

Improved General Attribute Reduction Algorithms

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Abstract

Attribute reduction is a critical issue in rough sets theory. In recent years, there are many kinds of attribute reduction proposed, such as positive region preservation reduction, generalized decision preservation reduction, distribution preservation reduction, maximum distribution preservation reduction, and relative discernibility relation preservation reduction. General reduction approaches to obtaining various types of reducts also have been explored, but they are computationally time-consuming in the condition of large-scale data processing. In this study, we focus on the efficient general reduction algorithm to obtain five typical reducts mentioned above. At first, we introduce a concept called granularity space to establish a unified representation of five typical reducts. Based on the unified representation, we construct two quick general reduction algorithms by extending the positive region approximation to the granularity space. Then, we conduct a series of comparisons with existing reduction algorithms in aspects of theoretical analysis and experiments to evaluate the performance of the proposed algorithms. The results of analysis and experiments indicate that the proposed algorithms are effective and efficient.

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1. Introduction

Rough sets theory, introduced by Z. Pawlak [24] in 1982, is an efficient tool for imprecise, incomplete and uncertain information processing [11, 28, 47]. Currently, rough sets theory has been successfully applied to many practical problems, including machine learning [1, 40], pattern recognition [9, 10], data mining [35], decision support systems [15], etc.

Attribute reduction is one of the core concepts in rough sets[32]. It represents the process of obtaining attribute reduct, *i.e.*, a minimal set of attributes that can preserve the same ability of classification as the entire attribute set.

- 10 Main studies of attribute reduction can be classified into two categories: the appropriate definition of attribute reduction and the efficient reduction algorithm.

The appropriate definition of attribute reduction is a prerequisite for the good performance of attribute reducts in classification. After analyzing the relation of the positive region and the classification rule in consistent decision tables,

- 15 Pawlak proposed the positive region preservation reduction [24]. Kryszkiewicz proposed two types of reduction for inconsistent decision tables: the generalized decision preservation reduction and the distribution preservation reduction [17], which guarantee the property of possible decisions of objects and the decision class membership distribution of objects unchanged respectively. After that, 20 Zhang et al. [48] presented the maximum distribution preservation reduction as a compromise between the capability of the generalized decision preservation reduction and the complexity of the distribution preservation reduction. Thereafter, it emerged as a mainstream that researchers design appropriate attribute reduction definitions based on understanding the relationship between different 25 attribute reductions. Liu et al. [21] presented the distribution preservation reduction, the maximum distribution preservation reduction, and the generalized decision preservation reduction in the way of the classic reduction. Furthermore, Ref [23] classified the existing reduction into three types: the region preservation reduction, the decision preservation reduction, and the relationship preservation 30 reduction. Meanwhile, the relative discernibility relation preservation reduction

was proposed. Then, Zhou et al. [49] reviewed the existing attribute reduction, and concluded that there were six different types of attribute reduction for complete inconsistent decision tables. On this basis, Jia et al. [14] explored the reduction definition from the user's perspective to alleviate the difficulties
35 of choosing appropriate attribute reduction for specific applications. After reviewing the discernibility relation of different reducts, Ge et al. [6] proposed a unified definition of five types of attribute reduction.

The efficient reduction algorithm [18, 22, 38] is the central focus of researchers' studies. Attribute reduction algorithms can be grouped into two classes [33]:
40 the discernibility matrix-based algorithm [30, 29] and the heuristic algorithm. Many researchers studied the discernibility matrix-based attribute reduction algorithms because it is easily understandable and can find all reducts [17, 48, 23, 49, 3, 37, 41, 44]. However, the discernibility matrix-based method is computationally expensive. Therefore, heuristic approaches are applied to
45 attribute reduction processes. The heuristic approach is composed of two parts: the heuristic function and the search strategy [45]. The heuristic function is the fitness function of a heuristic approach. Existing definitions of heuristics are mainly based on three aspects: dependency degree [8], entropy [34, 31, 43], and consistency [19, 2, 7]. The search strategy is the control structure of the heuristic approach. There are two basic search strategies in heuristic approaches [36]:
50 the directional search strategy and the non-directional search strategy. The directional search strategy contains three kinds of methods: the deletion method, the addition method, and the addition-deletion method, and it has been applied in mainstream heuristic reduction algorithms. The non-directional search
55 strategy is usually applied in evolutionary algorithms [13, 4, 16] and some optimization methods [46, 12]. To further increase the computational efficiency, many researchers studied acceleration mechanisms of the heuristic attribute reduction method. Ref [6, 42] computed equivalence classes using a classic sort algorithm, which improved the speed of attribute reduction algorithms. Qian
60 et al. [26] presented a counting sort algorithm to reduce the computation cost of positive regions and core attributes. Furthermore, Qian et al. [27] studied

an acceleration strategy for the positive region preservation reduction and three types of entropy reductions. Liang et al. [20] developed a new accelerator that simultaneously decreased the size of the universe and the number of attributes in each iteration process of attribute reduction.

There are five types of representative reduction for complete inconsistent decision tables, *i.e.*, positive region preservation reduction, generalized decision preservation reduction, distribution preservation reduction, maximum distribution preservation reduction, and relative discernibility relation preservation reduction. Ref [14] explored the general attribute reduction definition and Ref [6] researched the approaches to obtaining those five typical reducts. However, existing general reduction algorithms are computationally time-consuming in processing large-scale data due to the lack of an efficient framework of reduction theory. To alleviate this problem, we propose a new unified representation of five typical reducts, and on this basis, we propose two quick general reduction algorithms.

Firstly, to construct a new definition of general attribute reduction, we introduce the concept of granularity space and analyze the properties of its binary relation. By associating the indiscernibility relation of granularity space with the indiscernibility relation of five reducts, we construct the definition of general attribute reducts in the way of granularity space. Finally, by extending the positive region approximation to granularity space, we develop two quick general reduction algorithms. Meanwhile, a series of analyses in aspects of theory and experiments are conducted to evaluate the effectiveness and efficiency of proposed algorithms. Two major contributions of this study are listed as follows. (1) We introduce a concept named granularity space to represent five attribute reductions in a unified framework; (2) We extend the positive region approximation to the granularity space and design a new acceleration strategy for general attribute reduction algorithms, which can expand the acceleration domain from the positive region to the universe of decision tables.

The rest of this paper is organized as follows. In section 2, we briefly review preliminary notions related to five types of representative reduction and

the classic definition of the general reduct. Besides, we discuss the process of the discernibility matrix-based reduction method. In section 3, we analyze the
95 room for improvement in the classic general reduct definition and propose the granularity space as an example to support the analysis. After that, we present the quick general heuristic reduction algorithms based on granularity space. Besides, we explain the advantage of proposed algorithms and their relationship to existing reduction algorithms. In section 4, we conduct a series of experiments
100 with several UCI data sets to evaluate the performance of proposed reduction algorithms. Finally, section 5 concludes this paper and brings some remarks about the work of this paper.

2. Preliminaries

In this part, we briefly review the concepts of decision tables, the notions of
105 five representative attribute reducts, and the definition of general reducts. Next, we brush up on the discernibility matrix-based reduction algorithm, which is the kin of two proposed algorithms.

The research object of rough sets theory is called the information system, which can be expressed as a four-tuple, *i.e.*, (U, A, V, f) . Here U stands for
110 the universe of discourse, a non-empty finite set of instances. A is the set of attributes, $V = \bigcup_{a \in A} V_a$ is the set of all attribute values, and $f : U \times A \rightarrow V$ is an information function that maps an object in U to exactly one value in V_a . For $x \in U, a \in A$, we have $f(x, a) \in V_a$. In the classification problem, the information system contains two kinds of attributes, and it can be characterized
115 by a decision table $DT = (U, C \cup D, V, f)$ with $C \cap D = \emptyset$, where an element of C is called a condition attribute, C is called the condition attribute set, an element D is called a decision attribute, and D is called the decision attribute set [20].

For the condition attribute set $B \subseteq C$, the indiscernibility relation $\text{IND}(B)$
120 is defined by $\text{IND}(B) = \{(x, y) \mid x, y \in U, f(x, a) = f(y, a), \forall a \in B\}$. For an instance $x \in U$, the equivalence class of x , being represented as $[x]_B$, is described

by $\{y \mid y \in U, \langle x, y \rangle \in \text{IND}(B)\}$. The family of all equivalence classes of $\text{IND}(B)$, *i.e.*, the partition determined by B , is denoted by $U/\text{IND}(B)$ or simply U/B . X , a non-empty subset of U , is called a concept of U . The B -lower approximation
¹²⁵ $\underline{B}(X)$ and the B -upper approximation $\overline{B}(X)$ of the concept X are respectively defined by $\underline{B}(X) = \{x \in U \mid [x]_B \subseteq X\}$ and $\overline{B}(X) = \{x \in U \mid [x]_B \cap X \neq \emptyset\}$.

The classification ability of conditional attribute C is measured by the relation between $\text{IND}(C)$ and $\text{IND}(D)$, and there are some uncertain situations that objects x, y with the same value of conditional attributes perform differently in the decision attributes. The uncertain situations are represented as the difference between two notions, *i.e.*, positive region and boundary region which are induced from indiscernibility relation. The B -positive region $\text{POS}_B(D)$ and B -boundary region $\text{BND}_B(D)$ are defined as

$$\begin{aligned}\text{POS}_B(D) &= \bigcup_{X \in U/D} \underline{B}(X), \\ \text{BND}_B(D) &= \bigcup_{X \in U/D} (\overline{B}(X) - \underline{B}(X)).\end{aligned}$$

$\text{POS}_B(D)$ consists of objects which perform consistently in decision; $\text{BND}_B(D)$ is comprised of objects which perform inconsistently in decision. Generally,
¹³⁰ we take the difference between them as a measurement of the classification ability of B . In this viewpoint, an attribute set $B \subseteq C$ satisfying $\text{POS}_B(D) = \text{POS}_C(D)$ is meaningful for feature selection and knowledge representation. Exactly speaking, B , an attribute set satisfying $\text{POS}_B(D) = \text{POS}_C(D) \wedge (\forall B' \subset B, \text{POS}_{B'}(D) \neq \text{POS}_C(D))$, is called positive region preservation attribute reduct. After proposing the positive region preservation attribute reduct, many
¹³⁵ extensions of that are investigated. Considering the focus of paper, we list five typical reducts' definitions summarized in Ref [49] here.

Definition. *Given a decision table $DT = (U, C \cup D, V, f)$,*

- (1) *B is a positive region preservation reduct (denoted as PRPR) of C with respect to D if B satisfies $\text{POS}_B(D) = \text{POS}_C(D)$ and $\forall B' \subset B, \text{POS}_{B'}(D) \neq \text{POS}_C(D)$;*

- (2) B is a generalized decision preservation reduct (denoted as $GDPR$) of C with respect to D if B satisfies $\forall x \in U, \delta_B(x) = \delta_C(x)$ and $\forall B' \subset B, \exists x \in U, \delta_{B'}(x) \neq \delta_B(x)$, where $\delta_B(x) = \{f(y, D) | x \in U \wedge y \in [x]_B\}$;
- (3) B is a distribution preservation reduct (denoted as DPR) of C with respect to D if B satisfies $\forall x \in U, \mu_B(x) = \mu_C(x)$ and $\forall B' \subset B, \exists x \in U, \mu_{B'}(x) \neq \mu_B(x)$,
- 145 where $\mu_B(x) = (P(D_1|[x]_B), P(D_2|[x]_B), \dots, P(D_{|U/D|}|[x]_B)), P(D_j|[x]_B) = \frac{|D_j \cap [x]_B|}{|[x]_B|}, x \in U, D_j \in U/D (j = 1, 2, \dots, |U/D|)$;
- (4) B is a maximum distribution preservation reduct (denoted as $MDPR$) of C with respect to D if B satisfies $\forall x \in U, \phi_B(x) = \phi_C(x)$ and $\forall B' \subset B, \exists x \in U, \phi_{B'}(x) \neq \phi_B(x)$, where $\phi_B(x) = \{D_j | \frac{|[x]_B \cap D_j|}{|[x]_B|} = \max_{k=1}^{|U/D|} \{ \frac{|[x]_B \cap D_k|}{|[x]_B|} \}\}$;
- (5) B is a relative discernibility relation preservation reduct (denoted as $DR-PR$) of C with respect to D if B satisfies $IND(B|D) = IND(C|D)$ and $\forall B' \subset B, IND(B'|D) \neq IND(B|D)$, where $IND(B|D) = \{(x, y) \in U \times U | \wedge (\forall a \in B \rightarrow f(x, a) = f(y, a)) \vee f(x, D) = f(y, D)\}$.

