

Improved General Attribute Reduction Algorithms

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Abstract

Attribute reduction is a critical issue in rough sets theory. In recent years, there are many kinds of attribute reduction proposed, such as positive region preservation reduction, generalized decision preservation reduction, distribution preservation reduction, maximum distribution preservation reduction, and relative discernibility relation preservation reduction. General reduction approaches to obtaining various types of reducts also have been explored, but they are computationally time-consuming in the condition of large-scale data processing. In this study, we focus on the efficient general reduction algorithm to obtain five typical reducts mentioned above. At first, we introduce a concept called granularity space to establish a unified representation of five typical reducts. Based on the unified representation, we construct two quick general reduction algorithms by extending the positive region approximation to the granularity space. Then, we conduct a series of comparisons with existing reduction algorithms in aspects of theoretical analysis and experiments to evaluate the performance of the proposed algorithms. The results of analysis and experiments indicate that the proposed algorithms are effective and efficient.

Keywords: Attribute reduction, Granular computing, Inconsistent decision table, Rough sets

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1. Introduction

Rough sets theory, introduced by Z. Pawlak [1] in 1982, is an efficient tool to imprecise, incomplete and uncertain information processing [2, 3, 4]. Currently, rough set theory has been successfully applied to many practical problems,
5 including machine learning [5, 6], pattern recognition [7, 8], data mining [9], decision support systems [10], etc.

Attribute reduction is one of the core concepts in rough sets[11]. It represents the process of obtaining attribute reduct, *i.e.*, a minimal set of attributes that can preserve the same ability of classification as the entire attribute set.

- 10 Main studies of attribute reduction can be classified into two categories: the appropriate definition of attribute reduction and the efficient reduction algorithm.

The appropriate definition of attribute reduction is a prerequisite for the good performance of attribute reducts in classification. After analyzing the relation of the positive region and the classification rule in consistent decision tables,

- 15 Pawlak proposed the positive region preservation reduction [1]. Kryszkiewicz proposed two types of reduction for inconsistent decision tables: the generalized decision preservation reduction and the distribution preservation reduction [12], which guarantee the property of possible decisions of objects and the decision class membership distribution of objects unchanged respectively. After that,
20 Zhang et al. [13] presented the maximum distribution preservation reduction as a compromise between the capability of the generalized decision preservation reduction and the complexity of the distribution preservation reduction. Thereafter, it emerged as mainstream that researchers design appropriate attribute reduction definitions based on understanding the relationship between different
25 attribute reductions. Liu et al. [14] presented the distribution preservation reduction, the maximum distribution preservation reduction, and the generalized decision preservation reduction in the way of the classic reduction. Furthermore, Ref [15] classified the existing reduction into three types: the region preservation reduction, the decision preservation reduction, and the relationship preservation reduction.
30 Meanwhile, the relative discernibility relation preservation reduction

was proposed. Then, Zhou et al. [16] reviewed the existing attribute reduction, and concluded that there are six different types of attribute reduction for complete inconsistent decision tables. On this basis, Jia et al. [17] explored the reduction definition from the user’s perspective to alleviate the difficulties 35 of choosing appropriate attribute reduction for specific applications. After reviewing the discernibility relation of different reducts, Ge et al. [18] proposed a unified definition of five types of attribute reduction.

The efficient reduction algorithm [19, 20, 21] is the central focus of researchers’ studies. Attribute reduction algorithms can be grouped into two 40 classes [22]: the discernibility matrix-based algorithm [23, 24] and the heuristic algorithm. Many researchers studied attribute reduction algorithms using the discernibility matrix because it is easily understandable and can find all reducts [12, 13, 15, 16, 25, 26, 27, 28]. However, the discernibility matrix-based method is computationally expensive. Therefore, heuristic approaches are applied to 45 attribute reduction processes. The heuristic approach is composed of two parts: the heuristic function and the search strategy [29]. The heuristic function is the fitness function of a heuristic approach. Existing definitions of heuristics are mainly based on three aspects: dependency degree [30], entropy [31, 32, 33], and consistency [34, 35, 36]. The search strategy is the control structure of 50 the heuristic approach. There are two basic search strategies in heuristic approaches [37]: the directional search strategy and the non-directional search strategy. The directional search strategy contains three kinds of methods: the deletion method, the addition method, and the addition-deletion method. The non-directional search strategy is usually applied in evolutionary algorithms 55 [38, 39, 40] and some optimization methods [41, 42]. To further increase the computational efficiency, many researchers studied acceleration mechanisms of the heuristic attribute reduction method. Ref [18, 43] computed equivalence classes using a classic sort algorithm, which improved the speed of attribute reduction algorithms. Qian et al. [44] presented a counting sort algorithm 60 to reduce the computation cost of positive regions and core attributes. Furthermore, Qian et al. [45] studied an acceleration strategy for the positive

region preservation reduction and three types of entropy reductions. Liang et al. [46] developed a new accelerator that simultaneously decreased the size of the universe and the number of attributes in each iteration process of attribute reduction.

There are five types of representative reduction for complete inconsistent decision tables, *i.e.*, positive region preservation reduction, generalized decision preservation reduction, distribution preservation reduction, maximum distribution preservation reduction, and relative discernibility relation preservation reduction. The general attribute reduction definition [17] and related approaches to those five typical reducts [18] have been explored. However, existing general reduction algorithms are still computationally time-consuming in processing large-scale data due to the lack of an efficient framework of reduction theory. To alleviate this problem, we introduce the concept of granularity space to construct a unified representation of five typical reducts. Based on the unified representation, we propose two quick general reduction algorithms.

Firstly, to construct the general attribute reduction definition, we introduce the concept of granularity space and analyze the properties of its binary relation. Subsequently, by associating the indiscernibility relation of granularity space with the indiscernibility relation of five reducts, we construct the definition of general attribute reducts in the way of granularity space. Finally, by extending the positive region approximation to the granularity space, we develop two quick general reduction algorithms. Meanwhile, a series of analyses in aspects of theory and experiments are conducted to evaluate the effectiveness and efficiency of proposed algorithms. Two major contributions of this study are listed as follows. (1) We introduce a concept named granularity space to represent five attribute reductions in a unified framework; (2) We extend the positive region approximation to the granularity space and design a new acceleration strategy for general attribute reduction algorithms, which can expand the acceleration domain from the positive region to the universe of decision tables.

The rest of this paper is organized as follows. In section 2, we briefly review preliminary notions related to five types of representative reduction and

the classic definition of the general reduct. In addition, we review the process
 of the discernibility matrix-based reduction method. In section 3, we analyze
 95 flaws of the classic general reduct definition, and propose the granularity space
 as a candidate solution to those flaws. After that, we present the quick general
 heuristic reduction algorithms based on granularity space. Besides, we explain
 the advantage of proposed algorithms and their relationship to existing reduc-
 tion algorithms. In section 4, we conduct a series of experiments with several
 100 UCI data sets to evaluate the performance of proposed reduction algorithms.
 Finally, section 5 concludes this paper and brings some remarks about the work
 of this paper.

2. Preliminaries

In this part, we briefly review the concept of decision tables, the notions of
 105 five representative attribute reducts, and the definition of general reducts. Next,
 we brush up on the discernibility matrix-based reduction algorithm, which is the
 kin of two proposed algorithms.

The research object of rough sets theory is called the information system,
 which can be expressed as four-tuple, *i.e.*, (U, A, V, f) . Here U stands for the
 110 universe of discourse, a non-empty finite set of instances. A is the set of at-
 tributes, $V = \bigcup_{a \in A} V_a$ is the set of all attribute values, and $f : U \times A \rightarrow V$
 is an information function that maps an object in U to exactly one value in
 V_a . For $x \in U, a \in A$, we have $f(x, a) \in V_a$. In the classification problem, the
 information system contains two kinds of attributes, and it can be characterized
 115 by a decision table $DT = (U, C \cup D, V, f)$ with $C \cap D = \emptyset$, where an element
 of C is called a condition attribute, C is called the condition attribute set, an
 element D is called a decision attribute, and D is called the decision attribute
 set [46].

For the condition attribute set $B \subseteq C$, the indiscernibility relation $\text{IND}(B)$
 120 is defined by $\text{IND}(B) = \{(x, y) \mid x, y \in U, f(x, a) = f(y, a), \forall a \in B\}$. For an
 instance $x \in U$, the equivalence class of x , being represented as $[x]_B$, is described

by $\{y \mid y \in U, \langle x, y \rangle \in \text{IND}(B)\}$. The family of all equivalence classes of $\text{IND}(B)$, *i.e.*, the partition determined by B , is denoted by $U/\text{IND}(B)$ or simply U/B . X , a non-empty subset of U , is called a concept of U . The B -lower approximation
¹²⁵ $\underline{B}(X)$ and the B -upper approximation $\overline{B}(X)$ of the concept X are respectively defined by $\underline{B}(X) = \{x \in U \mid [x]_B \subseteq X\}$ and $\overline{B}(X) = \{x \in U \mid [x]_B \cap X \neq \emptyset\}$ respectively.

The classification ability of conditional attribute C is measured by the relation between $\text{IND}(C)$ and $\text{IND}(D)$, and there are some uncertain situations that objects x, y with the same value of conditional attributes perform differently in the decision attributes value. The uncertain situations are represented as the difference between two notions, *i.e.*, positive region and boundary region which are induced from indiscernibility relation. The B -positive region $\text{POS}_B(D)$ and B -boundary region $\text{BND}_B(D)$ are defined as

$$\begin{aligned}\text{POS}_B(D) &= \bigcup_{X \in U/D} \underline{B}(X), \\ \text{BND}_B(D) &= \bigcup_{X \in U/D} (\overline{B}(X) - \underline{B}(X)).\end{aligned}$$

$\text{POS}_B(D)$ consists of objects which perform consistently in decision; $\text{BND}_B(D)$ is comprised of objects which perform inconsistently in decision. The difference
¹³⁰ between them is taken as the measurement of the classification ability of B . In this viewpoint, an attribute set $B \subseteq C$ satisfying $\text{POS}_B(D) = \text{POS}_C(D)$ is meaningful for feature selection and knowledge representation. B , an attribute set satisfying $\text{POS}_B(D) = \text{POS}_C(D) \wedge (\forall B' \subset B, \text{POS}_{B'}(D) \neq \text{POS}_C(D))$, is called positive region preservation attribute reduct. After proposing the positive
¹³⁵ region preservation attribute reduct, many extensions of that are investigated. Considering the focus of paper, we list five typical reducts' definitions summarized in Ref [16] here.

