Discrete Mathematics

Chapter 1, Part II: Predicate Logic

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

Predicates and Quantifiers

Section 1.4

Propositional Logic Not Enough

If we have:

“All men are mortal.”

“Socrates is a man.”

Does it follow that “Socrates is mortal?”

Can’t be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.

Later we’ll see how to draw inferences.

Introducing Predicate Logic

Predicate logic uses the following new features:

- Variables: x, y, z
- Predicates: $P(x)$, $M(x)$
- Quantifiers (*to be covered in a few slides*):

Propositional functions are a generalization of propositions.

- They contain variables and a predicate, e.g., $P(x)$
- Variables can be replaced by elements from their *domain*.

Propositional Functions

Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).

The statement $P(x)$ is said to be the value of the propositional function P at x.

For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

P(-3) is false.

P(0) is false.

P(3) is true.

Often the domain is denoted by U. So in this example U is the integers.

Examples of Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5)

Solution: F

R(3,4,7)

Solution: T

R(x, 3, z)

Solution: Not a Proposition

Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

Q(2,-1,3)

Solution: T

Q(3,4,7)

Solution: F

Q(x, 3, z)

Solution: Not a Proposition

Compound Expressions

Connectives from propositional logic carry over to predicate logic.

If *P(x)* denotes "$x > 0$," find these truth values:

$P(3) \vee P(-1)$ **Solution**: T

$P(3) \wedge P(-1)$ **Solution**: F

$P(3) \rightarrow P(-1)$ **Solution**: F

$P(3) \rightarrow \neg P(-1)$ **Solution**: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \wedge P(y)$

$P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

Charles Peirce (1839-1914)

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All men are Mortal."
- "Some cats do not have fur."

The two most important quantifiers are:

- *Universal Quantifier,* **"**For all," symbol: $\forall$
- *Existential Quantifier*, "There exists," symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for every x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for some x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

$\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x$ $P(x)$ is false.

2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.

3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall$ $x\, P(x)$ is false.

Existential Quantifier

$\exists x\, P(x)$ is read as "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.

2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.

3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.

2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.

3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$

$\forall x\, (P(x) \vee Q(x))$ means something different.

Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating from English to Logic [1]

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, define a propositional function J(*x*) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from English to Logic $_2$

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *U* is all people, then translate as

$$\exists x\, (S(x) \wedge J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Equivalences in Predicate Logic

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

The notation $S \equiv T$ indicates that S and T are logically equivalent.

Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Thinking about Quantifiers as Conjunctions and Disjunctions

If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \land P(2) \land P(3)$$

$$\exists x P(x) \equiv P(1) \lor P(2) \lor P(3)$$

Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions [1]

Consider $\forall x\, J(x)$

“Every student in your class has taken a course in Java.”

Here $J(x)$ is “x has taken a course in Java” and

the domain is students in your class.

Negating the original statement gives “It is not the case that every student in your class has taken Java.” This implies that “There is a student in your class who has not taken Java.”

Symbolically $\neg \forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent

Negating Quantified Expressions [2]

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg\exists xP(x)$	$\forall x\neg P(x)$	For every *x*, *P*(*x*) is false.	There is *x* for which *P*(*x*) is true.
$\neg\forall xP(x)$	$\exists x\neg P(x)$	There is an *x* for which *P* (*x*) is false.	*P*(*x*) is true for every *x*.

The reasoning in the table shows that:

$$\neg\forall xP(x) \equiv \exists x\neg P(x)$$

$$\neg\exists xP(x) \equiv \forall x\neg P(x)$$

These are important. You will use these.

System Specification Example

Predicate logic is used for specifying properties that systems must satisfy.

For example, translate into predicate logic:

- "Every mail message larger than one megabyte will be compressed."
- "If a user is active, at least one network link will be available."

Decide on predicates and domains (left implicit here) for the variables:

- Let *L*(*m*, *y*) be "Mail message *m* is larger than *y* megabytes."
- Let *C*(*m*) denote "Mail message *m* will be compressed."
- Let *A*(*u*) represent "User *u* is active."
- Let *S*(*n, x*) represent "Network link *n* is state *x*.

Now we have: $\forall m \left(L(m,1) \rightarrow C(m) \right)$

$$\exists u\ A(u) \rightarrow \exists n\ S(n,\ available)$$

Lewis Carroll Example

Charles Lutwidge Dodgson (AKA Lewis Caroll) (1832-1898)

The first two are called *premises* and the third is called the *conclusion*.

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.

1. $\forall X \; (P(X) \rightarrow Q(X))$
2. $\exists X \; (P(X) \wedge \neg R(X))$
3. $\exists X \; (Q(X) \wedge \neg R(X))$

Later we will see how to prove that the conclusion follows from the premises

Nested Quantifiers

Section 1.4

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: “Every real number has an addictive inverse” is

$$\forall x\, \exists y(x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x\, \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x\, \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x\left(C(X)\vee\exists y\left(C(y)\wedge F(X,y)\right)\right)$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x\forall y\forall z\left(\left(F(x,y)\wedge F(x,z)\wedge(y\neq z)\right)\rightarrow\neg F(y,z)\right)$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. Introduce the variables *x* and *y*, and specify the domain, to obtain:
   "For all positive integers *x* and *y*, *x* + *y* is positive."
3. The result is:

$$\forall x \forall y \left(\left(x > 0 \right) \wedge \left(y > 0 \right) \right) \rightarrow \left(x + y > 0 \right)$$

where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let *P(w,f)* be "*w* has taken *f* " and *Q*(*f,a*) be "*f* is a flight on *a* ."
2. The domain of *w* is all women, the domain of *f* is all flights, and the domain of *a* is all airlines.
3. Then the statement can be expressed as:

$$\exists w \forall a \exists f \left(P\left(w, f\right) \wedge Q\left(f, a\right)\right)$$

Negating Nested Quantifiers

Example 1: Recall the previous logical expression developed the last slide:

$$\exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:a

1. $\neg \exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$
2. $\forall w \neg \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg \left(P(w, f) \wedge Q(f, a) \right)$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f \left(\neg P(w, f) \vee \neg Q(f, a) \right)$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"