155 Those five reducts can be taken as the special cases of the general reduct proposed by Yao et al. [45], which can be written as follows.

Definition. Given a decision table $DT = (U, C \cup D, V, f)$ and a certain property \mathbb{P} of DT , an attribute set $B \subseteq C$ is called a reduct of C if it satisfies the following three conditions:

- 160 (1) *Evaluability condition*: the property can be represented by an evaluation function $e: 2^C \rightarrow (L, \preceq)$;
- (2) *Jointly sufficient condition*: $e(A) \preceq e(B)$;
- (3) *Individually necessary condition*: for any $B' \subset B, \neg(e(A \preceq e(B')))$.

165 Here $e: 2^C \rightarrow (L, \preceq)$ is an evaluation or fitness function, which maps an attribute set to an element of a poset L equipped with the partial order relation \preceq , i.e., \preceq is reflexive, anti-symmetric and transitive.

Attribute reduction algorithms, approaches to obtaining a reduct or reducts, can be classified into two groups: the discernibility matrix-based algorithm and the heuristic algorithm. For the limitation of paper focus, here we only review 170 the discernibility matrix-based algorithm. A general discernibility matrix given

by Miao et al. [23] is defined as follows.

Definition. Given a decision table $DT = (U, C \cup D, V, f)$, its discernibility matrix $DM = (DM(x, y))$ is a $|U| \times |U|$ matrix. $DM(x, y)$ for an object pair (x, y) is $\{a \mid \langle x, y \rangle \in \text{DIS}(C|D), f(x, a) \neq f(y, a), a \in C\}$, where $\text{DIS}(C|D)$ is the relative discernibility relation.
¹⁷⁵

The relative discernibility relation of five typical reducts can be written as follows.

- (1) The relative discernibility relation of PRPR is defined by $\text{DIS}(C|D) = \{\langle x, y \rangle \mid x, y \in \text{POS}_C(D) \wedge f(x, D) \neq f(y, D) \vee x \in \text{POS}_C(D) \wedge y \notin \text{POS}_C(D)\};$
¹⁸⁰
- (2) The relative discernibility relation of GDPR is defined by $\text{DIS}(C|D) = \{\langle x, y \rangle \mid \delta_C(x) \neq \delta_C(y)\};$
- (3) The relative discernibility relation of DPR is defined by $\text{DIS}(C|D) = \{\langle x, y \rangle \mid \mu_C(x) \neq \mu_C(y)\};$
- (4) The relative discernibility relation of MDPR is defined by $\text{DIS}(C|D) = \{\langle x, y \rangle \mid \phi_C(x) \neq \phi_C(y)\};$
¹⁸⁵
- (5) The relative discernibility relation of DRPR is defined by $\text{DIS}(C|D) = \{\langle x, y \rangle \mid f(x, C) \neq f(y, C) \wedge f(x, D) \neq f(y, D)\}.$

Based on the discernibility matrix, one can get the reduct through the following discernibility function. $DF(DM) = \bigwedge \{\bigvee (DM(x, y)) \mid \forall x, y \in U, DM(x, y) \neq \emptyset\}.$ The expression $\bigwedge \{\bigvee (DM(x, y))\}$ is the conjunction of all $\bigvee (DM(x, y))$ while $\bigvee (DM(x, y))$ is the disjunction of condition attributes in $DM(x, y).$ The discernibility function can be transformed to a reduced disjunctive form, and each conjunct of the reduced disjunctive form is a reduct or a superset of a reduct.
¹⁹⁰

¹⁹⁵ 3. Granularity space and the quick general reduction algorithm

In this section, we represent five typical reducts in a unified way, and develop two quick general reduction algorithms.

In subsection 3.1, to show the reason why we introduce the granularity space, we firstly analyze some promotable points of the existing general reduct defini-

²⁰⁰ tions from aspects of the reduct definition construction and reduction algorithm designing. After that, we analyze the relationship between the indiscernibility relation of the granularity space and the indiscernibility relation of five typical reducts to perceive the association between granularity space and reducts.
²⁰⁵ Based on that association, we present five typical reducts in the way of granularity space. At the end of this subsection, we show the power and simplicity of the granularity space-based general reduct definition in the comparison to the existing general reduct definitions.

In subsection 3.2, to increase the efficiency of reduction algorithms, we extend the positive region approximation to the granularity space and design a
²¹⁰ new acceleration strategy called granularity approximation, which expands the acceleration domain from the positive region to the universe of decision tables. Based on granularity approximation, we develop two quick general reduction algorithms and present the relationship between the proposed reduction algorithms and the existing reduction algorithms.

²¹⁵ In subsection 3.3, we show the difference between the proposed algorithms and the related work in Ref [6]. For a better understanding of proposed algorithms, we provide a calculation example of obtaining positive region preservation reduct using proposed algorithms.

3.1. Granularity space: an alternative of general attribute reduct definition

²²⁰ One of the most important parts in rough sets is the answer to what the reduct is. The general reduct definition proposed by Yao et al. [45] has been proven as a remarkable candidate. It's such a powerful definition that there are so many researchers adopting it to accomplish the reduct definition construction and reduction algorithm designing. However, there is still some room for improvement in Yao's definition. Firstly, we could extend the focus on the subset of C to expand the information scale we take into account during data processing. Secondly, we can enhance the flexibility of reduction algorithm designing by abandoning the way of constructing reduct definition, *i.e.*, using evaluation or fitness function to measure the relationship between different attributes, which
²²⁵

²³⁰ results in the complexity of theory for that only one more fitness function was proposed before we could propose a new type of attribute reduction. To achieve these improvements, we would propose a new definition of general reduct.

Information granules naturally give rise to hierarchical structures: the same problem or system can be perceived at different levels of specificity (detail) depending on the complexity of the problem, available computing resources, and particular needs to be addressed [39]. In rough sets, information granules can be represented as a partition of the universe. For the convenience of writing, we denote the partition of the universe U as the granularity to emphasize the hierarchical structure of data. Given a non-empty finite set U , a granularity G of U is defined as

$$G = \{X_i \mid \bigcup_{X_i \subset G} X_i = U \wedge (X_i \cap X_j = \emptyset, i \neq j)\}.$$

The indiscernibility relation and the discernibility relation of G is defined as follows.

Definition 1. *Given a granularity G of a finite non-empty set U , the indiscernibility relation and the discernibility relation of G are defined by*

$$\text{IND}(G) = \{\langle x, y \rangle \mid \langle x, y \rangle \in X_i \times X_i \wedge X_i \in G\};$$

$$\text{DIS}(G) = \{\langle x, y \rangle \mid (\langle x, y \rangle \in X_i \times X_j, i \neq j) \wedge X_i, X_j \in G\}.$$

²³⁵ The indiscernibility relation of the granularity G of U is

- (1) symmetric, *i.e.*, for $\forall x \in U$, we have $\langle x, x \rangle \in \text{IND}(G)$;
- (2) transitive, *i.e.*, for $\forall x, y, z \in U$, if $\langle x, y \rangle, \langle y, z \rangle \in \text{IND}(G)$, we have $\langle x, z \rangle \in \text{IND}(G)$;
- (3) reflexive, *i.e.*, for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{IND}(G)$, we have $\langle y, x \rangle \in \text{IND}(G)$.

²⁴⁰ On the other side, the discernibility relation of a granularity G of U is reflexive, *i.e.*, for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{DIS}(G)$, one can get $\langle y, x \rangle \in \text{DIS}(G)$. Another point needed to be reminded of is that $\text{IND}(G) \cup \text{DIS}(G) = \{\langle x, y \rangle \mid \langle x, y \rangle \in U \times U\}$.

Definition 2. Granularity space can be represented in a two-tuple $GS = (G, O)$,
245 where G is a granularity of a finite non-empty set U , O is a set of operators on elements of G . The ancestor granularity space of G is defined by $\text{ANC}(G) = (G, \{\cup\})$; the granularity subspace of G is defined by $\text{SPR}(G) = (G, \{\infty\})$, where $\infty U = \{P, Q \mid P \cup Q = U \wedge P \cap Q = \emptyset\}$.

Granularity space is the set of granularity generated from doing several operations specified by O on G . In particular, we specify $G \in (G, \{\infty\})$ and $G \in (G, \{\cup\})$. As a result, we can construct the following lemma.
250

Lemma 1. Given a granularity G of a finite non-empty set U , for $\forall G' \in \text{ANC}(G)$, we have $\text{IND}(G') \supseteq \text{IND}(G), \text{DIS}(G') \subseteq \text{DIS}(G)$; for $\forall G' \in \text{SPR}(G)$, we have $\text{IND}(G') \subseteq \text{IND}(G), \text{DIS}(G') \supseteq \text{DIS}(G)$.

Proof. Taking into account $\text{IND}(G) \cup \text{DIS}(G) = U \times U, \text{IND}(G) \cap \text{DIS}(G) = \emptyset$, all we need to do for proving Lemma 1 true is proving true either $\forall G' \in \text{ANC}(G), \text{IND}(G') \supseteq \text{IND}(G)$ or $\forall G' \in \text{SPR}(G), \text{IND}(G') \subseteq \text{IND}(G)$. According to Definition 2, for $\forall G' \in \text{ANC}(G)$, it can be implied that for $\forall E \in G'$, there exists a set $PE \subseteq G$ such that $\bigcup_{P_i \in PE} P_i = E$. As a result, we can get $\emptyset \subseteq \{\langle x, y \rangle \mid x \in P_i \in PE, y \in \bigcup_{P_j \in PE - P_i} P_j, P_i, P_j \in PE, x, y \in E\} \Rightarrow \text{DIS}(G') \subseteq \text{DIS}(G)$. That is to say, Lemma 1 is true.
260

To enhance the understanding of concepts introduced above, here we provide a calculation example of the granularity space.

