Proof by Exhaustion

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1 A General Look

What is it? **Proof by Exhaustion** (also known as **Proof by Cases**) is a mathematical method of proving propositions [1] by considering all possible cases and demonstrating their truth. The name reflects the idea of exhaustively examining all cases that the proposition may take.

A general question Propositions regarding *"Proof by Exhaustion"* questions are generally along the lines: *"Prove that all squared even numbers are even."*

2 Steps of Solution

Generally, propositions are proven by exhaustion by one of two methods: We account for all of the cases algebraically, or we test all possible cases (values a variable can take) by plugging in all possible numbers and doing manual calculations .

2.1 Examples

2.1.1 An Algebraic Account for Cases

Proposition 1. *If n is an integer then*

$$5n^2 + n + 12$$

is always even.

Proof on the next page. Try the question! (Hint: Break the integers into odd numbers and even numbers)

In the exam, the question will mention that you should use "Proof by Exhaustion".

[1] A proposition is a mathematical statement such as "3 is greater than 4," "an infinite set exists," or "7 is prime."

Proof. Integers can be split into two groups:

- even numbers
- odd numbers

$$\text{Integers} \begin{cases} \text{Evens} \rightarrow 2,4,6,\ldots = 2n \ , \forall_n \in \mathbb{Z}^+ \\ \text{Odds} \rightarrow 1,3,5,\ldots = 2n{+}1 \ , \forall_n \in \mathbb{Z}^+ \end{cases}$$

$\therefore$ If we prove that $5n^2 + n + 12$ is even for both evens and odds, then $5n^2 + n + 12$ is even for all positive integers.

1. Evens: Any even number $= 2p$ for some positive integer value of p.
   $\therefore$ if $5(2p)^2 + (2p) + 12$ is even, then $5n^2 + n + 12$ is even when n is even.

$$\begin{aligned} &5(2p)^2 + (2p) + 12 \\ =&10p^2 + 2p + 12 \\ =&\underbrace{2(p^2 + 2p + 12)}_{\text{divisible by 2}} \end{aligned}$$

   $\implies 5n^2 + n + 12$ is even when n is even.

2. Odds: Any odd number $= 2q + 1$ for some positive integer value of q. $\therefore$ if $5(2q + 1)^2 + (2q + 1) + 12$ is even, then $5n^2 + n + 12$ is even when n is odd.

$$\begin{aligned} &5(2q + 1)^2 + (2q + 1) + 12 \\ =&5(4q^2 + 4q + 1) + 2q + 13 \\ =&20q^2 + 20q + 5 + 2q + 13 \\ =&20q^2 + 22q + 18 \\ =&\underbrace{2(10q^2 + 11q + 9)}_{\text{divisible by 2}} \end{aligned}$$

   $\implies 5n^2 + n + 12$ is even when n is odd.

$\therefore$ By *"Proof by Exhaustion"*, $5n^2 + n + 12$ is even for all integers as it is even for when n is both odd and even.

2.1.2 Plugging in All Possible Cases

How do we identify it? This method/approach is typically employed in cases where the question imposes constraints on the number of values that can be tested. For instance, the previous proposition cannot be proven using this approach since it is impractical to plug in an infinite number of both even and odd numbers.

Proposition 2. *Prove that when* n *is an integers and* $1 \leq \mathrm{n} \leq 6$, *then* m=n+2 *is* ***not*** *divisible by 10.*

Proof. We shall now consider all possible cases of n.

Value of n	Value of m	Is it divisible by 10?
1	1+2=3	No
2	2+2=4	No
3	3+2=5	No
4	4+2=6	No
5	5+2=7	No
6	6+2=8	No

$\therefore$ by *"Proof by Exhaustion"*, when n is an integers and $1 \leq n \leq 6$, then $m = n + 2$ is **not** divisible by 10.

3 Other General Methods of Proof

3.1 Proof by Counter-example

Being the easiest method of proof, all you have to do is show that a certain proposition is **not** true.

