假设对于概念A已知$P(A|[x])$，现有两种行动a_1和a_2，对应的损失为

$$\begin{cases} R(a_1|[x]) = \lambda_{11} P(A|[x]) + \lambda_{12} P(A^C|[x]) \\ R(a_2|[x]) = \lambda_{21} P(A|[x]) + \lambda_{22} P(A^C|[x]) \end{cases}$$

λ_{11}的物理含义为当样本为A类时采取行动a_1时的损失，λ_{12}为样本不为A类时采取行动a_1的损失。

注意：隐含假设只能采取一个行动，最完美的情形下，

$A \to a_1$；$A^C \to a_2$ || $A \to a_2$；$A^C \to a_1$

最优行动准则：

① $\min R(a_i|[x]) = R(a_1|[x])$ 采取a_1行动

$$(\lambda_{11} - \lambda_{12}) P(A|[x]) + \lambda_{12} \leq (\lambda_{21} - \lambda_{22}) P(A|[x]) + \lambda_{22}$$

$$[(\lambda_{11} - \lambda_{21}) + (\lambda_{22} - \lambda_{12})] P(A|[x]) \leq \lambda_{22} - \lambda_{12}$$

分情况讨论：$\Rightarrow \lambda_{11} < \lambda_{12}$ | $\lambda_{22} < \lambda_{21}$ | $\Rightarrow$分类正确的损失小于分类错误的

I. $\lambda_{11} < \lambda_{21}$，这样的情况下，我们倾向于说$a_1$就是为[x]属于A类时所应采取的行动。同时应有$\lambda_{22} < \lambda_{12}$，这意味着$a_2$为x属于$A^C$时采取的行动。这样的假设下，即有

$$[(\lambda_{21} - \lambda_{11}) + (\lambda_{12} - \lambda_{22})] P(A|[x]) \geq \lambda_{12} - \lambda_{22}$$

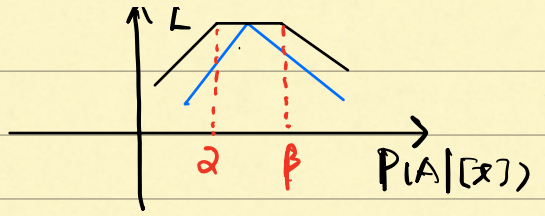
$$1 \geq P(A|[x]) \geq \frac{\lambda_{12} - \lambda_{22}}{(\lambda_{21} - \lambda_{11}) + (\lambda_{12} - \lambda_{22})} > 0$$

$\lambda_{11} < \lambda_{12}$

$\lambda_{21} > \lambda_{22}$

问题：在实际的观测A中，时常出现A无法从物理意义上指明或从数据统计上推断与行动a_1、a_2的关系。从观测中发现$P(A|[x]) \in [a,b] \subseteq [0,1]$时无法判定采取何种行动最优。

设 $L(P(A|[x])) = \min(R(a_1|[x]), R_2(a_2|[x]))$

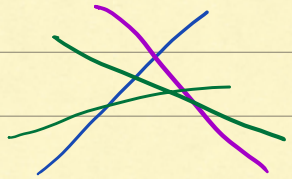


模糊

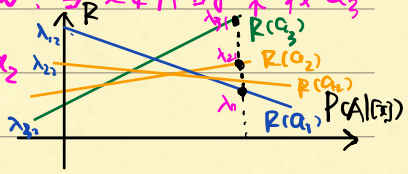
概率粗粒集：在许多情况下，知晓当前状况并不明确采取当下的手段应付能够大幅降低损失。例如患病的诊断（进一步的检查损失小于下判断患病与正常的损失）

设存在动作a_1, a_2, a_3与概念A. 已知$P(A|[x])$相应的风险为

$$R(a_i|x) = \lambda_{i1} P(A|[x]) + \lambda_{i2} P(A^c|[x])$$
$$= (\lambda_{i1} - \lambda_{i2}) P(A|[x]) + \lambda_{i2}$$
$$= (\lambda_{i2} - \lambda_{i1}) P(A^c|[x]) + \lambda_{i1}$$



