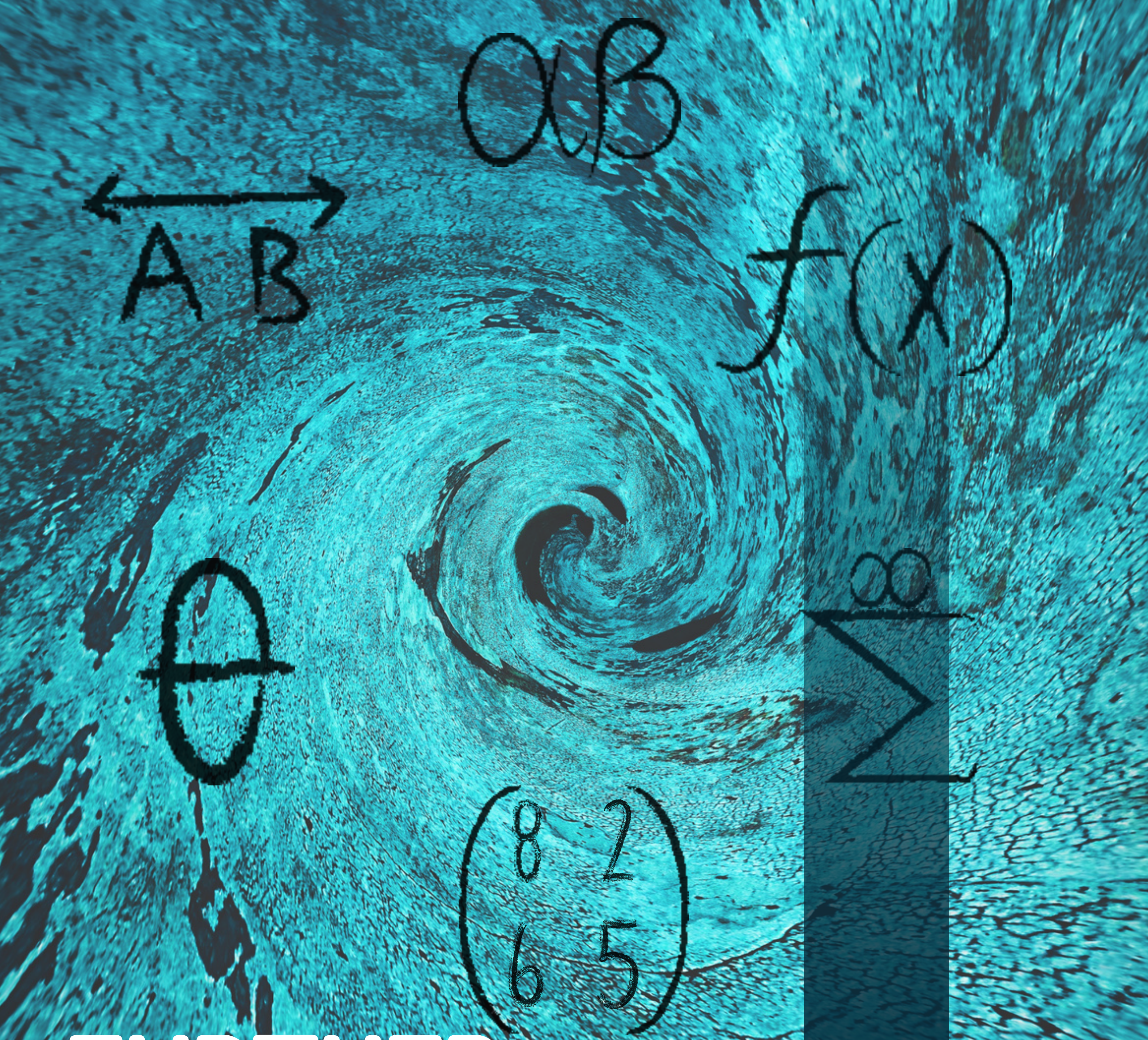

FURTHER MATHEMATICS 1

FOR STUDENTS
BY A STUDENT

YOUSEF IBRAHIM

Coursebook

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Further Pure Mathematics 1

For CAIE's 9231 syllabus

by Yousef Ibrahim

Preface

This book is intended to help self-studying students master the Further Pure Mathematics 1 syllabus (9231). I wrote this book because I found that the endorsed textbooks for this syllabus are designed for classroom use with a teacher. My goal was to write a book that provides self-studying students with clear and concise explanations of the material, as well as techniques for solving problems. The explanations are supported by solved examples that have been carefully chosen to illustrate key concepts.

To begin this book and the Further Pure Mathematics 1 syllabus, you must have completed all pure modules of the 9709 syllabus. You should also have a strong understanding of algebra, trigonometry, and calculus. If you plan on studying further mechanics or further statistics, you will also need to complete the A level Mechanics module and A level Statistics 1. While all of Statistics 2 is not required, it is strongly recommended that you complete the inferential statistics section.

This book is not a professional textbook, but it does provide detailed explanations of the material. The book is designed to be easy to read and understand, and it only includes information that you will need and use. Therefore, it is important that you read the entire book carefully, even if you feel like you already know some of the material.

I hope that you enjoy studying from this book and that you find it helpful in your mastery of Further Pure Mathematics 1. I believe that mathematics is a beautiful and powerful subject, and I hope that you will appreciate its beauty.

Please e-mail me on `yousef.n.ibrahim@terrasanta.edu.jo` in case of errors.

I wish you an experience that will further deepen your love for mathematics.

Yousef Ibrahim.

Acknowledgements

I would like to dedicate this page to the reader who picked up this book to give it a chance. This book is the single effort of a student that took approximately 350 hours to complete. The idea first began as a scan of my notes mixed with newly prepared notes before blooming into this. Thank you for giving me and this book a chance.

However, when it comes to help regarding my relationship with mathematics as a subject, I would dearly like to thank two teachers. Firstly, my 7^{th} grade math teacher Mrs. Duha Omat who really cared for my grades and performance when I didn't; getting scolded out of care during recess about a below expected test result was a monthly activity. Secondly, my O level/IGCSE math teacher Mr. Daoud Gammoh who really was the propeller behind everything in mathematics, and many things outside of it.

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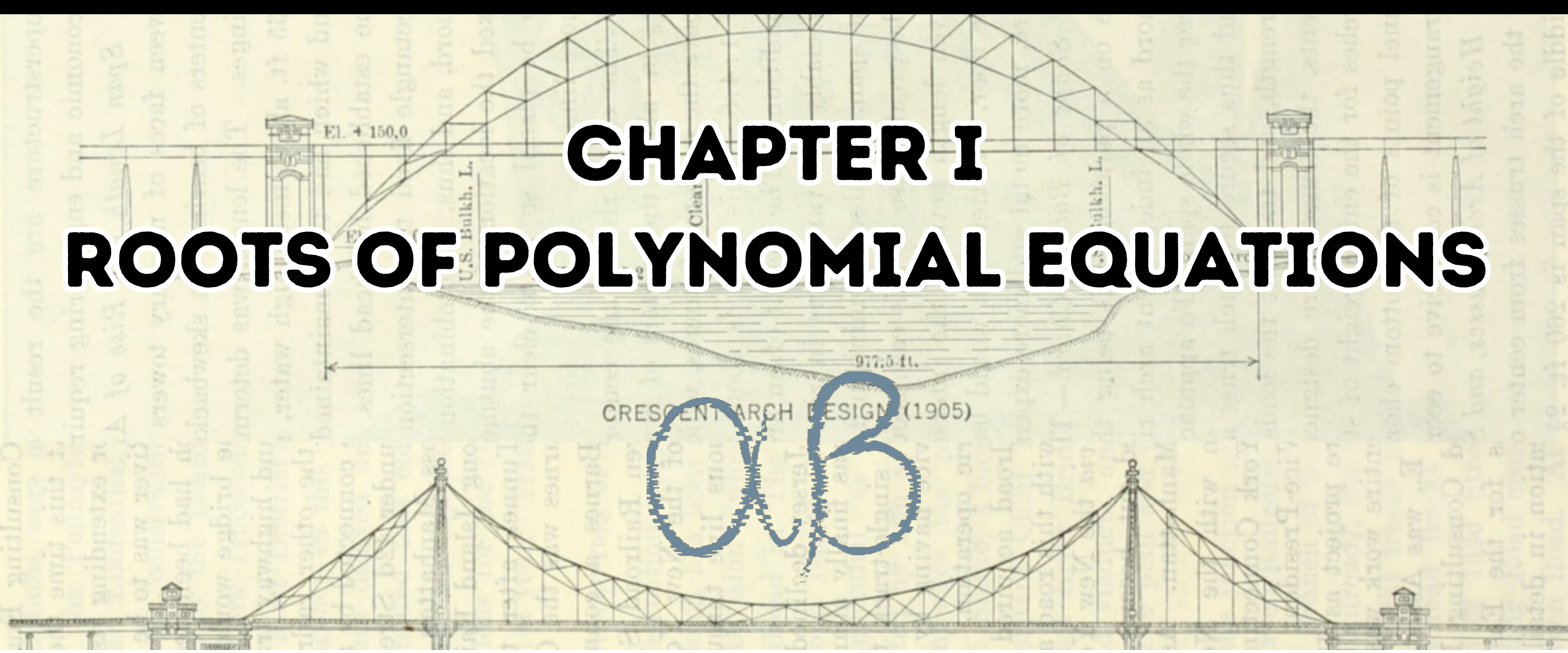
CHAPTER I
ROOTS OF POLYNOMIAL EQUATIONS



Chapter I

Roots of polynomial equations

What are polynomials?

