# Discrete Mathematics

Review - Chapter 2, Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Part 1

*Reference book: Discrete Mathematics and Its Applications 8$^{th}$ ed. (Rosen, 2019)*

# Sets

- A *set* is an unordered collection of objects (= elements, members).
  - $a \in A$ : $a$ is a member of the set $A$.
  - $a \notin A$ : $a$ is not a member of the set $A$

- Roaster Method
  - $V = \{a,e,i,o,u\}$ *vowels in English alphabet*
  - $O = \{1,3,5,7,9\}$ *odd positive integers less than 100*
- Set-Builder Notation
  - $S = \{x \mid x \text{ is a positive integer less than } 100\}$
  - $\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$
    *Positive rational numbers*

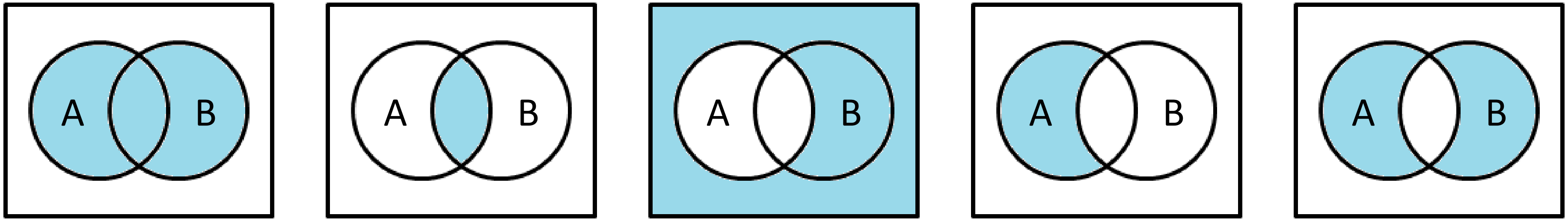
# Set properties

- Two sets are *equal* if and only if they have the same elements.
  - $\forall x(x \in A \leftrightarrow x \in B)$ or $\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$
  - $A \subseteq B$ and $B \subseteq A$
- The set $A$ is a *subset* of $B$, if and only if every element of $A$ is also an element of $B$.
  - $\forall x(x \in A \rightarrow x \in B)$ or $A \subseteq B$
  - $\emptyset \subseteq S$ and $S \subseteq S$ for every set $S$.
- If $A \subseteq B$, but $A \neq B$, then we say $A$ is a *proper subset* of $B$, denoted by $A \subset B$.
  - $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$
- The *cardinality* of a finite set $A$, denoted by $|A|$, is the number of (distinct) elements of $A$.
- *power set* $P(A)$ of $A$ is the set of all subsets of a set $A$
  - Given A = {a,b} P(A) = {ø, {a}, {b}, {a,b}}
  - |A| = 2, |P(A)| = 4

# Set operations

- Union $A \cup B = \{x | x \in A \vee x \in B\}$
- Intersection $A \cap B = \{x | x \in A \wedge x \in B\}$
- Complement $\bar{A} = \{x \in U | x \notin A\} = \mathrm{U} - \mathrm{A}$
- Difference $A - B = \{x | x \in A \wedge x \notin B\} = A \cap \overline{B}$
- Symmetric Difference $A \oplus B = (A - B) \cup (B - A)$

$A \cup B$ $A \cap B$ $\bar{A}$ $A - B$ $A \oplus B$




- Note that $|A \cup B| = |A| + |B| - |A \cap B|$

# Set Identities

| Name | Identity |
|---|---|
| Identity laws | $A \cup \emptyset = A \qquad A \cap U = A$ |
| Domination laws | $A \cup U = U \qquad A \cap \emptyset = \emptyset$ |
| Idempotent laws | $A \cup A = A \qquad A \cap A = A$ |
| Complementation laws | $\left(\overline{\bar{A}}\right) = A$ |
| Commutative laws | $A \cup B = B \cup A \qquad A \cap B = B \cap A$ |
| Associative laws | $A \cup (B \cup C) = (A \cup B) \cup C$<br>$A \cap (B \cap C) = (A \cap B) \cap C$ |
| Distributive laws | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$<br>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ |
| De Morgan's laws | $\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$ |
| Absorption laws | $A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$ |
| Complement laws | $A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$ |

# Proving Set Identities

Different ways to prove set identities:

1. **Subset method**. Prove that each set (side of the identity) is a subset of the other.