Definition. *Given a decision table $DT = (U, C \cup D, V, f)$,*

- (1) *B is a positive region preservation reduct (denoted as PRPR) of C with respect to D if B satisfies $\text{POS}_B(D) = \text{POS}_C(D)$ and $\forall B' \subset B, \text{POS}_{B'}(D) \neq$*
¹⁴⁰

$\text{POS}_C(D)$;

- (2) B is a generalized decision preservation reduct (denoted as GDPR) of C with respect to D if B satisfies $\forall x \in U, \delta_B(x) = \delta_C(x)$ and $\forall B' \subset B, \exists x \in U, \delta_{B'}(x) \neq \delta_B(x)$, where $\delta_B(x) = \{f(y, D) | x \in U \wedge y \in [x]_B\}$;
- ¹⁴⁵ (3) B is a distribution preservation reduct (denoted as DPR) of C with respect to D if B satisfies $\forall x \in U, \mu_B(x) = \mu_C(x)$ and $\forall B' \subset B, \exists x \in U, \mu_{B'}(x) \neq \mu_B(x)$, where $\mu_B(x) = (P(D_1|[x]_B), P(D_2|[x]_B), \dots, P(D_{|U/D|}|[x]_B)), P(D_j|[x]_B) = \frac{|D_j \cap [x]_B|}{|[x]_B|}, x \in U, D_j \in U/D (j = 1, 2, \dots, |U/D|)$;
- (4) B is a maximum distribution preservation reduct (denoted as MDPR) of C with respect to D if B satisfies $\forall x \in U, \phi_B(x) = \phi_C(x)$ and $\forall B' \subset B, \exists x \in U, \phi_{B'}(x) \neq \phi_B(x)$, where $\phi_B(x) = \{D_j | \frac{|[x]_B \cap D_j|}{|[x]_B|} = \max_{k=1}^{|U/D|} \{ \frac{|[x]_B \cap D_k|}{|[x]_B|} \}\}$;
- ¹⁵⁰ (5) B is a relative discernibility relation preservation reduct (denoted as DR-PR) of C with respect to D if B satisfies $\text{IND}(B|D) = \text{IND}(C|D)$ and $\forall B' \subset B, \text{IND}(B'|D) \neq \text{IND}(B|D)$, where $\text{IND}(B|D) = \{\langle x, y \rangle \in U \times U | \wedge (\forall a \in B \rightarrow f(x, a) = f(y, a)) \vee f(x, D) = f(y, D)\}$.

Those five reducts can be taken as the special cases of the general reduct proposed by Yao et al. [29], which can be written as follows.

Definition. Given a decision table $DT = (U, C \cup D, V, f)$ and a certain property \mathbb{P} of DT , an attribute set $B \subseteq C$ is called a reduct of C if it satisfies the following three conditions:

- (1) *Evaluability condition*: the property can be represented by an evaluation function $e: 2^C \rightarrow (L, \preceq)$;
- (2) *Jointly sufficient condition*: $e(A) \preceq e(B)$;
- (3) *Individually necessary condition*: for any $B' \subset B, \neg(e(A \preceq e(B')))$.

¹⁶⁵ Here $e: 2^C \rightarrow (L, \preceq)$ is an evaluation or fitness function, which maps an attribute set to an element of a poset L equipped with the partial order relation \preceq , i.e., \preceq is reflexive, anti-symmetric and transitive.

Attribute reduction algorithms, approaches to obtaining a reduct or reducts, can be classified into two groups: the discernibility matrix-based algorithm and ¹⁷⁰ the heuristic algorithm. For the limitation of paper focus, here we only review

the discernibility matrix-based algorithm. A general discernibility matrix given by Miao et al. [15] is defined as follows.

Definition. Given a decision table $DT = (U, C \cup D, V, f)$, its discernibility matrix $DM = (DM(x, y))$ is a $|U| \times |U|$ matrix. $DM(x, y)$ for an object pair (x, y) is $\{a \mid \langle x, y \rangle \in \text{DIS}(C|D), f(x, a) \neq f(y, a), a \in C\}$, where $\text{DIS}(C|D)$ is the relative discernibility relation.

The relative discernibility relation of five typical reducts can be written as follows.

- (1) The relative discernibility relation of PRPR is defined by $\text{DIS}(C|D) = \{ \langle x, y \rangle \mid x, y \in \text{POS}_C(D) \wedge f(x, D) \neq f(y, D) \vee x \in \text{POS}_C(D) \wedge y \notin \text{POS}_C(D) \};$
- (2) The relative discernibility relation of GDPR is defined by $\text{DIS}(C|D) = \{ \langle x, y \rangle \mid \delta_C(x) \neq \delta_C(y) \};$
- (3) The relative discernibility relation of DPR is defined by $\text{DIS}(C|D) = \{ \langle x, y \rangle \mid \mu_C(x) \neq \mu_C(y) \};$
- (4) The relative discernibility relation of MDPR is defined by $\text{DIS}(C|D) = \{ \langle x, y \rangle \mid \phi_C(x) \neq \phi_C(y) \};$
- (5) The relative discernibility relation of DRPR is defined by $\text{DIS}(C|D) = \{ \langle x, y \rangle \mid f(x, C) \neq f(y, C) \wedge f(x, D) \neq f(y, D) \}.$

Based on the discernibility matrix, one can get the reduct through the following discernibility function. $DF(DM) = \bigwedge \{\bigvee (DM(x, y)) \mid \forall x, y \in U, DM(x, y) \neq \emptyset\}.$ The expression $\bigwedge \{\bigvee (DM(x, y))\}$ is the conjunction of all $\bigvee (DM(x, y))$ while $\bigvee (DM(x, y))$ is the disjunction of condition attributes in $DM(x, y).$ The discernibility function can be transformed to a reduced disjunctive form, and each conjunct of the reduced disjunctive form is a reduct or a superset of a reduct.

3. Granularity space and the quick general reduction algorithm

In this section, we represent five typical reducts in a unified way, and develop two quick general reduction algorithms.

In subsection 3.1, to show the reason why we introduce the granularity space,
200 we firstly analyze the flaws of the existing general reduct definitions in aspects
of the reduct definition construction and reduction algorithms designing. Mean-
while, we analyze the relationship between the indiscernibility relation of the
granularity space and the indiscernibility relation of five typical reduts. Based
on the common ground of two indiscernibility relations, we present five typical
205 reduts in the way of granularity sapce. At the end of this subsection, we show
the power and simplicity of the granularity space based general reduct definition
in comparison to the existing general reduct definitions.

In subsection 3.2, we extend the positive region approximation to the gran-
ularity space and design a new acceleration strategy called granularity approxi-
210 mation, which expands the acceleration domain from the positive region to the
universe of decision tables. Based on granularity approximation, we develop t-
wo quick general reduction algorithms and present the relationship between the
proposed reduction algorihtms and the existing reduction algorithms.

In subsection 3.3, we show the difference between the proposed algorithm-
215 s and the related work in Ref [18]. For a better understanding of proposed
algorithms, we provide a calculation example of obtaining positive region p-
reservation reduct using proposed algorithms.

3.1. Granularity space: an alternative of general attribute reduct definition

One of the most important parts in rough sets is the answer to what the
220 reduct is. The general reduct definition proposed by Yao et al. [29] has been
proven as a remarkable candidate. It's such a powerful definition that there are
so many researchers adopting it to accomplish the reduct definition construc-
tion and reduction algorithms designing. However, there are two flaws in Yao's
definition. Firstly, the focus on the subset of C limits the scale of information
225 we considered during data processing. Secondly, the definition of reduct, using
evaluation or fitness function to measure the relationship between different at-
tributes, limits the flexibility of reduction algorithms designing. Meanwhile, it
results in the complexity of theory for that only one more fitness function was

proposed before we could propose a new type of attribute reduction. To address
²³⁰ the flaws mentioned above, we would propose a new definition of general reduct.

Information granules naturally give rise to hierarchical structures: the same problem or system can be perceived at different levels of specificity (detail) depending on the complexity of the problem, available computing resources, and particular needs to be addressed [47]. In rough sets, information granules can be represented as a partition of the universe. For the convenience of writing, we denote the partition of the universe U as the granularity to emphasize the hierarchical structure of data. Given a non-empty finite set U , a granularity G of U is defined as

$$G = \{X_i \mid \bigcup_{X_i \subset G} X_i = U \wedge (X_i \cap X_j = \emptyset, i \neq j)\}.$$

The indiscernibility relation and the discernibility relation of G is defined as follows.

Definition 1. *Given a granularity G of a finite non-empty set U , the indiscernibility relation and the discernibility relation of G are defined by*

$$\text{IND}(G) = \{\langle x, y \rangle \mid \langle x, y \rangle \in X_i \times X_i \wedge X_i \in G\};$$

$$\text{DIS}(G) = \{\langle x, y \rangle \mid (\langle x, y \rangle \in X_i \times X_j, i \neq j) \wedge X_i, X_j \in G\}.$$

The indiscernibility relation of the granularity G of U is

- (1) symmetric, *i.e.*, for $\forall x \in U$, we have $\langle x, x \rangle \in \text{IND}(G)$;
- ²³⁵ (2) transitive, *i.e.*, for $\forall x, y, z \in U$, if $\langle x, y \rangle, \langle y, z \rangle \in \text{IND}(G)$, we have $\langle x, z \rangle \in \text{IND}(G)$;
- (3) reflexive, *i.e.*, for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{IND}(G)$, we have $\langle y, x \rangle \in \text{IND}(G)$.

On the other side, the discernibility relation of a granularity G of U is reflexive, *i.e.*, for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{DIS}(G)$, one can get $\langle y, x \rangle \in \text{DIS}(G)$. Another point needed to be reminded of is that $\text{IND}(G) \cup \text{DIS}(G) = \{\langle x, y \rangle \mid \langle x, y \rangle \in U \times U\}$.
²⁴⁰

Definition 2. *Granularity space can be represented in 2-tuple $GS = (G, O)$,*

where G is a granularity of a finite non-empty set U , O is a set of operators on elements of G . The ancestor granularity space of G is defined by $\text{ANC}(G) = (G, \{\cup\})$; the granularity subspace of G is defined by $\text{SPR}(G) = (G, \{\infty\})$, where $\infty U = \{P, Q \mid P \cup Q = U \wedge P \cap Q = \emptyset\}$.

