Integration

A deep dive into identifying integrals

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Aim of this paper

The goal of this paper is to elucidate the process of identifying which integration technique/result you need to use for some integral, and illustrating the systematic approach to solving these integrals. With ample practice and using the identification techniques in this paper, you will, hopefully, guarantee all marks regarding integration. Note that this paper is concerned with EDEXCEL's P3 and P4 modules or Cambridge's P2 and P3 modules.

Limitations of this paper

This paper cannot come close to the experience you acquire by actually solving questions from past papers. The paper tries to smooth the process and nothing more. Since the paper is trying to accomplish a lot of things and given the lack of experience on my part writing such papers with a semi-unstructured approach, it will be some what messy. I would recommend anyone to read this paper swiftly, learning perhaps a unique bit of information; however, I would *not* recommend studying/carefully reading this paper to anyone capable of identifying the integration technique to be used with relative ease. I would not say that this is my best work nor my worst. There is definitely room for improvement in terms of structuring and having a clearer idea of information progress.

An Overview

This is an exhaustive list of all the integration techniques/results that you need to know:

1. $\dfrac{1}{x^n}$
2. reversing the chain rule/u-substitution
3. e^{kx}
4. a^x
5. $\ln(x)$ integrals

6. trigonometric functions and identities
7. integration by parts
8. using partial fraction decomposition

Note: We will *not* be discussing the proof of results. The paper exclusively focuses on computation and *not* theory

1 $\int \frac{1}{x^n}\,\mathrm{d}x$

Perhaps the easiest integral to compute, this is is just an application of the power rule.

When $n \neq 1$

$$\begin{aligned} \int \frac{1}{x^n}\,\mathrm{d}x &= \int x^{-n}\,\mathrm{d}x \\ &= \frac{x^{-n+1}}{-n+1} + C \\ \therefore \int \frac{1}{x^n}\,\mathrm{d}x &= \frac{1}{-n+1} \times \frac{1}{x^{n-1}} + C \end{aligned}$$

The rule fails when $n = 1$ as it generates an undefined results. (try it!) But what if the integral we are asked to find is

$$\int \frac{1}{x}\,\mathrm{d}x$$

This is explored in section 5.

Examples

Questions

1. $\int \frac{1}{x^4}\,\mathrm{d}x$
2. $\int \frac{x^2+2x+1}{x^4}\,\mathrm{d}x$
3. $\int \frac{x^2+4x+4}{(x+2)^2}\,\mathrm{d}x$
4. $\int \frac{x^2-4x+4}{(x-2)^2}\,\mathrm{d}x$

Solutions

1.

$$\begin{aligned}\int \frac{1}{x^4}\,\mathrm{d}x &= \int x^{-4}\,\mathrm{d}x \\ &= -\frac{1}{3x^3} + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{x^2+2x+1}{x^4}\,\mathrm{d}x &= \int \frac{1}{x^2} + \frac{2}{x^3} + \frac{1}{x^4}\,\mathrm{d}x \\ &= \int x^{-2} + 2x^{-3} + x^{-4}\,\mathrm{d}x \\ &= -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{3x^3} + C\end{aligned}$$

3.

$$\begin{aligned}\int \frac{x^2+4x+4}{(x+2)^2}\,\mathrm{d}x &= \int \frac{(x+2)^2}{(x+2)^2}\,\mathrm{d}x \\ &= \int 1\,\mathrm{d}x\end{aligned}$$

You can integrate directly, but for the sake of fun

$$\begin{aligned}&= \int x^0\,\mathrm{d}x \\ &= \frac{x^1}{1} + C \\ &= x + C\end{aligned}$$

4.

$$\begin{aligned}\int \frac{x^2-4x+4}{(x-2)^2}\,\mathrm{d}x &= \int \frac{(x+2)(x-2)}{(x-2)^2}\,\mathrm{d}x \\ &= \int \frac{(x+2)(x-2)}{(x+2)\,(x-2)}\,\mathrm{d}x \\ &= \int 1\,\mathrm{d}x \\ &= x + C\end{aligned}$$

2 Reversing the chain rule/u-substitution

Important! When u-substitution has to be used, the question signifies this by providing you with a substitution that you must use. When it is necessary to use u-substitution, the substitution itself will always be given.