Polynomials are algebraic expressions made up of terms added together. A term is a variable raised to a non-negative integers, multiplied with a real number; the real number is called the coefficient of that variable/ The variables in a polynomial are typically represented by symbols like x or y, while the coefficients are constants. The "degree" of a polynomial represents the highest power of the variable in the polynomial expression.

Polynomials play a crucial role in various areas of mathematics, including algebra, calculus, number theory, and more. They are used to model and solve a wide range of mathematical problems, such as finding roots (solutions) of equations, graphing functions, interpolation, and approximation. Additionally, polynomials have numerous applications in science, engineering, computer science, economics, and many other fields. In engineering polynomials are used to design and analyze structures, such as bridges, buildings, and airplanes. For example, a polynomial can be used to model the stress and strain on a bridge as it supports a load.

Polynomials will play a crucial role in creating Taylor and Maclaurin series for functions in the differentiation chapter for the Further Pure Mathematics 2 module.

A historical introduction

The use of polynomials dates back to the ancient Babylonians where polynomial equations, particularly quadratic equations, were solved geometrically. The ancient Egyptians applied this knowledge in practical areas like construction. After the Greeks laid out the foundation for algebraic concepts, Al-Khwarizmi's substantial advancements in the field of algebra developed methods to systematically solve linear and quadratic equations.

In this chapter, we will be exploring François Viète's work in relating the coefficients of polynomial equations and the roots

1.1 Coefficients of polynomial expressions and their roots

Quadratics

Let us consider the general quadratic equation:

$$ax^2 + bx + c = 0$$

We can divide both sides by a to get:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Suppose that the equation has roots α and β. It is then true that

$$\begin{aligned} (x-\alpha)(x-\beta) &= 0 \\ x^2 - (\alpha+\beta)\,x + \alpha\beta &= 0 \end{aligned} \quad \textit{Expand and factorise common terms in } x$$

If we compare coefficient we get the following equations:

Root-sum relationship

$$\begin{aligned} -(\alpha+\beta) &= \frac{b}{a} \\ \alpha+\beta &= -\frac{b}{a} \end{aligned} \tag{1.1}$$

Root-product relationship

$$\alpha\beta = \frac{c}{a} \tag{1.2}$$

In words:

- the sum of roots for a quadratic equation is equal to the negative of x's coefficient in the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
- the product of roots for a quadratic equation is equal to the coefficient of x^0 in the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Example: The sum of roots for $10x^2 + 40x + 40$ is equal to -$\frac{40}{10} = -4$

Solving the equation

$$10x^2 + 40x + 40 = 0$$

we get that the roots are -2 and -2 (repeated root) which have a sum of -4

We can also establish other useful results. But before we do that, it's time to introduce some useful notation that will make our lives easier. The notations will make use of the sigma function and recurrence relation.

Through out this <u>whole</u> chapter:

1. Using sigma function
   - $\sum \alpha$ denotes the sum of roots

     Example: for a quadratic expression with roots α and β

     $$\sum \alpha = \alpha + \beta$$

   - $\sum \alpha\beta$ denotes the sum of all possible distinct root products (discussed more later)

     Example: for a quadratic expression with roots α and β

     $$\sum \alpha\beta = \alpha\beta$$

2. Using recurrence relation
   - $S_n = \alpha^n + \beta^n + \dots$

     Example: for a quadratic expression with roots α and β

     $$S_1 = \alpha + \beta$$
     $$S_2 = \alpha^2 + \beta^2$$

We can now comfortably establish an important result that is periodically tested in exams.

Suppose we want to find $\frac{1}{\alpha} + \frac{1}{\beta}$ where α and β are the roots of a quadratic expression. It is impossible to directly find the result of the sum as our two equation relate the sum and product, so we must find a way to make use of our results.

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta}$$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{\sum \alpha}{\sum \alpha\beta}$$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\sum \alpha}{\sum \alpha\beta}$$

Which is a result that we can compute given the quadratic expression.

So far, we learned how to compute the sum, product and the reciprocal sum of roots. We will now learn how to find the sum of root to a power n where n is an integer. We will now make use of recurrence relation.

Suppose we have a quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Assume that the roots of the equation are α and β where α and β are real numbers $\implies$ α and β are solutions to the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

It its then true that

$$\alpha^2 + \frac{b}{a}\alpha + \frac{c}{a} = 0 \tag{1}$$

$$\beta^2 + \frac{b}{a}\beta + \frac{c}{a} = 0 \tag{2}$$

$$\xrightarrow[\text{equations}]{\text{add both}} (\alpha^2 + \beta^2) + \frac{b}{a}(\alpha + \beta) + \frac{c}{a} + \frac{c}{a} = 0$$

$$S_2 + \frac{b}{a}S_1 + 2 \times \frac{c}{a} = 0$$

$$\therefore S_2 = -\left(\frac{b}{a}S_1 + 2 \times \frac{c}{a}\right)$$

There is also a general formula for S_2 for all polynomials no matter what degree they are.

We know that $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$. So

$$\begin{aligned}(\alpha + \beta)^2 &= \alpha^2 + 2\alpha\beta + \beta^2 \\ \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ \underbrace{\sum \alpha^2 = \left(\sum \alpha\right)^2}_{\text{Note the difference!}} &- 2\sum \alpha\beta\end{aligned} \tag{1.3}$$

Example: Find the square and cubic sum of roots $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ for the equation $2x^2 + 6x + 4 = 0$

Solution:

$\underline{\alpha^2 + \beta^2}$:

$$2\alpha^2 + 6\alpha + 4 = 0 \tag{1}$$

$$2\beta^2 + 6\beta + 4 = 0 \tag{2}$$

$$2(\alpha+\beta)^2 + 6(\alpha+\beta) + 4 \times 2 = 0$$

$$2S_2 + 6S_1 + 8 = 0$$

$$2S_2 = -8 - 6S_1$$

$$\to S_1 = -\frac{6}{2} = -3$$

$$\therefore \alpha^2 + \beta^2 = \frac{-8 - 6 \times (-3)}{2}$$

$$= \frac{10}{2}$$

$$\alpha^2 + \beta^2 = 5$$

OR:

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

$$= \left(-\frac{6}{2}\right)^2 - 2\left(\frac{4}{2}\right)$$

$$= 9 - 2 \times 2$$

$$\sum \alpha^2 = 5$$

$$\therefore \alpha^2 + \beta^2 = 5$$

In general you always want to use the second method as it is faster and easier. Another reminder that this equation is universal for all polynomials.

When the question asks for the sum of roots to the power of $n \{n \in \mathbb{Z} \,|\, n \neq -1, 1, 2\}$, we always use recurrence relation, multiplying by a power of the variable to achieve the desired outcome. Obviously, this is because we have achieved formulas for the discarded values of n which are easier to work with.

$\underline{\alpha^3 + \beta^3}$

The steps of procedure are very similar to finding $\alpha^2 + \beta^2$; however, we must find a way to include S_3 in our equation. We shall do this by multiplying both sides of the equation by x

$$(2x^2 + 6x + 4 = 0) \times x$$

$$2x^3 + 6x^2 + 4x = 0$$

$$2\alpha^3 + 6\alpha^2 + 4\alpha = 0 \tag{1}$$

$$2\beta^3 + 6\beta^2 + 4\beta = 0 \tag{2}$$

$$2(\alpha^3 + \beta^3) + 6(\alpha^2 + \beta^2) + 4(\alpha + \beta) = 0$$

$$2S_3 + 6S_2 + 4S_1 = 0$$

$$2S_3 = -6S_2 - 4S_1$$

$$2S_3 = -6(5) - 4(-3)$$

$$S_3 = \frac{-18}{2}$$

$$\therefore S_3 = -9$$

The step of multiplying by a variable (x most commonly) raised to some power n where n is an integer to find the sum of roots raised to some integer value n is a common step among all of these questions. For example, if we want to find S_{-1} we would multiply the equation by x^{-1}. In general, all questions asked on the exam have a systematic approach.

Cubics

Suppose the cubic $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ has roots α, β and γ. It is then true that

$$\alpha + \beta + \gamma = \sum \alpha = -\frac{b}{a}\ , \text{ The sum of all distinct roots}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \sum \alpha\beta = \frac{c}{a}\ , \text{ The sum of all distinct double products of roots}$$

$$\alpha\beta\gamma = \sum \alpha\beta\gamma = -\frac{d}{a}\ , \text{ The sum of all distinct triple products of roots}$$

Proof for results is omitted as they are not required in the first place. In this case, the proof is quite simple; it is identical to that of the quadratic. However, instead of $(x-\alpha)(x-\beta) = 0$ we have $(x-\alpha)(x-\beta)(x-\gamma) = 0$ for obvious reasons. We then expand, collect terms and compare coefficients.

We also have the result

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\sum\left(\frac{1}{\alpha}\right)=\frac{\sum\alpha\beta}{\sum\alpha\beta\gamma}$$

Again proving the result is identical to proving the corresponding results for quadratics and is therefore omitted.

Questions regarding cubics are more or less the same as those for quadratics.