Granularity space is the set of granularity generated from doing several operations specified by O on G . In particular, we specify $G \in (G, \{\infty\})$ and $G \in (G, \{\cup\})$. As a result, we can construct following lemma.

Lemma 1. Given a granularity G of a finite non-empty set U , for $\forall G' \in \text{ANC}(G)$, we have $\text{IND}(G') \supseteq \text{IND}(G), \text{DIS}(G') \subseteq \text{DIS}(G)$; for $\forall G' \in \text{SPR}(G)$, we have $\text{IND}(G') \subseteq \text{IND}(G), \text{DIS}(G') \supseteq \text{DIS}(G)$.

Proof. Taking into account $\text{IND}(G) \cup \text{DIS}(G) = U \times U$, $\text{IND}(G) \cap \text{DIS}(G) = \emptyset$, all we need to do for proving Lemma 1 true is proving true either $\forall G' \in \text{ANC}(G), \text{IND}(G') \supseteq \text{IND}(G)$ or $\forall G' \in \text{SPR}(G), \text{IND}(G') \subseteq \text{IND}(G)$. According to Definition 2, for $\forall G' \in \text{ANC}(G)$, it can be implied that for $\forall E \in G'$, there exists a set $PE \subseteq G$ such that $\bigcup_{P_i \in PE} P_i = E$. As a result, we can get $\emptyset \subseteq \{\langle x, y \rangle \mid x \in P_i \in PE, y \in \bigcup_{P_j \in PE - P_i} P_j, P_i, P_j \in PE, x, y \in E\} \Rightarrow \text{DIS}(G') \subseteq \text{DIS}(G)$. That is to say, Lemma 1 is true.

To enhance the understanding of concepts introduced above, here we provide a calculation example of the granularity space.

Example 1. Given $U = \{x_1, x_2, x_3\}, G = \{\{x_1\}, \{x_2, x_3\}\}$, we know $\text{ANC}(G) = (G, \{\cup\}) = \{G, U\}$. let Q denotes as $\{\{x_1\}, \{x_2\}, \{x_3\}\}$, we have $\text{SPR}(G) = (G, \{\infty\}) = \{G, Q\}$, $\text{IND}(U) = U \times U, \text{DIS}(U) = \emptyset$;

$\text{IND}(G) = \{\langle x_1, x_1 \rangle, \langle x_2, x_2 \rangle, \langle x_3, x_3 \rangle, \langle x_2, x_3 \rangle, \langle x_3, x_2 \rangle\}, \text{DIS}(G) = \{\langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \langle x_1, x_3 \rangle, \langle x_3, x_1 \rangle\}, \text{IND}(Q) = \{\langle x_1, x_1 \rangle, \langle x_2, x_2 \rangle, \langle x_3, x_3 \rangle\}, \text{DIS}(Q) = \{\langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \langle x_1, x_3 \rangle, \langle x_3, x_1 \rangle, \langle x_2, x_3 \rangle, \langle x_3, x_2 \rangle\}$. It is apparent that $\forall G' \in \text{ANC}(G), \text{IND}(G) \subseteq \text{IND}(G')$ i.e., $\text{DIS}(G) \supseteq \text{DIS}(G')$ and $\forall G' \in \text{SPR}(G), \text{DIS}(G) \subseteq \text{DIS}(G')$ i.e., $\text{DIS}(G) \supseteq \text{DIS}(G')$.

Theorem 1. Given a decision table $DT = (U, C \cup D, V, f)$ and a granularity $G = U/C$, for any type of reduct B , we have $U/B \in \text{ANC}(G)$.

$U/B \in \text{ANC}(G)$ can be induced from $B \subseteq C$. In other words, for $\forall B \subseteq C$, if $y \in [x]_C$, we have $y \in [x]_B$. Here we summarize up the useful conclusions drawn by analyzing the granularity space. The first is that, for $\forall B \subseteq C$,
²⁷⁵ we have $U/B \in \text{ANC}(U/C)$, $\text{IND}(B) \supseteq \text{IND}(C)$ and $\text{DIS}(B) \subseteq \text{DIS}(C)$. Re-viewing on the discernibility relation of five typical reducts, we can imply that there are some redundant elements in existing discernibility relation. Second-
²⁸⁰ ly, $\text{ANC}(U/C)$ contains all granularity induced by $B \subseteq C$, and it indicates the possibility of designing the reduction algorithms by obtaining some special granularity. Here we re-construct the discernibility relation and the indiscernibility relation of five reducts from granularity perspective.

Definition 3. *Given a decision table $DT = (U, C \cup D, V, f)$, the discernibility relation and the indiscernibility relation of reducts are defined as follows.*

$$\text{DIS}(PRPR) = \{\langle x, y \rangle \mid y \notin [x]_C \wedge (x, y \in \text{POS}_C(D) \wedge f(x, D) \neq$$

$$f(y, D) \vee x \notin \text{POS}_C(D) \wedge y \in \text{POS}_C(D))\}$$

$$\text{IND}(PRPR) = \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge (x, y \notin \text{POS}_C(D)$$

$$\vee x, y \in \text{POS}_C(D) \wedge f(x, D) = f(y, D))\}$$

$$\text{DIS}(GDPR) = \{\langle x, y \rangle \mid y \notin [x]_C \wedge \delta_C(x) \neq \delta_C(y)\}$$

$$\text{IND}(GDPR) = \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge \delta_C(x) = \delta_C(y)\}$$

$$\text{DIS}(DPR) = \{\langle x, y \rangle \mid y \notin [x]_C \wedge \mu_C(x) \neq \mu_C(y)\}$$

$$\text{IND}(DPR) = \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge \mu_C(x) = \mu_C(y)\}$$

$$\text{DIS}(MDPR) = \{\langle x, y \rangle \mid y \notin [x]_C \wedge \phi_C(x) \neq \phi_C(y)\}$$

$$\text{IND}(MDPR) = \{\langle x, y \rangle \mid y \in [x]_C \vee y \notin [x]_C \wedge \phi_C(x) = \phi_C(y)\}$$

$$\text{DIS}(DRPR) = \{\langle x, y \rangle \mid y \notin [x]_C \wedge (x \in \text{BND}_C(D) \vee y \in \text{BND}_C(D)$$

$$\vee x, y \in \text{POS}_C(D) \wedge f(x, D) \neq f(y, D))\}$$

$$\text{IND}(DRPR) = \{\langle x, y \rangle \mid y \in [x]_C \vee x, y \in \text{POS}_C(D) \wedge f(x, D) = f(y, D)\}$$

For the discernibility relation and the indiscernibility relation of five typical reducts, it is worth stressing that for $\forall Red \in \{PRPR, GDPR, DPR, MDPR, DRPR\}$,

$\text{IND}(Red)$ is

- 285 (1) symmetric, i.e., for $\forall x \in U, \langle x, \text{we have } x \rangle \in \text{IND}(Red)$,
(2) transitive, i.e., for $\forall x, y, z \in U$, if $\langle x, y \rangle, \langle y, z \rangle \in \text{IND}(Red)$, we have $\langle x, z \rangle \in \text{IND}(Red)$,
(3) reflexive, i.e., for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{IND}(Red)$, we have $\langle y, x \rangle \in \text{IND}(Red)$.

On the other side, the discernibility relation of Red , i.e. $\text{DIS}(Red)$, is reflexive, which means that for $\forall x, y \in U$, if $\langle x, y \rangle \in \text{DIS}(Red)$, $\langle y, x \rangle$ is an element of $\text{DIS}(Red)$. In addition, we also have $\text{DIS}(Red) \cap \text{IND}(Red) = \emptyset$, $\text{IND}(Red) \cup \text{DIS}(Red) = U \times U$. As a result, we can imply the following theorem.

Theorem 2. *Given a decision table $DT = (U, C \cup D, V, f)$ and $B \subseteq C$, we have*

- 295 (1) $\text{DIS}(PRPR) \subseteq \text{DIS}(B) \Leftrightarrow \text{POS}_B(D) = \text{POS}_C(D)$;
(2) $\text{DIS}(GDPR) \subseteq \text{DIS}(B) \Leftrightarrow \forall x \in U, \delta_B(x) = \delta_C(x)$;
(3) $\text{DIS}(DPR) \subseteq \text{DIS}(B) \Leftrightarrow \forall x \in U, \mu_B(x) = \mu_C(x)$;
(4) $\text{DIS}(MDPR) \subseteq \text{DIS}(B) \Leftrightarrow \forall x \in U, \phi_B(x) = \phi_C(x)$;
(5) $\text{DIS}(DRPR) \subseteq \text{DIS}(B) \Leftrightarrow \text{IND}(B|D) = \text{IND}(C|D)$.

300

Proof. According to definitions of related reducts and the discernibility matrix-based reduction algorithm, it is easy to know that items (2), (3), and (4) of Theorem 2 are true. So here we just prove the correctness of items (1) and (5).

(1) **Sufficiency:** We make an assumption that if $\text{DIS}(PRPR) \subseteq \text{DIS}(B)$, there 305 is a object $x \in \text{POS}_C(D)$ satisfying $x \notin \text{POS}_B(D)$. It can be inferred that there exists a object $y \notin [x]_C, f(y, D) \neq f(x, D)$ satisfying $y \in [x]_B$. Thus, we have $\langle x, y \rangle \notin \text{DIS}(B), \langle x, y \rangle \notin \text{DIS}(PRPR)$ and this conflicts with the definition of $\text{DIS}(PRPR)$. As a result, our assumption is not true, and we know $\text{DIS}(PRPR) \subseteq \text{DIS}(B) \Rightarrow \text{POS}_B(D) = \text{POS}_C(D)$.

