Discrete Mathematics

Chapter 5, Induction and recursion

Part 1

Reference book: Discrete Mathematics and Its Applications 8th ed. (Rosen, 2019)

Mathematical Induction

Section 5.1

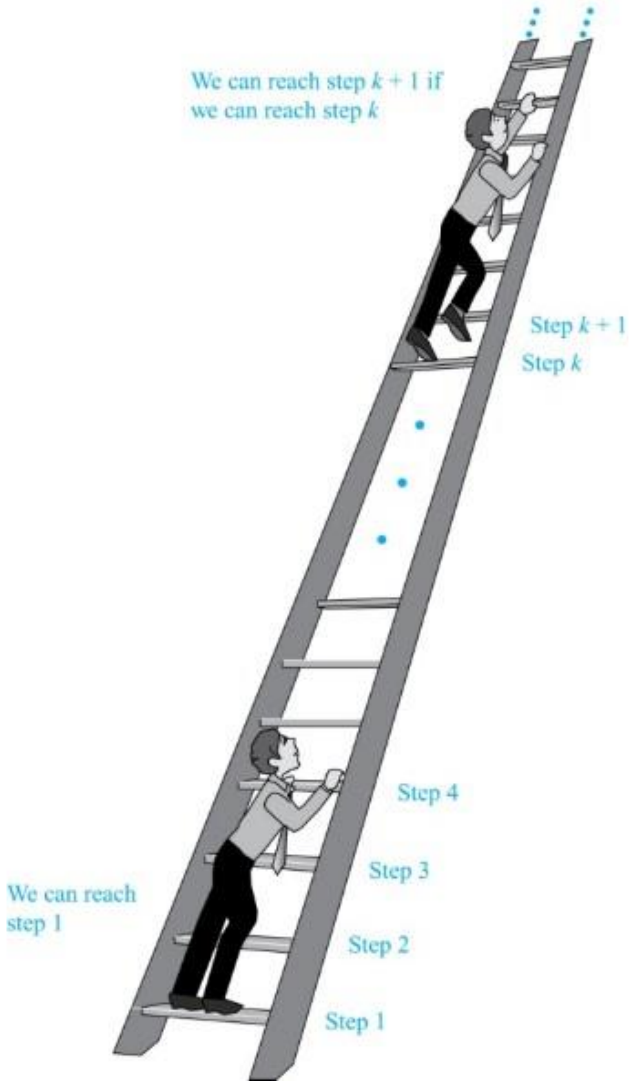
Climbing an Infinite Ladder

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.



Principle of Mathematical Induction

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \to P(k + 1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that must $P(k + 1)$ be true.

Climbing an Infinite Ladder Example:

- BASIS STEP: By (1), we can reach rung 1.
- INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k + 1$.

Hence, $P(k) \to P(k + 1)$ is true for all positive integers k. We can reach every rung on the ladder.

Important Points About Using Mathematical Induction

Mathematical induction can be expressed as the rule of inference

$$\big(P(1) \land \forall k\big(P(k) \to P(k+1)\big)\big) \to \forall n\, P(n),$$

where the domain is the set of positive integers.

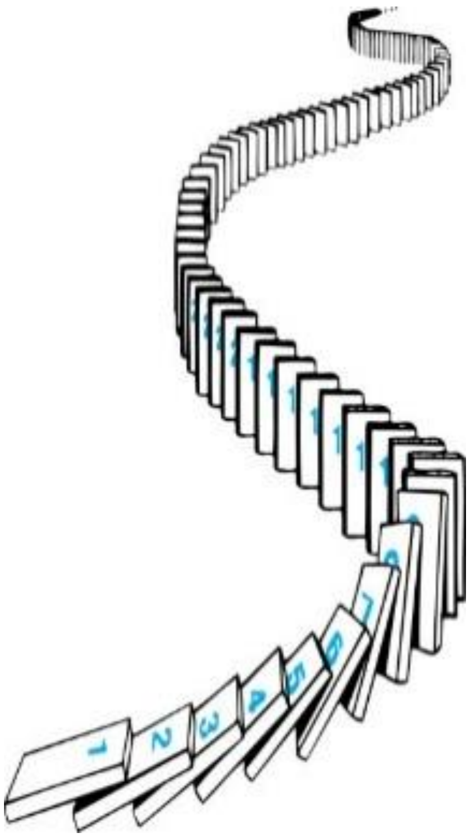
In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.

Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer. We will see examples of this soon.

Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$th domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Proving a Summation Formula by Mathematical Induction

Example: Show that: $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Solution:

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

- BASIS STEP: *P*(1) is true since 1(1 + 1)/2 = 1.
- INDUCTIVE STEP: Assume *P*(*k*) is true for *k*.

The inductive hypothesis is $$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Under this assumption,

$$\begin{aligned} 1+2+\ldots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Conjecturing and Proving Correct a Summation Formula

Example: Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture.

Solution: We have: $1 = 1,\ 1+3 = 4,\ 1+3+5 = 9,\ 1+3+5+7 = 16,\ 1+3+5+7+9 = 25.$

- We can conjecture that the sum of the first *n* positive odd integers is n^2,

$$1+3+5+\cdots+(2n-1) = n^2.$$

- We prove the conjecture with mathematical induction.
- BASIS STEP: $P(1)$ is true since $1^2 = 1$.
- INDUCTIVE STEP: $P(k) \rightarrow P(k+1)$ for every positive integer *k*.

  Assume the inductive hypothesis holds and then show that $P(k+1)$ holds has well.

Inductive Hypothesis: $1+3+5+\cdots+(2k-1) = k^2$

- So, assuming $P(k)$, it follows that:

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2k+1) &= [1+3+5+\cdots+(2k-1)]+(2k+1) \\ &= k^2+(2k+1) \ (\textit{by the inductive hypothesis}) \\ &= k^2+2k+1 \\ &= (k+1)^2 \end{aligned}$$

- Hence, we have shown that $P(k+1)$ follows from $P(k)$.
  Therefore the sum of the first *n* positive odd integers is n^2.

Proving Inequalities[1]

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

- BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.
- INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.
- Must show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ holds for all positive integers n.

Number of Subsets of a Finite Set[1]

Example: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

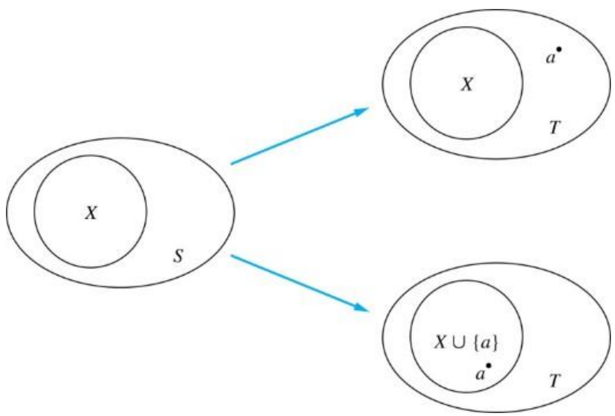
- Basis Step: $P(0)$ is true, because the empty set has only itself as a subset and $2^0 = 1$.
- Inductive Step: Assume $P(k)$ is true for an arbitrary nonnegative integer k.

Number of Subsets of a Finite Set$_2$

Inductive Hypothesis: For an arbitrary nonnegative integer k, every set with k elements has 2^k subsets.

Let T be a set with $k + 1$ elements. Then $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$. Hence $|S| = k$.

For each subset X of S, there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.



By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$.

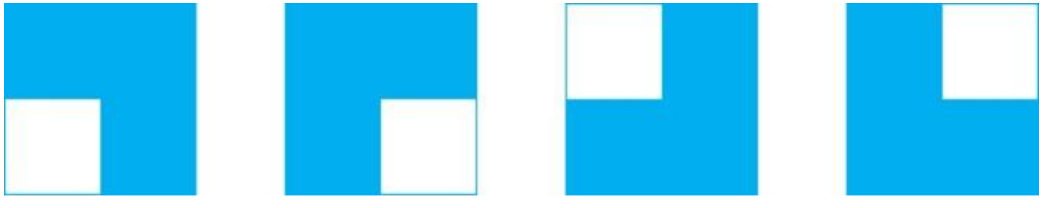
Tiling Checkerboards[1]

Example: Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.

A right triomino is an L-shaped tile which covers three squares at a time.

Solution: Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed (actually any) can be tiled using right triominoes. Use mathematical induction to prove that $P(n)$ is true for all positive integers n.

