some positive integer *k* and $f(k) = t$.

Showing that a Set is Countable 2

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

0, 1, − 1, 2, − 2, 3, − 3 ,..........

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$

The Positive Rational Numbers are Countable [1]

Definition: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.

- ¾ is a rational number
- √2 is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done.

The Positive Rational Numbers are Countable [2]

Constructing the List

First list p/q with $p + q = 2$.

Next list p/q with $p + q = 3$

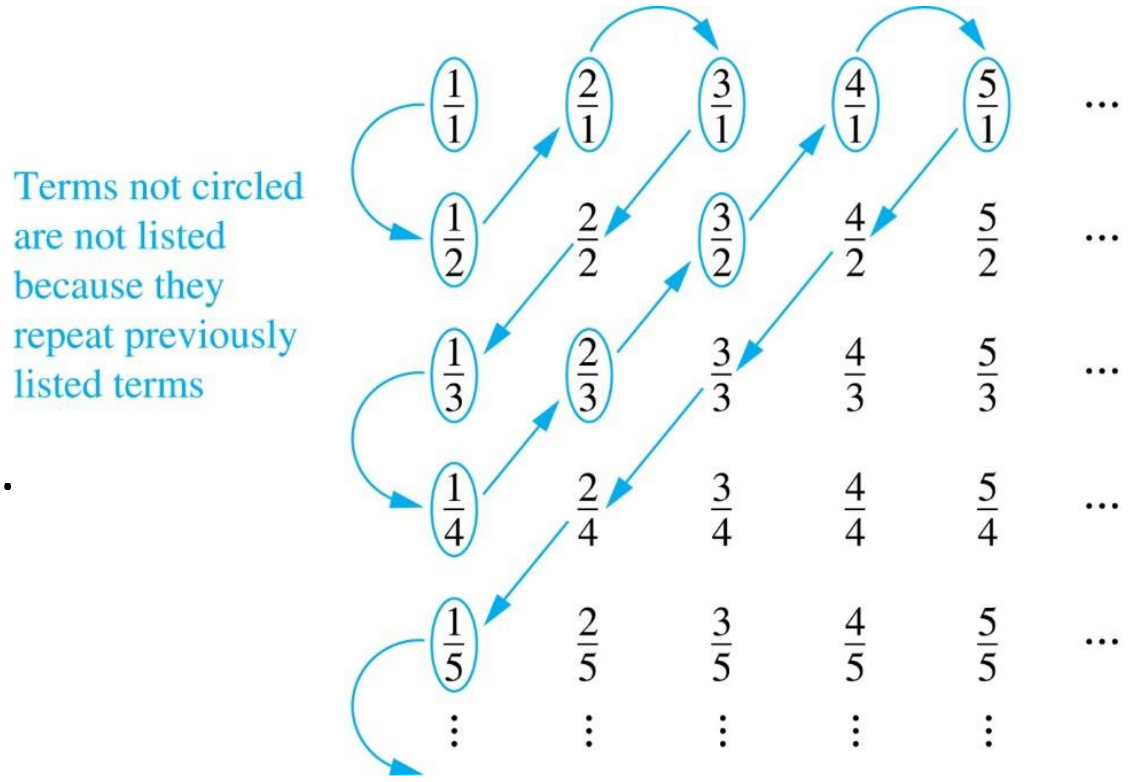
And so on.

1, ½, 2, 3, 1/3,1/4, 2/3, …

First row $q = 1$.

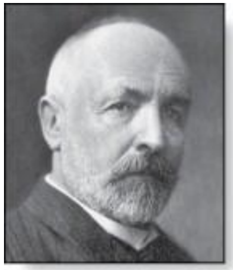
Second row $q = 2$.

etc.



The Real Numbers are Uncountable

Georg Cantor
(1845-1918)



Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagonalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be
$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots$$
$$\vdots$$
4. Form a new real number with the decimal expansion $r = .d_1d_2d_3d_4\ldots$

   where $d_i = 3$ if $d_{ii} \neq 3$ and $d_i = 4$ if $d_{ii} = 3$
5. *r* is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its *i*th position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

Matrices

Section 2.6

Matrix

Definition: A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

3×2 matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$

Notation

Let m and n be positive integers and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The ith row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2},\dots,a_{in}]$. The jth column of **A** is the $m \times 1$ *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$

The (i,j)th *element* or *entry* of **A** is the element a_{ij}. We can use **A** = $[a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

Matrix Arithmetic: Addition

Definition: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be $m \times n$ matrices. The sum of **A** and **B**, denoted by **A** + **B**, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, **A** + **B** = $[a_{ij} + b_{ij}]$.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note that matrices of different sizes can not be added.