310 **Necessity:** We make an assumption that if $\text{POS}_B(D) = \text{POS}_C(D)$, there exists a pair $\langle x, y \rangle \in \text{DIS}(PRPR)$ such that $\langle x, y \rangle \notin \text{DIS}(B)$. If $x \in \text{POS}_C(D) \wedge y \in \text{POS}_C(D) \wedge f(x, D) \neq f(y, D) \wedge \langle x, y \rangle \notin \text{DIS}(B)$, we know $x, y \notin \text{POS}_B(D)$ and $\text{POS}_B(D) \neq \text{POS}_C(D)$. That is conflicted with our assumption, i.e.,

$\text{POS}_B(C) = \text{POS}_C(D)$. If $x \in \text{POS}_C(D) \wedge y \notin \text{POS}_C(D)$, we know $x \in [y]_B$,
³¹⁵ and that implies $x \notin \text{POS}_B(D)$ and $\text{POS}_C(D) \neq \text{POS}_B(D)$. In summary, if
 $\text{POS}_B(D) = \text{POS}_C(D)$, then we have $\text{DIS}(PRPR) \subseteq \text{DIS}(B)$.

(5) **Sufficiency:** Assume that if $\text{DIS}(DRPR) \subseteq \text{DIS}(B)$, there exists a pair
 $\langle x, y \rangle \notin \text{IND}(C|D) \wedge \langle x, y \rangle \in \text{IND}(B|D)$. That means $\exists y \notin [x]_C, y \in [x]_B, f(x, D) \neq f(y, D)$. Then we know $\langle x, y \rangle \in \text{DIS}(DRPR), \langle x, y \rangle \notin \text{DIS}(B)$, and it conflicts with our assumption $\text{DIS}(DRPR) \subseteq \text{DIS}(B)$. Finally, we get $\text{DIS}(DRPR) \subseteq \text{DIS}(B) \Rightarrow \text{IND}(B|D) = \text{IND}(C|D)$.
³²⁰

Necessity: Assume that if $\text{IND}(B|D) = \text{IND}(C|D)$, we have $\exists \langle x, y \rangle \in \text{DIS}(RPR)$ such that $\langle x, y \rangle \notin \text{DIS}(B)$. If $y \notin [x]_C \wedge (x \in \text{BND}_C(D) \vee y \in \text{BND}_C(D))$, we have $\exists p \in [x]_C, \exists q \in [y]_C, f(p, C) \neq f(q, C), f(p, D) \neq f(q, D)$. Because $\langle x, y \rangle \in \text{IND}(B)$, we have $p \in [q]_B$. We get $\langle p, q \rangle \in \text{IND}(B|D)$ and $\langle p, q \rangle \notin \text{IND}(C|D)$. It is conflicted with $\text{IND}(B|D) = \text{IND}(C|D)$. If $x, y \in \text{POS}_C(D) \wedge f(x, d) \neq f(y, d)$, according to $\langle x, y \rangle \in \text{IND}(B)$, it is obvious that $\langle x, y \rangle \in \text{IND}(B|D)$. Noticing $\langle x, y \rangle \notin \text{IND}(C|D)$, we know that our assumption is not true. In summary, $\text{IND}(B|D) = \text{IND}(C|D) \Rightarrow \text{DIS}(DRPR) \subseteq \text{DIS}(B)$.
³²⁵

³³⁰

For the convenience of writing, we specify that *Red* is an element of $\{PRPR, GDPR, DPR, MDPR, DRPR, DRPR\}$ if there is no additional declaration. According to Theorem 1, 2 and Lemma 1, we can construct an unified definition of general attribute reduct using granularity space.

³³⁵ **Theorem 3.** *Given a decision table $DT = (U, C \cup D, V, f)$ and the target reduct Red, B is a Red of C iff B satisfies the following conditions.*

- (1) $U/B \in \text{ANC}(U/C) \cap \text{SPR}(\text{TGran}(Red))$;
- (2) $\forall B' \subset B, U/B'$ does not satisfy condition (1).

where $\text{TGran}(Red) = U/\text{IND}(Red)$.

³⁴⁰ In fact, item (1) can be written in short as $U/B \in \text{SPR}(\text{TGran}(Red))$ because of $\forall B \subseteq C, U/B \in \text{ANC}(U/C)$. Theorem 3 makes the reduct space more accurate by replacing $\text{ANC}(U/C) \cap \text{SPR}(U)$ with $\text{ANC}(U/C) \cap \text{SPR}(\text{TGran}(Red))$.

The core difference of Theorem 3 to the existing general reduct definitions is the attention paid toward the objects' distribution, *i.e.*, granularity space, instead of the relationship between different attribute sets. There
³⁴⁵ are three improvements of the general reduct defined by the objects' distribution. Firstly, the information space we take into consideration is greater, *i.e.*, $|\text{ANC}(U/C)| \geq |\{U/B | B \subseteq C\}|$. It is obvious that $\forall B \subseteq C, U/B \in \text{ANC}(U/C)$, and $\exists G \in \text{ANC}(U/C)$ satisfies $G \neq U/B$, where B is a subset of C . It can be
³⁵⁰ seen from the data in Table 1 that, the information space may vary when different reduct definitions are taken into account. *i.e.*, if we take Yao's definition as the reference of attribute reduction, the information space we take care of can be expressed as $\{U, U/\{a_1\}, U/\{a_2\}, U/\{a_1, a_2\}\}$; if we consider granularity space as the way of reducts representation, the information space should be
³⁵⁵ $\{U, U/\{a_1\}, U/\{a_2\}, U/\{a_1, a_2\}, \{\{x_1, x_3\}, \{x_2\}\}\}$. The expansion of information space makes it possible to design the more efficient reduction algorithms, and GS is a good example. Secondly, general reduct defined by granularity s-

Table 1. A decision table for indicating the information scale of granularity space

	U	a_1	a_2	d
x_1	0	0	0	
x_2	1	0	1	
x_3	1	1	0	

pace is independent of heuristic functions. That is to say, there is no need to design new heuristic functions for proposing new type of attribute reduction.
³⁶⁰ All we need for the construction of a new type of attribute reduction is extending or redefining the definition of target granularity space, *i.e.*, the definition of $\text{TGran}(\text{Red}) \cap \text{ANC}(U/C)$. When it comes to attribute reduction approaches, it is more flexible in heuristic functions during designing heuristic attribute reduction algorithms based on granularity space because the input of reduction algorithms is definite, *i.e.*, a given granularity space. For example, GS can work well for obtaining five typical reducts with three classical heuristic functions,
³⁶⁵ *i.e.*, dependency degree, consistency, and entropy. For the limitation of the fo-

cus of this paper, we do not further explore it here. Thirdly, we can expand the acceleration domain from positive region into the universe and it is helpful to increase the efficiency of reduction algorithms. We would explain it in Section 3.2.

Keeping in mind that $\text{IND}(Red)$ is an equivalence relation, we can construct Algorithm 3.1 for computing $\text{TGran}(Red)$. First step of CTGA is getting the

Algorithm 3.1 Calculating the target granularity algorithm(CTGA)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct $Red \in \{PRPR, GDPR, DPR, MDPR, DRPR\}$

Output: Target granularity of relative reduct.

```

1: Compute  $TG = U/C$ 
2: for  $ec1 \in TG$  do
3:   for  $ec2 \in TG - ec1$  do
4:     if  $\langle x, y \rangle \in \text{IND}(Red) : x \in ec1, y \in ec2$  then
5:        $ec1 := ec1 \cup ec2$ 
6:        $TG := TG - ec2$ 
7:     end if
8:   end for
9: end for
10: return  $TG$ 

```

partition U/C because $\text{TGran}(Red)$ is an element of $\text{ANC}(U/C)$. Then, CTGA scans all the combinations of $\langle e1, e2 \rangle, e1, e2 \in U/C$, and merges equivalence classes if $\langle x, y \rangle \in \text{IND}(Red) : x \in e1, y \in e2, e1 \neq e2$. After the merging step, the variable TG stores $\text{TGran}(Red)$. The upper bound of time complexity of computing $\text{TGran}(Red)$ using CTGA is $O(|U||C \cup D| + |U|^2)$. Noticing that the time complexity of merging step is $O(|U|^2)$, we design key converters to increase the efficiency of merging step, which generated keys are then hashed by a function implemented in python's dicitonary.

Definition 4. Given a decision table $(U, C \cup D, V, f)$ and $U/D = \{D_1, D_2, \dots, D_n\}$, some hash functions are defined as

$$(1) h(x, PRPR) = \begin{cases} i, & \text{if } x \in \text{POS}_C(D) \wedge f(x, D) = D_i \\ n+1, & \text{otherwise} \end{cases};$$

(2) $h(x, GDPR) = \delta_C(x);$

$$(3) \ h(x, DPR) = \mu_C(x);$$

$$(4) \ h(x, MDPR) = \phi_C(x);$$

$$(5) \ h(x, DRPR) = \begin{cases} i, & \text{if } x \in \text{POS}_C(D) \wedge f(x, D) = D_i \\ f(x, C), & \text{otherwise} \end{cases}.$$

Along this way, we can construct an auxiliary key converter $H_a(h(x, Red))$,
 390 which maps $h(x, Red)$ to a set of objects, for storing the target granularity TGran(Red). The key converters h and H_a can be constructed in the time complexity of $O(|U||C \cup D|)$ and $O(|U|)$. As a result, the upper bound of time complexity of computing TGran(Red) using CTGHF is $O(|U||C \cup D| + 2|U|)$.

Algorithm 3.2 Calculating target granularity using key converters(CTGKC)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct $Red \in \{PDR, GDPR, DPR, MDPR, DRPR\}$

Output: Target granularity of relative reduct in hash function way.

```

1: Compute  $U/C$ 
2: for  $ec \in U/C$  do
3:    $t := h(x, Red), H_a(t) := H_a(t) \cup \{x\}$ , where  $x \in ec$ 
4: end for
5: return  $H_a$ 
```

3.2. Granularity search: an efficient general reduction method

395 In this part, we focus on the efficient reduction algorithms. Based on the unified representation of five typical reducts, we present a heuristic function called granularity approximation for efficient attribute reduction algorithms. Subsequently, we develop two quick general attribute reduction algorithms.

Definition 5. Given a decision table $DT = (U, C \cup D, V, f)$ and a granularity G 400 of U , the granularity approximation of G in U/B is defined as $\text{GA}(U/B, G) = \bigcup \{[x]_B \mid x \in U, [x]_B \subseteq [x]_G\}$, where $[x]_G$ stands for a set of objects that belong to the same set in G .