As the first part of the name suggest, this method undoes the chain rule. When we differentiate $f(x)^n$ using the chain rule, $\frac{d}{dx}(f(x)^n) = n(f'(x))[f(x)]^{n-1}$. How can we adapt this for integrals?

Well if

$$\frac{d}{dx}(f(x)^n) = n(f'(x))[f(x)]^{n-1}$$

then

$$\int n(f'(x))[f(x)]^{n-1}\,\mathrm{d}x = (f(x)^n) + C$$

More generally

$$\int f'(x)[f(x)]^n\,\mathrm{d}x = \frac{(f(x))^{n+1}}{n+1} + C$$

This is called reversing the chain rule. It is *very important* to note that we can only reverse the chain rule if we the integrand is in the form of $f'(x)[f(x)]^n$.

There is another equivalent method known as the u-substitution. If you can reverse the chain rule, you can use u-substitution, and if you can use u-substitution, you can reverse the chain rule. However, it is not always clear if the integrand is in the required form to directly reverse the chain rule; this is where u-substitution comes into play. We use u-substitution when you have a hunch that reversing the chain rule might work, but the integrand is not obvious to be in the correct form. It would be best to show with an example. Remember, when assigning u, it's mostly about intuition and luck.

Suppose we want to find

$$\int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x$$

This is very messy and an obvious thing to do is to multiply by $\frac{e^x}{e^x}$. This will leave us with

$$\int \frac{e^x}{e^{2x}+1}\,\mathrm{d}x$$

When assigning u, we usually let u equal the most complex part in the integrand. In this case, it would be $e^{2x}+1$.

$$Let\ u = e^{2x} + 1$$

$$\rightsquigarrow \int \frac{e^x}{u}\,dx$$

We will leave the numerator for now. However, while our goal is to convert our integrand to be in terms of u, notice how we are still integrating with respect to x. To fix this, lets differentiate u with respect to x.

$$\mathrm{d}u = 2e^{2x}\mathrm{d}x \to \mathrm{d}x = \frac{1}{2e^{2x}}\mathrm{d}u$$

Substituting $\mathrm{d}x$, we soon realise this won't work. Let's try $u = e^x$

$$\int \frac{u}{u^2+1}\,\mathrm{d}x$$

Also,

$$\text{if } u = e^x \text{ then } \mathrm{d}u = e^x\,\mathrm{d}x = u\,\mathrm{d}x$$

$$\rightsquigarrow \int \frac{u}{u^2+1}\frac{1}{u}\,\mathrm{d}u$$

$$= \int \frac{1}{u^2+1}\,\mathrm{d}u$$

$$= \arctan u + C$$

$$= \arctan\left(e^x\right) + C$$

This substitution worked! Again, it's all about trial and error. With time, "guessing" (not solely based on chance) which substitution is correct becomes easier.

The following examples are definitely harder than what comes in the exam.

Questions

1. $\displaystyle\int \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$
2. $\displaystyle\int \frac{x^2}{x^2+1}\,\mathrm{d}x$
3. $\displaystyle\int x\sqrt{2x+3}\,\mathrm{d}x$

Solutions

1.

$$\int \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x$$

$$\textit{Let } u = \sqrt{x}\ ^{1}$$

$$\therefore \mathrm{d}u = \frac{1}{2\sqrt{x}}\mathrm{d}x$$

$$\to 2\sqrt{x}\,\mathrm{d}u = \mathrm{d}x$$

$$\begin{aligned} \rightsquigarrow \int \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x &= \int \frac{1}{\sqrt{x}(x+1)} \times 2\sqrt{x}\,\mathrm{d}u \\ &= \int \frac{1}{(u^2+1)}\,\mathrm{d}u \\ \therefore \int \frac{1}{\sqrt{x}(x+1)}\,\mathrm{d}x &= \arctan\sqrt{x} + C \end{aligned}$$

2. This one is definitely a brain teaser. No substitution is actually needed. We will leave this for you to figure out

3.

$$\begin{gathered} \textit{Let } u = 2x+3 \\ \to \mathrm{d}u = 2\ \mathrm{d}x \\ \frac{1}{2}\mathrm{d}u = \mathrm{d}x \end{gathered}$$

also

$$x = \frac{u-3}{2}$$

[1] Our options are letting u equal x, $x+1$ or $\sqrt{x}$. We chose the last option after going through the first two mentally.

