COMBATING THE GROWING DEMAND OF THE WORKING AGE: A POPULATION AGING MODEL

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Building on this foundation, we formulate PATM with a system of three differential equations to describe the transition between the defined population age groups:

$$\frac{dB(t)}{dt} = rB(t) - \alpha B(t), \tag{1}$$

$$\frac{dW(t)}{dt} = \gamma W(t) - \beta W(t) + \alpha B(t), \tag{2}$$

$$\frac{dR(t)}{dt} = -\delta R(t) + \beta W(t), \tag{3}$$

where,

- $B(t)$ is the number of individuals below working age,
- $W(t)$ is the number of individuals of working age,
- $R(t)$ is the number of individuals above working age (retired),
- $r, \alpha, \gamma, \beta, \delta$ are weighted factors that are defined as follows and are chosen because they significantly influence population inflows and outflows.

Parameter	**Description**
r	Growth rate of the population below working age
α	Proportion of individuals transitioning to working age per year
γ	Immigration factor of individuals per year
β	Proportion of individuals retiring per year
δ	Death rate factor of the retired population per year

1. Solution

To better understand the relationship between $B(t)$ and $R(t)$, we begin by solving the system of equations involving $B(t)$, $W(t)$, and $R(t)$.

1.1. Evaluate $B(t)$.

According to Equation 1 in our system, we have:

$$\frac{dB(t)}{dt} = rB(t) - \alpha B(t),$$

which is a separable differential equation. Rearranging and integrating:

$$\int \frac{1}{B(t)}\, dB(t) = \int (r - \alpha)\, dt,$$

$$\ln B(t) = (r - \alpha)t + C_B,$$

where C_B is a constant. Taking the exponential on both sides gives:

$$B(t) = e^{C_B} \cdot e^{(r-\alpha)t}.$$

Letting $B_0 = e^{C_B}$, the initial condition, we rewrite $B(t)$ as:

$$B(t) = B_0 e^{(r-\alpha)t}.$$

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In Ssection 3.1, we set $r \approx 0.05$ and $\alpha \approx 0.06$, which leads to:

$$B(t) = B_0 e^{-0.01t}.$$

In 2023, there were 72.832 million[1] people aged 17 and under. Setting this as the initial state, $B_0 = 72.832$, we have:

$$B(t) = 72.832 e^{-0.01t}.$$

1.2. **Evaluate $W(t)$.** According to Equation 2 in our system, we have:

$$\frac{dW(t)}{dt} = \gamma W(t) - \beta W(t) + \alpha B(t),$$

which simplifies to:

$$\frac{dW(t)}{dt} = (\gamma - \beta) W(t) + \alpha B(t).$$

Substituting $B(t) = B_0 e^{(r-\alpha)t}$ into the equation:

$$\frac{dW(t)}{dt} = (\gamma - \beta) W(t) + \alpha B_0 e^{(r-\alpha)t}.$$

The integrating factor is:

$$\mu(t) = e^{-\int (\gamma-\beta)\, dt} = e^{-(\gamma-\beta)t}.$$

Multiplying both sides by $\mu(t)$:

$$e^{-(\gamma-\beta)t} \frac{dW(t)}{dt} - (\gamma - \beta) e^{-(\gamma-\beta)t} W(t) = \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t}.$$

Simplifying the left-hand side:

$$\frac{d}{dt}\left(e^{-(\gamma-\beta)t} W(t)\right) = \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t}.$$

Integrating both sides with respect to t:

$$e^{-(\gamma-\beta)t} W(t) = \int \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t}\, dt.$$

Evaluating the integral:

$$\int \alpha B_0 e^{(r-\alpha-(\gamma-\beta))t}\, dt = \frac{\alpha B_0}{r - \alpha - (\gamma - \beta)} e^{(r-\alpha-(\gamma-\beta))t}.$$

