Discrete Mathematics

Chapter 2, Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Part 1

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

Sets

Section 2.1

Introduction

Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.

- Important for counting.
- Programming languages have set operations.

Set theory is an important branch of mathematics.

- Many different systems of axioms have been used to develop set theory.
- Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

A *set* is an unordered collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

$S = \{a,b,c,d\}$

Order not important

$S = \{a,b,c,d\} = \{b,c,a,d\}$

Each distinct object is either a member or not; listing more than once does not change the set.

$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

Or ‘Elipses (…)’ may be used to describe a set without listing all of the members when the pattern is clear.

$S = \{a,b,c,d, \ldots\ldots,z \}$

Some Important Sets

N = *natural numbers* = {0,1,2,3....}

Z = *integers* = {...,-3,-2,-1,0,1,2,3,...}

Z$^+$ = *positive integers* = {1,2,3,.....}

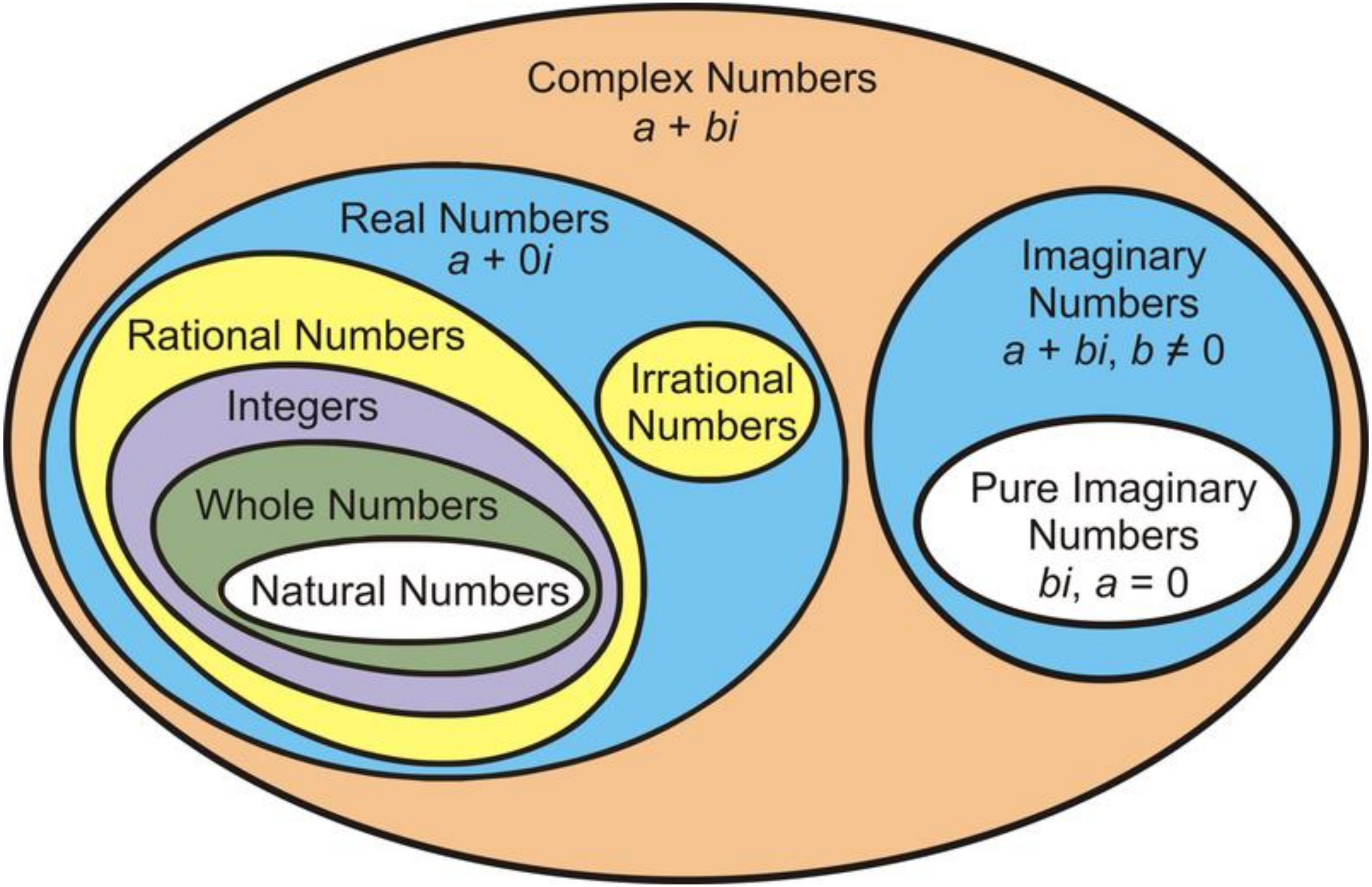
R = set of *real numbers*

R$^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Numbers



https://www.ck12.org/trigonometry/complex-numbers/lesson/Defining-Complex-Numbers-ALG-II/