Example 1. Given $U = \{x_1, x_2, x_3\}, G = \{\{x_1\}, \{x_2, x_3\}\}$, we know $\text{ANC}(G) = (G, \{\cup\}) = \{G, U\}$. let Q denotes as $\{\{x_1\}, \{x_2\}, \{x_3\}\}$, we have $\text{SPR}(G) = (G, \{\infty\}) = \{G, Q\}, \text{IND}(U) = U \times U, \text{DIS}(U) = \emptyset$;
 $\text{IND}(G) = \{\langle x_1, x_1 \rangle, \langle x_2, x_2 \rangle, \langle x_3, x_3 \rangle, \langle x_2, x_3 \rangle, \langle x_3, x_2 \rangle\}, \text{DIS}(G) = \{\langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \langle x_1, x_3 \rangle, \langle x_3, x_1 \rangle\}, \text{IND}(Q) = \{\langle x_1, x_1 \rangle, \langle x_2, x_2 \rangle, \langle x_3, x_3 \rangle\}, \text{DIS}(Q) = \{\langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \langle x_1, x_3 \rangle, \langle x_3, x_1 \rangle, \langle x_2, x_3 \rangle, \langle x_3, x_2 \rangle\}$. It is apparent that $\forall G' \in \text{ANC}(G), \text{IND}(G) \subseteq \text{IND}(G')$ i.e., $\text{DIS}(G) \supseteq \text{DIS}(G')$ and $\forall G' \in \text{SPR}(G), \text{DIS}(G) \subseteq \text{DIS}(G')$ i.e., $\text{DIS}(G) \supseteq \text{DIS}(G')$.
270

Theorem 1. *Given a decision table $DT = (U, C \cup D, V, f)$ and a granularity $G = U/C$, for any type of reduct B , we have $U/B \in ANC(G)$.*

$U/B \in ANC(G)$ can be induced from $B \subseteq C$. In other words, for $\forall B \subseteq C$, if
²⁷⁵ $y \in [x]_C$, we have $y \in [x]_B$. Here we summarize the useful conclusions drawn by analyzing the granularity space. The first is that for $\forall B \subseteq C$, we have $U/B \in ANC(U/C)$, $IND(B) \supseteq IND(C)$ and $DIS(B) \subseteq DIS(C)$. Reviewing on the discernibility relation of five typical reducts, we can imply that there are some redundant elements in existing discernibility relation. Secondly, $ANC(U/C)$
²⁸⁰ contains all granularity induced by $B \subseteq C$, and it indicates the possibility of designing the reduction algorithms by obtaining some special granularity. Here we re-construct the discernibility relation and the indiscernibility relation of five reducts from the perspective of granularity.

Definition 3. *Given a decision table $DT = (U, C \cup D, V, f)$, the discernibility relation and the indiscernibility relation of reducts are defined as follows.*

$$\begin{aligned} DIS(PRPR) &= \{\langle x, y \rangle \mid y \notin [x]_C \wedge (x, y \in POS_C(D) \wedge f(x, D) \neq \\ &\quad f(y, D) \vee x \notin POS_C(D) \wedge y \in POS_C(D))\} \\ IND(PRPR) &= \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge (x, y \notin POS_C(D) \\ &\quad \vee x, y \in POS_C(D) \wedge f(x, D) = f(y, D))\} \\ DIS(GDPR) &= \{\langle x, y \rangle \mid y \notin [x]_C \wedge \delta_C(x) \neq \delta_C(y)\} \\ IND(GDPR) &= \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge \delta_C(x) = \delta_C(y)\} \\ DIS(DPR) &= \{\langle x, y \rangle \mid y \notin [x]_C \wedge \mu_C(x) \neq \mu_C(y)\} \\ IND(DPR) &= \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge \mu_C(x) = \mu_C(y)\} \\ DIS(MDPR) &= \{\langle x, y \rangle \mid y \notin [x]_C \wedge \phi_C(x) \neq \phi_C(y)\} \\ IND(MDPR) &= \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge \phi_C(x) = \phi_C(y)\} \end{aligned}$$

$$\begin{aligned}\text{DIS}(DRPR) &= \{\langle x, y \rangle \mid y \notin [x]_C \wedge (x \in \text{BND}_C(D) \vee y \in \text{BND}_C(D) \\ &\quad \vee x, y \in \text{POS}_C(D) \wedge f(x, D) \neq f(y, D))\} \\ \text{IND}(DRPR) &= \{\langle x, y \rangle \mid y \in [x]_C \vee x, y \in \text{POS}_C(D) \wedge f(x, D) = f(y, D)\}\end{aligned}$$

For the discernibility relation and the indiscernibility relation of five typical
285 reducts, it is worth stressing that for $\forall Red \in \{PRPR, GDPR, DPR, MDPR, DRPR\}$,
 $\text{IND}(Red)$ is

- (1) symmetric, *i.e.*, for $\forall x \in U$, $\langle x, \text{we have } x \rangle \in \text{IND}(Red)$,
- (2) transitive, *i.e.*, for $\forall x, y, z \in U$, if $\langle x, y \rangle, \langle y, z \rangle \in \text{IND}(Red)$, we have $\langle x, z \rangle \in \text{IND}(Red)$,
- 290 (3) reflexive, *i.e.*, for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{IND}(Red)$, we have $\langle y, x \rangle \in \text{IND}(Red)$.

On the other side, the discernibility relation of Red , *i.e.* $\text{DIS}(Red)$, is reflexive, which means that for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{DIS}(Red)$, $\langle y, x \rangle$ is an element of $\text{DIS}(Red)$. In addition, we also have $\text{DIS}(Red) \cap \text{IND}(Red) = \emptyset$, $\text{IND}(Red) \cup \text{DIS}(Red) = U \times U$. As a result, we can imply the following theorem.
295

- Theorem 2.** *Given a decision table $DT = (U, C \cup D, V, f)$ and $B \subseteq C$, we have*
- (1) $\text{DIS}(PRPR) \subseteq \text{DIS}(B) \Leftrightarrow \text{POS}_B(D) = \text{POS}_C(D)$;
 - (2) $\text{DIS}(GDPR) \subseteq \text{DIS}(B) \Leftrightarrow \forall x \in U, \delta_B(x) = \delta_C(x)$;
 - (3) $\text{DIS}(DPR) \subseteq \text{DIS}(B) \Leftrightarrow \forall x \in U, \mu_B(x) = \mu_C(x)$;
 - 300 (4) $\text{DIS}(MDPR) \subseteq \text{DIS}(B) \Leftrightarrow \forall x \in U, \phi_B(x) = \phi_C(x)$;
 - (5) $\text{DIS}(DRPR) \subseteq \text{DIS}(B) \Leftrightarrow \text{IND}(B|D) = \text{IND}(C|D)$.

Proof. According to definitions of related reducts and the discernibility matrix-based reduction algorithm, it is easy to know that items (2), (3), and (4) of
305 Theorem 2 are true. So here we just prove the correctness of items (1) and (5).
(1) **Sufficiency:** We make an assumption that if $\text{DIS}(PRPR) \subseteq \text{DIS}(B)$, there is an object $x \in \text{POS}_C(D)$ satisfying $x \notin \text{POS}_B(D)$. It can be inferred that

there exists an object $y \notin [x]_C, f(y, D) \neq f(x, D)$ satisfying $y \in [x]_B$. Thus,
 we have $\langle x, y \rangle \notin \text{DIS}(B), \langle x, y \rangle \notin \text{DIS}(PRPR)$ and this conflicts with the def-
 310 definition of $\text{DIS}(PRPR)$. As a result, our assumption is not true, and we know
 $\text{DIS}(PRPR) \subseteq \text{DIS}(B) \Rightarrow \text{POS}_B(D) = \text{POS}_C(D)$.

Necessity: We make an assumption that if $\text{POS}_B(D) = \text{POS}_C(D)$, there exists
 a pair $\langle x, y \rangle \in \text{DIS}(PRPR)$ such that $\langle x, y \rangle \notin \text{DIS}(B)$. If $x \in \text{POS}_C(D) \wedge y \in$
 $\text{POS}_C(D) \wedge f(x, D) \neq f(y, D) \wedge \langle x, y \rangle \notin \text{DIS}(B)$, we know $x, y \notin \text{POS}_B(D)$
 315 and $\text{POS}_B(D) \neq \text{POS}_C(D)$. That is conflicted with our assumption, *i.e.*,
 $\text{POS}_B(C) = \text{POS}_C(D)$. If $x \in \text{POS}_C(D) \wedge y \notin \text{POS}_C(D)$, we know $x \in [y]_B$,
 and that implies $x \notin \text{POS}_B(D)$ and $\text{POS}_C(D) \neq \text{POS}_B(D)$. In summary, if
 $\text{POS}_B(D) = \text{POS}_C(D)$, then we have $\text{DIS}(PRPR) \subseteq \text{DIS}(B)$.

(5) **Sufficiency:** Assume that if $\text{DIS}(DRPR) \subseteq \text{DIS}(B)$, there exists a pair
 320 $\langle x, y \rangle \notin \text{IND}(C|D) \wedge \langle x, y \rangle \in \text{IND}(B|D)$. That means $\exists y \notin [x]_C, y \in [x]_B, f(x, D)$
 $\neq f(y, D)$. Then we know $\langle x, y \rangle \in \text{DIS}(DRPR), \langle x, y \rangle \notin \text{DIS}(B)$, and it conflict-
 s with our assumption $\text{DIS}(DRPR) \subseteq \text{DIS}(B)$. Finally, we get $\text{DIS}(DRPR) \subseteq$
 $\text{DIS}(B) \Rightarrow \text{IND}(B|D) = \text{IND}(C|D)$.

Necessity: Assume that if $\text{IND}(B|D) = \text{IND}(C|D)$, we have $\exists \langle x, y \rangle \in \text{DIS}(RPR)$
 325 such that $\langle x, y \rangle \notin \text{DIS}(B)$. If $y \notin [x]_C \wedge (x \in \text{BND}_C(D) \vee y \in \text{BND}_C(D))$, we
 have $\exists p \in [x]_C, \exists q \in [y]_C, f(p, C) \neq f(q, C), f(p, D) \neq f(q, D)$. Because $\langle x, y \rangle \in$
 $\text{IND}(B)$, we have $p \in [q]_B$. We get $\langle p, q \rangle \in \text{IND}(B|D)$ and $\langle p, q \rangle \notin \text{IND}(C|D)$.
 It is conflicted with $\text{IND}(B|D) = \text{IND}(C|D)$. If $x, y \in \text{POS}_C(D) \wedge f(x, d) \neq$
 $f(y, d)$, according to $\langle x, y \rangle \in \text{IND}(B)$, it is obvious that $\langle x, y \rangle \in \text{IND}(B|D)$.
 330 Noticing $\langle x, y \rangle \notin \text{IND}(C|D)$, we know that our assumption is not true. In sum-
 mary, $\text{IND}(B|D) = \text{IND}(C|D) \Rightarrow \text{DIS}(DRPR) \subseteq \text{DIS}(B)$.

For the convenience of writing, we specify that *Red* is an element of $\{PRPR, GDPR, DPR, MDPR, DRPR, DRPR\}$ if there is no additional declaration.
 335 According to Theorem 1, 2 and Lemma 1, we can construct a unified definition of general attribute reduct using granularity space.

Theorem 3. *Given a decision table $DT = (U, C \cup D, V, f)$ and the target reduct*

Red, B is a Red of C iff B satisfies the following conditions.

- (1) $U/B \in \text{ANC}(U/C) \cap \text{SPR}(\text{TGran}(Red))$;
- (2) $\forall B' \subset B$, U/B' does not satisfy condition (1).

Here $\text{TGran}(Red)$ stands for $U/\text{IND}(Red)$.

In fact, item (1) can be written in short as $U/B \in \text{SPR}(\text{TGran}(Red))$ because of $\forall B \subseteq C, U/B \in \text{ANC}(U/C)$. Theorem 3 makes the reduct space more accurate by replacing $\text{ANC}(U/C) \cap \text{SPR}(U)$ with $\text{ANC}(U/C) \cap \text{SPR}(\text{TGran}(Red))$.