If we take G as equivalence classes determined by an attributes set Z , granularity approximation can be written as $\text{POS}_B(Z)$. It is consistent with the uncertainty processing measurement in Pawlak attribute reduction. Obviously,
 405

if $\text{GA}(U/B, \text{TGran}(\text{Red})) = U$, we have $U/B \in \text{SPR}(\text{TGran}(\text{Red}))$. Thus, the general reduct definition 3 can be re-written as follows.

Theorem 4. *Given a decision table $DT = (U, C \cup D, V, f)$ and the target granularity $\text{TGran}(\text{Red})$, B is a Red of C iff B satisfies*

- (1) $\text{GA}(U/B, \text{TGran}(\text{Red})) = U$
- (2) $\forall B' \subset B, \text{GA}(U/B', \text{TGran}(\text{Red})) \neq U$.

For the granularity G of U , it is easy to know that $\forall P \subseteq Q \subseteq C, \text{GA}(U/P, G) \subseteq \text{GA}(U/Q, G)$. According to Definition 5, the attribute significance for given attribute sets can be defined as follows.

Definition 6. *Given $DT = (U, C \cup D, V, f)$ and target granularity G , the weeded significance of $a \in B \subseteq C$ in B for G is defined as $\text{Sig}^-(a, B, G) = |\text{GA}(U/B, G) - \text{GA}(U/B - \{a\}, G)|$; the joined significance of $a \in C - B$ in B for G is defined as $\text{Sig}^+(a, B, G) = |\text{GA}(U/B \cup \{a\}, G) - \text{GA}(U/B, G)|$, where $|S|$ represents the cardinality of set S .*

To increase the computation efficiency of the significance function, we explore the faster approach with the deletion of objects unrelated to the calculation. For the convenience of writing, we use $\text{Sig}^+(a, B, G, U)$ to represent the attribute significance, which denotes the value of the significance measure on the universe U . One can prove the following theorem of rank preservation.

Theorem 5. *Given a decision table $DT = (U, C \cup D, V, f)$, an attribute set $B \subseteq C$, a granularity G of U and a set $U' = U - \text{GA}(U/B, G)$, for $\forall a, b \in C - B$, if $\text{Sig}^+(a, B, G, U) \leq \text{Sig}^+(b, B, G, U)$, we have $\text{Sig}^+(a, B, G, U') \leq \text{Sig}^+(b, B, G, U')$.*

Proof. *From the definition of $\text{Sig}^+(a, B, G, U)$, we know that its value only depends on the function $\text{GA}(U/B, G)$. Since $U' = U - \text{GA}(U/B, G)$, one can know $\text{GA}(U'/B, G) = \text{GA}(U/B \cup \{a\}, G) - \text{GA}(U/B, G)$. Therefore, we have*

$$\frac{\text{Sig}^+(a, B, G, U)}{\text{Sig}^+(a, B, G, U')} = \frac{|\text{GA}(U/B \cup \{a\}, G, U) - \text{GA}(U/B, G, U)|}{|\text{GA}(U/B \cup \{a\}, G, U') - \text{GA}(U/B, G, U')|} = 1$$

So $\text{Sig}^+(a, B, G, U) > \text{Sig}^+(b, B, G, U) \Leftrightarrow \text{Sig}^+(a, B, G, U') > \text{Sig}^+(b, B, G, U')$.

430 Now we can construct the attribute reduction algorithm based on granularity approximation as Algorithm 3.3, *i.e.*, GS¹. The reason why we termed

Algorithm 3.3 Granularity Search(GS)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct Red

Output: A reduct of C

```

1: Compute the target granularity  $G$  according to algorithm 3.2;
2: Compute  $\text{Sig}^-(a_k, C, G, U), k \leq |C|$ 
3: Put  $a_k$  into  $core$ , where  $\text{Sig}^-(a_k, C, G, U) > 0$ 
4:  $red := core, i := 0, U_0 := U$ 
5: while  $|U_i| > 0$  do
6:   Calculate  $a_{max} : a_{max} = \arg \max_{a \in C - red} \text{Sig}^+(a, red, G, U_{i+1})$ 
7:    $red := red \cup \{a_{max}\}$ 
8:   Compute  $U_{i+1} := U_i - GA(U_i / red, G, U_i)$ 
9:    $i := i + 1;$ 
10: end while
11: return  $red$ 

```

Algorithm 3.3 as granularity search is to emphasize the intuition of finding a granularity in the given space. GS consists of four parts: the computation of target granularity, the calculation of core attributes, the selection of attribute with maximal significance, termination judgment. The complexity of computing TGran(Red) is $O(|U||C \cup D| + 2|U|)$; the complexity of calculating core attributes is $O(|U||C|^2)$; the complexity of remaining steps is $O(\sum_{i=1}^{|C|-cr} |U_i|(|cr| + i + 1))$, where cr denotes the set of core attributes. According to the attribute reduction algorithms in Ref [34, 18], the granularity search algorithm can be simplified to further increase the time efficiency by removing the part of computing core attributes and the attribute with maximal significance. For convenience, we denote granularity search without computing core attributes and the attribute with maximal significance as GSV.

We present the comparison of the upper bound of relevant algorithms time complexity in Table 2, in which “/” denotes that the step do not exist in this algorithm. QGARA-FS and QGARA-BS are general reduction algorithms pro-

¹ This algorithm is not related to the granularity search in granularity computing area

Algorithm 3.4 A Granularity Search Variant(GSV)

Input: decision table $DT = (U, C \cup D, V, f)$ and target reduct Red
Output: A reduct of C

```
1: Compute the target granularity  $G$  according to algorithm 3.2;  
2:  $red := \emptyset, i := 0$  and  $U_0 := U$   
3: while  $|U_i| > 0$  do  
4:    $flag := False$   
5:   for  $a \in C - red$  do  
6:     if  $Sig^+(a, red, G, U_i) > 0$  then  
7:        $flag := True$   
8:        $red := red \cup \{a\}$   
9:       Compute  $U_{i+1} := U_i - GA(U_i/red, G, U_i)$   
10:       $i := i + 1$   
11:    end if  
12:   end for  
13:   if  $flag$  is False and  $|U_i| > 0$  then  
14:      $red := red \cup \{a_0\}$ , where  $a_0$  is an arbitrary attribute of  $C$   
15:   end if  
16: end while  
17: return  $red$ 
```

posed in Ref [18]. Taking into consideration the reduction results of QGARA-BS and GSV are partly determined by the attribute order scanned, we divide four algorithms into two groups for comparison. One is GS *vs.* QGARA-FS;
450 another is GSV *vs.* QGARA-BS. The upper bound of time complexity of GS is $O(|U||C| + |U||C|^2 + \sum_{i=1}^{|C|-cr} |U_i|(|cr| + i + 1))$, where cr denotes core attributes. However, the upper bound of time complexity of QGARA-FS is $O(|U||C| + |U||C|^2 + \sum_{i=1}^{|C|-cr} |U|(|cr| + i + 1))$. Obviously, the time complexity of GS is lower than that of QGARA-FS. Meanwhile, the upper bound of time
455 complexity of GSV is $O(|U||C| + (\sum_{i=1}^{|C|} |U_i| \times i))$, which is lower than the upper bound of time complexity of QGARA-BS, *i.e.*, $O(|U||C| + \sum_{i=1}^{|C|} |U|(|C| - i + 1))$.

When it comes to the relationship to the existing heuristic functions, reduction algorithms and acceleration mechanisms of the heuristic reduction algorithm, the granularity approximation is an extension of a dependency degree; two proposed algorithms, *i.e.*, GS and GSV, can be taken as the improved edition
460 of algorithms proposed in Ref. [18], and the reason why GS and GSV are more

Table 2. The complexity description

Algorithms	Compute TGran	Compute core	Iteration structure
QGARA-FS	$O(U C)$	$O(C ^2 U)$	$O(\sum_{i=1}^{ C -cr} U (cr + i))$
QGARA-BS	$O(U C)$	/	$O(\sum_{i=1}^{ C } U (C - i + 1))$
GS	$O(U C)$	$O(C ^2 U)$	$O(\sum_{i=1}^{ C -cr} U_i (cr + i))$
GSV	$O(U C)$	/	$O(\sum_{i=1}^{ C } U_i \times i)$

efficient is the reduct defined from granularity perspective; as an acceleration strategy, the granularity approximation can be taken as the extension or the
465 special case of the positive region approximation. Besides, GS and GSV are kin to discernibility matrix-based reduction algorithms, which pay attention to the discernibility relation of reducts instead of the indiscernibility relation of reducts. It is worth trying to apply GS and GSV to address the problems where discernibility matrix-based reduction algorithms have got applied.

470 3.3. The analysis of granularity space

In this part, the relation and difference between the perspective of granularity and the relative discernibility relation are analyzed. Furthermore, attribute reduction processes related to two perspectives are also compared. In the end,
475 to provide principled intuition of proposed algorithms, we show the process of GS for obtaining the positive region preservation reduct.

For the convenience of comparison and the consistency of writing with Ref [18], let $\Delta \in \{PRPR, GDPR, DPR, MDPR, DRPR\}$ denote as the specific type of reduct, general attribute reduct can be defined in the way of relative discernibility relation as follows.

480 **Definition.** Given the general decision table $DT_\Delta = (U, C \cup D, V_\Delta, f_\Delta)$, the attribute subset $B \subseteq C$ is a Δ -reduct of C with respect to D , iff it satisfies the following two conditions:

- (1) $\text{IND}(B|D_\Delta) = \text{IND}(C|D_\Delta)$
- (2) $\forall B' \subset B, U/B'$ does not satisfy condition (1).