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$S = \{x \mid \mathrm{P}(x)\}$

Example: $S = \{x \mid \mathrm{Prime}(x)\}$

Positive rational numbers:

$\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

$$[a, b] = \{x | a \leq x \leq b\}$$
$$[a,b) = \{x | a \leq x < b\}$$
$$(a, b] = \{x | a < x \leq b\}$$
$$(a, b) = \{x | a < x < b\}$$

closed interval [a,b]

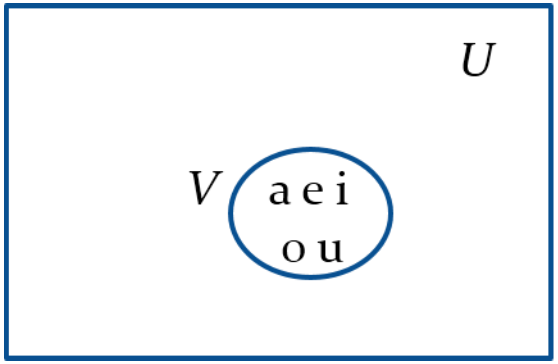
open interval (a,b)

Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

Venn Diagram



The *empty set* is the set with no elements. Symbolized *∅*, but *{}* also used.

John Venn (1834-1923)
Cambridge, UK

Some things to remember

Sets can be elements of sets.

$\{\{1,2,3\},a, \{b,c\}\}$

$\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x\left(x \in A \leftrightarrow x \in B\right)$
- We write $A = B$ if A and B are equal sets.

$\{1,3,5\} = \{3, 5, 1\}$

$\{1,5,5,5,3,3,1\} = \{1,3,5\}$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Another look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x \left(x \in A \leftrightarrow x \in B \right)$$

Using logical equivalences we have that $A = B$ iff

$$\forall x \left[\left(x \in A \rightarrow x \in B \right) \wedge \left(x \in B \rightarrow x \in A \right) \right]$$

This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

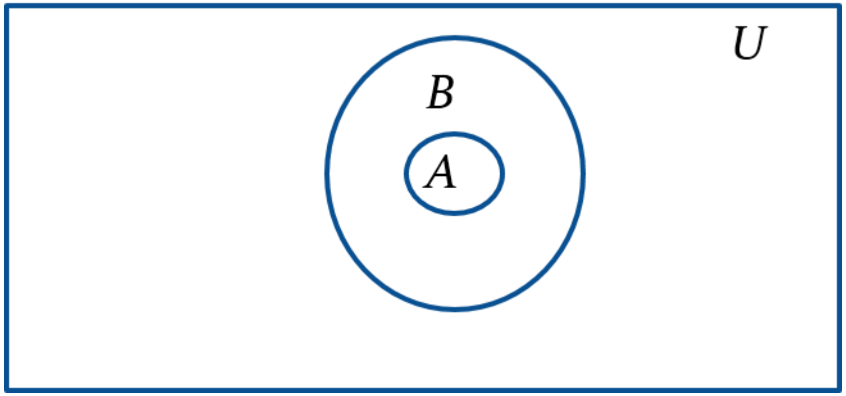
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted $P(A)$, is called the *power set* of A.

Example: If $A = \{a,b\}$ then

$$P(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

If a set has n elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product[1]

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset *R* of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product 2

Definition: The Cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for i = 1, … n.

$$A_1 \times A_2 \times \cdots \times A_n =$$

$$\{(a_1, a_2 \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\left\{ x \in D \mid P\left(x \right) \right\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set {-1,1}

Set Operations

Section 2.2

Union

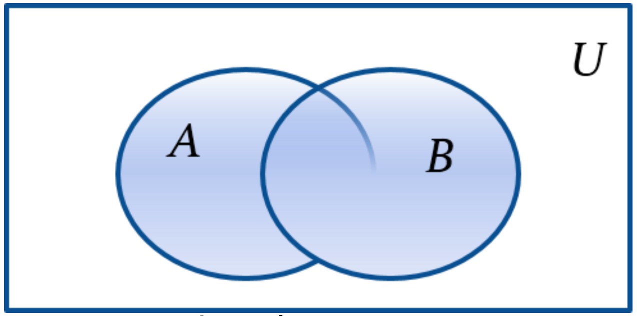
Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is {1,2,3} ∪ {3, 4, 5}?

Solution: {1,2,3,4,5}

Venn Diagram for *A* ∪ *B*



We assume that all sets are assumed to be subsets of *U (a universal set)*

Intersection

Definition: The *intersection* of sets *A* and *B*, denoted by *A* ∩ *B*, is

$$\{x | x \in A \wedge x \in B\}$$

Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

Example: What is? {1,2,3} ∩ {3,4,5} ?

