3.4 Measurement of a Quantum State - Part 2

We have discussed some of the basis of measurement in Sect. 2.3.3. When a quantum state is measured, the state will collapse to one of the basis states. For example, if we perform a spin measurement on a spin qubit, the quantum state will collapse to either spin-up, $|\uparrow\rangle$, or spin-down, $|\downarrow\rangle$. Experminetally, we will also obtain a real number in the maeasurement (e.g., $\frac{1}{2}$or $-\frac{1}{2}$).

In general, depending on what we are measuring, the basis states it will collapse to and the real values measured are the eigenvectors and eigenvalues, respectively, of a Hermitian matrix, $\boldsymbol{A}$. For example, if we rare perfroming a spin measurement of and electron, this measurement corresponds to the Hermitian matrix, $\frac{1}{2}\sigma_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, which has eigenvalues of $\frac{1}{2}$ and $-\frac{1}{2}$. It should be noted that the corresponding Hermitian matrix is **NOT an operator to perform the measurement**. It is only that its eigenvectors are the states it will collapse to and its eigenvalues are the numbers being measured experimentally.

It should also be clear to the reader why the corresponding operators must be Hermitian. This is because the Hermitian matrix has real eigenvalues which are what will be measured expermientally.

We mentioned that the probability of a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ collapsing to one of the basis states (which are the eigenstates of the corresponding Hermitian matrix) is the square of the modulus of the corresponding coefficient. Here, we will give a more versatile definition of the probability, $Prob(|i\rangle)$,it will collapse to basis state $|i\rangle$. That is,

$$Prob(|i\rangle) = \langle\psi|\boldsymbol{P}_{|i\rangle}|\psi\rangle\,, \tag{3.29}$$

where the projection operator to $|i\rangle$ is used.

Example 3.5 Derive Eq. (2.21) using Eq.(3.29).

$$\begin{aligned} Prob(|0\rangle) &= \langle\psi|P_{|0\rangle}|\psi\rangle\,, \\ &= (\alpha^*\langle 0| + \beta^*\langle 1|)(|0\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle), \\ &= \alpha^*\langle 0|0\rangle\langle 0|\ \alpha\ |0\rangle\,, \\ &= \alpha^*\alpha = |\alpha|^2, \end{aligned} \tag{3.30}$$

where from line 2 to line 3, we have used the fact that $\langle 0|1\rangle = 0$.

3.4.1 Expectation Value in a Measurement

If $\boldsymbol{A}$ is the Hermitian matrix corresponding to a measurement and has eigenvectors $|0\rangle$ and $|1\rangle$, then the **expectation value** or the average value obtained by performing the measurement on many identically prepared state $|\psi\rangle$ is the sum of the eigenvalues (λ_0, λ_1) of each eigenvector weighted by the probability of the eigenvector to which $|\psi\rangle$ will collapse. Therefore, the expectation value of $\boldsymbol{A}$ (or the average measured value) for the given state $|\psi\rangle$ is

$$
\begin{aligned}
\langle \boldsymbol{A} \rangle &= Prob(|0\rangle)\lambda_0 + Prob(|1\rangle)\lambda_1, \\
&= \langle \psi | P_{|0\rangle} | \psi \rangle \lambda_0 + \langle \psi | P_{|1\rangle} | \psi \rangle \lambda_1, \\
&= \langle \psi | 0 \rangle \langle 0 | \psi \rangle \lambda_0 + \langle \psi | 1 \rangle \langle 1 | \psi \rangle \lambda_1, \\
&= \langle \psi | \left(|0\rangle \langle 0| \lambda_0 + |1\rangle \langle 1| \lambda_1 \right) |\psi\rangle, \\
&= \langle \psi | \ \boldsymbol{A} \ | \psi \rangle .
\end{aligned}
\tag{3.31}
$$

In the last line, we used the fact that working in $\boldsymbol{A}$'s eigenbasis, $\boldsymbol{A}$ is a diagonal matrix with the eigenvalues along the diagonal which is $\boldsymbol{A} = |0\rangle \langle 0| \lambda_0 + |1\rangle \langle 1| \lambda_1$ (see Eqs. (3.8) and (3.9)).

