# 4 Recurrences

- **Difficulty of Algorithms Research**
  - **Model 建模**
  - **Specify 描述**
  - **Correctness 正确性**
    - **Verify 验证**
    - **Proof 证明**
  - **Complex 复杂度 （Efficiency 有效性）**
    - **Actual computing 实际可计算性**

Design 设计

Correctness Analysis
正确性分析

Computing Analysis
可计算性分析

- **Recurrence is a basic method to analyze algorithm**

## Algorithms analysis

- **sum**

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}(t_j - 1) + c_7\sum_{j=2}^{n}(t_j - 1) + c_8(n-1)$$

| INSERTION-SORT(*A*) | cost | times |
|---|---|---|
| for( *j* = 2; *j* <=*length*[*A*]; *j*++) | $c_1$ | $n$ |
| { *key* = *A*[*j*] | $c_2$ | $n$-1 |
| // Insert *A*[*j*] into the sorted sequence *A*[1 .. *j*-1] | 0 | $n$-1 |
| *i* = *j*-1 | $c_4$ | $n$-1 |
| while( *i* > 0 && *A*[*i*] > *key*) | $c_5$ | $\sum_{j=2}^{n} t_j$ |
| { *A*[*i*+1] = *A*[*i*] | $c_6$ | $\sum_{j=2}^{n}(t_j - 1)$ |
| *i* = *i*-1 | $c_7$ | $\sum_{j=2}^{n}(t_j - 1)$ |
| } | | |
| *A*[*i*+1] = *key* | $c_8$ | $n$-1 |
| } | | |

# 4 Recurrences

**Algorithms analysis**

- **recursion**

$$T(n)=\begin{cases}1 & \text{, if } n=1\\ 2T(n/2)+n & \text{, if } n>1\end{cases} \qquad (4.1)$$

| | | cost |
|---|---|---|
| **MERGE-SORT(*A*, *p*, *r*)** | | ***T*(*n*)** |
| **1** | **if *p* < *r*** | |
| **2** | $q \leftarrow \lfloor (p+r)/2 \rfloor$ | |
| **3** | **MERGE-SORT(*A*, *p*, *q*)** | ***T*(*n*/2)** |
| **4** | **MERGE-SORT(*A*, *q*+1, *r*)** | ***T*(*n*/2)** |
| **5** | **MERGE(*A*, *p*, *q*, *r*)** | ***n*** |

# 4 Recurrences

**Algorithms analysis**

- **recursion**

$$T(n)=\begin{cases}1 & ,\text{ if } n\le 2\\ T(n-1)+T(n-2) & ,\text{ if } n>2\end{cases}$$

```
f(n)
{
   if(n<=2)
      return 1;
   else
      return f(n-1)+f(n-2);
}
```

# 4 Recurrences

## Algorithms analysis

◆ **recursion**

$$h(n) = h(0)*h(n-1) + h(1)*h(n-2) + \ldots + h(n-1)h(0)$$

(其中 n>=2) 另类递归式:

$$h(n) = ((4*n-2)/(n+1))*h(n-1)$$

该递推关系的解为:

$$h(n) = C(2n, n)/(n+1)$$

(n = 1, 2, 3,...)

## E Zexal的二叉树（签到）

时间限制：1000ms 内存限制：65536kb

通过率：200/209 (95.69%) 正确率：200/596 (33.56%)

### 题目

知识点：树，数论，dp，递归（都可以做）

上学期我们学习了二叉树，也都知道3个结点的二叉树有5种， 现给你二叉树的结点个数$n$，要你输出不同形态二叉树的种数。

### 输入

第一个数为一个整数$n(n <= 30)$

### 输出

对于每组数据，输出一行，不同形态二叉树的种数。

### 输入样例

```
3
```

### 输出样例

```
5
```

# 4 Recurrences

**A recurrence is an equation or inequality in terms of**

- **one or more base cases, and**
- **itself, with smaller arguments.**

**Examples:**

(1) $T(n) = \begin{cases} 1 & \text{, if } n=1, \\ T(n-1)+1 & \text{, if } n>1. \end{cases}$

