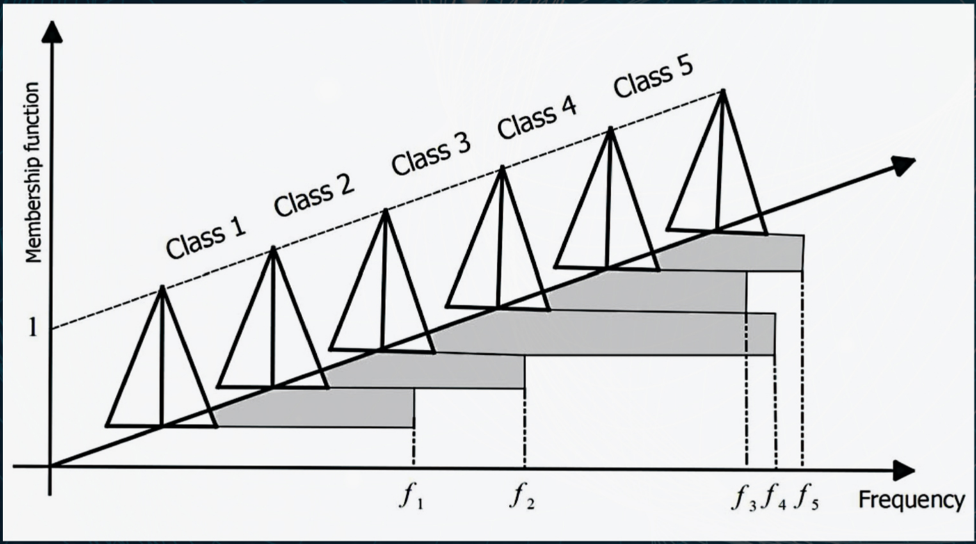
FUZZY STATISTICAL INFERENCES BASED ON FUZZY RANDOM VARIABLES



GHOLAMREZA HESAMIAN

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To my parents,
Sedigheh and Mohammad

And special thanks to my wife, Samin

Contents

Foreword

Probability and statistics are concerned with the chance to analyze events and analysis of data and stochastic models. Therefore, the quantitative description of events and data is essential for statistical inferences.

This book tries to extend some common statistical inferences adopted with new ideas and simple techniques as much as possible for the probability of fuzzy events and statistics with fuzzy data. For simplicity in theoretic and calculations, all fuzzy statistical procedures were conducted on triangular fuzzy numbers. These methods essentially rely on some fuzzy statistical methods recently published by the author. Many of these methods also modified and developed as much as possible.

Compared to similar fuzzy statistics books, we tried to cover a set of frequently used statistical techniques used in applied sciences. Unlike other existing fuzzy statistics books, this book tried to gather almost all essential statistical methods such as exact and fuzzy probabilities, probabilistic inequalities, limit theorems, hypothesis tests, quality control, reliability theory, analysis of variance, and descriptive statistics for fuzzy data. In each section, the proposed techniques were also compared with other existing ones. Each section was then ended via some theoretical and applied exercises. The required optimization procedure can be done by any mathematical software. However, in this book the Mathematica and MATLAB software were utilized in calculations and plotting the figures. The available programs can be obtained from the author.

It is my hope that this book will serve as a solid starting point for analyzing fuzzy quantities and thus enlarging the domain of applicability of the field of probability and statistics.

Preface

Probability and statistics are two fascinating topics on the relationship between mathematics and real-life applications. The probabilistic methods provide a rigorous framework for modeling uncertainty due to randomness. Statistics can be classified into two different categories namely descriptive statistics and statistical inference. Descriptive statistics describe the data, whereas statistical inference consists in the use of statistics to draw conclusions about some unknown aspect of a population based on a random sample from that population. It also helps to assess the relationship between the dependent and independent variables. In statistical inference, the data are taken from the sample and allow you to generalize the population.

However, the methods of probability theory and statistics typically involve processing imprecision of two distinct types arising from a lack of knowledge due to inherent vagueness in concepts themselves which, in the sense of classical probability and statistical inferences, may be well defined. Therefore, the conventional methods are usually inadequate for dealing with certain kinds of imprecision called fuzzy sets. There are many situations in the conventional probability theory that an event and/or a parameter are fuzzy quantities. Moreover, there are many practical problems that require dealing with observations that represent inherently imprecise or linguistic characteristics. In such cases, fuzzy sets may be more effective in encoding such quantities rather than precise ones.

It is now generally accepted that statistical theory and the theory of fuzzy sets are both used to study uncertainties where both types of uncertainties including randomness and fuzziness have occurred. Thus, to produce suitable statistical inferences dealing with imprecise information, we need to model the imprecise information and extend the usual probability theory and statistics to imprecise environments.

This book is an honest attempt to produce an unified techniques for the elementary probability theory based on fuzzy events induced by a family of density functions induced by fuzzy parameters. The main attempt of this book is to introduce some new techniques for the most commonly used statistical inferences based on fuzzy data. This book is designed as a practical guide for scientists to help them solve statistical problems involving fuzzy data and fuzzy events. Consequently, the book has a practical bent and contains lots of examples. A familiarity with probability and statistics is assumed for this book. The book is intended for a statistical scientist at the Bachelor of Science level or above for a specific course on probability and statistics. It also has applications to other applied sciences such as engineering and social sciences.

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Symbols

Symbol Description

Symbol	Description
A	Crisp set
$\widetilde{A}$	Fuzzy set
FN	Fuzzy number
$\widetilde{A}$	Fuzzy set
TFN	Triangular fuzzy number
STFN	Symmetric triangular fuzzy number
$\widetilde{A}[\alpha]$	α-cut of a fuzzy number $\widetilde{A}$
$\widetilde{A}_{\alpha}^{L}$	Lower bound of the α-cut of a fuzzy number $\widetilde{A}$
$\widetilde{A}_{\alpha}^{U}$	Upper bound of the α-cut of a fuzzy number $\widetilde{A}$
$\{x_1, x_2, ...\}$	Set of elements $x_1, x_2, ...$
$\mathbb{R}$	Set of real numbers
$\mathbb{X}$	Universal set
$\mathbb{Z}$	Set of all integer numbers
$\mathbb{N}$	Set of all natural numbers
$\mathbb{F}(\mathbb{X})$	Set of all fuzzy sets on $\mathbb{X}$
$\mathbb{F}(\mathbb{R})$	Set of all fuzzy numbers on $\mathbb{R}$
$\widetilde{A}_{\alpha}$	α-value of a fuzzy number $\widetilde{A}$
$\widetilde{A}(x)$	Membership degree of $\widetilde{A}$ at x
$\lvert\widetilde{A}\rvert$	Cardinal number of a fuzzy set $\widetilde{A}$
$\widetilde{A} \cup \widetilde{B}$	Union of fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} \cap \widetilde{B}$	Intersection of fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} - \widetilde{B}$	Difference between two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} \odot \widetilde{B}$	Product of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} \triangle \widetilde{B}$	Symmetric difference of $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A}^c$	Complement of $\widetilde{A}$
$\widetilde{A} \oplus \widetilde{B}$	Summation of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} \ominus \widetilde{B}$	Difference of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} \otimes \widetilde{B}$	Multiply of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$\widetilde{A} \oslash \widetilde{B}$	Deviation of two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$
$g(\widetilde{A})$	Image of $\widetilde{A}$ under the mapping $g(.)$
$\liminf \widetilde{A}_n$	Limes inferior of a countable collection of fuzzy sets $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$

$\limsup \widetilde{A}_n$	Limes superior of a countable collection of fuzzy sets $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$	P_X	Probability measure induced by X
$\widetilde{A}_1 \times \widetilde{A}_2 \times ... \times \widetilde{A}_n$	Cartesian product of the fuzzy sets $\widetilde{A}_1, \widetilde{A}_2, , ..., \widetilde{A}_n$	$(\mathbb{R}, \mathbb{F}_m(\mathbb{R}))$	Fuzzy measurable space
$S(\widetilde{A}, \widetilde{B})$	Similarity measure between two triangular fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$	$P_X(\widetilde{A})$	Exact probability of a fuzzy event of $\widetilde{A}$
$P_d(\widetilde{A} \succ \widetilde{B})$	Preference degree that '$\widetilde{A}$ is greater than $\widetilde{B}$'	$f_X^{\boldsymbol{\theta}}$	Parametric density function of X
$d(\widetilde{A}, \widetilde{B})$	Exact distance between two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$	$=^d$	identical distribution function
$\widetilde{D}(\widetilde{A}, \widetilde{B})$	Fuzzy distance between two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$	E_X	Expectation with respect to X
$\widetilde{A} \ominus_G \widetilde{B}$	Generalized difference between two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$	$\widetilde{\boldsymbol{\theta}}$	Fuzzy parameter
$\widetilde{\max}(\widetilde{A}, \widetilde{B})$	Maximum of the two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$	$P_X^{\widetilde{\boldsymbol{\theta}}}(\widetilde{A})$	Exact probability of $\widetilde{A}$ with fuzzy parameter
$\widetilde{\min}(\widetilde{A}, \widetilde{B})$	Minimum of the two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$	$\widetilde{P}_X^{\widetilde{\boldsymbol{\theta}}}(\widetilde{A})$	Fuzzy probability of $\widetilde{A}$ with fuzzy parameter
$\|\|\widetilde{A}\|\|$	absolute value a fuzzy number of $\widetilde{A}$	$P_{\boldsymbol{X}}(\widetilde{\boldsymbol{A}})$	Probability of a fuzzy product event $\widetilde{\boldsymbol{A}}$
P	Probability measure	$P_X(\widetilde{A}\|\widetilde{B})$	Conditional probability of $\widetilde{A}$ given $\widetilde{B}$
Ω	Set of outcomes of an experiment	$\widetilde{\overline{x}}$	Fuzzy arithmetic mean
$\mathcal{A}$	A σ-filed of Ω	$\widetilde{\overline{x}}_g$	Fuzzy geometric mean
$(\Omega, \mathcal{A}, P)$	Probability space	$\widetilde{\overline{x}}_h$	Fuzzy harmonic mean
$\mathcal{B}(\mathbb{R})$	Set of all Borel subsets of $\mathbb{R}$	$\widetilde{m}$	Fuzzy median
X	Random variable	$\widetilde{H}_p$	Fuzzy quantile
		$\widetilde{R}$	Fuzzy sample range
		$\widetilde{IQR}$	Fuzzy interquartile range
		S_D	Mean absolute deviation from the fuzzy mean
		S^2	Standard deviation of a set of fuzzy data

$\widetilde{S}_k$	Fuzzy (Pearson) measure of skewness
m_t	Exact measure of skewness
g	Measure of kurtosis of fuzzy data
CDF	Cumulative distribution function
$\widetilde{X}$	Fuzzy random variable (**FRV**)
$\widetilde{E}(\widetilde{X})$	Fuzzy expectation of $\widetilde{X}$
FRS	Ruzzy random sample
$\mathbf{var}(\widetilde{X})$	Exact variance of $\widetilde{X}$
$\mathbf{Cov}(\widetilde{X}, \widetilde{Y})$	Covariance of two **FRV**s of $\widetilde{X}$ and $\widetilde{Y}$
$\boldsymbol{\rho}(\widetilde{X}, \widetilde{Y})$	Covariance of two **FRV**s of $\widetilde{X}$ and $\widetilde{Y}$
$\widetilde{P}(\widetilde{X} \in (a,b))$	Fuzzy probability that $\widetilde{X} \in (a,b)$
$\liminf_{n\to\infty} \widetilde{X}_n$	lim inf of a sequence of **FRV**s $\{\widetilde{X}_n\}_{n\geq 1}$
$\limsup_{n\to\infty} \widetilde{X}_n$	lim sup of a sequence of **FRV**s $\{\widetilde{X}_n\}_{n\geq 1}$
$\widetilde{F}_{\widetilde{X}}(x)$	Fuzzy cumulative distribution function of $\widetilde{X}$ at x (**FCDF**)
LSFRV	Location-scale fuzzy random variable
SFRV	Scale fuzzy random variable
$\widetilde{X}_n \to^c \widetilde{X}$	Converges completely
$\widetilde{X}_n \to^p \widetilde{X}$	Converges in probability
$\widetilde{X}_n \to^{a.s} \widetilde{X}$	Converges almost surely
$\widetilde{X}_n \to^d \widetilde{X}$	Converges in distribution
$\widetilde{X}_n \to^{\widetilde{L}^p} \widetilde{X}$	converges in $\widetilde{L}^p$
SLLN	Strong Low of Large Numbers
GFRV	Generalized fuzzy random variable
TFRV	Triangular fuzzy random variable
TNFRV	Triangular normal fuzzy random variable
$\widetilde{H}_0$	Fuzzy null hypothesis
$\widetilde{H}_1$	Fuzzy alternative hypothesis
$\widetilde{T}$	Fuzzy sign-test Statistics
D_R	Degree of rejection of $\widetilde{H}_0$
D_A	Degree of acceptance of $\widetilde{H}_0$
δ	Significance level
$\widetilde{\widetilde{M}}$	Fuzzy median
$\widetilde{\widetilde{W}}$	Fuzzy Wilcoxon test statistics
$\sqrt{n}\widetilde{D}_n$	Fuzzy one-sample Kolmogorov-Smirnov test
FECDF	Fuzzy empirical cumulative distribution function
$\sqrt{\frac{mn}{m+n}}\widetilde{D}_{mn}$	Fuzzy Two-sample Kolmogorov-Smirnov test
$\widetilde{H}$	Fuzzy Kruskal-Wallis test
Cr	Credibility measure
ANOVA	Analysis of variance
SST	Sum of squares total of fuzzy data
SSR	Sum of squares

	between variations of fuzzy data
SSE	Sum of squares within variations of fuzzy data
χ^2_r	Chi-square distribution with r degrees of freedom
t_r	t-student distribution with r degrees of freedom
F_{ν_1,ν_2}	F-fisher distribution with r degrees of freedom of ν_1 and ν_2
$\chi^2_{r,\delta}$	δ-quantile of the chi-squared distribution with r degrees of freedom
$t_{r,\delta}$	δ-quantile of the t-distribution with r degrees of freedom
$F_{\nu_1,\nu_2,\delta}$	δ-quantile of the f-distribution with ν_1 and ν_2 degrees of freedom
$\widetilde{LCL}$	Fuzzy lower central limit
$\widetilde{CL}$	Fuzzy central limit
$\widetilde{LCL}$	Fuzzy upper central limit
$\widetilde{\overline{x}}$-chart	Fuzzy $\overline{x}$-chart
s-chart	s-chart based on fuzzy random variables
D_v	Degree of violence
EWMA	Exponentially-weighted moving average
$\widetilde{C}(u,v)$	Fuzzy process capability
$\widetilde{d}_a(\widetilde{A},\widetilde{B})$	Fuzzy triangular absolute error distance
$\widetilde{\widehat{C}}(u,v)$	Fuzzy estimated value of $\widetilde{C}(u,v)$
LTIFRV	Life time fuzzy random variable
FRF	Fuzzy reliability function
IMTTF	Fuzzy mean time to failure
FMRLF	Fuzzy mean residual life function
$\widetilde{R}_S$	Fuzzy system reliability
$\widetilde{\widehat{\beta}}_0$	Fuzzy intercept
β_j	Regression coefficient
MSE	Mean square error
RMSE	Root mean square error
MSM	Mean similarity measure
$K(.)$	Kernel function
h	Bandwidth
GCV	Generalized cross-validation criterion
ACF	Autocorrelation function
$\widetilde{\boldsymbol{X}}_{\boldsymbol{T}}$	Fuzzy time series data
$\mathbf{cov}(\widetilde{X}_t,\widetilde{X}_{t-k})$	Autocovariance of the generating fuzzy time series
$\mathbf{cov}(\widetilde{X}_t,\widetilde{X}_{t-k})$	Autocorrelation of the generating fuzzy time series
$\widehat{\boldsymbol{c}}_k$	sample autocovariance function of the generating fuzzy time series

