# Products of Binomial Expansions

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## Introduction

This paper will *not* cover the whole idea of the binomial expansion; it will, however, tackle the recurring questions regarding the coefficient of $x^n$ in binomial expansions. It is assumed that you know your way around the binomial expansion.

## 1 The coefficient of $x^n$ in the binomial expansion

### 1.1 The Idea

Suppose we have the product

$$\left(1+x^2\right)^2 \times \left(1+\frac{1}{x}\right)^5,$$

and we are asked to find the coefficient of $x^{-3}$. It would be very cumbersome to fully expand this product. In fact, if you chose to expand and simplify this product, you will end up with the following large expression:

$$\frac{1}{x^5}+x^4+\frac{5}{x^4}+5x^3+\frac{12}{x^3}+12x^2+\frac{20}{x^2}+20x+\frac{26}{x}+26$$

Pretty ugly no?

Thankfully, we *do not* need to find the whole expansion, as we have an easier way.

If we refer to the full expansion, you can quickly realize that we only need one term, and the rest of the terms are irrelevant. How can we only find the required term? Lets "untangle" the $x^{-3}$ term.

$$\left(1+x^2\right)^2 \times \left(1+\frac{1}{x}\right)^5 = \left(1+2x^2+x^4\right)\left(\frac{1}{x^5}+\frac{5}{x^4}+\frac{10}{x^3}+\frac{10}{x^2}+\frac{5}{x}+1\right)$$

$$= \cdots + 1 \times 10x^{-3} + 2x^2 \times \frac{1}{x^5} + \cdots$$

Notice that $10x^{-3}$ and $2x^{-3}$ are the two terms that add up to $12x^{-3}$. Hence, we are only interested in the terms that have $x^{-3}$; therefore, we are interested in *all* the result of *any* two terms which give $x^{-3}$. We should think this through *before* the binomial expansion of any bracket.

Let's test this with our previous example

| | Powers of $x$ after expanding |
|---|---|
| $(1+x^2)^2$ | $x^0$ ; $x^2$ ; $x^4$ |
| $\left(1+\dfrac{1}{x}\right)^5$ | $x^0$ ; $x^{-1}$ ; $x^{-2}$ ; $x^{-3}$ ; $x^{-4}$ ; $x^{-5}$ |

Note that in the expansions every terms in a bracket after the expansion is multiplied by every term in the other bracket after the expansion. For example, $x^0$ in the $(1+x^2)^2$ bracket will be multiplied by $x^0$, $x^{-1}$, $x^{-2}$, $\cdots$ in the $\left(1+\dfrac{1}{x}\right)^5$ bracket.

Now, all we have to do is think of the multiplications which give $x^{-3}$. The possible ways we can attain $x^{-3}$ is by multiplying $x^0$ in the first bracket with $x^{-3}$ in the second bracket and $x^2$ in the first bracket with $x^{-5}$ in the second bracket. Hence, we can obtain the $x^{-3}$ term by multiplying the $x^0$ term in the first bracket with $x^{-3}$ term in the second bracket, and multiplying the $x^2$ term in the first bracket with $x^{-5}$ term in the second bracket followed by adding the two results.

*Solution:*

Finding relevant terms for the expansion of $(1+x^2)^2$:

$x^0$ term: $1^2 \times \binom{2}{0} \times (x^2)^0 = 1$

$x^2$ term: $1^1 \times \binom{2}{1} \times (x^2)^1 = 2x^2$

Finding relevant terms for the expansion of $\left(1+\dfrac{1}{x}\right)^5$

$x^{-3}$ term: $1^2 \times \binom{5}{3} \times \left(\dfrac{1}{x}\right)^3 = 10x^{-3}$

$x^{-5}$ term: $1^0 \times \binom{5}{5} \times \left(\dfrac{1}{x}\right)^5 = x^{-5}$

Finding relative results:

$$1x^0 \times 10x^{-3} + 2x^2 \times \frac{1}{x^5} = 12x^{-3}$$

Therefore, the coefficient of $x^{-3}$ in the expansion of $\left(1+x^2\right)^2 \times \left(1+\dfrac{1}{x}\right)^5$ is 12.

## 2 Exercises:

1. **a)** Find the first three terms in the expansion, in ascending powers of $x$, of $(2+3x)^4$

   **b)** Find the first three terms in the expansion, in ascending powers of $x$, of $(1-2x)^5$

   **c)** Hence find the coefficient of $x^2$ in the expansion of $(2+3x)^4(1-2x)^5$

2. **a)** Give the complete expansion of $\left(x+\dfrac{2}{x}\right)^5$.

   **b)** In the expansion of $(a+bx^2)\left(x+\dfrac{2}{x}\right)^5$, the coefficient of x is zero and the coefficient of $\dfrac{1}{x}$ is 80.

   Find the values of the constants $a$ and $b$.

3. Find the term independent of x in each of the following expansions.

   **a)** $\left(3x+\dfrac{2}{x^2}\right)^6$

   **b)** $\left(3x+\dfrac{2}{x^2}\right)^6(1-x^3)$