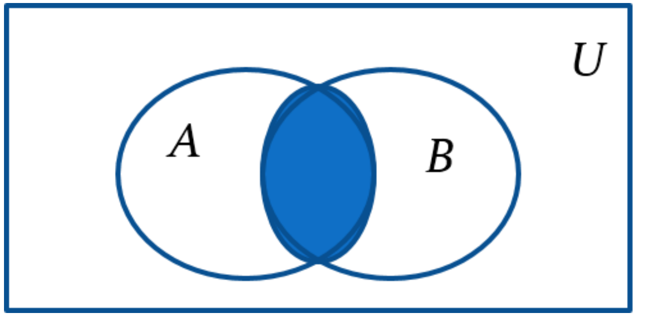
Solution: {3}

Example: What is?

{1,2,3} ∩ {4,5,6} ?

Solution: Ø

Venn Diagram for A ∩B



Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set *U - A*

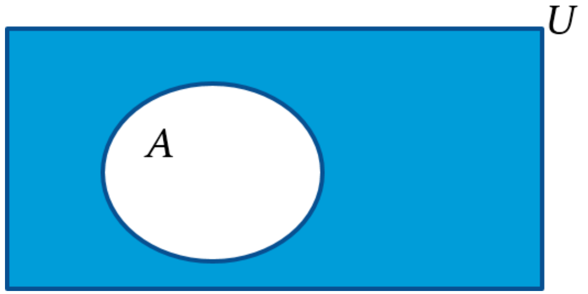
$$\bar{A} = \{x \in U | x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

$$\text{Solution}: \{x \mid x \leq 70\}$$

Venn Diagram for Complement

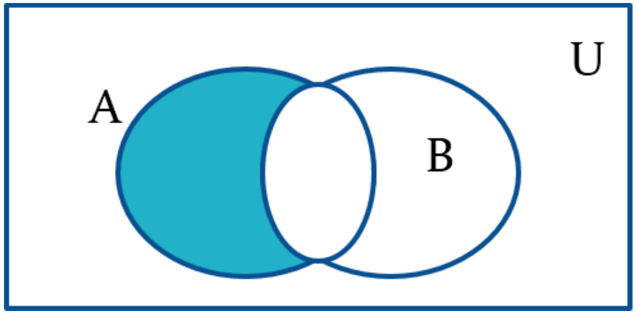


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



Set Identities [1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities 2

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities 3

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

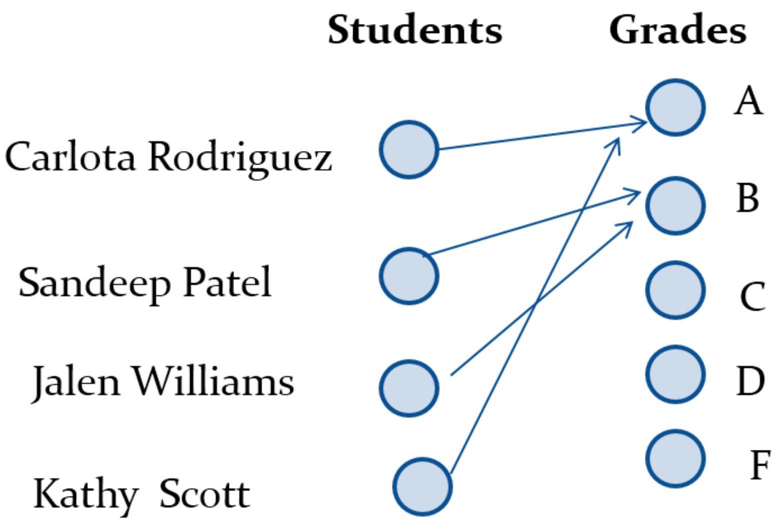
Functions

Section 2.3

Functions[1]

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions $_2$

A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation, cartesian product). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

$$\forall x\left[x \in A \rightarrow \exists y\left[y \in B \wedge (x, y) \in f\right]\right]$$

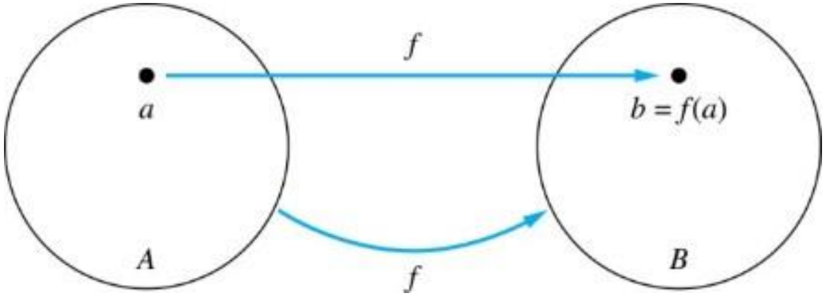
and

$$\forall x, y_1, y_2\left[\left[(x, y_1) \in f \wedge (x, y_2) \in f\right] \rightarrow y_1 = y_2\right]$$