Part I

Fuzzy Statistical Inferences Based on Fuzzy Random Variables

1

Introduction

In this chapter, the basic definitions and properties from fuzzy sets and fuzzy numbers needed for the book are recalled and discussed. We utilize such definitions and properties to produce some probabilistic reasoning and statistical inferences in the fuzzy domain the next chapter.

1.1 Fuzzy Set

Let $\mathbb{X}$ be an universal in the universe of discourse and it includes all possible elements related with the given problem. The classical set theory is built on the fundamental concept of 'set'. A set is defined as collection of objects, which share certain characteristics. A set A (of $\mathbb{X}$) can be represented a set by enumerating its elements as $A = \{x_1, x_2, ..., x_n\}$. There exists a membership function, which may be also used to define a set as a mapping of $\mathbb{X}$ into $\{0, 1\}$. Such membership function for a set A can be represents by

$$A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{1.1}$$

Therefore, each individual element in a set is a member or an element of the set. However, in real-life we come across many situations where membership and non-membership in a set are not clearly defined. For instance, consider the set of $A = \{\text{all real numbers near to 1}\}$ in the context 'positive values less than 5'. In classical set theory, one may imagine a membership function to describe A as

$$A(x) = \begin{cases} 1, & x \in [1-a, 1+a], \\ 0, & x \notin (1-a, 1+a). \end{cases} \tag{1.2}$$

where $a \in (0, 1]$. As a result, a value which is exactly a belongs to A and is considered to be 'all real numbers near to 1', but the values of $[0, a - 0.001]$ do not belong to A. In addition, all values in $[1-a, 1+a]$ may provide different memberships to describe 'near to 1'. The value of 1 is completely consistence to 'near to 1' but the other values partially inherit the imprecise concept of 'all real numbers near to 1'. These distinction are mathematically correct, but practically unreasonable. Therefore, A is not well defined in the sense

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of classical set theory and cannot be precisely represented by 0 or 1. The ambiguity in this situation follows from that the boundaries of A is imprecise so as to characterize the A in a precise and rigorous way. For this purpose, instead of a sharp cut at the exact values of 0.9 and 1.1 (as well as all values in $[1-a, 1+a]$), different memberships of 'near to 1' can be given via a membership function on $[0,1]$ instead of $\{0,1\}$. In this regard, one may utilize the following membership function to describe 'All numbers near to 1' as

$$A(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 1-x, & 1 \leq x \leq 2, \\ 0, & \mathbb{R} - [1,2]. \end{cases} \tag{1.3}$$

For example, the value of 0.25 is considered to be 'all real numbers near to 1' with 'degree 0.25' and with 'degree 0.95' for the value of 0.25 and 0.95 according to the membership function of f. Therefore, we cannot exclude the all values in $[0,1]$ from A, nor include them completely. Thus A is a generalization of the classical characteristic function, which can be used to conclude that a value 'is' or 'is not' a member of 'all real numbers near to 1'. Thus, such membership functions can provide a suitable mathematical tool for modeling such imprecision situations. Unlike the classical sets, the fuzzy set allows flexible membership degrees to each element of sets. That means that a fuzzy set contains elements that have varying degrees of membership in the set. Therefore, it provides a potential procedure to appreciate how uncertainty originating from human thinking can affect scientific problems.

Fuzzy sets were introduced by Zadeh [219] for the first time as an extension of classical sets. During the last two decades, fuzzy set theory has been successfully employed in working with numerous practical applications (for instance, see [12, 157, 185, 187, 223, 215]).

Definition 1.1 ([41, 157, 187]) *A fuzzy set of $\mathbb{X}$ (universe of discourse) is a mapping $\widetilde{A} : \mathbb{X} \to [0,1]$, which assigns to each $x \in \mathbb{X}$ a degree of membership $0 \leq \widetilde{A}(x) \leq 1$. The set of all fuzzy sets on $\mathbb{X}$ is denoted by $\mathbb{F}(\mathbb{X})$.*

Definition 1.2 ([41, 114, 157]) *$\widetilde{A}$ is said to be a convex fuzzy set on $\mathbb{X}$ if and only if for all $x, y \in \mathbb{X}$ and $\lambda \in [0,1]$, $\widetilde{A}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{A}(x), \widetilde{A}(y)\}$. A fuzzy set $\widetilde{A}$ is called normal fuzzy set if there exist $x_0 \in \mathbb{X}$ such that $\widetilde{A}(x_0) = 1$.*

Definition 1.3 ([41, 114, 157]) *For each $\alpha \in (0,1]$, $\{x \in \mathbb{X} \mid \widetilde{A}(x) \geq \alpha\}$ of $\mathbb{X}$ is called the α-cut of $\widetilde{A}$ and is denoted by $\widetilde{A}[\alpha]$. In addition, $\widetilde{A}[0] = Cl\{x \in \mathbb{X} : \widetilde{A}(x) > 0\}$ is called the support of $\widetilde{A}$ where $Cl(A)$ is the closure of A [140].*

For every $\alpha \in (0,1]$, the lower and upper bounds of $\widetilde{A}[\alpha]$ are denoted by $\widetilde{A}^L_\alpha$ and $\widetilde{A}^U_\alpha$, respectively.

Lemma 1.1 *A fuzzy set $\widetilde{A}$ is convex if and only if $\{\widetilde{A}[\alpha]\}_{\alpha \in [0,1]}$ is a collection of non-empty nested sets on $\mathbb{X}$.*

Proof *For a proof see [114].*

Having a collection of non-empty nested sets $\{\widetilde{A}[\alpha]\}_{\alpha\in[0,1]}$ on $\mathbb{X}$, the membership function of $\widetilde{A}(x)$ can be evaluated by the following lemma.

Lemma 1.2 *Let $\{\widetilde{A}[\alpha]\}_{\alpha\in[0,1]}$ be a collection of non-empty nested sets on $\mathbb{X}$. Then:*

$$\widetilde{A}(x) = \sup\{\alpha \in [0,1] : x \in \widetilde{A}[\alpha]\}. \tag{1.4}$$

Proof *See [114, 123, 157, 187] to prove this result.*

Definition 1.4 *We say $\widetilde{A}$ is a discrete fuzzy set if it is convex, normal and $\widetilde{A}(x)$ is not zero only at a finite number of $x \in \mathbb{X} = \{x_1, x_2, \ldots, x_n\}$. For such cases, $\widetilde{A}$ is typically denoted by $\widetilde{A} = \{\frac{a_1}{x_1}, \frac{a_2}{x_2}, \ldots, \frac{a_n}{x_n}\}$ where $\widetilde{A}(x_i) = a_i \in (0,1]$ are the membership degrees of x_i.*

Definition 1.5 *A fuzzy set $\widetilde{A}$ is called positive fuzzy set ($\widetilde{A} > 0$) if $\inf \widetilde{A}[0] > 0$, and it is negative fuzzy set ($\widetilde{A} < 0$) if $\sup \widetilde{A}[0] < 0$.*

Example 1.1 *Consider the set 'small' in $\mathbb{X}$ consisting of natural numbers less than 10. Such an imprecise set can be described via the following fuzzy set:*

$$\widetilde{A} = \{\frac{1}{1}, \frac{0.9}{2}, \frac{0.8}{3}, \frac{0.5}{4}, \frac{0.3}{5}, \frac{0.1}{6}, \frac{0}{\{7,8,9\}}\}.$$

For all $x, y \in \mathbb{X}$ and $\lambda \in [0,1]$, it is easy to check that $\widetilde{A}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{A}(x), \widetilde{A}(y)\}$. $\widetilde{A}$ is also a normal fuzzy set. Accordingly, $\widetilde{A}$ is a (discrete) fuzzy set.

Since an element can partially belong to a fuzzy set, a natural generalization of the classical notion of cardinality is to weigh each element by its membership degree, which resulted in the following definition for cardinality of a fuzzy set.

Definition 1.6 *[41, 114] For any fuzzy set $\widetilde{A}$ defined on a finite universal set $\mathbb{X}$, the cardinality of $\widetilde{A}$ is defined by $|\widetilde{A}| = \sum_{x\in\mathbb{X}} \widetilde{A}(x)$.*

Example 1.2 *Let $\mathbb{X} = \{1,2,3,4,5\}$ and $\widetilde{A} = \{\frac{0.3}{1}, \frac{0.5}{2}, \frac{0.7}{3}, \frac{1}{4}, \frac{0.8}{5}\}$. Then $|\widetilde{A}| = \sum_x \widetilde{A}(x) = 0.3 + 0.5 + 0.7 + 1 + 0.8 = 3.3$.*

The inclusion measure is an important concept in the area of fuzzy sets and indicates the degree to which a given fuzzy set belongs to a non-fuzzy set [34].

Definition 1.7 *Let $\widetilde{A} \in \mathbb{F}(\mathbb{X})$ be a fuzzy set and I be a subset of $\mathbb{R}$. Then, the degree to which $\widetilde{A}$ belongs to I is defined by $d(\widetilde{A} \in I) = \frac{\sum_{x\in I} \widetilde{A}(x)}{|\widetilde{A}|}$.*

Therefore, $d(\widetilde{A} \in I)$ is a relation between fuzzy sets $\widetilde{A}$ and classical set of I that indicates the degree to which $\widetilde{A}$ belongs to I.

Lemma 1.3 *Let $\widetilde{A} \in \mathbb{F}(\mathbb{X})$ and $I \subseteq \mathbb{R}$. Then:*

1) If $\{I_j\}_{j=1}^k$ is a collection of disjoint sets on $\mathbb{X}$, then $\sum_{j=1}^k d(\widetilde{A} \in I_j) = 1$.

2) $d(\widetilde{A} \in I) = 1$ if and only if $\widetilde{A}[0] \subseteq I$.

Proof *Assuming $\{I_j\}_{j=1}^k$ is a sequence of disjoint sets on $\mathbb{X}$, implies that*

$$\begin{aligned}\sum_{j=1}^{k} d(\widetilde{A} \in I_j) &= \frac{1}{|\widetilde{A}|}\sum_{j=1}^{k}\sum_{x\in I_j}\widetilde{A}(x)\\ &= \frac{1}{|\widetilde{A}|}\sum_{x\in\cup_{j=1}^k I_j}\widetilde{A}(x)\\ &= \frac{1}{|\widetilde{A}|}|\widetilde{A}| = 1.\end{aligned}$$

Now, if $\widetilde{A}[0] \subseteq I$ then:

$$d(\widetilde{A} \in I) = \frac{\sum_{x\in\widetilde{A}[0]}\widetilde{A}(x) + \sum_{x\in(I-\widetilde{A}[0])}\widetilde{A}(x)}{|\widetilde{A}|} = \frac{|\widetilde{A}|}{|\widetilde{A}|} = 1.$$

Conversely, $d(\widetilde{A} \in I) = 1$ yields $\sum_{x\in I^c}\widetilde{A}(x) = 0$ where I^c shows the compliment set of I. This simply concludes that $\widetilde{A}[0] \subseteq I$ and verifies the proof.

Remark 1.1 *It should be pointed out that $d(\widetilde{A} \in I)$ may be interpreted as the probability that $\widetilde{A}$ belongs to I. Moreover, based on a given collection of disjoint sets $\{I_j\}_{j=1}^k$, from Lemma 1.3, it is natural to say that $\widetilde{A} \in I_{j*}$ if $d(\widetilde{A} \in I_{j*}) = \max_{j=1}^k d(\widetilde{A} \in I_j)$. In particular, assume that I and J are two sets of $\mathbb{X}$ in which $I \cup J = \mathbb{X}$ and $I \cap J = \emptyset$. Therefore, if $d(\widetilde{A} \in I) \geq 0.5$ then $d(\widetilde{A} \in J) < 0.5$. In this case, one can say $\widetilde{A}$ belongs to I.*

Example 1.3 *Let $\widetilde{A} = \{\frac{0.4}{3}, \frac{0.6}{4}, \frac{1}{5}, \frac{0.8}{6}, \frac{0.5}{7}\}$ be a fuzzy set on $\mathbb{X} = \{1,2,3,4,5,6\ ,7,8,9,10\}$. Then, assuming $I_1 = \{1,2,3,4,\}$, $I_2 = \{5,6\}$, and $I_3 = \{7,8,9,10\}$ implies that $d(\widetilde{A} \in I_1) = 0.303$, $d(\widetilde{A} \in I_2) = 0.545$, and $d(\widetilde{A} \in I_1) = 0.152$. Since $d(\widetilde{A} \in I_2) \geq 0.5$, we get that $\widetilde{A} \in I_2$.*

1.1.1 Operations of fuzzy sets

Although the set-theoretic operations (union, intersection, and complement) possess some rigorous axiomatic properties, the conventional operations of sets can be extended for fuzzy sets [12, 157, 185, 187, 215, 223]. In this section, some well-established operations of fuzzy sets are recalled and discussed.