Example For the cubic equation $x^3-3x^2+4=0$ with roots α, β and γ, find the value of S_{-3} and S_3

Solution:

Let us first find S_3. We will skip adding the equations satisfied by the roots and jump into the result first hand.

$$S_3-3S_2+4\times 3=0$$

$$S_3=3S_2-12$$

$$S_3=3\left[\left(\sum\alpha\right)^2-2\sum\alpha\beta\right]-12$$

$$\begin{aligned} S_3 &= 3\left[(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)\right]-12 \\ &= 3[(-(-3))^2-2(0)]-12 \\ &= 3(9)-12 \\ &= 15 \end{aligned}$$

where 0 is the coefficient of x in $x^3-3x^2+4=0$

$$\therefore S_3=15$$

To find S_{-3} we have to multiply the equation by x^{-3} to obtain a term in x^{-3}, and thus obtain S_{-3} in the sum of our equations where $x=$ the roots

$\therefore$ for S_{-3}

$$(x^3 - 3x^2 + 4 = 0) \times x^{-3}$$

$$1 + 3x^{-1} + 4x^{-3} = 0$$

$$\rightsquigarrow 1 \times 3 + 3S_{-1} + 4S_{-3} = 0$$

$$4S_{-3} = -3 - 3S_{-1}$$

$$4S_{-3} = -3 - 3\left(\frac{\sum \alpha\beta}{\sum \alpha\beta\gamma}\right)$$

$$4S_{-3} = -3 - 3 \times \frac{0}{-4}$$

$$S_{-3} = -\frac{3}{4}$$

Quartics

Again, the questions follow the same ideas but using different results since we are dealing with a 4$^{\text{th}}$ degree polynomial

Suppose the quartic equation $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$ has roots α, β, γ and δ. It is then true that

$$\alpha + \beta + \gamma + \delta = \sum \alpha = -\frac{b}{a}\ , \ \textit{Previous definition holds}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \sum \alpha\beta = \frac{c}{a}\ , \ \textit{Previous definition holds}$$

$$\alpha\beta\gamma\delta = \sum \alpha\beta\gamma = -\frac{d}{a}\ , \ \textit{Previous definition holds}$$

$$\alpha\beta\gamma\delta = \sum \alpha\beta\gamma\delta = \frac{e}{a}\ , \ \text{The sum of all distinct quadruple product of roots}$$

We also have this result

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \sum\left(\frac{1}{\alpha}\right) = \frac{\sum \alpha\beta\gamma}{\sum \alpha\beta\gamma\delta}$$

We can now see why we introduced the root-sigma notation.

You may now see the pattern for the sum of reciprocal roots for different polynomial degrees. This pattern will aid in memorization:

Type of Polynomial	Result
Quadratic	$\frac{\sum \alpha}{\sum \alpha\beta}$
Cubic	$\frac{\sum \alpha\beta}{\sum \alpha\beta\gamma}$
Quartic	$\frac{\sum \alpha\beta\gamma}{\sum \alpha\beta\gamma\delta}$

Example: Given that $x^4 + 3x^2 - x + 5 = 0$ has roots α, β, γ and δ, find $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$

Solution:

To obtain a term in x^3, we must multiply by x^{-1}

$$(x^4 + 3x^2 - x + 5 = 0) \times x^{-1}$$

$$x^3 + 3x - 1 + 5x^{-1} = 0$$

Now, we use recurrence relation

$$S_3 + 3S_1 - 1 \times 4 + 5S_{-1} = 0$$

$$\begin{aligned} S_3 &= 4 - 3S_1 - 5S_{-1} \\ &= 4 - 3(-3) - 5\left(\frac{-(-1)}{5}\right) \\ S_3 &= 12 \end{aligned}$$

1.2 Substitutions

Some problems arise when trying to find S_8 for any given degree of a polynomial. This is where we employ substitution techniques. Substitution techniques allow us to construct new polynomial equations that have roots altered from the original (it will make sense in a bit).

Forming new equations

Suppose we have the quadratic equation $x^2 + 5x + 8 = 0$ with roots α and β, and we are asked to find a quadratic equation with roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$. What can we do?

Method 1:

We want a polynomial equation in terms of a variable (we will use y as x is already taken) that has the roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$.

Since

$$(\alpha)^2 + 5(\alpha) + 8 = 0$$

then

$$\left(2 \times \frac{\alpha}{2}\right)^2 + 5\left(2 \times \frac{\alpha}{2}\right) + 8 = 0$$

$$2^2\left(\frac{\alpha}{2}\right)^2 + 10\left(\frac{\alpha}{2}\right) + 8 = 0$$

$$4\left(\frac{\alpha}{2}\right)^2 + 10\left(\frac{\alpha}{2}\right) + 8 = 0$$

$$\therefore 4y^2 + 10y + 8 = 0 \text{ has the root } \frac{\alpha}{2}$$

Similarly

$$(\beta)^2 + 5(\beta) + 8 = 0$$

then

$$\left(2 \times \frac{\beta}{2}\right)^2 + 5\left(2 \times \frac{\beta}{2}\right) + 8 = 0$$

$$2^2\left(\frac{\beta}{2}\right)^2 + 10\left(\frac{\beta}{2}\right) + 8 = 0$$

$$4\left(\frac{\beta}{2}\right)^2 + 10\left(\frac{\beta}{2}\right) + 8 = 0$$

$$\therefore 4y^2 + 10y + 8 = 0 \text{ has the root } \frac{\beta}{2}$$

Therefore, the equation $4y^2 + 10y + 8 = 0$ has the roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$.

Although it was pretty simple and fast, this method becomes cumbersome when we involve addition and subtraction; for example, it would be very awkward to find an equation with the roots $\frac{\alpha+5}{2}$ and $\frac{\beta+5}{2}$, Let us develop a new method using the same equation.

Method 2:

Let $y = \frac{x}{2}$ when $x = \alpha$ or $x = \beta \to x = 2y$. We will now substitute this result into our polynomial equation to get

$$(2y)^2 + 5(2y) + 8 = 0$$

$$4y^2 + 10y + 8 = 0$$

$\therefore$ When $y = \frac{\alpha}{2}$ or $y = \frac{\beta}{2}$

$$x^2 + 5x + 8$$

$$=4y^4 + 10y + 8$$

$$=0$$

Where $y = \frac{\alpha}{2}$ or $y = \frac{\beta}{2}$ is equivalent to $x = \alpha$ or $x = \beta$ respectively, by our definition.

$$\implies 4y^2 + 10y + 8 = 0 \text{ has the roots } \frac{\alpha}{2} \text{ and } \frac{\beta}{2}.$$

In the exam, you only need to show the following steps:

1. State the substitution

   $\hookrightarrow$ *Let* $y = \frac{x}{2}$

2. Make x subject of the formula and substitute it in

   $\hookrightarrow x = 2y \to (2y)^2 + 5(2y) + 8 = 0$

3. Simplify and state the finishing statement

   $\hookrightarrow 4y^2 + 10y + 8 = 0$ has the roots has the roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$

This is the method that you should use when doing these types of questions in the exam. We will use it for the remaining of the chapter.

Example The cubic equation $x^3 + x^2 + 5 = 0$ has roots α, β and γ. Find a cubic equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Solution:

Let

$$y = \frac{1}{x} \to x = \frac{1}{y}$$

Substitute this result into the equation $x^3 + x^2 + 5 = 0$

$$\xrightarrow{x=\frac{1}{y}} \left(\frac{1}{y}\right)^3 + \left(\frac{1}{y}\right)^2 + 5 = 0$$

$$\left(\frac{1}{y^3} + \frac{1}{y^2} + 5 = 0\right) \times y^3$$

$$5y^3 + y + 1 = 0$$

$\therefore 5y^3 + y + 1 = 0$ has the roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$

We will now tackle finding polynomial equations with powers of roots. The procedure is a bit more challenging but it is nothing to worry about.

Suppose we have the cubic equation $x^3 + 7x^2 - 1 = 0$ with roots α, β and γ. Find a cubic equation with roots α^2, β^2 and γ^2

Solution.

It is apparent that the substitution we need is $y = x^2$. While we can use the substitution $x = \sqrt{y}$, it will be very annoying to deal with but not impossible (you can give it a try). We will try a different approach.

Let us try to make all the x variables raised to an even power so that we can substitute $y = x^2$ easily. This can be done by shifting some terms to the right hand side and raise both sides to the power of 2. By your mathematical intuition and mental trials, it will become apparent that what we must do is the following

$$x^3 + 7x^2 - 1 = 0$$

$$x^3 = 1 - 7x^2$$

$$\to \left(x^3\right)^2 = \left(1 - 7x^2\right)^2$$

$$x^6 = 49x^4 - 14x^2 + 1$$

$$\left(x^2\right)^3 - 49\left(x^2\right)^2 + 12\left(x^2\right) - 1 = 0$$

$$\xrightarrow{y=x^2} y^3 - 49y^2 + 12y - 1 = 0$$

$\therefore$ the equation $y^3 - 49y^2 + 12y - 1 = 0$ has the roots α^2, β^2 and γ^2

Example: Given that the cubic equation $x^3 + 5x^2 + 1 = 0$ has roots α, β and γ, find the cubic equation with roots α^3, β^3 and γ^3.

