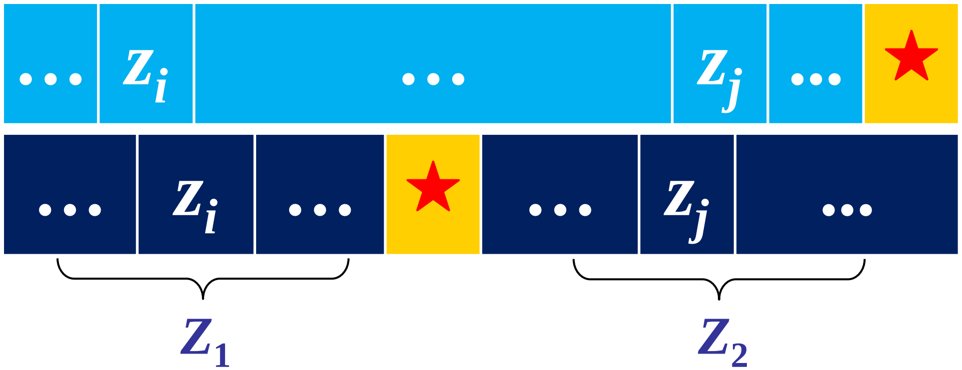
Chapter 5

Probabilistic Analysis and Randomized Algorithms

Why is QUICKSORT quick?

```
QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
```



$T(n) = T(x) + T(n\text{-}x\text{-}1) + \Theta(n)$?

(1) $T(n) = T(n/2) + T(n/2 - 1) + \Theta(n)$ （分区位置是中间时）

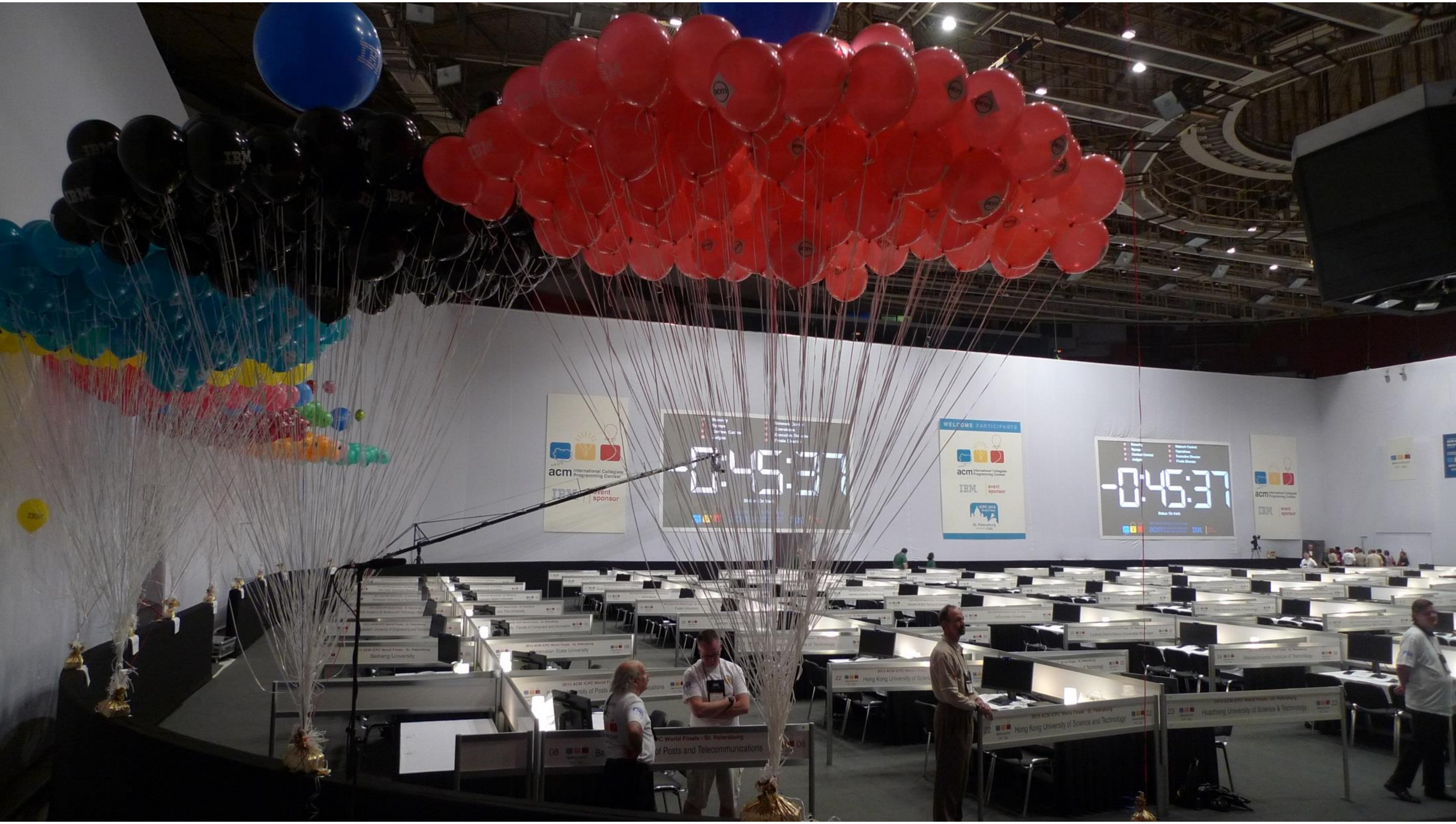
(2) $T(n) = T(a) + T(n\text{-}a\text{-}1) + \Theta(n)$ （a是常数）

(3) ……

……

5 Probabilistic Analysis and Randomized Algorithms

- **Explain the difference between probabilistic analysis & randomized algorithms.**
  （概率分析与随机算法）
- **Present the technique of indicator random variables.**
  （用指示(器)随机变量来对算法进行概率分析）
- **Give another example of a randomized algorithm.**
  （一个随机算法实例）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St. Petersburg

2013 ACM-ICPC World Final, St. Petersburg, Russia

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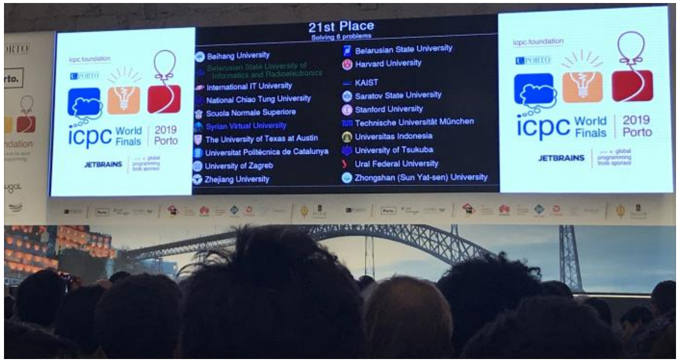
2019 ICPC World Final, Porto, Portugal