Definition 1.8 *For two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, we say:*

1. $\widetilde{A}$ *is included in* $\widetilde{B}$, *say* $\widetilde{A} \subseteq \widetilde{B}$, *if* $\widetilde{A}(x) \leq \widetilde{B}(x)$, *for all* $x \in \mathbb{X}$.
2. $\emptyset$ *is the empty (fuzzy) set if* $\emptyset(x) = 0$ *for any* $x \in \mathbb{X}$.
3. $\widetilde{A}^c$ *is the complement of fuzzy set* $\widetilde{A}$ *with the membership function* $\widetilde{A}^c(x) = 1 - \widetilde{A}(x)$, *for all* $x \in \mathbb{X}$.
4. $\widetilde{A} \cup \widetilde{B}$ *is the union of fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *with the membership function* $(\widetilde{A} \cup \widetilde{B})(x) = \max\{\widetilde{A}(x), \widetilde{B}(x)\}$, *for all* $x \in \mathbb{X}$.
5. $\widetilde{A} \cap \widetilde{B}$ *is the intersection of fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *with the membership function* $(\widetilde{A} \cap \widetilde{B})(x) = \min\{\widetilde{A}(x), \widetilde{B}(x)\}$, *for all* $x \in \mathbb{X}$.
6. $\widetilde{A} - \widetilde{B}$ *is the difference between two fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *with the membership function of* $(\widetilde{A} - \widetilde{B})(x) = \max\{0, \widetilde{A}(x) - \widetilde{B}(x)\}$.
7. $\widetilde{A} \odot \widetilde{B}$ *is the product of two fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *with the membership function of* $(\widetilde{A} \odot \widetilde{B})(x) = \widetilde{A}(x)\widetilde{B}(x)$.
8. $\widetilde{A} \triangle \widetilde{B}$ *is the symmetric difference of* $\widetilde{A}$ *and* $\widetilde{B}$ *with the membership function of* $(\widetilde{A} \triangle \widetilde{B})(x) = |\widetilde{A}(x) - \widetilde{B}(x)|$.

Example 1.4 *Let* $\mathbb{X} = \{1, 2, 3, 4\}$, $\widetilde{A} = \{\frac{0.2}{1}, \frac{0.7}{2}, \frac{1}{3}, \frac{0.6}{4}\}$, *and* $\widetilde{B} = \{\frac{0.9}{1}, \frac{1}{2}, \frac{0.8}{3}, \frac{0.5}{4}\}$. *Then, it can be easy to verify that:*

$$\begin{aligned}
\widetilde{A}^c &= \left\{\frac{0.8}{1}, \frac{0.3}{2}, \frac{0}{3}, \frac{0.4}{4}\right\}, \\
\widetilde{A} \cup \widetilde{B} &= \left\{\frac{\max\{0.2, 0.9\}}{1}, \frac{\max\{0.7, 1\}}{2}, \frac{\max\{1, 0.8\}}{3}, \frac{\max\{0.6, 0.5\}}{4}\right\} \\
&= \{\frac{0.9}{1}, \frac{1}{2}, \frac{1}{3}, \frac{0.6}{4}\},
\end{aligned}$$

$$\begin{aligned}
\widetilde{A} \cap \widetilde{B} &= \left\{\frac{\min\{0.2, 0.9\}}{1}, \frac{\min\{0.7, 1\}}{2}, \frac{\min\{1, 0.8\}}{3}, \frac{\min\{0.6, 0.5\}}{4}\right\} \\
&= \left\{\frac{0.2}{1}, \frac{0.7}{2}, \frac{0.8}{3}, \frac{0.5}{4}\right\},
\end{aligned}$$

$$\begin{aligned}
\widetilde{A} - \widetilde{B} = \Bigg\{&\frac{\max\{0, 0.2 - 0.9\}}{1}, \frac{\max\{0, 0.7 - 1\}}{2}, \frac{\max\{0, 1 - 0.8\}}{3}, \\
&\frac{\max\{0, 0.6 - 0.5\}}{4}\Bigg\} \\
= &\left\{\frac{0}{1}, \frac{0}{2}, \frac{0.2}{3}, \frac{0.1}{4}\right\},
\end{aligned}$$

$$\widetilde{A} \odot \widetilde{B} = \left\{\frac{0.2 \times 0.9}{1}, \frac{0.7 \times 1}{2}, \frac{1 \times 0.8}{3}, \frac{0.6 \times 0.5}{4}\right\}$$
$$= \left\{\frac{0.18}{1}, \frac{0.7}{2}, \frac{0.8}{3}, \frac{0.03}{4}\right\},$$

$$\widetilde{A} \triangle \widetilde{B} = \left\{\frac{|0.2 - 0.9|}{1}, \frac{|0.7 - 1|}{2}, \frac{|1 - 0.8|}{3}, \frac{|0.6 - 0.5|}{4}\right\}$$
$$= \left\{\frac{0.7}{1}, \frac{0.3}{2}, \frac{0.2}{3}, \frac{0.1}{4}\right\}.$$

Lemma 1.4 *For three fuzzy sets $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$,*

1. *Distributivity:*

$$\widetilde{A} \cap (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \cap \widetilde{B}) \cup (\widetilde{A} \cap \widetilde{C}), \tag{1.5}$$

$$\widetilde{A} \cup (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \cup \widetilde{B}) \cap (\widetilde{A} \cup \widetilde{C}). \tag{1.6}$$

2. *De Morgan Laws:*

$$(\widetilde{A} \cup \widetilde{B})^c = \widetilde{A}^c \cap \widetilde{B}^c, \tag{1.7}$$

$$(\widetilde{A} \cap \widetilde{B})^c = \widetilde{A}^c \cup \widetilde{B}^c. \tag{1.8}$$

Proof *To see a proof of these results see [41, 114].*

Theorem 1.1 *For three fuzzy sets $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, the following hold.*

1) $\widetilde{A} \odot (\widetilde{B} \cup \widetilde{C}) = (\widetilde{A} \odot \widetilde{B}) \cup (\widetilde{A} \odot \widetilde{C})$.

2) $\widetilde{A} \odot (\widetilde{B} \cap \widetilde{C}) = (\widetilde{A} \odot \widetilde{B}) \cap (\widetilde{A} \odot \widetilde{C})$.

3) $\widetilde{A} \odot (\widetilde{B} - \widetilde{C}) = (\widetilde{A} \odot \widetilde{B}) - (\widetilde{A} \odot \widetilde{C})$.

Proof *Note that* $\widetilde{A} \odot (\widetilde{B} \cup \widetilde{C})(x) = \widetilde{A}(x) \max\{\widetilde{B}(x), \widetilde{C}(x)\} = \max\{\widetilde{A}(x)\widetilde{B}(x), \widetilde{A}(x)\widetilde{C}(x)\}$. *This verifies part (1). The second part can be shown similarly. Further, we have* $(\widetilde{A} \odot (\widetilde{B} - \widetilde{C}))(x) = \widetilde{A}(x) \max\{0, \widetilde{B}(x) - \widetilde{C}(x)\} = \max\{0, \widetilde{A}(x)\widetilde{B}(x) - \widetilde{A}(x)\widetilde{C}(x)\}$. *This concludes that* $\widetilde{A} \odot (\widetilde{B} - \widetilde{C}) = (\widetilde{A} \odot \widetilde{B}) - (\widetilde{A} \odot \widetilde{C})$.

Theorem 1.2 *For two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ and for every $\alpha \in (0, 1]$, the following results show the connections between fuzzy sets and their α-cuts:*

1. *If $\widetilde{A} \subseteq \widetilde{B}$ then $\widetilde{A}[\alpha] \subseteq \widetilde{B}[\alpha]$.*

2. $\widetilde{A}^c[\alpha] = (\widetilde{A}[\overline{1 - \alpha}])^c$ *where* $\widetilde{A}[\overline{1 - \alpha}] = \{x : \widetilde{A}(x) > 1 - \alpha\}$.

3. $(\widetilde{A} \cup \widetilde{B})[\alpha] = \widetilde{A}[\alpha] \cup \widetilde{B}[\alpha]$.
4. $(\widetilde{A} \cap \widetilde{B})[\alpha] = \widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]$.

Proof *For a comprehensive proof see [41, 114].*

Lemma 1.5 *For two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, $|\widetilde{A} \cup \widetilde{B}| = |\widetilde{A}| + |\widetilde{B}| - |\widetilde{A} \cap \widetilde{B}|$.*

Proof *According to the definition of $\widetilde{A} \cup \widetilde{B}$, we easily have:*

$$\begin{aligned} |\widetilde{A} \cup \widetilde{B}| &= \sum_{x \in \mathbb{R}} (\widetilde{A} \cup \widetilde{B})(x) \\ &= \sum_{x \in \mathbb{R}} (\widetilde{A}(x) + \widetilde{B}(x) - (\widetilde{A} \cap \widetilde{B})(x)) \\ &= |\widetilde{A}| + |\widetilde{B}| - |\widetilde{A} \cap \widetilde{B}|. \end{aligned}$$

Definition 1.9 *(Extension principle [41, 114]) Suppose that $g : \mathbb{X} \to \mathbb{Y}$ is a real-valued function and $\widetilde{A}$ is a fuzzy set on $\mathbb{F}(\mathbb{X})$. The image of the fuzzy set $\widetilde{A}$ under the mapping $g(.)$ can be expressed as a fuzzy set $g(\widetilde{A}) \in \mathbb{F}(\mathbb{Y})$ with the following membership function:*

$$g(\widetilde{A})(y) = \begin{cases} \sup_{x:y=g(x)} \widetilde{A}(y), & g^{-1}(y) \neq \emptyset, \\ 0, & g^{-1}(y) = \emptyset. \end{cases} \tag{1.9}$$

Example 1.5 *Consider a fuzzy set on $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ stands for 'about 1' with the membership function of $\widetilde{A} = \{\frac{0.4}{-2}, \frac{0.6}{-1}, \frac{0.8}{0}, \frac{1}{1}, \frac{0.8}{2}, \frac{0.6}{3}, \frac{0.4}{4}\}$. Letting $g(x) = x^4$, we have $y \in \{0, 1, 16, 81, 256\}$ and thus:*

$$\begin{aligned} g(\widetilde{A})(0) &= \widetilde{A}(0) = 0.8, \\ g(\widetilde{A})(1) &= \max\{\widetilde{A}(-1), \widetilde{A}(1)\} = \max\{0.6, 1\} = 1, \\ g(\widetilde{A})(16) &= \max\{\widetilde{A}(-2), \widetilde{A}(2)\} = \max\{0.4, 0.8\} = 0.8, \\ g(\widetilde{A})(81) &= \widetilde{A}(3) = 0.6, \\ g(\widetilde{A})(256) &= \widetilde{A}(4) = 0.4. \end{aligned}$$

From above, the membership function of $g(\widetilde{A}) = (\widetilde{A})^4$ can be obtained as:

$$(\widetilde{A})^4 = \{\frac{0.8}{0}, \frac{1}{1}, \frac{0.8}{16}, \frac{0.6}{81}, \frac{0.4}{256}\}.$$

In order to use fuzzy sets in any real-life one should be able to perform arithmetic operations including addition, subtraction, multiplication, and division in computational process, which is called fuzzy arithmetic. The usual arithmetic operations on real numbers can be extended to the ones defined on fuzzy sets based on the Extension principle.

Definition 1.10 *[41, 114] The arithmetic operations between two fuzzy sets of* $\widetilde{A}$ *and* $\widetilde{B}$ *is defined as a fuzzy set* $\widetilde{A} \circ \widetilde{B}$ *with the following membership function:*

$$(\widetilde{A} \circ \widetilde{B})(z) = \sup_{x,y:z=x\bullet y} \min\{\widetilde{A}(x), \widetilde{B}(y)\}. \tag{1.10}$$

The symbols $\circ = \oplus, \ominus, \otimes, \oslash$ *denotes the extended arithmetic operations* $\bullet = +, -, \times, /$ *in a fuzzy domain.*

Next, a notion of convergence a collection of fuzzy sets is given.

Definition 1.11 *Let* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ *be a countable collection of fuzzy sets and* $\widetilde{A}$ *be a fuzzy set. Then, we say:*

1. $\widetilde{A}_n \to \widetilde{A}$ *if* $\lim_{n\to\infty} \widetilde{A}_n(x) = \widetilde{A}(x)$ *for all* $x \in \mathbb{R}$.
2. $\widetilde{A}_n \uparrow \widetilde{A}$ *if* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ *is an increasing countable collection of fuzzy sets and* $\lim_{n\to\infty} \widetilde{A}_n(x) = \widetilde{A}(x)$ *for all* $x \in \mathbb{R}$.
3. $\widetilde{A}_n \downarrow \widetilde{A}$ *if* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ *is a decreasing countable collection of fuzzy sets and* $\lim_{n\to\infty} \widetilde{A}_n(x) = \widetilde{A}(x)$ *for all* $x \in \mathbb{R}$.