Solution:

Let $y = x^3$

$$x^3 + 5x^2 + 1 = 0$$

$$5x^2 = -1 - x^3$$

$$\left(5x^2\right)^3 = (-1 - x^3)^3 125x^6 = -(x^9 + 3x^6 + 3x^3 + 1)$$

$$\left(x^3\right)^3 + 128\left(x^3\right)^2 + 3(x^3) + 1 = 0$$

$$y^3 + 128y^2 + 3y + 1 = 0$$

$\therefore$ $y^3 + 128y^2 + 3y + 1 = 0$ has roots α^3, β^3 and γ^3

Using the new equations

While forming these new equations may seem useless, they simplify a certain problem.

Example Given that the cubic equation $x^3 + 5x^2 + 1 = 0$ has roots α, β and γ, Find:

i) the cubic equation with roots α^3, β^3 and γ^3.

ii) $\alpha^3 + \beta^3 + \gamma^3$

iii) $\alpha^6 + \beta^6 + \gamma^6$

iv) $\alpha^9 + \beta^9 + \gamma^9$

Solution:

i) Solved , $y^3 + 128y^2 + 3y + 1 = 0$

ii) $\alpha^3 + \beta^3 + \gamma^3$ is the sum of roots of the equation $y^3 + 128y^2 + 3y + 1 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = -128$

iii) $\alpha^6 + \beta^6 + \gamma^6$ is the sum of the squared roots of the equation $y^3 + 128y^2 + 3y + 1 = 0$

$\therefore \alpha^6 + \beta^6 + \gamma^6 = (-128)^2 - 2(3) = 16378$

iv) $\alpha^9 + \beta^9 + \gamma^9$ is the sum of the cubed roots of the equation $y^3 + 128y^2 + 3y + 1 = 0$. We will use recurrence relation for this

$$\left(\alpha^3\right)^3 + 128\left(\alpha^3\right)^2 + 3\left(\alpha^3\right) + 1 = 0 \tag{1}$$

$$\left(\beta^3\right)^3 + 128\left(\beta^3\right)^2 + 3\left(\beta^3\right) + 1 = 0 \tag{2}$$

$$\left(\gamma^3\right)^3 + 128\left(\gamma^3\right)^2 + 3\left(\gamma^3\right) + 1 = 0 \tag{3}$$

Adding the equations

$$S_9 + 128S_6 + 3S_1 + 1 \times 3 = 0$$

$$S_9 = -(128S_6 + 3S_1 + 3)$$

$$S_9 = -(128 \times 16378 + 3 \times -128 + 3)$$

$$S_9 = -2096003$$

These techniques can be adjusted for each question of this type.

In the exam, you will be asked to find the equation with altered roots and then for S_n where n is some integer; you will use your new equation to calculate required results

1.3 Important Exercises

Questions:

1. The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ.

   i) Use the substitution $y = (1 + x)^{-1}$ to show that the equation

   $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $(\alpha + 1)^{-1}$, $(\beta + 1)^{-1}$ and $(\gamma + 1)^{-1}$

   ii) For the cases $n = 1$ and $n = 2$, find the value of $(\alpha+1)^{-n}+(\beta+1)^{-n}+(\gamma+1)^{-n}$

   iii) Deduce the value of $(\alpha + 1)^{-3} + (\beta + 1)^{-3} + (\gamma + 1)^{-3}$

iv) Hence show that $\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\alpha+1)(\gamma+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36}$

9231 Paper 1 Q_7 November 2010

2. The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots α, β and γ.

   i) *Let* $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma$,$\alpha - \beta + \gamma$ and $\alpha + \beta - \gamma$

   ii) State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $(\beta\gamma)^{-1}$,$(\gamma\alpha)^{-1}$ and $(\alpha\beta)^{-1}$

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3. The equation $x^3 - x + 1 = 0$ has roots α,β and γ

   i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation $y^3 + 3y^2 + 2y + 1 = 0$ has roots α^3,β^3 and γ^3. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$.

Let $S_n = \alpha^n + \beta^n + \gamma^n$

   ii) Find the value of S_{-3}

   iii) Show that $S_6 = 5$ and find the value of S_9

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DETAILED SOLUTIONS ON THE NEXT PAGE. ATTEMPT BEFORE PROCEEDING

Solutions:

1. The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ.

   i)

$$\textit{Let } y = \frac{1}{1+x}$$

$$1 + x = \frac{1}{y}$$

$$x = \frac{1}{y} - 1 = \frac{1-y}{y}$$

Substituting our result

$$\left(\frac{1-y}{y}\right)^3 + 4\left(\frac{1-y}{y}\right) - 1 = 0$$

$$\left[\left(\frac{1-3y+3y^2-y^3}{y^3}\right) + 4\left(\frac{1-y}{y}\right) - 1 = 0\right] \times -y^3$$

$$y^3 - 3y^2 + 3y - 1 - 4y^2(1-y) + y^3 = 0$$

$$2y^3 - 3y^2 + 3y - 1 - 4y^2 + 4y^3$$

$$6y^3 - 7y^2 + 3y - 1 = 0 \quad \square$$

   ii)

$$\frac{1}{(\alpha+1)} + \frac{1}{(\beta+1)} + \frac{1}{(\gamma+1)}$$

and

$$\frac{1}{(\alpha+1)^2} + \frac{1}{(\beta+1)^2} + \frac{1}{(\gamma+1)^2}$$

are the sum of roots and the sum of squared roots of the equation $6y^3 - 7y^2 + 3y - 1 = 0$ respectively

$$\therefore S_1 = \frac{1}{(\alpha+1)} + \frac{1}{(\beta+1)} + \frac{1}{(\gamma+1)} = -\frac{-7}{6} = \frac{7}{6}$$

and

$$\therefore S_2 = \frac{1}{(\alpha+1)^2} + \frac{1}{(\beta+1)^2} + \frac{1}{(\gamma+1)^2} = \left(\frac{7}{6}\right)^2 - 2\left(\frac{3}{6}\right)$$

$$= \frac{13}{36}$$

iii) $(\alpha+1)^{-3}+(\beta+1)^{-3}+(\gamma+1)^{-3}$ is the sum of cubed roots of $6y^3-7y^2+3y-1=0$. Using recurrence relation we get

$$6S_3 - 7S_2 + 3S_1 - 1 \times 3 = 0$$

$$6S_3 = 3 + 7S_2 - 3S_1$$

$$S_3 = \frac{1}{6}\left(3 + 7 \times \frac{13}{36} - 3 \times \frac{7}{6}\right)$$

$$= \frac{1}{6}\left(\frac{73}{36}\right)$$

$$\therefore S_3 = \frac{73}{216} \quad \square$$

iv) We will attempt to manipulate the expression algebraically to obtain parts that can be substituted in

$$\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\alpha+1)(\gamma+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2}$$

$$= \frac{(\alpha+1)(\beta+1)(\gamma+1)}{(\alpha+1)^3} + \frac{(\alpha+1)(\beta+1)(\gamma+1)}{(\beta+1)^3} + \frac{(\alpha+1)(\beta+1)(\gamma+1)}{(\gamma+1)^3}$$

$$= (\alpha+1)(\beta+1)(\gamma+1)\left(\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}\right)$$

$$= \left(\frac{1}{(\alpha+1)(\beta+1)(\gamma+1)}\right)^{-1}\left(\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}\right)$$

$$= \left(-\frac{-1}{6}\right)^{-1} \times \frac{73}{216}$$

$$= 6 \times \frac{73}{216}$$

$$= \frac{73}{36}$$

$$\therefore \frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\alpha+1)(\gamma+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36} \quad \square$$

2. i)

$$\begin{aligned} u + 2\alpha &= -\alpha + 2\alpha + \beta + \gamma \\ &= \alpha + \beta + \gamma \\ &= -(-1) \\ &= 1 \\ \therefore u + 2\alpha &= 1 \quad \square \end{aligned}$$

For the second part of the question, it is a bit more tricky.
We know that

$$\alpha + \beta + \gamma = 1$$

We can then deduce the 3 following results

$$1 - 2\alpha = -\alpha + \beta + \gamma \tag{1}$$

$$1 - 2\beta = \alpha - \beta + \gamma \tag{2}$$

$$1 - 2\gamma = \alpha + \beta - \gamma \tag{3}$$

Therefore, we want a cubic equation with roots $1 - 2\alpha$, $1 - 2\beta$ and $1 - 2\gamma$

$$Let \;\; y = 1 - 2x \to x = \frac{1-y}{2}$$

Substituting our result in, we get

$$\left(\frac{1-y}{2}\right)^3 - \left(\frac{1-y}{2}\right)^2 - 3\left(\frac{1-y}{2}\right) - 10 = 0$$

$$\left[\left(\frac{(1-y)^3}{8}\right) - \left(\frac{(1-y)^2}{4}\right) - 3\left(\frac{1-y}{2}\right) - 10 = 0\right] \times 8$$

$$\left[(1 - 3y + 3y^2 - y^3) - 2(y^2 - 2y + 1) - 12(1 - y) - 80 = 0\right] \times -1$$

$$y^3 - 3y^2 + 3y - 1 + 2y^2 - 4y + 2 + 12 - 12y + 80 = 0$$

$$y^3 - y^2 - 13y + 93 = 0$$

$\therefore y^3 - y^2 - 13y + 93 = 0$ has the roots $1 - 2\alpha = -\alpha + \beta + \gamma$, $1 - 2\beta = \alpha - \beta + \gamma$ and $1 - 2\gamma = \alpha + \beta - \gamma$

ii) For the first part of the question

$$\alpha\beta\gamma = -(-10) = 10$$

For the second part of the question, we must manipulate the roots algebraically to get a result that we can apply some of the results that we have found. Notice how the first part of the question asked for the value of $\alpha\beta\gamma$, so we probably have to achieve $\alpha\beta\gamma$ when manipulating $\frac{1}{\beta\gamma}$, $\frac{1}{\alpha\gamma}$ and $\frac{1}{\alpha\beta}$

$$\frac{1}{\beta\gamma} = \frac{\alpha}{\alpha\beta\gamma} = \frac{\alpha}{10} \qquad \text{(root 1)}$$

$$\frac{1}{\alpha\gamma} = \frac{\beta}{\alpha\beta\gamma} = \frac{\beta}{10} \qquad \text{(root 2)}$$

$$\frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = \frac{\gamma}{10} \qquad \text{(root 3)}$$