Solution: $T(n) = n.$

(2) $T(n) = \begin{cases} 1 & \text{, if } n=1, \\ 2T(n/2)+n, & \text{if } n>1. \end{cases}$

Solution: $T(n) = n \lg n + n.$

(3) $T(n) = \begin{cases} 0 & \text{, if } n=2, \\ T(\sqrt{n})+1 & \text{, if } n>2. \end{cases}$

Solution: $T(n) = \lg \lg n.$

(4) $T(n) = \begin{cases} 1 & \text{, if } n=1, \\ T(n/3)+T(2n/3)+n, & \text{if } n>1. \end{cases}$

Solution: $T(n) = \Theta(n \lg n)$

# 4 Recurrences

**How to obtain asymptotic "Θ" or "*O*" bounds on the recurrence solution?**

- **Substitution method（置换法）: guesses a bound and then use mathematical induction to prove our guess correct.**
- **Iteration method（迭代法）: converts the recurrence into a summation and then relies on techniques for bounding summations to solve the recurrence.**
- **Recursion-tree method ( a kind of iteration method )**
- **Master method（主方法，母函数法）: provides bounds for recurrences of the form $T(n) = aT(n/b)+f(n)$, where $a \geq 1$, $b > 1$, and $f(n)$ is a given function.**

# Technicalities

**In practice, we neglect certain technical details when we state and solve recurrences.（忽略技术细节）**

1) **Assumption of integer arguments to functions**

- **Normally, $T(n)$ is only defined when $n$ is an integer**
- **Example, the worst-case running time of MERGE-SORT**

$$T(n)=\begin{cases}\Theta(1) & \text{, if } n=1,\\ T(\lceil n/2\rceil)+T(\lfloor n/2\rfloor)+\Theta(n) & \text{, if } n>1,\end{cases} \qquad (4.2)$$

# Technicalities

## 2) Ignore **boundary conditions**

- **Omit statements of the boundary conditions of recurrences, assume that $T(n)$ is constant for small $n$, that is $T(n) = \Theta(1)$ for sufficiently small $n$.**

  **（$n$ 较小时，$T(n)$为常数）**

$$T(n) = \begin{cases} \cancel{1} & \cancel{, \text{if } n = 1} \\ 2T(n/2) + n & , \text{if } n > 1 \end{cases} \qquad (4.1)$$

- **Example, state recurrence (4.1) as**

  **$T(n) = 2T(n/2) + \Theta(n)$, ( Omit $T(1)=\Theta(1)$ ) (4.3)**

  **without explicitly giving values for small $n$.**

  - **The reason is that although changing the value of $T(1)$ changes the solution to the recurrence, the order of growth is unchanged.（改变边界值，可能改变递归式的解，但不改变解的函数增长率）**

# Technicalities

neglect certain technical details

1) Assumption of integer arguments to functions
2) Ignore boundary conditions
3) **Omit floors, ceilings**

- **These details don't affect the asymptotic bounds of many recurrences encountered in the analysis of algorithms** （忽略细节不会影响算法的渐近分析）

# 4.1 The substitution method

**Key points**

**1) guessing the form of the solution;**

**2) using mathematical induction to show the solution works.**

Example: $T(n)=\begin{cases}1 & \text{, if } n=1,\\ 2T(n/2)+n, & \text{if } n>1.\end{cases}$

(1) Guess: $T(n)=n\lg n+n$

(2) Induction

**Basic**: $n=1 \Rightarrow n\lg n+n=1=T(n)$

**Inductive step**: Inductive hypothesis is that $T(k)=k\lg k+k$ for all $k<n$.

Use the inductive hypothesis of $T(n/2)$ to prove $T(n)$

$$\begin{aligned}T(n) &= 2T(n/2)+n\\ &= 2((n/2)\lg(n/2)+(n/2))+n \quad \text{(by inductive hypothesis)}\\ &= n\lg(n/2)+n+n = n(\lg n-\lg 2)+2n = n\lg n+n.\end{aligned}$$

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Use the inductive hypothesis of $T(n/2)$ to prove $T(n)$.