- BASIS STEP: P(1) is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino.

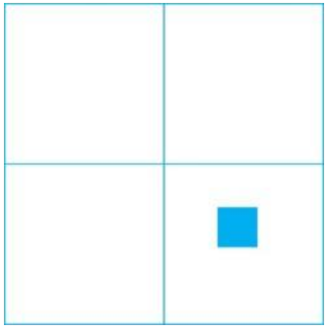


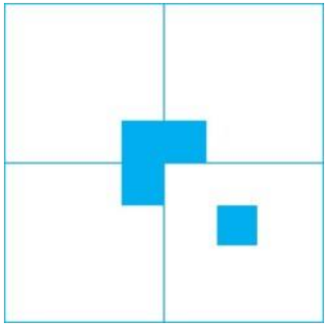
- INDUCTIVE STEP: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k.

Tiling Checkerboards$_2$

Inductive Hypothesis: Every $2^k \times 2^k$ checkerboard, for some positive integer k, with one square removed can be tiled using right triominoes.

Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions.





Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triomino.

Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

Strong Induction and Well-Ordering

Section 5.2

Strong Induction

Strong Induction: To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:

- *Basis Step*: Verify that the proposition $P(1)$ is true.
- *Inductive Step*: Show the conditional statement

  $$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$$

  holds for all positive integers k.

> Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Strong Induction and the Infinite Ladder

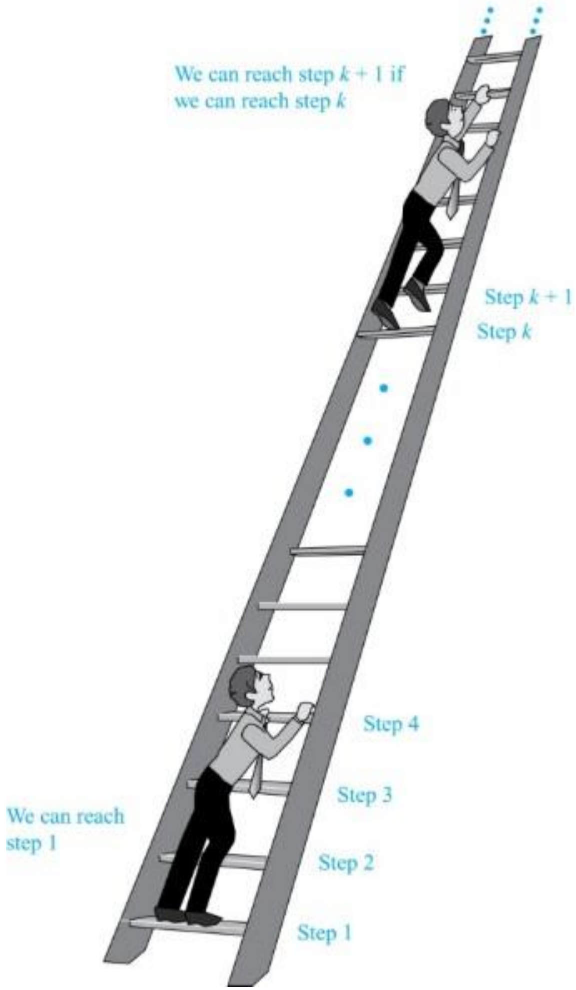
Strong induction tells us that we can reach all rungs if:

1. We can reach the first rung of the ladder.
2. For every integer k, if we can reach the first k rungs, then we can reach the $(k + 1)$th rung.

To conclude that we can reach every rung by strong induction:

- BASIS STEP: $P(1)$ holds
- INDUCTIVE STEP: Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for an arbitrary integer k, and show that $P(k + 1)$ must also hold.

We will have then shown by strong induction that for every positive integer n, $P(n)$ holds, i.e., we can reach the nth rung of the ladder.



Completion of the proof of the Fundamental Theorem of Arithmetic

Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as a product of primes.

- BASIS STEP: $P(2)$ is true since 2 itself is prime.
- INDUCTIVE STEP: The inductive hypothesis is that $P(j)$ is true for all integers j with $2 \leq j \leq k$. To show that $P(k + 1)$ must be true under this assumption, two cases need to be considered:
  - If $k + 1$ is prime, then $P(k + 1)$ is true.
  - Otherwise, $k + 1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k + 1$ can also be written as the product of those primes.