The core difference of Theorem 3 to the existing general reduct definitions is the attention paid toward the objects' distribution, *i.e.*, granularity space, instead of the relationship between different attribute sets. There are three improvements of the general reduct defined by the objects' distribution. Firstly, the information space we take into consideration is greater, *i.e.*, $|\text{ANC}(U/C)| \geq |\{U/B | B \subseteq C\}|$. It is obvious that we have $\forall B \subseteq C, U/B \in \text{ANC}(U/C)$ and $\exists G \in \text{ANC}(U/C)$ satisfies $G \neq U/B$. It can be seen from the data in Table 1 that, the information space may vary when different reduct definitions are taken into account, *i.e.*, if we take Yao's definition as the reference of attribute reduction, the information space we take care of can be expressed as $\{U, U/\{a_1\}, U/\{a_2\}, U/\{a_1, a_2\}\}$; if we consider granularity space as the way of reducts representation, the information space should be $\{U, U/\{a_1\}, U/\{a_2\}, U/\{a_1, a_2\}, \{\{x_1, x_3\}, \{x_2\}\}\}$. The expansion of information space makes it possible to design the more efficient reduction algorithms, and GS is a good example. Secondly, general reducts defined by granularity

Table 1. A decision table for indicating the information scale of granularity space

	U	a_1	a_2	d
x_1	0	0	0	
x_2	1	0	1	
x_3	1	1	0	

space is independent of heuristic functions. That is to say, there is no need to design new heuristic functions for proposing new type of attribute reduction. All we need for the construction of a new type of attribute reduction is extend-

ing or redefining the definition of target granularity space, *i.e.*, the definition of $T\text{Gran}(Red) \cap \text{ANC}(U/C)$. When it comes to attribute reduction approaches,
365 it is more flexible in heuristic functions during designing heuristic attribute reduction algorithms based on granularity space because the input of reduction algorithms is definite, *i.e.*, a given granularity space. For example, GS can work well for obtaining five typical reducts with three classical heuristic functions,
370 *i.e.*, dependency degree, consistency, and entropy. For the limitation of the focus of this paper, we do not further explore it here. Thirdly, we can expand the acceleration domain from positive region into the universe and it is helpful to increase the efficiency of reduction algorithms. We would explain it in Section 3.2.

Keeping in mind that $\text{IND}(Red)$ is an equivalence relation, we can construct Algorithm 3.1 for computing $T\text{Gran}(Red)$. First step of CTGA is getting the

Algorithm 3.1 Calculating the target granularity algorithm(CTGA)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct $Red \in \{\text{PRPR}, \text{GDPR}, \text{DPR}, \text{MDPR}, \text{DRPR}\}$

Output: Target granularity of relative reduct.

```

1: Compute  $TG = U/C$ 
2: for  $ec1 \in TG$  do
3:   for  $ec2 \in TG - ec1$  do
4:     if  $\langle x, y \rangle \in \text{IND}(Red) : x \in ec1, y \in ec2$  then
5:        $ec1 := ec1 \cup ec2$ 
6:        $TG := TG - ec2$ 
7:     end if
8:   end for
9: end for
10: return  $TG$ 
```

375 partition U/C because $T\text{Gran}(Red)$ is an element of $\text{ANC}(U/C)$. Then, CTGA scans all the combinations of $\langle e1, e2 \rangle, e1, e2 \in U/C$, and merges equivalence classes if $\langle x, y \rangle \in \text{IND}(Red) : x \in e1, y \in e2, e1 \neq e2$. After the merging step, the variable TG stores $T\text{Gran}(Red)$. The upper bound of time complexity of
380 computing $T\text{Gran}(Red)$ using CTGA is $O(|U||C \cup D| + |U|^2)$. Noticing that the time complexity of merging step is $O(|U|^2)$, we design key converters, whose generated keys are then hashed by a function implemented in Python's dictionary,

to increase the efficiency of merging step

Definition 4. Given a decision table $(U, C \cup D, V, f)$ and $U/D = \{D_1, D_2, \dots, D_n\}$,
 some key converters are defined as

- 385 (1) $\text{Key}(x, PRPR) = \begin{cases} i, & \text{if } x \in \text{POS}_C(D) \wedge f(x, D) = D_i \\ n+1, & \text{otherwise} \end{cases};$
- (2) $\text{Key}(x, GDPR) = \delta_C(x);$
- (3) $\text{Key}(x, DPR) = \mu_C(x);$
- (4) $\text{Key}(x, MDPR) = \phi_C(x);$
- 390 (5) $\text{Key}(x, DRPR) = \begin{cases} i, & \text{if } x \in \text{POS}_C(D) \wedge f(x, D) = D_i \\ f(x, C), & \text{otherwise} \end{cases}.$

Along this way, we can construct an auxiliary key converter $\text{Key}_a(\text{Key}(x, Red))$, which maps $\text{Key}(x, Red)$ to a set of objects, for storing the target granularity $T\text{Gran}(Red)$. The key converters Key and Key_a can be constructed in the time complexity of $O(|U||C \cup D|)$ and $O(|U|)$. As a result, the upper bound of time complexity of computing $T\text{Gran}(Red)$ using CTGKC is $O(|U||C \cup D| + 2|U|)$.

Algorithm 3.2 Calculating target granularity using key converters(CTGKC)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct $Red \in \{PRPR, GDPR, DPR, MDPR, DRPR\}$

Output: Target granularity of relative reduct in hash function way.

- 1: Compute U/C
 - 2: **for** $ec \in U/C$ **do**
 - 3: $t := \text{Key}(x, Red)$, $\text{Key}_a(t) := \text{Key}_a(t) \cup \{x\}$, where $x \in ec$
 - 4: **end for**
 - 5: **return** Key_a
-

395

3.2. Granularity search: an efficient general reduction method

In this part, we focus on the efficient reduction algorithms. Based on the unified representation of five typical reducts, we present a heuristic function called granularity approximation for efficient attribute reduction algorithms.

400 Subsequently, we develop two quick general attribute reduction algorithms.

Definition 5. Given a decision table $DT = (U, C \cup D, V, f)$ and a granularity G of U , the granularity approximation of G in U/B is defined as $\text{GA}(U/B, G) = \bigcup\{[x]_B \mid x \in U, [x]_B \subseteq [x]_G\}$, where $[x]_G$ stands for a set of objects that belong to the same set in G .

405 If we take G as equivalence classes determined by an attributes set Z , granularity approximation can be written as $\text{POS}_B(Z)$. It is consistent with the uncertainty processing measurement in Pawlak attribute reduction. Obviously, if $\text{GA}(U/B, \text{TGran}(Red)) = U$, we have $U/B \in \text{SPR}(\text{TGran}(Red))$. Thus, the general reduct definition 3 can be re-written as follows.

410 **Theorem 4.** Given a decision table $DT = (U, C \cup D, V, f)$ and the target granularity $\text{TGran}(Red)$, B is a Red of C iff B satisfies

- (1) $\text{GA}(U/B, \text{TGran}(Red)) = U$
- (2) $\forall B' \subset B, \text{GA}(U/B', \text{TGran}(Red)) \neq U$.

For the granularity G of U , it is easy to know that $\forall P \subseteq Q \subseteq C, \text{GA}(U/P, G) \subseteq 415 \text{GA}(U/Q, G)$. According to Definition 5, the attribute significance for given attribute sets can be defined as follows.

420 **Definition 6.** Given $DT = (U, C \cup D, V, f)$ and target granularity G , the weeded significance of $a \in B \subseteq C$ in B for G is defined as $\text{Sig}^-(a, B, G) = |\text{GA}(U/B, G) - \text{GA}(U/B - \{a\}, G)|$; the joined significance of $a \in C - B$ in B for G is defined as $\text{Sig}^+(a, B, G) = |\text{GA}(U/B \cup \{a\}, G) - \text{GA}(U/B, G)|$, where $|S|$ represents the cardinality of set S .

To increase the computation efficiency of the significance function, we explore the faster approach with the deletion of objects unrelated to the calculation. For the convenience of writing, we use $\text{Sig}^+(a, B, G, U)$ to represent the attribute 425 significance, which denotes the value of the significance measure on the universe U . One can prove the following theorem of rank preservation.

Theorem 5. Given a decision table $DT = (U, C \cup D, V, f)$, an attribute set $B \subseteq C$, a granularity G of U and a set $U' = U - \text{GA}(U/B, G)$, for $\forall a, b \in$

$C - B$, if $Sig^+(a, B, G, U) \leq Sig^+(b, B, G, U)$, we have $Sig^+(a, B, G, U') \leq Sig^+(b, B, G, U')$.

Proof. From the definition of $Sig^+(a, B, G, U)$, we know that its value only depends on the function $GA(U/B, G)$. Since $U' = U - GA(U/B, G)$, one can know $GA(U'/B, G) = GA(U/B \cup \{a\}, G) - GA(U/B, G)$. Therefore, we have

$$\frac{Sig^+(a, B, G, U)}{Sig^+(a, B, G, U')} = \frac{|GA(U/B \cup \{a\}, G, U) - GA(U/B, G, U)|}{|GA(U/B \cup \{a\}, G, U') - GA(U/B, G, U')|} = 1$$

So $Sig^+(a, B, G, U) > Sig^+(b, B, G, U) \Leftrightarrow Sig^+(a, B, G, U') > Sig^+(b, B, G, U')$.

Now we can construct the attribute reduction algorithm based on granularity approximation as Algorithm 3.3, i.e., GS¹. The reason why we termed

Algorithm 3.3 Granularity Search(GS)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct Red

Output: A reduct of C

- 1: Compute the target granularity G according to algorithm 3.2;
 - 2: Compute $Sig^-(a_k, C, G, U), k \leq |C|$
 - 3: Put a_k into $core$, where $Sig^-(a_k, C, G, U) > 0$
 - 4: $red := core, i := 0, U_0 := U$
 - 5: **while** $|U_i| > 0$ **do**
 - 6: Calculate $a_{max} : a_{max} = \arg \max_{a \in C - red} Sig^+(a, red, G, U_{i+1})$
 - 7: $red := red \cup \{a_{max}\}$
 - 8: Compute $U_{i+1} := U_i - GA(U_i/red, G, U_i)$
 - 9: $i := i + 1$;
 - 10: **end while**
 - 11: **return** red
-

Algorithm 3.3 as granularity search is to emphasize the intuition of finding a granularity in the given space. GS consists of four parts: the computation of target granularity, the calculation of core attributes, the selection of attribute with maximal significance, termination judgment. The complexity of computing $TGran(Red)$ is $O(|U||C \cup D| + 2|U|)$; the complexity of calculating core attributes is $O(|U||C|^2)$; the complexity of remaining steps is $O(\sum_{i=1}^{|C|-cr} |U_i|(|cr| + i + 1))$,

¹ This algorithm is not related to the granularity search in granularity computing area

⁴⁴⁰ where cr denotes the set of core attributes. According to the attribute reduction algorithms in Ref [19, 6], the granularity search algorithm can be simplified to further increase the time efficiency by removing the part of computing core attributes and the attribute with maximal significance. For convenience, we denote granularity search without computing core attributes and the attribute with maximal significance as GSV.