- **The method is powerful, but it can be applied only in cases when it is easy to guess the form of the answer.**

# 4.1 The substitution method

- **The substitution method is powerful, but it can be applied only in cases when it is easy to guess the form of the answer.**

$$h(n) = h(0)*h(n-1) + h(1)*h(n-2) + \ldots + h(n-1)h(0)$$

**(其中 n>=2) 另类递归式:**

$$h(n) = ((4*n-2)/(n+1))*h(n-1)$$

$$h(n) = C(2n, n)/(n+1)$$
$$(n = 1, 2, 3, \ldots)$$

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**输出样例**

```
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```

# 4.1 The substitution method

- **The substitution method can be used to establish either upper ($O$) or lower bounds ($\Omega$) on a recurrence.**
- **example, determining an upper bound on the recurrence**

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, \qquad \textbf{(4.4)}$$

**(1) Guessing that the solution is $T(n) = O(n \lg n)$.**

**(2) Proving $T(n) \le cn \lg n$ for a some constant $c > 0$.**

- **Assume that this bound holds for $n/2$, that is, that $T(\lfloor n/2 \rfloor) \le c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$. Substituting into the recurrence yields**

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \le 2(c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n \\ &\le cn\lg(n/2) + n = cn\lg n - cn\lg 2 + n = cn\lg n - cn + n \le cn\lg n, \end{aligned}$$

**where the last step holds as long as $c \ge 1$.**

# 4.1 The substitution method

- **Mathematical induction now requires us to show that our solution holds for the boundary conditions (some small $n$).**
- **Typically, the boundary conditions are suitable as base cases for the inductive proof.**

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, \qquad (4.4)$$

$$T(n) = O(n \lg n) \quad , \quad T(n) \le c\, n \lg n$$

- **This requirement can sometimes lead to problems.**
- **Assume that $T(1) = 1$ is the sole boundary condition of the recurrence. Then, we can't choose $c$ large enough, since $T(1) \le c\ 1 \lg 1 = 0$, which is at odds with $T(1)=1$. The case of our inductive proof fails to hold.（递归结果与初始情况矛盾，即递归证明失败？）**

## (1)To extend boundary conditions

$$T(n) = 2T(\lfloor n/2 \rfloor) + n\ ; \quad T(n) = O(n \lg n) \quad , \ T(n) \le c\ n \lg n$$

- **An inductive hypothesis inconsistent with specific boundary condition, How to overcome the difficulty?**
  **（如何克服递归结果与边界条件不一致的问题？）**
  - **asymptotic notation only requires us to prove $T(n) \le cn \lg n$ for $n \ge n_0$, where $n_0$ is a constant.**
  - **to remove the difficult boundary condition $T(1) = 1$**
  - **Impose $T(2)$ and $T(3)$ as boundary conditions for the inductive proof.**
  - **From the recurrence, we derive**
    **$T(2) = 2*T(1) + 2 = 4$ , $T(3) = 2*T(1) + 3 = 5$.**
  - **The inductive proof that $T(n) \le cn \lg n$ can now be completed by choosing any $c \ge 2$ so that**
    **$T(2) = 4 \le c*2 \lg 2$ and $T(3) = 5 \le c\ 3 \lg 3$.**

## 4.1.1 Making a good guess

- **Unfortunately, there is no general way to guess the correct solutions to recurrences.（猜想不是一种方法）**
- **Guessing a solution takes experience and, occasionally, creativity.**