2. **Membership Tables**. Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

3. **Apply existing identities**. Start with one side, transform it into the other side using a sequence of steps by applying an established identity

*Exercise*

# Membership Table

**Example**: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

| *A* | *B* | *C* | *B∩C* | *A∪ (B∩C)* | *A∪B* | *A∪C* | *(A∪B) ∩ (A∪C)* |
|---|---|---|---|---|---|---|---|
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# Proof of Second De Morgan Law 1

**Example**: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

We prove this identity by showing that:

$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

# Proof of Second De Morgan Law 2

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

| | |
|---|---|
| $x \in \overline{A \cap B}$ | by assumption |
| $x \notin A \cap B$ | defn. of complement |
| $\neg\left(\left(x \in A\right) \wedge \left(x \in B\right)\right)$ | by defn. of intersection |
| $\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$ | 1st De Morgan law for Prop Logic |
| $x \notin A \vee x \notin B$ | defn. of negation |
| $x \in \overline{A} \vee x \in \overline{B}$ | defn. of complement |
| $x \in \overline{A} \cup \overline{B}$ | by defn. of union |

# Proof of Second De Morgan Law [3]

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

| | |
|---|---|
| $x \in \overline{A} \cup \overline{B}$ | by assumption |
| $\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$ | by defn. of union |
| $(x \notin A) \vee (x \notin B)$ | defn. of complement |
| $\neg(x \in A) \vee \neg(x \in B)$ | defn. of negation |
| $\neg\big((x \in A) \wedge (x \in B)\big)$ | 1st De Morgan law for Prop Logic |
| $\neg(x \in A \cap B)$ | defn. of intersection |
| $x \in \overline{A \cap B}$ | defn. of complement |

# Set-Builder Notation: Second De Morgan Law

$\overline{A \cap B} = x \in \overline{A \cap B}$ by defn. of complement

$= \{x \mid \neg(x \in (A \cap B))\}$ by defn. of does not belong symbol

$= \{x \mid \neg(x \in A \wedge x \in B\}$ by defn. of intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ by 1st De Morgan law for Prop Logic

$= \{x \mid x \notin A \vee x \notin B\}$ by defn. of not belong symbol

$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$ by defn. of complement

$= \{x \mid x \in \overline{A} \cup \overline{B}\}$ by defn. of union

$= \overline{A} \cup \overline{B}$ by meaning of notation

*Exercise*

Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Solution?

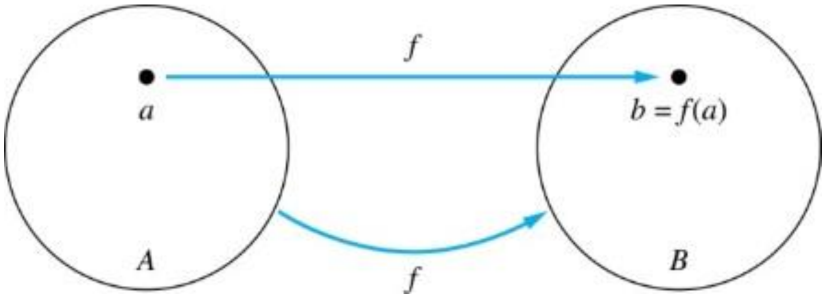
$\overline{A \cup (B \cap C)} \quad =$

# Functions[1]

A *function f* from *A* to *B*, denoted *f*: *A* → *B* is an assignment of each element of *A* to exactly one element of *B*. (Nonempty set A and B)

Mapping: $\forall x\left[x \in A \rightarrow \exists y[y \in B \land (x, y) \in f]\right]$