Our substitution is more apparent now

$$\textit{Let } y = \frac{x}{10} \rightarrow x = 10y$$

$$\xrightarrow{x=10y} (10y)^3 - (10y)^2 - 3(10y) - 10 = 0$$

$$1000y^3 - 100y^2 - 30y - 10 = 0$$

$$100y^3 - 10y^2 - 3y - 1 = 0$$

3. i) $x = y^{\frac{1}{3}} \iff x^3 = y$
We will now start setting up our equation for substitution

$$x^3 - x + 1 = 0$$

$$x^3 + 1 = x$$

$$(x^3 + 1)^3 = x^3$$

$$x^9 + 3x^6 + 3x^3 + 1 - x^3 = 0$$

$$(x^3)^3 + 3(x^3)^2 + 2(x^3) + 1 = 0$$

$$\xrightarrow{y=x^3} y^3 + 3y^2 + 2y + 1 = 0 \quad \square$$

The sum of cubed roots of $x^3 - x + 1$ is the sum of roots of $y^3 + 3y^2 + 2y + 1 = 0$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = -3$$

ii) S_{-3} is the sum of reciprocal roots of the equation $y^3 + 3y^2 + 2y + 1 = 0$

$$S_{-3} = \frac{2}{-1} = -2$$

iii) S_6 is the sum of squared roots of the equation $y^3 + 3y^2 + 2y + 1 = 0$

$$S_6 = (-3)^2 - 2(2) = 5 \quad \square$$

Using recurrence relation with the equation $y^3 + 3y^2 + 2y + 1 = 0$

$$S_9 + 3S_6 + 2S_3 + 1 \times 3 = 0$$

$$\begin{aligned} S_9 &= -(3S_6 + 2S_3 + 3) \\ &= -\left(3 \times 5 + 2 \times -3 + 3\right) \end{aligned}$$

$$S_9 = -12$$

Key Results

Quadratics

$$\sum \alpha = -\frac{b}{a}$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\sum \left(\frac{1}{\alpha}\right) = \frac{\sum \alpha}{\sum \alpha\beta}$$

Cubics

$$\sum \alpha = -\frac{b}{a}$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\sum \alpha\beta\gamma = -\frac{d}{a}$$

$$\sum \left(\frac{1}{\alpha}\right) = \frac{\sum \alpha\beta}{\sum \alpha\beta\gamma}$$

Quartics

$$\sum \alpha = -\frac{b}{a}$$

$$\sum \alpha\beta = \frac{c}{a}$$

$$\sum \alpha\beta\gamma = -\frac{d}{a}$$

$$\sum \alpha\beta\gamma\delta = \frac{e}{a}$$

$$\sum \left(\frac{1}{\alpha}\right) = \frac{\sum \alpha\beta\gamma}{\sum \alpha\beta\gamma\delta}$$

General

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

For a polynomial

$$x^n + kx^{n-1} + tx^{n-2} + sx^{n-3} + \cdots + c = 0$$

of the n^{th} degree, it is true that

$$S_n + kS_{n-1} + tS_{n-2} + sS_{n-3} + \cdots + c \times n = 0$$

Where k, t, s and c are real constants

CHAPTER II
RATIONAL FUNCTIONS

$f(x)$

Chapter II

Rational functions

What are rational functions?

A rational function is a function that can be written as the ratio of two polynomials, where the denominator polynomial is not equal to zero. In other words, a rational function is a fraction in which the numerator and denominator are both polynomials.

Rational functions are a powerful tool that can be used to model and solve a variety of problems in many different fields. They are used in economics, engineering, finance, physics, and chemistry. In economics and finance, rational functions are used to model cost and revenue functions and utility functions. In engineering, rational functions can be used to design and analyze electrical circuits, mechanical systems, and other engineering systems. In chemistry, rational functions can be used to model reaction rates and equilibrium constants. This is just a shallow look into the applications of rational functions in *some* fields.

A historical introduction

Rational functions were first used by mathematicians in ancient Greece to solve polynomial equations. In the 9^{th} century, Al-Khwarizmi began using rational functions to solve linear and quadratic equations.

During the 17^{th} century, René Descartes made a breakthrough when he introduced the Cartesian or Rectangular coordinate system; this system made it possible to represent rational functions in a more convenient way, which led to a great deal of progress in the study of rational functions.

Enter Leonard Euler and Carl Gauss, two of the most important mathematicians of all time.Euler was the first to show that rational functions could be used to represent many different kinds of mathematical objects, such as curves, surfaces, and volumes. He also developed new methods for studying the properties of rational functions. Gauss was the first to develop a systematic way of classifying rational functions. He also showed that rational functions could be used to solve many different kinds of problems in diverse fields of mathematics and science. The contributions of Euler and Gauss laid the foundation for much of the subsequent work on rational functions. Their work is still used today by mathematicians and scientists all over the world.

Steps of sketching graphs

To sketch curves effectively, we must find some key aspects of the function. In the exam, the question will progress linearly, asking you to find these crucial features before sketching. In short, to sketch a rational function, you must find:

1. any vertical asymptotes if present
2. the horizontal or oblique asymptote (if present)
3. intersections with the coordinate axes
4. turning points or points of maxima and minima

We will go through each component before learning how to sketch.

2.1 Vertical asymptotes

What is a vertical asymptote? A vertical asymptote is a *discontinuity* in the graph of a function. Rational functions very frequently have these discontinuities, due to the polynomial denominator. A discontinuity is formed due to the function being undefined at a point; when the denominator is 0, the function is undefined and there is a vertical asymptote for that input value.[1]The input can never be equal to the value that produced a 0 denominator/an undefined result, it can only approach this value, which is synonomous to saying that the curve approaches the vertical asymptote.

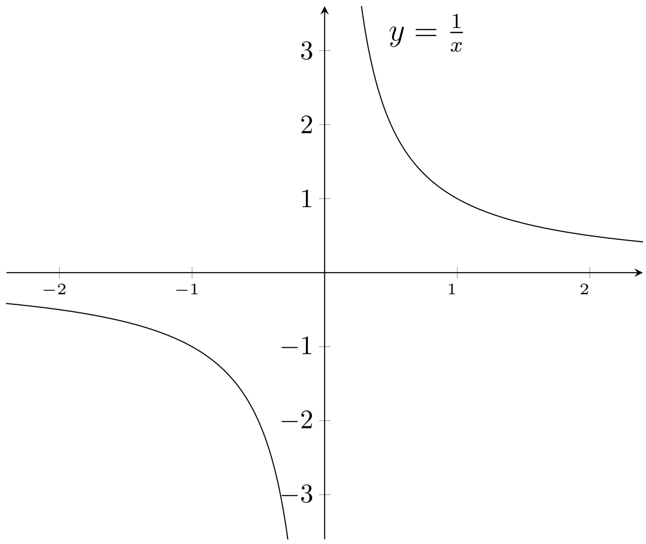
In this book, any asymptotes, vertical or horizontal, will be drawn as a dashed line. You should already have a rough idea about vertical and horizontal asymptotes, though any previous knowledge will be ignored.

[1]Although we can get a hole in our curve instead of a vertical asymptote, this case is not covered in the Further Mathematics syllabus. Therefore, always assume that we have a vertical asymptote for the input value by which the denominator is equal to 0

Consider the function

$$y = \frac{1}{x} \quad \text{(see diagram below)}$$

We know that we will have a vertical asymptote at $x = 0$ because when $x = 0$, the denominator is equal to 0 which produces and undefined result. To test behaviour near the vertical asymptote, we must take values that only slightly larger and smaller than 0. Plugging in $x = 0.0001$ and $x = -0.0001$, we get $y = 10000$ and $y = -10000$ respectively. Although we did not discuss how we identified the horizontal asymptote, you should know the graph of this function from IGCSE Mathematics.



