COMBATING THE GROWING DEMAND OF THE WORKING AGE: A POPULATION AGING MODEL

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Abstract. Population aging is becoming a significant problem in the modern century, due to rising life expectancy and declining fertility rates in the United States. This causes an increasing demand for a sustainable workforce in order to take care of these people. We propose the Population Aging Transition Model (PATM), a model that consists of a system of differential equations to simulate demographic shifts through the three age groups: individuals below working age, individuals of working age, and individuals above working age. By leveraging differential equations and key parameters such as birth rate, immigration factor, death rate, and transition rates between categories, our model evaluates the labor force's sustainability with population aging. Based on our model, we predict the critical impact of adjusting the birth rate in order to mitigate the age imbalance between individuals under working age and individuals above working age. We propose an optimal rate to balance the demographic needs. Future work includes considering dynamically modeling the birth rate in our model and adaptations to other national contexts.

1. Problem description

As global life expectancy continues to rise and global birth rates are in decline, modern society faces a problem never seen before: population aging. This aging of the population leads to an increasing number of individuals entering retirement age without sufficient replacement from the younger generation that is stepping into working age. According to the CDC National Center for Health Statistics[2], the fertility rate in the United States has been declining at a rate of about 2% consistently every year since 2014. Consequently, the demand for a growing workforce has reached an unprecedented high.

In this paper, we propose the Population Aging Transition Model (PATM) to simulate the demographic shifts. We attempt to model the shift between these three population groups: *individuals below working age, individuals of working age, and individuals above working age.* This model aims to capture the current age population dynamics if birth rates are continuously at the current low rate and analyze the gap between the younger generation and retired generation, thereby evaluating the current severity of population aging. After modeling the current situation, we attempt to identify an optimal birth rate in order to close the population gap between the younger generation and the retired generation to maintain a sustainable workforce.

2. Assumptions

Population changes are affected by numerous factors. Making predictions while accounting for all these complex, often interdependent, and unseen factors can be challenging, especially when considering every specific case individually. To create a manageable model, we simplify the model to focus on the transitions between three age groups: individuals below working age, individuals of working age, and individuals above working age in the United States. Immigration is also considered, simplifying the factor by making the assumption that immigrants remain in the U.S. after immigrating and contributing only to the working population. The death rate is only considered for individuals above working age, symbolizing individuals aging above retirement age.

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In this paper, we define the age categories as follows:

- **Individuals under working age**: Individuals aged 0 to 17 years.
- **Individuals of working age**: Individuals aged 18 to 66 years.
- **Individuals over working age (retired)**: Individuals aged 67 years and above.

We define individuals of working age based on the legal minimum age to work without restrictions in the U.S., and individuals over working age are defined using the current full retirement age in the U.S. These definitions will be applied consistently throughout this paper. All data in our paper are based on the official 2023 U.S. population statistics.

3. Mathematical Model

The logistic growth model[5] describes population growth under the assumption that the population initially grows rapidly but is constrained by limited resources as it approaches the carrying capacity K. The general form of the logistic differential equation is:

$$\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right),$$

where $P(t)$ represents the population size at time t, r is the intrinsic growth rate (per unit time), and K is the carrying capacity of the environment. Inspired by the original model and its evolved versions with additional parameters, as discussed in the literature we reviewed [9, 11], we propose a modified framework to account for the transitions between different age population groups. Our fundamental modeling principle is:

$$\frac{dF}{dt} = \text{Rate In} - \text{Rate Out},$$

where F represents the net population flows. Building on this foundation, we formulate PATM with a system of three differential equations to describe the transition between the defined population age groups:

$$\frac{dB(t)}{dt} = rB(t) - \alpha B(t), \tag{1}$$

$$\frac{dW(t)}{dt} = \gamma W(t) - \beta W(t) + \alpha B(t), \tag{2}$$

$$\frac{dR(t)}{dt} = -\delta R(t) + \beta W(t), \tag{3}$$

where,

- $B(t)$ is the number of individuals below working age,
- $W(t)$ is the number of individuals of working age,
- $R(t)$ is the number of individuals above working age (retired),
- $r, \alpha, \gamma, \beta, \delta$ are weighted factors that are defined as follows and are chosen because they significantly influence population inflows and outflows.