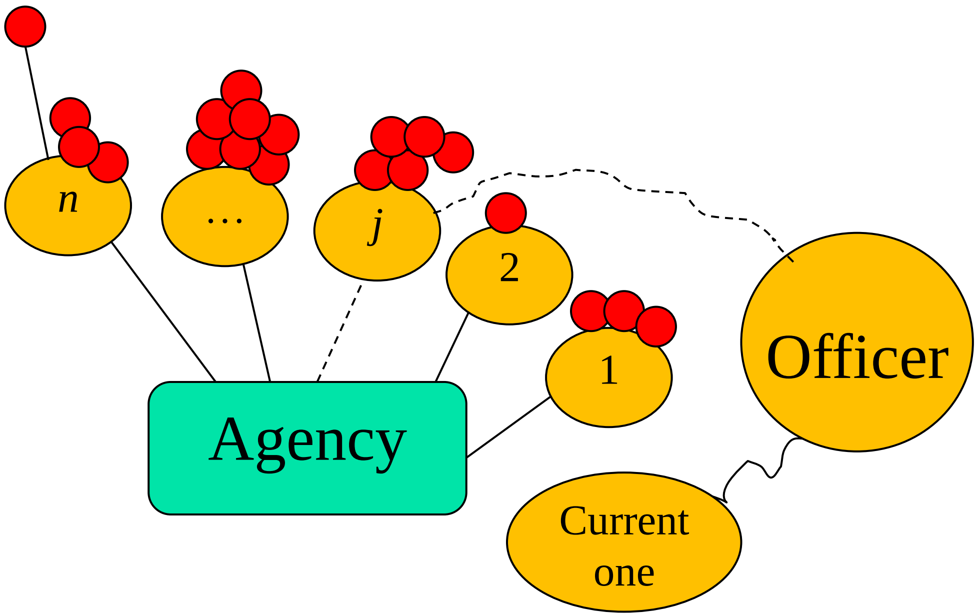


5.1 The hiring problem (雇佣问题，or 助手问题)

Scenario（情景）:hire the best office assistant in a month

- **You are using an employment agency to hire a new office assistant.（猎头（代理）公司帮你物色办公助理候选人）**

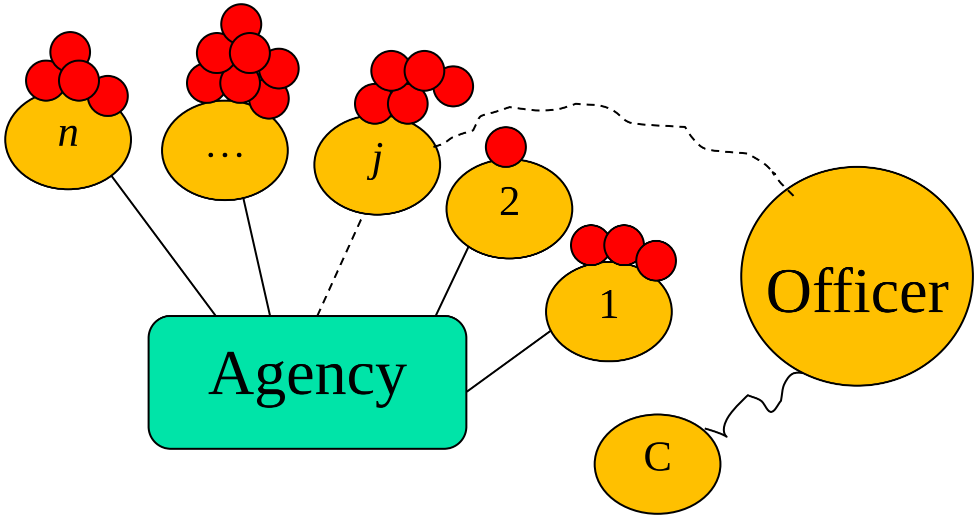
5.1 The hiring problem



Scenario（情景）:hire the best office assistant in a month

- **You are using an employment agency to hire a new office assistant.**
  （猎头（代理）公司帮你物色办公助理候选人）
- **The agency sends you one candidate each day.**
  (1, 2, ... , *n*表示候选人的编号，红色圆圈候选人的工作能力)
- **You interview the candidate and must immediately decide whether or not to hire him. But if you hire, you must also fire your current one.**
- **……**

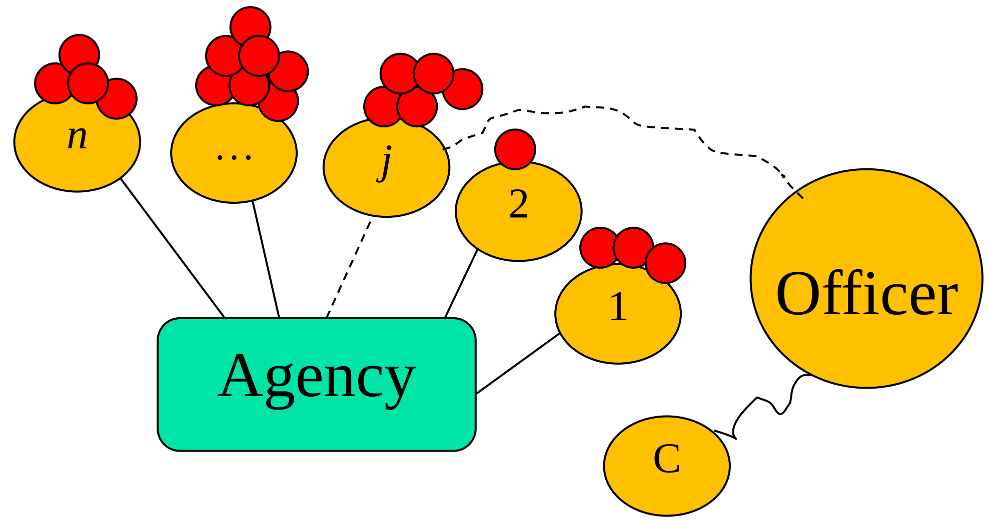
5.1 The hiring problem



- You are using an employment agency to hire a new office assistant.
- The agency sends you one candidate each day.
- You interview the candidate and must immediately decide whether or not to hire him. But if you hire, you must also fire your current one.
- **Cost to interview is 1K per candidate (interview fee paid to agency). （面试一个候选人支付代理公司 1K）**
- **Cost to hire is 10K per candidate, includes:**
  cost to fire current office assistant + hiring fee paid to agency

***Goal*: Determine what the price of this strategy will be？**

5.1 The hiring problem



Pseudocode to model this scenario: Assumes that the candidates are numbered 1 to *n*. Uses a dummy candidate 0 that is worse than all others, so that the first candidate is always hired.

HIRE-ASSISTANT(*n*)	**cost**	**times**
best **← 0 // candidate 0 is a least-qualified dummy candidate**		
for *i* ← 1 to *n*		
interview candidate *i*	c_i	n
if candidate *i* is better than candidate *best*		
best **←** ***i***		
hire candidate *i*	c_h	m

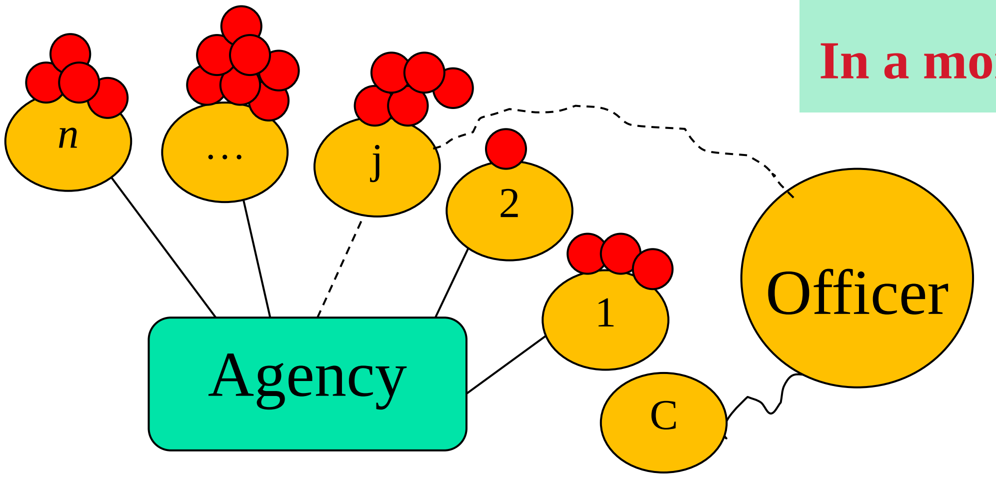
Cost: *n* candidates, we hire *m* of them, cost is T(*n*) = $nc_i + mc_h$