Functions [3]

Given a function $f: A \rightarrow B$**:**

- We say *f* *maps* *A* to *B* *or f* is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.
- If $f(a) = b$,
  - then *b* is called the *image* of *a* under *f*.
  - *a* is called the *preimage* of *b.*



- The range of *f* is the set of all images of points in **A** under *f*.
  We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Questions

$f(a) = ?$ z

The image of d is ? z

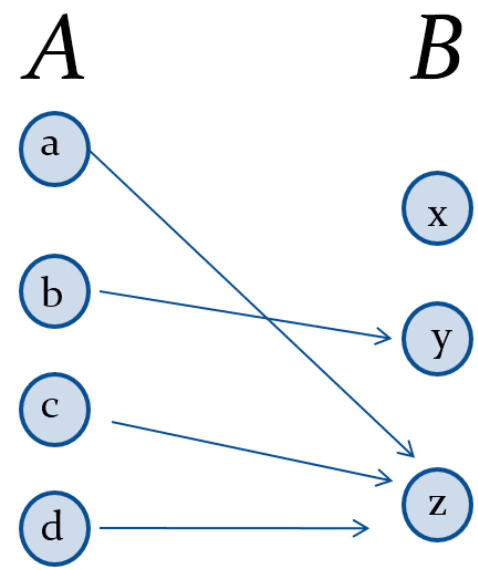
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A) = ?$ {y,z}

