1 Questions

1. Yes. With any initial belief, the second strategy is always prior to the first. Thus the (2,2) NE is always achieved.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	0 1.5	0 1.5
1	2	2	1 2	1 2	0.33 2	0.33 2
2	2	2	1 3	1 3	0.5 2.2	0.5 2.2
3	2	2	1 4	1 4	0.6 2.4	0.6 2.4
4	2	2	1 5	1 5	0.67 2.5	0.67 2.5
5	2	2	1 6	1 6	0.71 2.6	0.71 2.6
6	2	2	1 7	1 7	0.75 2.6	0.75 2.6
7	2	2	1 8	1 8	0.78 2.7	0.78 2.7
8	2	2	1 9	1 9	0.8 2.7	0.8 2.7
9	2	2	1 10	1 10	0.82 2.7	0.82 2.7
10	2	2				

2. Yes. Both can be achieved.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	1.5 1.5	1.5 1.5
1	1	1	2 1	2 1	1.7 1	1.7 1
2	1	1	3 1	3 1	1.8 0.75	1.8 0.75
3	1	1	4 1	4 1	1.8 0.6	1.8 0.6
4	1	1	5 1	5 1	1.8 0.5	1.8 0.5
5	1	1	6 1	6 1	1.9 0.43	1.9 0.43
6	1	1	7 1	7 1	1.9 0.38	1.9 0.38
7	1	1	8 1	8 1	1.9 0.33	1.9 0.33
8	1	1	9 1	9 1	1.9 0.3	1.9 0.3
9	1	1	10 1	10 1	1.9 0.27	1.9 0.27
10	1	1				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	1.5 1.5	1.5 1.5
1	2	2	1 2	1 2	1.3 2	1.3 2
2	2	2	1 3	1 3	1.2 2.2	1.2 2.2
3	2	2	1 4	1 4	1.2 2.4	1.2 2.4
4	2	2	1 5	1 5	1.2 2.5	1.2 2.5
5	2	2	1 6	1 6	1.1 2.6	1.1 2.6
6	2	2	1 7	1 7	1.1 2.6	1.1 2.6
7	2	2	1 8	1 8	1.1 2.7	1.1 2.7
8	2	2	1 9	1 9	1.1 2.7	1.1 2.7
9	2	2	1 10	1 10	1.1 2.7	1.1 2.7
10	2	2				

3. No, just one of them. If any of then prefer strategy 1, the other will also prefer it and converges to (1,1). Otherwise, Both strategy get payoff 0 and we randomly select, so the probability that both of them still act 2 after r round is $2^{-r} \to 0$ as $r \to \infty$. So we can only converge to (1,1).

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	0.5 0	0.5 0
1	1	1	2 1	2 1	0.67 0	0.67 0
2	1	1	3 1	3 1	0.75 0	0.75 0
3	1	1	4 1	4 1	0.8 0	0.8 0
4	1	1	5 1	5 1	0.83 0	0.83 0
5	1	1	6 1	6 1	0.86 0	0.86 0
6	1	1	7 1	7 1	0.88 0	0.88 0
7	1	1	8 1	8 1	0.89 0	0.89 0
8	1	1	9 1	9 1	0.9 0	0.9 0
9	1	1	10 1	10 1	0.91 0	0.91 0
10	1	1				

4. Yes. The distribution of strategy choices follows the mix-strategy NE.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	1 1	0.5 2
1	1	2	1 2	2 1	1.3 0.67	0.33 2.7
2	1	2	1 3	3 1	1.5 0.5	0.25 3
3	1	2	1 4	4 1	1.6 0.4	0.2 3.2
4	1	2	1 5	5 1	1.7 0.33	0.17 3.3
5	1	1	2 5	6 1	1.4 0.57	0.29 2.9
6	1	1	3 5	7 1	1.2 0.75	0.38 2.5
7	1	1	4 5	8 1	1.1 0.89	0.44 2.2
8	1	1	5 5	9 1	1 1	0.5 2
9	1	1	6 5	10 1	0.91 1.1	0.55 1.8
10	2	1	7 5	10 2	0.83 1.2	0.58 1.7
11	2	1	8 5	10 3	0.77 1.2	0.62 1.5
12	2	2	8 6	10 4	0.86 1.1	0.57 1.7
13	2	2	8 7	10 5	0.93 1.1	0.53 1.9
14	2	2	8 8	10 6	1 1	0.5 2
15	1	2	8 9	11 6	1.1 0.94	0.47 2.1
16	1	2	8 10	12 6	1.1 0.89	0.44 2.2
17	1	2	8 11	13 6	1.2 0.84	0.42 2.3
18	1	2	8 12	14 6	1.2 0.8	0.4 2.4
19	1	2	8 13	15 6	1.2 0.76	0.38 2.5
20	1	2	8 14	16 6	1.3 0.73	0.36 2.5
21	1	2	8 15	17 6	1.3 0.7	0.35 2.6
22	1	2	8 16	18 6	1.3 0.67	0.33 2.7
23	1	2	8 17	19 6	1.4 0.64	0.32 2.7
24	1	2	8 18	20 6	1.4 0.62	0.31 2.8
25	1	2	8 19	21 6	1.4 0.59	0.3 2.8
26	1	2	8 20	22 6	1.4 0.57	0.29 2.9
27	1	2	8 21	23 6	1.4 0.55	0.28 2.9
28	1	2	8 22	24 6	1.5 0.53	0.27 2.9
29	1	1	9 22	25 6	1.4 0.58	0.29 2.8
30	1	1	10 22	26 6	1.4 0.62	0.31 2.8
31	1	1	11 22	27 6	1.3 0.67	0.33 2.7
32	1	1	12 22	28 6	1.3 0.71	0.35 2.6
33	1	1				