  **( why we study the course? It's a training for us to get experience, to catch occasion, to have creativity. )**

- **Fortunately, though, there are some heuristics (recursion tress) that can help you become a good guesser.**

## 4.1.1 Making a good guess

- **If a recurrence is similar to one you have seen before, then guessing a similar solution is reasonable. For example,**

$$T(n) = 2T(\lfloor n/2 \rfloor + 17) + n,$$

  - **which looks difficult because of the added "17".**
  - **Intuitively, this additional term cannot substantially affect the solution to the recurrence.**（该附加项不会从本质上影响递归解）
  - **When $n$ is large, the difference between $T(\ n/2\ )$ and $T(\ n/2\ + 17)$ is not that large. Consequently, we make the guess that $T(n) = O(n \lg n)$, which you can verify as correct by using the substitution method.**

## 4.1.1 Making a good guess

- **Another way to make a good guess is to prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty.**

  （寻找松的上、下渐近界，范围缩小，逐步逼近）

  **For example**

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \qquad (4.4)$$

  - **we might start with a lower bound of $T(n) = \Omega(n)$, since we have the term $n$ in the recurrence,**
  - **and we can prove an initial upper bound $T(n) = O(n^2)$.**
  - **Then, we can gradually lower the upper bound and raise the lower bound until we converge on the correct, asymptotically tight solution of $T(n) = \Theta(n\lg n)$.**

## 4.1.2 Subtleties

- **Sometimes, guess correctly, but somehow the math doesn't seem to work out in the induction.**

**For example** $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1.$

- **Guess the solution is $O(n)$ , then try to show that $T(n) \le cn$ for an appropriate constant $c$. Substituting .. , then**

$$\begin{aligned} T(n) &= T(n/2) + T(n/2) + 1 \\ &\le c \cdot n/2 + c \cdot n/2 + 1 = cn + 1, \end{aligned}$$

**which does not imply $T(n) \le cn$ for any choice of $c$.**

## 4.1.2 Subtleties

- **Sometimes, guess correctly, but somehow the math doesn't seem to work out in the induction.**

  **For example** $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1.$

  guess $T(n) = cn$, thne $T(n) \le c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 = cn + 1$,

  contradiction.

- **Usually, it is that the inductive assumption isn't strong enough to prove the detailed bound. How to overcome?**

  **（递归假设条件不强）**

## 4.1.2 Subtleties

$$T(n) \le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \qquad \text{Solution: } T(n) = O(n)$$

- **try a larger guess $T(n) = O(n^2)$, which can work.**
- **But the guess that the solution is $T(n) = O(n)$ is correct.**
- **Intuitively, our guess is nearly right: we're only off by the constant 1, a lower-order term.**
- **Nevertheless, mathematical induction doesn't work!**
- **(2)Subtracting a lower-order term from our previous guess. New guess is $T(n) \le cn - b$, where $b \ge 0$ is constant, then**

$$T(n) \le (c\lfloor n/2 \rfloor - b) + (c\lceil n/2 \rceil - b) + 1 = cn - 2b + 1 \le cn - b\ ,$$

**as long as $b \ge 1$. As before, the constant $c$ must be chosen large enough to handle the boundary conditions.**

## 4.1.2 Subtleties

- **Most people find the idea of subtracting a lower-order term counterintuitive.** **（违反直觉）**