The preimage(s) of z is (are) ? {a,c,d}



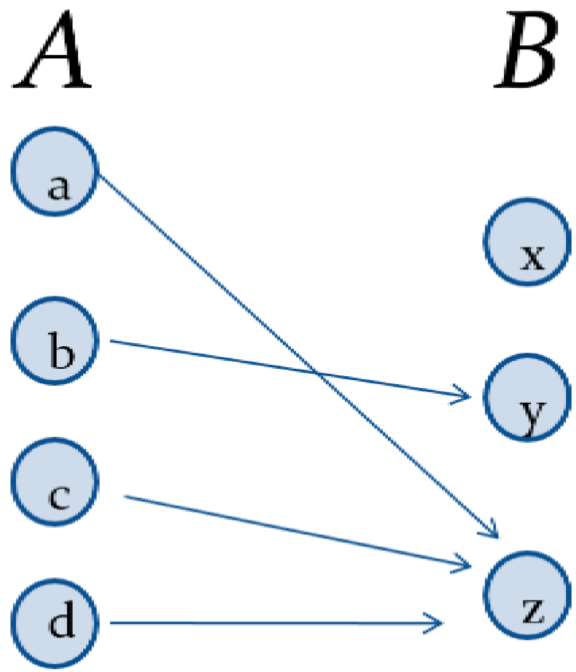
Question on Functions and Sets

If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{ f(s) \mid s \in S \}$$

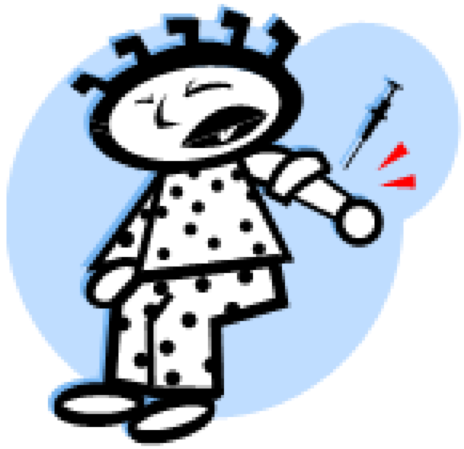
f {a,b,c,} is ? {y,z}

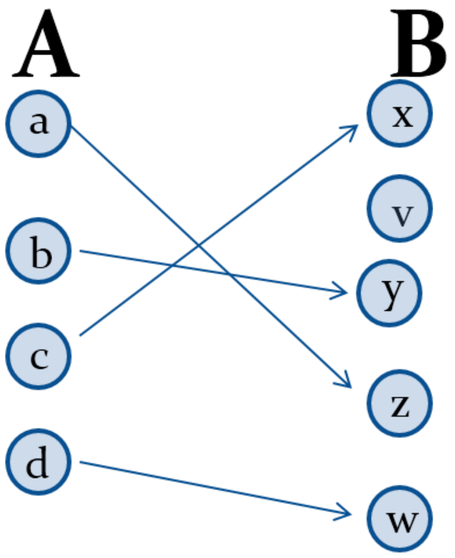
f {c,d} is ? {z}



Injections

Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

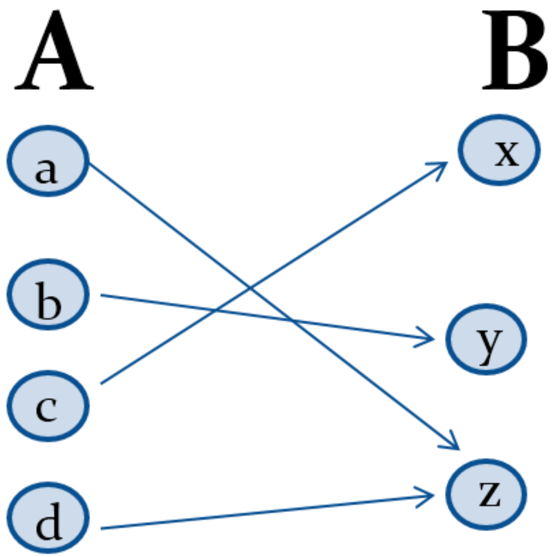




Surjections

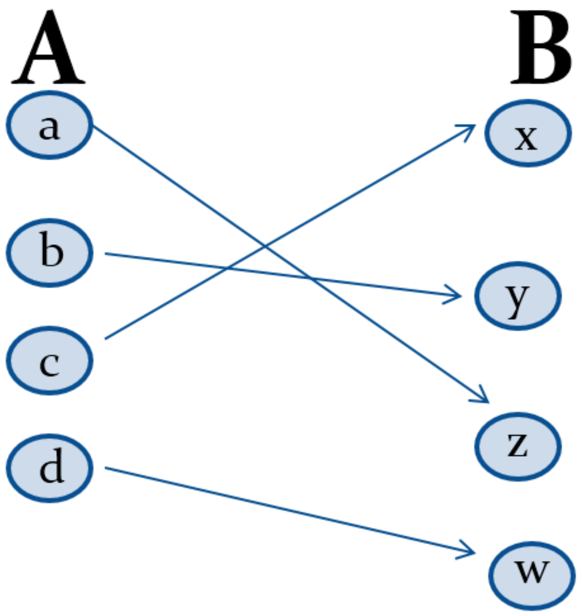
Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

A function f is called a *surjection* if it is *onto*.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto [1]

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

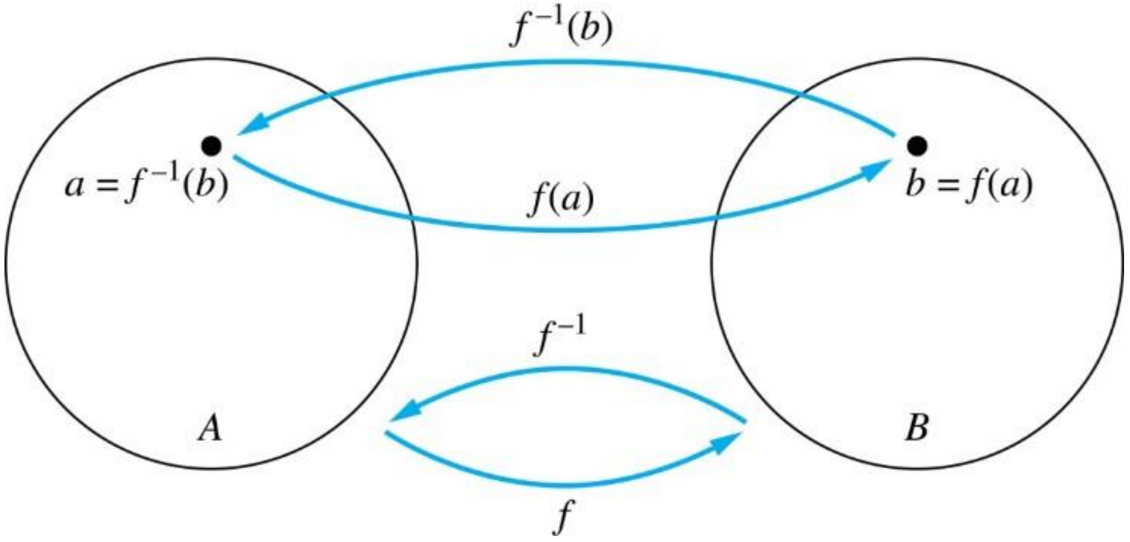
To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