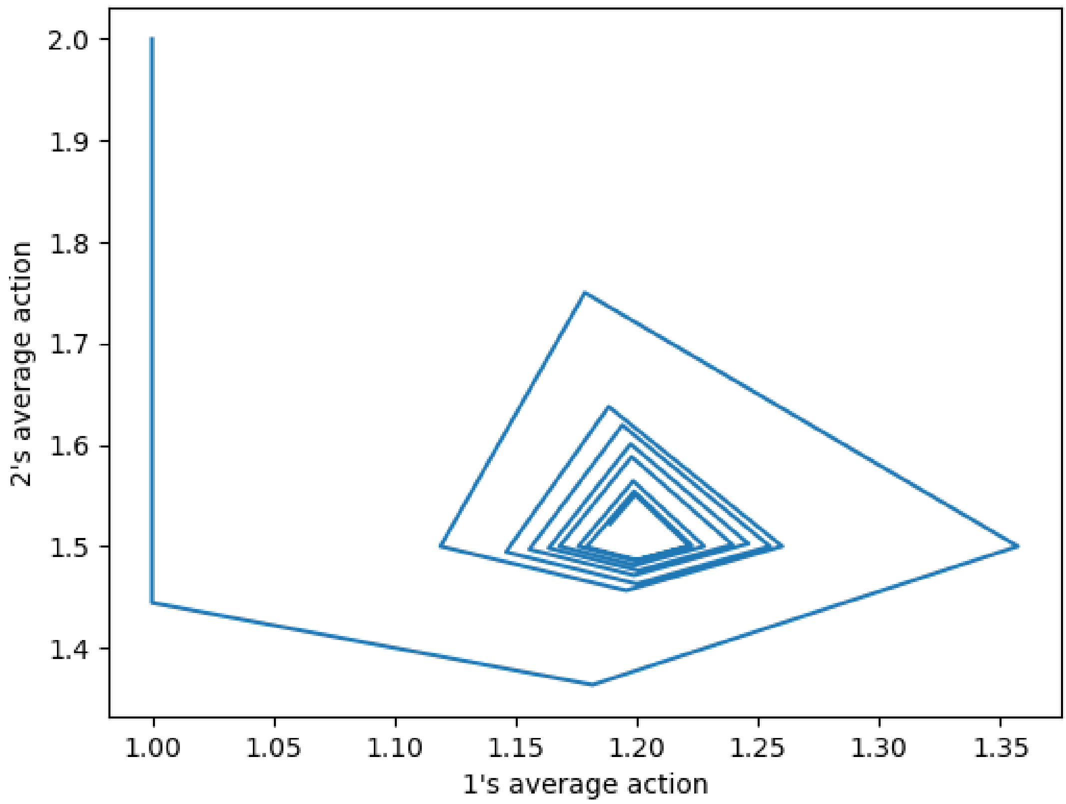

Figure 1: Question 4

5. Yes. The distribution of strategy choices follows the mix-strategy NE.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	0.5 0.5	0.5 0.5
1	1	2	1 2	2 1	0.67 0.33	0.33 0.67
2	1	1	2 2	3 1	0.5 0.5	0.5 0.5
3	1	1	3 2	4 1	0.4 0.6	0.6 0.4
4	2	1	4 2	4 2	0.33 0.67	0.67 0.33
5	2	1	5 2	4 3	0.29 0.71	0.71 0.29
6	2	1	6 2	4 4	0.25 0.75	0.75 0.25
7	2	2	6 3	4 5	0.33 0.67	0.67 0.33
8	2	2	6 4	4 6	0.4 0.6	0.6 0.4
9	2	2	6 5	4 7	0.45 0.55	0.55 0.45
10	2	2	6 6	4 8	0.5 0.5	0.5 0.5
11	1	2	6 7	5 8	0.54 0.46	0.46 0.54
12	1	2	6 8	6 8	0.57 0.43	0.43 0.57
13	1	2	6 9	7 8	0.6 0.4	0.4 0.6
14	1	2	6 10	8 8	0.62 0.38	0.38 0.62
15	1	1	7 10	9 8	0.59 0.41	0.41 0.59
16	1	1	8 10	10 8	0.56 0.44	0.44 0.56
17	1	1	9 10	11 8	0.53 0.47	0.47 0.53
18	1	1	10 10	12 8	0.5 0.5	0.5 0.5
19	2	1	11 10	12 9	0.48 0.52	0.52 0.48
20	2	1				