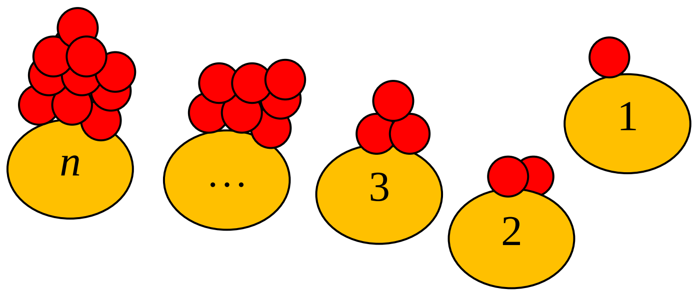
5.1.1 Worst-case analysis

Cost: n candidates, we hire m of them, cost is $T(n) = nc_i + mc_h$

- **We focus on analyzing the hiring cost mc_h?** **($c_h >> c_i$)**
- **mc_h varies with each run of the algorithm: it depends on the order in which we interview the candidates.**
- **Worst-case analysis**
  - **We hire all n candidates. $T(n) = nc_i + nc_h$? When?**
  - **The candidates appear in increasing order of qulity.**

In a month, $T(30) = 30 + 30*10 =$ 330 K

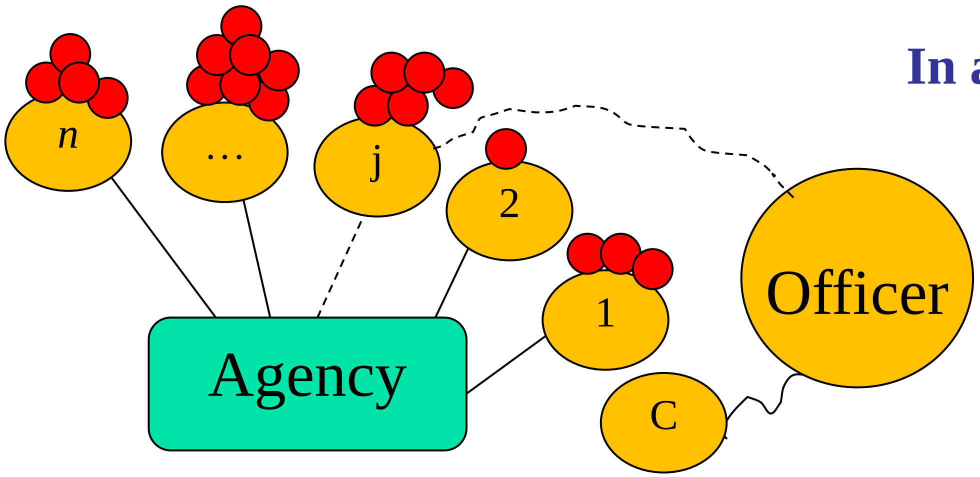




Cost: n candidates, we hire m of them, cost is $T(n) = nc_i + mc_h$

- **Worst-case analysis**
  - **We hire all n candidates. $T(n) = nc_i + nc_h$.**

- **Average-case ?**
- **Best-case ?**



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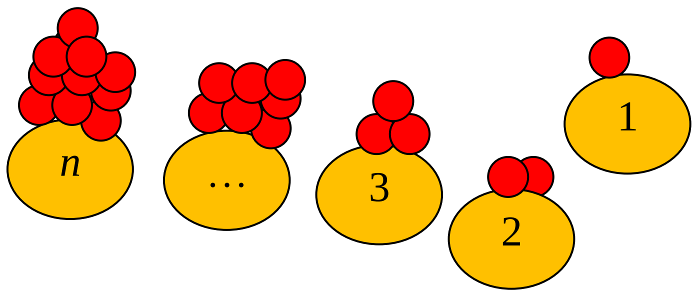
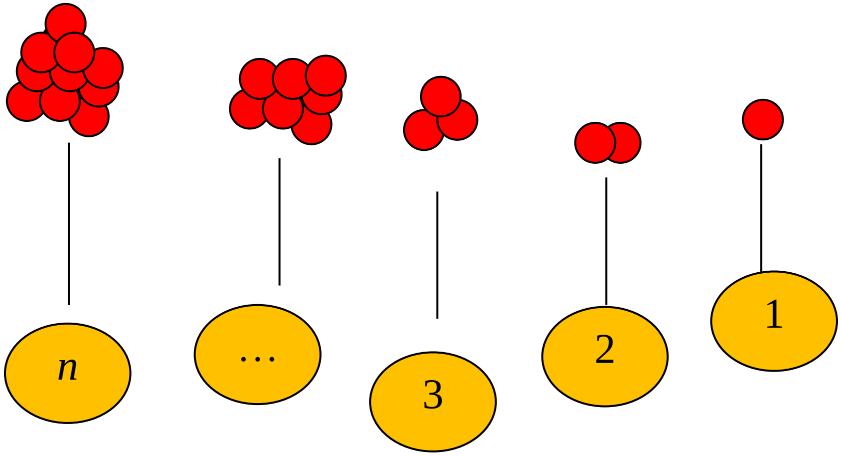
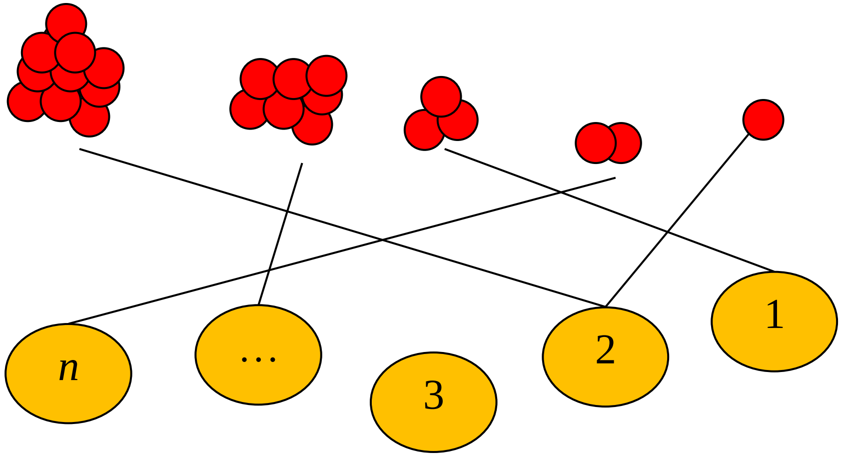


5.1.2 Propbabilistic analysis

Assume that candidates come in a random order:

- **Assign a rank to each candidate: *rank*(*i*) is a unique integer in the range 1 to *n*.** **（对每一个候选人分配一个“名次”）**
- **The list <*rank*(1), …, *rank*(*n*)> is a permutation of the candidate numbers <1, …, *n*>, such as <5, 2, 1, 28, 9, … , 11>**
- **The ranks form a uniform random permutation: each of the possible *n*! permutations appears with equal probability**





