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Iterative Residual Dynamic Mode Decomposition for Prediction and Control of Dynamical Systems

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*: Equal Contribution

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Motivation
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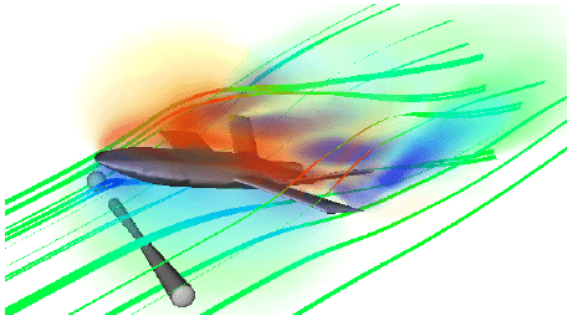
Preliminaries

Modeling is Difficult

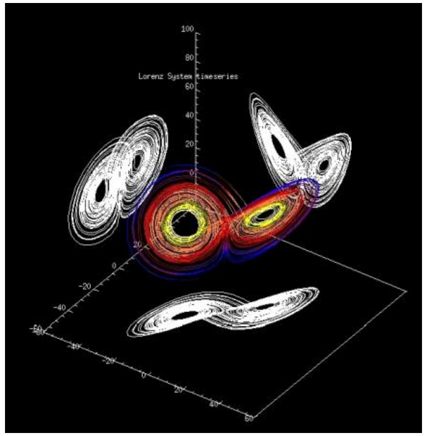
- Accurately modeling dynamical systems is one of the fundamental problems in control engineering.

- Some systems have known mathematical models. However, many real world systems…



too complex to model

no accurate model



change unpredictably

Modeling is Difficult

- Accurately modeling dynamical systems is one of the fundamental problems in the field of control engineering.
- Several systems have known mathematical models. However, many real world systems…

- Koopman operator theory + Extended dynamic mode decomposition (EDMD) [1]

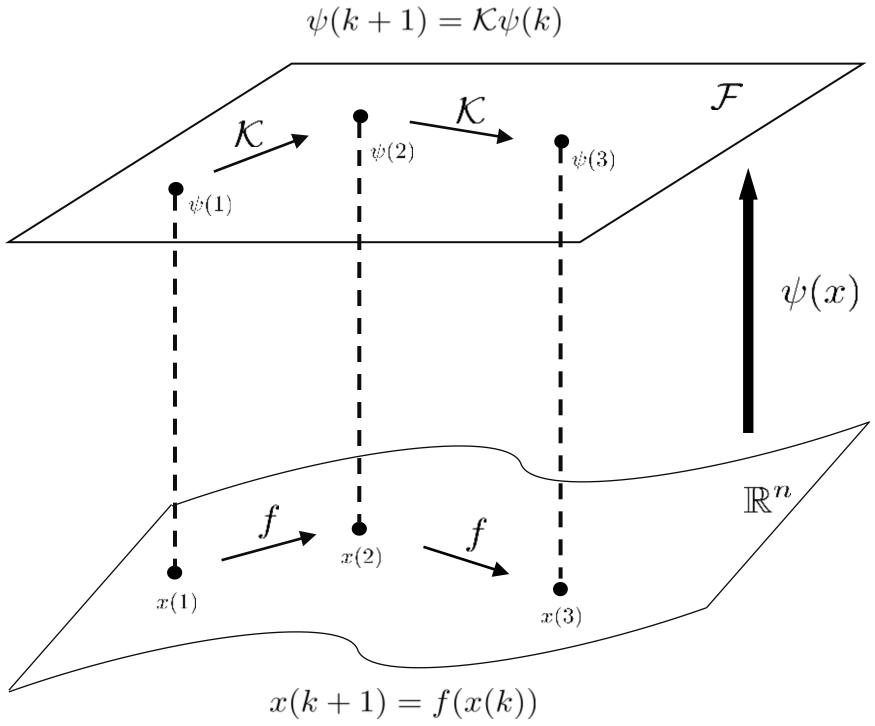
[1] Williams, Matthew O., Ioannis G. Kevrekidis, and Clarence W. Rowley. "A data–driven approximation of the koopman operator: Extending dynamic mode decomposition." *Journal of Nonlinear Science* 25.6 (2015): 1307-1346.

