Discrete Mathematics

Review - Chapter 6, Counting

Part 1 and 2

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

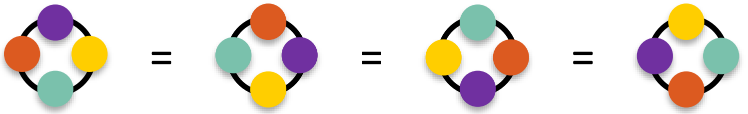
Basic Counting Principles

- **Product Rule**: A sequence of multiple tasks => $n_1 \cdot n_2$
  - In Set: $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$.
  - ex) How many bit strings of length 3 are there?
    => 2*2*2 = 8

- **Sum Rule**: One of n_1 or n_2, where none of the set of n_1 ways is the same as any of the n_2 ways => $n_1 + n_2$
  - In Set: $$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$ $$\text{when } A_i \cap A_j = \emptyset \text{ for all } i, j.$$
  - ex) How many IDs are there (ID can be a letter or 1-digit number)?
    => 26 (letters) + 10 (numbers) = 36

Basic Counting Principles

- **Subtraction Rule**: One of n_1 or n_2 ways => $n_1 + n_2$ - Common
  - In Set: $|A \cup B| = |A| + |B| - |A \cap B|$
  - ex) How many bit strings of length 8 either start with a 1 bit or end with the two bits 00?
    => 128 (=2^7) + 64(=2^6) − 32(=2^5) = 160.
- **Division Rule**: Exactly *d* of the *n* ways correspond to each *w*
  => n/d ways
  - ex) How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?
    => 4! = 24 ways but for every choice for seat 1, we get the same seating.
    => 24/4=6 n=24, d=4

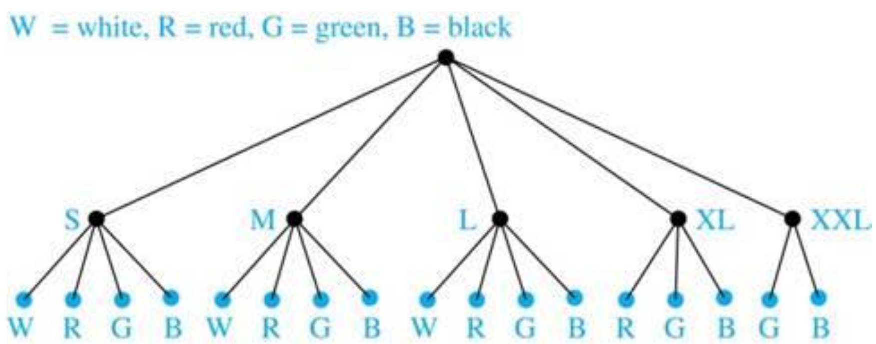


Tree Diagrams

Tree Diagrams: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice, and the leaves represent possible outcomes.

Example: Suppose that "*I Love Discrete Math*" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus bookstore needs to stock to have one of each size and color available?

Solution: Draw the tree diagram.



The store must stock 17 T-shirts.

The Pigeonhole Principle $_{1}$

- **Pigeonhole Principle**:
  If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

- **Corollary 1**:
  A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

- **The Generalized Pigeonhole Principle**:
  If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Permutations and combinations

- **Permutations**: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.
- $P(n, r) = n(n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$ for $1 \le r \le n$

- **Combinations**: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set.
- $C(n, r) = \frac{n!}{(n-r)!r!}$.
- $C(n, r) = C(n, n-r)$.

Binomial Theorem

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Corollary 1: With $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

Pascal's Identity:

If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

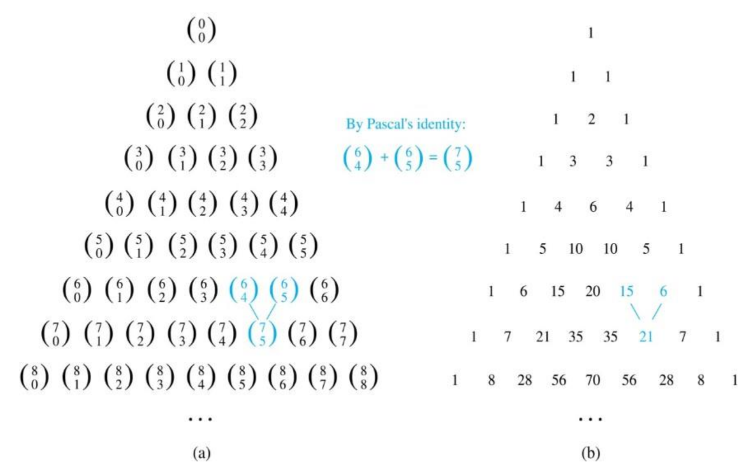
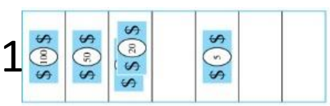
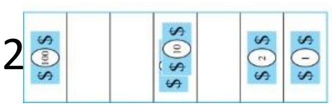


TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$P(n,r) = \dfrac{n!}{(n-r)!}$
r-combinations	No	$C(n,r) = \dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$C(n+r-1,r) = \dfrac{(n+r-1)!}{r!\,(n-1)!}$

r-combinations

case 1  * | * | * * | | * | |

case 2  * | | | * * | | * | *

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case M

Need 5 positions for 5 bills.

1	2	3	4	5	6	7	8	9	10	11

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Distributing Objects into Boxes 2

Distinguishable objects and *distinguishable boxes*.

- There are $n!/(n_1!n_2! \cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
- ex): How many ways are there to distribute hands of 5 cards to each of four players from the 52 cards?
  => 52!/(5!5!5!5!32!)

Indistinguishable objects and *distinguishable boxes*.

- There are $C(n + r - 1, r)$ ways to place r indistinguishable objects into n distinguishable boxes.
- ex) Place 10 indistinguishable objects into 8 distinguishable boxes.
  => $C(8 + 10 - 1, 10)$ = C(17,10) = 19,448

Distinguishable objects and *indistinguishable boxes*.

Indistinguishable objects and *indistinguishable boxes*.

- *No simple closed formula*

Exercise

Telephone Numbering Plan

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let *X* denote a digit from 0 through 9.
- Let *N* denote a digit from 2 through 9.
- Let *Y* denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was *NYX-NNX-XXXX*.
- In the new plan, the format is *NXX-NXX-XXXX*.

How many different telephone numbers are possible under the old plan and the new plan?

Exercise

Counting Internet Addresses

Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8 16	24 31
Class A	0	netid					hostid
Class B	1	0	netid				hostid
Class C	1	1	0	netid			hostid
Class D	1	1	1	0	Multicast Address		
Class E	1	1	1	1	0	Address	

Class A Addresses: used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.
Class B Addresses: used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.
Class C Addresses: used for the smallest networks, a 110,followed by a 21-bit netid and a 8-bit hostid.

- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0s and all 1s are not available in any network.

Example: How many different IPv4 addresses are available for computers on the internet?

Exercise

Tree Diagrams

Example: Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

Pigeonhole Principle

Example: Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.