Algorithm 3.4 A Granularity Search Variant(GSV)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct Red
Output: A reduct of C

```

1: Compute the target granularity  $G$  according to algorithm 3.2;
2:  $red := \emptyset, i := 0$  and  $U_0 := U$ 
3: while  $|U_i| > 0$  do
4:    $flag := False$ 
5:   for  $a \in C - red$  do
6:     if  $Sig^+(a, red, G, U_i) > 0$  then
7:        $flag := True$ 
8:        $red := red \cup \{a\}$ 
9:       Compute  $U_{i+1} := U_i - GA(U_i / red, G, U_i)$ 
10:       $i := i + 1$ 
11:    end if
12:   end for
13:   if  $flag$  is False and  $|U_i| > 0$  then
14:      $red := red \cup \{a_0\}$ , where  $a_0$  is an arbitrary attribute of  $C$ 
15:   end if
16: end while
17: return  $red$ 

```

⁴⁴⁵ We present the comparison of the upper bound of relevant algorithms time complexity in Table 2, in which “/” denotes that the step does not exist in this algorithm. QGARA-FS and QGARA-BS are general reduction algorithms proposed in Ref [6]. Taking into consideration the reduction results of QGARA-BS and GSV are partly determined by the attribute order scanned, we divide four algorithms into two groups for comparison. One is GS *vs.* QGARA-FS; another is GSV *vs.* QGARA-BS. The upper bound of time complexity of GS is $O(|U||C| + |U||C|^2 + \sum_{i=1}^{|C-cr|} |U_i|(|cr| + i + 1))$, where cr denotes core attributes. However, the upper bound of time complexity of QGARA-FS is $O(|U||C| + |U||C|^2 + \sum_{i=1}^{|C-cr|} |U|(|cr| + i + 1))$. Obviously, the time complexity of GS is

lower than that of QGARA-FS. Meanwhile, the upper bound of time complexity of GSV is $O(|U||C| + (\sum_{i=1}^{|C|} |U_i| \times i))$, which is lower than the upper bound of time complexity of QGARA-BS, *i.e.*, $O(|U||C| + \sum_{i=1}^{|C|} |U|(|C| - i + 1))$.

Table 2. The complexity description

Algorithms	Compute TGran	Compute core	Iteration structure
QGARA-FS	$O(U C)$	$O(C ^2 U)$	$O(\sum_{i=1}^{ C -cr} U (cr + i))$
QGARA-BS	$O(U C)$	/	$O(\sum_{i=1}^{ C } U (C - i + 1))$
GS	$O(U C)$	$O(C ^2 U)$	$O(\sum_{i=1}^{ C -cr} U_i (cr + i))$
GSV	$O(U C)$	/	$O(\sum_{i=1}^{ C } U_i \times i)$

When it comes to the relationship to the existing heuristic functions, reduction algorithms and acceleration mechanisms of the heuristic reduction algorithm, the granularity approximation is an extension of a dependency degree; two proposed algorithms, *i.e.*, GS and GSV, can be taken as the improved edition of algorithms proposed in Ref. [6], and the reason why GS and GSV are more efficient is that the reduct defined from the perspective of granularity; as an acceleration strategy, the granularity approximation can be taken as the extension or the special case of the positive region approximation. Besides, GS and GSV are kin to discernibility matrix-based reduction algorithms, which pay attention to the discernibility relation of reducts instead of the indiscernibility relation of reducts. It is worth trying to apply GS and GSV to address the problems where discernibility matrix-based reduction algorithms have got applied.

3.3. The analysis of granularity space

In this part, the relation and difference between the perspective of granularity and the relative discernibility relation are analyzed. Furthermore, attribute reduction processes related to two perspectives are also compared. In the end, to provide principled intuition of proposed algorithms, we show the process of GS for obtaining the positive region preservation reduct.

For the convenience of comparison and the consistency of writing with Ref [6], let $\Delta \in \{PRPR, GDPR, DPR, MDPR, DRPR\}$ denote as the specific

type of reduct, general attribute reduct can be defined in the way of relative discernibility relation as follows.

Definition. Given the general decision table $DT_{\Delta} = (U, C \cup D, V_{\Delta}, f_{\Delta})$, the attribute subset $B \subseteq C$ is a Δ -reduct of C with respect to D , iff it satisfies the following two conditions:

- (1) $\text{IND}(B|D_{\Delta}) = \text{IND}(C|D_{\Delta})$
- (2) $\forall B' \subset B, U/B'$ does not satisfy condition (1).

D_{Δ} is an attribute satisfying $U/D_{\Delta} = \text{TGran}(\Delta)$.

In comparison to $\text{IND}(B|D_{\Delta}) = \text{IND}(C|D_{\Delta})$, $\text{GA}(U/B, \text{TGran}(Red)) = U$ is simpler because of $|U/C| \leq |U|$ and $|\text{IND}(C|D)| \leq |U|^2$. In addition, the granularity based reduct definition is more intuitive. For the calculation of general reduct, based on relative discernibility relation, $B \subseteq C$ is a superset of Δ -reduct if $W_{\Delta}(B|D) = W_{\Delta}(C|D)$, where $W_{\Delta}(B|D) = |U|^2 - \overline{W_{\Delta}(D)} - \overline{W_{\delta}(B)} + \overline{W_{\Delta}(D \cup B)} = |U|^2 - \sum_{X_i \in U/D} |X_i|^2 - \sum_{X_j \in U/B} |X_j|^2 + \sum_{X_k \in U/B \cup D} |X_k|^2$; based on the granularity space, it is $\text{GA}(U/B, U/D_{\Delta}) = U$ for the determination of the superset of Δ -reduct, where $\text{GA}(U/B, U/D) = \bigcup \{[x]_B | [x]_B \subseteq [x]_{D_{\Delta}}\} = \text{POS}_B(D_{\Delta})$. Here we provide an example of presenting the difference between two general reduct definitions and reduction process.

Example 2. A calculation example of the positive region preservation attribute reduction.

Table 3. A decision table

U	a_1	a_2	a_3	a_4	a_5	a_6	d
x_1	0	0	0	0	0	0	1
x_2	1	0	0	0	0	0	2
x_3	1	1	0	0	0	0	1
x_4	1	1	0	0	0	0	2
x_5	1	1	0	0	1	1	1
x_6	1	1	0	0	1	1	2
x_7	1	1	0	1	1	1	1
x_8	1	1	0	1	1	1	2
x_9	1	1	0	1	1	1	2
x_{10}	1	1	1	1	1	1	2
x_{11}	1	1	1	1	1	1	3

For Table 3, we have $U = \{x_1, x_2, \dots, x_{11}\}$, $C = \{a_1, a_2, \dots, a_6\}$, $D = \{d\}$,
500 and $U/C = \{X_1, X_2, X_3, X_4, X_5, X_6\} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$, $U/D = \{P_1, P_2, P_3\} = \{\{x_1, x_3, x_5, x_7\}, \{x_2, x_4, x_6, x_8, x_9, x_{10}\}, \{x_{11}\}\}$.

Assume that we know $\text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, \dots, x_6\}\}$, and
505 we want to express $B = \{a_1, a_2\}$ is a superset of the positive region preservation
reduct, it can be written in relative discernibility realtion way, *i.e.*,
 $\text{IND}(B|D_\Delta) = \{\langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \langle x_1, x_3 \rangle, \langle x_3, x_1 \rangle, \dots\} = \text{IND}(C|D_\Delta)$, or in the
way of granularity space, *i.e.*, $U/B \in \text{SPR}(\text{TGran}(Red))$, $U/B = \text{TGran}(Red)$.
Let U/P denote as $\text{TGran}(PRPR)$, $U/C, U/P$ and U/B can be drawn as Figure1, in which objects contained in a rectangle belong to the same equivalence
510 class. From the relation of $U/P, U/B$, we can easily know B is a positive region preservation reduct. As mentioned above, the general reduct defined by
granularity space is simpler and more intuitive than that defined by the relative
indiscernibility relation.

x_1	x_1	x_1
x_2	x_2	x_2
x_3	x_3	x_3
x_4	x_4	x_4
x_5	x_5	x_5
x_6	x_6	x_6
x_7	x_7	x_7
x_8	x_8	x_8
x_9	x_9	x_9
x_{10}	x_{10}	x_{10}
x_{11}	x_{11}	x_{11}
U/C	U/P	U/B

Fig 1. Granularity comparison

To help the understanding of the proposed algorithms, here we show the
515 process of algorithm 3.3 for obtaining a positive region preservation reduct.

An example of algorithm 3.2 for computing the target granularity.

Considering $X_1 \subset \text{POS}_C(D)$, we have $\forall x \in X_1, \text{Key}(x, PRPR) = f(x_1, d) =$

1, $\text{Key}_a(1).\text{add}(x_1) = \{x_1\}$.

Considering $X_2 \subset \text{POS}_C(D)$, we have $\forall x \in X_2, \text{Key}(x, PRPR) = f(x_2, d) = 2, \text{Key}_a(2).\text{add}(x_2) = \{x_2\}$.

Considering $X_3 \subset \text{BND}_C(D)$, we have $\forall x \in X_3, \text{Key}(x, PRPR) = 4, \text{Key}_a(4).\text{add}(X_3) = \{x_3, x_4\}$.

Considering $X_4 \subset \text{BND}_C(D)$, we have $\forall x \in X_4, \text{Key}(x, PRPR) = 4, \text{Key}_a(4).\text{add}(X_4) = \{x_3, x_4, x_5, x_6\}$.

525 Considering $X_5 \subset \text{BND}_C(D)$, we have $\forall x \in X_5, \text{Key}(x, PRPR) = 4, \text{Key}_a(4).\text{add}(X_5) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

Considering $X_6 \subset \text{BND}_C(D)$, we have $\forall x \in X_6, \text{Key}(x, PRPR) = 4, \text{Key}_a(4).\text{add}(X_6) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$.

Finally, we get $\text{Key}_a(1) = \{x_1\}, \text{Key}_a(2) = \{x_2\}, \text{Key}_a(4) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$ and $\text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}\}$.

An example for calculating core attributes.

According to Algorithm 3.2, we get $G = \text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, \dots, x_{11}\}\}$. Firstly, we compute the partition of U determined by C , i.e., $U/C = \{X_1, X_2, X_3, X_4, X_5, X_6\} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$.

535 Secondly, we calculate the significance of every element of C to determine whether the element is a core attribute or not. For attribute a_1 , we have $U/C - \{a_1\} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $\text{Sig}^-(a_1, C, G) = |U - \{x_3, \dots, x_{11}\}| = 2$. According to $\text{Sig}^-(a_1, C, G) > 0$, we know that a_1 is a core attribute. For attribute a_2 , we have $U/C - \{a_2\} = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $\text{Sig}^-(a_2, C, G) = |U - \{x_1, x_5, x_6, \dots, x_{11}\}| = 3$.

According to $\text{Sig}^-(a_2, C, G) > 0$, we know that a_2 is a core attribute. For attribute a_3 , we have $U/C - \{a_3\} = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9, x_{10}, x_{11}\}\}$ and $\text{Sig}^-(a_3, C, G) = 0$. As a result, we know a_3 is not a core attribute. For attribute a_4 , we have $U/C - \{a_4\} = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6, x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $\text{Sig}^-(a_4, C, G) = 0$. So we know a_4 is not a core attribute. For attributes a_5 and a_6 , we have $U/C - \{a_5\} = U/C - \{a_6\} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $\text{Sig}^-(a_5, C, G) = \text{Sig}^-(a_6, C, G) = 0$. Thus, neither a_5 nor a_6 is not a core attribute. Finally we get $\text{core} = \{a_1, a_2\}$

**Example of algorithm3.3 for obtaining a positive region preservation
reduct.**

Now we get $G = \text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, \dots, x_6\}\}$ and $\text{reduct} = \text{core} = \{a_1, a_2\}$. As a result, we can get $U = U - \text{GA}(\text{reduct}) = \emptyset$. Because U is \emptyset , Algorithm3.3 outputs $\text{reduct} = \{a_1, a_2\}$.