Koopman Operator Theory

Definition 1 (Koopman operator)**.** *Consider a discrete-time dynamical system $x(k+1) = f(x(k), u(k))$ with $u(k) \equiv 0$, i.e., the autonomous system. For the function of state space $\psi \in \mathcal{F}$ with $\psi : \mathcal{X} \to \mathbb{C}$, which is called an observable, the Koopman operator $\mathcal{K} : \mathcal{F} \to \mathcal{F}$ is defined by*

$$\mathcal{K}\psi(x(k)) = \psi \circ f(x(k)),$$

where $\mathcal{F}$ denotes a infinite-dimensional function space.

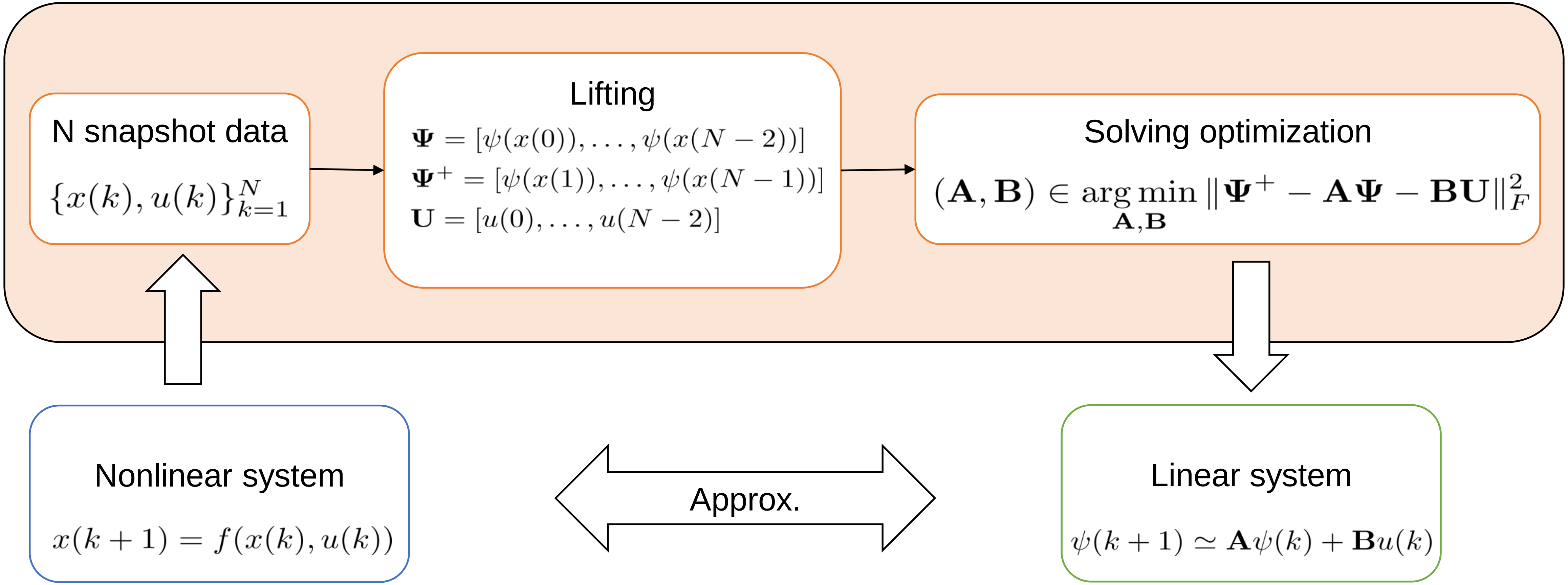


- There always exists a linear operator for every autonomous system that propagates its state on the infinite dimensional function space.

[2] Koopman, Bernard O. "Hamiltonian systems and transformation in Hilbert space." *Proceedings of the National Academy of Sciences* 17.5 (1931): 315-318.