Parameter	**Description**
r	Growth rate of the population below working age
α	Proportion of individuals transitioning to working age per year
γ	Immigration factor of individuals per year
β	Proportion of individuals retiring per year
δ	Death rate factor of the retired population per year

3.1. **Determining r and α.** We define n_i as the percentage[1] of people at age i, where $i \in \{0, 1, 2, 3, \ldots, 101\}$. Here, n_1 represents the percentage of 1-year-olds, and n_{101} represents the percentage of individuals aged 101 and older.

In Equation 1, we know that r directly affects $B(t)$. r is the growth rate of the population below working age. Thus, by definition, we need to calculate the proportion of n_0 in $\sum_{i=0}^{17} n_i$, which can be determined with the following equation:

$$r = \frac{n_0}{\sum\limits_{i=0}^{17} n_i} = \frac{1.09}{21.76} \approx 0.05.$$

The parameter α represents the proportion of individuals below working age who transition into working age. By definition, we only need to consider the population aged 17 this year, as they will become 18 years old in a year and enter working age. We calculate α as:

$$\alpha = \frac{n_{17}}{\sum\limits_{i=0}^{17} n_i} = \frac{1.31}{21.76} \approx 0.06.$$

3.2. **Determining γ and β.** Similar to the approach in Section 3.1, β, which represents the proportion of individuals retiring per year, can be expressed as:

$$\beta = \frac{n_{66}}{\sum\limits_{n=18}^{66} n} = \frac{1.22}{63.16} \approx 0.0193.$$

However, here, γ is not particularly easy to determine because, based on the data we obtained from [10], we know that immigration numbers fluctuated significantly after the COVID-19 pandemic (2020). Furthermore, we have only found total immigration numbers up to 2022 so far. Therefore, we used linear regression to predict the total immigration number $N_{\text{immigrant}}$ for 2023. After excluding the immigration numbers during the exceptional period of 2020 and 2021 from [10], we performed a linear regression analysis on U.S. immigration data from 2002 to 2022, as shown in Figure 1. Finally, we estimated the value of $N_{\text{immigrant}}$ for 2023, which is approximately 2.9 million.

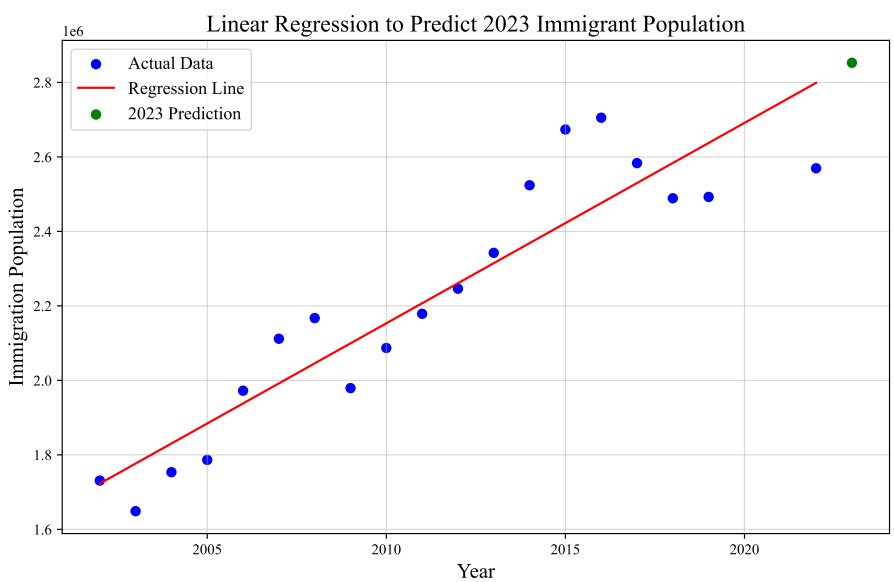

Figure 1. Linear regression to predict $N_{\text{immigrant}}$ for 2023

[1] Data based on 2023 US Census Data [7].

