Discrete Mathematics

Review-Chapter 5, Induction and recursion

Part 1

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

Mathematical Induction

- Mathematical induction
  - Basis step: verify that $P(1)$ is true.
  - Inductive step: show that $\forall k\big(P(k) \to P(k+1)\big)$ is true.

- Strong induction
  - Basis step: verify that $P(1)$ is true.
  - Inductive step: show that $\forall k\big([P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1)\big)$ is true.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Solution:

> Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$.
- INDUCTIVE STEP: Assume $P(k)$ is true for k.

The inductive hypothesis is $$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Under this assumption,

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Example problem

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Proof using Strong Induction 2

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

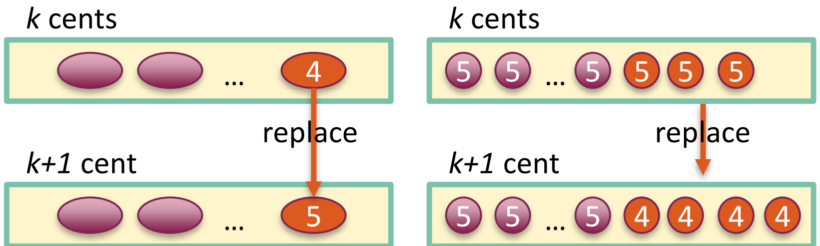
- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses three 4-cent stamps. $12 = 4 + 4 + 4$
  - $P(13)$ uses two 4-cent stamps and one 5-cent stamp. $13 = 4 + 4 + 5$
  - $P(14)$ uses one 4-cent stamp and two 5-cent stamps. $14 = 4 + 5 + 5$
  - $P(15)$ uses three 5-cent stamps. $15 = 5 + 5 + 5$
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \le j \le k$, where $k \ge 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds. => $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$
- Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \ge 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents. Hence, $P(n)$ holds for all $n \ge 12$.

Proof of Same Example using Mathematical Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer k is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show $P(k + 1)$ where $k \geq 12$, we consider two cases:
  - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of $k + 1$ cents.
  - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of $k + 1$ cents.

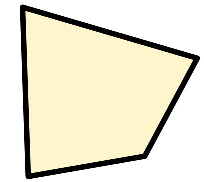


Hence, $P(n)$ holds for all $n \geq 12$.

Triangulation of Polygons

A polygon is a closed geometric figure consisting of a sequence of line segments s_1 to s_n called *sides.*

- An end point of a side is called a vertex.



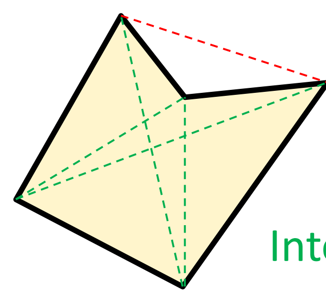
A polygon is simple when no two nonconsecutive sides intersect.

- A simple polygon divides the plane into interior and exterior.

A polygon is called *convex* if every line connecting two points in the interior lies entirely in the interior.

A diagonal is a line connecting two nonconsecutive vertices.

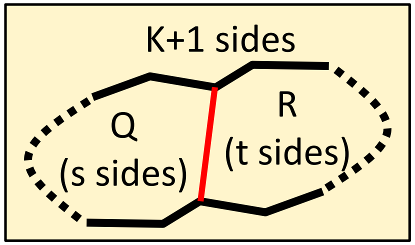
- An interior diagonal lies in the interior entirely.



Triangulation of Polygons

- Triangulation is a process to divide a polygon into triangles by adding nonintersecting diagonals
- ***Theorem.*** A simple polygon with n sides for $n \geq 3$ can be triangulated into n-2 triangles
- ***Lemma.*** A simple polygon with at least 4 sides has an interior diagonal

Proof. T(n): a simple polygon with n sides can be triangulated into n-2 triangles



- BASE STEP: $T(3)$ holds, obviously.
- INDUCTIVE STEP: $[T(3) \wedge T(4) \wedge \cdots \wedge T(k)] \rightarrow T(k+1)$?
  - By Induction hypotheses, $T(j)$ holds for $3 \leq j \leq k$.
  - By the lemma, a simple polygon with $k+1$ sides has an interior diagonal that divides the polygon into another two simple polygons Q with s sides and R with t sides. ($3 \leq s \leq k$ and $3 \leq t \leq k$ and $k+1 = s+t$-2)
  - Each of Q and R can be triangulated since the number of sides in Q or R is less than $k+1$.
  - There will be s-2 and t-2 triangles for Q and R. Thus, the original figure $(k+1)$ will have s-$2+t$-$2 = s+t$-2-$2 = (k+1)$-2 triangles.

Which Form of Induction Should Be Used?

We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction. (*See page* 335 *of text*.)

In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent. (*Exercises* 41-43)

Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Proving Inequalities$_2$

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.