5.1.2 Propbabilistic analysis

	cost	times
HIRE-ASSISTANT(*n*)	**cost**	**times**
best **← 0 // candidate 0 is a least-qualified dummy candidate**		
for *i* ← 1 to *n*		
interview candidate *i*	c_i	n
if candidate *i* is better than candidate *best*		
best **←** ***i***		
hire candidate *i*	c_h	m

Essential idea of probabilistic analysis: **We must use knowledge of, or make assumptions about, the distribution of inputs.**

（概率分析的本质：需已知或假定输入的概率分布）

- **The expectation is taken over this distribution.**
  （依据概率分布来求期望，期望值即是平均 hire 的人数）
- **Section 5.2 contains a probabilistic analysis of the hiring problem.**

5.1.3 Randomized algorithms

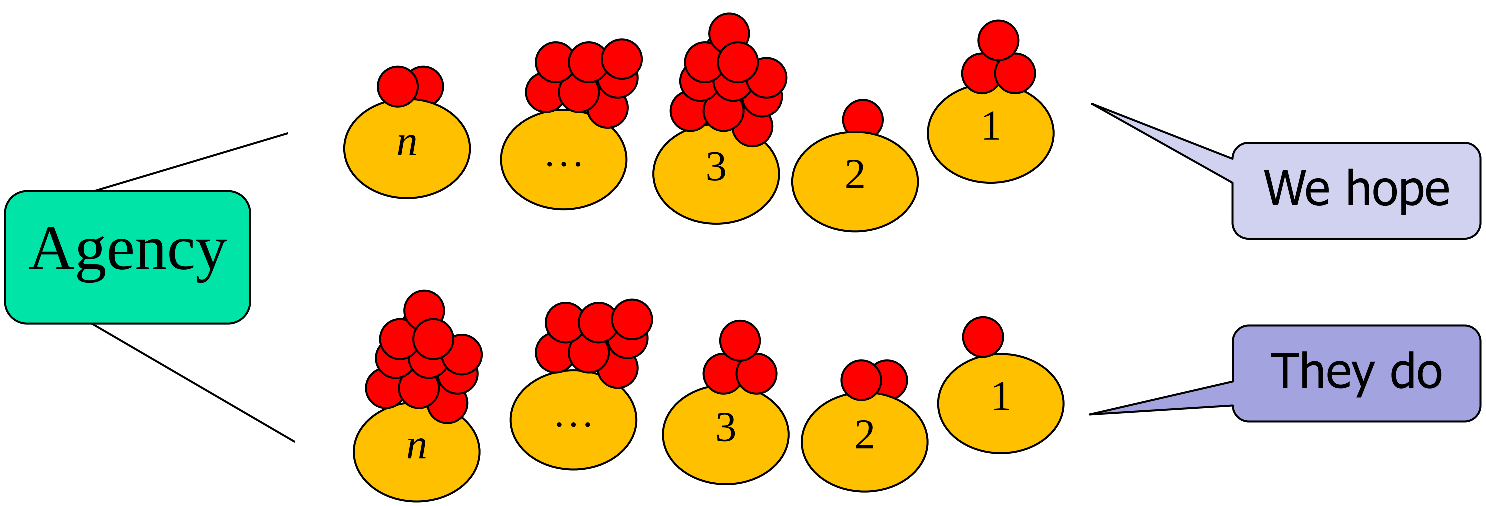
Idea

- **We might not know the distribution of inputs, or we might not be able to model it computationally.**

  （我们不知道输入的分布，也不可能为输入的分布进行可计算的建模）

- **Instead, we use randomization within the algorithm in order to impose a distribution on the inputs.**

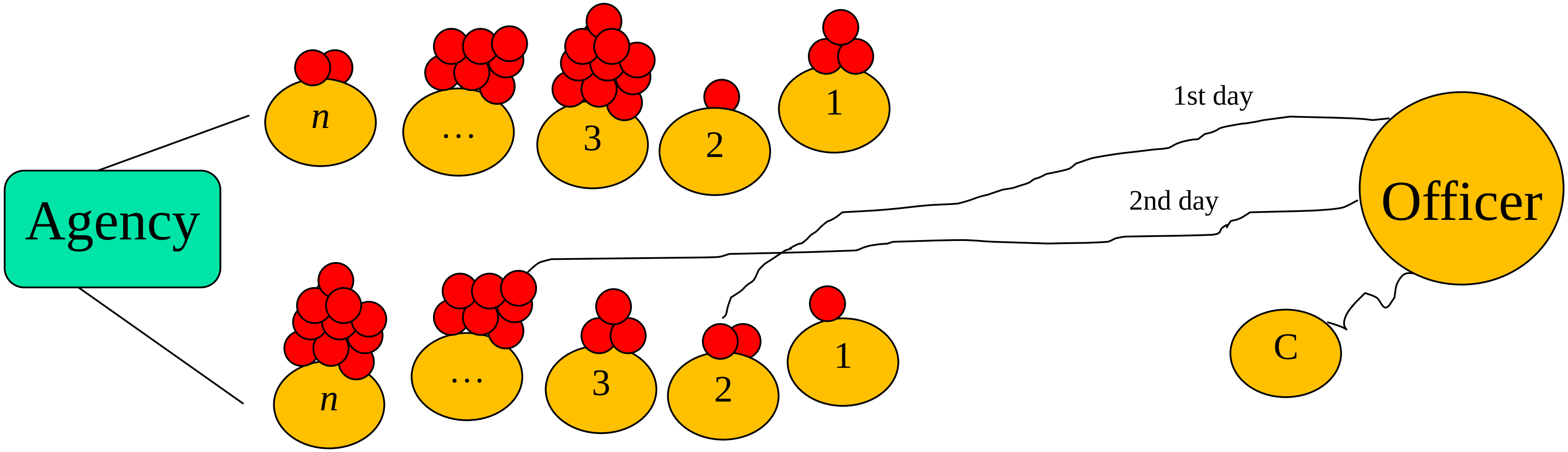
  （在算法中通过对输入进行随机化处理，从而为输入强加一种分布）



5.1.3 Randomized algorithms

For the hiring problem, **change the scenario:**

- **The employment agency sends us a list of all *n* candidates in advance.**
  （猎头公司预先提供 *n* 个候选人列表）
- **On each day, we randomly choose a candidate from the list to interview.**
  （每天，我们随机选取一人进行面试）
- **Instead of relying on the candidates being presented to us in a random order, we take control of the process and enforce a random order. (Chap 5.3)**
  （无须担心候选人是否被随机提供，我们通过随机运算预处理可以控制候选人的随机顺序）



5.1.3 Randomized algorithms

What makes an algorithms randomized: An algorithm is randomized if its behavior is determined in part by values produced by a *random-number generator*.