Then, the calculation of γ is similar to the process used to determine the previous coefficients: we divide the number of immigrants $N_{\text{immigrant}}$ by the number of individuals of working age. This approach is justified because, as stated in our assumptions in Section 2, our model considers that all immigrants impact this specific population group. Here, γ can be expressed as:

$$\gamma = \frac{N_{\text{immigrant}}}{\sum_{n=18}^{66} n_i \times N_{\text{total}}} = \frac{2.90}{63.16\% \times 334.9} \approx 0.0137.$$

Here, N_{total} represents the total population (in millions), with $N_{\text{total}} = 334.9$ million as stated in [4]. The total number of working individuals is determined by multiplying the proportion of working-age individuals by the total population.

3.3. **Determining δ.** In Equation 3, we know that δ directly affects $R(t)$. δ represents the proportion of deaths among the retired population, which can be determined by the following equation:

$$\delta = \frac{N_{\text{death}}}{\sum_{n=67}^{101} n_i \times N_{\text{total}}},$$

where N_{death} represents the number of deaths[2] (in millions) among the retired population. The total retired population is calculated by multiplying the proportion of retired individuals by the total population. Substituting with the given values:

$$\delta = \frac{2.310745}{0.1648 \times 334.9} \approx 0.041867754.$$

4. Solution

To better understand the relationship between $B(t)$ and $R(t)$, we begin by solving the system of equations involving $B(t)$, $W(t)$, and $R(t)$.

4.1. **Evaluate $B(t)$.** According to Equation 1 in our system, we have:

$$\frac{dB(t)}{dt} = rB(t) - \alpha B(t),$$

which is a separable differential equation. Rearranging and integrating:

$$\begin{aligned} \int \frac{1}{B(t)}\, dB(t) &= \int (r-\alpha)\, dt \\ \ln B(t) &= (r-\alpha)t + C_B \\ B(t) &= e^{C_B} \cdot e^{(r-\alpha)t} \\ B(t) &= B_0 e^{(r-\alpha)t} \end{aligned}$$

where $B_0 = e^{C_B}$ is a constant. In Section 3.1, we set $r \approx 0.05$ and $\alpha \approx 0.06$, so $r - \alpha = -0.01$. In 2023, there were 72.832 million[3] people aged 17 and under. Setting this as the initial state, $B_0 = 72.832$, we have:

$$B(t) = 72.832e^{-0.01t}.$$

[2]Data based on CDC data [1].

[3]Data based on 2023 US Census Data [7, 4]

4.2. **Evaluate $W(t)$.** According to Equation 2 in our system, we have:

$$\frac{dW(t)}{dt} = (\gamma - \beta)W(t) + \alpha B(t).$$

Substituting $B(t) = B_0 e^{(r-\alpha)t}$ into the equation:

$$\frac{dW(t)}{dt} = (\gamma - \beta)W(t) + \alpha B_0 e^{(r-\alpha)t}.$$

Multiplying both sides by integrating factor $\mu(t) = e^{-(\gamma-\beta)t}$ yields:

$$\begin{aligned}
e^{-(\gamma-\beta)t}\frac{dW(t)}{dt} - (\gamma-\beta)e^{-(\gamma-\beta)t}W(t) &= \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t} \\
\frac{d}{dt}\left(e^{-(\gamma-\beta)t}W(t)\right) &= \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t} \\
e^{-(\gamma-\beta)t}W(t) &= \int \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t}\,dt \\
e^{-(\gamma-\beta)t}W(t) &= \frac{\alpha B_0}{r-\alpha-(\gamma-\beta)} e^{(r-\alpha-(\gamma-\beta))t} + C_W \\
W(t) &= \frac{\alpha B_0}{r-\alpha-(\gamma-\beta)} e^{(r-\alpha)t} + C_W,
\end{aligned}$$

where C_W is a constant. Based on the coefficients we chose in Section 3.1 and Section 3.2: $r - \alpha = -0.01, \gamma - \beta = -0.0056, r - \alpha - (\gamma - \beta) = -0.0044$. Thus, $W(t)$ becomes:

