Extension of the Collatz Conjecture

An exploritory paper

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The collatz conjecture creates challenge from its behaivioral complexity, which makes comprehending it nearly impossible, and a proof even more illusive. The collatz conjecture is extremely simple to initially understand, and is defined as follows. Given an positive integer n, repeat the following rules. If n is even, divide the number by two, if n is odd, multiply by three and add one. Following these steps, the conjecture states that if iterated a finite amount of times, all numbers will eventually reach 1, at which point they will loop along the path $1 \to 4 \to 2 \to 1$. The challenge in proving it is not the even case, the challenge is the odd case. I hope that this paper might shine a little light on the behavior of numbers in the odd case through the use of a heuristic describing the distance a number is from the 2^n tree.

This idea came from conversation between Seth and I, where upon my explaination of the collatz conjecture, asked if any other numbers work as the multiple, add, and division (instead of 3, 1, and 2). This led me to write a program to test which numbers seem to work, and it came to the conclusion that numbers in the following groups work:

$$0n + 2^a$$

$$2^a n + 2^a$$

$$2^a 3n + 2^a$$

The first case makes complete sense. If the number is odd and you mulitply by 0 and add 2^a, it is clear that dividing by two over and over will have it reach 1. The second case is slightly less clear, but I have a proof that shows why it works. First lets look at case 2 when $a = 0$. This means that if n is even, you still divide by two, and if n is odd then you multiply by 1 and add 1 (just add one). First lets define a function that represents the difference between n and the next power of two:

$$H(n) = 2^{ceil(log_2(n))} - n$$

Notice that if we can show that this function is strictly decreasing as we iterate collatz, then that shows that this case is true because if $H(n)$ is always decreasing, then once it reaches 0 it is on the 2^a tree. Once a number is on the 2^a tree it will keep decreasing to 1. Now lets prove that $H(a)$ is strictly decreasing for the $1-1$ case (this notation will be used from now on, standard collatz is the 3-1 case). To start the proof lets notice that for some integer p, $H(p)$ can be written as $2^x - H(p)$. $2^x = p + H(p)$. For now lets just say that $H(p)$ can be written as $2^x - a$. First lets look at the case when $p\%2 = 0$. This means that $a\%2 = 0$. This means that the next step in the collatz sequence is $\frac{p}{2}$ and $H(\frac{p}{2})$ is $2^{x-1} - \frac{a}{2}$. Now lets look at the relationship between $2^{x-1} - \frac{a}{2}$ and $2^x - a$. Lets see if $2^x - a > 2^{n-1} - \frac{a}{2}$. This simplifies to $a < 2^{x+1} - 2^x$. Since we know that 2^x is the next power of two after a, $a < 2^x$, therefore this is true. The other case is that $p\%2 = 1$. In this case, we know that we add one, and since $H(a)$ is the distance to the next highest power of two, adding one will decrease this value. These two facts show that $H(a)$ is strictly decreasing as collatz is iterated for the $1-1$ case, completeing this proof.