Extended Dynamic Mode Decomposition (with Control) [3],[4]

- Infinite dimensional operator → Finite dimensional approx.



[3] Proctor, Joshua L., Steven L. Brunton, and J. Nathan Kutz. "Dynamic mode decomposition with control." SIAM Journal on Applied Dynamical Systems 15.1 (2016): 142-161.
[4] Korda, Milan, and Igor Mezić. "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control." *Automatica* 93 (2018): 149-160.

Data-Driven Modeling

- Koopman operator theory + Extended dynamic mode decomposition (EDMD)

Limitations
- Observables selection
- Curse of dimensionality

Q. Can we design a DMD algorithm that automatically selects observables while maintaining a lower computational cost?

Other DMD approaches

- **Neural network-based EDMD** [5]
  - Pros: Automatically select observables using deep neural networks.
  - Cons: Long computational time, difficult to analyze.

- **Kernel DMD** [6]
  - Pros: Implicitly captures observables by defining kernel functions.
  - Cons: Must select appropriate kernel functions.
    Computational time increases exponentially as the # of data points grows.

- **Time-delay DMD (HAVOK)** [7]
  - Pros: Automatically select observables using time-delay coordinates.
  - Cons: Quadratic growth of computational time due to Hankel matrix construction.

[5] Yeung, Enoch, Soumya Kundu, and Nathan Hodas. "Learning deep neural network representations for Koopman operators of nonlinear dynamical systems." 2019 American Control Conference (ACC). IEEE, 2019.
[6] I. Kevrekidis, C. W. Rowley, and M. Williams, “A kernel-based method for data-driven Koopman spectral analysis,” Journal of Computational Dynamics, vol. 2, no. 2, pp. 247–265, 2016.
[7] Brunton, Steven L., et al. "Chaos as an intermittently forced linear system." Nature communications 8.1 (2017): 19.

Iterative Residual Dynamic Mode Decomposition (IRDMD)

Contributions

- IRDMD automatically construct observables by iteratively performs linear approximations of the residuals that arise in basic DMD.
- For the same number of observables, IRDMD is more computationally efficient than EDMD.
- We numerically show that IRDMD captures nonlinearities more efficiently than EDMD.

Algorithm

Algorithm Overview

Nonlinear system

$$x(k+1) = f(x(k), u(k))$$

Affine form approx.

$$x(k+1) \simeq \mathbf{A}x(k) + \mathbf{B}u(k) + b$$

The first residual

$$r_1(k+1) = x(k+1) - (\mathbf{A}x(k) + \mathbf{B}u(k) + b)$$

Affine form approx.

$$r_1(k+1) \simeq \mathbf{F}_1 r_1(k) + g_1$$

The second residual

$$r_2(k+1) = r_1(k+1) - (\mathbf{F}_1 r_1(k) + g_1)$$

Affine form approx.

$$r_2(k+1) \simeq \mathbf{F}_2 r_2(k) + g_2$$

$\cdots$

Algorithm

- Consider $x(k+1) = f(x(k), u(k))$ with $\mathcal{D} = \{x(k), u(k)\}_{k=1}^{N}$.

Affine-form approximation (DMD with control [3])

$$x(k+1) \simeq \mathbf{A}x(k) + \mathbf{B}u(k) + b$$

where $$[\mathbf{A}\ \mathbf{B}\ b] = \underset{\mathbf{A},\mathbf{B},b}{\arg\min}\, \mathbb{E}_{(x,u,x^+)\sim\mathcal{D}} \|x^+ - (\mathbf{A}x + \mathbf{B}u + b)\|^2 \quad (1)$$

[3] Proctor, Joshua L., Steven L. Brunton, and J. Nathan Kutz. "Dynamic mode decomposition with control." SIAM Journal on Applied Dynamical Systems 15.1 (2016): 142-161.