- **After all, if the math doesn't work out, shouldn't we be increasing our guess?**
- **The key to understand this step is to remember that we are using mathematical induction: we can prove something stronger for a given value by assuming something stronger for smaller values.** **（假设更强的条件，可证明更强的结论）**

# 4.1.3 (3)Avoiding pitfalls (陷阱)

- **It is easy to err in the use of asymptotic notation.**

**For example, in the recurrence (4.4)**

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \qquad (4.4)$$

**we can falsely prove $T(n) = O(n)$ by guessing $T(n) \leq cn$ and then arguing**

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \leq 2(c\lfloor n/2 \rfloor) + n$$

$$\leq cn + n$$

$$= O(n)\ , \qquad \Leftarrow wrong\ !!!$$

**since $c$ is a constant. The error is that we haven't proved the exact form of the inductive hypothesis, that is, that $T(n) \leq cn$.**

## 4.1.4 Changing variables

- **algebraic manipulation: sometimes solute an unknown recurrence similar to one you have seen before.**

**Example,** $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \lg n,$

**which looks difficult. Simplify the recurrence with a change of variables. For convenience, we shall not worry about rounding off values, such as** $\sqrt{n}$ **, to be integers.**

**Let** $m = \lg n$ **, then** $T(2^m) = 2T(2^{m/2})+m$.

**Thus rename** $S(m)=T(2^m)$ => $S(m)=2S(m/2)+m$,

**which is very much like recurrence (4.4) and has the same solution:** $S(m)=O(m \lg m)$ **. Changing back from** $S(m)$ **to** $T(n)$**, we obtain** $T(n)=T(2^m)=S(m)=O(m \lg m)=O(\lg n \lg\lg n)$ **.**

# 4.1 The substitution method: Notes

**(1) To extend boundary conditions**

**(2) Subtracting a lower-order term**

**(3) Avoiding pitfalls**

## 4.2 The iteration method

- **Substitution: It is difficult to come up with a good guess**
- **The iteration method**
  - **doesn't require us to guess the answer**
  - **may require more algebra**（迭代法对代数能力的要求较高）
  - **to expand (iterate) the recurrence and express it as a summation of terms, and the initial conditions**
  - **to evaluate summations.**（不断迭代展开为级数，并求和）

**For example,** $T(n) = 3T(\lfloor n/4 \rfloor) + n$

$$
\begin{aligned}
T(n) &= n + 3T(\lfloor n/4 \rfloor) \\
&= n + 3\ (\lfloor n/4 \rfloor + 3T(\lfloor n/16 \rfloor)) \\
&= n + 3(\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/64 \rfloor))) \\
&= n + 3\lfloor n/4 \rfloor + 9\lfloor n/16 \rfloor + 27T(\lfloor n/64 \rfloor),
\end{aligned}
$$

# 4.2 The iteration method

$$T(n)=n+3T(\lfloor n/4\rfloor)=n+3\ (\lfloor n/4\rfloor+3T(\lfloor n/16\rfloor))$$
$$=n+3(\lfloor n/4\rfloor+3(\lfloor n/16\rfloor+3T(\lfloor n/64\rfloor)))$$
$$=n+3\lfloor n/4\rfloor+9\lfloor n/16\rfloor+27T(\lfloor n/64\rfloor),$$

- **How far must we iterate the recurrence?**
  - **The *i*th term in the series is** $3^iT(\lfloor n/4^i\rfloor)$ **.**
  - **The iteration halts when** $\lfloor n/4^i\rfloor=1$**. By continuing the iteration until this point and using the bound** $\lfloor n/4^i\rfloor\le n/4^i$ **, we get a decreasing geometric series:**