Example 1.6 *Let* $\mathbb{X} = \{-3, -2, -1, 0, 1, 2, 3\}$ *and consider a sequence of discrete fuzzy set on* $\mathbb{X}$ *with* $\widetilde{A}_n(x) = (1 - (1/nx^2))^n$, $x \in \mathbb{X}$. *Since* $\lim_{n\to\infty} \widetilde{A}_n(x) = e^{-x^2}$, *we have* $\lim_{n\to\infty} \widetilde{A}_n(x) = \widetilde{A}(x) = e^{-x^2}$ *with the following membership function:*

$$\widetilde{A} = \{\frac{-3}{e^{-9}}, \frac{-2}{e^{-4}}, \frac{-1}{e^{-1}}, \frac{0}{1}, \frac{1}{e^{-1}}, \frac{2}{e^{-4}}, \frac{3}{e^{-9}}\}.$$

Proposition 1.1 *For a a countable collection of fuzzy sets* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$, $\widetilde{A}_n \to \widetilde{A}$ *if and only if* $\widetilde{A}_n[\alpha] \to \widetilde{A}[\alpha]$ *for any* $\alpha \in (0, 1]$.

Proof *The proof is left for the reader.*

Definition 1.12 *The limes superior and limes inferior of a countable collection of fuzzy sets* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ *are defined to be fuzzy sets* $\limsup \widetilde{A}_n = \cap_{n\geq 1} \cup_{m\geq n} \widetilde{A}_m$ *and* $\liminf \widetilde{A}_n = \cup_{n\geq 1} \cap_{m\geq n} \widetilde{A}_m$, *respectively.*

Example 1.7 *Let* $\mathbb{X} = \{-3/4, -2/4, -1/4, 0, 1/4, 2/4, 3/4\}$ *and consider a sequence of discrete fuzzy set on* $\mathbb{X}$ *with* $\widetilde{A}_n(x) = \max\{0, 1 - (n/(n+1))|x|\}$, $x \in \mathbb{X}$. *Then, for a fixed* $x \in \mathbb{X}$, *we have*

$$\begin{aligned}(\cup_{m\geq n}\widetilde{A}_m)(x) &= \max_{m\geq n} \widetilde{A}_m(x) \\ &= \max\{0, \max_{m\geq n}(1 - (m/(m+1))|x|\} \\ &= \max\{0, 1 - (n/(n+1))|x|\}.\end{aligned}$$

This concludes that

$$\begin{aligned}(\limsup \widetilde{A}_n)(x) &= (\cap_{n\geq 1} \cup_{m\geq n} \widetilde{A}_m)(x)\\ &= \min_{n\geq 1}(\cup_{m\geq n}\widetilde{A}_m)(x)\\ &= \min_{n\geq 1} \max\{0, 1-(n/(n+1))|x|\}\\ &= \max\{0, 1-|x|\} = \widetilde{A}(x).\end{aligned}$$

Therefore,

$$\limsup \widetilde{A}_n = \{\frac{-3/4}{1/4}, \frac{-2/4}{2/4}, \frac{-1/4}{3/4}, \frac{0}{1}, \frac{1/4}{3/4}, \frac{2/4}{2/4}, \frac{3/4}{1/4}\}.$$

Theorem 1.3 *Let* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ *be a countable collection of fuzzy sets. Then, the following are valid.*

1. $\liminf \widetilde{A}_n \subseteq \limsup \widetilde{A}_n$.
2. $(\liminf \widetilde{A}_n)^c = \limsup \widetilde{A}_n^c$.
3. $(\limsup \widetilde{A}_n)^c = \liminf \widetilde{A}_n^c$.

Proof *First note that* $(\liminf \widetilde{A}_n)[\alpha] = \cup_{n\geq 1} \cap_{m\geq n} \widetilde{A}_n[\alpha]$. *That concludes that* $(\liminf \widetilde{A}_n)[\alpha] \subseteq \cap_{n\geq 1} \cup_{m\geq n} \widetilde{A}_n[\alpha] = (\limsup \widetilde{A}_n)[\alpha]$ *for any* $\alpha \in [0,1]$ *or* $\liminf \widetilde{A}_n \subseteq \limsup \widetilde{A}_n$. *In addition, for any* $\alpha \in [0,1]$, $(\liminf \widetilde{A}_n)^c[\alpha] = (\cup_{n\geq 1} \cap_{m\geq n} \widetilde{A}[\overline{1-\alpha}])^c = \cap_{n\geq 1} \cup_{m\geq n} (\widetilde{A}_n[\overline{1-\alpha}])^c = \cap_{n\geq 1} \cup_{m\geq n} \widetilde{A}_n^c[\alpha]$. *This means* $(\liminf \widetilde{A}_n)^c = \limsup \widetilde{A}_n^c$. *Similarly, it can be shown that* $(\limsup \widetilde{A}_n)^c = \liminf \widetilde{A}_n^c$.

Theorem 1.4 *Let* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ *and* $\{\widetilde{B}_n\}_{n\in\mathbb{N}}$ *be two countable collections of fuzzy sets. Then:*

1) $\limsup(\widetilde{A} \oplus \widetilde{B})_n \subseteq (\limsup \widetilde{A}_n \oplus \limsup \widetilde{B}_n)$.

2) $\liminf(\widetilde{A} \oplus \widetilde{B})_n \supseteq (\liminf \widetilde{A}_n \oplus \liminf \widetilde{B}_n)$.

Proof *For any* $\alpha \in (0,1]$, $\limsup(\widetilde{A}_n \oplus \widetilde{B}_n)[\alpha] = \cap_{n\geq 1} \cup_{m\geq n} (\widetilde{A}_n[\alpha] + \widetilde{B}_n[\alpha])$. *Now, assume that* x *is an arbitrary element of* $\limsup(\widetilde{A}_n \oplus \widetilde{B}_n)[\alpha]$. *Therefore, for any* $n \geq 1$, *there exist a* $m \geq n$ *such that* $x = y + z$ *where* $y \in \widetilde{A}_n[\alpha]$ *and* $z \in \widetilde{B}_n[\alpha]$. *This concludes that* $x \in (\limsup \widetilde{A}_n \oplus \limsup \widetilde{B}_n)[\alpha]$ *which providing (1). The second part can be verified similarly.*

Proposition 1.2 *For a countable collection of fuzzy sets* $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$*:*

1) $\lim_{n\to\infty} \cup_{k\geq n}\widetilde{A}_k = \limsup \widetilde{A}_n$.

2) $\lim_{n\to\infty} \cap_{k\geq n}\widetilde{A}_k = \liminf \widetilde{A}_n$.

Proof *Let $\widetilde{B}_k^n = \cup_{k\geq n}\widetilde{A}_k$, for any $\alpha \in (0,1]$. For (1), since $\widetilde{B}_k^n[\alpha] \downarrow \cap_{n\geq 1} \cup_{k\geq n} \widetilde{A}_k[\alpha] = \limsup \widetilde{A}_n[\alpha]$, we easily have $\lim_{n\to\infty} \cup_{k\geq n}\widetilde{A}_k = \limsup \widetilde{A}_n$ by Lemma 1.1. The second part can be verified similarly.*

Proposition 1.3 *Let $\{\widetilde{A}_n\}_{n\in\mathbb{N}}$ be a countable collection of fuzzy sets. Then $\widetilde{A}_n \to \widetilde{A}$ if and only if $\liminf \widetilde{A}_n = \limsup \widetilde{A}_n = \widetilde{A}$.*

Proof *First assume that $\widetilde{A}_n \to \widetilde{A}$. This simply concludes that $\widetilde{A}_n[\alpha] \to \widetilde{A}[\alpha]$ for any $\alpha \in (0,1]$ which implies that $\liminf \widetilde{A}_n[\alpha] = \limsup \widetilde{A}_n = \widetilde{A}[\alpha]$ for any $\alpha \in [0,1]$. Thus $\liminf \widetilde{A}_n = \limsup \widetilde{A}_n = \widetilde{A}$ and thus the 'if part' is verified. The 'only if part' can be shown similarly.*

Next, a common notion of a real-valued function of a fuzzy set is given.

1.1.2 Cartesian product of fuzzy sets

The Cartesian products of some sets mean the product of some non-empty sets in a systematic way. Here, a common definition of the fuzzy product of (discrete) fuzzy is recalled, and some of its main properties are investigated. All such concepts were gathered from [100].

Definition 1.13 *Let $\widetilde{A}_i \in \mathbb{F}(\mathbb{X}_i)$, $i = 1, 2, \ldots, n$. A fuzzy product set on $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times ... \times \mathbb{X}_n$ is defined as a Cartesian product of the fuzzy set $\widetilde{\boldsymbol{A}} = \widetilde{A}_1 \times \widetilde{A}_2 \times ... \times \widetilde{A}_n$ with the following membership function:*

$$\widetilde{\boldsymbol{A}}(\boldsymbol{x}) = \min_{i=1}^{n}\{\widetilde{A}_i(x_i)\}, \ x_i \in \mathbb{X}_i. \tag{1.11}$$

Example 1.8 *Let $\mathbb{X}_1 = \mathbb{X}_2 = \{2, 4, 6\}$, $\widetilde{A}_1 = \{\frac{0.7}{2}, \frac{1}{4}, \frac{0.6}{6}\}$ and $\widetilde{A}_2 = \{\frac{1}{2}, \frac{0.8}{4}\}$. Then, the membership function of fuzzy product of $\widetilde{A}_1$ and $\widetilde{A}_2$ can be shown by:*

$$\widetilde{A}_1 \times \widetilde{A}_2 = \{\frac{0.7}{(2,2)}, \frac{0.7}{(2,4)}, \frac{1}{(4,2)}, \frac{0.8}{(4,4)}, \frac{0.6}{(6,2)}, \frac{0.6}{(6,4)}\}.$$

Proposition 1.4 *If $\{\widetilde{\boldsymbol{A}}_j = \widetilde{A}_{1j} \times \widetilde{A}_{2j} \times ... \times \widetilde{A}_{nj}\}_{j=1}^m$ is a sequence of fuzzy product sets then $\bigcup_{j=1}^m \widetilde{\boldsymbol{A}}_j = (\bigcup_{j=1}^m \widetilde{A}_{1j}) \times \ldots \times (\bigcup_{j=1}^m \widetilde{A}_{nj})$.*

Proof *For a fixed $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{X}^n$, we have:*

$$\begin{aligned}
(\bigcup_{j=1}^{m} \widetilde{\boldsymbol{A}}_j)(\boldsymbol{x}) &= \max_{j=1}^{m} \min\{\widetilde{A}_{1j}(x_1), \widetilde{A}_{2j}(x_2), ..., \widetilde{A}_{nj}(x_n)\} \qquad (1.12)\\
&= \min\{\max_{j=1}^{m} \widetilde{A}_{1j}(x_1), \max_{j=1}^{m} \widetilde{A}_{2j}(x_2), ..., \max_{j=1}^{m} \widetilde{A}_{nj}(x_n)\}\\
&= (\bigcup_{j=1}^{m} \widetilde{A}_{1j})(x_1) \times (\bigcup_{j=1}^{m} \widetilde{A}_{2j})(x_2) \times ... \times (\bigcup_{j=1}^{m} \widetilde{A}_{nj})(x_n).
\end{aligned}$$

It completes the proof.

Proposition 1.5 *If* $\{\widetilde{\boldsymbol{A}}_j = \widetilde{A}_{1j} \times \widetilde{A}_{2j} \times ... \times \widetilde{A}_{nj}\}_{j=1}^m$ *is a sequence of fuzzy product sets then* $\bigcap_{j=1}^m \widetilde{\boldsymbol{A}}_j = (\bigcap_{j=1}^m \widetilde{A}_{1j}) \times \ldots \times (\bigcap_{j=1}^m \widetilde{A}_{nj})$.

Proof *For a fixed* $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{X}^n$, *it is easily seen that:*

$$(\bigcap_{j=1}^m \widetilde{\boldsymbol{A}}_j)(\boldsymbol{x}) = \min_{j=1}^m \min\{\widetilde{A}_{1j}(x_1), \widetilde{A}_{2j}(x_2), ..., \widetilde{A}_{nj}(x_n)\} =$$

$$\min\{\min_{j=1}^m \widetilde{A}_{1j}(x_1), \min_{j=1}^m \widetilde{A}_{2j}(x_2), ..., \min_{j=1}^m \widetilde{A}_{nj}(x_n)\} =$$

$$(\bigcap_{j=1}^m \widetilde{A}_{1j})(x_1) \times (\bigcap_{j=1}^m \widetilde{A}_{2j})(x_2) \times ... \times (\bigcap_{j=1}^m \widetilde{A}_{nj})(x_n).$$

Lemma 1.6 *Let* $\{\widetilde{A}_i\}_{i=1}^n$ *be a finite collection of fuzzy sets. For a given fuzzy set* $\widetilde{B}$, *we have the following results:*

1. $(\cup_{i=1}^n \widetilde{A}_i) \times \widetilde{B} = \cup_{i=1}^n (\widetilde{A}_i \times \widetilde{B})$.
2. $\widetilde{B} \times (\cup_{i=1}^n \widetilde{A}_i) = \cup_{i=1}^n (\widetilde{B} \times \widetilde{A}_i)$.
3. $(\cap_{i=1}^n \widetilde{A}_i) \times \widetilde{B} = \cap_{i=1}^n (\widetilde{A}_i \times \widetilde{B})$.
4. $\widetilde{B} \times (\cap_{i=1}^n \widetilde{A}_i) = \cap_{i=1}^n (\widetilde{B} \times \widetilde{A}_i)$.

Proof *The first part is proved only. The others are left for readers. For this purpose, a fixed* $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{X}^n$ *and* $x \in \mathbb{X}$, *we have:*

$$((\cup_{i=1}^n \widetilde{A}_i) \times \widetilde{B})(\boldsymbol{x}, x) = \min\{\max\{\widetilde{A}_1(x_1), \widetilde{A}_2(x_2), ..., \widetilde{A}_n(x_n)\}, \widetilde{B}(x)\} =$$

$$\max\{\min\{\widetilde{A}_1(x_1), \widetilde{B}(x)\}, ..., \min\{\widetilde{A}_n(x_n), \widetilde{B}(x)\}\} = (\cup_{i=1}^n (\widetilde{A}_i \times \widetilde{B}))(\boldsymbol{x}, x).$$

This confirm that $(\cup_{i=1}^n \widetilde{A}_i) \times \widetilde{B} = \cup_{i=1}^n (\widetilde{A}_i \times \widetilde{B})$.