6. It can converges to all three of them. In Fig 3, the radius of disk is given by 0.7^t, t is the number of rounds past. The center is the average action between 1 to t.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	5 5	5 5
1	1	1	2 1	2 1	6.7 3.3	6.7 3.3
2	1	1	3 1	3 1	7.5 2.5	7.5 2.5
3	1	1	4 1	4 1	8 2	8 2
4	1	1	5 1	5 1	8.3 1.7	8.3 1.7
5	1	1	6 1	6 1	8.6 1.4	8.6 1.4
6	1	1	7 1	7 1	8.8 1.2	8.8 1.2
7	1	1	8 1	8 1	8.9 1.1	8.9 1.1
8	1	1	9 1	9 1	9 1	9 1
9	1	1	10 1	10 1	9.1 0.91	9.1 0.91
10	1	1				

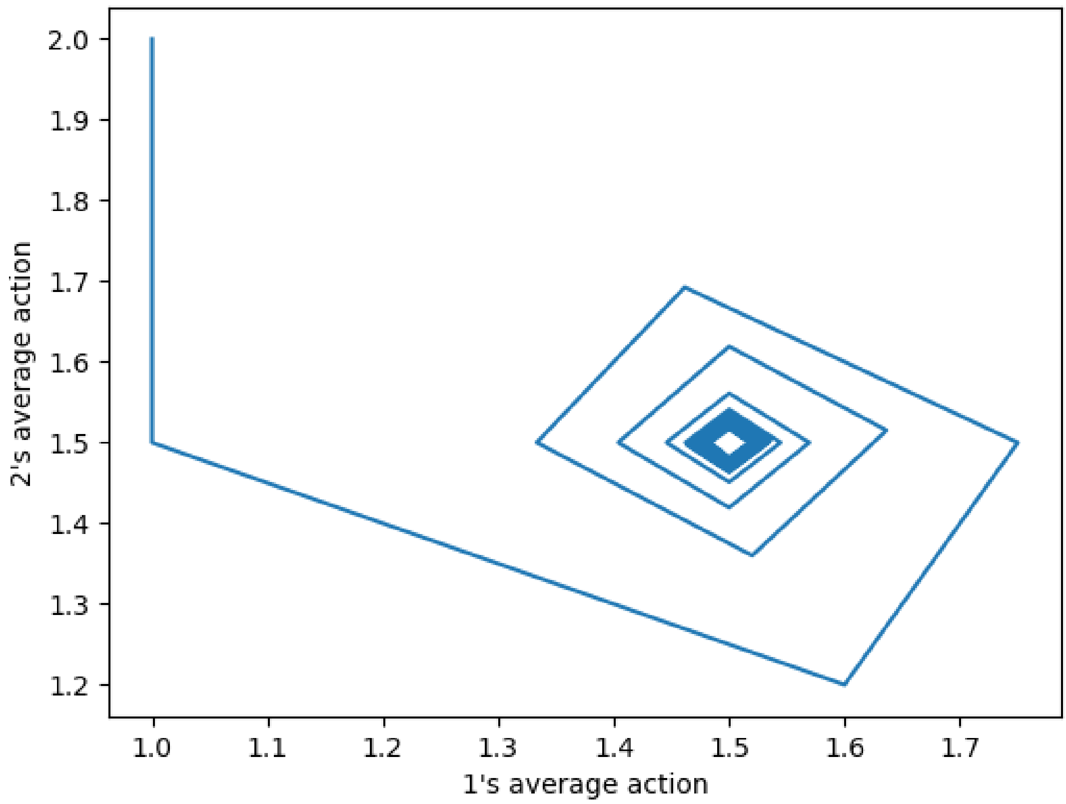

Figure 2: Question 5

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	5 5	5 5
1	2	2	1 2	1 2	3.3 6.7	3.3 6.7
2	2	2	1 3	1 3	2.5 7.5	2.5 7.5
3	2	2	1 4	1 4	2 8	2 8
4	2	2	1 5	1 5	1.7 8.3	1.7 8.3
5	2	2	1 6	1 6	1.4 8.6	1.4 8.6
6	2	2	1 7	1 7	1.2 8.8	1.2 8.8
7	2	2	1 8	1 8	1.1 8.9	1.1 8.9
8	2	2	1 9	1 9	1 9	1 9
9	2	2	1 10	1 10	0.91 9.1	0.91 9.1
10	2	2				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1.5	1.5 1	4 6	4 6
1	2	1	2 1.5	1.5 2	5.7 4.3	5.7 4.3
2	1	2	2 2.5	2.5 2	4.4 5.6	4.4 5.6
3	2	1	3 2.5	2.5 3	5.5 4.5	5.5 4.5
4	1	2	3 3.5	3.5 3	4.6 5.4	4.6 5.4
5	2	1	4 3.5	3.5 4	5.3 4.7	5.3 4.7
6	1	2	4 4.5	4.5 4	4.7 5.3	4.7 5.3
7	2	1	5 4.5	4.5 5	5.3 4.7	5.3 4.7
8	1	2	5 5.5	5.5 5	4.8 5.2	4.8 5.2
9	2	1	6 5.5	5.5 6	5.2 4.8	5.2 4.8
10	1	2				