（算法随机化：由随机数产生器生成数值……）

- **RANDOM(*a*, *b*) returns an integer *r*, where $a \le r \le b$ and each of the *b*-*a*+1 possible values of *r* is equally likely.**
- **In practice, RANDOM is implemented by a pseudorandom-number generator, which is a deterministic method returning numbers that "look" random and pass statistical tests.（RANDOM实际上由一个确定的算法〔伪随机产生器〕产生，其结果表面上看上去像是随机数）**

5.1.3 Randomized algorithms

Random number generator（随机数产生器）

- **Most random number generators generate a sequence of integers by the following recurrence**
- **X_0= a given integer (seed), $X_{i+1} = aX_i \bmod M$**
- **For example, for X_0=1, a=5, M=13, we have X_1=5%13=5, X_2=25%13=12, X_3= 60%13=8. Each integer in the sequence lies in the range [0, M - 1].**

A probabilistic analysis of the hiring problem

5.2 Indicator random variables（指示随机变量）

Given a sample space and an event *A*, we define the indicator random variable:

$$\mathrm{I}\{A\}=\begin{cases}1 & \text{if } A \text{ occur,}\\ 0 & \text{if } A \text{ does not occur.}\end{cases}$$

- ***Lemma***

For an event, let $X_A = \mathrm{I}\{A\}$. Then $\mathrm{E}[X_A] = \Pr\{A\}$.

Proof **Letting $\sim A$ be the complement of *A*, we have**

$$\begin{aligned}\mathrm{E}[X_A] &= \mathrm{E}[\mathrm{I}\{A\}]\\ &= 1.\Pr\{A\} + 0.\Pr\{\sim A\} \qquad \{\text{definition of expected value}\}\\ &= \Pr\{A\}.\end{aligned}$$

5.2 Indicator random variables（指示随机变量）

- ***Lemma***

  For an event A, let $X_A = \mathrm{I}\{A\}$.
  Then $\mathrm{E}[X_A]=\Pr\{A\}$.

- **Simple example: Determine the expected number of heads when we flip a fair coin one time.**

  （投一次硬币，正面向上的期望〔平均数〕）

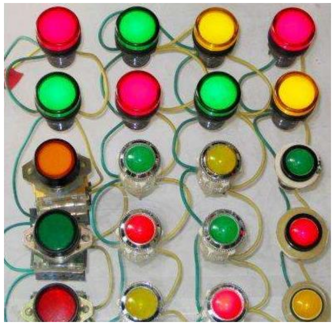
  - Sample space is $\{H, T\}$
  - $\Pr\{H\} = \Pr\{T\} = 1/2$
  - Define indicator random variable $X_H = \mathrm{I}\{H\}$.

    X_H counts the number of heads in one flip.
  - Since $\Pr\{H\} =1/2$, lemma says that $\mathrm{E}[X_H] = 1/2$.

5.2 Indicator random variables（指示随机变量）

- *Lemma*

  For an event A, let X_A = I{A}.
  Then E[X_A]=Pr{A}.

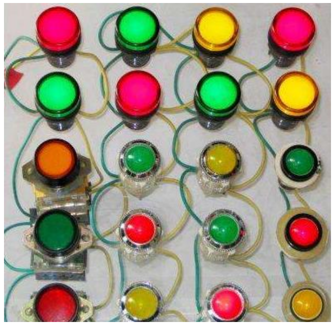


- **Slightly more complicated example: Determine the expected number of heads when in *n* coin flips.**
  （投 *n* 次硬币，正面向上的期望〔平均数〕）
  - **Let *X* be a random variable for the number of heads in *n* flips.**（令随机变量 *X* 表示投 *n* 次硬币正面向上的数）
  - **Then,** $$\mathrm{E}[X]=\sum_{k=0}^{n} k\cdot \Pr\{X=k\}$$
    a slightly more complicated? Yes!
  - **Instead, we'll use indicator random variables……**

5.2 Indicator random variables（指示随机变量）

- ***Lemma***

  For an event *A*, let X_A = I{*A*}.
  Then E[X_A]=Pr{*A*}.



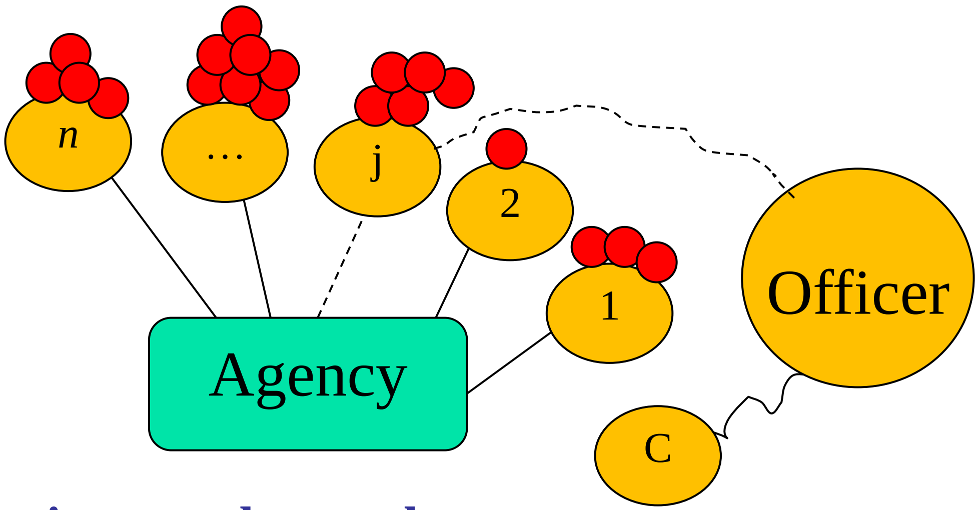
- **Slightly more complicated example: Determine the expected number of heads when in *n* coin flips.**

  （投 *n* 次硬币，正面向上的期望〔平均数〕）

  - **For *i*=1, …, *n*, define X_i = I{the *i*th flip results in event *H*}**
  - **Then,** $X = \sum_{i=1}^{n} X_i$
  - **Lemma says that E[X_i] = Pr{*H*} = 1/2 for i=1, 2, …, n**
  - **Expected number of heads is**

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i] = \sum_{i=1}^{n} 1/2 = n/2$$

Analysis of the hiring problem



Assume that the candidates arrive in a random order.

Let *X* be a random variable that equals the number of times we hire new office assistant.

（令随机变量 *X* 表示我们雇用新助手的人数）

- **Use a probabilistic analysis**

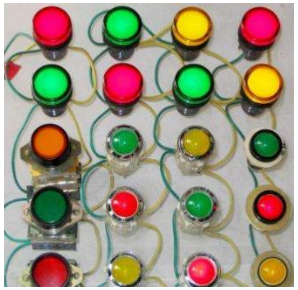
  Then

$$E[X]=\sum_{i=1}^{n} i \Pr\{X=i\}$$

  The calculation would be cumbersome（计算烦琐）

Analysis of the hiring problem

Assume that the candidates arrive in a random order.



random variable X = the number of times we hire new office assistant（随机变量 *X* 表示我们雇用新助手的人数）