Algorithm

- The first residual: $r_1(k+1) := x(k+1) - (\mathbf{A}x(k) + \mathbf{B}u(k) + b)$

$\big\downarrow$ Affine-form approximation

$$r_1(k+1) \simeq \underbrace{\mathbf{F}_{01}x(k) + \mathbf{F}_{11}r_1(k) + \mathbf{F}_{u1}u(k) + g_1}_{=:\hat{r}_1(k+1)} \quad (2)$$

where $$[\mathbf{F}_{(\cdot)}\ \ g_1] = \operatorname*{arg\,min}_{\mathbf{F}_{(\cdot)}\ \ g_1} \mathbb{E}_{(x,u,r_1,r_1^+)\sim\mathcal{D}} \|r_1^+ - (\mathbf{F}_{01}x + \mathbf{F}_{11}r_1 + \mathbf{F}_{u1}u + g_1)\|^2 \quad (3)$$

$$x(k+1) \simeq \mathbf{A}x(k) + \mathbf{B}u(k) + b + \hat{r}_1(k+1)$$: More accurate predictor!

Algorithm

- We can let $\mathbf{F}_{01}, \mathbf{F}_{u1} = 0.$

$$\because \quad [\mathbf{A}\ \mathbf{B}\ b] = \arg\min_{\mathbf{A},\mathbf{B},b} \mathbb{E}_{(x,u,x^+)\sim\mathcal{D}} \|\underbrace{x^+ - (\mathbf{A}x + \mathbf{B}u + b)}_{=r_1^+}\|^2 \quad (1)$$

: Least-square regression

r_1^+ is linearly independent of x and u (orthogonal).

$$r_1(k+1) \simeq \mathbf{F}_{01}x(k) + \mathbf{F}_{11}r_1(k) + \mathbf{F}_{u1}u(k) + g_1 \xrightarrow{\mathbf{F}_{01}, \mathbf{F}_{u1} = 0.}$$

$$\begin{aligned} r_1(k+1) &= \mathbf{F}_1 r_1(k) + g_1 + r_2(k+1) \quad (4) \\ &\simeq \mathbf{F}_1 r_1(k) + g_1 \\ r_2(k+1) &\simeq \mathbf{F}_2 r_2(k) + g_2 \\ &\vdots \\ r_M(k+1) &\simeq \mathbf{F}_M r_M(k) + g_M \end{aligned}$$

Algorithm

$$\begin{bmatrix} x(k+1) \\ r_1(k+1) \\ r_2(k+1) \\ \cdots \\ r_M(k+1) \end{bmatrix} \simeq \begin{bmatrix} \mathbf{A} & \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_M \\ \mathbf{0} & \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_M \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_2 & \cdots & \mathbf{F}_M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} x(k) \\ r_1(k) \\ r_2(k) \\ \vdots \\ r_M(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} u(k) + \begin{bmatrix} b + g_1 + g_2 + \cdots + g_M \\ g_1 + g_2 + \cdots + g_M \\ g_2 + g_3 + \cdots + g_M \\ \vdots \\ g_M \end{bmatrix} \quad (5)$$

Simplify

$$\chi(k+1) \simeq \mathbf{A}_{\mathrm{IRDMD}}\chi(k) + \mathbf{B}_{\mathrm{IRDMD}}u(k) + l \quad (6)$$

- Note that, $\chi(k)$ is a linear transformation of the delay coordinates $[x(k)^\top, x(k-1)^\top, \cdots, x(k-M)^\top]^\top$.

In Feedback Control Task

$$x(2) \simeq \mathbf{A}x(1) + \mathbf{B}u(1) + b$$

$$\begin{bmatrix} x(3) \\ r_1(3) \end{bmatrix} \simeq \begin{bmatrix} \mathbf{A} & \mathbf{F}_1 \\ \mathbf{0} & \mathbf{F}_1 \end{bmatrix} \begin{bmatrix} x(2) \\ r_1(2) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u(2) + \begin{bmatrix} b + g_1 \\ g_1 \end{bmatrix} \qquad (7)$$

