2.3.4 Orthonormal Basis and Vector Normalization

An **orthonormal basis** a is basis with an orthonormal basis vector. Let it be a n-dimensional basis and thus it has n basis vectors, $|x_0\rangle, |x_1\rangle, \cdots, |x_{n-1}\rangle$. If each basis vector is normalized (with a length of 1) and orthogonal to the others (with 0 overlap or inner product with other basis vectors), it is called an orthonormal basis. This can be written as

$$\begin{aligned} \langle x_i|x_j\rangle &= \begin{cases} 0 & \text{if i} \neq \text{j, (orthogonal)} \\ 1 & \text{if i = j, (normalized)} \end{cases} \\ &= \delta_{ij} \end{aligned} \tag{2.23}$$

where we use the **Kronecker delta** in the last line. Note that when i = j, $\langle x_i|x_j\rangle = \langle x_i|x_i\rangle$ and this is just the suqare of the norm of the basis vector, $|x_i\rangle$ (Eq.(2.7)). If it is one, then it means that the length is also one. Working on an orthonormal basis probides a lot of convenience in calculations due to the fact that $\langle x_i|x_j\rangle$ results in either 0 or 1. We can also thus write the vector in a clolum or row form as in Eqs. (2.18) and (2.19).
For example, if a vector $\langle V|$ represented in an orthonormal basis, $|x_i\rangle$, is given by

$$|V\rangle = a_0 |x_0\rangle + \cdot\cdot + a_{n-1} |x_{n-1}\rangle \tag{2.24}$$

to find its norm squared, $||v||^2$ (Eq.(2.7)), we have

$$\begin{aligned} \langle v|v\rangle &= (a_0^* a_1^* \cdots a_{n-1}^*) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \\ &= a_0^* a_0 + a_1^* a_1 + \cdots + a_{n-1}^* a_{n-1}, \\ &= |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2, \end{aligned} \tag{2.25}$$

If $\langle v|v\rangle = 1$, then vector $|v\rangle$ is a **a normalized vector**. As shown in Eq.(2.25), this means that a normalized vector has the sum of the coefficient modulus squared equals one. Recalling that upon the measurement of a quantum state, the probability is the corresponding coefficient modulus squared (Eqs.(2.21)) and (2.22), then a quantunm state must be normalized so that the probability of collapsing to any ofthe basis states is one. In other words, this ensures the sum of the probabilities ofmeasureing one of the basis states to be one. Therefore, any quantum state must be normalized vector.

2.4 Tensor product

We can combine two or more vector spaces through **tensor product**. For a more detailed discussion , please refer to Chapters 11 and 12 in [1]. What is the meaning of *combing two vector spaces* and why do we want to do that? For example, we can describe the spin of an electron using a 2D Hilbert space. It has two basis vectors, $|0\rangle_1$ and $|1\rangle_1$. Here I used subsript 1 to indicate that this is the vector scpace belonging to the first electron. The state of any possible spin of the electron is a vector, $|\psi_1\rangle$, in this Hilbert space and is a linear combination of the basis states,

$$|\psi_1\rangle = \alpha_1 |0\rangle_1 + \beta_1 |1\rangle_1 \tag{2.26}$$

Similarly, if there si a second electron, its basis vectors are $|0\rangle_2$ and $|1\rangle_2$. Its state is given by.

$$|\psi_2\rangle = \alpha_2 |0\rangle_2 + \beta_1 |1\rangle_2 \tag{2.27}$$

If we want to describe the two electrons together or treat the two electrons as a *single* physical system, then the tensor product is the mathematical tool for us to do so.