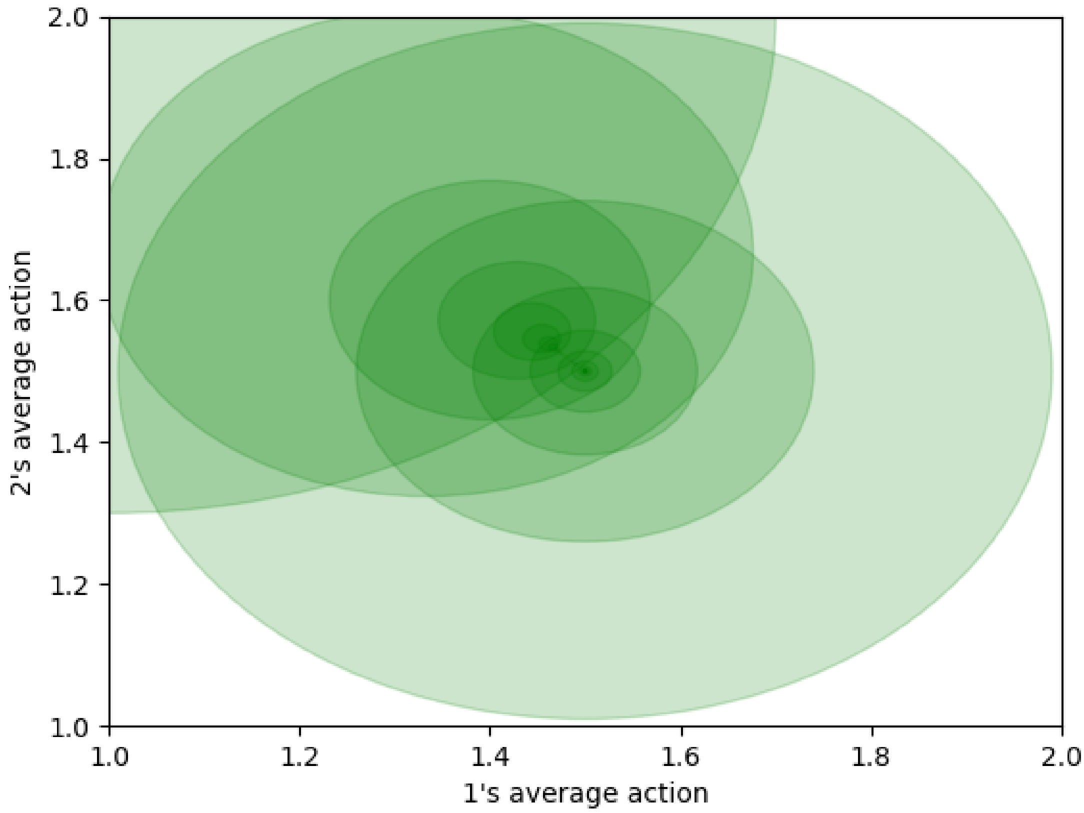

Figure 3: Question 6

7. It can converges to all three of them.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	0.5 0.5	0.5 0.5
1	1	1	2 1	2 1	0.33 0.67	0.33 0.67
2	2	2	2 2	2 2	0.5 0.5	0.5 0.5
3	2	2	2 3	2 3	0.6 0.4	0.6 0.4
4	1	1	3 3	3 3	0.5 0.5	0.5 0.5
5	1	2	3 4	4 3	0.57 0.43	0.57 0.43
6	1	2	3 5	5 3	0.62 0.38	0.62 0.38
7	1	2	3 6	6 3	0.67 0.33	0.67 0.33
8	1	2	3 7	7 3	0.7 0.3	0.7 0.3
9	1	2	3 8	8 3	0.73 0.27	0.73 0.27
10	1	2				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	0.5 0.5	0.5 0.5
1	1	1	2 1	2 1	0.33 0.67	0.33 0.67
2	2	2	2 2	2 2	0.5 0.5	0.5 0.5
3	2	2	2 3	2 3	0.6 0.4	0.6 0.4
4	1	1	3 3	3 3	0.5 0.5	0.5 0.5
5	2	1	4 3	3 4	0.43 0.57	0.43 0.57
6	2	1	5 3	3 5	0.38 0.62	0.38 0.62
7	2	1	6 3	3 6	0.33 0.67	0.33 0.67
8	2	1	7 3	3 7	0.3 0.7	0.3 0.7
9	2	1	8 3	3 8	0.27 0.73	0.27 0.73
10	2	1				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1.5	1 1.5	0.6 0.4	0.6 0.4
1	1	1	2 1.5	2 1.5	0.43 0.57	0.43 0.57
2	2	2	2 2.5	2 2.5	0.56 0.44	0.56 0.44
3	1	1	3 2.5	3 2.5	0.45 0.55	0.45 0.55
4	2	2	3 3.5	3 3.5	0.54 0.46	0.54 0.46
5	1	1	4 3.5	4 3.5	0.47 0.53	0.47 0.53
6	2	2	4 4.5	4 4.5	0.53 0.47	0.53 0.47
7	1	1	5 4.5	5 4.5	0.47 0.53	0.47 0.53
8	2	2	5 5.5	5 5.5	0.52 0.48	0.52 0.48
9	1	1	6 5.5	6 5.5	0.48 0.52	0.48 0.52
10	2	2				

