Part III
Superconducting Qubit architecture and Hardware

Chapter 13
Lagrangian Mechanics and Hamiltonian Mechanics

13.1 Introduction

Most of us have learned the basics of Newtonian mechanics. Newtonian mechanics is also called *vectorial mechanics* because it studies the motion of bodies under the influence of vector euantities such as $force$. However, Newtonian mechanics is not convenient in solving certain problems. There are other frameworks in theoretical physics called analytical mechanics such as **Lagrangian mechanics** and **Hamiltonian mechanics**. They are equivalent to Newtonian mechanics and they use scalar quantities such as the kinetic energy and potential energy of a system to derive the equations of motion of the system. In many problems, they appear to be more elegant and succincy than Newtonian mechanics. More importantly, the concpets in analytical mechanics can be *generalized* to hyperspace/phase space, in which we do not live. Moreover, Hamiltonian mechanics allow us to transition from classical mechanics to quantum mechanics more "smoothly." In this chapter, we will learn the *skills* of using Lagrangian and Hamiltonian mechanics. Readers are expected to learn the rules only. Readers may refer to [1] if they are interested in having a deeper appreciation of analytical mechanics.

13.1.1 Learning Outcomes

Be able to write down the Lagrangian and Hamiltonian of a given physical system; be able to derive the equation of motion of a system based on its Lagrangian and Hamiltonian.

13.1.2 Teaching Videos

- Search for Ch13 in this playlist
  - `https://tinyurl.com/3yhze3jn`

- Other Videos
  - `https://youtu.be/Ydj2hintCkc`
  - `https://youtu.be/u2SgXmf2SvQ`
  - `https://youtu.be/IdSF_064ZSo`

13.2 Lagrangian Mechanics

13.2.1 Generalized Coordinates and Belocities

For the purposes in the following chapters, we only consider point particles, conservative forces, and non-relativistic mechanics. Let us consider a system comprised of N particle. We know that if the coordinates of each particle and the velocity of each particle are known at a given time, the system has a well-defined state. This is because the acceleration of a particle depends on the force exerted on it. And the force is the spatial derivative of its potential, which is a function of its coordinates. Therefore, if we know their positions and velocities, we know their accelerations and, thus, can deduce their past and future states.

For N particles, inour real space, there are $3N$ independent coordinates due to the three orthogonal directions. Therefore, they are the collection of $\vec{q} = \{q_1, q_2, \cdots, q_{3N}\}$, where we write it as a $3N$-dimensional vector. Similarly, it has $3N$ independent velocities, $\dot{\vec{q}} = \{\dot{q_1}, \dot{q_2}, \cdots, \dot{q_{3N}}\}$, where

$$\dot{q_i} = \frac{dq_i}{dt}. \tag{13.1}$$

Besides using real spatial coordinates and velocities to uniquely determine the state of a system, one may also use other $3N$ quantities to determine its coordinates as long as they also give the system $3N$ degrees of freedom [2]. Such quantities are called the **generalized coordinates**. The time derivatives (Eq.(13.1)) of the generalized coordinates are called the **generalized velocities**. For the formalism we will discuss later, it is easier to think and understand using spatial coordinates and velocities but we need to keep in mind and accept the fact that they are applicable to generalized coordinates and velocities, too.

13.2.2 Lagrangian and Lagrange's Equations

We will first introduce a neqy quantity called **Lagrangian**, $\mathcal{L}$, which is defined as,

$$\mathcal{L} = T - V, \tag{13.2}$$

where T and V and the **kinetic energy** and **potential energy** of the system, respectively. Naturally, $\mathcal{L}$ has the unit of energy. Since T is a function of velocities, $\dot{\vec{q}}$, and V is a function of coordinates, $\vec{q}$, $\mathcal{L}$ is a function of both velocities and coordinates. Of course, they are all functions of time, t. Therefore,

$$\begin{aligned}\mathcal{L} &= T(\dot{\vec{q}}, t) - V(\vec{q}, t), \\ &= \mathcal{L}(\vec{q}, \dot{\vec{q}}, t), \\ &= \mathcal{L}(q_1, q_2, \cdots, \dot{q_1}, \dot{q_2}, \cdots, t). \end{aligned} \tag{13.3}$$

It is given that one can derive the **equations of motion** of the system by solving the **Lagrangian's equation** [1],

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_i}}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \tag{13.4}$$

fot i from 1 to $3N$. Lagrange's equation contain the Lagrangian of the system with the *coordinates and velocities being the independent variables.* This forms the basics of Lagrangian mechanics. It should be noted that when working with Lagrangian mechanics, one needs to make sure to **express the Lagrangian explicitly as a function of coordinates and velocities**. Of course, this includes the *generalized* coordinates and velocities.