Let us now take

$$y = \frac{1}{x-1}$$

Setting the denominator to 0, we get that we have a vertical asymptote at $x = 1$. Taking points around $x = 1$, we establish that the function approaches $+\infty$ as $x \to 1$ from the right hand side and it approaches $-\infty$ as $x \to 1$ from the left hand side. In more formal notation we have

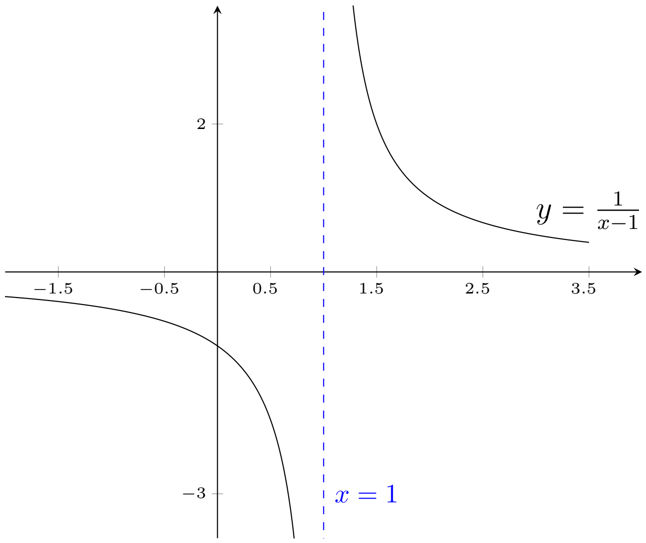
$$\lim_{\substack{x\to 1\\ x>1}} \frac{1}{x-1} = +\infty$$

and

$$\lim_{\substack{x\to 1\\ x<1}} \frac{1}{x-1} = -\infty$$

We should also consider any intersection with the coordinates axis as we have one with the y-axis here. Setting $x = 0$ we get $y = -1$ which is the y-intercept. Again the horizontal asymptote will not be considered here but from previous work you should tell that it is $y = 0$. Note that we are not attempting to find the turning points as we already know that this graph and the one above don't have any.

We are now ready to sketch. You should get something like the diagram below



Finally lets look at a case where the denominator is a quadratic expression.

Suppose we want to graph

$$y = \frac{1}{3x^2 + 2x - 5}$$

Though we still haven't tackled how to find horizontal asymptotes, you may guess that it is $y = 0$. For the vertical asymptotes, we must set our denominator equal to 0 and solve for the x values. Solving $3x^2 + 2x - 5 = 0$, we get that $x = 1$ and $x = -\frac{5}{3}$ which are the equations of our vertical asymptotes. We must then attempt to find the intersections with the coordinate axis.

It is obvious that there are no intersections with the x-axis as $y \neq 0$

For our y intercept, by letting $x = 0$, we get that $y = -\frac{1}{5}$

Now we move on to find any turning points.

We already know that $y' = 0$ at all turning points, so we will start things off by finding the derivative of our function.

By the quotient rule.

$$y' = \frac{-6x - 2}{(3x^2 + 2x - 5)^2}$$

Letting $y' = 0$, we get the x-coordinate of our turning point

$$-6x - 2 = 0$$

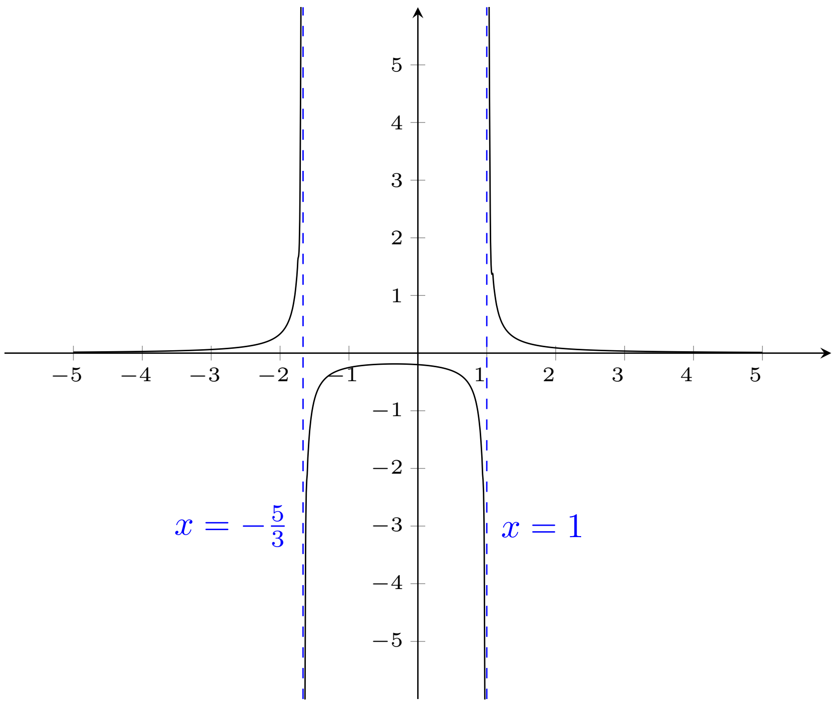
$$x = -\frac{1}{3}$$

for our y-coordinate

$$y = \frac{1}{3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) - 5} = -\frac{1}{4}$$

$\therefore$ the coordinates of our turning point is $\left(-\frac{1}{3}, -\frac{1}{4}\right)$

We can now sketch the graph to get



2.2 Horizontal and oblique asymptotes

Unlike vertical asymptotes, horizontal and oblique asymptotes are not input values where undefined results are given out. They are horizontal or slanted lines that the function approaches as $x \to +/-\infty$. In other terms, they describe the end behaviour of a function i.e what happens as $x \to +/-\infty$.

You may have noticed that we have grouped horizontal and oblique asymptotes together; this is because we cannot have both. The reason why will become more apparent.

I would like to stress how horizontal asymptotes and oblique asymptotes are *not* horizontal/slanted lines that cannot be crossed like vertical asymptotes, as they are not related to illegal inputs/x-values; they are lines that the curve approaches as $x \to +/-\infty$. However, horizontal and oblique asymptotes can only be crossed for regions that are close, relative to ∞.

2.2.1 Horizonal asymptotes

Horizontal asymptotes occur if and only if the following is true:
Polynomial degree of the numerator $\leq$ Polynomial degree of the denominator

To find the horizontal asymptote, we will think of the end behaviour and either compute or deduce the horizontal asymptote

Consider we want to find the horizontal asymptote of the functions

i) $y = \dfrac{1}{x+1}$

ii) $y = \dfrac{x+4}{x-1}$

iii) $y = \dfrac{5x^2+2x-3}{4x^2-x+3}$

Computing with limits

Before beginning, we have to get one thing straight; we can *never* substitute in infinity as infinity is *not* a number yet a concept/idea. We will substitute infinity here to make the idea more clear. Moreover, the notion of a limit is *not* to substitute in the value that x approaches; that defies the idea of a limit. However the limit reads as follows: "What happens as x approaches infinity?" To further solidify our understanding of limits (though this is not required), to find

$$\lim_{x\to 3} \frac{1}{x},$$

we don't substitute in $x = 3$, even though it is a valid input and gives the correct result; we have to consider what value we approach as x tends to 3 from the right and left side of the limit (we can go into more detail but this is unnecessary).

i) We will now consider what happens as $x \to +/-\infty$

$$\lim_{x\to+\infty} \frac{1}{x+1} = \frac{1}{\infty+1} = 0$$

In less formal notation, as $x \to +\infty$, $y \to 0$

$$\lim_{x\to-\infty} \frac{1}{x+1} = 0$$

In less formal notation, as $x \to -\infty$, $y \to 0$

$\therefore$ our horizontal asymptote is $y = 0$

ii) Again we will consider what happens as $x \to +/-\infty$

$$\lim_{x\to\infty} \frac{x+4}{x-1} = \frac{\infty+4}{\infty-1} = ???$$

The result isn't clear. To determine the limit we can do one of two things

Using algebraic manipulations:

$$\lim_{x\to\infty} \frac{x+4}{x-1}$$

$$= \lim_{x\to\infty} \frac{x}{x} \times \frac{1+\frac{4}{x}}{1-\frac{1}{x}}$$

$$=1 \times \frac{1+\frac{4}{\infty}}{1-\frac{1}{\infty}}$$

$$=1 \times \frac{1}{1} = 1$$

In less formal notation, as $x \to \infty, y \to 1$

$\therefore$ our horizontal asymptote is $y = 1$

The notion of positive and negative infinity is generally not important here as the result will always be the same for positive and negative infinity

Polynomial long division

By long polynomial division we get that

$$y = \frac{x+4}{x-1} = 1 + \frac{5}{x-1}$$

Now taking the limit of this new form, we get the following

$$\lim_{x\to\infty} 1 + \frac{5}{x-1} = 1 + \frac{5}{\infty-1} = 1$$

$\therefore$ our horizontal asymptote is $y = 1$

iii) We cannot "substitute" infinity to get a limit that we can find the value for. We will opt for algebraic manipulation

$$\lim_{x\to\infty} \frac{5x^2+2x-3}{4x^2-x+3}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2} \times \frac{5+\frac{2}{x}-\frac{3}{x^2}}{4-\frac{1}{x}+\frac{3}{x^2}}$$

$$=1 \times \frac{5+\frac{2}{\infty}-\frac{3}{\infty^2}}{4-\frac{1}{\infty}+\frac{3}{\infty^2}}$$

$$=1 \times \frac{5}{4} = \frac{5}{4}$$

$\therefore$ our horizontal asymptote is $y = \frac{5}{4}$

Thinking of the dominant term and end behaviour

We can also think of the end behaviour of a function. The end behaviour of a function is dictated by the terms where the variable is raised to the highest power; in other words, when we are dealing with very large positive and negative values of x, the term with the highest power of x dictates what happens and the other terms have a "negligible effect".