3.2 Proofs Using Mathematical Properties

The main idea is to some mathematical properties to show that a statement is true.

Proposition 3. *Prove that* $x^2 + 8x + 20 \geq 4$ *for all values of* x.

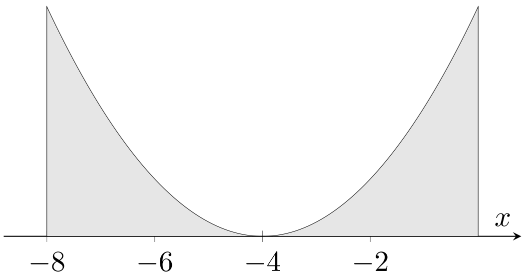
Proof. As a rule of thumb, we always like to have a zero side in quadratic equations/inequalities; the proof is as follows: *NEXT PAGE*

We should simplify the quadratic in equality to establish a certain mathematical property:

$$\begin{aligned} x^2 + 8x + 20 &\geq 4 \\ x^2 + 8x + 16 &\geq 0 \\ \underbrace{(x+4)^2}_{\textbf{a squared bracket}} &\geq 0 \end{aligned}$$

since $x+4$ is real, $(x+4)^2$ must be greater than or equal to 0 as any squared real number is greater than or equal to zero.

We can also see this if we plot the graph; it is always above the x axis (or touching it at $x = -4$)



In general, when we are dealing with quadratics, we want to do one of two things: we wither factorise as previously; and if the result of factoring was not significant, we get two factors and not a single squared factor, we almost always complete the square. When completing the square, the usual result is

$$a(x+b)^2 + c.$$

if we want the quadratic to always be greater than 0, then a and c must both be positive numbers.

Proposition 4. *Prove that for all positive values of x and y:*

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

Proof. To get an idea on what to do, we should dedicate a 'scratch' portion of the solution; this is called jotting.

<u>Jotting:</u>

$$\begin{aligned} \tfrac{x}{y} + \tfrac{y}{x} &\geq 2 \\ x^2 - 2xy + y^2 &\geq 0 \\ \underbrace{(x-y)^2 \geq 0}_{\text{Which is true}} \end{aligned}$$

we want to cancel the denominators

Actual Proof:

Consider $(x-y)^2 \geq 0$ which is true for all positive values of x and y as $x-y$ is real, and any (real number)$^2 \geq 0$. In a more mathematical manner, since $x-y \in \mathbb{R}$ then $(x-y)^2 \geq 0$ as $k^2 \geq 0$ is true $\forall_k \in \mathbb{R}$.

$$(x-y)^2 \geq 0$$
$$x^2 - 2xy + y^2 \geq 0$$
$$x^2 + y^2 \geq 2xy$$
$$\underbrace{\frac{x}{y} + \frac{y}{x} \geq 2}_{\text{which was given}}$$

Hence, Since

$$(x-y)^2 \geq 0$$

is true for all positive values of x and y and

$$(x-y)^2 \geq 0 \implies \frac{x}{y} + \frac{y}{x} \geq 2,$$

then

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

must be true for all positive values of x and y.

Proposition 5. *Give that a is a positive real numbers, prove that*

$$a + \frac{1}{a} \geq 2$$

Proof. Let's start with jotting down some ideas.

Jotting:

$$a + \frac{1}{a} \geq 2$$

we multiply by a, which won't change the inequality sign since a is positve.

$$a^2 - 2a + 1 \geq 0$$
$$(a-1)^2 \geq 0$$

which is true.

Now, we will neatly write down out proof.

Proof. [2]
If $a \in \mathbb{R}$, then $a - 1 \in \mathbb{R}$. Hence,

$$\begin{aligned}
(a-1)^2 &\geq 0 \\
a^2 - 2a + 1 &\geq 0 \\
\frac{a^2 - 2a + 1}{a} &\geq 0 \\
a - 2 + \frac{1}{a} &\geq 0 \\
a + \frac{1}{a} &\geq 2 \quad \square
\end{aligned}$$

[2] We will use a slightly different approach.