Plugging into our integral,

$$\begin{aligned}\int x\sqrt{2x+3}\,\mathrm{d}x &= \int \frac{u-3}{2} \times \sqrt{u}\,\mathrm{d}u \\ &= \frac{1}{2}\int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)\,\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}}\right) + C \\ &= \frac{u^{\frac{5}{2}}}{5} - u^{\frac{3}{2}} + C\end{aligned}$$

3 e^{kx}

We know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x$$

and

$$\int e^x\,\mathrm{d}x = e^x + C$$

When it comes to e^{kx} for some real number k, it's just reversing the chain rule, or using the substitution $u = kx$.

Questions with solutions

1. $\displaystyle\int e^{\pi x}\,\mathrm{d}x = \frac{1}{\pi}e^{\pi x} + C$
2. if $k \in \mathbb{R}$ and is constant, then $\displaystyle\int e^{\frac{k^2+2k-\pi}{2k+e^k}x}\,\mathrm{d}x = \frac{2k+e^k}{k^2+2k-\pi}e^{\frac{k^2+2k-\pi}{2k+e^k}x} + C$

Pretty simple if you ask me.

4 a^x

This is also a pretty simple application of the rule. We will just show the derivation without examples

$$\begin{aligned}\int a^x\,\mathrm{d}x &= \int e^{\ln a^x}\,\mathrm{d}x \\ &= \int e^{x\ln a}\,\mathrm{d}x\end{aligned}$$

since $\ln a$ is a constant real number

$$\int a^x \, \mathrm{d}x = \frac{1}{\ln a} e^{x \ln a} + C$$

5 $\ln x$ integrals

This is not referring to the integral where the integrand is $\ln x$, but the integral where the result is $\ln x$ or some variation of it.

We are given the fact that

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln f(x) + C$$

hence,

$$\int \frac{1}{x} \, \mathrm{d}x = \ln x + C$$

However, the integrals are made to be not so obvious; what can we do? To test if we have a "$\ln x$" integral (that's just a term I coined and nothing more), we first *differentiate the numerator* and then equate it to k(numerator). This will be clearer through the examples.

Questions

1. $\displaystyle\int \frac{1}{3x+2} \, \mathrm{d}x$
2. $\displaystyle\int \frac{3x+2}{6x^2+8x+3} \, \mathrm{d}x$
3. $\displaystyle\int \frac{32x^3+16x}{x^4+x^2+33} \, \mathrm{d}x$

Solutions

1. We will first test if we are dealing with a "$\ln x$" integral

   If there exists some $k \in \mathbb{R}$ such that

   $$\frac{\mathrm{d}}{\mathrm{d}x}(3x+2) = k(1)$$

   then we have a "$\ln x$" integral which we will show how to solve shortly

   $$\frac{d}{dx}(3x+2) = 3$$

   hence,

   $$3 \stackrel{?}{=} k(1)$$

   $$\therefore k = 3$$

Therefore, we have a "$\ln x$" integral, with $k = 3$. As for our next steps, observe the following

$$\begin{aligned}\int \frac{1}{3x+2}\,\mathrm{d}x &= \frac{1}{3}\int \frac{3}{3x+2}\,\mathrm{d}x \\ &= \frac{1}{3}\ln(3x+2) + C\end{aligned}$$

We can see that we multiply the integral by $\frac{k}{k}$ if we have the "$\ln x$" integral.

2. Let k be a constant real number.

$$\begin{aligned}\frac{\mathrm{d}}{dx}\left(6x^2+8x+3\right) &\stackrel{?}{=} k(3x+2) \\ 12x+8 &\stackrel{?}{=} k(3x+2) \\ 4(3x+2) &\stackrel{?}{=} k(3x+2) \\ k &= 4\end{aligned}$$

This leads to

$$\begin{aligned}\int \frac{3x+2}{6x^2+8x+3}\,\mathrm{d}x &= \frac{1}{4}\int \frac{4(3x+2)}{6x^2+8x+3}\,\mathrm{d}x \\ &= \frac{1}{4}\int \frac{12x+8}{6x^2+8x+3}\,\mathrm{d}x \\ \therefore \int \frac{3x+2}{6x^2+8x+3}\,\mathrm{d}x &= \frac{1}{4}\ln(6x^2+8x+3)\end{aligned}$$