8. It can converges to all three of them.

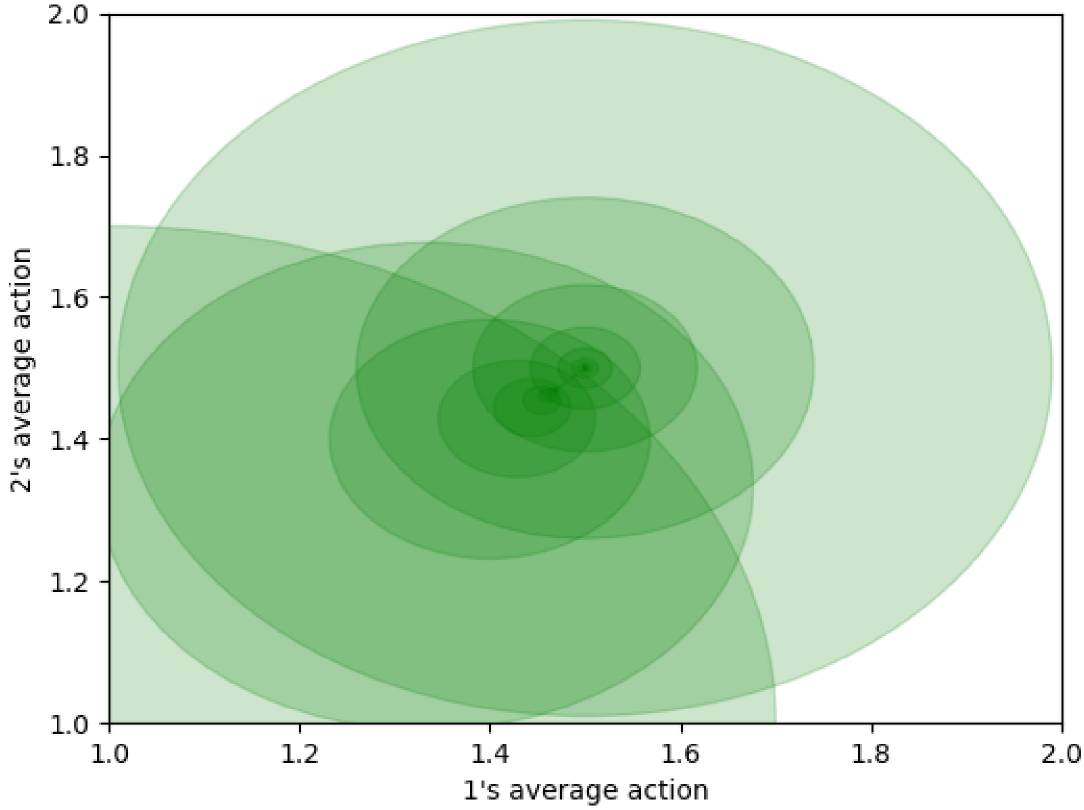

Figure 4: Question 7

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	1.5 1	1 1.5
1	1	2	1 2	2 1	1 1.3	0.67 2
2	2	1	2 2	2 2	1.5 1	1 1.5
3	1	2	2 3	3 2	1.2 1.2	0.8 1.8
4	1	1	3 3	4 2	1.5 1	1 1.5
5	1	1	4 3	5 2	1.7 0.86	1.1 1.3
6	1	1	5 3	6 2	1.9 0.75	1.2 1.1
7	1	1	6 3	7 2	2 0.67	1.3 1
8	1	1	7 3	8 2	2.1 0.6	1.4 0.9
9	1	1	8 3	9 2	2.2 0.55	1.5 0.82
10	1	1				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	1.5 1	1 1.5
1	1	2	1 2	2 1	1 1.3	0.67 2
2	2	1	2 2	2 2	1.5 1	1 1.5
3	1	2	2 3	3 2	1.2 1.2	0.8 1.8
4	2	2	2 4	3 3	1 1.3	0.67 2
5	2	2	2 5	3 4	0.86 1.4	0.57 2.1
6	2	2	2 6	3 5	0.75 1.5	0.5 2.2
7	2	2	2 7	3 6	0.67 1.6	0.44 2.3
8	2	2	2 8	3 7	0.6 1.6	0.4 2.4
9	2	2	2 9	3 8	0.55 1.6	0.36 2.5
10	2	2				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1.5 1	1 1.5	1.8 0.8	1.2 1.2
1	1	2	1.5 2	2 1.5	1.3 1.1	0.86 1.7
2	1	2	1.5 3	3 1.5	1 1.3	0.67 2
3	2	1	2.5 3	3 2.5	1.4 1.1	0.91 1.6
4	1	2	2.5 4	4 2.5	1.2 1.2	0.77 1.8
5	2	1	3.5 4	4 3.5	1.4 1.1	0.93 1.6
6	1	2	3.5 5	5 3.5	1.2 1.2	0.82 1.8
7	1	2	3.5 6	6 3.5	1.1 1.3	0.74 1.9
8	2	1	4.5 6	6 4.5	1.3 1.1	0.86 1.7
9	1	2	4.5 7	7 4.5	1.2 1.2	0.78 1.8
10	2	1				

