Discrete Mathematics

Chapter 6, Counting

Part 2

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

Binomial Coefficients and Identities

Section 6.4

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from of the sums and a y from the other two. There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$.

Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$\left(2x+(-3y)\right)^{25} = \sum_{j=0}^{25}\binom{25}{j} 2x^{25-j}(-3y)^{j}.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = \frac{25!}{13!12!} 2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With *n* ≥0, $\sum_{k=0}^{n}\binom{n}{k}=2^n.$

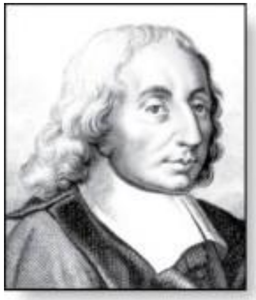
Proof (*using binomial theorem*): With *x* = 1 and *y* = 1, from the binomial theorem we see that:

$$2^n=(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^k1^{(n-k)}=\sum_{k=0}^{n}\binom{n}{k}.$$

Proof (*combinatorial*): Consider the subsets of a set with *n* elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with *n* elements. Therefore the total is $\sum_{k=0}^{n}\binom{n}{k}.$

Since, we know that a set with *n* elements has 2^n subsets, we conclude: $\sum_{k=0}^{n}\binom{n}{k}=2^n.$

Pascal's Identity



Blaise Pascal
(1623-1662)

Pascal's Identity: If *n* and *k* are integers with $n \geq k \geq 0$, then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Proof (*combinatorial*): Let *T* be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of *T* containing *k* elements.

Each of these subsets either:

- contains *a* with $k - 1$ other elements, or
- contains *k* elements of *S* and not *a*.

There are

- $\binom{n}{k-1}$ subsets of *k* elements that contain *a*, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of *S*,
- $\binom{n}{k}$ subsets of *k* elements of *T* that do not contain *a*, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Pascal’s Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\ldots
\end{array}$$

(a)

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\ldots
\end{array}$$

(b)

By Pascal’s identity, adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.

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Generalized Permutations and Combinations

Section 6.5

Permutations with Repetition

Theorem 1: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Proof: There are *n* ways to select an element of the set for each of the *r* positions in the *r*-permutation when repetition is allowed. Hence, by the product rule there are n^r *r*-permutations with repetition.

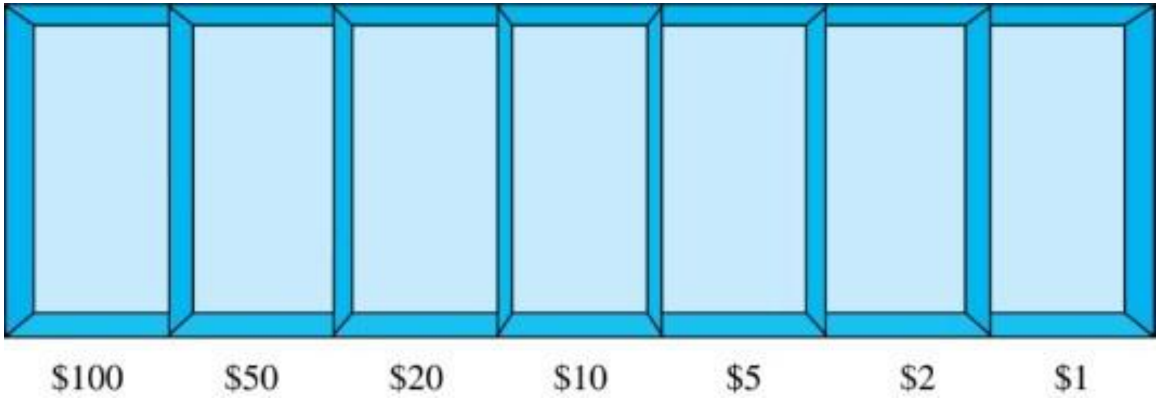
Example: How many strings of length *r* can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of *r*-permutations of a set with 26 elements.