最优行动准则：[预设：当$x \in A$时采取a_1行动，当$x \notin A$时采取a_3行动，当无法确定x与A之间的关系时采取a_2



比较明确为是$(\lambda_{11} - \lambda_{12}) \le (\lambda_{21} - \lambda_{22}) \le (\lambda_{31} - \lambda_{32})$

$\lambda_{32} \le \lambda_{22} \le \lambda_{12}$； $\lambda_{31} \ge \lambda_{21} \ge \lambda_{11}$

① 采取a_1行动，当$R(a_1|x) \le R(a_2|x) \wedge R(a_1|x) \le R(a_3|x)$

也即有$[(\lambda_{11} - \lambda_{12}) - (\lambda_{21} - \lambda_{22})] P(A|[x]) \le \lambda_{22} - \lambda_{12}$

$[(\lambda_{11} - \lambda_{21}) + (\lambda_{22} - \lambda_{12})] P(A|[x]) \le \lambda_{22} - \lambda_{12}$

$[(\lambda_{21} - \lambda_{11}) + (\lambda_{12} - \lambda_{22})] P(A|[x]) \le \lambda_{12} - \lambda_{22}$

$$P(A|[x]) \ge \frac{\lambda_{12} - \lambda_{22}}{[(\lambda_{21} - \lambda_{11}) + (\lambda_{12} - \lambda_{22})]} = \alpha$$

$[(\lambda_{11} - \lambda_{31}) + (\lambda_{32} - \lambda_{12})] P(A|[x]) \le \lambda_{32} - \lambda_{12}$

$[(\lambda_{31} - \lambda_{11}) + (\lambda_{12} - \lambda_{32})] P(A|[x]) \le \lambda_{12} - \lambda_{32}$

$$P(A|[x]) \ge \frac{\lambda_{12} - \lambda_{32}}{[(\lambda_{31} - \lambda_{11}) + (\lambda_{12} - \lambda_{32})]} = \gamma$$

② 采取a_2行动，当$R(a_2|x) \le R(a_1|x)$； $R(a_2|x) \le R(a_3|x)$

$[(\lambda_{21} - \lambda_{11}) + (\lambda_{12} - \lambda_{22})] P(A|x) \le \lambda_{12} - \lambda_{22}$

$$P(A|[x]) \le \frac{\lambda_{12} - \lambda_{22}}{[(\lambda_{21} - \lambda_{11}) + (\lambda_{12} - \lambda_{22})]} = \alpha$$

$[(\lambda_{21} - \lambda_{31}) + (\lambda_{32} - \lambda_{22})] P(A|x) \le \lambda_{32} - \lambda_{22}$

$[(\lambda_{31} - \lambda_{21}) + (\lambda_{22} - \lambda_{32})] P(A|x) \ge \lambda_{22} - \lambda_{32}$

$$P(A|[x]) \geq \frac{\lambda_{22}-\lambda_{32}}{[(\lambda_{31}-\lambda_{21})+(\lambda_{22}-\lambda_{32})]} = \beta$$

② 采取 a_3 行动，当 $R(a_3|x) \leq R(a_1|x)$；$R(a_3|x) \leq R(a_2|x)$

$$[(\lambda_{31}-\lambda_{11})+(\lambda_{12}-\lambda_{32})]P(A|x) \leq \lambda_{12}-\lambda_{32}$$

$$P(A|[x]) \leq \frac{\lambda_{12}-\lambda_{32}}{[(\lambda_{31}-\lambda_{11})+(\lambda_{12}-\lambda_{32})]} = \beta$$

$$[(\lambda_{31}-\lambda_{21})+(\lambda_{22}-\lambda_{32})]P(A|x) \leq \lambda_{22}-\lambda_{32}$$

$$P(A|[x]) \leq \frac{\lambda_{22}-\lambda_{32}}{[(\lambda_{31}-\lambda_{21})+(\lambda_{32}-\lambda_{12})]} = \gamma$$