9. It can converges to all three of them.

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 1	1 1	1.5 1.5	1.5 1.5
1	1	1	2 1	2 1	2 1.7	2 1.7
2	1	1	3 1	3 1	2.2 1.8	2.2 1.8
3	1	1	4 1	4 1	2.4 1.8	2.4 1.8
4	1	1	5 1	5 1	2.5 1.8	2.5 1.8
5	1	1	6 1	6 1	2.6 1.9	2.6 1.9
6	1	1	7 1	7 1	2.6 1.9	2.6 1.9
7	1	1	8 1	8 1	2.7 1.9	2.7 1.9
8	1	1	9 1	9 1	2.7 1.9	2.7 1.9
9	1	1	10 1	10 1	2.7 1.9	2.7 1.9
10	1	1				

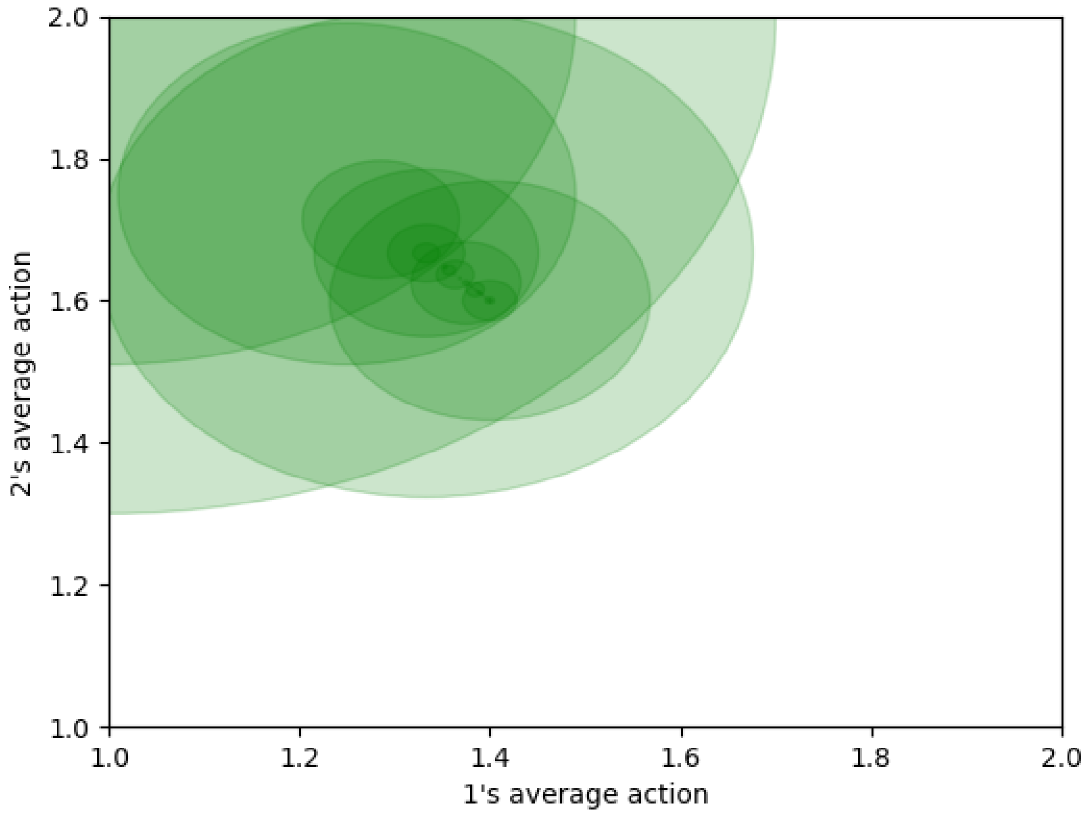

Figure 5: Question 8

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1 2	1 2	1 1.3	1 1.3
1	2	2	1 3	1 3	0.75 1.2	0.75 1.2
2	2	2	1 4	1 4	0.6 1.2	0.6 1.2
3	2	2	1 5	1 5	0.5 1.2	0.5 1.2
4	2	2	1 6	1 6	0.43 1.1	0.43 1.1
5	2	2	1 7	1 7	0.38 1.1	0.38 1.1
6	2	2	1 8	1 8	0.33 1.1	0.33 1.1
7	2	2	1 9	1 9	0.3 1.1	0.3 1.1
8	2	2	1 10	1 10	0.27 1.1	0.27 1.1
9	2	2	1 11	1 11	0.25 1.1	0.25 1.1
10	2	2				

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0			1.5 1	1 1.5	1.8 1.6	1.8 1.6
1	1	2	1.5 2	2 1.5	1.3 1.4	1.3 1.4
2	2	1	2.5 2	2 2.5	1.7 1.6	1.7 1.6
3	1	2	2.5 3	3 2.5	1.4 1.5	1.4 1.5
4	2	1	3.5 3	3 3.5	1.6 1.5	1.6 1.5
5	1	2	3.5 4	4 3.5	1.4 1.5	1.4 1.5
6	2	1	4.5 4	4 4.5	1.6 1.5	1.6 1.5
7	1	2	4.5 5	5 4.5	1.4 1.5	1.4 1.5
8	2	1	5.5 5	5 5.5	1.6 1.5	1.6 1.5
9	1	2	5.5 6	6 5.5	1.4 1.5	1.4 1.5
10	2	1				