1.2 Fuzzy Numbers

Fuzzy numbers generalize classical real numbers and roughly speaking a fuzzy number is a fuzzy set of the real line that has some additional properties. That is the domain of a fuzzy number is a specified set of real numbers and its range is the span of non-negative real numbers between 0 and 1. Each numerical value in the domain is assigned a specific degree of membership where 0 represents the smallest possible grade, and 1 is the largest possible degree. A unified definition of a fuzzy number is given below.

Definition 1.14 ([41, 114, 157]) *A fuzzy set* $\widetilde{A}$ *of* $\mathbb{R}$ *is called a fuzzy number (**FN**) if it is normal and the set* $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \widetilde{A}(x) \geq \alpha\}$ *is a non-empty nested closed interval in* $\mathbb{R}$, *for every* $\alpha \in [0, 1]$.

This interval is denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. For this book, take $\mathcal{F}(\mathbb{R})$ as denoting the set of all **FN**s.

The arithmetic operations between two **FN**s $\widetilde{A}$ and $\widetilde{B}$ can also be evaluated using arithmetic operations of their α-cuts.

Theorem 1.5 *Let $\widetilde{A}$ and $\widetilde{B}$ be two **FN**s. Then, for every $\alpha \in (0, 1]$:*

1. $(\widetilde{A} \oplus \widetilde{B})[\alpha] = [\widetilde{A}^L_\alpha + \widetilde{B}^L_\alpha, \widetilde{A}^U_\alpha + \widetilde{B}^U_\alpha]$.
2. $(\widetilde{A} \ominus \widetilde{B})[\alpha] = [\widetilde{A}^L_\alpha - \widetilde{B}^U_\alpha, \widetilde{A}^U_\alpha - \widetilde{B}^L_\alpha]$.
3. $(\widetilde{A} \otimes \widetilde{B})[\alpha] = [(\widetilde{A} \otimes \widetilde{B})^L_\alpha, (\widetilde{A} \otimes \widetilde{B})^U_\alpha]$ *where*
$$(\widetilde{A} \otimes \widetilde{B})^L_\alpha = \min\{\widetilde{A}^L_\alpha \widetilde{B}^L_\alpha, \widetilde{A}^L_\alpha \widetilde{B}^U_\alpha, \widetilde{A}^U_\alpha \widetilde{B}^L_\alpha, \widetilde{A}^U_\alpha \widetilde{B}^U_\alpha\}, \tag{1.13}$$
$$(\widetilde{A} \otimes \widetilde{B})^U_\alpha = \max\{\widetilde{A}^L_\alpha \widetilde{B}^L_\alpha, \widetilde{A}^L_\alpha \widetilde{B}^U_\alpha, \widetilde{A}^U_\alpha \widetilde{B}^L_\alpha, \widetilde{A}^U_\alpha \widetilde{B}^U_\alpha\}, \tag{1.14}$$
4. $(\widetilde{A} \oslash \widetilde{B})[\alpha] = [(\widetilde{A} \oslash \widetilde{B})^L_\alpha, (\widetilde{A} \oslash \widetilde{B})^U_\alpha]$ *where*
$$(\widetilde{A} \oslash \widetilde{B})^L_\alpha = \min\{\widetilde{A}^L_\alpha / \widetilde{B}^L_\alpha, \widetilde{A}^L_\alpha / \widetilde{B}^U_\alpha, \widetilde{A}^U_\alpha / \widetilde{B}^L_\alpha, \widetilde{A}^U_\alpha / \widetilde{B}^U_\alpha\}, \tag{1.15}$$
$$(\widetilde{A} \oslash \widetilde{B})^U_\alpha = \max\{\widetilde{A}^L_\alpha / \widetilde{B}^L_\alpha, \widetilde{A}^L_\alpha / \widetilde{B}^U_\alpha, \widetilde{A}^U_\alpha / \widetilde{B}^L_\alpha, \widetilde{A}^U_\alpha / \widetilde{B}^U_\alpha\}, \tag{1.16}$$
provided that $0 \notin \widetilde{B}[0]$.

Proof *See [41, 114, 157, 187], for instance.*

By the above results and Lemma 1.2, the membership functions of $(\widetilde{A} \circ \widetilde{B})$ can be evaluated as:

$$(\widetilde{A} \circ \widetilde{B})(x) = \sup\{\alpha \in [0, 1] : x \in (\widetilde{A} \circ \widetilde{B})[\alpha]\}. \tag{1.17}$$

where $\circ = \oplus, \ominus, \otimes, \oslash$ denotes the extended arithmetic operations $\bullet = +, -, \times, /$ in a fuzzy domain.

Several definitions have been proposed to capture the information contained in a (unimodal) **FN** to simplify the **FN**'s representation using a functional parametric form known as LR-fuzzy numbers (LR-**FN**) $\widetilde{A} = (a; l_a, r_a)_{LR}$ [41, 114, 123]. The membership function of a LR-**FN** $\widetilde{A}$ is defined by:

$$\widetilde{A}(x) = \begin{cases} L(\dfrac{a-x}{l_a}), & a - l_a \leq x \leq a, \\ R(\dfrac{x-a}{r_a}), & a \leq x \leq a + r_a, \\ 0, & x \in \mathbb{R} - [a - l_a, a + r_a], \end{cases} \tag{1.18}$$

where $l_a > 0$ is the left spread, $r_a > 0$ is the right spread, and L and R represent reference functions defining the left and the right shapes of the **FN**, respectively. The left and the right shapes $L, R : [0, 1] \to [0, 1]$ should also satisfy the following conditions:

1. $L(1) = R(1) = 0$,
2. $L(0) = R(0) = 1$, and
3. $L(x)$ and $R(x)$ are continuous and monotone decreasing functions on $[0, 1]$.

Moreover, a LR-**FN** becomes an L-**FN** when $L(x) = R(x)$. It is easy to check that the α-cut of a LR-**FN** $\widetilde{A}$ is:

$$\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha] = [a - L^{-1}(\alpha)l_a, a + R^{-1}(\alpha)r_a], \quad \alpha \in [0, 1].$$

Some common operations between two LR-**FN**s of are given as follows.

Theorem 1.6 *For two LR-**FN**s $\widetilde{A} = (a; l_a, r_a)_{LR}$ and $\widetilde{B} = (b; l_b, r_b)_{LR}$,*

1) Addition: $\widetilde{A} \oplus \widetilde{B} = (a + b; l_a + l_b, r_a + r_b)_{LR}$.

2) Difference: $\widetilde{A} \ominus \widetilde{B} = (a - b; l_a + r_b, r_a + l_b)_{RL}$.

3) The scalar multiplication:

$$\lambda \otimes \widetilde{A} = \begin{cases} (\lambda a; \lambda l_a, \lambda r_a)_{LR}, & \lambda > 0, \\ (\lambda a; -\lambda r_a, -\lambda l_a)_{RL}, & \lambda < 0. \end{cases}$$

Proof *See [41, 114, 123] to have a proof.*

In next the chapters, we employ the most commonly used LR-**FN**s so-called triangular fuzzy numbers (**TFN**s) to handle imprecision included in data set. Note that the membership function of a **TFN**, denoted by $\widetilde{A} = (a; l_a, r_a)_T$, is given by:

$$\widetilde{A}(x) = \begin{cases} \frac{x-(a-l_a)}{l_a}, & a - l_a \leq x \leq a, \\ \frac{a+r_a-x}{r_a}, & a \leq x \leq a + r_a, \\ 0, & x \in \mathbb{R} - [a - l_a, a + r_a]. \end{cases} \tag{1.19}$$

It is worth noting that a **TFN** $\widetilde{A} = (a; l_a, r_a)_T$ can also be rewritten as $\widetilde{A} = (a^L, a, a^U)_T$ where $a^L = a - l_a$ and $a^U = a + r_a$. In this regards, the membership function of $\widetilde{A}$ is:

$$\widetilde{A}(x) = \begin{cases} \frac{x-a^L}{a-a^L}, & a^L \leq x \leq a, \\ \frac{a^U-x}{a^U-a}, & a \leq x \leq a^U, \\ 0, & x \in \mathbb{R} - [a^L, a^U]. \end{cases} \tag{1.20}$$

For simplicity, a symmetric **TFN** is represented by **STFN** throughout this book.
Here, an example is provided to obtain the function of a LR-**FN**.

Example 1.9 *Let $g(x) = x^2$ and $\widetilde{A} = (0; 2, 1)_{LR}$ with $L(x) = \max\{0, 1-x\}$ and $R(x) = \sqrt{1-x^2}$. First note that the membership function of $\widetilde{A}$ is:*

$$\widetilde{A}(x) = \begin{cases} \frac{x+2}{2}, & -2 \leq x \leq 0, \\ \sqrt{1-x^2}, & 0 \leq x \leq 1, \\ 0, & x \in \mathbb{R} - [-2, 1]. \end{cases} \tag{1.21}$$

Then, it is easy to check that:

$$\begin{aligned} g(\widetilde{A})(y) &= \sup_{x:y=x^2} \widetilde{A}(y) \\ &= \begin{cases} \max\{\widetilde{A}(\sqrt{y}), \widetilde{A}(-\sqrt{y})\}, & 0 \leq y \leq 1, \\ \widetilde{A}(-\sqrt{y}), & 1 < y \leq 4, \end{cases} \\ &= \begin{cases} \max\{\frac{2-\sqrt{y}}{2}, \sqrt{1-y}\}, & 0 \leq y \leq 1, \\ \frac{2-\sqrt{y}}{2} & 1 < y \leq 4, \end{cases} \\ &= \begin{cases} \sqrt{1-y}, & 0 \leq y \leq 0.64, \\ \frac{2-\sqrt{y}}{2}, & 0.64 < y \leq 4. \end{cases} \end{aligned}$$

The membership function of $g(\widetilde{A}) = \widetilde{A}^2$ is plotted in Fig. 1.1.

An example of investigating of limit superior of a sequence of **TFN**s is given by the following example

Example 1.10 *Consider a countable sequence of **TFN**s defined as $\widetilde{A}_n = (0; l_{a_n}, r_{a_n})_T$ where $l_{a_n} = 1/n$ and $r_{a_n} = 2 - 1/n$. Based on Proposition 1.3, we have*

$$\begin{aligned} \limsup \widetilde{A}_n[\alpha] &= \cap_{n\geq 1} \cup_{m\geq n} [-(1/m)(1-\alpha), (2-1/m)(1-\alpha)] \\ &= \cap_{n\geq 1}[-(1/n)(1-\alpha), 2(1-\alpha)] \\ &= [0, 2(1-\alpha)] \\ &= \liminf \widetilde{A}_n[\alpha], \end{aligned}$$

which concludes that $\lim_{n\to\infty} \widetilde{A}_n = (0; 0, 2)_T$.

1.2.1 A similarity measure

Similarity measures are very important and useful tool to determine the amount of similarity between two or more fuzzy quantities. In almost every field of science and engineering, the concept of similarity measure has important significance [8]. Such measures are used to compare different kinds of objects in real-life applications [16, 20, 26, 76, 124, 171, 176, 193]. Here, a similarity measure between two **TFN**s is introduced. We will employ such criteria as a performance measure in Chapter 7.

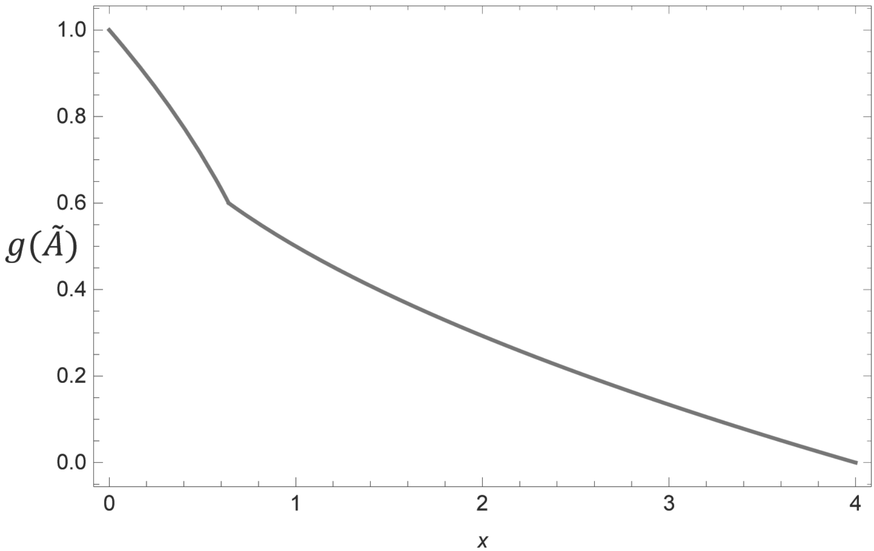

FIGURE 1.1
Plot of $g(\widetilde{A})$ in Example 1.9.

Definition 1.15 *Let $\widetilde{A} = (a^L, a, a^U)_T$ and $\widetilde{B} = (b^L, b, b^U)_T$ be two **TFN**s. The similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is defined as:*

$$S(\widetilde{A}, \widetilde{B}) = 1 - \frac{|a^L - b^L| + |a - b| + |a^U - b^U|}{3(\max\{a^U, b^U\} - \min\{a^L, b^L\})}. \tag{1.22}$$

Proposition 1.6 *Let $\widetilde{A} = (a^L, a, a^U)_T$ and $\widetilde{B} = (b^L, b, b^U)_T$ be two **TFN**s. Then:*

1) $S(\widetilde{A}, \widetilde{B}) \in [0, 1]$.

2) $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$.

3) $S(\widetilde{A}, \widetilde{B}) = 1$ *if and only if* $\widetilde{A} = \widetilde{B}$.

Proof *The proof is left for the reader.*

According to Proposition 1.6, we can say that the values of $S(\widetilde{A}, \widetilde{B}) < 0.5$ show the degree of dissimilarity while the values of $S(\widetilde{A}, \widetilde{B}) \geq 0.5$ represent the degree of similarity. In this regard, one can say that:

1) If $S(\widetilde{A}, \widetilde{B}) \in [0.5, 0.6)$ then $\widetilde{A}$ and $\widetilde{B}$ are more or less similar.