3.5 Tensor Product of Matrices

In Sect. 2.4, we discussed how to construct a larger space by combining smaller spaces using the **tensor product**. The state/vector of the combined system can be described by the tensor product of the states/vectors of the smaller systems (Eq. (2.28)). Note that it can also be a linear combination of the tensor products if they are **entangled** which will be discussed later. We also need to create an operator for the combined system so that it is equivalent to the individual operators i nthe subsystems. For example, if $\boldsymbol{M_1}$ is applied to $|\psi_1\rangle$ and $\boldsymbol{M_2}$ is applied to $|\psi_2\rangle$, what is the equivalent operator $\boldsymbol{M}$ applied to state of the combined system, i.e., $|\psi_1\rangle \otimes |\psi_2\rangle$?

We construct $\boldsymbol{M}$ using a tensor product of $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$,

$$
\boldsymbol{M} = \boldsymbol{M_1} \otimes \boldsymbol{M_2}. \tag{3.32}
$$

At a result, we have

$$
\begin{aligned}
\boldsymbol{M} |\psi\rangle &= \boldsymbol{M}(|\psi_1\rangle \otimes |\psi_2\rangle), \\
&= (\boldsymbol{M_1} \otimes \boldsymbol{M_2})(|\psi_1\rangle \otimes |\psi_2\rangle), \\
&= (\boldsymbol{M_1} |\psi_1\rangle) \otimes (\boldsymbol{M_2} |\psi_2\rangle).
\end{aligned}
\tag{3.33}
$$

Note that the operator in each subsystem only applies to the state in that system. For example, a magnetic pulse to rotate the spin state of electron 1 ($\boldsymbol{M_1}$) is only physically applied to electron 1 and shoul not have an effect on electron 2 . If it has, this is already an operator in the combined system.

When the operators are expressed in their matrix form, we follow the approach in Eq. (2.29) to perform the tensor product.

Example 3.6 If $\boldsymbol{M_1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\boldsymbol{M_2} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, find $\boldsymbol{M}=\boldsymbol{M_1} \otimes \boldsymbol{M_2}$.

$$\begin{aligned}
\boldsymbol{M} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \\
&= \begin{pmatrix} a \begin{pmatrix} e & f \\ g & h \end{pmatrix} & b \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ \\ c \begin{pmatrix} e & f \\ g & h \end{pmatrix} & d \begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{pmatrix}, \\
&= \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.
\end{aligned} \tag{3.34}$$

■

3.6 Summary

In this chapter, we have reviewed some fundamental concepts of matrix. A Hermitian matrix has real eigenvalues. Therefore, all measurements must be corresponding to a Hermitian matrix. However, it is emphasized that the Hermitian matrix is not an operator that results in a measurement and its eigenvalues are the xperimenatlly measured values. A unitary matrix preserves the inner products of vectors and preserves the vector norms. Therefore, the transformation matrix for basis changign must be unitary. We also learn how to create the operators of a combined system using a tensor porduct of the oprators in the subsystems. Now, we have reviewed most of the essential basic linear algebra and we can start studying the physics of quantum computers, namely, the Schrödinger equation in the nex chapter.

Problems

3.1 Dual Correspondesnce

Prove Eq. (3.6) by using Eq. (2.6)

3.2 Adjoint Martix

find the adjoint matrix of $\begin{pmatrix} i & 1 \\ 0 & i \end{pmatrix}$. Is it Hermitian? Is it unitary?

3.3 Transformation

How is a general vector $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ transformed in the example in Fig. 2.2?

3.4 Tensor Product

Transfrom $|0\rangle_1 \otimes |0\rangle_2$ using the matrices in Eq. (3.34). Firstly, transform each qubit individually in its own space and then find the combined vector using a tensor product. Secondly, transform $|0\rangle_1 \otimes |0\rangle_2$ in the combined space using the corresponding matrix in the combined space. Show that both methods give the same result.

3.5 Diagonal Matrix

Show that this is a diagoanl matrix by performing approproated substitutions: $\boldsymbol{A} = |0\rangle\langle 0|\,\lambda_0 + |1\rangle\langle 1|\,\lambda_1$. See also Eq. (3.31).

Reference

1. Hiu-Yang Wong. *Introduction to Quantum Computing*. Springer, 2024.