It should also be noted that the Lagrangian of a system is *not unique.* As long as it gives the correct equations of motion through Eq. (13.4), it is a valid Lagrangian.

In this book, we take Lagrangian's equation as given like how we trust $F = ma$ when we study **Newtonian mechanics**. But Lagrangian's equation can be derived from a more fundamental principle, namely, the **principle of least action** or **Hamiltonian's principle** [2]. The principle defines **action** S, as,]

$$S = \int_{t_1}^{t_2} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)\, dt. \tag{13.5}$$

The action, S, integrates the Lagrangian of a system from time t_1 to time t_2 and mandates that the system should evolve from time t_1 to time t_2 in a way such that S is minimal based on which the Lagrange's equation to Eq. (13.4) are derived [2].

Again, for the purpose of this book, we just need to learn the skills to construct the Lagrangian and solve the Lagrange's equation of a given system. Let us look at two examples.

Example 13.1 Find the equations of motion of a free particle with mass, m. Assume that the particle is moving in the $\hat{x}$-direction with speed v at time t_0.

We already know from Newton's first law that a free particle will keep moving at a constant speed. Let us see if we will obtain the same result by using Lagrangian mechanics.

Set $\vec{q} = \{q_1 = x, q_2 = y, q_3 = z\}$. Its velocity is $\dot{\vec{q}} = \{\dot{q_1}, \dot{q_2}, \dot{q_3}\}$. As a free particle, there is no external force and thus it experiences a constant potential energy that can be set to a constant ϕ. Therefore, its kinetic energy is

$$T = \frac{1}{2}m\dot{q_1}^2 + \frac{1}{2}m\dot{q_2}^2 + \frac{1}{2}m\dot{q_3}^2, \tag{13.6}$$

and its potential energy is

$$V = \phi. \tag{13.7}$$

The Lagrangian of the system is

$$\begin{aligned} \mathcal{L} &= T - V, \\ &= \frac{1}{2}m\dot{q_1}^2 + \frac{1}{2}m\dot{q_2}^2 + \frac{1}{2}m\dot{q_3}^2 - \phi. \end{aligned} \tag{13.8}$$

To find its equation of motion, we solve Lagrange's equations in Eq. (13.4). Note that there are three coordinates (for $i = 1$ to $i = 3$ with $q_1 = x, q_2 = y, q_3 = z$).

Therefore, we have three equations,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\right) - \frac{\partial \mathcal{L}}{\partial q_1} = 0, \tag{13.9}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}}\right) - \frac{\partial \mathcal{L}}{\partial q_2} = 0, \tag{13.10}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_3}}\right) - \frac{\partial \mathcal{L}}{\partial q_3} = 0, \tag{13.11}$$

The equation involve partial derivatives with respect to $q_1, q_2, q_3, \dot{q_1}, \dot{q_2}$, and $\dot{q_3}$. But none of the erms in Eq. (13.8) depends on q_1, q_2, and q_3 as ϕ is a constant. $\mathcal{L}$ only depends on $\dot{q_1}, \dot{q_2}$, and $\dot{q_3}$. Therefore, the three Lagrange's equation becomes,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\right) - 0 = 0, \tag{13.12}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_2}}\right) - 0 = 0, \tag{13.13}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_3}}\right) - 0 = 0, \tag{13.14}$$

Let us only solve Eq. (13.12),

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q_1}}\right) &= 0, \\
\frac{d}{dt}\left(\frac{\partial(\frac{1}{2}m\dot{q_1}^2 + \frac{1}{2}m\dot{q_2}^2 + \frac{1}{2}m\dot{q_3}^2 - \phi)}{\partial \dot{q_1}}\right) &= 0, \\
\frac{d(m\dot{q_1})}{dt} &= 0, \\
\ddot{q_1} &= 0.
\end{aligned} \tag{13.15}$$

$\ddot{q_1}$ is the time derivative of velocity, which is the acceleration in the $\hat{x}$ direction. It means the particle will move at a constant velocity in the $\hat{x}$ direction. Since at $t = t_0, \dot{q_1} = v$, then it will be moving at speed v in the $\hat{x}$ direction forever. Similarly, $\ddot{q_2} = \ddot{q_3} = 0$; therefore, $q_2 = q_3 = 0$ at all time.

Using Lagrangian mechanics, we obtain the same conclusion as Newton's first law.

Example 13.2 Find the euqation of motion of a mass, m, attached to a fixed wall through a spring with a spring with a spring constant of k (Fig. 13.1). This is the famous **simple harmonic oscillator (SHO)** problem.

This is a 1D system

References

1. Goldstein, H., Poole, C. & Safko, J. *Classical Mechanics* (Pearson, 2001).
2. Landau, L. & Lifshitz, E. *Mechanics: Volume 1 (Course of Theoretical Physics S)* (Butterworth-Heinemann, 1976).