485 D_Δ is an attribute satisfying $U/D_\Delta = \text{TGran}(\Delta)$.

In comparison to $\text{IND}(B|D_\Delta) = \text{IND}(C|D_\Delta)$, $\text{GA}(U/B, \text{TGran}(Red)) = U$ is simpler because of $|U/C| \leq |U|$ and $|\text{IND}(C|D)| \leq |U|^2$. In addition, the granularity based reduct definition is more intuitive. For the calculation of general reduct, based on relative discernibility relation, $B \subseteq C$ is a superset of Δ -reduct if $W_\Delta(B|D) = W_\Delta(C|D)$, where $W_\Delta(B|D) = |U|^2 - \overline{W_\Delta(D)} - \overline{W_\delta(B)} + \overline{W_\Delta(D \cup B)} = |U|^2 - \sum_{X_i \in U/D} |X_i|^2 - \sum_{X_j \in U/B} |X_j|^2 + \sum_{X_k \in U/B \cup D} |X_k|^2$; based on the granularity space, it is $\text{GA}(U/B, U/D_\Delta) = U$ for the determination of the superset of Δ -reduct, where $\text{GA}(U/B, U/D) = \bigcup\{[x]_B | [x]_B \subseteq [x]_{D_\Delta}\} = \text{POS}_B(D_\Delta)$. Here we provide an example of presenting the difference between two general reduct definitions and reduction process.

Example 2. A calculation example of the positive region preservation attribute reduction.

Table 3. A decision table

U	a_1	a_2	a_3	a_4	a_5	a_6	d
x_1	0	0	0	0	0	0	1
x_2	1	0	0	0	0	0	2
x_3	1	1	0	0	0	0	1
x_4	1	1	0	0	0	0	2
x_5	1	1	0	0	1	1	1
x_6	1	1	0	0	1	1	2
x_7	1	1	0	1	1	1	1
x_8	1	1	0	1	1	1	2
x_9	1	1	0	1	1	1	2
x_{10}	1	1	1	1	1	1	2
x_{11}	1	1	1	1	1	1	3

For Table 3, we have $U = \{x_1, x_2, \dots, x_{11}\}$, $C = \{a_1, a_2, \dots, a_6\}$, $D = \{d\}$, and $U/C = \{X_1, X_2, X_3, X_4, X_5, X_6\} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$, $U/D = \{P_1, P_2, P_3\} = \{\{x_1, x_3, x_5, x_7\}, \{x_2, x_4, x_6, x_8, x_9, x_{10}\}, \{x_{11}\}\}$.

Assume that we know $\text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, \dots, x_6\}\}$, and we want to express $B = \{a_1, a_2\}$ is a superset of the positive region preservation reduct, it can be written in relative discernibility realtion way, *i.e.*, $\text{IND}(B|D_\Delta) = \{\langle x_1, x_2 \rangle, \langle x_2, x_1 \rangle, \langle x_1, x_3 \rangle, \langle x_3, x_1 \rangle, \dots\} = \text{IND}(C|D_\Delta)$, or in the

way of granularity space, *i.e.*, $U/B \in \text{SPR}(\text{TGran}(Red))$, $U/B = \text{TGran}(Red)$. For the intuition, $U/P = \text{TGran}(PRPR)$, $U/C, U/P$ and U/B can be drawn as Figure1, in which objects contained in a rectangle belong to the same equivalence class. From the relation of $U/P, U/B$, we can easily know B is a positive region preservation reduct. As mentioned above, the general reduct defined by granularity space is simpler and more intuitive than that defined by the relative indiscernibility relation.

x_1	x_1	x_1
x_2	x_2	x_2
x_3	x_3	x_3
x_4	x_4	x_4
x_5	x_5	x_5
x_6	x_6	x_6
x_7	x_7	x_7
x_8	x_8	x_8
x_9	x_9	x_9
x_{10}	x_{10}	x_{10}
x_{11}	x_{11}	x_{11}
U/C	U/P	U/B

Fig 1. Granularity comparison

To help the understanding of the proposed algorithms, here we show the process of algorithm 3.3 for obtaining a positive region preservation reduct.

515 An example of algorithm 3.2 for computing the target granularity.

Considering $X_1 \subset \text{POS}_C(D)$, we have $\forall x \in X_1, h(x, PRPR) = f(x_1, d) = 1, H_a(1).add(x_1) = \{x_1\}$.

Considering $X_2 \subset \text{POS}_C(D)$, we have $\forall x \in X_2, h(x, PRPR) = f(x_2, d) = 2, H_a(2).add(x_2) = \{x_2\}$.

520 Considering $X_3 \subset \text{BND}_C(D)$, we have $\forall x \in X_3, h(x, PRPR) = 4, H_a(4).add(X_3) = \{x_3, x_4\}$.

Considering $X_4 \subset \text{BND}_C(D)$, we have $\forall x \in X_4, h(x, PRPR) = 4, H_a(4).add(X_4) = \{x_3, x_4, x_5, x_6\}$.

525 Considering $X_5 \subset \text{BND}_C(D)$, we have $\forall x \in X_5, h(x, PRPR) = 4, H_a(4).add(X_5) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

Considering $X_6 \subset \text{BND}_C(D)$, we have $\forall x \in X_6, h(x, PRPR) = 4, H_a(4).add(X_6) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$.

Finally, we get $H_a(1) = \{x_1\}, H_a(2) = \{x_2\}, H_a(4) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$ and $\text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}\}$.

530 An example for calculating core attributes.

According to Algorithm 3.2, we get $G = \text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, \dots, x_{11}\}\}$. Firstly, we compute the partition of U determined by C , i.e., $U/C = \{X_1, X_2, X_3, X_4, X_5, X_6\} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$. Secondly, we calculate the significance of every element of C to determine whether the element is a core attribute or not. For attribute a_1 , we have $U/C - \{a_1\} = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $Sig^-(a_1, C, G) = |U - \{x_3, \dots, x_{11}\}| = 2$. According to $Sig^-(a_1, C, G) > 0$, we know that a_1 is a core attribute. For attribute a_2 , we have $U/C - \{a_2\} = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $Sig^-(a_2, C, G) = |U - \{x_1, x_5, x_6, \dots, x_{11}\}| = 3$.

540 According to $Sig^-(a_2, C, G) > 0$, we know that a_2 is a core attribute. For attribute a_3 , we have $U/C - \{a_3\} = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9, x_{10}, x_{11}\}\}$ and $Sig^-(a_3, C, G) = 0$. As a result, we know a_3 is not a core attribute. For attribute a_4 , we have $U/C - \{a_4\} = \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_5, x_6, x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $Sig^-(a_4, C, G) = 0$. So we know a_4 is not a core attribute. For attributes a_5 and a_6 , we have $U/C - \{a_5\} = U/C - \{a_6\} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_7, x_8, x_9\}, \{x_{10}, x_{11}\}\}$ and $Sig^-(a_5, C, G) = Sig^-(a_6, C, G) = 0$. Thus, neither a_5 nor a_6 is not a core attribute. Finally we get $core = \{a_1, a_2\}$

Example of algorithm3.3 for obtaining a positive region preservation reduct.

550 Now we get $G = \text{TGran}(PRPR) = \{\{x_1\}, \{x_2\}, \{x_3, x_4, \dots, x_6\}\}$ and $reduct = core = \{a_1, a_2\}$. As a result, we can get $U = U - \text{GA}(reduct) = \emptyset$. Because U is \emptyset , Algorithm3.3 outputs $reduct = \{a_1, a_2\}$.

4. Experiments and analyses

The objective of following experiments in this section is to show the effectiveness and the efficiency of the attribute reduction algorithms, *i.e.*, GS and GSV. Experiments are divided into three aspects. First, we employed 12 data sets in Table 4 to verify the performance of time consumption of GS, GSV, QGARA-FS, and QGARA-BS. Then, the computational time of algorithms GS, GSV, QGARA-FS and QGARA-BS with the increase of the size of objects and attributes was compared. Finally, we evaluated the classification accuracy of reducts generated by general attribute reduction algorithms using the naive bayes classifier and the decision tree classifier.

We carried out all the attribute reduction algorithms in experiments on a personal computer with Windows 10, Intel(R) Core(TM) CPU i5-8265U 1.60GHZ and 8GB RAM memory. The software used was Visual Studio Code 1.3.8, and the programming language was python 3.7.

Table 4. UCI Data Sets

ID	Data sets	cases	attributes	classes	$\gamma_C(D)$
1	Shuttle	58000	10	7	0.230
2	Mushroom	5644	23	6	0.536
3	Tic	9822	86	2	0.968
4	Segmentation	2310	20	7	0.989
5	Pima-indians-diabetes	768	9	2	0.995
6	Splice	3190	61	3	0.999
7	Dermatology	358	34	6	1
8	Wdbc	569	31	2	1
9	CNAE9	1080	856	9	1
10	Semeion	1593	267	10	1
11	DNA	2000	181	3	1
12	Connect4	67557	43	3	1

The data sets used in experiments are all downloaded from UCI repository of machine learning data sets [48] whose basic information is outlined in Table 4. For the sake that reduction algorithms can address only symbolic data, data sets containing continuous attributes ,*i.e.*, Segmentation, Pima-indians-

diabetes, and Wdbc, were preprocessed by equal-width discretization algorithms. For data sets with missing values, *i.e.*, Mushroom, we removed the objects with missing values to achieve uniform treatment of all data sets. The last column of Table 4, *i.e.*, $\gamma_C(D)$, stands for the positive region dependency degree
575 $|\text{POS}_C(D)|/|U|$. The data set is consistent when $\gamma_C(D) = 1$; otherwise, the data set is inconsistent. As shown in Table 4, Shuttle, Mushroom, Tic, Segmentation, Pima-indians-diabetes, and Splice are inconsistent and the other 6 data sets are consistent. Taking into consideration the similar results of five types of reducts under the general reduction algorithms, we mainly taken positive region
580 preservation reduction and relative discernibility relation preservation reduction results to verify the difference of four reduction algorithms.

4.1. Efficiency comparison of four general attribute reduction algorithms

In this subsection, to show the time efficiency of proposed algorithms, we presented the time consumption of four attribute reduction algorithms in obtaining reducts, and experiments results were shown in Tables 5 and 6.
585

Table 5 indicated the computational time of QGARA-FS, QGARA-BS, GS, and GSV for obtaining a positive region preservation reduct on 12 data sets. We can see that GSV was the fastest of four attribute reduction algorithms, and GS was faster than QGARA-FS. From results of experiments on both consistent
590 and inconsistent decision tables, the computational time of four algorithms in obtaining positive region preservation reduct followed this order: $\text{QGARA-FS} \geq \text{GS}$, $\text{QGARA-FS} > \text{GSV}$. GS performed better than QGARA-BS on small data sets. However, in processing the large scale data, it consumed more time than QGARA-BS for that the computation of the attribute with maximal significance
595 was time-consuming. For most of the cases in experiments, the computational time of GS can reduce over half the computation time of QGARA-FS, such as data sets 2(Mushroom), 3(Tic), 4(Segmentation), etc. In the same condition, GSV can reduce over half of the computation time of QGARA-BS, such as data sets 5(Pima-indians-diabetes), 7(Dermatology), 9(CNAE9), etc. In summary,
600 for calculating positive region preservation reducts on the consistent and in-

Table 5. Time consumption for obtaining PRPR.