4. Experiments and analyses

The objective of following experiments in this section is to show the effectiveness and the efficiency of the attribute reduction algorithms, *i.e.*, GS and GSV. Experiments are divided into three aspects. First, we employed 12 data sets in Table 4 to verify the performance of time consumption of GS, GSV, QGARA-FS, and QGARA-BS. Then, the computational time of algorithms GS, GSV, QGARA-FS and QGARA-BS with the increase of the size of objects and attributes was compared. Finally, we evaluated the classification accuracy of reducts generated by general attribute reduction algorithms using the naive bayes classifier and the decision tree classifier.

We carried out all the attribute reduction algorithms in experiments on a personal computer with Windows 10, Intel(R) Core(TM) CPU i5-8265U 1.60GHZ and 8GB RAM memory. The software used was Visual Studio Code 1.3.8, and the programming language was Python 3.7.

The data sets used in experiments are all downloaded from UCI repository of machine learning data sets [5] whose basic information is outlined in Table 4. For the sake that reduction algorithms can address only symbolic data, data sets containing continuous attributes, *i.e.*, Segmentation, Pima-indians-diabetes, and Wdbc, were preprocessed by equal-width discretization algorithms. For data sets with missing values, *i.e.*, Mushroom, we removed the objects with missing values to achieve uniform treatment of all data sets. The last column of Table 4, *i.e.*, $\gamma_C(D)$, stands for the positive region dependency degree $|\text{POS}_C(D)|/|U|$. The data set is consistent when $\gamma_C(D) = 1$; otherwise, the data set is inconsistent. As shown in Table 4, Shuttle, Mushroom, Tic, Segmen-

Table 4. UCI Data Sets

ID	Data sets	cases	attributes	classes	$\gamma_C(D)$
1	Shuttle	58000	10	7	0.230
2	Mushroom	5644	23	6	0.536
3	Tic	9822	86	2	0.968
4	Segmentation	2310	20	7	0.989
5	Pima-indians-diabetes	768	9	2	0.995
6	Splice	3190	61	3	0.999
7	Dermatology	358	34	6	1
8	Wdbc	569	31	2	1
9	CNAE9	1080	856	9	1
10	Semeion	1593	267	10	1
11	DNA	2000	181	3	1
12	Connect4	67557	43	3	1

tation, Pima-indians-diabetes, and Splice are inconsistent and the other 6 data sets are consistent. Taking into consideration the similar results of five types of
580 reducts under the general reduction algorithms, we mainly took positive region preservation reduction and relative discernibility relation preservation reduction results to verify the difference of four reduction algorithms.

4.1. Efficiency comparison of four general attribute reduction algorithms

In this subsection, to show the time efficiency of proposed algorithms, we
585 presented the time consumption of four attribute reduction algorithms in obtaining reducts, and experiments results were shown in Tables 5 and 6.

Table 5 indicated the computational time of QGARA-FS, QGARA-BS, GS, and GSV for obtaining a positive region preservation reduct on 12 data sets.
590 We can see that GSV was the fastest in four attribute reduction algorithms, and GS was faster than QGARA-FS. From results of experiments on both consistent and inconsistent decision tables, the computational time of four algorithms in obtaining positive region preservation reduct followed this order: QGARA-FS \geq GS, QGARA-FS $>$ GSV. GS performed better than QGARA-BS on small-scale data sets. However, in processing the large-scale data, it consumed more time than QGARA-BS for that the computation of the attribute with maximal
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significance was time-consuming. For most of the cases in experiments, the computational time of GS can reduce over half the computation time of QGARA-FS, such as data sets 2(Mushroom), 3(Tic), 4(Segmentation), etc. In the same condition, GSV can reduce over half of the computation time of QGARA-BS, such as data sets 5(Pima-indians-diabetes), 7(Dermatology), 9(CNAE9), etc.

600 In summary, for calculating positive region preservation reducts on the consis-

Table 5. Time consumption for obtaining PRPR.

ID	Time of QGARA-FS(s)	Time of QGARA-BS(s)	Time of GS(s)	Time of GSV(s)
1	9.326	7.967	6.562	1.676
2	11.788	1.990	1.816	0.213
3	192.414	18.784	17.890	1.581
4	1.215	0.772	0.521	0.117
5	0.123	0.109	0.067	0.018
6	27.328	3.790	5.638	0.406
7	1.139	0.216	0.233	0.024
8	1.533	0.272	0.259	0.040
9	1095.294	67.414	129.634	2.199
10	162.241	13.839	35.478	2.247
11	94.026	9.592	21.678	3.702
12	889.698	61.010	137.170	17.500

tent and inconsistent decision tables, the general attribute reduction algorithms proposed in this paper, *i.e.* GS and GSV, were more efficient than the existing general attribute reduction algorithms, *i.e.*, QGARA-FS and QGARA-BS.

605 Table 6 shows the time consumption of four general reduction algorithms for obtaining a relative discernibility relation preservation reduct. For the consistent decision table, a positive region preservation reduct is also a relative discernibility relation preservation reduct. Thus, the results of the time consumption of four general reduction algorithms on six consistent data sets were similar to the statistics of Table 5. For the time consumption of general reduction algorithms on six inconsistent data sets, we can know that the computational time of GS was less than that of QGARA-FS, and the same condition to GSV and QGARA-BS. In brief, the results of Table 6 were consistent with the observations of Table 5. In summary, for calculating relative discernibility

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Table 6. Time consumption for obtaining DRPR.

ID	Time of QGARA-FS(s)	Time of QGARA-BS(s)	Time of GS(s)	Time of GSV(s)
1	10.885	9.831	7.489	4.103
2	3.640	2.419	2.064	1.864
3	200.923	19.734	19.231	3.168
4	1.423	0.925	0.648	0.244
5	0.157	0.140	0.099	0.044
6	29.326	4.279	6.016	0.794
7	1.145	0.263	0.258	0.051
8	1.570	0.357	0.312	0.065
9	1152.034	70.652	129.052	2.061
10	175.911	14.870	36.988	2.057
11	95.743	10.190	20.596	3.522
12	894.983	60.119	132.542	16.472

615 relation preservation reduct on the consistent and inconsistent decision tables, the general reduction algorithms proposed in this paper were more efficient than the existing general reduction algorithms.

4.2. Comparison of reduction algorithms for different proportion data sets

In this subsection, to further compare the efficiency of general reduction algorithms, we compared the computational time of QGARA-FS, QGARA-BS, GS, and GSV for obtaining a positive region preservation reduct with the increase of the size of objects and the size of attributes.

Figure 2 shows detailed change trends of the time consumption of each algorithm for obtaining a positive region preservation reduct with the number of objects increasing. In Figure 2, the x-coordinate denotes the percentages of objects to the universe, while the y-coordinate concerns the time consumption of algorithms. We employed 8 data sets with different scale (Mushroom, Tic, Splice, Dermatology, Wdbc, CNAE9, DNA, and Connect4) to verify the performance of time consumption of QGARA-FS, QGARA-BS, GS, and GSV. 620 Generally speaking, the computational time of four algorithms increased with the increase of the percentages of objects to the universe. The same as tables 5 and 6, GS was more efficient than QGARA-FS and GSV was faster than 625

QGARA-BS. When dealing with the same UCI data sets, it is often the case that the computational time of GS was less than that of QGARA-FS, and equal to that of QGARA-BS for small-scale data. But in presence of large-scale data 635 sets, QGARA-BS performed better than GS. It can be observed in many data sets, such as Figure 2 (c), (e), (g). The computational time of GSV was less than that of the other three general reduction algorithms. The computational time of QGARA-FS increased distinctly in comparison to GS when the number 640 of objects was increasing; the computational time of QGARA-BS increased distinctly in comparison to GSV when the number of objects was increasing.

In Figure 3, the x-coordinate pertains to the percentages of attributes to the conditional attributes of the data set, while the y-coordinate concerns the time 645 consumption of algorithms. We took 8 data sets (Mushroom, Tic, Segmentation, Splice, Dermatology, Wdbc, CNAE9, and DNA) to verify the performance of the computational time of QGARA-FS, QGARA-BS, GS, and GSV for obtaining a relative discernibility realtion preservation reduct. The result of QGARA-FS, QGARA-BS, GS, and GSV was similar to the result induced from Figure 2.

4.3. Comparison of classification accuracy for general attribute reduction algorithms

As we know, there are many factors to the diversity of reducts obtained by 650 reduction algorithms, such as reduction criterion, search strategy, and heuristic functions used, etc. That is to say, the different general reduction algorithms with the same reduction criterion may generate different reducts. To evaluate the effect of reduct obtained by different general reduction algorithms, we 655 randomly selected 10 data sets as test objects from Table 4. We utilized the original data and the reduced data, which is generated by five algorithms QGARA-FS, QGARA-BS, GS, GSV, and chi-square feature selection(CSFS for short), to train naive bayes classifier and decision tree classifier based on the 10-fold cross-validation method. For chi-square feature selection, naive bayes classifier, 660 and decision tree classifier, we used its implementation in [25]. Regarding the parameter K in CSFS, determining how many attributes are contained in re-