3. Let k be a constant real number.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4+x^2+33\right) &\stackrel{?}{=} k\left(32x^3+16x\right) \\ 4x^3+2x &\stackrel{?}{=} k\left(32x^3+16x\right) \\ \frac{1}{8}\left(32x^3+16x\right) &\stackrel{?}{=} k\left(32x^3+16x\right) \\ k &= \frac{1}{8}\end{aligned}$$

This leads to

$$\begin{aligned}\int \frac{32x^3+16x}{x^4+x^2+33}\,\mathrm{d}x &= \frac{1}{\frac{1}{8}}\int \frac{\frac{1}{8}\left(32x^3+16x\right)}{x^4+x^2+33}\,\mathrm{d}x \\ &= 8\int \frac{4x^3+2x}{x^4+x^2+33}\,\mathrm{d}x \\ \therefore \int \frac{32x^3+16x}{x^4+x^2+33}\,\mathrm{d}x &= 8\ln\left(x^4+x^2+33\right) + C\end{aligned}$$

6 Trigonometric functions and identities

This is where people freak out when it comes to integration questions. However, it is extremely easy to know which identities to use after analyzing the integrand. I will assume that you are aware of the standard integrals given in the formula sheet. That include results such as $\int \sec^2 x \, \mathrm{d}x \cdots$

There are *so many* possible tricks that can be implemented using trigonometric functions and identities. However, I will try my best to illustrate everything i know.

6.1 Sine and cosine variants

If we have the integrand as a product of sine and cosine where sine and cosine are raised to the same power, we use the identity

$$\sin 2x = 2 \sin x \cos x$$

If we have the integrand as a squared trigonometric functions (it doesn't matter if the function is housing x, $2x$, $\cdots$), we use the identity

$$\cos 2x = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

If we have a product of $\sin(kx)\cos^n(kx)$ or $\cos(kx)\sin^n(kx)$ where $n \in \mathbb{Q}$ and $k \in \mathbb{Z}$, we reverse the chain rule, or use u-substitution where u is the function raised to the power n.

Let us try an example[2]

$$\begin{aligned}
\int \cos(2x)\sin^9 x \cos^9 x \, \mathrm{d}x &= \int \cos(2x)(\sin x \cos x)^9 \, \mathrm{d}x \\
&= \int \cos(2x)\left(\frac{1}{2} \times 2 \sin x \cos x\right)^9 \, \mathrm{d}x \\
&= \int \cos 2x \left(\frac{1}{2}\right)^9 (2 \sin x \cos x)^9 \, \mathrm{d}x \\
&= \frac{1}{512} \int \cos 2x (\sin 2x)^9 \, \mathrm{d}x
\end{aligned}$$

Reversing the chain rule, we get

$$\begin{aligned}
\int \cos(2x)\sin^9 x \cos^9 x \, \mathrm{d}x &= \frac{1}{512} \times \sin^{10}(2x) \times \frac{1}{10} \times \frac{1}{2} \\
&= \frac{1}{10240}(\sin 2x)^{10} + C
\end{aligned}$$

[2]You won't face such an example in the exam; however, it shows how we can use 2 of the mentioned techniques.

One more

$$\int \sin^2 x \, \mathrm{d}x$$

But $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$$\therefore \int \sin^2 x \, \mathrm{d}x = \frac{1}{2} \int (1 - \cos 2x) \, \mathrm{d}x$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

6.2 $\tan x$ related integrals

I could only think of 3 possible instances. We will explore them using examples only.

