Linear Algebra_Quiz 5 (Oct 12(Mon) / 2020)

1)P.163

6. Let $P: V \to V$ be a **projection**, that is a linear map such that $P^2 = P$.
   (a) Show that $V = \operatorname{Ker} P + \operatorname{Im} P$.
   (b) Show that the intersection of $\operatorname{Im} P$ and $\operatorname{Ker} P$ is $\{O\}$. In other words, if $v \in \operatorname{Im} P$ and $v \in \operatorname{Ker} P$, then $v = O$.

2)P.170

13. Let V be a vector space, and let U, W be two subspaces. Assume that V is the direct sum of U and W, that is

$$V = U + W \qquad \text{and} \qquad U \cap W = \{O\}.$$

Let $L: U \times W \to V$ be the mapping such that

$$L(u, w) = u + w.$$

Show that L is a bijective linear map. (So prove that L is linear, L is injective, L is surjective.)

3)P.178

2. Let A be a symmetric $n \times n$ matrix. Given two column vectors X, $Y \in \mathbf{R}^n$, define

$$\langle X, Y \rangle = {}^tXAY.$$

   (a) Show that this symbol satisfies the first three properties of a scalar product.
   (b) Give an example of a 1×1 matrix and a non-zero 2×2 matrix such that the fourth property is not satisfied. If this fourth property is satisfied, that is ${}^tXAX > 0$ for all $X \neq O$, then the matrix A is called **positive definite**.
   (c) Give an example of a 2×2 matrix which is symmetric and positive definite.
   (d) Let $a > 0$, and let

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

   Prove that A is positive definite if and only if $ad - b^2 > 0$. [*Hint*: Let $X = {}^t(x, y)$ and complete the square in the expression ${}^tXAX$.]
   (e) If $a < 0$ show that A is not positive definite.

4)P.190

7. Let V be a finite dimensional vector space with a positive definite scalar product. Let W be a subspace. Show that

$$V = W + W^{\perp} \qquad \text{and} \qquad W \cap W^{\perp} = \{0\}.$$

In the terminology of the preceding chapter, this means that V is the direct sum of W and its orthogonal complement. [Use Theorem 2.3.]

5)P.190

10. Let A be a symmetric $n \times n$ matrix. Let $X, Y \in \mathbf{R}^n$ be eigenvectors for A, that is suppose that there exist numbers a, b such that $AX = aX$ and $AY = bY$. Assume that $a \neq b$. Prove that X, Y are perpendicular.