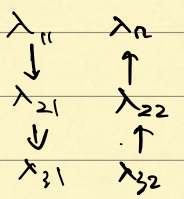
已知 $(\lambda_{11}-\lambda_{12}) \leq (\lambda_{21}-\lambda_{22}) \leq (\lambda_{31}-\lambda_{32})$

对于 $\alpha = \frac{\lambda_{12}-\lambda_{22}}{[(\lambda_{21}-\lambda_{11})+(\lambda_{12}-\lambda_{22})]}$；$\gamma = \frac{\lambda_{12}-\lambda_{32}}{[(\lambda_{31}-\lambda_{11})+(\lambda_{12}-\lambda_{32})]}$

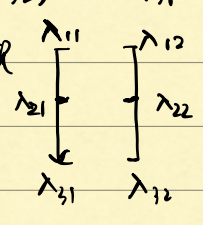
$$\beta = \frac{\lambda_{22}-\lambda_{32}}{[(\lambda_{31}-\lambda_{21})+(\lambda_{22}-\lambda_{32})]}$$

$$\beta-\alpha = (\lambda_{22}-\lambda_{32})[(\lambda_{21}-\lambda_{11}+(\lambda_{12}-\lambda_{22})]$$
$$+ (\lambda_{22}-\lambda_{12})[(\lambda_{31}-\lambda_{21})+(\lambda_{22}-\lambda_{32})]$$
$$= \lambda_{22}[(\lambda_{31}-\lambda_{11})+(\lambda_{12}-\lambda_{32})]$$
$$+\lambda_{32}[(\lambda_{11}-\lambda_{21})+(\lambda_{22}-\lambda_{12})]$$
$$+\lambda_{12}[(\lambda_{21}-\lambda_{31})+(\lambda_{32}-\lambda_{22})]$$
$$= \lambda_{22}(\lambda_{31}-\lambda_{11})+\lambda_{22}(\lambda_{12}-\lambda_{32})-\lambda_{32}(\lambda_{21}-\lambda_{11})-\lambda_{32}(\lambda_{12}-\lambda_{22})$$
$$-\lambda_{12}(\lambda_{31}-\lambda_{21})-\lambda_{12}(\lambda_{22}-\lambda_{32})$$
$$= \lambda_{22}\lambda_{31}-\lambda_{22}\lambda_{11}+\lambda_{22}\lambda_{12}+\lambda_{12}\lambda_{32}-\lambda_{32}\lambda_{21}+\lambda_{32}\lambda_{11}-\lambda_{32}\lambda_{12}+\lambda_{32}\lambda_{22}$$
$$-\lambda_{12}\lambda_{31}+\lambda_{12}\lambda_{21}-\lambda_{12}\lambda_{22}+\lambda_{12}\lambda_{32}$$
$$= \lambda_{11}(\lambda_{32}-\lambda_{22})+(\lambda_{12}-\lambda_{32})\lambda_{21}+\lambda_{31}(\lambda_{22}-\lambda_{12})$$
$$= \lambda_{11}(\lambda_{32}-\lambda_{22})+(\lambda_{12}-\lambda_{22}+\lambda_{22}-\lambda_{32})\lambda_{21}+\lambda_{31}(\lambda_{22}-\lambda_{12})$$
$$= (\lambda_{21}-\lambda_{11})(\lambda_{22}-\lambda_{32})+(\lambda_{21}-\lambda_{31})(\lambda_{12}-\lambda_{22})$$
$$= (\lambda_{21}-\lambda_{11})(\lambda_{22}-\lambda_{32})-(\lambda_{31}-\lambda_{21})(\lambda_{12}-\lambda_{22})$$

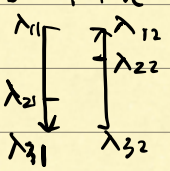
用几何画板来理解简单



相等情况



大于0情况



$\lambda_{11}+\lambda_{22}$

$$\lambda_{21} > \frac{\lambda_{11} + \lambda_{31}}{2}$$

$$\lambda_{22} > \frac{\lambda_{12} + \lambda_{32}}{2}$$