ID	Time of QGARA-FS(s)	Time of QGARA-BS(s)	Time of GS(s)	Time of GSV(s)
1	9.326	7.967	6.562	1.676
2	11.788	1.990	1.816	0.213
3	192.414	18.784	17.890	1.581
4	1.215	0.772	0.521	0.117
5	0.123	0.109	0.067	0.018
6	27.328	3.790	5.638	0.406
7	1.139	0.216	0.233	0.024
8	1.533	0.272	0.259	0.040
9	1095.294	67.414	129.634	2.199
10	162.241	13.839	35.478	2.247
11	94.026	9.592	21.678	3.702
12	889.698	61.010	137.170	17.500

consistent decision tables, the general attribute reduction algorithms proposed in this paper, *i.e.* GS and GSV, were more efficient than the existing general attribute reduction algorithms, *i.e.*, QGARA-FS and QGARA-BS.

Table 6. Time consumption for obtaining DRPR.

ID	Time of QGARA-FS(s)	Time of QGARA-BS(s)	Time of GS(s)	Time of GSV(s)
1	10.885	9.831	7.489	4.103
2	3.640	2.419	2.064	1.864
3	200.923	19.734	19.231	3.168
4	1.423	0.925	0.648	0.244
5	0.157	0.140	0.099	0.044
6	29.326	4.279	6.016	0.794
7	1.145	0.263	0.258	0.051
8	1.570	0.357	0.312	0.065
9	1152.034	70.652	129.052	2.061
10	175.911	14.870	36.988	2.057
11	95.743	10.190	20.596	3.522
12	894.983	60.119	132.542	16.472

Table 6 shows the time consumption of four general reduction algorithms for obtaining a relative discernibility relation preservation reduct. For the consistent decision table, a positive region preservation reduct is also a relative discernibility relation preservation reduct. Thus, the results of the time

consumption of four general reduction algorithms on six consistent data sets were similar to the statistics of Table 5. For the time consumption of general reduction algorithms on six inconsistent data sets, we can know that the computational time of GS was less than that of QGARA-FS, and the same condition to GSV and QGARA-BS. In brief, the results of Table 6 were consistent with the observations of Table 5. In summary, for calculating relative discernibility relation preservation reduct on the consistent and inconsistent decision tables, the general reduction algorithms proposed in this paper were more efficient than the existing general reduction algorithms.

4.2. Comparison of reduction algorithms for different proportion data sets

In this subsection, to further compare the efficiency of general reduction algorithms, we compared the computational time of QGARA-FS, QGARA-BS, GS, and GSV for obtaining a positive region preservation reduct with the increase of the size of objects and the size of attributes.

Figure 2 shows detailed change trends of the time consumption of each algorithm for obtaining a positive region preservation reduct with the number of objects increasing. In Figure 2, the x-coordinate denotes the percentages of objects to the universe, while the y-coordinate concerns the time consumption of algorithms. We employed 8 data sets with different scale (Mushroom, Tic, Splice, Dermatology, Wdbc, CNAE9, DNA, and Connect4) to verify the performance of time consumption of QGARA-FS, QGARA-BS, GS, and GSV. Generally speaking, the computational time of four algorithms increased with the increase of the percentages of objects to the universe. The same as tables 5 and 6, GS was more efficient than QGARA-FS and GSV was faster than QGARA-BS. When dealing with the same UCI data sets, it is often the case that the computational time of GS was less than that of QGARA-FS, and equal to that of QGARA-BS for small scale data. But in presence of large-scale data sets, QGARA-BS performed better than GS. It can be observed in many data sets, such as Figure 2 (c), (e), (g). The computational time of GSV was less than that of the other three general reduction algorithms. The computational

time of QGARA-FS increased distinctly in comparison to GS when the number of objects was increasing; the computational time of QGARA-BS increased distinctly in comparison to GSV when the number of objects was increasing.

In Figure 3, the x-coordinate pertains to the percentages of attributes to the conditional attributes of the data set, while the y-coordinate concerns the time consumption of algorithms. We taken 8 data sets (Mushroom, Tic, Segmentation, Splice, Dermatology, Wdbc, CNAE9, and DNA) to verify the performance of the computational time of QGARA-FS, QGARA-BS, GS, and GSV for obtaining a relative discernibility realtion preservation reduct. The result of QGARA-FS, QGARA-BS, GS, and GSV was similar to the result induced from Figure 2.

4.3. Comparison of classification accuracy for general attribute reduction algorithms

As we know, there are many factors to the diversity of reducts obtained by reduction algorithms, such as reduction criterion, search strategy, and heuristic functions used, etc. That is to say, the different general reduction algorithms with the same reduction criterion may generate different reducts. To evaluate the effect of reduct obtained by different general reduction algorithms, we randomly selected 10 data sets as test objects from Table 4. We utilized the original data and the reduced data, which generated by five algorithms QGARA-FS, QGARA-BS, GS, GSV, and chi-square feature selection(CSFS for short), to train naive bayes classifier and decision tree classifier based on the 10-fold cross-validation method. For chi-square feature selection, naive bayes classifier, and decision tree classifier, we used its implementation in [49]. Regarding the parameter K in CSFS, determining how many attributes are contained in reduced data, we assign K as the cardinality of the reduct generated by GS. It is worth reminding that CSFS is not related to attribute reduction in theory and the reason why we put it into comparisons is to do the evaluation of GS and GSV in feature selection perspective. For convenience of comparison, we take the output of CSFS as a PRPR when K is assigned with the cardinality

