Linear Algebra_Quiz 4 (Oct 5 (Mon) / 2020)

1)P.127

In each case, to prove that the image is equal to a certain set S, you must prove that the image is contained in S, and also that every element of S is in the image.

12. Let F be the mapping defined by $F(x, y) = (x/3, y/4)$. What is the image under F of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1?$$

2) P.135

Convex set definition(https://wikidocs.net/17413)

15. Let $L: V \to W$ be a linear map. Let T be a convex set in W and let S be the set of elements $v \in V$ such that $L(v) \in T$. Show that S is convex.

Remark. Why do these exercises give a more general proof of what you should already have worked out previously? For instance: Let $A \in \mathbf{R}^n$ and let c be a number. Then the set of all $X \in \mathbf{R}^n$ such that $X \cdot A \geqq c$ is convex. Also if S is a convex set and c is a number, then cS is convex. How do these statements fit as special cases of Exercises 14 and 15?

3) p.136

Eigenvectors and eigenvalues. Let V be a vector space, and let $L: V \to V$ be a linear map. An **eigenvector** v for L is an element of V such that there exists a scalar c with the property.

$$L(v) = cv.$$

The scalar c is called an **eigenvalue** of v with respect to L. If $v \neq 0$ then c is uniquely determined. When V is a vector space whose elements are functions, then an eigenvector is also called an **eigenfunction**.

18. Let $L: V \to V$ be a linear map, and let W be the subset of elements of V consisting of all eigenvectors of L with a given eigenvalue c. Show that W is a subspace.

4) P.143

13. A function (real valued, of a real variable) is called **even** if $f(-x) = f(x)$. It is called **odd** if $f(-x) = -f(x)$.
(a) Verify that $\sin x$ is an odd function, and $\cos x$ is an even function.
(b) Let V be the vector space of all functions. Define the map

$$P: V \to V$$

by $(Pf)(x) = (f(x) + f(-x))/2$. Show that P is a linear map.
(c) What is the kernel of P?
(d) What is the image of P? Prove your assertions.

5) p.143

15. **The product space.** Let U, W be vector spaces. We let the **direct product**, simply called the **product**, $U \times W$ be the set of all pairs (u, w) with $u \in U$ and $w \in W$. This should not be confused with the product of numbers, the scalar product, the cross product of vectors which is sometimes used in physics to denote a different type of operation. It is an unfortunate historical fact that the word product is used in two different contexts, and you should get accustomed to this. For instance, we can view $\mathbf{R}^4$ as a product,

$$\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}^1 = \mathbf{R}^3 \times \mathbf{R}$$

by viewing a 4-tuple (x_1, x_2, x_3, x_4) as putting side by side the triple (x_1, x_2, x_3) and the single number x_4. Similarly,

$$\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2,$$

by viewing (x_1, x_2, x_3, x_4) as putting side by side (x_1, x_2) and (x_3, x_4).
If (u_1, w_1) and (u_2, w_2) are elements of $U \times W$, so

$$u_1, u_2 \in U \qquad \text{and} \qquad w_1, w_2 \in W,$$

we define their **sum** componentwise, that is we define

$$(u_1, w_1) + (u_2, w_2) = (u_1 + u_2, w_1 + w_2).$$

If c is a number, define $c(u, w) = (cu, cw)$.

(a) Show that $U \times W$ is a vector space with these definitions. What is the zero element?

(b) Show that $\dim(U \times W) = \dim U + \dim W$. In fact, let $\{u_i\}$ $(i = 1, \ldots, n)$ be a basis of U and $\{w_j\}$ $(j = 1, \ldots, m)$ be a basis of W. Show that the elements $\{(u_i, 0)\}$ and $\{(0, w_j)\}$ form a basis of $U \times W$.

(c) Let U be a subspace of a vector space V. Show that the subset of $V \times V$ consisting of all elements (u, u) with $u \in U$ is a subspace of $V \times V$.

(d) Let $\{u_i\}$ be a basis of U. Show that the set of elements (u_i, u_i) is a basis of the subspace in (c). Hence the dimension of this subspace is the same as the dimension of U.

6) P.149

8. Let U be a subspace of $\mathbf{R}^n$. Prove that $U^\perp$ is also a subspace.

7) p.157

6. Let $L: V \to V$ be a linear map. Let $v \in V$. We say that v is an eigenvector for L if there exists a number c such that $L(v) = cv$. Suppose that V has a basis $\{v_1, \ldots, v_n\}$ consisting of eigenvectors, with $L(v_i) = c_i v_i$ for $i = 1, \ldots, n$. What is the matrix representing L with respect to this basis?