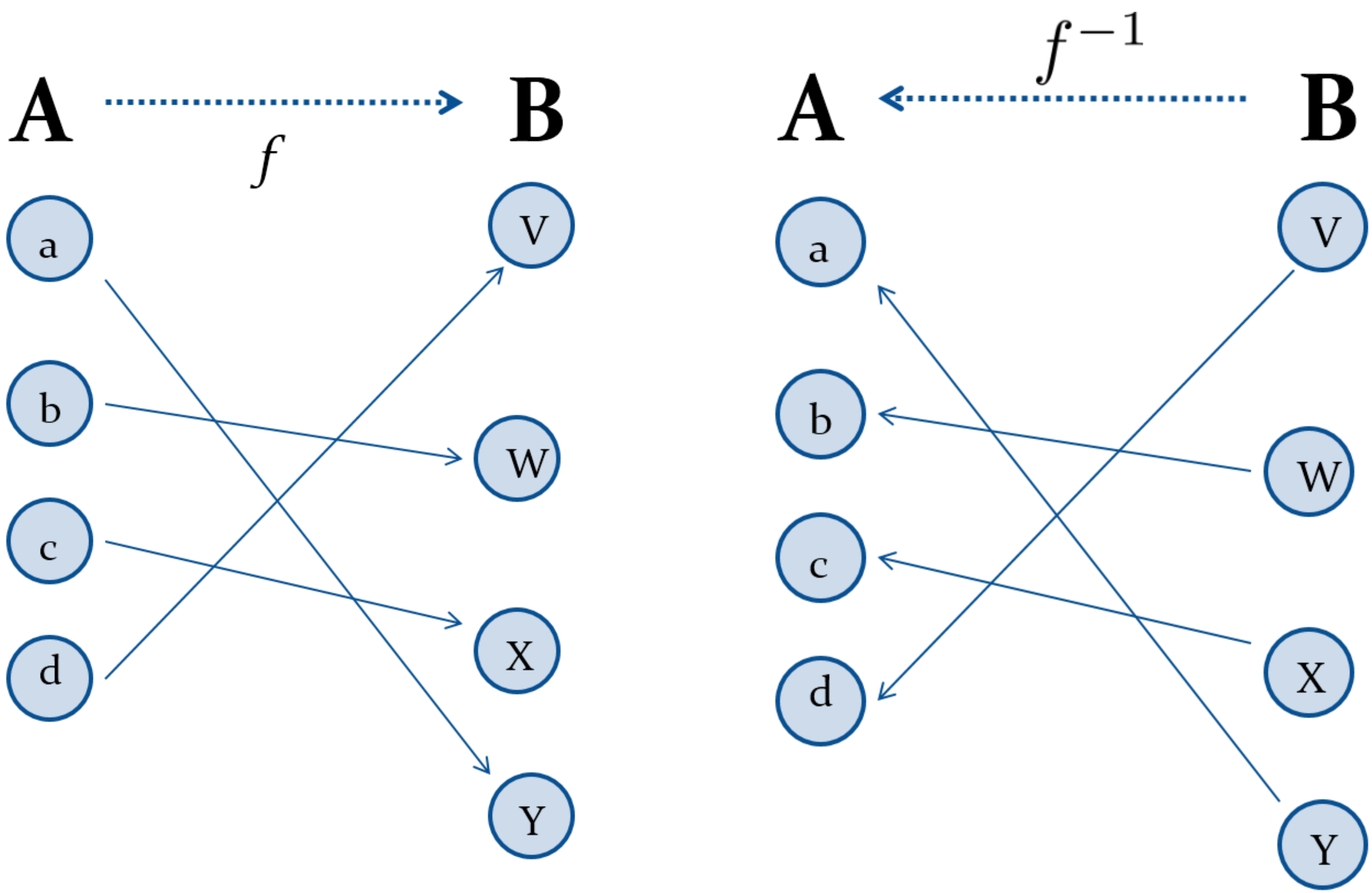
Inverse Functions 1

Definition: Let f be a bijection from *A* to *B*. Then the *inverse* of f, denoted f^{-1} is the function from *B* to *A* defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless f is a bijection. Why?



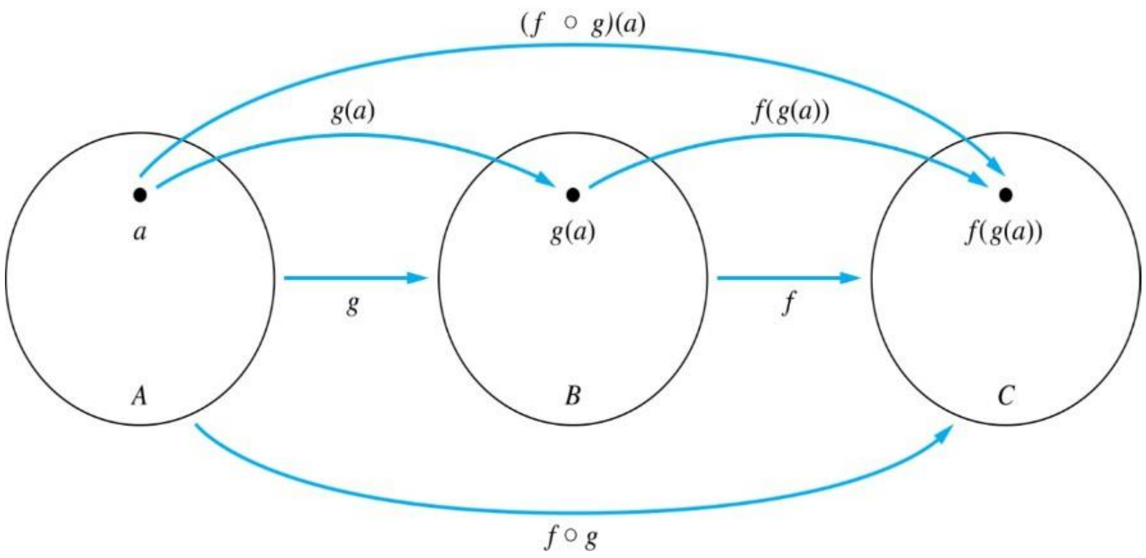
Inverse Functions [2]