1.
$$\int \sec^2 x \tan^6 x \, \mathrm{d}x = \frac{\tan^7 x}{7} + C$$

   This is just an instance of reversing the chain rule and nothing more $\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x\right) = \sec^2 x$

2.
$$\int 10 + \tan^2 x \, \mathrm{d}x$$

   while we could use integration by parts after splitting the integral, there is a much simpler way:

$$\int 10 + \tan^2 x \, \mathrm{d}x = \int 9 + 1 + \tan^2 x \, \mathrm{d}x$$
$$= \int 9 \, \mathrm{d}x + \int 1 + \tan^2 x \, \mathrm{d}x$$
$$= 9x + C' + \int \sec^2 x \, \mathrm{d}x$$
$$\therefore \int 10 + \tan^2 x \, \mathrm{d}x = 9x + \tan x + C$$

In general, we use $1 + \tan^2 x \equiv \sec^2 x$ whenever we have an integral in the form

$$\int k + \tan^2 x \, \mathrm{d}x$$

where k is a real number **including zero**.

Remember! For the integral of $\tan x$ *and* $\cot x$*, we convert both into their* $\sin x$ *and* $\cos x$ *definition, where the numerator is a scalar multiple of the derivative of the denominator.*

7 Integration by parts

Integrals where integration by parts has to be employed are easy to spot. Whenever we have a product where reversing the chain, trigonometric identities don't work or it isn't a standard integral, we use integration by parts. When it comes to choosing our u in the selection of u and $\mathrm{d}v$, we follow the ILATE rule. The ILATE rule is simply states the priority of function types when picking u.

- **I** $\rightarrow$ Inverse trigonometric functions: we always use them as our choice of u if present.
- **L** $\rightarrow$ Logarithmic functions: we use them as our choice of u if inverse trigonometric function are not present. In our case, this always represents $\ln x$ for some constant k
- **A** $\rightarrow$ Algebraic functions: we use them as our choice of u if neither **I** nor **L** are present. This usually represents x^n for some integer n.
- **T** $\rightarrow$ Trigonometric functions: we use them as our choice of u in-case I,L or A are not present.
- **E** $\rightarrow$ The exponential function e^x. We use this as our last choice of u

I	Inverse Trigonometric ($\mathrm{Sin}^{-1}x$, $\tan^{-1}x$,etc)
L	Logarithmic ($\log_3 x$, log x, ln x,etc)
A	Algebraic (x^3, $3\sqrt{x}$, etc)
T	Trigonometric (sin x , csc x , etc)
E	Exponential (3^x , e^x , etc)

NOTE : Priority is from top to bottom

Figure 1: This shows the ILATE rule in a clearer way.

We will start with a rather To compute the integral of $\arctan(x)$ using integration by parts, we'll use the formula:

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

Let:

$$\begin{aligned} u &= \arctan(x) && \Rightarrow && \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{1+x^2} \\ \frac{\mathrm{d}v}{\mathrm{d}x} &= 1 && \Rightarrow && v = x \end{aligned}$$

Now, applying the integration by parts formula:

$$\int \arctan(x) \times 1\,dx = x\arctan(x) - \int x \cdot \frac{1}{1+x^2}\,\mathrm{d}x.$$

To proceed, we'll simplify the second integral by using a substitution. Let:

$$u = x^2 + 1 \quad \Rightarrow \quad du = 2x\,dx.$$

Therefore:

$$\begin{aligned} \int x \cdot \frac{1}{1+x^2}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u \\ &= \frac{1}{2}\ln|u| + C_1 \\ &= \frac{1}{2}\ln|x^2+1| + C_1, \end{aligned}$$

where C_1 is the constant of integration from the second integral.

Putting it all together:

$$\begin{aligned}\int \arctan(x)\ \mathrm{d}x &= x\arctan(x) - \frac{1}{2}\ln|x^2+1| + C + C_1\\ &= x\arctan(x) - \frac{1}{2}\ln|x^2+1| + C_2,\end{aligned}$$

where $C_2 = C + C_1$ is the constant of integration.

This popped up several times in P3 Cambridge

We will now see a case where we need to use integration by parts twice. We do this when our functions are cyclic e.g. trigonometric functions.

Suppose we want to find $\displaystyle\int e^x \sin 2x\ \mathrm{d}x$

Following the ILATE rule where trigonometric functions (T) come before exponential functions (E),

$$\begin{aligned} u &= \sin 2x & \frac{\mathrm{d}u}{\mathrm{d}x} &= 2\cos 2x\\ \frac{\mathrm{d}v}{\mathrm{d}x} &= e^x & v &= e^x \end{aligned}$$

$$\leadsto \int e^x \sin 2x\ \mathrm{d}x = e^x \sin 2x - 2\int e^x \cos 2x\ \mathrm{d}x$$

We have a product where reversing the chain and trigonometric identities don't work, and it isn't a standard integral. We must use integration by parts again! Let us take the new integral alone.