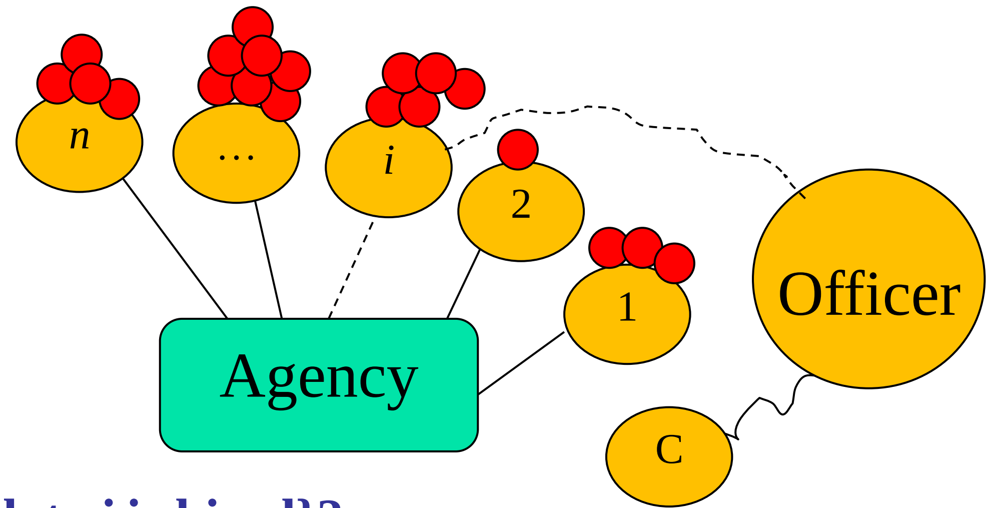
- **Use indicator random variables**
  - **Define indicator random variables $X_1, X_2, \ldots, X_n$, where X_i = I{ candidate *i* is hired}**
  - **Useful properties:**

$$X = X_1 + X_2 + \ldots + X_n$$

Lemma => E[X_i] = Pr{ candidate *i* is hired}.

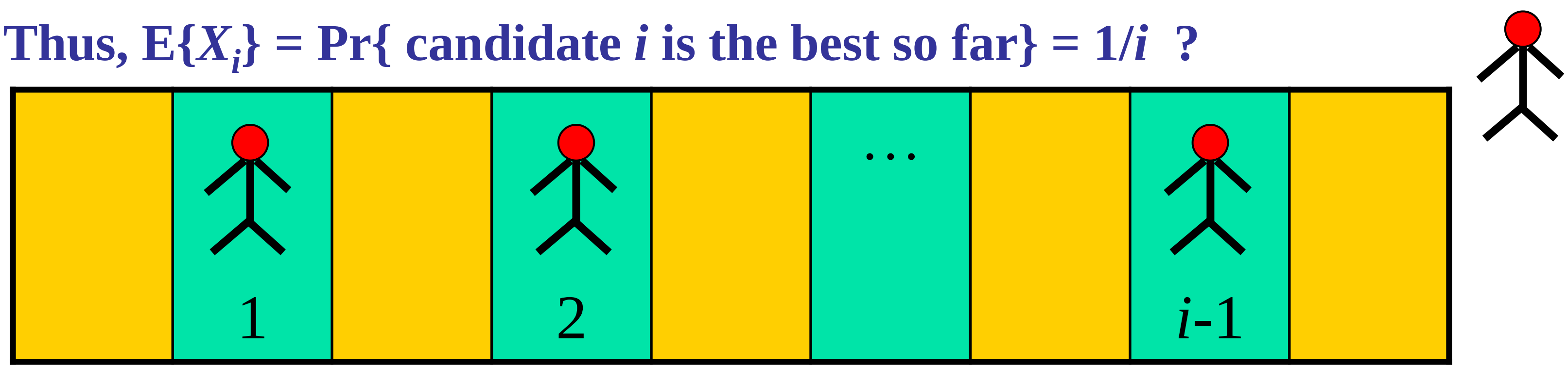
We need to compute Pr{ candidate *i* is hired}?

Analysis of the hiring problem



We need to compute Pr{ candidate *i* is hired}?

- ***i* is hired ⇔ *i* is better than each of candidates 1, 2, … , *i*-1.**
- **Assumption that the candidates arrive in random order => any one of these candidates is equally likely to be the best one.**
  （若候选人随机到来，则每一个候选人为最佳人选的概率相等）
- **Thus, E{X_i} = Pr{ candidate *i* is the best so far} = 1/*i* ?**



Analysis of the hiring problem

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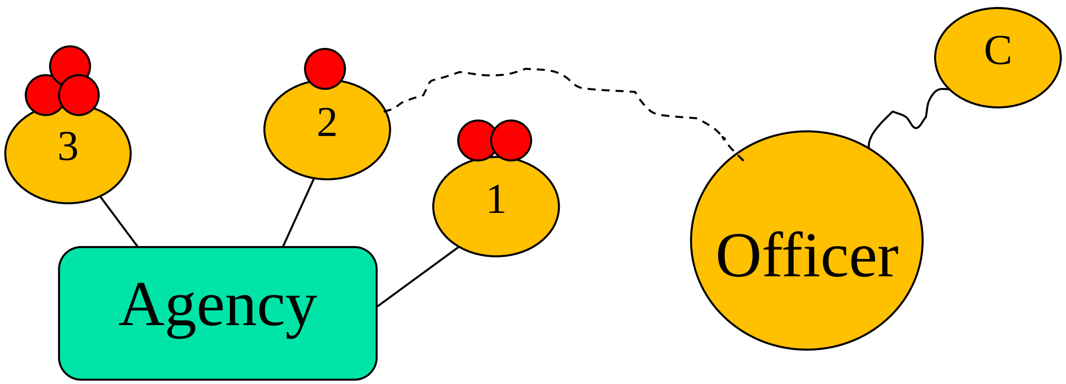


viewed : viewing , Pr of hired
已面试：待面试，待面试人被雇佣的概率
0 : {1,2,3} , 1

1:{2, 3}, 1/3*1
2:{1, 3}, 1/3*1/2
3:{1, 2}, 1/3*0

{1, 2}:3, 1/3*1
{1, 3}:2, 1/3*0
{2, 3}:1, 1/3*0

第一天面试的人的资历可能是1or2or3，每种情况的Pr是1/3；若是第一种情况，第二天面试的人被雇佣的Pr是1，则条件概率 Pr=1/3*1；其他情况相似。

Analysis of the hiring problem

HIRE-ASSISTANT(n)	cost	times
best ← 0 // candidate 0 is a least-qualified dummy candidate		
for *i* ← 1 to *n*		
interview candidate *i*	c_i	n
if candidate *i* is better than candidate *best*		
best ← *i*		
hire candidate *i*	c_h	m

$T(n) = nc_i + mc_h$

We need to compute Pr{ candidate *i* is hired}?

- *i* is hired ⇔ *i* is better than each of candidates 1, 2, … , *i*-1.
- Assumption that the candidates arrive in random order => any one of these candidates is equally likely to be the best one. （若候选人随机到来，则每一个候选人为最佳人选的概率相等）
- Thus, $E\{X_i\} = \Pr\{$ candidate i is the best so far$\} = 1/i$?

Then

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i = \ln n$$

Thus, the expected hiring cost is $nc_i+(\ln n)c_h$, which is much better than the worst-case cost of nc_i+nc_h .

(nc_i+mc_h): 30+3.4*10 = 64 vs 30+30*10 = 330

(ln30 = 3.4…)

ln*n* vs *n*

Analysis of the hiring problem

Harmonic series

For positive integers n, the nth ***harmonic number*** is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1) . \quad \text{(A.7)}$$

(We shall prove a related bound in Section A.2.)