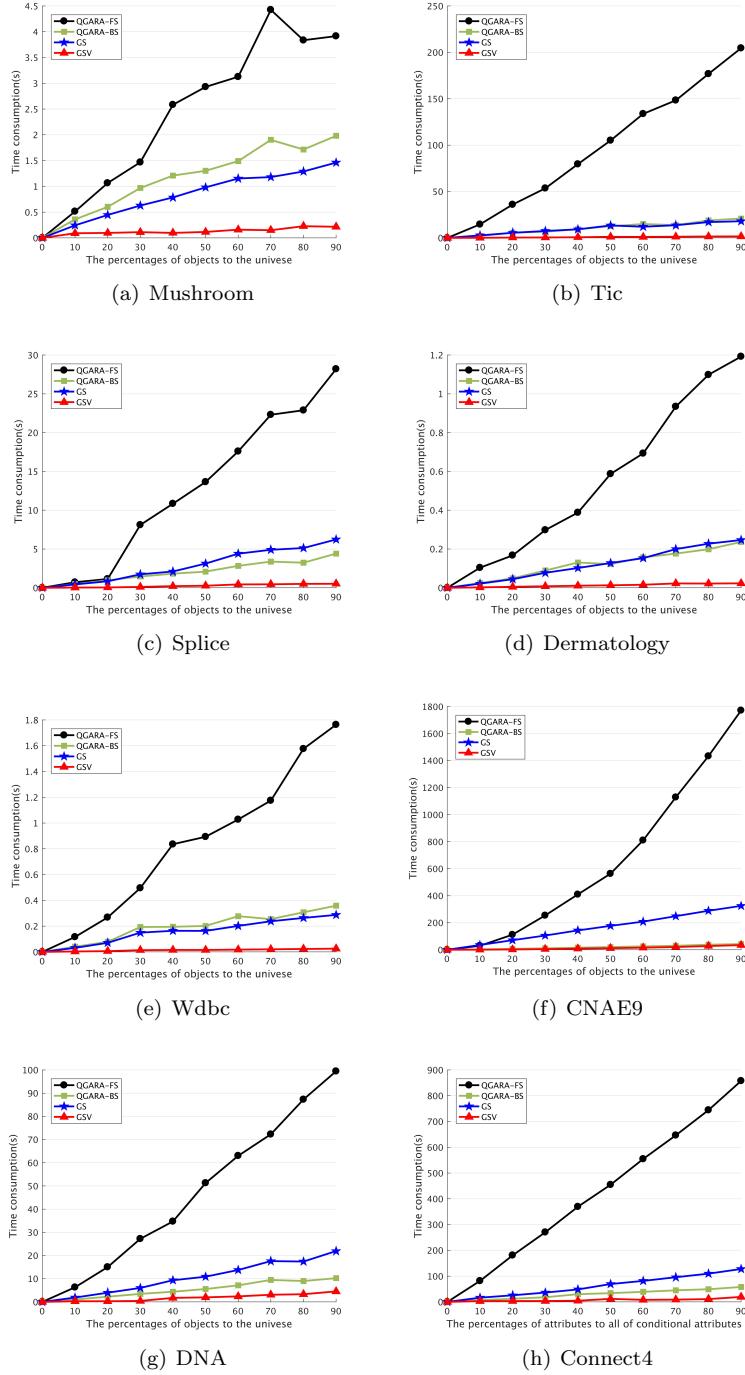


Fig 2. The time of general reduction algorithms versus the size of objects

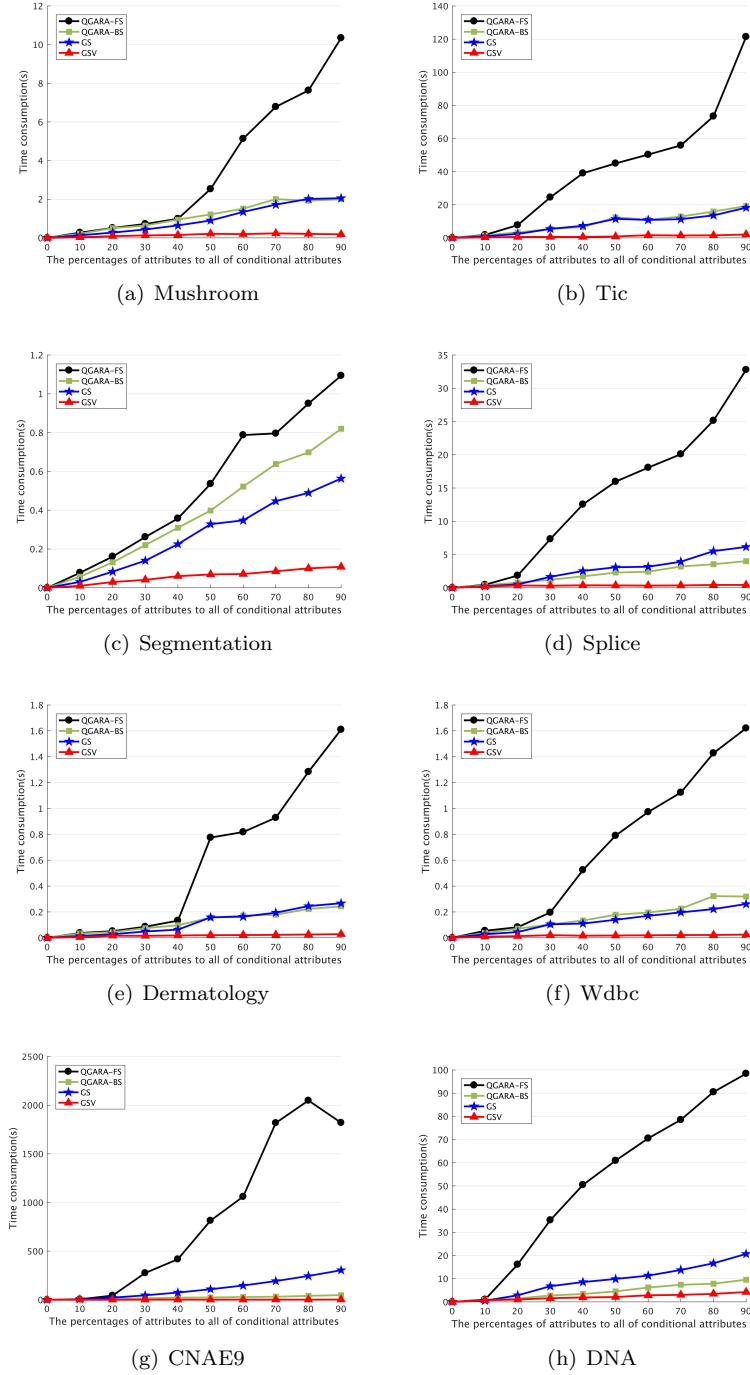


Fig 3. The time of general reduction algorithms versus the size of attributes

duced data, we assigned K as the cardinality of the reduct generated by GS.
 It is worth noticing that CSFS is not related to attribute reduction in theory
 665 and the reason why we put it into comparisons is to do the evaluation of GS
 and GSV in feature selection perspective. For convenience of comparison, we
 take the output of CSFS as a PRPR when K is assigned with the cardinality
 670 of the PRPR generated by GS; we take the output of CSFS as a DRPR when
 K is assigned with the cardinality of the DRPR generated by GS. The classi-
fication accuracy to the raw data and the reduced data generated by different
algorithms were shown in Tables 7 to 10, where the column "Raw" represents
the classification accuracies of the classifier trained on raw data sets, the bold-
face highlights the highest accuracy among different algorithms, and the row
"Average" represents average classification accuracy of reduction algorithms on
675 10 data sets, which can be interpreted as an estimated value of classification
accuracy obtained by the output of related reduction algorithm over unknown
data sets. Obviously, for most of reduced data sets, reduced data can retain

Table 7. The classification accuracy of decision tree with PRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.551	0.603	0.608	0.614	0.574	0.599
3	0.896	0.892	0.898	0.897	0.896	0.901
4	0.943	0.938	0.937	0.937	0.939	0.825
5	0.685	0.687	0.689	0.682	0.689	0.695
6	0.900	0.656	0.449	0.760	0.804	0.872
7	0.936	0.604	0.735	0.789	0.867	0.685
8	0.926	0.940	0.905	0.916	0.933	0.935
9	0.856	0.873	0.859	0.870	0.874	0.867
11	0.901	0.857	0.494	0.871	0.580	0.932
12	0.476	0.478	0.462	0.481	0.475	0.471
Average	0.807	0.753	0.704	0.782	0.763	0.778

similar classification accuracy as the entire data set.

For Table 7, the order of algorithms in the number of achieving the most clas-
 680 sification accuracy is CSFS(4) > GSV(3) > GS(2)Q > GARA-FS(1) > QGARA-
 BS(0). The order of algorithms in the average of classification accuracy on 10

Table 8. The classification accuracy of decision tree with DRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.551	0.554	0.551	0.558	0.551	0.584
3	0.896	0.896	0.898	0.897	0.894	0.900
4	0.943	0.938	0.940	0.938	0.942	0.819
5	0.685	0.682	0.685	0.685	0.691	0.684
6	0.900	0.651	0.455	0.762	0.809	0.873
7	0.936	0.590	0.746	0.787	0.878	0.685
8	0.926	0.938	0.902	0.907	0.931	0.933
9	0.856	0.872	0.853	0.870	0.871	0.862
11	0.901	0.861	0.502	0.868	0.580	0.933
12	0.476	0.477	0.466	0.477	0.477	0.472
Average	0.807	0.746	0.700	0.775	0.762	0.775

data sets is $GS(0.782) > CSFS(0.778) > GSV(0.763) > QGARA-FS(0.753) > QGARA-BS(0.704)$. GS achieves the best average classification accuracy on 10 data sets. That is to say, the steadiness of algorithms QGARA-FS, QGARA-BS, 685 CSFS, and GSV in classification accuracy is not as good as GS. Furthermore, the PRPR obtained by GS and GSV performs better than that obtained by QGARA-FS and QGARA-BS in the average classification accuracy of decision tree classifier. When it comes to the reduced data generated in the criterion of relative discernibility relation preservation reduction, GSV and CSFS obtain the highest classification accuracy 4 times; QGARA-FS obtains the highest classification accuracy 3 times; GS obtains the highest classification accuracy 1 time; QGARA-BS obtains the highest classification accuracy 0 times. GSV and CSFS perform the best in times of achieving the best classification accuracy. 690 Observing the average classification accuracy on ten data sets, GS is also the best of five, *i.e.*, $GS(0.775) \geq CSFS(0.775) > GSV(0.762) > QGARA-FS(0.746) > QGARA-BS(0.700)$. As a result, the DRPR obtained by GS and GSV performs better than that obtained by QGARA-FS and QGARA-BS in the average 695 classification accuracy of decision tree classifier.

In the classification accuracy results of naive bayes classifier on reduced 700 data generated in the criterion of positive region preservation reduction and

Table 9. The classification accuracy of naive bayes with PRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.649	0.557	0.650	0.565	0.582	0.668
3	0.777	0.885	0.895	0.896	0.896	0.805
4	0.603	0.575	0.570	0.575	0.605	0.519
5	0.682	0.682	0.682	0.682	0.682	0.682
6	0.792	0.545	0.519	0.657	0.576	0.778
7	0.977	0.602	0.769	0.763	0.933	0.682
8	0.902	0.804	0.865	0.889	0.874	0.753
9	0.949	0.909	0.901	0.907	0.933	0.887
11	0.924	0.767	0.535	0.827	0.594	0.894
12	0.597	0.601	0.599	0.606	0.597	0.599
Average	0.785	0.693	0.698	0.737	0.727	0.727

relative discernibility relation preservation reduction, GS was the best one in the classification accuracy average on 10 data sets, and GSV was the second.

Table 10. The classification accuracy of naive bayes with DRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.649	0.595	0.642	0.595	0.642	0.649
3	0.777	0.891	0.895	0.896	0.896	0.805
4	0.603	0.575	0.570	0.575	0.605	0.519
5	0.682	0.682	0.682	0.682	0.682	0.682
6	0.792	0.545	0.519	0.657	0.576	0.778
7	0.977	0.602	0.769	0.763	0.933	0.682
8	0.902	0.804	0.865	0.889	0.874	0.753
9	0.949	0.909	0.901	0.907	0.933	0.887
11	0.924	0.767	0.535	0.827	0.594	0.894
12	0.597	0.601	0.599	0.606	0.597	0.599
Average	0.785	0.697	0.698	0.740	0.733	0.725

Furthermore, we also used the t-test to compare the average 10-fold cross-validation based accuracies over each dataset. Taking the classification accuracies of reducts obtained by GS, GSV, QGARA-FS, and QGARA-BS as the sampling results of four random variables V_{GS} , V_{GSV} , V_{QFS} and V_{QBS} , we set up original hypothesis as $H_0 : V_{GS} = V_{QFS} \vee V_{GSV} = V_{QBS}$ and assigned 0.05 as the significance level α . For convenience of evaluation, we took as a com-

parison case the average 10-fold cross-validation based accuracies of the same
710 classification algorithm (naive bayes or decision trees) with two reducts (PRPR or DRPR) generated by the comparative algorithms (GS *v.s.* QGARA-FS or GSV *v.s.* QGARA-BS) over a dataset, and there are 80 (2 classification algorithms \times 2 types of reduct \times 2 comparisons \times 10 datasets) comparison cases in experiment. In experimental result, there are only 19 cases rejecting the original hypothesis. That is to say, from the perspective of statistics inference, there
715 are only 23.75% of cases actually supporting that the proposed algorithms are better than the existing algorithms; the remaining 76.25% of cases support the original hypothesis. As a result, four algorithms tie in the aspect of classification accuracy, and if time permitted, it is a reasonable choice of selecting the
720 existing general reduction algorithms for data processing.