i) as $|x| \to \infty$, $y \approx \dfrac{1}{x} \therefore y \to 0 \implies$ our horizontal asymptote is $y = 0$

ii) as $|x| \to \infty$, $y \approx \dfrac{x}{x} = 1 \therefore y \to 1 \implies$ our horizontal asymptote is $y = 1$

iii) as $|x| \to \infty$, $y \approx \dfrac{5x^2}{4x^2} = \dfrac{5}{4} \therefore y \to \dfrac{5}{4} \implies$ our horizontal asymptote is $y = \dfrac{5}{4}$

ii) We have the vertical asymptote $x = 1$ and the horizontal asymptote $y = 1$

To find the x-intercept, let $x+4 = 0 \to x = -4 \therefore$ our x-intercept occurs at $(-4, 0)$

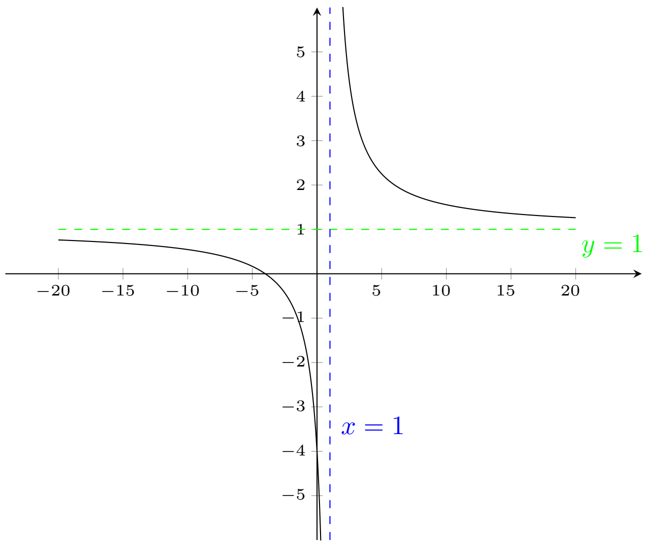
To find the y-intercept, let $x = 0 \to y = -4$

Finally to find the turning points, we first need to find y'

$$y' = \frac{(x-1)(1)-(x+4)(1)}{(x-1)^2} = -\frac{5}{(x-1)^2}$$

Since $y' \neq 0$, we have no turning points

Finally, we can sketch the graph to get



iii) First, to find the vertical asymptotes,

$$Let \quad 4x^2 - x + 3 = 0$$

$$\rightarrow \Delta = (-1)^2 - 4(4)(3) < 0 \therefore \text{no real solution} \implies \text{no vertical asymptotes}$$

Our horizontal asymptote is $y = \frac{5}{4}$
To find our turning point

$$y' = \frac{-13x^2 + 54x + 3}{(4x^2 - x + 3)^2}$$

Let $y' = 0$

$$-13x^2 + 54x + 3 = 0$$

Solving by the quadratic formula

$$x_1 = \frac{27 - 16\sqrt{3}}{13} \approx 4.2 \qquad x_2 = \frac{27 + 16\sqrt{3}}{13} \approx -0.05$$

Plug into equation to find corresponding y values

$$y_1 \approx 1.3 \qquad y_2 \approx -1$$

$$\therefore \text{our turning points are} (4.2, 1.3) \text{ and } (-0.05, -1)$$

To find our x axis intercepts

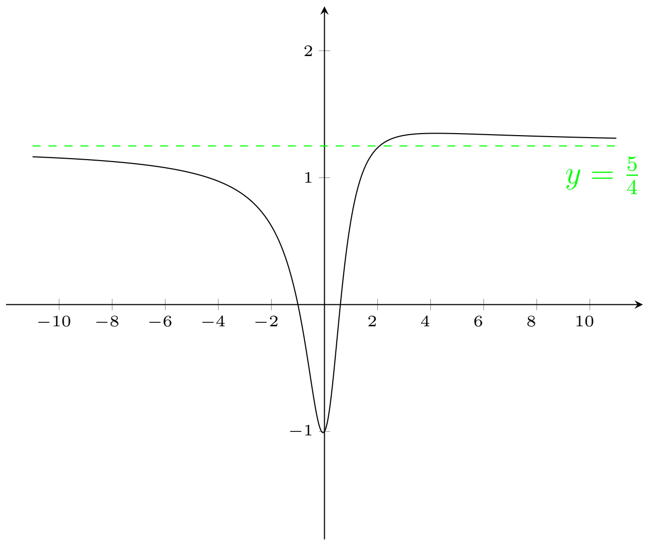
$$\textit{Let} \quad 5x^2 + 2x - 3 = 0$$

solving for x we get

$$x_1 = \frac{3}{5} \quad x_2 = -1$$

$$\therefore \text{our } x \text{ intercepts coordinates are } (0.6, 0) \quad \text{and} \quad (-1, 0)$$

For our y-axis intercept, let $x = 0 \rightsquigarrow y = -1$ We can now sketch the graph to get



2.2.2 Oblique asymptotes

Oblique asymptotes appear when the polynomial degree of the numerator exceeds that of the denominator.

To think about why we get an oblique asymptote, when the degree of the numerator exceeds that of the denominator by 1 we will use long division.

Suppose we have a rational function

$$\frac{p(x)}{q(x)}$$

Where the degree of $p(x)$ is one more than the degree of $q(x)$

Using long division to simplify the rational function, we achieve the following

$$\frac{p(x)}{q(x)} = r(x) + \frac{k}{q(x)}$$

Where $r(x)$ is a linear function.

As $|x| \to \infty$, $\dfrac{k}{q(x)} \to 0 \therefore$ as $|x| \to \infty$, $y \approx r(x)$

Lets sketch the follow graphs

i) $y = \dfrac{x^2 - 1}{2x - 3}$

ii) $y = \dfrac{x^2 - 9}{1 - x}$

Sketching:

i) To find the vertical asymptote, let $2x - 3 = 0 \to x = 1.5$

Since the degree of the polynomial in the numerator is one more than the denominator, we will have an oblique asymptote, not a horizontal one. To find the oblique asymptote we will commence long division to get the following result

$$y = \frac{x^2 - 1}{2x - 3} = \frac{1}{2}x + \frac{3}{4} + \frac{5}{4(2x - 3)}$$

$\therefore$ our oblique asymptote is $y = \dfrac{1}{2}x + \dfrac{3}{4}$

Next we find intersections with the coordinate axis

x-axis: $x^2 - 1 = 0 \to x = \pm 1$

y-axis: let $x = 0 \to y = \frac{1}{3}$

For our turning points

$$y' = \frac{1}{2} - \frac{5}{2(2x - 3)^2}$$

Where we differentiated the result from our long polynomial division

$$\to y' = 0$$

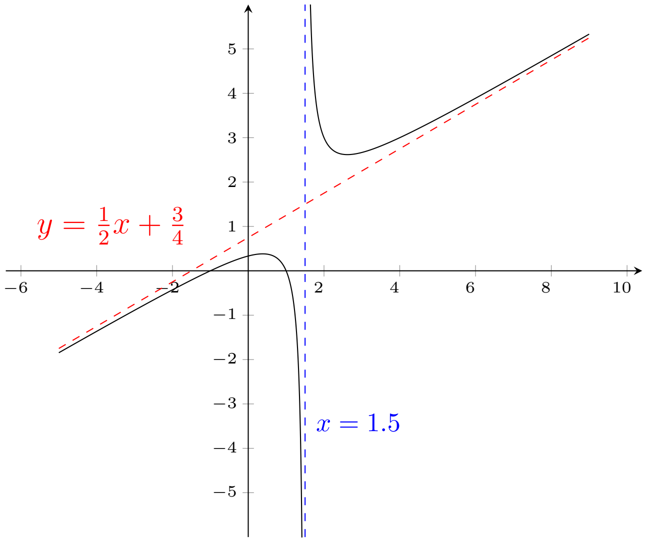
$$\leadsto x_1 = \frac{3 + \sqrt{5}}{2} \approx 2.6 \qquad x_2 = \frac{3 - \sqrt{5}}{2} \approx 0.4$$

Plugging in our x values to get corresponding y values

$$y_1 = \frac{3 + \sqrt{5}}{2} \approx 2.6 \qquad y_2 = \frac{3 - \sqrt{5}}{2 \approx 0.4}$$

$\therefore$ our turning points are $(2.6, 2.6)$ and $(0.4, 0.4)$

Sketching the function, we get the follow graph



ii) Vertical asymptote: $x = 1$

For our oblique asymptote

$$y = \frac{x^2 - 9}{1 - x} = -x - 1 - \frac{8}{1 - x}$$

$\therefore$ our oblique asymptote is $y = -x - 1$

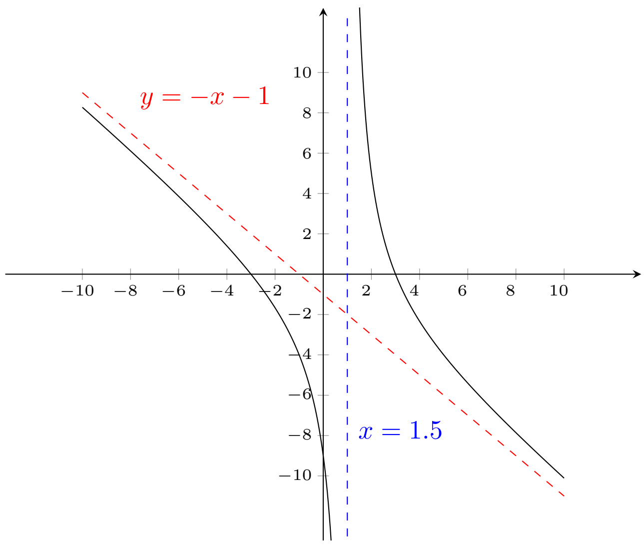
x-axis intersection: (3,0) and (-3,0)
y-axis intersection: (0,-9)