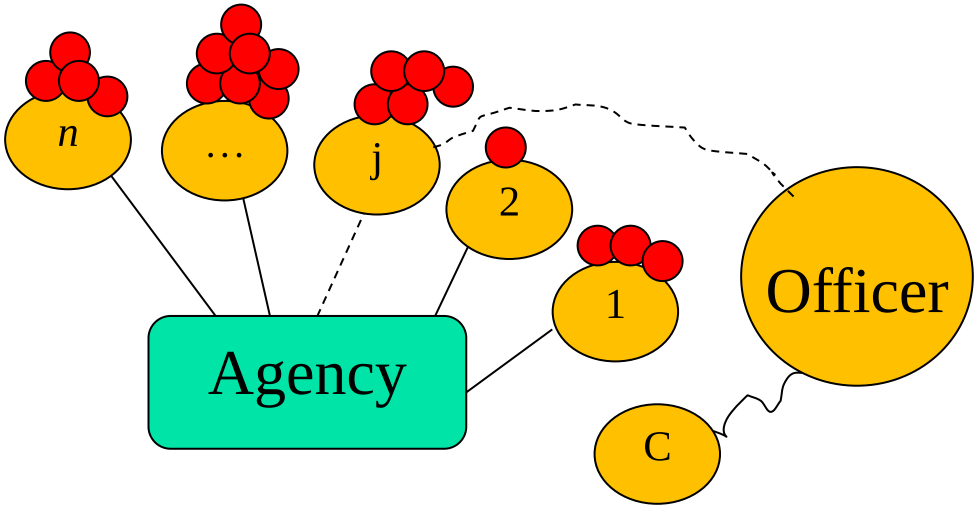
$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i] = \sum_{i=1}^{n} 1/i = \ln n$$

Thus, the expected hiring cost is $nc_i + (\ln n)c_h$, which is much better than the worst-case cost of $nc_i + nc_h$.

($nc_i + mc_h$): 30+3.4*10=64 vs 30+30*10=330. (ln30 = 3.4…)

$\ln n$ vs n

5.3 Randomized algorithms

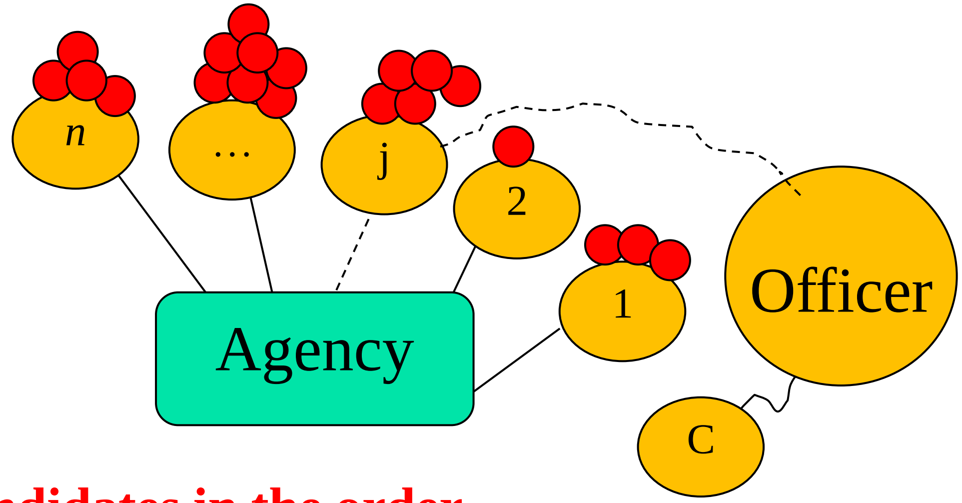


5.3.1 The hiring problem

For the hiring problem, the algorithm is deterministic.

- **For any given input, the number of times we hire a new office assistant will always be the same.（给定输入，则雇用的人数确定）**
- **The number of times we hire a new office assistant depends only on the input.（雇用的人数〔资源消费〕依赖于输入）**
  - **Some rank orderings will always produce a high hiring cost. Example: <1,2,3,4,5,6>, where each is hired.**
  - **Some will always produce a low hiring cost. Example: <6,*,*,*,*,*>, where only the first is hired.**
  - **Some may be in between.**

5.3.1 The hiring problem



Instead of always interviewing the candidates in the order presented, what if we first randomly permuted this order.

- **The randomization is now in the algorithm, not in the input distribution.**
  （随机化过程在算法中，而不是在输入中体现）
- **Given a particular input, we can no longer say what its hiring cost will be. Each time we run the algorithm, we can get a different hiring cost.**
  （算法的运行时间与输入无关）
- **No particular input always elicits worst-case behavior.**
  （算法的最坏运算时间不取决于特定的输入）
- **Bad behavior occurs only if we get "unlucky" numbers from the random-number generator.**
  （只有当随机数产生器产生很不幸运的数时，算法的运算时间最坏）

5.3.1 The hiring problem

Algorithm for randimized hiring problem

RANDOMIZED-HIRE-ASSISTANT(*n*)
    Randomly permute the list of candidates
    HIRE-ASSISTANT(*n*)

- ***Lemma***

  The expected hiring cost of RANDOMIZED-HIRE-ASSISTANT is $nc_i+(\ln n)c_h$.

Proof

After permuting the input array, we have a situation identical to the probabilistic analysis of deterministic HIRE-ASSISTANT.

（对输入矩阵进行随机置换后，情况同HIRE-ASSISTANT相同）

5.3.2 Randomly permuting an array

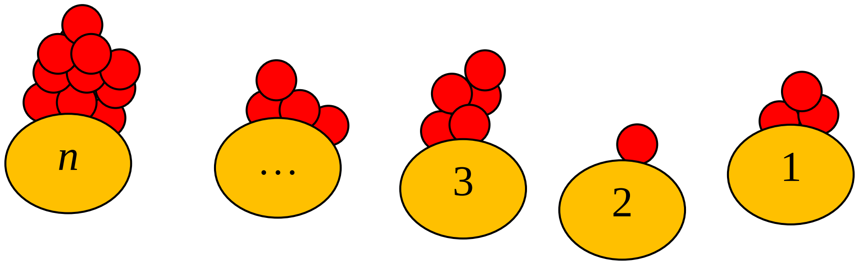
Goal:

Produce a uniform random permutation (each of the n! permutations is equally likely to be produced),

that is

A = <1, 2, 3, … , n>

The numbers of permutation of A is $P_n^n = n!$, each of that is equally likely to be produced.



5.3.2 Randomly permuting an array

(1) Permute-by-sorting

The method is not very good

- **Assume the given array *A* contains the element 1 through *n*.**

  （设数组 A = <1, 2, … , n>）

- **Assign each element *A*[*i*] a random priority *P*[*i*] , then sort the elements of *A* according to these priorities. Example**

  （为每个元素 $A[i]$ 分配一个随机数 $P[i]$ 作为其优先权，然后依据这些优先权对数组 A 进行排序。）

  - **If initial array is *A*=<1,2,3,4>, choose random priorities *P*=<36,3,97,19>, then produce an array *B*=<2,4,1,3>**

```
PERMUTE-BY-SORTING(A)
n = length[A]
for(i=1; i<=n; i++)
        P[i] = RANDOM(1, n^3)
sort A, using P as sort keys
return A
```

We use a range of 1 to n^3 in RANDOM to make it likely that all the priorities in *P* are unique.(Exercise 5.3-5)

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

The method is better

```
RANDOMIZE-IN-PLACE(A, n)
for(i=1; i<=n; i++)
      swap(A[i], A[RANDOM(i, n)])
```

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

Idea:

- **In iteration *i*, choose *A*[*i*] randomly from *A*[*i* .. *n*].**
- **Will never alter *A*[*i*] after iteration *i*.** （第 *i* 次迭代后不再改变 *A*[*i*]）

Merit：

- **It runs in linear time without requiring sorting (*O*(*n*)).**
- **It needs fewer random bits (*n* random numbers in the range 1 to *n* rather than the range 1 to n^3)**（仅需更小范围的随机数产生器）
- **No auxiliary array is required.** （不需要辅助空间）

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

```
RANDOMIZE-IN-PLACE(A, n)
for(i=1; i<=n; i++)
      swap(A[i], A[RANDOM(i, n)])
```

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

Correctness:

- **Given a set of n elements, a k-permutation is a sequence containing k of the n elements. There are $n!/(n-k)!$ possible k-permutations ?** （给定 n 个元素，从其中任取 k 个元素进行排列，则有 $n!/(n-k)!$ 种不同的 k-排列，或 k-置换 ?）

$$\mathrm{P}_n^k = \mathrm{C}_n^k \cdot \mathrm{P}_k^k = \frac{n!}{k!(n-k)!} \cdot k! = \frac{n!}{(n-k)!}$$

- ***Lemma***

RANDOMIZE-IN-PLACE computes a uniform random permutation.