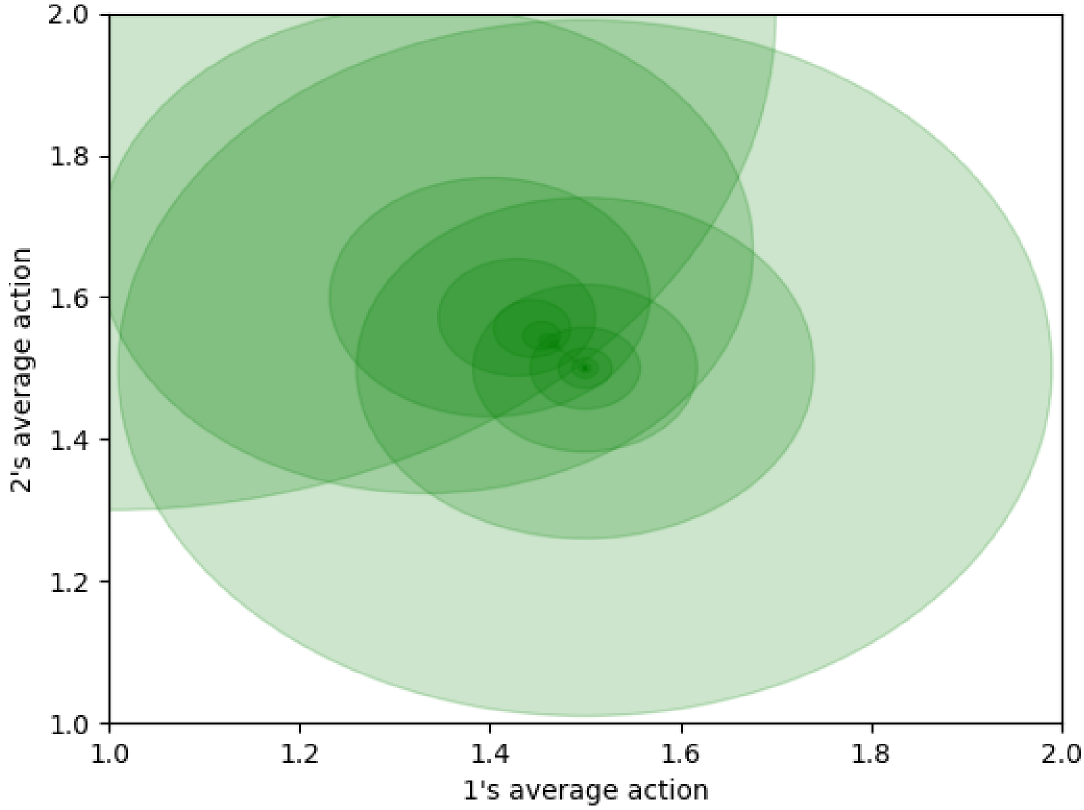

Figure 6: Question 9

10. No. Consider the chicken game:

	c_1	c_2
r_1	0, 0	10, 0
r_2	0, 10	-100, -100

It has two pure NE: (r_1, c_2) and (r_2, c_1) and one mix-strategy NE $P(r_1) = P(c_1) = 0.909$, $P(r_2) = P(c_2) = 0.091$. In a 500 independent round test, because we should have no prior knowledge about where the mixed-NE is, I set initial belief uniform in $U(0, 10)$. Each independent test goes 500 rounds, and we record the average action for both player. The result is Fig 7, we can see that the mix-NE is hard to achieve.

