# Discrete Mathematics

Review - Chapter 2, Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Part 2

*Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)*

# Sequences[1]

- A *sequence(수열)* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set *S*.
- A *geometric progression:* $a, ar^2, \ldots, ar^n, \ldots$
- A *arithmetic progression*: $a, a+d, a+2d, \ldots, a+nd, \ldots$
- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation.*
  => solution (or closed formula)

$a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$
$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$
$$\vdots$$
$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

# Summations [1]

Sum of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \qquad \sum\nolimits_{j=m}^{n} a_j \qquad \sum\nolimits_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

# Product Notation (*optional*)

Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$

The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \leq j \leq n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

# Some Useful Summation Formulae

**TABLE 2** Some Useful Summation Formulae.

| ***Sum*** | ***Closed From*** |
|---|---|
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1}-a}{r-1}, \ r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \lvert \mathrm{x} \rvert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=0}^{\infty} kx^{k-1}, \lvert \mathrm{x} \rvert < 1$ | $\frac{1}{(1-x)^2}$ |

# Cardinality [1]

- one-to-one from $A$ to $B$: $|A| \leq |B|$
- one-to-one correspondence from $A$ to $B$: $|A| = |B|$

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
  ex) Integer (countable), rational number (countable)
  real number (uncountable)

- A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called *countable*.

# Showing that a Set is Countable 1

**Example 1:** Show that the set of positive even integers $E$ is countable set.

**Solution**: Let $f(x) = 2x$

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
2 & 4 & 6 & 8 & 10 & 12
\end{array}
$$

Then $f$ is a bijection from **N** to $E$ since $f$ is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that $t$ is an even positive integer. Then $t = 2k$ for some positive integer $k$ and $f(k) = t$.

# The Positive Rational Numbers are Countable [1]

**Definition**: A *rational number* can be expressed as the ratio of two integers $p$ and $q$ such that $q \neq 0$ and $p, q \in \mathbf{Z}^+$

- ¾ is a rational number
- √2 is not a rational number.

**Example 3**: Show that the positive rational numbers are countable.

**Solution**: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done.

# The Positive Rational Numbers are Countable 2

**Constructing the List**

First list $p/q$ with $p + q = 2$.

Next list $p/q$ with $p + q = 3$

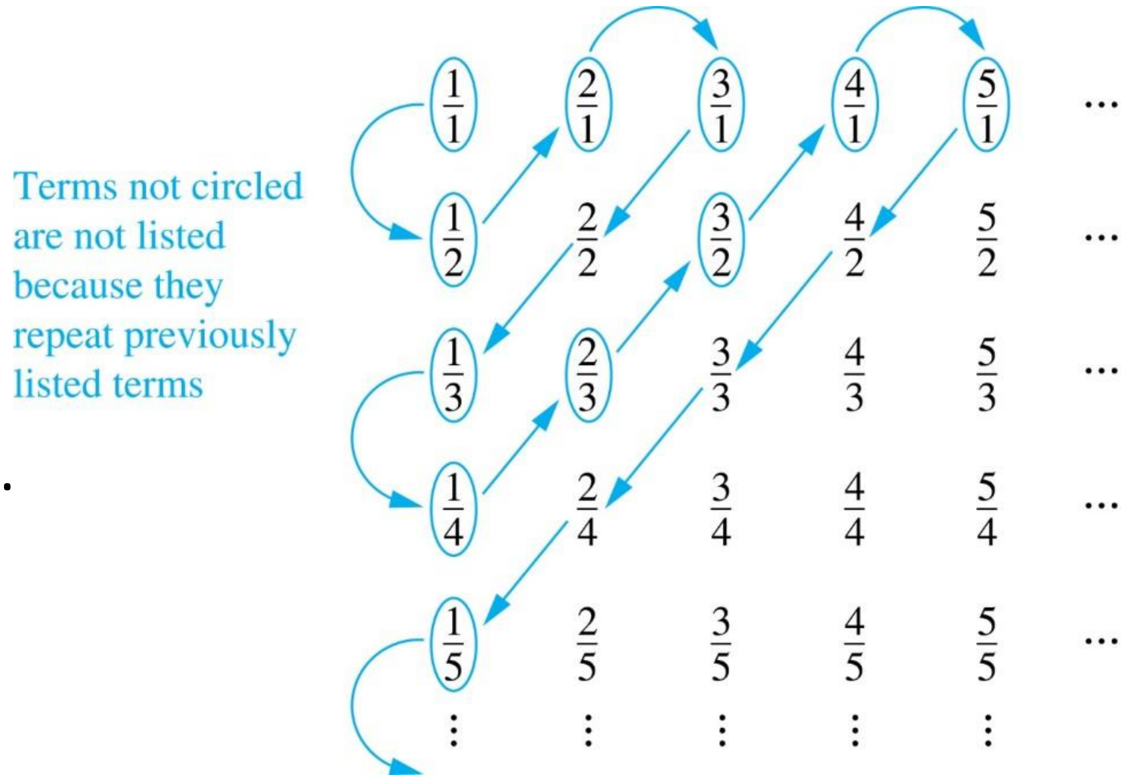
And so on.

$1, ½, 2, 3, 1/3, 1/4, 2/3, \ldots$

First row $q = 1$.

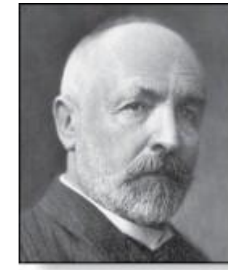
Second row $q = 2$.

etc.



# The Real Numbers are Uncountable

Georg Cantor
(1845-1918)



**Example**: Show that the set of real numbers is uncountable.

**Solution**: The method is called the Cantor diagonalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$ .
3. Let the decimal representation of this listing be
$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots$$
$$\vdots$$
4. Form a new real number with the decimal expansion $r = .d_1d_2d_3d_4\ldots$
where $d_i = 3$ if $d_{ii} \neq 3$ and $d_i = 4$ if $d_{ii} = 3$
5. *r* is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from $r_i$ in its *i*th position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.