Proof **Use a loop invariant:**

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

```
RANDOMIZE-IN-PLACE(A, n)
for(i=1; i<=n; i++)
      swap(A[i], A[RANDOM(i, n)])
```

- ***Lemma***

  RANDOMIZE-IN-PLACE computes a uniform random permutation.

$$1/\mathrm{P}_n^k = 1/\frac{n!}{(n-k)!} = \frac{(n-k)!}{n!}$$

Proof **Use a loop invariant:**

Loop invariant: Just prior to the *i*th iteration of the for loop, for each possible (*i*-1)-permutation, subarray *A*[1 .. *i*-1] contains this (*i*-1)-permutation with probability (*n*-*i*+1)!/*n*! ?

（第 *i* 次迭代之前，对 (*i*-1)-置换，任意一个(*i*-1)-置换*A*[1 .. *i*-1]的概率为(*n*-*i*+1)!/*n*! ？）

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

RANDOMIZE-IN-PLACE(*A*, *n*)
for(*i*=1; *i*<=*n*; *i*++)
swap(*A*[*i*], *A*[RANDOM(*i*, *n*)])

- ***Lemma*** **RANDOMIZE-IN-PLACE computes a uniform random permutation.**

Proof **Use a loop invariant:**

Loop invariant: **A[1 .. i-1] contains each (i-1)-permutation with probability (n-i+1)!/n! .**

- **Initialization:** **Just before first iteration, i=1. Loop invariant says for each possible 0-permutation, subarray A[1 .. 0] contains this 0-permutation with probability n!/n!=1. A[1 .. 0] is an empty subarray, and a 0-permutation has no elements. So, A[1 .. 0] contains any 0-permutation with probability 1.**

（空集包含空置换的概率为1）

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

```
RANDOMIZE-IN-PLACE(A, n)
for(i=1; i<=n; i++)
      swap(A[i], A[RANDOM(i, n)])
```

- ***Lemma*** RANDOMIZE-IN-PLACE computes a uniform random permutation.

Proof **Loop invariant:** Pr{A[1 .. i-1] contains each (i-1)-permutation} = (n-i+1)!/n! .

- **Maintenance:** Assume that prior to the ith iteration, Pr{A[1 .. i-1] contains each (i-1)-permutation} = (n-i+1)!/n! , we will show that after the ith iteration, Pr(A[1 .. i] contains each i-permutation) = (n-i)!/n! .

  （第 i 次迭代前，设(i-1)-置换 A[1 .. i-1] 中，任一置换发生的概率为 (n-i+1)!/n!，则需证明在第i 次迭代后，任一 i-置换 的概率为(n-i)!/n! ）

  Consider a particular i-permutation R=<x_1, x_2, … , x_i>. It consists of an (i-1)-permutation R'=<x_1, x_2, … , x_{i-1}>, followed by x_i . （考虑一个特别的 i-置换 R，其前 i-1 个元素组成 (i-1)-置换 R'，最后一个元素为 x_i ）. ……

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

RANDOMIZE-IN-PLACE(A, n)
for(i=1; i<=n; i++)
swap($A[i]$, A[RANDOM(i, n)])

- ***Lemma*** RANDOMIZE-IN-PLACE computes a uniform random permutation.

Proof **Loop invariant:** Pr{A[1 .. i-1] contains each (i-1)-permutation} = $(n-i+1)!/n!$.

- **Maintenance: … …**

i-permutation R=$\langle x_1, x_2, \ldots, x_i \rangle = \langle x_1, x_2, \ldots, x_{i-1} \rangle \cup x_i$ = R' $\cup x_i$.

Let E_1 be the event that the algorithm actually puts R' into A[1 .. i-1] . By the loop invariant, Pr{E_1}=$(n-i+1)!/n!$.

Let E_2 be the event that the ith iteration puts x_i into $A[i]$.

We get the i-Permutation R in A[1 .. i] if and only if both E_1 and E_2 occur => the probability that the algorithm produces R in A[1 .. i] is Pr{$E_2 \cap E_1$} = ?. …（令事件E_1表示算法实际输出 (i-1)-置换R'为A[1 .. i-1]，根据循环不变量， Pr{E_1}=$(n-i+1)!/n!$，令事件E_2表示第 i 次迭代后输出 $A[i]$ 为x_i，则当且仅当E_1 和 E_2同时发生时我们得到 i-置换 R 为A[1 .. i]，其概率为Pr{$E_2 \cap E_1$} = ? ）…

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

> **RANDOMIZE-IN-PLACE(A, n)**
> **for(i=1; i<=n; i++)**
> **swap($A[i]$, A[RANDOM(i, n)])**

- ***Lemma*** **RANDOMIZE-IN-PLACE computes a uniform random permutation.**

Proof **Loop invariant:** **Pr{A[1 .. i-1] contains each (i-1)-permutation} = (n-i+1)!/n! .**

- **Maintenance:**

i	i+1	…	n
$A(j_i)$	$A(j_{i+1})$	…	$A(j_n)$

… …

$\Pr\{E_2 \cap E_1\} = \Pr\{E_2 | E_1\} \Pr\{E_1\}$.

The algorithm choose x_i randomly from the n-i+1 possibilities in $A[i .. n]$ => $\Pr\{E_2|E_1\} = 1/(n-i+1)$? Thus,

$$\Pr\{E_2 \cap E_1\} = \Pr\{E_2 \mid E_1\}\Pr\{E_1\}$$
$$= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} = \frac{(n-i)!}{n!}$$

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

```
RANDOMIZE-IN-PLACE(A, n)
for(i=1; i<=n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

- ***Lemma*** **RANDOMIZE-IN-PLACE computes a uniform random permutation.**

Proof **Loop invariant:** **Pr{$A[1 .. i\text{-}1]$ contains each (i-1)-permutation} = $(n\text{-}i+1)!/n!$.**

- **Termination:**

  At termination, $i=n+1$, it is true prior to the ith iteration, so we conclude that $A[i .. n]$ is a given n-permutation with probability $(n\text{-n})!/n! = 1/n!$. ■ **(lemma)**

*5.4 Further applications

- 自学

Exercises and problems

- **Exercises**

  所有课后习题。

  编程验证习题。

- **Class-exercises**

  雇用问题中，仅雇用1个人的概率，即 Pr(X=1)？

  雇用问题中，n 个人都会被雇佣的概率，即 Pr(X=n)？

  雇用问题中，仅雇用2个人的概率，即 Pr(X=2)？

  （数学推导和编程验证相结合）