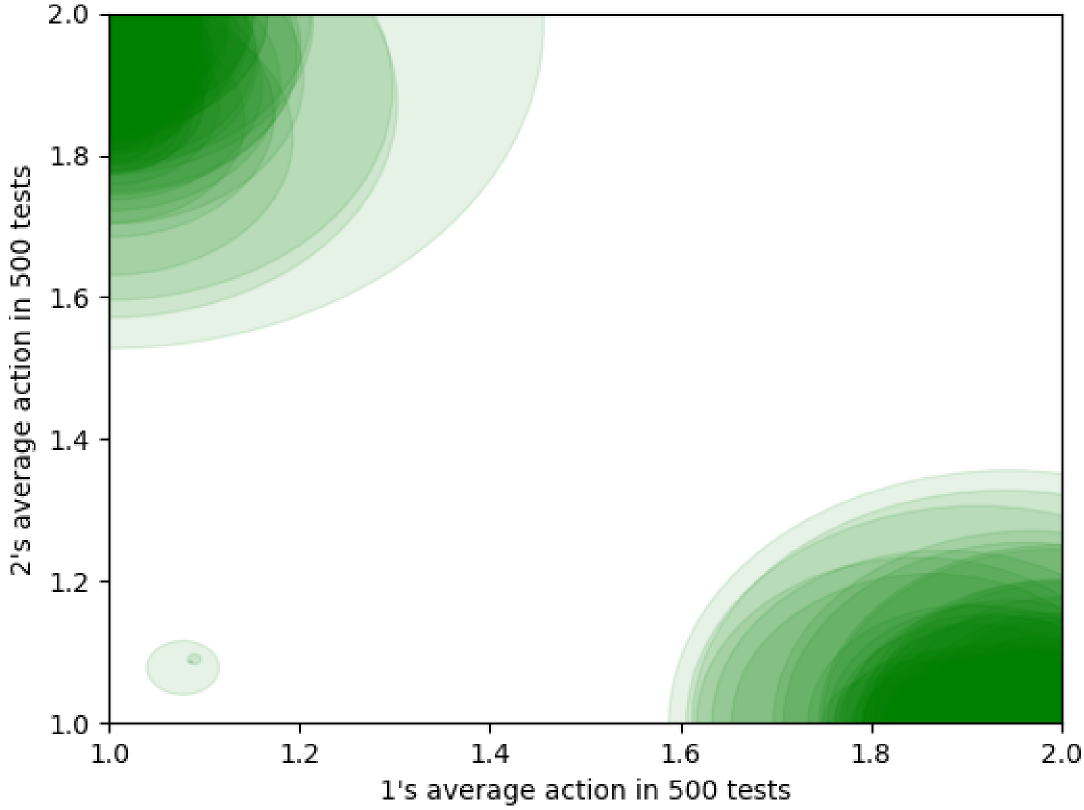

Figure 7: Question 10

2 Code Explanation

The normal form is stored in a $2 \times 2 \times 2$ matrix

```
typedef float payoff;
typedef std::array<std::array<std::array<payoff, 2>, 2>, 2> matrix;
```

Then we can calculate the utility of a player based on the other player's distribution of strategy. Also, obtain the nest response. If the utility of both strategy is identical, randomly choose one.

```cpp
payoff utility(player p, strategy s, const std::array<float, 2>& prob) const {
    // utility of p given p's strat s and (p^1)'s prob of strat
    return prob[0] * utility(p, make_profile(p, s, 0)) + prob[1] * utility(p, make_profile(p, s, 1));
}

strategy best_response(player p, const std::array<float, 2>& prob) {
    payoff p0 = utility(p, 0, prob), p1 = utility(p, 1, prob);
    if (p0 != p1) return p0 < p1? 1 : 0;
    std::uniform_int_distribution<> dis(0, 1);
    strategy s[] = {0, 1};
    return s[dis(gen)];
}
```

On simulating the fictitious play, it iteratively calculates the probability(distribution) of each player's choice of strategy. Then record their utility for choosing each strategy and their best response. Then add each of their belief. This function returns a CSV format.

```cpp
CsvFormatter play(unsigned round) {
    CsvFormatter cf(7, round + 1);
    cf() = CsvFormatter::row{"Round", "1\'s action", "2\'s action", "1\'s belief", "2\'s belief", "1\'s payoff", "2\'s payoff"};

    cf[0][0] = "0";
    for (unsigned i = 0; i < round; ++i) {
        cf[i + 1][0] = std::to_string(i + 1);
        cf[i][3] = pair_float(bp[1][0], bp[1][1]);
        cf[i][4] = pair_float(bp[0][0], bp[0][1]);

        std::array<float, 2> prob0, prob1; // probi[s] is the prob of i using strat s
        prob0[0] = bp[0][0] / (bp[0][0] + bp[0][1]);
        prob0[1] = bp[0][1] / (bp[0][0] + bp[0][1]);
        prob1[0] = bp[1][0] / (bp[1][0] + bp[1][1]);
        prob1[1] = bp[1][1] / (bp[1][0] + bp[1][1]);
        cf[i][5] = pair_float(nf.utility(0, 0, prob1), nf.utility(0, 1, prob1));
        cf[i][6] = pair_float(nf.utility(1, 0, prob1), nf.utility(1, 1, prob1));

        NormalForm::strategy br0 = nf.best_response(0, prob1), br1 = nf.best_response(1, prob0);

        cf[i + 1][1] = std::to_string(br0 + 1);
        cf[i + 1][2] = std::to_string(br1 + 1);

        ++bp[0][br0];
        ++bp[1][br1];
    }
    return cf;
}
```

The CSV formatter is simply a header with a 2D array.

```cpp
typedef std::string element;
typedef std::vector<element> row;
typedef std::vector<row> grid;
```

```cpp
private:
    row header;
    grid content;
```

In the main function, we store all the game payoff information. Then we play the fictitious game based on the argv.

```
NormalForm::make_matrix({-1, -1, 1, 0, 0, 1, 3, 3}),
NormalForm::make_matrix({2, 2, 1, 0, 0, 1, 3, 3}),
NormalForm::make_matrix({1, 1, 0, 0, 0, 0, 0, 0}),
NormalForm::make_matrix({0, 1, 2, 0, 2, 0, 0, 4}),
NormalForm::make_matrix({0, 1, 1, 0, 1, 0, 0, 1}),
NormalForm::make_matrix({10, 10, 0, 0, 0, 0, 10, 10}),
NormalForm::make_matrix({0, 0, 1, 1, 1, 1, 0, 0}),
NormalForm::make_matrix({3, 2, 0, 0, 0, 0, 2, 3}),
NormalForm::make_matrix({3, 3, 0, 2, 2, 0, 1, 1}),
NormalForm::make_matrix({0, 0, 0, 10, 10, 0, -100, -100})
```

```
FictitiousPlay::belief_profile bp{{{dis(gen), dis(gen)}, {dis(gen), dis(gen)}}};
FictitiousPlay fp(games[std::atoi(argv[2])], bp);
fout << fp.play(200);
```