$$T(n)\le n+3n/4+9n/16+27n/64+\ldots+3^iT(n/4^i)$$

$$\le n\sum_{i=0}^{\infty}(3/4)^i+\Theta(n^{\log_4 3})=4n+o(n)=O(n).$$

$$(n/4^i=1\ \Rightarrow\ i=\log_4 n\ \Rightarrow\ 3^i=3^{\log_4 n}=n^{\log_4 3})$$

## 4.2 The iteration method

- **The iteration method usually leads to lots of algebra. It can be a challenge. The key points:**
  - **the number of times the recurrence needs to be iterated to reach the boundary condition,**（迭代次数）
  - **and the sum of the terms arising from each level of the iteration process.**（级数求和）

- **Sometimes, in the process of iterating a recurrence, you can guess the solution without working out all the math. Then, the iteration can be abandoned in favor of the substitution method, which usually requires less algebra.**

  （在展开递归式为迭代求和的过程中，有时只需要部分展开，然后根据其规律来猜想递归式的解，接着用置换法进行证明。）

# 4.2 The iteration method

- **When a recurrence contains floor and ceiling functions, the math can become especially complicated.**

- **Often, it helps to assume that the recurrence is defined only on exact powers of a number.**

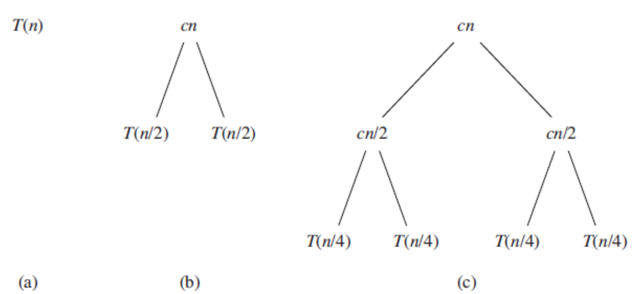
**Example,** $T(n) = 3T(\lfloor n/4 \rfloor) + n$

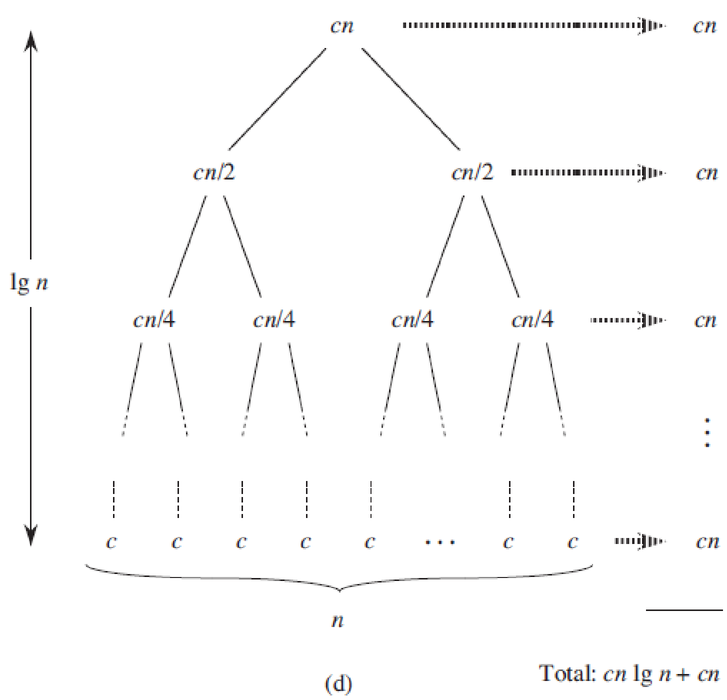
**if we had assumed that $n = 4^k$ for some integer $k$, the floor functions could have been conveniently omitted.**

# * 4.3 The recursion-tree method

- **Drawing out a recursion tree, is a straightforward way to devise a good guess, and to show the iteration method intuitively.**（画递归树可以从直观上表示迭代法，也有助于猜想递归式的解）
- **Recursion trees are particularly useful when the recurrence describes the running time of a divide-and-conquer algorithm.**

$$T(n) = 2T(n/2) + n$$



# 4.4 The master method（主方法，母函数法）

- **solving recurrences of the form**

$$T(n) = aT(n/b)+f(n), \tag{4.5}$$

**where $a \geq 1$ and $b>1$, and $f(n)$ is asymptotically positive.**

- **The master method requires memorization of three cases, but then the solution of many recurrences can be determined quite easily, often without pencil and paper.**
- **$n/b$ might not be an integer. Replacing each of $T(n/b)$ with either $T(\lfloor n/b \rfloor)$ or $T(\lceil n/b \rceil)$ doesn't affect the asymptotic behavior .**
- **Normally, it is convenient to omit the floor and ceiling functions when writing divide-and-conquer recurrences of this form.**

# 4.4.1 The master theorem

- **Theorem 4.1**

  **Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence**

  $$T(n) = aT(n/b) + f(n),$$

  **where we interpret $n/b$ to mean either $\lfloor n/b \rfloor$ *or* $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.**

  1. If $f(n) = O(n^{(\log_b a) - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
  2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
  3. If $f(n) = \Omega(n^{(\log_b a) + \varepsilon})$ for some constant $\varepsilon > 0$,

     and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ ?

     and all sufficiently large $n$, then $T(n) = \Theta(f(n))$.

? 的意思是 $a.\ f(n/b) > f(n)$ 不能得出 $T(n) = d.f(n)$

proof: Let $T(n) = d.f(n)$, then $T(n) = a.T(n/b) + f(n)$ ➔ $d.\ f(n) = a.d.f(n/b) + f(n)$

If $a.\ f(n/b) > f(n)$, then $d.\ f(n) = a.d.\ f(n/b) + f(n) > d.\ f(n) + f(n) = (d+1).\ f(n)$ ➔ $d > d+1$