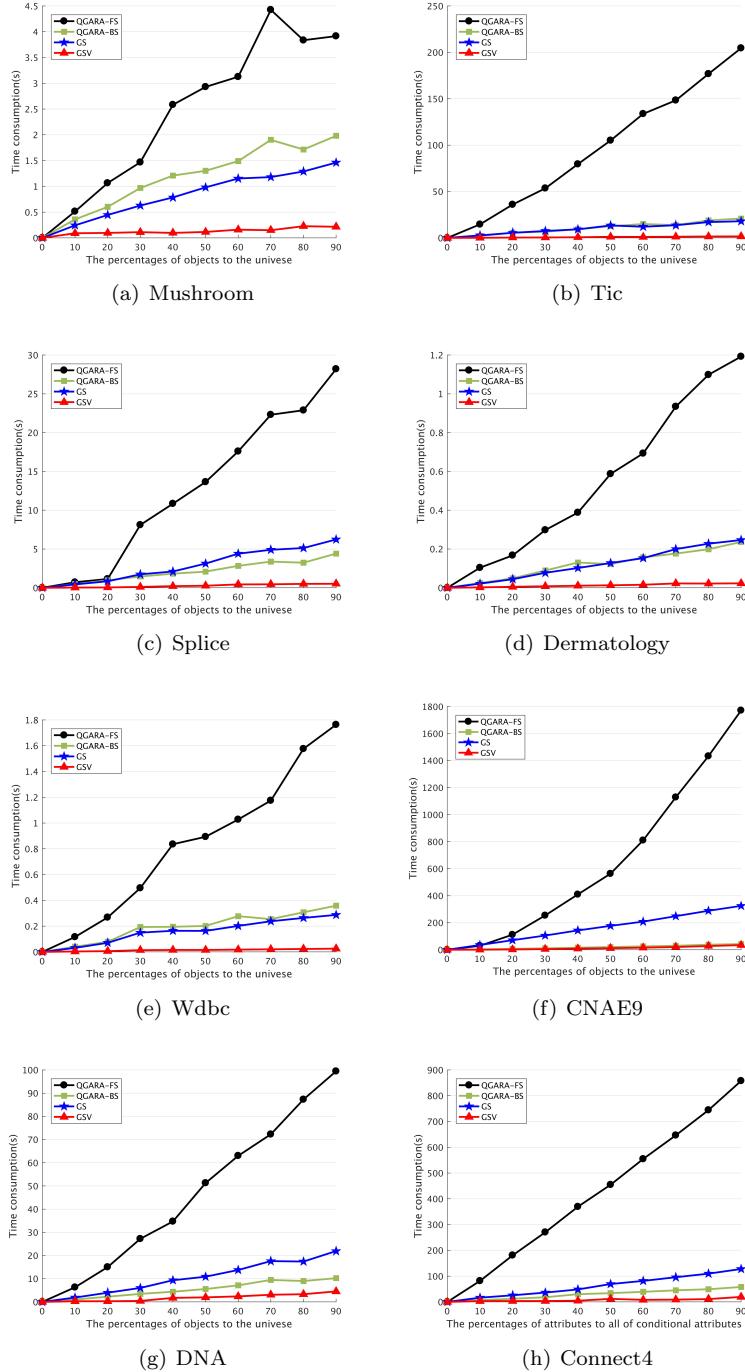


Fig 2. The time of general reduction algorithms versus the size of objects

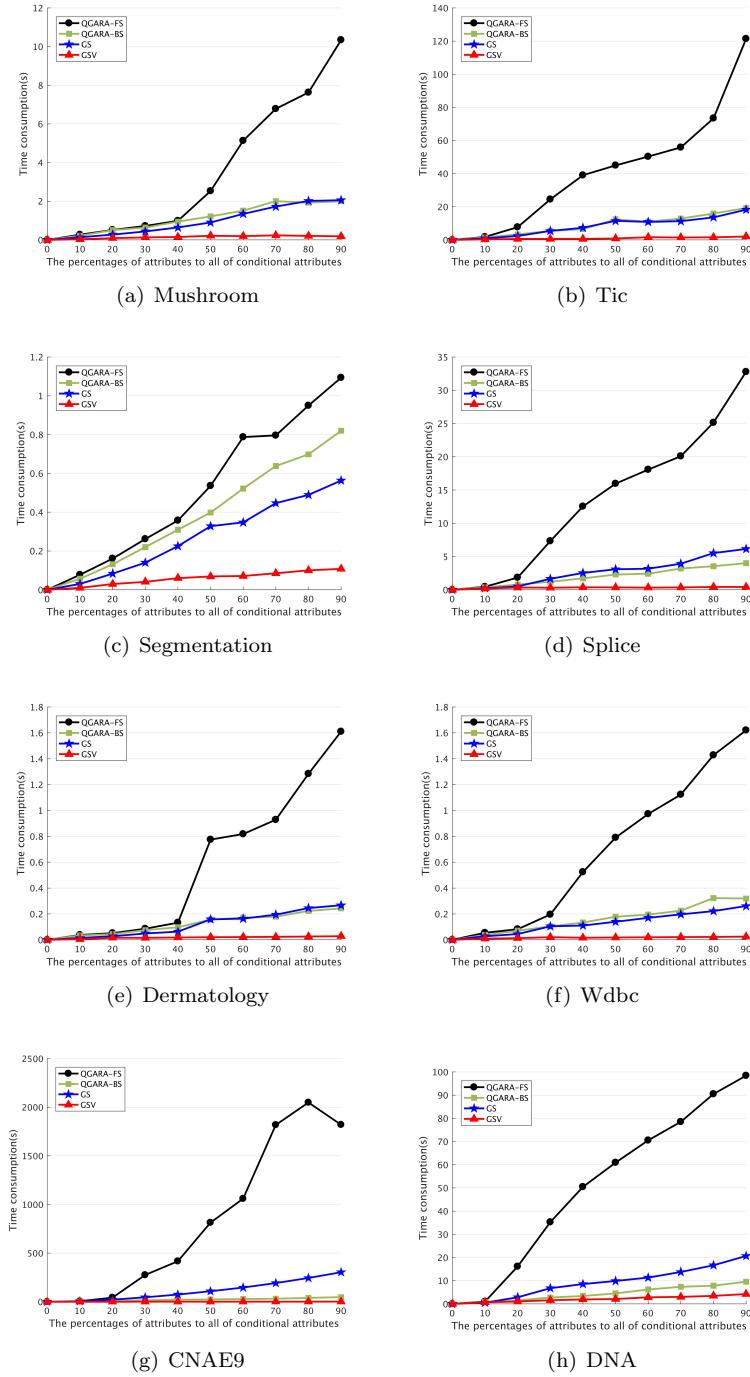


Fig 3. The time of general reduction algorithms versus the size of attributes

of the PRPR generated by GS; we take the output of CSFS as a DRPR when K is assigned with the cardinality of the DRPR generated by GS. The classification accuracy to the raw data and the reduced data generated by different algorithms were shown in Tables 7 to 10, where the column "Raw" represents the classification accuracies of the classifier trained on raw data sets, the bold-face highlights the highest accuracy among different algorithms, and the row "Average" represents average classification accuracy of reduction algorithms on 10 data sets, which can be interpreted as an estimated value of classification accuracy obtained by the output of related reduction algorithm over unknown data sets. Obviously, for most of reduced data sets, reduced data can retain

Table 7. The classification accuracy of decision tree with PRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.551	0.603	0.608	0.614	0.574	0.599
3	0.896	0.892	0.898	0.897	0.896	0.901
4	0.943	0.938	0.937	0.937	0.939	0.825
5	0.685	0.687	0.689	0.682	0.689	0.695
6	0.900	0.656	0.449	0.760	0.804	0.872
7	0.936	0.604	0.735	0.789	0.867	0.685
8	0.926	0.940	0.905	0.916	0.933	0.935
9	0.856	0.873	0.859	0.870	0.874	0.867
11	0.901	0.857	0.494	0.871	0.580	0.932
12	0.476	0.478	0.462	0.481	0.475	0.471
Average	0.807	0.753	0.704	0.782	0.763	0.778

similar classification accuracy as the entire data set.

For Table 7, the order of algorithms in the number of achieving the most classification accuracy is CSFS(4) > GSV(3) > GS(2) > QGARA-FS(1) > QGARA-BS(0). The order of algorithms in the average of classification accuracy on 10 data sets is GS(0.782) > CSFS(0.778) > GSV(0.763) > QGARA-FS(0.753) > QGARA-BS(0.704). GS achieves the best average classification accuracy on 10 data sets. That is to say, the steadiness of algorithms QGARA-FS, QGARA-BS, CSFS, and GSV in classification accuracy is not as good as GS. Furthermore, the PRPR obtained by GS and GSV performs better than that obtained by

Table 8. The classification accuracy of decision tree with DRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.551	0.554	0.551	0.558	0.551	0.584
3	0.896	0.896	0.898	0.897	0.894	0.900
4	0.943	0.938	0.940	0.938	0.942	0.819
5	0.685	0.682	0.685	0.685	0.691	0.684
6	0.900	0.651	0.455	0.762	0.809	0.873
7	0.936	0.590	0.746	0.787	0.878	0.685
8	0.926	0.938	0.902	0.907	0.931	0.933
9	0.856	0.872	0.853	0.870	0.871	0.862
11	0.901	0.861	0.502	0.868	0.580	0.933
12	0.476	0.477	0.466	0.477	0.477	0.472
Average	0.807	0.746	0.700	0.775	0.762	0.775

QGARA-FS and QGARA-BS in the average classification accuracy of decision tree classifier. When it comes to the reduced data generated in the criterion of relative discernibility relation preservation reduction, GSV and CSFS obtain the highest classification accuracy 4 times; QGARA-FS obtains the highest classification accuracy 3 times; GS obtains the highest classification accuracy 1 time; QGARA-BS obtains the highest classification accuracy 0 times. GSV and CSFS perform the best in times of achieving the best classification accuracy. Observing the average classification accuracy on ten data sets, GS is also the best of five, *i.e.*, $GS(0.775) \geq CSFS(0.775) > GSV(0.762) > QGARA-FS(0.746) > QGARA-BS(0.700)$. As a result, the DRPR obtained by GS and GSV performs better than that obtained by QGARA-FS and QGARA-BS in the average classification accuracy of decision tree classifier.

In the classification accuracy results of naive bayes classifier on reduced data generated in the criterion of positive region preservation reduction and relative discernibility relation preservation reduction, GS was the best one in the classification accuracy average on 10 data sets, and GSV was the next one.

Furthermore, we also used the t-test to compare the average 10-fold cross-validation based accuracies over each dataset. Taking the classification accuracies of GS, GSV, QGARA-FS, and QGARA-BS as the sampling results of

Table 9. The classification accuracy of naive bayes with PRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.649	0.557	0.650	0.565	0.582	0.668
3	0.777	0.885	0.895	0.896	0.896	0.805
4	0.603	0.575	0.570	0.575	0.605	0.519
5	0.682	0.682	0.682	0.682	0.682	0.682
6	0.792	0.545	0.519	0.657	0.576	0.778
7	0.977	0.602	0.769	0.763	0.933	0.682
8	0.902	0.804	0.865	0.889	0.874	0.753
9	0.949	0.909	0.901	0.907	0.933	0.887
11	0.924	0.767	0.535	0.827	0.594	0.894
12	0.597	0.601	0.599	0.606	0.597	0.599
Average	0.785	0.693	0.698	0.737	0.727	0.727

Table 10. The classification accuracy of naive bayes with DRPR found by five algorithms

ID	Raw	QGARA-FS	QGARA-BS	GS	GSV	CSFS
2	0.649	0.595	0.642	0.595	0.642	0.649
3	0.777	0.891	0.895	0.896	0.896	0.805
4	0.603	0.575	0.570	0.575	0.605	0.519
5	0.682	0.682	0.682	0.682	0.682	0.682
6	0.792	0.545	0.519	0.657	0.576	0.778
7	0.977	0.602	0.769	0.763	0.933	0.682
8	0.902	0.804	0.865	0.889	0.874	0.753
9	0.949	0.909	0.901	0.907	0.933	0.887
11	0.924	0.767	0.535	0.827	0.594	0.894
12	0.597	0.601	0.599	0.606	0.597	0.599
Average	0.785	0.697	0.698	0.740	0.733	0.725

four variables V_{GS}, V_{GSV}, V_{QFS} and V_{QBS} , we set up original hypothesis as $H_0 : V_{GS} = V_{QFS}, V_{GSV} = V_{QBS}$. There are only 23.75% of results declining the original hypothesis. That is to say, there are only 23.75% of cases in experiments supporting that the proposed algorithms are better than the existing algorithms; the remaining 76.25% of cases support the original hypothesis. As a result, it is also a reasonable choice of selecting the existing general reduction algorithms for data processing if time permitted.

In the experimental part, we made a series of comparisons between the proposed general attribute reduction algorithms and the existing general attribute

⁷¹⁵ reduction algorithms, and we had got conclusion listed as follows.

(1) In time consumption of the algorithms to obtain reducts, GS performed well in dealing with small-scale data. When processing large-scale data sets, GSV and QGARA-BS were good choices for attribute reduction.

(2) According to the experiments, the classification accuracy of reducts generated by GS and GSV is better than that generated by QGARA-FS and QGARA-BS. However, the existing general reduction algorithms are also good choices for data processing if time permitted.

5. Conclusion

In this study, we focus on the effective and efficient general reduction approach to obtain five types of reducts on the complete decision tables. We introduce a concept termed as granularity space and represent five typical reducts with granularity space. Based on the unified representation, we develop two quick general reduction algorithms. In comparison to the existing general reduction algorithms, the proposed algorithms have two advantages as follows.

(1) The proposed algorithms are more efficient. In the process of attribute reduction, GS can reduce one half of the computation time of QGARA-FS, and GSV can reduce over one half of the computation time of QGARA-BS.

(2) The reducts generated by proposed algorithms perform well as the existing reduction algorithms in the classification accuracy of decision tree classifier and naive bayes classifier. Over the data sets in experiments, the average classification accuracy of decision tree classifier and naive bayes classifier trained on reduced data generated by GS and GSV is equal to or higher than that trained on reduced data generated by QGARA-FS and QGARA-BS.

However, it is notable that the general reduction definitions and algorithms proposed in this paper are only suitable for the complete decision table. There exist many generalized decision tables, such as incomplete decision tables, interval-valued decision tables. Research on the extension of granularity space for the generalized decision tables will be investigated in future work.

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