2) If $S(\widetilde{A}, \widetilde{B}) \in [0.6, 0.7)$ then the similarity of $\widetilde{A}$ and $\widetilde{B}$ is moderate.

3) If $S(\widetilde{A},\widetilde{B}) \in [0.7, 0.8)$ then the similarity of $\widetilde{A}$ and $\widetilde{B}$ is strong.

4) If $S(\widetilde{A},\widetilde{B}) \in [0.8, 0.9)$ then the similarity of $\widetilde{A}$ and $\widetilde{B}$ is very strong.

3) If $S(\widetilde{A},\widetilde{B}) \in [0.9, 0.1]$ then the similarity of $\widetilde{A}$ and $\widetilde{B}$ is complete.

Example 1.11 *Let $\widetilde{A} = (2, 3, 4)_T$ and $\widetilde{B} = (2.5, 3.5, 3.8)_T$ be two **TFN**s. Then, the similarity measure between $\widetilde{A}$ and $\widetilde{B}$ is:*

$$S(\widetilde{A},\widetilde{B}) = 1 - \frac{|2.5-2| + |3.5-3| + |3.8-4|}{3(\max\{4, 3.7\} - \min\{2, 2.5\})} = 1 - \frac{1.2}{3(4-2)} = 0.8.$$

Therefore, the similarity of $\widetilde{A}$ and $\widetilde{B}$ is very strong.

Lemma 1.7 *If $\{S_i\}_{i=1}^n$ is a sequence of similarity measures then $S = (1/n)\sum_{i=1}^n S_i$ is also a similarity measure.*

Proof *The verification of this lemma is left as an exercise.*

1.2.2 A criteria for ranking FNs

The problem of ordering **FN**s has received considerable attention in real-life applications. For instance, in many statistical hypothesis tests with fuzzy data, fuzzy hypotheses are based on comparing observed fuzzy test statistics and critical values . Therefore there is a need for a criterion to compare such quantities. Because of the nature of the measurement, many different strategies have been proposed for ranking of **FN**s [14, 22, 27, 42, 155, 178, 192, 203, 200, 214]. Another method of ranking **FN**s relies on preference degree. This method provides a decision-maker with a preference degree to measure how a **FN** is greater than another [216, 143, 87]. Here, a common preference criterion to rank **FN**s is recalled [143].

Definition 1.16 *For two **FN**s $\widetilde{A}$ and $\widetilde{B}$, define:*

$$\Delta_{\widetilde{A}\widetilde{B}} = \int_{\{\alpha:\widetilde{A}_\alpha^L \geq \widetilde{B}_\alpha^L\}} (\widetilde{A}_\alpha^L - \widetilde{B}_\alpha^L)d\alpha + \int_{\{\alpha:\widetilde{A}_\alpha^U \geq \widetilde{B}_\alpha^U\}} (\widetilde{A}_\alpha^U - \widetilde{B}_\alpha^U)d\alpha. \tag{1.23}$$

The preference degree that '$\widetilde{A}$ is greater than $\widetilde{B}$' is defined by:

$$P_d(\widetilde{A} \succ \widetilde{B}) = \begin{cases} 0.5, & \widetilde{A} = \widetilde{B}, \\ \frac{\Delta_{\widetilde{A}\widetilde{B}}}{\Delta_{\widetilde{A}\widetilde{B}} + \Delta_{\widetilde{B}\widetilde{A}}}, & \widetilde{A} \neq \widetilde{B}. \end{cases} \tag{1.24}$$

Definition 1.17 *For two **FN**s $\widetilde{A}$ and $\widetilde{B}$, it is said that:*

1. *$\widetilde{A}$ is greater than $\widetilde{B}$, denoting by $\widetilde{A} \succ_{P_d} \widetilde{B}$, if $P_d(\widetilde{A} \succ \widetilde{B}) > 0.5$.*

2. *$\widetilde{A}$ is equivalent to $\widetilde{B}$, denoting by $\widetilde{A} \simeq_{P_d} \widetilde{B}$, if $P_d(\widetilde{A} \succ \widetilde{B}) = P_d(\widetilde{B} \succ \widetilde{A}) = 0.5$.*

3. $\widetilde{A}$ *is greater than or equal to* $\widetilde{B}$*, denoting by* $\widetilde{A} \succeq_{P_d} \widetilde{B}$*, if* $\widetilde{A} \succ_{P_d} \widetilde{B}$ *or* $\widetilde{A} \simeq_{P_d} \widetilde{B}$.

The preference criterion P_d meets the following properties.

Proposition 1.7 *let* $\widetilde{A}$*,* $\widetilde{B}$*, and* $\widetilde{C}$ *be three* ***FN****s. Then:*

1) P_d *is reciprocal, i.e.,* $P_d(\widetilde{A} \succ \widetilde{B}) = 1 - P_d(\widetilde{B} \succ \widetilde{A})$,

2) P_d *is reflexive, i.e.,* $\widetilde{A} \succeq_{P_d} \widetilde{A}$,

3) P_d *is transitive, i.e.,* $\widetilde{A} \succeq_{P_d} \widetilde{B}$ *and* $\widetilde{B} \succeq_{P_d} \widetilde{C}$ *imply* $\widetilde{A} \succeq_{P_d} \widetilde{C}$,

4) $P_d(\widetilde{A} \succeq \widetilde{B}) = 1$ *if and only if* $\widetilde{B}^L_\alpha \leq \widetilde{A}^L_\alpha$ *and* $\widetilde{B}^U_\alpha \leq \widetilde{A}^U_\alpha$ *for all* $\alpha \in [0, 1]$.

Proof *The parts (1), (2), and (4) are immediately followed. To prove (3), note that* $P_d(\widetilde{A} \succeq \widetilde{B}) \geq 0.5$ *if and only if* $\int_0^1 (M_{\widetilde{A}}[\alpha] - M_{\widetilde{B}}[\alpha])d\alpha \geq 0$ *where* $M_{\widetilde{A}}[\alpha] = (\widetilde{A}^L_\alpha + \widetilde{A}^U_\alpha)/2$ *and* $M_{\widetilde{B}}[\alpha] = (\widetilde{B}^L_\alpha + \widetilde{B}^U_\alpha)/2$*. Therefore,* $\widetilde{A} \succeq_{P_d} \widetilde{B}$ *and* $\widetilde{B} \succeq_{P_d} \widetilde{C}$ *concludes that:*

$$\int_0^1 (M_{\widetilde{A}}[\alpha] - M_{\widetilde{C}}[\alpha])d\alpha = \int_0^1 (M_{\widetilde{A}}[\alpha] - M_{\widetilde{B}}[\alpha])d\alpha + \int_0^1 (M_{\widetilde{B}}[\alpha] - M_{\widetilde{C}}[\alpha])d\alpha \geq 0.$$

That is $\widetilde{A} \succeq_{P_d} \widetilde{C}$.

Example 1.12 *Consider two* ***TFN****s* $\widetilde{A} = (a^L, a, a^U)_T$ *and* $\widetilde{B} = (b^L, b, b^U)_T$ *for three specific cases as shown in Fig 1.2. First, assume that* $b^L < a^L < a < b < b^U < a^U$ *((A) in Fig. (1.2)). To compute* $P_d(\widetilde{B} \succeq \widetilde{A})$*,* $\Delta_{\widetilde{A}\widetilde{B}}$ *and* $\Delta_{\widetilde{B}\widetilde{A}}$ *can evaluate as follows:*

$$\begin{aligned}
\Delta_{\widetilde{A}\widetilde{B}} &= \int_0^{\frac{a^L - b^L}{b - a + a^L - b^L}} \big(a - b + ((b - b^L) - (a - a^L))(1 - \alpha)\big)d\alpha \\
&+ \int_0^{\frac{a^U - b^U}{b - a + a^U - b^U}} \big((b - a) + ((b^U - b) - (a^U - a))(1 - \alpha)\big)(1 - \alpha)d\alpha \\
&= \frac{(a^L - b^L)^2}{2(b - a + a^L - b^L)} + \frac{(a^U - b^U)^2}{2(b - a + a^U - b^U)},
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{\widetilde{B}\widetilde{A}} &= \int_{\frac{a^L - b^L}{b - a + a^L - b^L}}^1 \big(b - a + ((a - a^L) - (b - b^L))(1 - \alpha)d\alpha \\
&+ \int_{\frac{a^U - b^U}{b - a + a^U - b^U}}^1 \big((b - a) + ((b^U - b) - (a^U - a))(1 - \alpha)\big)(1 - \alpha)d\alpha \\
&= \frac{(a - b)^2}{2(b - a + a^L - b^L)} + \frac{(a - b)^2}{2(b - a + a^U - b^U)}.
\end{aligned}$$

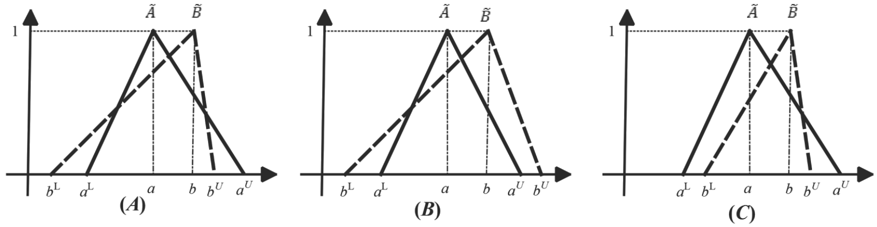

FIGURE 1.2
Membership functions of $\widetilde{A}$ and $\widetilde{B}$ in three cases in Example 1.12.

Therefore:

$$P_d(\widetilde{B} \succeq \widetilde{A}) = \frac{\Delta_{\widetilde{B}\widetilde{A}}}{\Delta_{\widetilde{A}\widetilde{B}} + \Delta_{\widetilde{B}\widetilde{A}}},$$

where

1) $\Delta_{\widetilde{B}\widetilde{A}} = (a-b)^2(2(b-a) + a^L - b^L + a^U - b^U)$.

2) $\Delta_{\widetilde{A}\widetilde{B}} = (b-a+a^U-b^U)(a^L-b^L)^2 + (b-a+a^L-b^L)(a^U-b^U)^2$.

Similarly, the preference degree to which $\widetilde{A}$ *is greater that* $\widetilde{B}$ *for cases* (1) *and* (2) *can be summarized as follows:*

(1) *If* $b^L < a^L < a < b < a^U < b^U$ *((B) in Fig.* (1.2)*), then:*

$$P_d(\widetilde{B} \succeq \widetilde{A}) = \frac{\frac{(b-a+b^L-a^L)(b-a+a^U-b^U)+(a-b)^2}{2(b-a+a^U-b^U)}}{\frac{(b-a+b^L-a^L)(b-a+a^U-b^U)+(a-b)^2}{2(b-a+a^U-b^U)} + \frac{(a^U-b^U)^2}{2(b-a+a^U-b^U)}}.$$

(2) *If* $a^L < b^L < a < b < b^U < a^U$ *((C) in Fig.* (1.2)*), then:*

$$P_d(\widetilde{B} \succeq \widetilde{A}) = \frac{\frac{(b-a+b^U-a^U)(b-a+a^L-b^L)+(a-b)^2}{2(b-a+a^L-b^L)}}{\frac{(b-a+b^U-a^U)(b-a+a^L-b^L)+(a-b)^2}{2(b-a+a^L-b^L)} + \frac{(a^L-b^L)^2}{2(b-a+a^L-b^L)}},$$

For all cases (1), (2), *and* (3), *it should be noted that* $P_d(\widetilde{B} \succeq \widetilde{A}) \in (0.5, 1)$.

Remark 1.2 *Using the preference degree* P_d *defined for each ordered pair of* ***FN****s, it should be noted that we can sort a set of* n ***FN****s* $\widetilde{A}_i$ *by sorting* $\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ *via calculating* $\binom{n}{2}$ *preference degrees into* $\{\widetilde{A}_{k_1}, \widetilde{A}_{k_2}, \ldots, \widetilde{A}_{k_n}\}$ *so that for any* $i < j$, $P_d(\widetilde{A}_j \succeq \widetilde{A}_i) \geq 0.5$. *The feasibility of the sorting is guaranteed by Proposition 1.7. Based on the sorting results, therefore,* $\widetilde{A}_{k_n}$ *is the most preferred choice,* $\widetilde{A}_{k_{(n-1)}}$ *is the second, etc.*

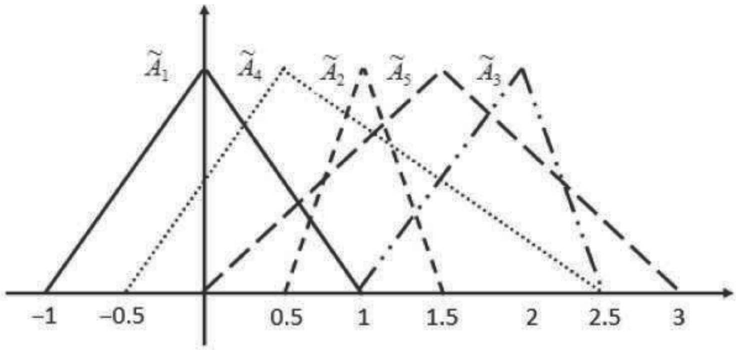

FIGURE 1.3
Membership functions of $\widetilde{A}_i$, $i = 1, 2, ..., 5$ in Example 1.13.

Example 1.13 *Consider five* ***TFN****s of* $\widetilde{A}_1 = (-1, 0, 1)_T$, $\widetilde{A}_2 = (0.5, 1, 1.5)_T$, $\widetilde{A}_3 = (1, 2, 2.5)_T$, $\widetilde{A}_4 = (-0.5, 0.5, 2.5)_T$, *and* $\widetilde{A}_5 = (0, 1.5, 3)_T$. *The plots of* $\widetilde{A}_i$, $i = 1, 2, .., 5$ *are shown in Fig. 1.3. The preference degree* $P_d(\widetilde{A}_j \succeq \widetilde{A}_i)$, $i, j = 1, 2, ..., 5$ *are listed in Table 1.1. Accordingly, we can get a sorted result as* $\widetilde{A}_5 \succeq_{P_d} \widetilde{A}_3 \succeq_{P_d} \widetilde{A}_4 \succeq_{P_d} \widetilde{A}_2 \succeq_{P_d} \widetilde{A}_1$.