# 4.4.1 The master theorem: understand what it says

$$T(n) = aT(n/b) + f(n),$$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right), & f(n) = O\left(n^{(\log_b a)-\varepsilon}\right) \\ \Theta\left(n^{\log_b a} \lg n\right), & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta\left(f(n)\right), & f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) \text{ and } af(n/b) \le cf(n) \text{ for large } n \end{cases} \Bigg\} \exists \varepsilon > 0, c < 1$$

- **Comparing the function $f(n)$ with $n^{\log_b a}$. Intuitively, the solution is determined by the larger of the two functions.**
  - **Case 1, $n^{\log_b a}$ larger, then the solution is $T(n) = \Theta(n^{\log_b a})$.**
  - **Case 3, $f(n)$ larger, then the solution is $T(n) = \Theta(f(n))$.**
  - **Case 2, the two functions are the same size, we multiply by a logarithmic factor, and the solution is**

$$T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(f(n) \lg n) \quad .$$

Case 1: $f(n) \le n^{(\log_b a)-\varepsilon} = n^{(\log_b a)} / n^{\varepsilon} < n^{(\log_b a)}$

## 4.4.1 The master theorem: special cases

$$T(n) = aT(n/b) + f(n),$$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right), & f(n) = O\left(n^{(\log_b a)-\varepsilon}\right) \\ \Theta\left(n^{\log_b a}\lg n\right), & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta(f(n)), & f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) \text{ and } af(n/b) \le cf(n) \text{ for large } n \end{cases} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \begin{matrix}\exists\varepsilon > 0 \\ , c < 1\end{matrix}$$

- **Polynomially**
  - **Case 1, *f*(*n*) must be polynomially smaller than** $n^{\log_b a}$ **.**
  - **Case 3, *f*(*n*) must be polynomially larger than** $n^{\log_b a}$ **.**
- **Gap**

  Example: $f(n) = n\lg n$ , $n^{\log_b a} = n$

  - **There is a gap between cases 1 and 2 when *f*(*n*) is smaller than** $n^{\log_b a}$ **but not polynomially smaller.**
  - **Similarly, there is a gap between cases 2 and 3 when *f*(*n*) is larger than** $n^{\log_b a}$ **but not polynomially larger.**

# 4.4.2 Using the master method: some examples

$$T(n) = aT(n/b) + f(n),$$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right), & f(n) = O\left(n^{(\log_b a)-\varepsilon}\right) \\ \Theta\left(n^{\log_b a} \lg n\right), & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta\left(f(n)\right), & f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) \text{ and } af(n/b) \le cf(n) \text{ for large } n \end{cases} \left. \vphantom{\begin{cases} a \\ b \\ c \end{cases}} \right\} \begin{array}{l} \exists \varepsilon > 0 \\ , c < 1 \end{array}$$

- $T(n)=9T(n/3)+n$

- $T(n)=T(2n/3)+1$

# 4.4.2 Using the master method: some examples

$$T(n) = aT(n/b) + f(n),$$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right), & f(n) = O\left(n^{(\log_b a)-\varepsilon}\right) \\ \Theta\left(n^{\log_b a}\lg n\right), & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta(f(n)), & f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) \text{ and } af(n/b) \le cf(n) \text{ for large } n \end{cases} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \begin{matrix}\exists\varepsilon > 0\\ ,c < 1\end{matrix}$$