$$W(t) = \frac{4.36992}{-0.0044} e^{-0.01t} + C_W e^{-0.0056t},$$

In 2023, there were 211.52284 million people[4] aged between 18 and 66, setting it as the initial state $W(0)$, thus we have:

$$W(0) = \frac{4.36992}{-0.0044} + C_W = 211.52284.$$

Thus, $C_W = 1204.69$, and the solution for $W(t)$ is:

$$W(t) = \frac{4.36992}{-0.0044} e^{-0.01t} + 1204.69 e^{-0.0056t}.$$

4.3. **Evaluate $R(t)$.** To make the process less complex, we substitute the coefficients before solving the equation. According to Equation 3 in our system, we have:

$$\frac{dR(t)}{dt} + 0.041867754R(t) = 23.2505e^{-0.0056t} - 19.1681e^{-0.01t}.$$

Multiplying both sides by integrating factor $\mu(t) = e^{0.041867754t}$ yields:

$$e^{0.041867754t}\frac{dR(t)}{dt} + 0.041867754e^{0.041867754t}R(t) = e^{0.041867754t}\left(23.2505e^{-0.0056t} - 19.1681e^{-0.01t}\right).$$

Finally, we obtain:

$$R(t) = e^{-0.041867754t}\left(641.08e^{0.0362678t} - 601.488e^{0.0318678t} + C_R\right),$$

where C_R is a constant. In 2023, there were 55.19152 million people[5] aged above 66, setting it as the initial state $R(0)$, then we have:

$$R(0) = e^0\left(641.08e^0 - 601.488e^0 + C_R\right) = 55.19152.$$

[4] Data based on 2023 US CENSUS Data [7]

[5] Data based on 2023 US CENSUS Data [7]

Thus, $C_R = 15.59952$, the solution for $R(t)$ is:

$$R(t) = e^{-0.041867754t}\left(641.08e^{0.0362678t} - 601.488e^{0.0318678t} + 15.59952\right).$$

5. Results[6]

First, we plot $B(t)$, $W(t)$, and $R(t)$ as shown in Figures 2–4 below. Figure 5 illustrates the relationship between $B(t)$ and $R(t)$.