TABLE 1.1
Preference degrees $P_d(\widetilde{A}_i \succeq \widetilde{A}_j)$ in Example 1.13.

j	1	2	3	4	5
$P_d(\widetilde{A}_1 \succeq \widetilde{A}_j)$	1	0	0	0	0
$P_d(\widetilde{A}_2 \succeq \widetilde{A}_j)$	1	1	0	$\frac{3}{8}$	$\frac{1}{2}$
$P_d(\widetilde{A}_3 \succeq \widetilde{A}_j)$	1	1	1	1	$\frac{3}{8}$
$P_d(\widetilde{A}_4 \succeq \widetilde{A}_j)$	1	$\frac{5}{8}$	0	1	0
$P_d(\widetilde{A}_5 \succeq \widetilde{A}_j)$	1	$\frac{1}{2}$	$\frac{5}{8}$	1	1

Remark 1.3 *Let* $\widetilde{A} = (a^L, a, a^U)_T$ *and* $\widetilde{B} = (b^L, b, b^U)_T$ *be two* ***TFN****s. According to Proposition 1.7, we have* $P_d(\widetilde{A} \succeq \widetilde{B}) = 1$ *if and only if* $a^L \leq b^L$, $a \leq b$ *and* $a^U \leq b^U$.

Further, a different type of ranking for **TFN**s is introduced based on an exact criterion. We employed this criterion to construct a fuzzy hypothesis in Chapter 5.

Definition 1.18 *Let* $\widetilde{A} = (a^L, a, a^U)_T$ *and* $\widetilde{B} = (b^L, b, b^U)_T$ *be two* ***TFN****s. Then, we say that:*

1. $\widetilde{A} \simeq_M \widetilde{B}$, *if* $M_{\widetilde{A}} = M_{\widetilde{B}}$.

2. $\widetilde{A} \succ_M \widetilde{B}$, *if* $M_{\widetilde{A}} > M_{\widetilde{B}}$.

3. $\widetilde{A} \succeq_M \widetilde{B}$, *if* $M_{\widetilde{A}} > M_{\widetilde{B}}$ *or* $M_{\widetilde{A}} = M_{\widetilde{B}}$.

where $M_{\widetilde{A}} = (a^L + a + a^U)/3$ *and* $M_{\widetilde{B}} = (b^L + b + b^U)/3$.

Lemma 1.8 *Let* $\widetilde{A}$, $\widetilde{B}$, *and* $\widetilde{C}$ *be three* ***TFN****s. Then:*

1) $\succeq_M$ *is reflexive, i.e.,* $\widetilde{A} \succeq_M \widetilde{A}$.

2) $\succeq_M$ *is transitive, i.e.,* $\widetilde{A} \succeq_M \widetilde{B}$ *and* $\widetilde{B} \succeq_M \widetilde{C}$ *imply* $\widetilde{A} \succeq_M \widetilde{C}$.

3) $\succeq_M$ *is antisymmetric, i.e.,* $\widetilde{A} \succeq_M \widetilde{B}$ *and* $\widetilde{B} \succeq_M \widetilde{A}$ *imply* $\widetilde{A} \simeq_M \widetilde{B}$.

Proof *The proof is left as an exercise.*

1.2.3 Distance measure

In several fields, the necessity to determine the distance that separates two points arises. When there is imprecision on the location of these points, the calculation of the distance has to take this uncertainty into consideration. The methods for measuring the distance between fuzzy sets have become important due to their significant applications in various areas [17, 49, 50, 79, 136, 192, 199]. Particularly, many statistical inferences such as hypothesis test, decision making, regression analysis, and non-parametric inferences rely on distance measures. Therefore, when we speak statistical inferences based on fuzzy data, there is a need to extend the classic distance measures where the observed data are reported by fuzzy quantities.

Definition 1.19 *It is said that d is a distance measure between two fuzzy sets if it meets the following conditions for any fuzzy sets* $\widetilde{A}, \widetilde{B}$, *and* $\widetilde{C}$*:*

1) $d(\widetilde{A}, \widetilde{B}) = 0$ *if and only if* $\widetilde{A} = \widetilde{B}$,

2) $d(\widetilde{A}, \widetilde{B}) = d(\widetilde{B}, \widetilde{A})$,

3) $d(\widetilde{A}, \widetilde{C}) \leq d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C})$.

Here, a non-fuzzy distance measures between two **TFN**s are introduced and discussed.

Lemma 1.9 *Consider two* ***TFN****s* $\widetilde{A} = (a^L, a, a^U)_T$ *and* $\widetilde{B} = (b^L, b, b^U)_T$. *Let:*

1) $d_1(\widetilde{A}, \widetilde{B}) = \frac{|a^L - b^L| + |a - b| + |a^U - b^U|}{3}$,

2) $d_2(\widetilde{A}, \widetilde{B}) = \max\{|a^L - b^L|, |a - b|, |a^U - b^U|\}$,

3) $d_3(\widetilde{A},\widetilde{B}) = \sqrt{|a^L - b^L|^2 + |a - b|^2 + |a^U - b^U|^2}$.

Then, d_1, d_2, and d_3 are distance measures.

Proof *The proof is left for the reader.*

Example 1.14 *Consider two **TFN**s $\widetilde{A} = (-3,-1,3)_T$ and $\widetilde{B} = (-2,1,4)_T$. Then:*

1) $d_1(\widetilde{A},\widetilde{B}) = \frac{|a^L-b^L|+|a-b|+|a^U-b^U|}{3} = 4/3,$

2) $d_2(\widetilde{A},\widetilde{B}) = \sqrt{\frac{|a^L-b^L|^2+|a-b|^2+|a^U-b^U|^2}{3}} = \sqrt{2},$

3) $d_3(\widetilde{A},\widetilde{B}) = \max\{|a^L - b^L|, |a-b|, |a^U - b^U|\} = 2.$

The following definition extends the distance between two **FN**s of $\widetilde{A}$ and $\widetilde{B}$ as a **FN**.

Definition 1.20 *[199] For three fuzzy sets $\widetilde{A}, \widetilde{B}$, and $\widetilde{C}$, it is said that $\widetilde{D}$ is a fuzzy distance measure (**FDM**) if it meets the following conditions:*

1) $\widetilde{D}(\widetilde{A},\widetilde{B}) = I\{0\}$ *if and only if* $\widetilde{A} = \widetilde{B}$.

2) $\widetilde{D}(\widetilde{A},\widetilde{B}) = \widetilde{D}(\widetilde{B},\widetilde{A})$.

3) $\widetilde{D}(\widetilde{A},\widetilde{C}) \preceq_{P_d} \widetilde{D}(\widetilde{A},\widetilde{B}) \oplus \widetilde{D}(\widetilde{B},\widetilde{C})$.

Theorem 1.7 *For two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ define:*

$$\widetilde{D}(\widetilde{A},\widetilde{B})(d) = \sup_{x,y:|x-y|=d} \min\{\widetilde{A}(x), \widetilde{B}(y)\}. \tag{1.25}$$

*Then, $\widetilde{D}$ is a **FDM**.*

Proof *To prove, see [199].*

Example 1.15 *Let $\mathbb{X} = \{-1,0,2,3,4\}$, $\widetilde{A} = \{\frac{0.3}{-1}, \frac{0.5}{0}, \frac{0.7}{2}, \frac{1}{3}, \frac{0.8}{4}\}$, and $\widetilde{B} = \{\frac{0.8}{-1}, \frac{1}{0}, \frac{0.9}{2}, \frac{0.6}{3}, \frac{0.4}{4}\}$. Then, the membership functions of $\widetilde{D}(\widetilde{A},\widetilde{B})$ can be evaluated as follows:*

$$\begin{aligned}
\widetilde{D}(\widetilde{A},\widetilde{B})(0) &= \sup_{x,y:|x-y|=0} \min\{\widetilde{A}(x), \widetilde{B}(y)\} \\
&= \max\Big\{\min\{\widetilde{A}(-1), \widetilde{B}(-1)\}, \min\{\widetilde{A}(0), \widetilde{B}(0)\} \\
&, \min\{\widetilde{A}(2), \widetilde{B}(2)\}, \min\{\widetilde{A}(3), \widetilde{B}(3)\}, \min\{\widetilde{A}(4), \widetilde{B}(4)\}\Big\} \\
&= \max\Big\{\min\{0.3, 0.8\}, \min\{0.5, 1\}, \min\{0.7, 0.9\}, \min\{1, 0.6\}, \\
&\min\{0.4, 0.8\}\Big\} = 0.7.
\end{aligned}$$

$$\widetilde{D}(\widetilde{A},\widetilde{B})(1) = \sup_{x,y:|x-y|=1} \min\{\widetilde{A}(x),\widetilde{B}(y)\}$$
$$= \max\Big\{\min\{\widetilde{A}(-1),\widetilde{B}(0)\},\min\{\widetilde{A}(0),\widetilde{B}(-1)\},\min\{\widetilde{A}(2),$$
$$\widetilde{B}(3)\},\min\{\widetilde{A}(3),\widetilde{B}(2)\},\min\{\widetilde{A}(4),\widetilde{B}(3)\},\min\{\widetilde{A}(3),\widetilde{B}(4)\}\Big\}$$
$$= \max\Big\{\min\{0.3,1\},\min\{0.8,0.3\},\min\{0.7,0.6\},\min\{0.6,0.7\},$$
$$\min\{1,0.9\},\min\{0.8,0.6\},\min\{0.4,1\}\Big\} = 0.9.$$

Similarly, it can be shown that $\widetilde{D}(\widetilde{A},\widetilde{B})(3) = 1$, $\widetilde{D}(\widetilde{A},\widetilde{B})(2) = \widetilde{D}(\widetilde{A},\widetilde{B})(4) = \widetilde{D}(\widetilde{A},\widetilde{B})(5) = 0.8$ *and thus:*

$$\widetilde{D}(\widetilde{A},\widetilde{B}) = \{\frac{0.7}{0},\frac{0.9}{1},\frac{0.8}{2},\frac{1}{3},\frac{0.8}{\{4,5\}}\}.$$

Here, the proposed fuzzy distance measures between two **TFN**s are developed in the fuzzy domain.

Proposition 1.8 *Consider two* ***TFNs*** $\widetilde{A} = (a^L, a, a^U)_T$ *and* $\widetilde{B} = (b^L, b, b^U)_T$. *Let:*

$$\widetilde{d}_a(\widetilde{A},\widetilde{B}) = (d_1(\widetilde{A},\widetilde{B}), d_2(\widetilde{A},\widetilde{B}), d_3(\widetilde{A},\widetilde{B}))_T, \tag{1.26}$$

where d_1, d_2, *and* d_3 *are as in Lemma 1.9. Then,* $\widetilde{d}_a$ *is* ***FDM***.

Proof *First note that* d_1, d_2, *and* d_3 *are (exact) distance measures and* $d_1(\widetilde{A},\widetilde{B}) \leq d_2(\widetilde{A},\widetilde{B}) \leq d_3(\widetilde{A},\widetilde{B})$ *for any* $\widetilde{A},\widetilde{B} \in \mathcal{F}(\mathbb{R})$. *Therefore, it can be checked that*

1) $\widetilde{d}_a(\widetilde{A},\widetilde{B}) = I\{0\}$ *if and only if* $\widetilde{A} = \widetilde{B}$,

2) $\widetilde{d}_a(\widetilde{A},\widetilde{B}) = \widetilde{d}_a(\widetilde{B},\widetilde{A})$,

and

$$d_1^L(\widetilde{A},\widetilde{C}) \leq d_1^L(\widetilde{A},\widetilde{B}) + d_1^L(\widetilde{B},\widetilde{C}),$$
$$d_2(\widetilde{A},\widetilde{C}) \leq d_2(\widetilde{A},\widetilde{B}) + d_2(\widetilde{B},\widetilde{C}),$$
$$d_3^U(\widetilde{A},\widetilde{C}) \leq d_3^U(\widetilde{A},\widetilde{B}) + d_3^U(\widetilde{B},\widetilde{C}).$$

From Remark 1.3, these conclude that $P_d(\widetilde{d}_a(\widetilde{A},\widetilde{C}) \preceq (\widetilde{d}_a(\widetilde{A},\widetilde{B}) \oplus \widetilde{d}_a(\widetilde{B},\widetilde{C}))) = 1$ *or* $\widetilde{d}_a(\widetilde{A},\widetilde{C}) \preceq_{P_d} \widetilde{d}_a(\widetilde{A},\widetilde{B}) \oplus \widetilde{d}_a(\widetilde{B},\widetilde{C})$. *Therefore,* $\widetilde{d}_a$ *is* ***FDM***.

Example 1.16 *Recall Example 1.14. Therefore, the fuzzy distance between* $\widetilde{A}$ *and* $\widetilde{B}$ *can be evaluated via a* ***TFN*** $\widetilde{d}_a(\widetilde{A},\widetilde{B}) = (4/3, \sqrt{2}, 2)_T$.

1.3 α-values of FNs

Here, the notion of α-values of **FN**s is recalled. We will utilize such a notion to rewrite a common notion of the fuzzy random variable in the next section [78].