$$\begin{bmatrix} x(4) \\ r_1(4) \\ r_2(4) \end{bmatrix} \simeq \begin{bmatrix} \mathbf{A} & \mathbf{F}_1 & \mathbf{F}_2 \\ \mathbf{0} & \mathbf{F}_1 & \mathbf{F}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_2 \end{bmatrix} \begin{bmatrix} x(3) \\ r_1(3) \\ r_2(3) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u(3) + \begin{bmatrix} b + g_1 + g_2 \\ g_1 + g_2 \\ g_2 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} x(k+1) \\ r_1(k+1) \\ r_2(k+1) \\ \cdots \\ r_M(k+1) \end{bmatrix} \simeq \begin{bmatrix} \mathbf{A} & \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_M \\ \mathbf{0} & \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_M \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_2 & \cdots & \mathbf{F}_M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} x(k) \\ r_1(k) \\ r_2(k) \\ \vdots \\ r_M(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} b + g_1 + g_2 + \cdots + g_M \\ g_1 + g_2 + \cdots + g_M \\ g_2 + g_3 + \cdots + g_M \\ \vdots \\ g_M \end{bmatrix}$$

Control input is designed using the augmented state at that time.

IRDMD gradually improves model accuracy.

Computational Cost Analysis

Computational Efficiency of IRDMD

n_x : dimension of state
d : dimension of augmented state χ $(d = n_x(M+1))$
N : # of data points
M : max order of residuals

	EDMD	IRDMD
Time Complexity	$\mathcal{O}(Nd^2)$ $(= \mathcal{O}(NM^2 n_x^2))$	$\mathcal{O}(NMn_x^2)$
Space Complexity	$\mathcal{O}(Nd)$ $(= \mathcal{O}(NMn_x))$	$\mathcal{O}(Nn_x)$

- IRDMD is more suitable for high-dimensional systems.

Numerical Examples

Prediction Task: Lotka-Volterra model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 (1 - \frac{x_1}{N}) - \beta x_1 x_2 \\ \delta x_1 x_2 - \gamma x_2 \end{bmatrix}$$

- Large singular value of the Koopman matrix $\rightarrow$ Principal mode of the dynamics [8].

Matrix S (Singular values):

1.7226	0	0	0	0	0
0	1.2739	0	0	0	0
0	0	1.0071	0	0	0
0	0	0	0.9778	0	0
0	0	0	0	0.7884	0
0	0	0	0	0	0.5663

EDMD

Matrix S (Singular values):

3.1333	0	0	0	0	0
0	1.8063	0	0	0	0
0	0	0.9227	0	0	0
0	0	0	0.7493	0	0
0	0	0	0	0.6749	0
0	0	0	0	0	0.3470

IRDMD

$\rightarrow$ IRDMD is more efficient to capture the dominant dynamics.

[8] Tu, Jonathan H.Dynamic mode decomposition: Theory and applications. Diss. Princeton University,2013.

Prediction Task: Lotka-Volterra model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1(1 - \frac{x_1}{N}) - \beta x_1 x_2 \\ \delta x_1 x_2 - \gamma x_2 \end{bmatrix}$$

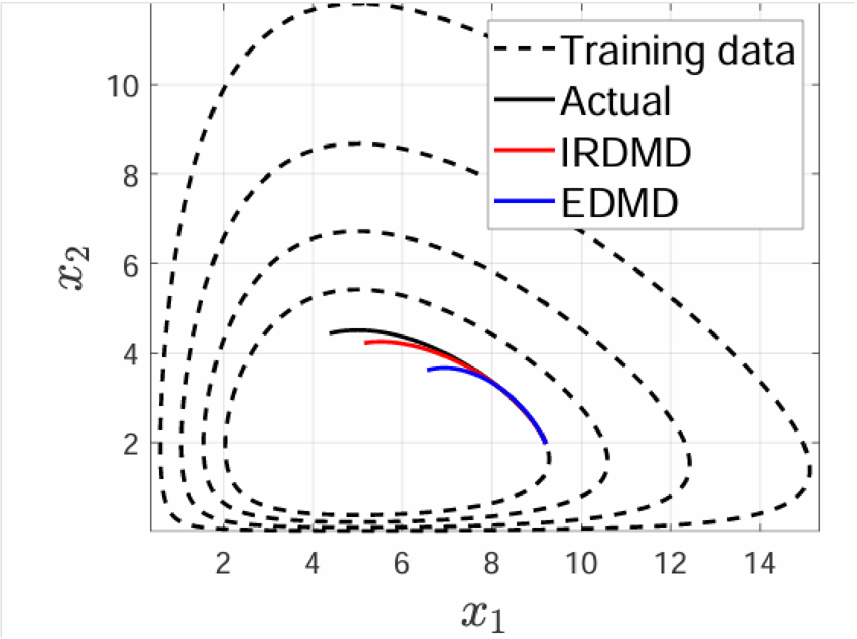


Fig. 1. Phase plot of the Lotka-Volterra model which is approximated using EDMD and IRDMD.