Thus, we have:

$$e^{-(\gamma-\beta)t} W(t) = \frac{\alpha B_0}{r - \alpha - (\gamma - \beta)} e^{(r-\alpha-(\gamma-\beta))t} + C_W,$$

where C_W is a constant. Multiplying both sides by $e^{(\gamma-\beta)t}$:

$$W(t) = \frac{\alpha B_0}{r - \alpha - (\gamma - \beta)} e^{(r-\alpha)t} + C_W e^{(\gamma-\beta)t}.$$

Based on the coefficients we chose in Section 3.1 and Section 3.2:

$$r - \alpha = -0.01, \quad \gamma - \beta = -0.0056,$$

[1]Data based on 2023 US Census Data [2, 1]

$$r - \alpha - (\gamma - \beta) = -0.0044.$$

Thus, $W(t)$ becomes:

$$W(t) = \frac{4.36992}{-0.0044} e^{-0.01t} + C_W e^{-0.0056t}.$$

In 2023, there were 211.52284 million people[2] aged between 18 and 66, setting it as the initial state W_0, thus we have:

$$W(0) = \frac{4.36992}{-0.0044} + C_W = 211.52284$$

We solve for C_W:

$$C_W = 1204.69.$$

Thus, the solution for $W(t)$ is:

$$W(t) = \frac{4.36992}{-0.0044} e^{-0.01t} + 1204.69 e^{-0.0056t}.$$

1.3. **Evaluate $R(t)$.** To make the process less complex, we substitute the coefficients before solving the equation. According to Equation 3 in our system, we have:

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = 0.0193 \left(1204.69 e^{-0.0056t} - 993.164 e^{-0.01t}\right) - 0.041867754 R(t),$$

Rewrite it as:

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} + 0.041867754 R(t) = 23.2505 e^{-0.0056t} - 19.1681 e^{-0.01t}.$$

The integrating factor is:

$$\mu(t) = e^{\int 0.041867754 \, \mathrm{d}t} = e^{0.041867754t}.$$

Multiplying both sides by $\mu(t)$:

$$e^{0.041867754t} \frac{\mathrm{d}R(t)}{\mathrm{d}t} + 0.041867754 e^{0.041867754t} R(t) = e^{0.041867754t} \left(23.2505 e^{-0.0056t} - 19.1681 e^{-0.01t}\right).$$

Using the substitution $0.041867754 e^{0.041867754t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{0.041867754t}\right)$, the left-hand side becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{0.041867754t} R(t)\right) = e^{0.041867754t} \left(23.2505 e^{-0.0056t} - 19.1681 e^{-0.01t}\right).$$

Integrating both sides with respect to t:

$$\int \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{0.041867754t} R(t)\right) \mathrm{d}t = \int e^{0.041867754t} \left(23.2505 e^{-0.0056t} - 19.1681 e^{-0.01t}\right) \mathrm{d}t.$$

Evaluating the integrals:

$$e^{0.041867754t} R(t) = 641.08 e^{0.0362678t} - 601.488 e^{0.0318678t} + C_R,$$

Finally, dividing both sides by $\mu(t) = e^{0.041867754t}$, we obtain:

$$R(t) = e^{-0.041867754t} \left(641.08 e^{0.0362678t} - 601.488 e^{0.0318678t} + C_R\right).$$

In 2023, there were 55.19152 million people[3] aged above 66, setting it as the initial state $R(0)$. We solve for C_R:

$$R(0) = e^0 \left(641.08 e^0 - 601.488 e^0 + C_R\right) = 55.19152.$$

[2]Data based on 2023 US CENSUS Data [2]

[3]Data based on 2023 US CENSUS Data [2]

We solve for C_R:

$$C_R = 15.59952$$

Thus, the solution for $R(t)$ is:

$$R(t) = e^{-0.041867754t}\left(641.08e^{0.0362678t} - 601.488e^{0.0318678t} + 15.59952\right).$$

References

[1] Data Commons. *United States of America*. Accessed November 29, 2024. 2024. URL: https://datacommons.org/place/country/USA?utmmedium=exploremprop=countpopt (visited on 11/29/2024).

[2] U.S. Census Bureau. *U.S. and World Population Clock*. Accessed November 29, 2024. 2024. URL: https://www.census.gov/popclock/ (visited on 11/29/2024).

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