Discrete Mathematics

Chapter 1, Part I and II: Review

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

Propositions

A *proposition* is a declarative sentence that is either true or false.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
		Negation	*Negation*	*Conjunction*	*Disjunction*	*XOR*	*Implication*	*Biconditional*
T	T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T

From $p \rightarrow q$ we can form new conditional statements .

- $q \rightarrow p$ is the **converse** of $p \rightarrow q$
- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
- $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$

Exercise

Example Truth Table

Construct a truth table for $p \lor q \rightarrow \neg r$

p	q	r	$\neg r$	$p \lor q$	$p \lor q \rightarrow \neg r$

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\land$	2
$\lor$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \lor q \rightarrow \neg r$ is equivalent to $(p \lor q) \rightarrow \neg r$

If the intended meaning is $p \lor (q \rightarrow \neg r)$

then parentheses must be used.

Tautologies, Contradictions, and Contingencies

A *tautology* is a proposition which is always true. (Ex: $p \lor \neg p$)

A *contradiction* is a proposition which is always false.(Ex: $p \land \neg p$)

A *contingency* is a proposition which is neither a tautology nor a contradiction, such as p

Two compound propositions p and q are *logically equivalent* if $p \leftrightarrow q$ is a *tautology*. ($p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.), (ex: $\neg p \lor q \equiv p \rightarrow q$)

De Morgan's Laws

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

Constructing New Logical Equivalences

We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.

To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

Equivalence Proofs [1]

Example: Show that $\neg\left(p \vee (\neg p \wedge q)\right)$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg\left(p \vee (\neg p \wedge q)\right) \equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge \left[\neg(\neg p) \vee \neg q\right]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
$\equiv (\neg p \wedge \neg q)$	By the identity law for **F**

Exercise

Equivalence Proofs $_2$

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Propositional Satisfiability

A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true. When no such assignments exist, the compound proposition is *unsatisfiable*.

A compound proposition is unsatisfiable if and only if its negation is a tautology.

$$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$$

T to *p, q,* and *r*

$$(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$$

T to *p* and ***F*** to *q*

Exercise

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

p	*q*	*r*	*¬p*	*¬q*	*¬r*	*p ∨ ¬q*	*q ∨ ¬r*	*r ∨ ¬p*	*p ∨ q ∨ r*	*¬p ∨ ¬q ∨ ¬r*	*Q*

Discrete Mathematics

Chapter 1, Part II: Predicate Logic

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Predicate Logic

Predicate logic uses the following new features:

- Variables: *x*, *y*, *z* Predicates: *P*(*x*), *M*(*x*)
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., *P*(*x*)
  - Variables can be replaced by elements from their *domain*.
- Quantifiers
  - *Universal Quantifier,* **"**For all," symbol: ∀
  - *Existential Quantifier*, "There exists," symbol: ∃
  - ∀ and ∃ have higher precedence than all the logical operators.
  - $\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge P(x_3) \ldots$
  - $\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee P(x_3) \ldots$

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.

2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.

3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Translating from English to Logic [1]

Example 1: Translate the following sentence into predicate logic: “Every student in this class has taken a course in Java.”

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(*x*) denoting “x has taken a course in Java” and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting “x is a student in this class” and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \land J(x))$ is not correct. What does it mean?

Translating from English to Logic $_2$

Example 2: Translate the following sentence into predicate logic: “Some student in this class has taken a course in Java.”

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\, (S(x) \wedge J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Equivalences and Negation

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value

- The notation $S \equiv T$ indicates that S and T are logically equivalent. (**Ex**: $\forall x\, \neg\neg S(x) \equiv \forall x\, S(x)$)

De Morgan's Laws for Quantifiers

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an addictive inverse" is

$$\forall x\, \exists y(x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x\, \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is
$\exists y\, P(x, y)$ where $P(x, y)$ *is* $(x + y = 0)$

Quantifications of Two Variables

Statement	**When True?**	**When False**
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x, y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x, y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f " and $Q(f,a)$ be "f is a flight on a ."

2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.

3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P(w,f) \wedge Q(f,a) \right)$$

Negating Nested Quantifiers

"There is a woman who has taken a flight on every airline in the world."

Example 1: Recall the previous logical expression:

$$\exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$
2. $\forall w \neg \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f \left(\neg P(w, f) \vee \neg Q(f, a) \right)$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Questions on Order of Quantifiers [1]

Let U be the real numbers,

Define $P(x,y)$: $x \cdot y = 0$

Define $Q(x,y)$: $x / y = 1$

What is the truth value of the following:

1. $\forall x\, \forall y\, P(x,y)$
2. $\forall x\, \exists y\, P(x,y)$
3. $\exists x\, \forall y\, P(x,y)$
4. $\exists x\, \exists y\, P(x,y)$
5. $\forall x\, \forall y\, Q(x,y)$
6. $\forall x\, \exists y\, Q(x,y)$
7. $\exists x\, \forall y\, Q(x,y)$
8. $\exists x\, \exists y\, Q(x,y)$

Exercise

Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: “Brothers are siblings.”

Example 2: “Siblinghood is symmetric.”

Example 3: “Everybody loves somebody.”

Example 4: “There is someone who is loved by everyone.”

Example 5: “There is someone who loves someone.”

Example 6: “Everyone loves himself”