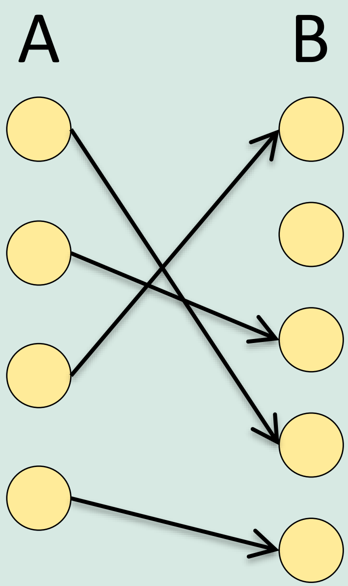
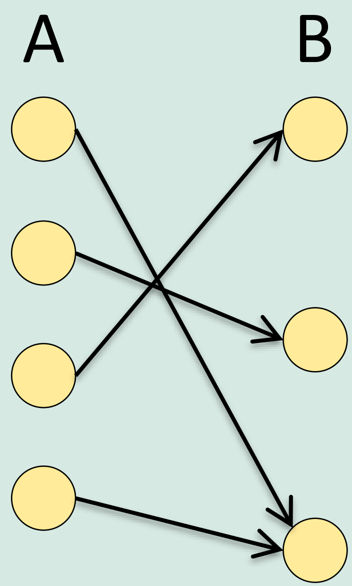
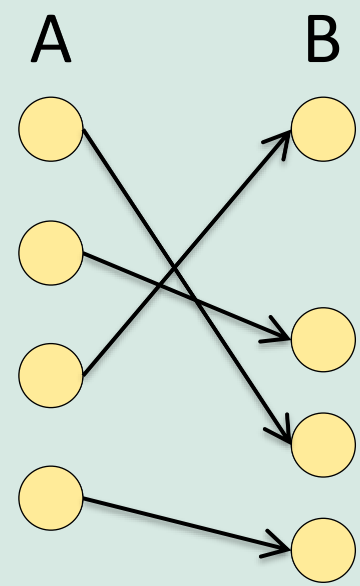
Uniqueness: $\forall x, y_1, y_2\left[[(x, y_1) \in f \land (x, y_2) \in f] \rightarrow y_1 = y_2\right]$




$\boldsymbol{A}$: domain of $f$ $\boldsymbol{B}$: codomain of $f$

$\boldsymbol{a}$: preimage of $b$ $\boldsymbol{b}$: image of $a$ under $f$

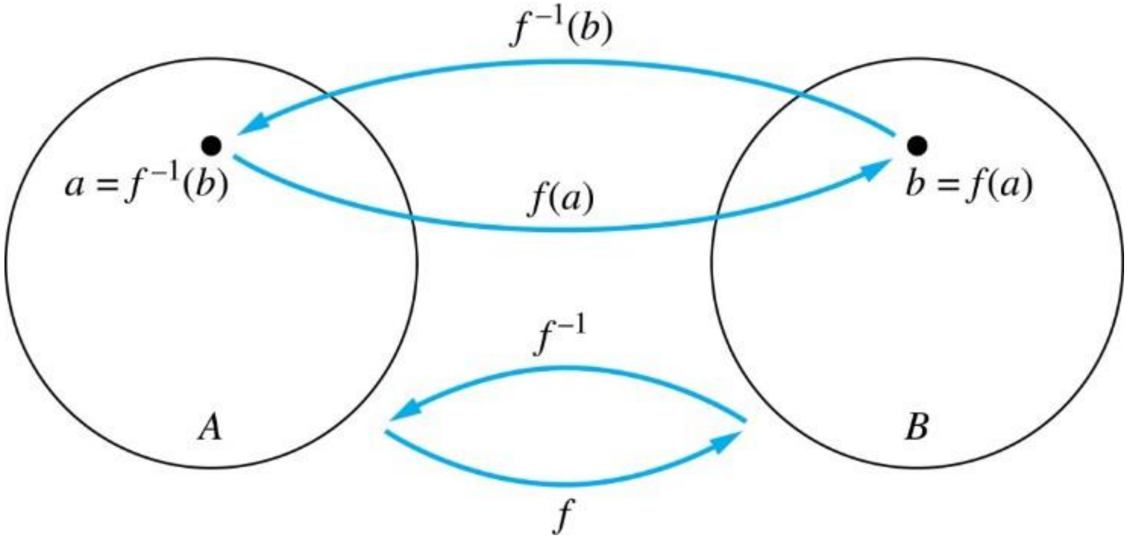
# One-to-One and Onto functions

| Injections | Surjections | Bijections |
|---|---|---|
| One-to-one | Onto | One-to-One correspondence |
| $f(a1) = f(a2)$ implies that $a1 = a2$ for all $a1$ and a2 in the domain of $f$ | for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$ | Both one-to-one and onto |
|  |  |  |

# Inverse Functions [1]

**Definition**: Let $f$ be a bijection from $A$ to $B$. Then the *inverse* of $f$, denoted $f^{-1}$ is the function from $B$ to $A$ defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless $f$ is a bijection. Why?



# Composition 1

**Definition**: Let *g*: *A*→*B* and *f*: *B*→*C*,

The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $f \circ g(x) = f\big(g(x)\big)$