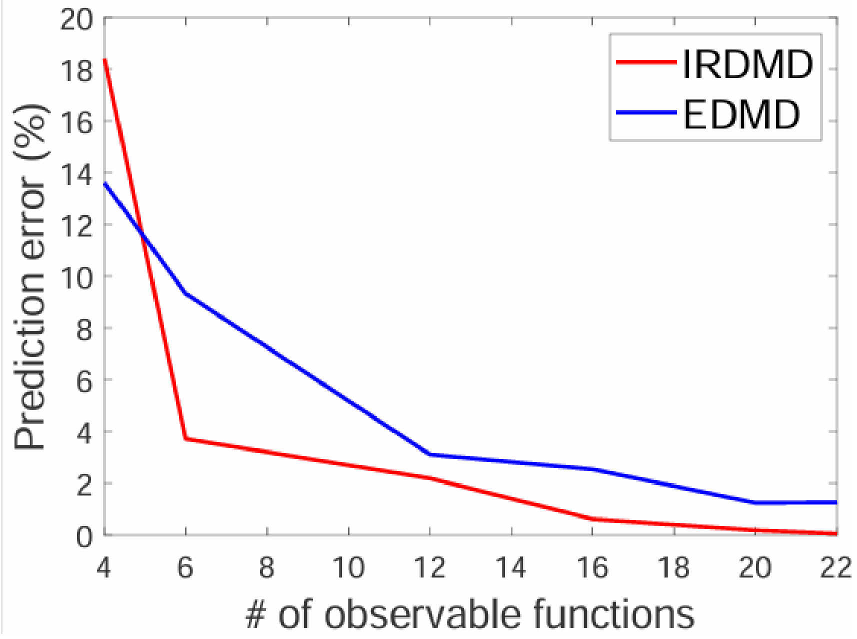


Fig. 2. Prediction error with respect to the number of observables.

Prediction Task: Lotka-Volterra model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 (1 - \frac{x_1}{N}) - \beta x_1 x_2 \\ \delta x_1 x_2 - \gamma x_2 \end{bmatrix}$$

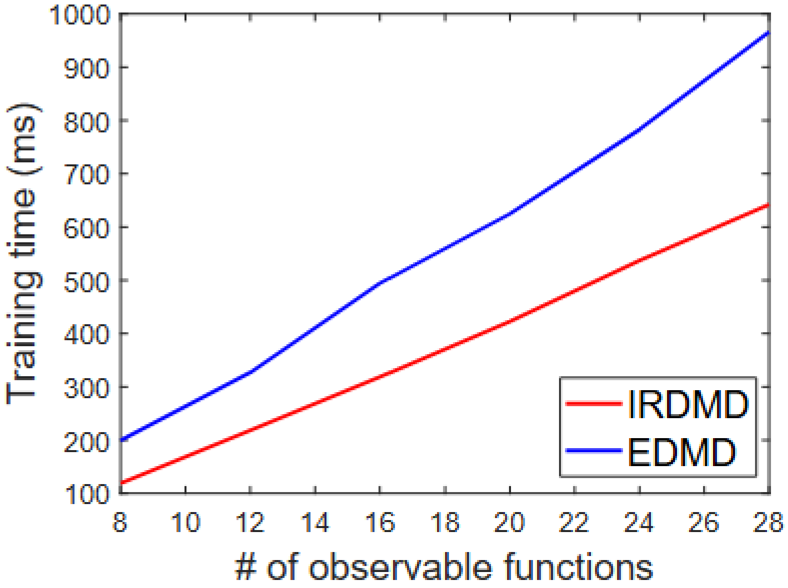
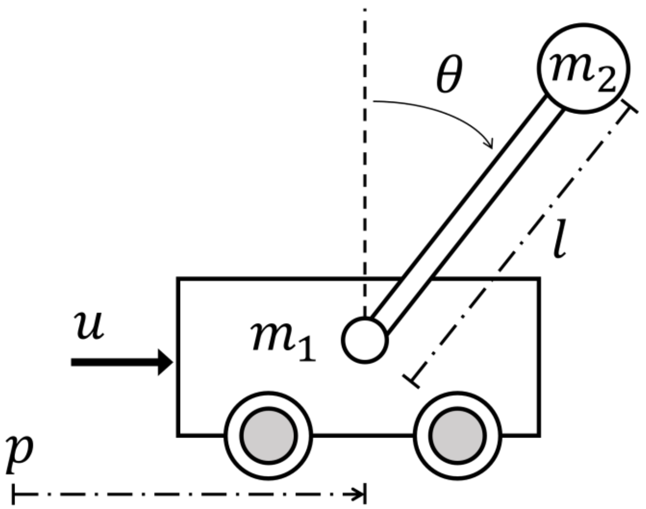