As aligned with our common sense, the new system must have a larger vector space. Here, *let me emphasize* that a larger space is *NOT* obtained through a simple extension of a lower-dimension one to a higher-dimension one (e.g., adding a time dimension to the 3D space to become a 4D space-time). It is a result of the tensor product, $\otimes$, of the lower space basis states. The number of the new basis states is the electron system as a whole has four basis states, $|0\rangle_1 \otimes |0\rangle_2\,, |0\rangle_1 \otimes |1\rangle_2\,, |1\rangle_1 \otimes |0\rangle_2\,, and\, |1\rangle_1 \otimes |1\rangle_2$. We may also omit $\otimes$ by writing it as $|0\rangle_1 |0\rangle_2\,, |0\rangle_1 |1\rangle_2\,, |1\rangle_1 |0\rangle_2\,, and\, |1\rangle_1 |1\rangle_2$. And if we agree with each other that the first (second) number refers to the first (second) electron, we can also succinctly write it as $|00\rangle\,, |01\rangle\,, |10\rangle\,, and\, |11\rangle$.

With the new sapce, we also expect that a state in the combined system must be a linear combination of the new basis vectors. This can be seen clearly by considering the tensor product of the tow-eletron system. The following demonstrates how to perform tensor products without explaining the background. Readers can treat it as a result of the definitions and should apprecieate its similarity of a regular algebraic product.

For example, if the first electron is in state $|\psi_1\rangle$ and the second electron is in state $|\psi_2\rangle$, then the state of the whole system, $|\psi\rangle$, is obtained through the tensor product of $|\psi_1\rangle\, and\, |\psi_2\rangle$ (Eqs. (2.26) and (2.27)):

$$\begin{aligned}
|\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle\,, \\
&= (\alpha_1 |0\rangle_1 + \beta_1 |1\rangle_1) \otimes (\alpha_2 |0\rangle_2 + \beta_2 |1\rangle_2), \\
&= \alpha_1\alpha_2 |0\rangle_1 |0\rangle_2 + \alpha_1\beta_2 |0\rangle_1 |1\rangle_2 + \beta_1\alpha_2 |1\rangle_1 |0\rangle_2 + \beta_1\beta_2 |1\rangle_1 |1\rangle_2\,, \\
&= \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle\,.
\end{aligned} \tag{2.28}$$

We can also do this in matrix form,

$$
\begin{aligned}
|psi\rangle &= |psi_1\rangle \otimes |psi_2\rangle\,, \\
&= \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}, \\
&= \begin{pmatrix}\alpha_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\\ \beta_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\end{pmatrix}, \\
&= \begin{pmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_1\\ \beta_1\beta_2\end{pmatrix}. \begin{matrix}|00\rangle\\ |01\rangle\\ |10\rangle\\ |11\rangle\end{matrix}
\end{aligned}
\tag{2.29}
$$

where in the last line, the corresponding basis states are indicated. The same methodology is used for higher-dimensional spaces.

If more than 2 spaces need to be combined, we can do this one after another.

2.5 Summary

We review the basic properties of vectors in various vector spaces. In quantum computing, we will work in the Hilbert space. Therefore, the inner product which is an important component of the Hilbert space plays an important role in all calculations. We also dicuss the measurement of a quantum state. Although measurement is not a part of linear algebra, it requires that all quantum state needs to be normalized. We also practice how to combine two subststems into a larger one using tensor product. In the next chapter, we will discuss more advanced linear algebra. We will discuss martices and operators and their applications in quantum computing.

Problems

2.1 Vector space

α is a scalar and $|W\rangle$ is a vector. Given that,

$$|aW\rangle = a\,|W\rangle \tag{2.30}$$

using also Eq. (2.1), prove the following equations:

$$\langle V|aW\rangle = a\,\langle V|W\rangle \tag{2.31}$$

$$\langle aV|W\rangle = a^*\,\langle V|W\rangle \tag{2.32}$$

2.2 Orthonormal Basis

Prove Eq (2.25) using bra-ket notation (e.g., Eq.(2.24))instead of using matrix form.

2.3 Tensor product

Find the tensor product of $|a\rangle \, and \, |b\rangle$ in Example 2.3.

References

1. Hiu-yung Wong. Introductino to Quantum Computing. Springer, 2024.
2. Vector space. `Http://en.wikipedia.org/wiki/Vector_space`. Accessed:2024-01-08.