Combinations with Repetition 1

Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

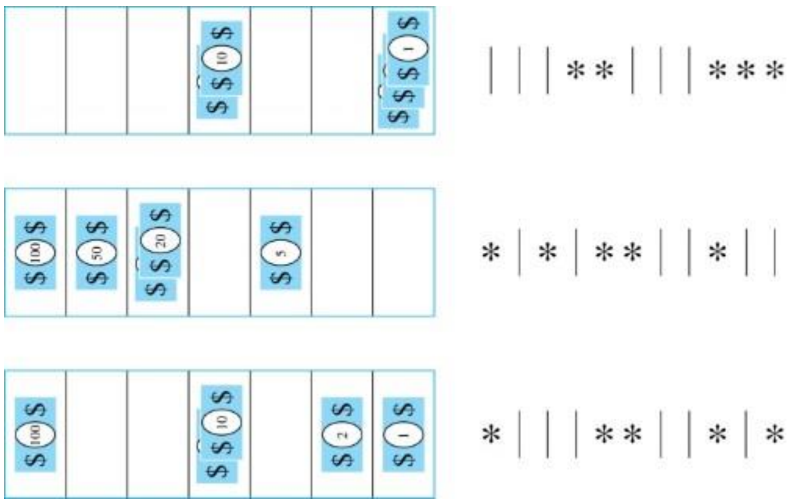
Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



Combinations with Repetition 2

Some possible ways of placing the five bills:

The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.



This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

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Combinations with Repetition

Theorem 2: The number 0f *r*-combinations from a set with *n* elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof: Each *r*-combination of a set with *n* elements with repetition allowed can be represented by a list of $n - 1$ bars and *r* stars. The bars mark the *n* cells containing a star for each time the *i*th element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the *r* positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars. This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars.

Combinations with Repetition

Example: How many solutions does the equation

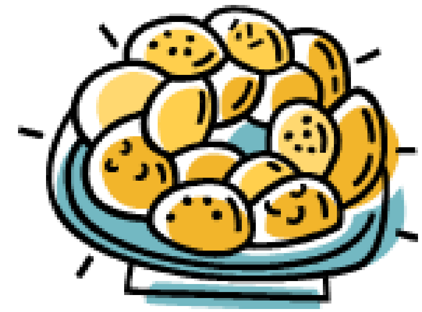
$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?
Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.
By Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13\cdot 2}{1\cdot 2} = 78$$

solutions.

Combinations with Repetition



Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	***Repetition Allowed?***	***Formula***
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Permutations with Indistinguishable Objects [1]

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!\,4!} \cdot \frac{4!}{2!\,2!} \cdot \frac{2!}{1!\,1!} \cdot \frac{1!}{1!\,0!} = 420.$$

The reasoning can be generalized to the following theorem. $\rightarrow$

Permutations with Indistinguishable Objects [2]

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\ C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\ \frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!}=\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Distributing Objects into Boxes 1

Many counting problems can be solved by counting the ways objects can be placed in boxes.

- The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
- The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes 2

Distinguishable objects and *distinguishable boxes*.

- There are $n!/(n_1!n_2! \cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
- Example: How many ways are there to distribute hands of 5 cards to each of four players from the 52 cards?
- Answer: There are 52!/(5!5!5!5!32!) ways to distribute hands of 5 cards each to four players.

Indistinguishable objects and *distinguishable boxes*.

- There are $C(n + r - 1, r)$ ways to place r indistinguishable objects into n distinguishable boxes.
- Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19{,}448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes 3

Distinguishable objects and *indistinguishable boxes*.

- Example: There are 14 ways to put four employees into three indistinguishable offices (*see Example* 10).
- There is *no simple closed formula* for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.
- See the text for a formula involving *Stirling numbers of the second kind*. (need to check chap. 8)

Indistinguishable objects and *indistinguishable boxes*.

- Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example* 11).
- The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.
- *No simple closed formula* exists for this number.