Definition 1.21 *The α-values of a **FN** $\widetilde{A}$ is a mapping $\widetilde{A}_\alpha : [0,1] \to \mathbb{R}$ defined by:*

$$\widetilde{A}_\alpha = \begin{cases} \widetilde{A}^L_{2\alpha}, & \alpha \in [0, 0.5], \\ \widetilde{A}^U_{2(1-\alpha)}, & \alpha \in [0.5, 1]. \end{cases} \tag{1.27}$$

Lemma 1.10 *The α-values a **FN** $\widetilde{A}$ is an increasing function of α on $[0,1]$.*

Proof *Since $\widetilde{A}^L_\alpha$ and $\widetilde{A}^U_\alpha$ are increasing and decreasing functions, for any $\alpha_1 < \alpha_2$ in $[0,1]$, we have $\widetilde{A}^L_{2\alpha_1} \leq \widetilde{A}^L_{2\alpha_2}$ for $\alpha \in [0, 0.5]$ and $\widetilde{A}^U_{2(1-\alpha_1)} \leq \widetilde{A}^L_{2(1-\alpha_2)}$ for $\alpha \in [0.5, 1]$. These verify the proof.*

Example 1.17 *Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ be a LR-**FN**. From Definition 1.21, one finds that:*

$$\widetilde{A}_\alpha = \begin{cases} a - l_a L^{-1}(2\alpha), & 0 \leq \alpha \leq 0.5, \\ a + r_a R^{-1}(2(1-\alpha)), & 0.5 \leq \alpha \leq 1. \end{cases}$$

For instance,

1. *If $\widetilde{A} = (a; l_a, r_a)_T$ is a **TFN**, then:*

$$\widetilde{A}_\alpha = \begin{cases} (a - l_a) + 2l_a\alpha, & 0 \leq \alpha \leq 0.5, \\ a + r_a - 2r_a(1-\alpha), & 0.5 \leq \alpha \leq 1. \end{cases}$$

2. *Let $\widetilde{A} = (a; l_a, r_a)_{LR}$ with $L(x) = \sqrt{1 - x^3}$ and $R(x) = 1 - x^5$ then:*

$$\widetilde{A}_\alpha = \begin{cases} a - l_a \sqrt[3]{1 - 4\alpha^2}, & 0 \leq \alpha \leq 0.5, \\ a + r_a \sqrt[5]{2\alpha - 1}, & 0.5 \leq \alpha \leq 1. \end{cases}$$

The relationship between α-values and α-cuts of a **FN** is given below.

Lemma 1.11 *Let $\widetilde{A}[\alpha]$ and $\widetilde{A}_\alpha$ be the α-cuts and α-values of a **FN** $\widetilde{A}$, respectively. Then:*

$$\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha] = [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]. \tag{1.28}$$

Proof *The proof is a simple consequence of Eq.* (1.27) *since* $\widetilde{A}_{\alpha/2} = \widetilde{A}^L_\alpha$ *and* $\widetilde{A}_{1-\alpha/2} = \widetilde{A}^U_\alpha$ *for any* $\alpha \in [0,1]$.

Therefore, the membership degree of $\widetilde{A}$ at $x \in \mathbb{R}$ can be written as:

$$\widetilde{A}(x) = \sup\{\alpha \in [0,1] : x \in [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]\}.$$

The following Lemma shows that the interval-valued arithmetic operations of **FN**s can be simplified using their α-values.

Lemma 1.12 *Let* $\lambda \in \mathbb{R}$ *and* $\widetilde{A}$ *and* $\widetilde{B}$ *be two* ***FN****s. Then, for any* $\alpha \in [0,1]$*:*

1. $(\widetilde{A} \oplus \widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha$.

2. $(\lambda \otimes \widetilde{A})_\alpha = \begin{cases} \lambda \widetilde{A}_\alpha, & \lambda \geq 0, \\ -\lambda \widetilde{A}_{1-\alpha}, & \lambda < 0. \end{cases}$

3. $(\widetilde{A} \otimes \widetilde{B})_\alpha = \begin{cases} \widetilde{A}_\alpha \widetilde{B}_\alpha, & if \quad \widetilde{A}, \widetilde{B} > 0, \\ \widetilde{A}_{1-\alpha} \widetilde{B}_{1-\alpha}, & if \quad \widetilde{A}, \widetilde{B} < 0, \\ \widetilde{A}_{1-\alpha} \widetilde{B}_\alpha, & if \quad \widetilde{A} > 0, \widetilde{B} < 0. \end{cases}$

Proof *From Definition 1.21 and the arithmetic operation of* ***FN****s, it is easy to check that:*

$$\begin{aligned} (\widetilde{A} \oplus \widetilde{B})_\alpha &= \begin{cases} (\widetilde{A} \oplus \widetilde{B})^L_{2\alpha}, & 0 \leq \alpha \leq 0.5, \\ (\widetilde{A} \oplus \widetilde{B})^U_{2(1-\alpha)}, & 0.5 < \alpha \leq 1. \end{cases} \\ &= \begin{cases} \widetilde{A}^L_{2\alpha} + \widetilde{B}^L_{2\alpha}, & 0 \leq \alpha \leq 0.5, \\ \widetilde{A}^U_{2(1-\alpha)} + \widetilde{B}^U_{2(1-\alpha)}, & 0.5 < \alpha \leq 1. \end{cases} \\ &= \widetilde{A}_\alpha + \widetilde{B}_\alpha, \end{aligned} \tag{1.29}$$

and

$$\begin{aligned} (\lambda \otimes \widetilde{A})_\alpha &= \begin{cases} (\lambda \otimes \widetilde{A})^L_{2\alpha}, & 0 \leq \alpha \leq 0.5, \\ (\lambda \otimes \widetilde{A})^U_{2(1-\alpha)}, & 0.5 < \alpha \leq 1. \end{cases} \\ &= \lambda \widetilde{A}_\alpha I(\lambda \geq 0) - \lambda \widetilde{A}_{1-\alpha} I(\lambda < 0). \end{aligned} \tag{1.30}$$

These verify (1) and (2). To prove (3), first assume that $\widetilde{A}$ *and* $\widetilde{B}$ *are two positive* ***FN****s. Since* $(\widetilde{A}^2)_\alpha = (\widetilde{A}_\alpha)^2$ *we get:*

$$\begin{aligned} ((\widetilde{A} \oplus \widetilde{B})^2)_\alpha &= ((\widetilde{A} \oplus \widetilde{B}) \otimes (\widetilde{A} \oplus \widetilde{B}))_\alpha \\ &= (\widetilde{A}_\alpha + \widetilde{B}_\alpha)^2 \\ &= \widetilde{A}^2_\alpha + \widetilde{B}^2_\alpha + 2\widetilde{A}_\alpha \widetilde{B}_\alpha. \end{aligned}$$

By (1) and (2), therefore:

$$\begin{aligned}((\widetilde{A} \oplus \widetilde{B})^2)_\alpha &= (\widetilde{A}^2 \oplus \widetilde{B}^2 \oplus 2\widetilde{A} \otimes \widetilde{B})_\alpha \\ &= \widetilde{A}_\alpha^2 + \widetilde{B}_\alpha^2 + 2(\widetilde{A} \otimes \widetilde{B})_\alpha.\end{aligned}$$

This shows that $(\widetilde{A} \otimes \widetilde{B})_\alpha = \widetilde{A}_\alpha \widetilde{B}_\alpha$. *Now, if* $\widetilde{A}$ *and* $\widetilde{B}$ *are two negative* ***FN*s***, then we have* $\widetilde{A} = (-1) \otimes \widetilde{A}'$ *and* $\widetilde{B} = (-1) \otimes \widetilde{B}'$ *for two positive* ***FN*s** *of* $\widetilde{A}'$ *and* $\widetilde{B}'$. *So, it concludes that:*

$$\begin{aligned}(\widetilde{A} \otimes \widetilde{B})_\alpha &= \Big((-1) \otimes \widetilde{A}' \otimes (-1) \otimes \widetilde{B}'\Big)_\alpha \\ &= (\widetilde{A}' \otimes \widetilde{B}')_\alpha = \widetilde{A}'_\alpha \widetilde{B}'_\alpha = \widetilde{A}_{1-\alpha} \widetilde{B}_{1-\alpha}.\end{aligned}$$

This provides (3). These results can also be verified in cases where $\widetilde{A} > 0$ *and* $\widetilde{B} < 0$.

According to Lemma 1.12, it should be noted that the arithmetic operations of **FN**s can be rewritten according to their α-values.

Lemma 1.13 *Consider a mapping* $\widetilde{A}_\alpha : [0,1] \to \mathbb{R}$ *such that: 1)* $\widetilde{A}_\alpha$ *strictly increasing with respect to* α *for any* $\alpha \in [0,1]$ *and 2)* $\widetilde{A}_{0.5}$ *is a constant number. Then,* $\{\widetilde{A}_\alpha\}_{\alpha \in [0,1]}$ *provides a* ***FN*** $\widetilde{A}$ *with the following* α*-cut:*

$$\widetilde{A}[\alpha] = [\widetilde{A}_{\alpha/2}, \widetilde{A}_{1-\alpha/2}]. \tag{1.31}$$

Proof *Let* $\widetilde{A}_\alpha : [0,1] \to \mathbb{R}$ *be a mapping such that: 1)* $\widetilde{A}_\alpha$ *strictly increasing with respect to* α *for any* $\alpha \in [0,1]$ *and 2)* $\widetilde{A}_{0.5}$ *is a constant number. It is easy to verify that* $[\widetilde{A}_{1-\alpha_2/2},\, \widetilde{A}_{\alpha_2/2}] \subseteq [\widetilde{A}_{1-\alpha_1/2},\, \widetilde{A}_{\alpha_1/2}]$ *for any* $\alpha_1 < \alpha_2$. *Therefore,* $\widetilde{A}[\alpha] = [\widetilde{A}_{1-\alpha/2},\, \widetilde{A}_{\alpha/2}]$ *construct a sequence of* α*-cuts of a* ***FN***.

This is the reason why we focused on α-values since a **FN** of $\widetilde{A}$ can be traceable via its α-values ($\{\widetilde{A}_\alpha\}$) instead of its α-cuts ($\{\widetilde{A}[\alpha]\}$).

Lemma 1.14 *Let* $\widetilde{A} \in \mathcal{F}(\mathbb{R})$ *and* g *be a continuous and strictly monotone real-valued function. Then, for every* $\alpha \in [0,1]$

$$(g(\widetilde{A}))_\alpha = \begin{cases} g(\widetilde{A}_\alpha), & g \text{ is increasing}, \\ g(\widetilde{A}_{1-\alpha}), & g \text{ is decreasing}. \end{cases} \tag{1.32}$$

Proof *Without loss of generality, assume that* g *is a strictly increasing function. Regarding the extension principle and Definition 1.21, first note that* $g(\widetilde{A})(y) = \widetilde{A}(g^{-1}(y))$ *and therefore:*

$$\begin{aligned}g(\widetilde{A})[\alpha] &= g(\widetilde{A}[\alpha]) \\ &= [g(\widetilde{A}_\alpha^L), g(\widetilde{A}_\alpha^U)] \\ &= [g(\widetilde{A}_{\alpha/2}), g(\widetilde{A}_{1-\alpha/2})].\end{aligned}$$

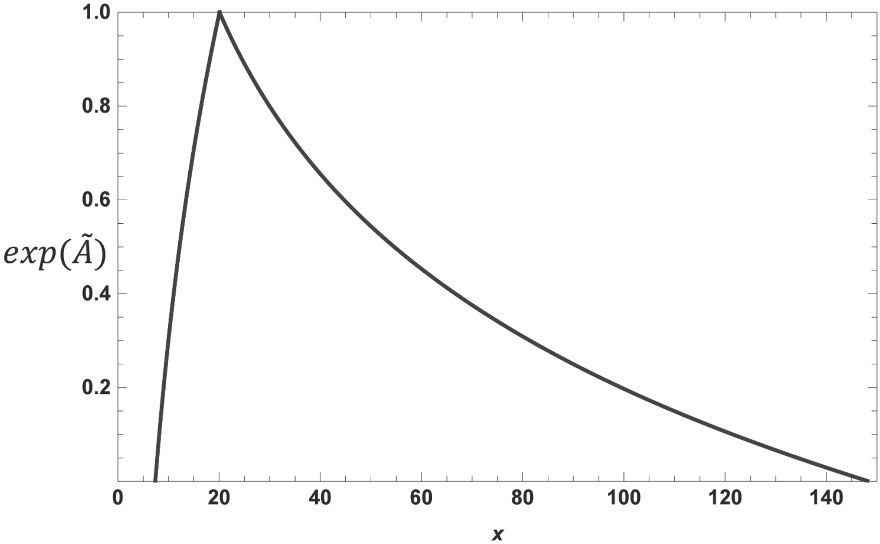

FIGURE 1.4
Membership function of $\exp(\widetilde{A})$ in Example 1.18.

Now, by the relationship between α-values and α-cuts given in Definition 1.21, it can be seen that $(g(\widetilde{A}))_\alpha = g(\widetilde{A}_\alpha)$ for every $\alpha \in [0,1]$.

Remark 1.4 *By the relationship between α-values and α-cuts of **FN**s, note that the preference degree that '$\widetilde{A}$ is greater than $\widetilde{B}$' can be rewritten as follows:*

$$P_d(\widetilde{A} \succ \widetilde{B}) = \begin{cases} 0.5, & \widetilde{A} = \widetilde{B}, \\ \frac{\Delta_{\widetilde{A}\widetilde{B}}}{\Delta_{\widetilde{A}\widetilde{B}} + \Delta_{\widetilde{B}\widetilde{A}}}, & \widetilde{A} \neq \widetilde{B}. \end{cases}$$

where

$$\Delta_{\widetilde{A}\widetilde{B}} = \int_{\{\alpha : \widetilde{A}_\alpha \geq \widetilde{B}_\alpha\}} (\widetilde{A}_\alpha - \widetilde{B}_\alpha) d\alpha. \tag{1.33}$$

Therefore $P_d(\widetilde{A} \succeq \widetilde{B}) = 1$ if and only if $\widetilde{A}_\alpha \geq \widetilde{B}_\alpha$ for any $\alpha \in [0,1]$.

From Lemma 1.14, we can observe that the conventional function of a **FN** (based on the extension principle) can also be rewritten based on the α-values of the **FN**s.

Example 1.18 *Let $g(x) = \exp(x)$ and $\widetilde{A} = (a; l_a, r_a)_T$ be a **TFN**. Therefore, the α-cuts of $g(\widetilde{A})$ can be evaluated as:*

$$(g(\widetilde{A}))_\alpha = \begin{cases} \exp((a - l_a) + 2l_a\alpha), & 0 \leq \alpha \leq 0.5, \\ \exp(a + r_a - 2r_a(1-\alpha)), & 0.5 < \alpha \leq 1. \end{cases}$$

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