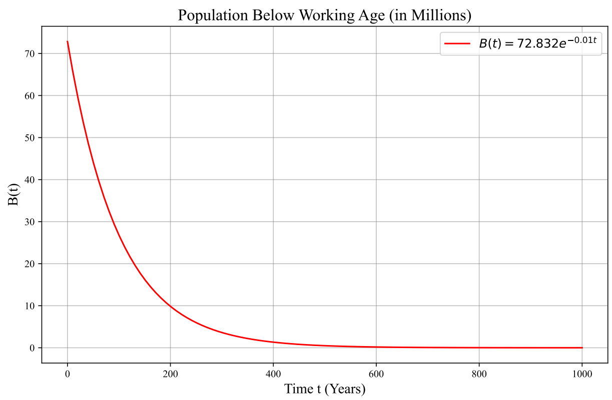

Figure 2. Plot of $B(t)$

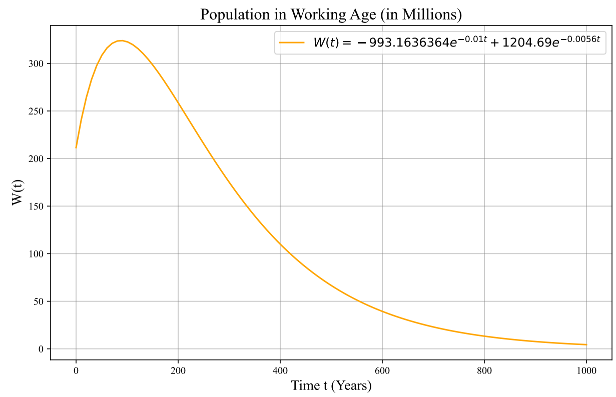

Figure 3. Plot of $W(t)$

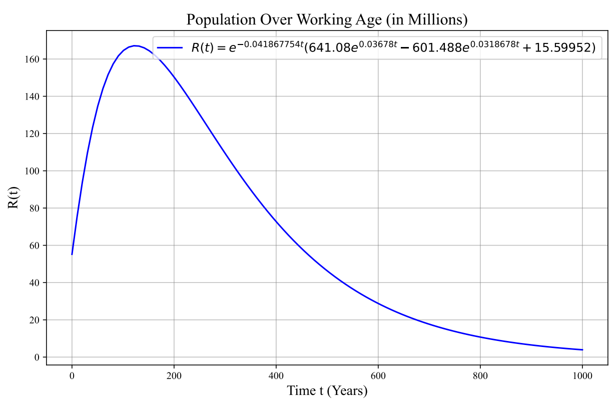

Figure 4. Plot of $R(t)$

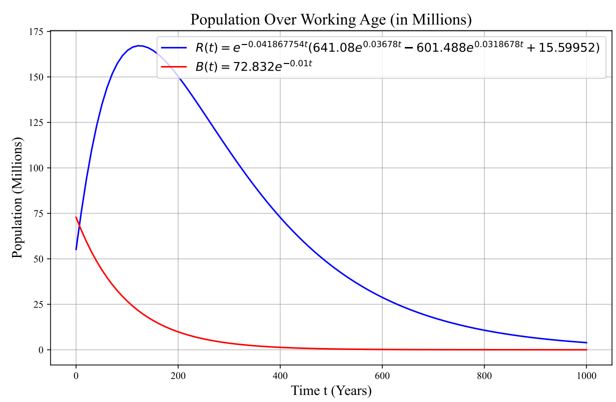

Figure 5. Plot of $R(t)$ v.s $B(t)$

From the plots above, we can see that $B(t)$, the population of individuals under working age is in exponential decline approaching 0 as t approaches ∞. $W(t)$, the population of individuals of working age, initially rises and then declines approaching 0 as t approaches ∞. $R(t)$, the population of individuals over working age, follows the trend of $W(t)$, but spiking at a t greater than $W(t)$, and declining approaching 0 as t approaches ∞.

Our concerns are valid. As shown in Figure 5, we plotted two exponentially decreasing graphs together to illustrate the current relationship between the population under working age $B(t)$ and the population over working age $R(t)$. Under the current circumstances, our workforce will face a significant shortage due to not having enough people join the workforce because of declining birth rates, a situation that might take up to 1000 years to recover from — an outcome we are unwilling to see.

[6] All codes can be found in [3].

Thus we propose to adjust the birth rate r, which is an effective solution to the phenomenon observed in Figure 6. After testing various values, we identified $r = 0.075$ as the optimal choice. Under this condition, $B(t)$ and $R(t)$ are observed to completely overlap. At this rate, the number of retired people appears almost equal to the number of individuals below working age, ensuring there are nearly enough people to adequately replace them in the workforce. At the same time, we also observe that when $r > 0.075$, $B(t)$ gradually surpasses $R(t)$, indicating that the population aged 0–17 is increasingly capable of offsetting the decline in the labor force.