Matrix Multiplication

Definition: Let **A** be an $m \times k$ matrix and **B** be a $k \times n$ matrix. The *product* of **A** and **B**, denoted by **AB**, is the $m \times n$ matrix that has its (i,j)th element equal to the sum of the products of the corresponding elements from the ith row of **A** and the jth column of **B**. In other words, if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

Illustration of Matrix Multiplication

The Product of **A** = [a$_{ij}$] and **B** = [b$_{ij}$]

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \dots & a_{1k} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} b_{11} & a_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix}$$

$$\mathrm{AB} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & b_{22} & \dots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$$

Identity Matrix and Powers of Matrices

Definition: The *identity matrix of order n* is the $m \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$
I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ 0 & 0 & \dots & 1 \end{bmatrix}
$$

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$$

when $\mathbf{A}$ is an $m \times n$ matrix

Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$A^0 = I_n \qquad A^r = \underbrace{AAA\cdots A}_{\text{r times}}$$

Transposes of Matrices [1]

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}^t$,is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$.

If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for i =1,2,…,n and j = 1,2, …,m.

$$\text{The transpose of the matrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is the matrix } \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Transposes of Matrices 2

Definition: A square matrix **A** is called symmetric if **A** = **A**^{t}. Thus **A** = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

$$\text{The matrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{is square.}$$

Square matrices do not change when their rows and columns are interchanged.

Zero-One Matrices [1]

Definition: A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*. (These will be used in Chapters 9 and 10.)

Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Zero-One Matrices 2

Definition: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be an $m \times n$ zero-one matrices.

- The *join* of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The *join* of **A** and **B** is denoted by **A** $\vee$ **B**.

- The *meet* of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The *meet* of **A** and **B** is denoted by **A** $\wedge$ **B**.

Joins and Meets of Zero-One Matrices

Example: Find the join and meet of the zero-one matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: The join of **A** and **B** is

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The meet of **A** and **B** is

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices [1]

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ zero-one matrix with (i,j)th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{\mathrm{i}2} \wedge b_{2\mathrm{j}}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

Boolean Product of Zero-One Matrices [2]

Example: Find the Boolean product of **A** and **B**, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$A \odot B = \begin{bmatrix} (1 \wedge 1) & \vee & (0 \wedge 0) & (1 \wedge 1) & \vee & (0 \wedge 1) & (1 \wedge 0) & \vee & (0 \wedge 1) \\ (0 \wedge 1) & \vee & (1 \wedge 0) & (0 \wedge 1) & \vee & (1 \wedge 1) & (0 \wedge 0) & \vee & (1 \wedge 1) \\ (1 \wedge 1) & \vee & (0 \wedge 0) & (1 \wedge 1) & \vee & (0 \wedge 1) & (1 \wedge 0) & \vee & (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \vee & 0 & 1 & \vee & 0 & 0 & \vee & 0 \\ 0 & \vee & 0 & 0 & \vee & 1 & 0 & \vee & 1 \\ 1 & \vee & 0 & 1 & \vee & 0 & 0 & \vee & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Powers of Zero-One Matrices [1]

Definition: Let **A** be a square zero-one matrix and let *r* be a positive integer. The *r*th Boolean power of **A** is the Boolean product of *r* factors of **A**, denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathrm{A}^{[r]} = \underbrace{\mathrm{A} \odot \mathrm{A} \odot \cdots \odot \mathrm{A}.}_{\text{r times}}$$

We define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

(The Boolean product is well defined because the Boolean product of matrices is associative.)

Boolean Powers of Zero-One Matrices 2

Example: Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find $\mathbf{A}^n$ for all positive integers n.

Solution:

$$A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[4]} = A^{[3]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A^{[n]} = A^5 \text{ for all positive integers } n \text{ with } n \geq 5$$