In the experimental part, we made a series of comparisons between the proposed general attribute reduction algorithms and the existing general attribute reduction algorithms. We could draw conclusions listed as

- (1) In time consumption of the algorithms to obtain reducts, GS performed well
725 in dealing with small-scale data. When processing large-scale data sets, GSV and QGARA-BS were good choices for attribute reduction;
- (2) According to the experiments, the classification accuracies of reducts generated by GS and GSV are competitive as that generated by QGARA-FS and QGARA-BS.

730 5. Conclusion

In this study, we focus on the effective and efficient general reduction approach to obtain five types of reducts on the complete decision tables. We introduce a concept termed as granularity space and represent five typical reducts with granularity space. Based on the unified representation, we develop two
735 quick general reduction algorithms. In comparison to the existing general reduction algorithms, the proposed algorithms have two advantages as follows.

- (1) The proposed algorithms are more efficient. In the process of attribute re-

duction, GS can reduce one half of the computation time of QGARA-FS, and GSV can reduce over one half of the computation time of QGARA-BS.

740 (2) The reducts generated by proposed algorithms perform well as that generated by the existing reduction algorithms in the classification accuracy of decision tree classifier and naive bayes classifier. Over the data sets in experiments, the average classification accuracy of decision tree classifier and naive bayes classifier trained on reduced data generated by GS and GSV is equal to or higher than
745 that trained on reduced data generated by QGARA-FS and QGARA-BS.

750 However, it is notable that the general reduction definitions and algorithms proposed in this paper are only suitable for the complete decision table. There exist many generalized decision tables, such as incomplete decision tables, interval-valued decision tables. Research on the extension of granularity space for the generalized decision tables will be investigated in future work.

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References

- [1] Ron Tor Das, Kai Keng Ang, and Chai Quek. ierspop: a novel incremental rough set-based pseudo outer-product with ensemble learning. *Applied Soft Computing*, 46:170–186, 2016.
- 760 [2] Manoranjan Dash and Huan Liu. Consistency-based search in feature selection. *Artificial Intelligence*, 151(1):155–176, 2003.
- [3] Da Yong Deng and Hou Kuan Huang. A new discernibility matrix and function. In *International Conference on Rough Sets and Knowledge Technology*, pages 114–121. Springer, 2006.

- 765 [4] Ting Quan Deng, Cheng Dong Yang, Yue Tong Zhang, and Xin Xia Wang.
 An improved ant colony optimization applied to attributes reduction. In
Fuzzy Information and Engineering, pages 1–6. Springer, 2009.
- [5] Dheeru Dua and Casey Graff. UCI machine learning repository, 2017.
- 770 [6] Hao Ge, Long Shu Li, Yi Xu, and Chuan Jian Yang. Quick general reduc-
 tion algorithms for inconsistent decision tables. *International Journal of
 Approximate Reasoning*, 82:56–80, 2017.
- [7] Qing Hua Hu, Hui Zhao, Zong Xia Xie, and Da Ren Yu. Consistency based
 attribute reduction. In *Pacific-Asia Conference on Knowledge Discovery
 and Data Mining*, pages 96–107. Springer, 2007.
- 775 [8] Xiao Hua Hu and Nick Cercone. Learning in relational databases: a rough
 set approach. *Computational intelligence*, 11(2):323–338, 1995.
- [9] YiChung Hu. Flow-based tolerance rough sets for pattern classification.
Applied Soft Computing, 27:322–331, 2015.
- 780 [10] Kuang Yu Huang. An enhanced classification method comprising a genetic
 algorithm, rough set theory and a modified pbmf-index function. *Applied
 Soft Computing*, 12(1):46–63, 2012.
- [11] Kuang Yu Huang and I-Hui Li. A multi-attribute decision-making model
 for the robust classification of multiple inputs and outputs datasets with
 uncertainty. *Applied Soft Computing*, 38:176–189, 2016.
- 785 [12] Richard Jensen and Qiang Shen. Semantics-preserving dimensionality re-
 duction: rough and fuzzy-rough-based approaches. *IEEE Transactions on
 knowledge and data engineering*, 16(12):1457–1471, 2004.
- [13] Xiu Yi Jia, Wen He Liao, Zhen Min Tang, and Lin Shang. Minimum cost
 attribute reduction in decision-theoretic rough set models. *Information
 Sciences*, 219:151–167, 2013.

- [14] Xiu Yi Jia, Lin Shang, Bing Zhou, and Yi Yu Yao. Generalized attribute reduct in rough set theory. *Knowledge-Based Systems*, 91:204–218, 2016.
- [15] Yilmaz Kaya and Murat Uyar. A hybrid decision support system based on rough set and extreme learning machine for diagnosis of hepatitis disease. *Applied Soft Computing*, 13(8):3429–3438, 2013.
- [16] Liang Jun Ke, Zu Ren Feng, and Zhi Gang Ren. An efficient ant colony optimization approach to attribute reduction in rough set theory. *Pattern Recognition Letters*, 29(9):1351–1357, 2008.
- [17] Marzena Kryszkiewicz. Comparative study of alternative types of knowledge reduction in inconsistent systems. *International journal of intelligent systems*, 16(1):105–120, 2001.
- [18] Fei Li and Yun Qiang Yin. Approaches to knowledge reduction of covering decision systems based on information theory. *Information Sciences*, 179(11):1694–1704, 2009.
- [19] Min Li, Chang Xing Shang, Sheng Zhong Feng, and Jian Ping Fan. Quick attribute reduction in inconsistent decision tables. *Information Sciences*, 254:155–180, 2014.
- [20] Ji Ye Liang, Jun Rong Mi, Wei Wei, and Feng Wang. An accelerator for attribute reduction based on perspective of objects and attributes. *Knowledge-Based Systems*, 44:90–100, 2013.
- [21] Qi He Liu, Leiting Chen, Jian Zhong Zhang, and Fan Min. Knowledge reduction in inconsistent decision tables. In Xue Li, Osmar R. Zaïane, and Zhanhuai Li, editors, *Advanced Data Mining and Applications*, pages 626–635, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg.
- [22] Zu Qiang Meng and Zhong Zhi Shi. A fast approach to attribute reduction in incomplete decision systems with tolerance relation-based rough sets. *Information Sciences*, 179(16):2774–2793, 2009.

- [23] Duo Qian Miao, Yan Zhao, Yi Yu Yao, H X Li, and Fei Fei Xu. Relative
reducts in consistent and inconsistent decision tables of the pawlak rough
set model. *Information Sciences*, 179(24):4140–4150, 2009.
- [24] Zdzisław Pawlak. Rough sets. *International Journal of Computer & Information Sciences*, 11(5):341–356, Oct 1982.
- [25] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel,
M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Pas-
825 soss, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay. Scikit-
learn: Machine learning in Python. *Journal of Machine Learning Research*,
12:2825–2830, 2011.
- [26] Jin Qian, Duo Qian Miao, Z H Zhang, and W Li. Hybrid approaches
to attribute reduction based on indiscernibility and discernibility relation.
830 *International Journal of Approximate Reasoning*, 52(2):212–230, 2011.
- [27] Yu Hua Qian, Ji Ye Liang, Witold Pedrycz, and Chuang Yin Dang. Positive
approximation: an accelerator for attribute reduction in rough set theory.
Artificial Intelligence, 174(9-10):597–618, 2010.
- [28] Jiao Shi, Yu Lei, Ying Zhou, and Maoguo Gong. Enhanced rough-fuzzy c-
means algorithm with strict rough sets properties. *Applied Soft Computing*,
835 46:827–850, 2016.
- [29] Andrzej Skowron and Cecylia Rauszer. The discernibility matrices and
functions in information systems. In *Intelligent decision support*, pages
331–362. Springer, 1992.
- [30] Andrzej Skowron and Jaroslaw Stepaniuk. Towards an approximation the-
ory of discrete problems, part i. *Fundam. Inform.*, 15(2):187–207, 1991.
- [31] Hai Rong Sun, Wang Rui, Bi Xia Xie, and Tian Yao. Continuous attribute
reduction method based on an automatic clustering algorithm and decision
entropy. In *Control Conference*, 2017.

- 845 [32] K Thangavel and A Pethalakshmi. Dimensionality reduction based on
rough set theory: A review. *Applied Soft Computing*, 9(1):1–12, 2009.
- [33] K Thangavel and A Pethalakshmi. Dimensionality reduction based on
rough set theory: A review. *Applied Soft Computing*, 9(1):1–12, 2009.
- 850 [34] Jing Tian, Quan Wang, Bing Yu, and Dan Yu. A rough set algorithm
for attribute reduction via mutual information and conditional entropy.
In *2013 10th International Conference on Fuzzy Systems and Knowledge
Discovery (FSKD)*, pages 567–571. IEEE, 2013.
- [35] Feng Wang, Jiye Liang, and Chuangyin Dang. Attribute reduction for
dynamic data sets. *Applied Soft Computing*, 13(1):676–689, 2013.
- 855 [36] Jue Wang and Duo Qian Miao. Analysis on attribute reduction strategies
of rough set. *Journal of computer science and technology*, 13(2):189–192,
1998.
- 860 [37] Jue Wang and Ju Wang. Reduction algorithms based on discernibility
matrix: the ordered attributes method. *Journal of computer science and
technology*, 16(6):489–504, 2001.
- [38] Ling Wei and Jian Jun Qi. Relation between concept lattice reduction and
rough set reduction. *Knowledge-Based Systems*, 23(8):934–938, 2010.
- 865 [39] Pedrycz Witold. Granular computing for data analytics: a manifesto
of human-centric computing. *IEEE/CAA Journal of Automatica Sinica*,
5(6):1025–1034.
- [40] Xiaojun Xie, Xiaolin Qin, Chunqiang Yu, and Xingye Xu. Test-cost-
sensitive rough set based approach for minimum weight vertex cover prob-
lem. *Applied Soft Computing*, 64:423–435, 2018.
- 870 [41] Zhang Yan Xu, Li Yu Huang, and Bing Ru Yang. Efficient attribute re-
duction algorithm based on skowron discernibility matrix. In *2009 Interna-
tional Workshop on Intelligent Systems and Applications*, pages 1–4. IEEE,
2009.

- [42] Zhang Yan Xu, Zuo peng Liu, Bing Ru Yang, and Wei Song. A quick attribute reduction algorithm with complexity of $\max(o(cu), o(c^2u/c))$. *Jisuanji Xuebao(Chinese Journal of Computers)*, 29(3):391–399, 2006.
- [43] Tao Yan and Chong Zhao Han. Entropy based attribute reduction approach for incomplete decision table. In *International Conference on Information Fusion*, 2017.
- [44] Yi Yu Yao and Yan Zhao. Discernibility matrix simplification for constructing attribute reducts. *Information sciences*, 179(7):867–882, 2009.
- [45] Yiyu Yao, Yan Zhao, and Jue Wang. On reduct construction algorithms. In *Transactions on computational science II*, pages 100–117. Springer, 2008.
- [46] Hong Yu, Guo Yin Wang, and Fakuan Lan. Solving the attribute reduction problem with ant colony optimization. In *Transactions on rough sets XIII*, pages 240–259. Springer, 2011.
- [47] Jianming Zhan, Muhammad Irfan Ali, and Nayyar Mehmood. On a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods. *Applied Soft Computing*, 56:446–457, 2017.
- [48] Wen Xiu Zhang, Ju Sheng Mi, and Wei Zhi Wu. Knowledge reductions in inconsistent information systems. *Chinese Journal of Computers-Chinese Edition*, 26(1):12–18, 2003.
- [49] Jie Zhou, Duo Qian Miao, Witold Pedrycz, and Hong Yun Zhang. Analysis of alternative objective functions for attribute reduction in complete decision tables. *Soft Computing*, 15(8):1601–1616, 2011.