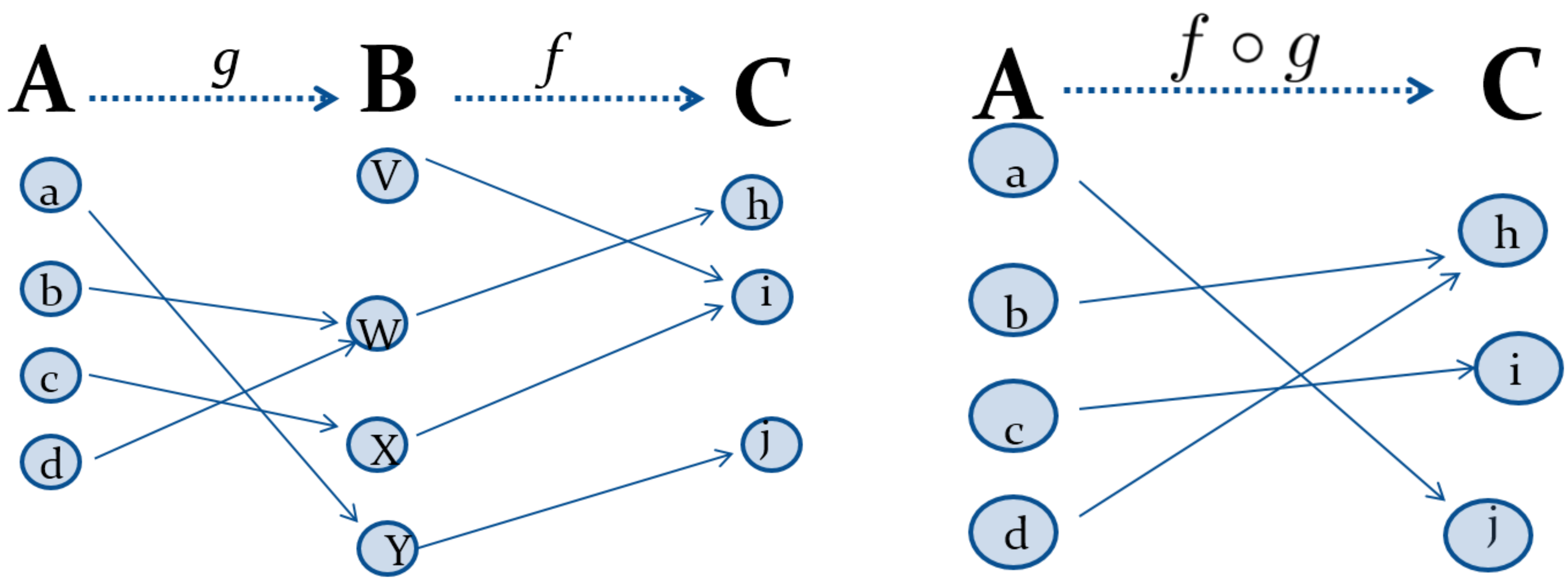
Composition[1]

Definition: Let g: $A \to B$ and f: $B \to C$,

The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $f \circ g(x) = f\big(g(x)\big)$