Hence, it has been shown that every integer greater than 1 can be written as the product of primes.

Proof using Strong Induction 2

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: $P(12)$, $P(13)$, $P(14)$, and $P(15)$ hold.
  - $P(12)$ uses three 4-cent stamps.
  - $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
  - $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
  - $P(15)$ uses three 5-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k + 1)$ holds. => $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$
- Using the inductive hypothesis, $P(k - 3)$ holds since $k - 3 \geq 12$. To form postage of $k + 1$ cents, add a 4-cent stamp to the postage for $k - 3$ cents. Hence, $P(n)$ holds for all $n \geq 12$.

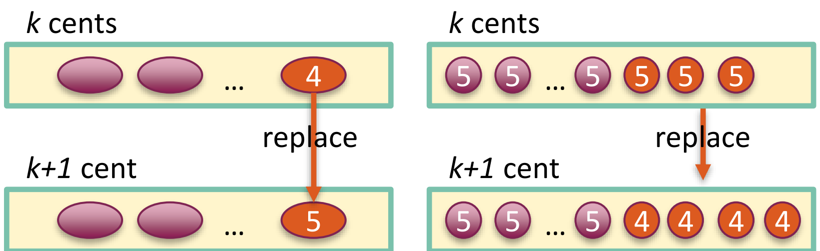
Proof of Same Example using Mathematical Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer k is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show $P(k + 1)$ where $k \geq 12$, we consider two cases:
  - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of $k + 1$ cents.
  - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of $k + 1$ cents.

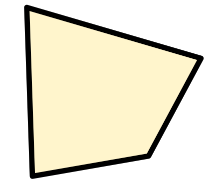
Hence, $P(n)$ holds for all $n \geq 12$.



Triangulation of Polygons

A polygon is a closed geometric figure consisting of a sequence of line segments s_1 to s_n called *sides.*

- An end point of a side is called a vertex.

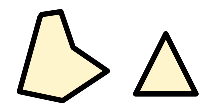


4 sides and
4 vertices

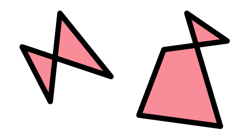
A polygon is simple when no two nonconsecutive sides intersect.

- A simple polygon divides the plane into interior and exterior.

Simple

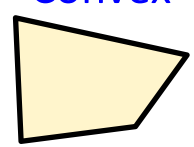


Non-Simple

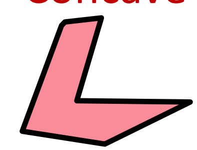


A polygon is called *convex* if every line connecting two points in the interior lies entirely in the interior.

Convex

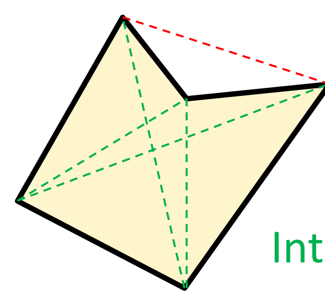


Concave



A diagonal is a line connecting two nonconsecutive vertices.

- An interior diagonal lies in the interior entirely.



Exterior Diagonal

Interior Diagonal

Triangulation of Polygons

- Triangulation is a process to divide a polygon into triangles by adding nonintersecting diagonals
- Theorem. A simple polygon with *n* sides for $n \geq 3$ can be triangulated into *n*-2 triangles
- Lemma. A simple polygon with at least 4 sides has an interior diagonal

Proof. T(n): a simple polygon with n sides can be triangulated into n-2 triangles

- BASE STEP: T(3) holds, obviously.
- INDUCTIVE STEP: $[T(3) \wedge T(4) \wedge \cdots \wedge T(k)] \rightarrow T(k+1)$?
  - By Induction hypotheses, T(j) holds for $3 \leq j \leq k$.
  - By the lemma, a simple polygon with $k+1$ sides has an interior diagonal that divides the polygon into another two simple polygons Q with s sides and R with t sides. ($3 \leq s \leq k$ and $3 \leq t \leq k$ and $k+1 = s+t-2$)
  - Each of Q and R can be triangulated since the number of sides in Q or R is less than $k+1$.
  - There will be s-2 and t-2 triangles for Q and R. Thus the original figure (k+1) will have s-2+t-2 = s+t-2-2 = (k+1)-2 triangles.