- ***T(n)=9T(n/3)+n***

$$a = 9, b = 3, f(n) = n \quad \Rightarrow \quad n^{\log_b a} = n^{\log_3 9} = n^2 = \Theta(n^2)$$

$$\Rightarrow \quad f(n) = O(n^{\log_3 9-\varepsilon}), \text{where } \varepsilon = 1 \quad \Rightarrow \quad T(n) = \Theta(n^2)$$

- ***T(n)=T(2n/3)+1***

$$a = 1, b = 3/2, f(n) = 1 \quad \Rightarrow \quad n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$$

$$\Rightarrow \quad f(n) = \Theta(n^{\log_3 a}) = \Theta(1) \quad \Rightarrow \quad T(n) = \Theta(\lg n)$$

# 4.4.2 Using the master method: some examples

$$T(n) = aT(n/b) + f(n),$$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right), & f(n) = O\left(n^{(\log_b a)-\varepsilon}\right) \\ \Theta\left(n^{\log_b a}\lg n\right), & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta(f(n)), & f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) \text{ and } af(n/b) \le cf(n) \text{ for large } n \end{cases} \left.\vphantom{\begin{cases}a\\b\\c\end{cases}}\right\} \begin{matrix} \exists \varepsilon > 0 \\ , c < 1 \end{matrix}$$

- ***T*(*n*)=3*T*(*n*/4)+*n*lg*n***

$$a = 3, b = 4, f(n) = n\lg n \quad \Rightarrow \quad n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$$

$$\Rightarrow \quad f(n) = \Omega(n^{(\log_4 3)+\varepsilon}), \text{where } \varepsilon \approx 0.2, \text{ and for sufficiently large } n,$$

$$af(n/b) = 3(n/4)\lg(n/4) \le (3/4)n\lg n = cf(n) \text{ for } c = 3/4$$

$$\Rightarrow \quad T(n) = \Theta(n\lg n)$$

# 4.4.2 Using the master method: some examples

$$T(n) = aT(n/b) + f(n),$$

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right), & f(n) = O\left(n^{(\log_b a)-\varepsilon}\right) \\ \Theta\left(n^{\log_b a} \lg n\right), & f(n) = \Theta\left(n^{\log_b a}\right) \\ \Theta(f(n)), & f(n) = \Omega\left(n^{(\log_b a)+\varepsilon}\right) \text{ and } af(n/b) \le cf(n) \text{ for large } n \end{cases} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \begin{matrix} \varepsilon > 0 \\ c < 1 \end{matrix}$$

- ***T(n)=2T(n/2)+nlgn***

$$a = 2, b = 2, f(n) = n \lg n \quad \Rightarrow \quad n^{\log_b a} = n^{\log_2 2} = n^1 = n,$$

but $f(n)/n = \lg n$, which is asymptotically less than $n^{\varepsilon}$ for any positive constant $c$, that is $f(n)$ is not polynomially larger than $n$. Consequently, the recurrence falls into the gap between case 2 and case 3.

# Exercises and problems

**Please give bounds for the following recurrences.**

$T(n) = 4T(n/2) + n$

$T(n) = 4T(n/2) + n^2$

$T(n) = 4T(n/2) + n^3$

$T(n) = T(n\text{-}1) + T(n\text{-}2)$

$T(n) = 2*T(n\text{-}1) + 1$

$T(n) = T(n\text{-}1) + T(n\text{-}5)$

# Exercises and problems

$$h(n) = h(0)*h(n-1) + h(1)*h(n-2) + \ldots + h(n-1)h(0)$$

**(其中 n>=2) 另类递归式:**

$$h(n) = ((4*n-2)/(n+1))*h(n-1)$$

**请证明，该递推关系的解为:**

$$h(n) = C(2n, n)/(n+1)$$

**(n = 1, 2, 3,...)**

## E Zexal的二叉树（签到）

时间限制：1000ms 内存限制：65536kb

通过率：200/209 (95.69%) 正确率：200/596 (33.56%)

### 题目

知识点：树，数论，dp，递归（都可以做）

上学期我们学习了二叉树，也都知道3个结点的二叉树有5种， 现给你二叉树的结点个数$n$，要你输出不同形态二叉树的种数。

### 输入

第一个数为一个整数$n(n <= 30)$

### 输出

对于每组数据，输出一行，不同形态二叉树的种数。

### 输入样例

```
3
```

### 输出样例

```
5
```