Composition ₂



Composition 3

Example 1: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

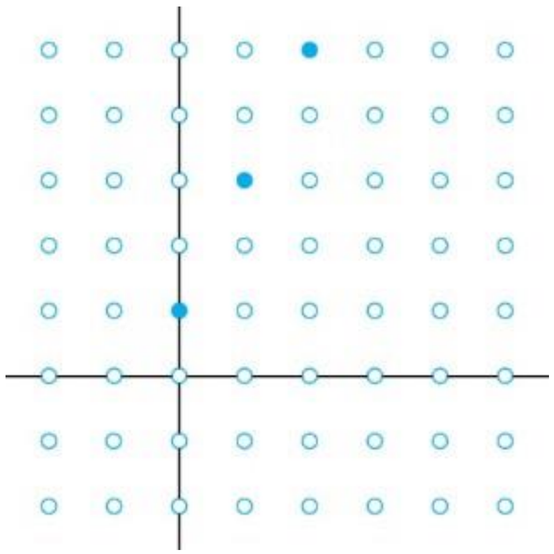
$$f(g(x)) = (2x+1)^2$$

and

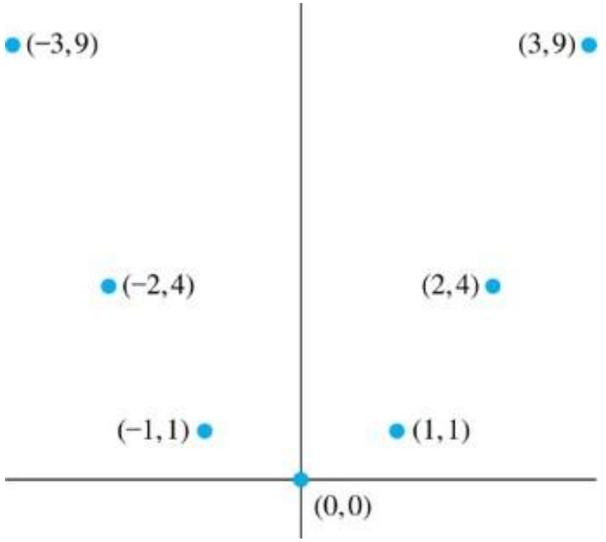
$$g(f(x)) = 2x^2 + 1$$

Graphs of Functions

Let f be a function from the set A to the set B. The *graph of the function f* is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

The *ceiling* function, denoted

$$f(x) = \lceil x \rceil$$

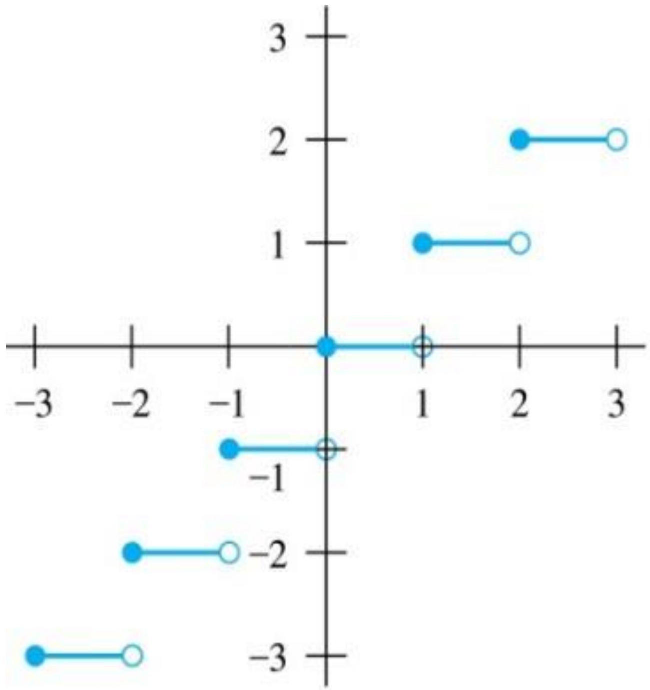
is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

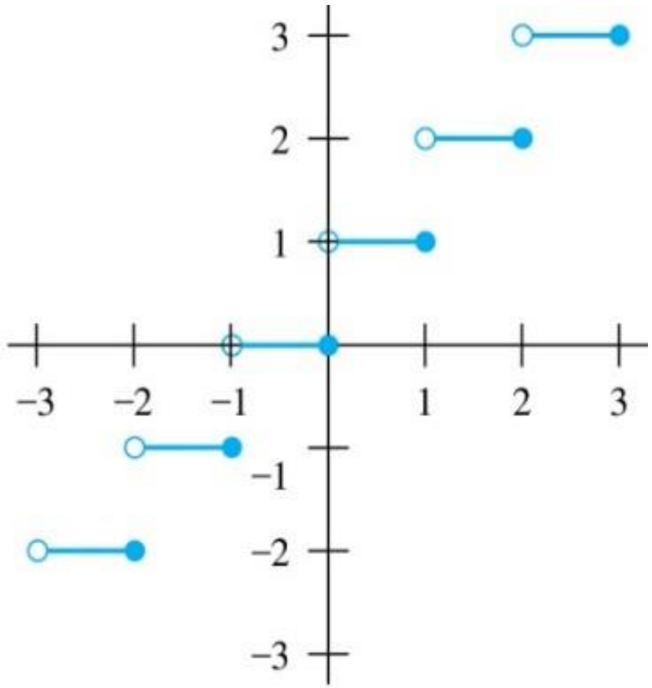
Floor and Ceiling Functions [1]

Graph of Floor function



(a) $y = [x]$

Graph of Ceiling function



(b) $y = [x]$

Floor and Ceiling Functions 2

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n = x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x = n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n = x$
(1d)	$\lceil x \rceil = n$ if and only if $x = n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Factorial Function

Definition: f: **N** $\rightarrow$ **Z**$^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Stirling's Formula:

$$\boldsymbol{n! \sim \sqrt{2\pi n}(n/e)^n}$$

With the definition of the operator $\sim$ as $f(n) \sim g(n) \doteq \lim_{n\to\infty} f(n)/g(n) = 1$ and Euler's Number e