To find $\displaystyle\int e^x \cos 2x\ \mathrm{d}x$, we use integration by parts. Using the ILATE rule,

$$\begin{aligned} u &= \cos 2x & \frac{\mathrm{d}u}{\mathrm{d}x} &= -2\sin 2x\\ \frac{\mathrm{d}v}{\mathrm{d}x} &= e^x & v &= e^x \end{aligned}$$

$$\leadsto \int e^x \cos 2x\ \mathrm{d}x = e^x \cos 2x + 2\int e^x \sin 2x\ \mathrm{d}x$$

The resulting integral again requires integration by parts. However, it would be pointless to reiterate the process as we are basically going in a loop! But wait: the resulting integral here is equal to the integral we had to find at the beginning of the question! Lets substitute this integral into our main equation

$$\int e^x \sin 2x \, \mathrm{d}x = e^x \sin 2x - 2\left(e^x \cos 2x + 2\int e^x \sin 2x \, \mathrm{d}x\right)$$

$$\int e^x \sin 2x \, \mathrm{d}x = e^x \sin 2x - 2e^x \cos 2x - 4\int e^x \sin 2x \, \mathrm{d}x$$

$$5\int e^x \sin 2x \, \mathrm{d}x = e^x \sin 2x - 2e^x \cos 2x$$

$$\int e^x \sin 2x \, \mathrm{d}x = \frac{1}{5}\left(e^x \sin 2x - 2e^x \cos 2x\right)$$

In these questions, getting it right is all about keeping your solution neat and tidy.

8 Partial fraction decomposition

Integrals where partial fraction decomposition is required are usually given directly after using partial fraction decomposition to break the integrand into simpler fractions; hence, it would be fairly easy to find those integrals. However, we will assume that is not the case.

Our cues to use partial fraction decomposition is when the integrand is a fraction where the denominator is a degree 2 and above polynomial that we cannot integrate directly by reversing the chain rule, integration by parts or anything else; the fraction has a polynomial denominator that is already factored; the faction is not in the "$\ln x$" integral form, and, very importantly, *cannot* be simplified by long division.

Remember that the goal of this paper is to aid in the process of identifying which integration technique we should use for some integral; consequentially, we will not go over how to use partial fraction decomposition.

We will cut the examples short as it can get pretty lengthy here.

Example 1:

$$\int \frac{1}{(x-1)(x-2)^2} \, \mathrm{d}x$$

This is obviously a partial decomposition question.

Example 2:

$$\int \frac{x^3}{x^2+2x-15} \, \mathrm{d}x$$

First of all, this is definitely not in our "$\ln x$" integral form. We can start things off by simplifying the fraction using long polynomial division to get the integral

$$\int \left(x+2+\frac{19x+30}{x^2+2x-15}\right) \, \mathrm{d}x = \int (x+2) \, \mathrm{d}x + \int \left(\frac{19x+30}{x^2+2x-15}\right) \, \mathrm{d}x$$

The first integral in the RHS is trivial. The second integral on the other hand, not so much. Let us first test if this is a "$\ln x$" integral.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+2x-15\right) &\stackrel{?}{=} k(19x+30)\\ 2x+2 &\stackrel{?}{=} k(19x+30)\\ 2x+2 &\neq k(19x+30)\end{aligned}$$

Hence, we don't have a "$\ln x$" integral on our hand. We obviously can't reverse the chain rule or use integration by parts. Therefore, we only have partial fraction decomposition as our only possible technique to try.

$$\begin{aligned}\int\left(\frac{19x+30}{x^2+2x-15}\right)\,\mathrm{d}x &= \int\left(\frac{19x+30}{(x+5)(x-3)}\right)\,\mathrm{d}x\\ &= \frac{1}{8}\int\left(\frac{65}{x+5}+\frac{87}{x-3}\right)\,\mathrm{d}x\end{aligned}$$

which is an integral that we can solve. However, it is highly unlikely you would receive this integral like this: usually, you would have the quadratic polynomial factored already to hint for the use of partial fraction decomposition.