Fig. 3. Training time of each algorithm with respect to the number of observables, given 1,000,000 data samples.

Control Task: Cart-Pole



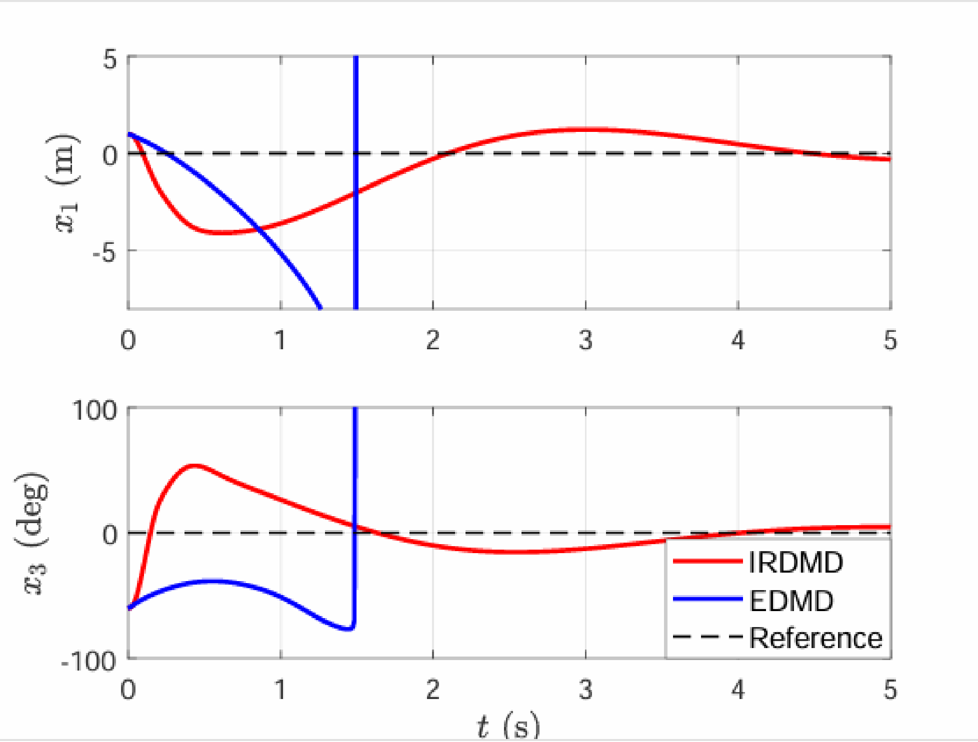
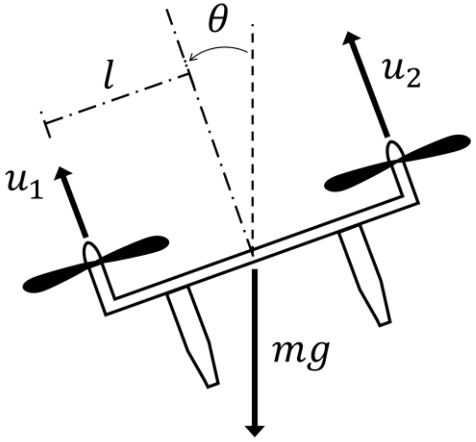


Fig. 4. Position and pole angle of the cart-pole system overtime, using IRDMD and EDMD with 10 observables.