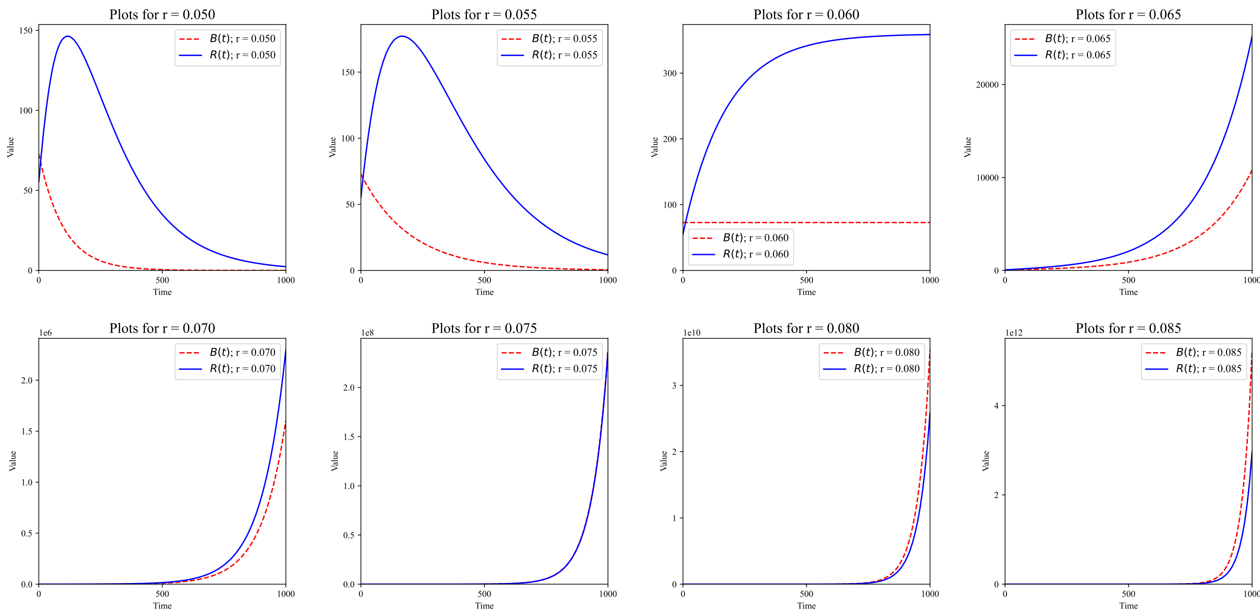

Figure 6. Population Plots with Various Growth Rates

Therefore, the best way to address the growing demand for a sustainable workforce is by adjusting the birth rate r to 0.075, an optimal birth rate to negate the effects of population aging, where $B(t)$ and $R(t)$ approximately overlap in the future.

6. Future Work

6.1. **Integrating a Dynamic System.** In Section 3.1, we considered a static r but assumed different r values to make sure $B(t)$ gradually surpass $R(t)$. However, in real-life scenarios, r is more realistically treated as a dynamic variable because statistics show that the birth rate in the U.S. has been decreasing. To account for this, we define r' as $r \cdot 0.98^t$, a dynamic variable dependent on time t, reflecting a phenomenon described in [2]: a continuous annual 2% decrease in the birth rate. Therefore we have:

$$r' = 0.05 \cdot 0.98^t.$$

In this case, we reconstruct our model:

$$\frac{d\mathcal{B}(t)}{dt} = (0.05 \cdot 0.98^t)\mathcal{B}(t) - \alpha\mathcal{B}(t), \tag{4}$$

$$\frac{d\mathcal{W}(t)}{dt} = \gamma\mathcal{W}(t) - \beta\mathcal{W}(t) + \alpha\mathcal{B}(t), \tag{5}$$

$$\frac{d\mathcal{R}(t)}{dt} = -\delta\mathcal{R}(t) + \beta\mathcal{W}(t). \tag{6}$$

Solving For $\mathcal{B}(t)$ in our new system:

$$\frac{d\mathcal{B}(t)}{dt} = (0.05 \cdot 0.98^t)\mathcal{B}(t) - \alpha\mathcal{B}(t),$$

which is a separable equation. Rearranging and integrating:

$$\begin{aligned}\int \frac{1}{\mathcal{B}(t)}\, d\mathcal{B}(t) &= \int (0.05 \cdot 0.98^t - \alpha)\, dt \\ \ln \mathcal{B}(t) &= \frac{0.05 \cdot 0.98^t}{\ln(0.98)} - \alpha t + C_{\mathcal{B}} \\ \mathcal{B}(t) &= e^{C_{\mathcal{B}}} \cdot 0.98 \cdot e^{0.05 \cdot 0.98^t - \alpha t} \\ &= \mathcal{B}_0 \cdot 0.98 \cdot e^{0.05 \cdot 0.98^t - \alpha t},\end{aligned}$$

where $\mathcal{B}_0$ is a constant. Followed by 3.1, we set $\alpha \approx 0.06$, thus:

$$\mathcal{B}(t) = 0.98\mathcal{B}_0 \cdot e^{0.05 \cdot 0.98^t - 0.06t}.$$

Combined with the initial condition, which is $\mathcal{B}(0) = 72.832$ million, we finally have:

$$\mathcal{B}(t) = 70.934 \cdot e^{0.05 \cdot 0.98^t - 0.06t}.$$

In the future, we will solve $\mathcal{W}(t)$ and $\mathcal{R}(t)$. However, based on the currently solved $\mathcal{B}(t)$, we observe that the situation may worsen more rapidly. This suggests that with an annual 2% decrease in the birth rate, the number of births could fail to meet the labor force demand even faster, as shown in Figure 7.