Control Task: Planar Multirotor



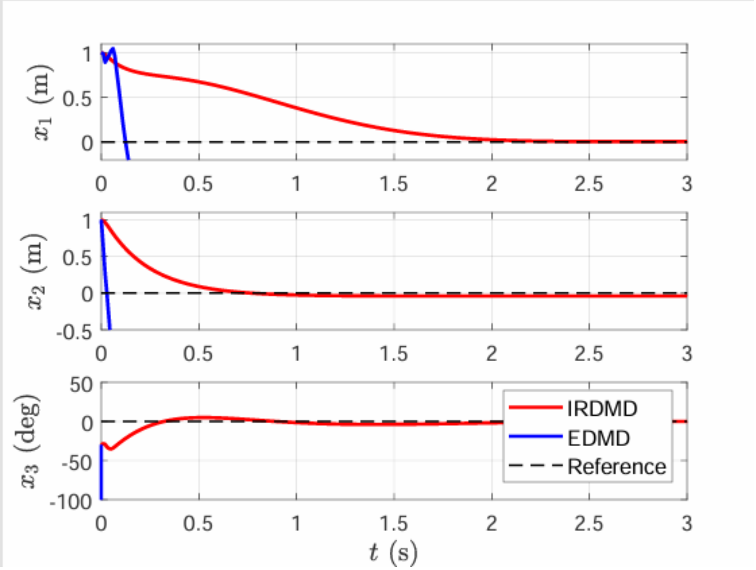


Fig. 5. Position and pitch angle of the planar multirotor, using IRDMD and EDMD with 15 observables.

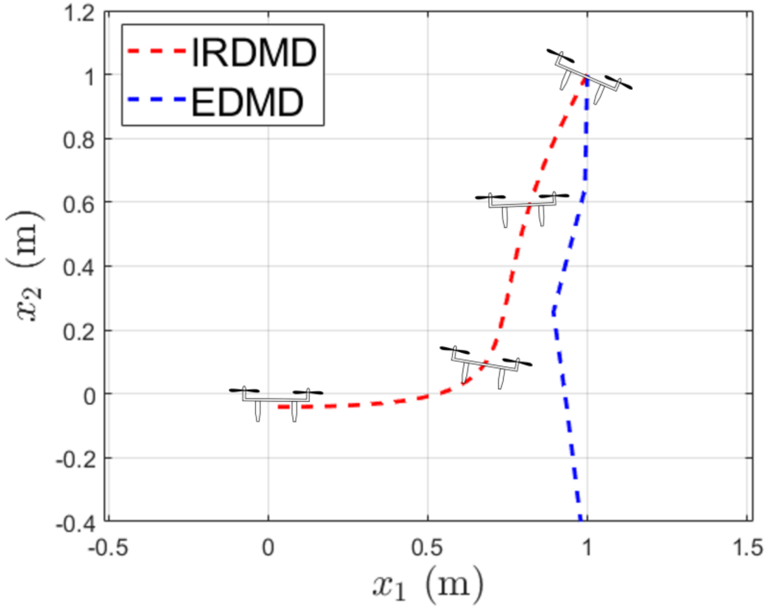


Fig. 6. Trajectory of the planar multirotor, using IRDMD and EDMD with 15 observables.

Conclusions

Conclusions

- We proposed IRDMD, a DMD algorithm that utilizes residuals as observables.

- IRDMD automatically construct observables.

- IRDMD offers better computational efficiency than EDMD.

- We numerically showed that IRDMD captures dominant modes efficiently than EDMD.

Future works : Error bound analysis and automatic truncation method of IRDMD.

Thank you!

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