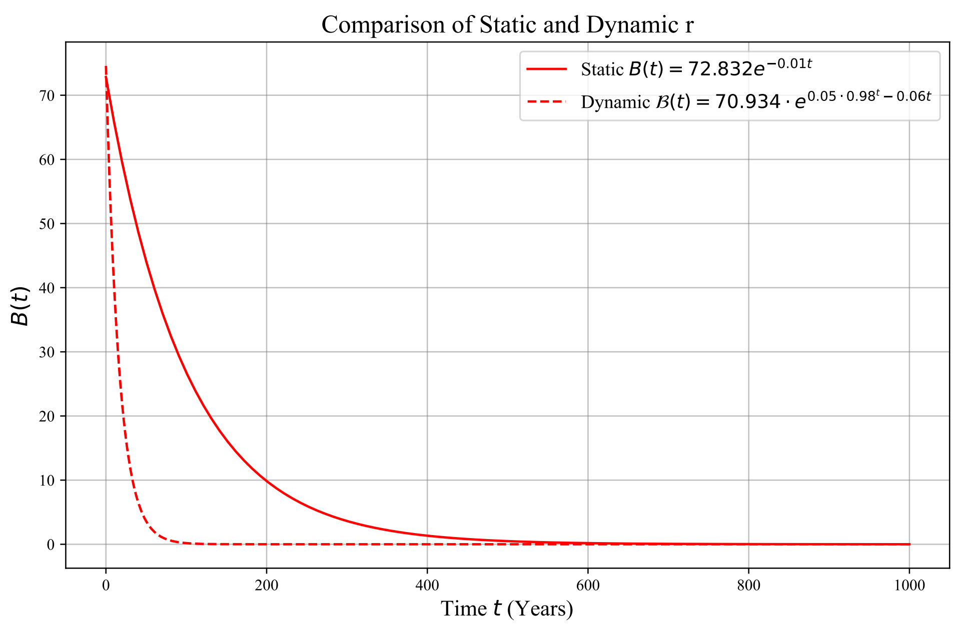

Figure 7. $\mathcal{B}(t)$ v.s. $B(t)$

6.2. **Applications.** As 2024 comes to an end, this presents an opportunity to validate the accuracy of our model using 2024 data. This is a crucial step in mathematical modeling. Based on the 2024 data, we will make further refinements to our model.

At the same time, our population model can be adapted to model the population transition between age groups in countries beyond the United States, however, our parameters need to be adjusted with the data collected from other counties and calculated in a similar manner to reflect differences in demographics in other countries and include other country-specific factors. For instance, γ the immigration factor for individuals of working age, is a big factor that we considered because of the prevalence of immigrants in the United States. This factor might not be relevant in countries with minimal immigration, while other factors may be relevant. This adaptation would help refine the model and identify optimal values for critical parameters like the growth rate r, to account for the unique demographics of each country.

7. Conclusion

In this paper, we employed a rate-in-rate-out model inspired by the Logistic model to develop a system of differential equations which we named the Population Aging Transition Model (PATM). The model takes into consideration essential parameters representing the major factors that cause population growth and decline in different age groups in the United States to simulate the population flow.

We calculated the current values of our parameters using the data from 2023. This allows us to analyze the current trends and severity of population aging. Then we adjusted the growth rate in our model and determined the optimal growth rate to be $r = 0.075$ for the younger generation to be able to surpass the retired population to ensure that there can exist a sustainable workforce. In the future, we plan to enhance our model by incorporating dynamic modeling techniques, to change our growth rate r into a variable dependent on t to make the simulation more realistic.

In the creation and evaluation of the model, we learned how to utilize modeling and data analytic skills in order to simulate and attempt to solve a real-life problem. We used real-world data and assumptions to simplify our model and consider the most relevant and effective coefficients.

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