For our turning points

$$y' = -1 - \frac{8}{(1 - x)^2}$$

$$y' \neq 0 \implies \text{no turning points}$$

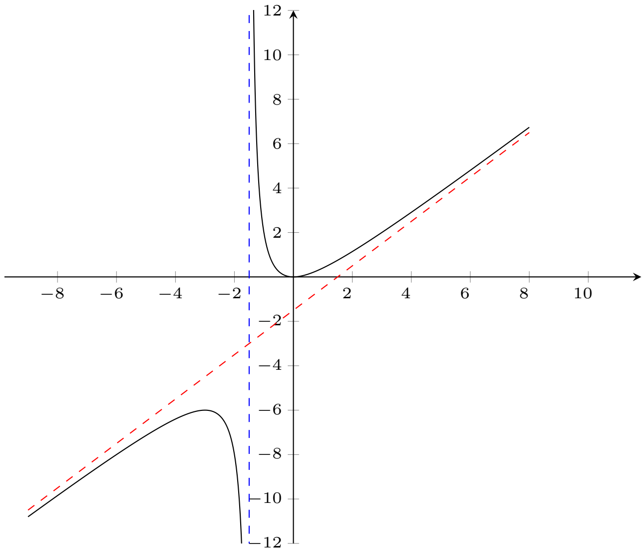
Sketching the function we get



2.3 Inequalities

Consider the curve $y = \dfrac{2x^2}{2x+3}$

If we want to sketch the curve, we end up with something like this



Suppose we want to find the solutions for $\frac{2x^2}{2x+3} < 2$.

From the graph, we can tell that $x < -1.5$ is one of our inequalities.

For the second inequality we first have to find where $\frac{2x^2}{2x+3} = 2$

$$\to \frac{2x^2}{2x+3} = 2$$

$$\rightsquigarrow x^2 - 2x - 3 = 0$$

$$x_1 = -1 \quad , \quad x_2 = 3$$

$\therefore$ our second inequality is $-1 < x < 3$

Notice that without the sketch, we couldn't have detected our first inequality!

Suppose you ark asked for the range of values of $y = \dfrac{x^2 + 2x - 1}{2x - 1}$

While sketching is a viable option, we can do one of these two methods

Method 1: Differentiation Turning points are either points of local maxima or minima; hence, locating the turning points and investigating around the asymptotes enables us to deduce the range of the function

$$y = \frac{x}{2} + \frac{5}{4} + \frac{1}{4(2x-1)}$$

$$y' = \frac{1}{2} - \frac{1}{2(2x-1)^2}$$

$$\xrightarrow{y'=0} x_1 = 0 \quad , \quad x_2 = 1$$

finding the corresponding y values

$$y_1 = 1 \quad , \quad y_2 = 2$$

Investigating around $x = 0.5$, we get that the function approaches $+\infty$ to the right of $x = 0.5$ and $-\infty$ to the left of $x = 0.5$ $\therefore$ we can conclude that $y \geq 2$ and $y \leq 1$

Method 2: Discriminant

$$y = \frac{x^2 + 2x - 1}{2x - 1}$$

$$y(2x - 1) = x^2 + 2x - 1$$

$$2xy - y = x^2 + 2x - 1$$

$$x^2 + (2 - 2y)x + (y - 1) = 0$$

We are looking for y-values with 1 or more corresponding x-values. In other words, we are looking for existing values of $y \therefore \Delta \geq 0$

$$\rightarrow \Delta = (2-2y)^2 - 4(1)(y-1) \geq 0$$

$$4y^2 - 12y + 8 \geq 0$$

$$(y-1)(y-2) \geq 0$$

$$y \geq 2 \quad , \quad y \leq 1$$

Reminder: questions develop linearly in the exam. For example, you will usually be asked to state the asymptotes, then in another part you will be asked to find the turning points or the range of values. At last, you will be asked to sketch.

Let's try one more example

Lets find the range of values of $y = \dfrac{1}{x^2 + 3x + 6}$

$$y(x^2 + 3x + 6) = 1$$

$$(y)x^2 + (3y)x + (6y - 1) = 0$$

$$\Delta = (3y)^2 - 4(y)(6y-1) \geq 0$$

$$15y^2 - 4y \leq 0$$

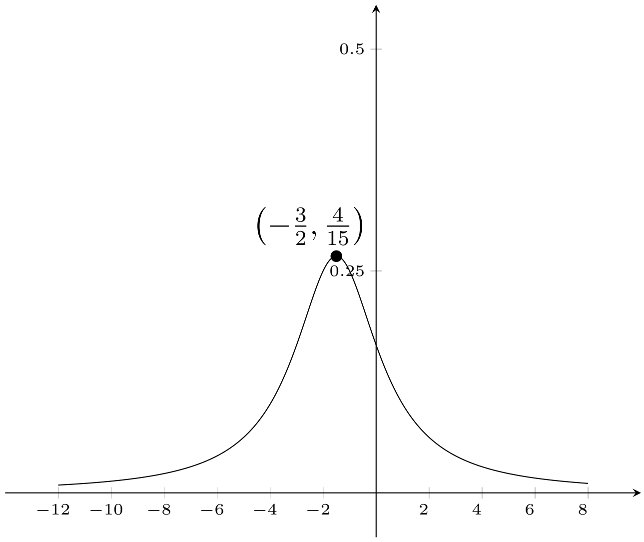
$$y(15y - 4) \leq 0$$

$$0 \leq y \leq \frac{4}{15}$$

But $y \neq 0$ as $y = 0$ is a horizontal asymptote that is not crossed as we can have one turning point (numerator of the derivative will be linear) and that turning point belongs to our maximum point since there are no vertical asymptotes (see graph below)

$$\therefore 0 < y \leq \frac{4}{15}$$

Here is the graph of $y = \dfrac{1}{x^2 + 3x + 6}$



2.4 Relationships between curves

We will now investigate the relationship between $y = f(x)$, $y^2 = f(x)$, $y = \dfrac{1}{f(x)}$, $y = f(|x|)$ and $y = |f(x)|$

$y = f(x)$ and $y = \dfrac{1}{f(x)}$

We will be dealing with the case where $f(x)$ is a second degree polynomial. Higher degree polynomials will follow as such; however, they have never been tested.

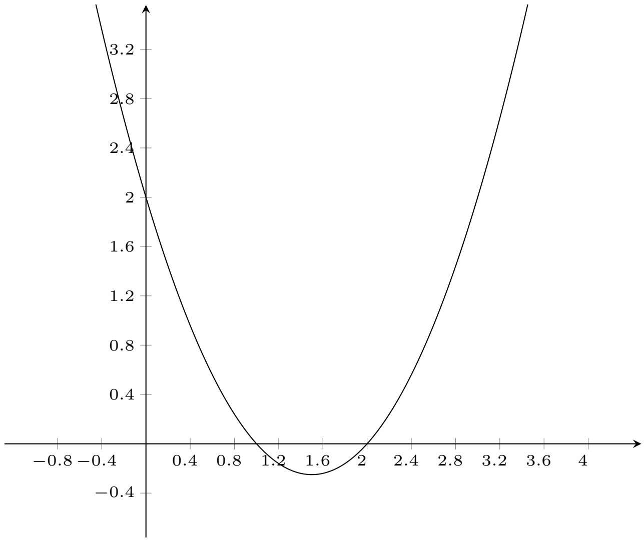
Consider $f(x) = x^2 - 3x + 2$

Solving for the roots of the function we get $x_1 = 1$ and $x_2 = 2$

Our y intercept has the coordinates $(0, 2)$

Our vertex has the coordinates $(1.5, -0.25)$

Sketching the graph, we get something like this



Lets now consider $\dfrac{1}{f(x)} = \dfrac{1}{x^2 - 3x + 2}$

We know that we have vertical asymptotes for x-values such that $x^2 - 3x + 2 = 0$; these are the roots of $f(x) \therefore$ we will have the vertical asymptotes $x = 1$ and $x = 2$ for $\frac{1}{f(x)}$

For our y-axis intersection, it will occur at the somewhere on $x = 0$. Since

$$f(0) = y$$

then

$$\frac{1}{f(0)} = \frac{1}{y}$$

So our intersection with the y-axis will be $\left(0, \dfrac{1}{2}\right)$

The x-coordinate of our turning points will remain unchanged.

$$\frac{d}{dx}(f(x)) = f'(x)$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{f(x) \times 0 - 1 \times f'(x)}{(f(x))^2} = -\frac{f'(x)}{(f(x))^2}$$

When finding the turning points, we set the derivative to 0. We notice that we will end up with the same equation
For $f(x)$

$$\frac{d}{dx}(f(x)) = 0$$

$$f'(x) = 0$$

For $\dfrac{1}{f(x)}$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = 0$$

$$-\frac{f'(x)}{(f(x))^2} = 0$$

$$f'(x) = 0$$

Our y coordinate, however, will be raised to the power of -1
Assume that the x coordinate of the turning point for $f(x)$ and thus its reciprocal is x and the y coordinates of the turning point of $f(x)$ is y.

Therefore

$$f(x) = y$$

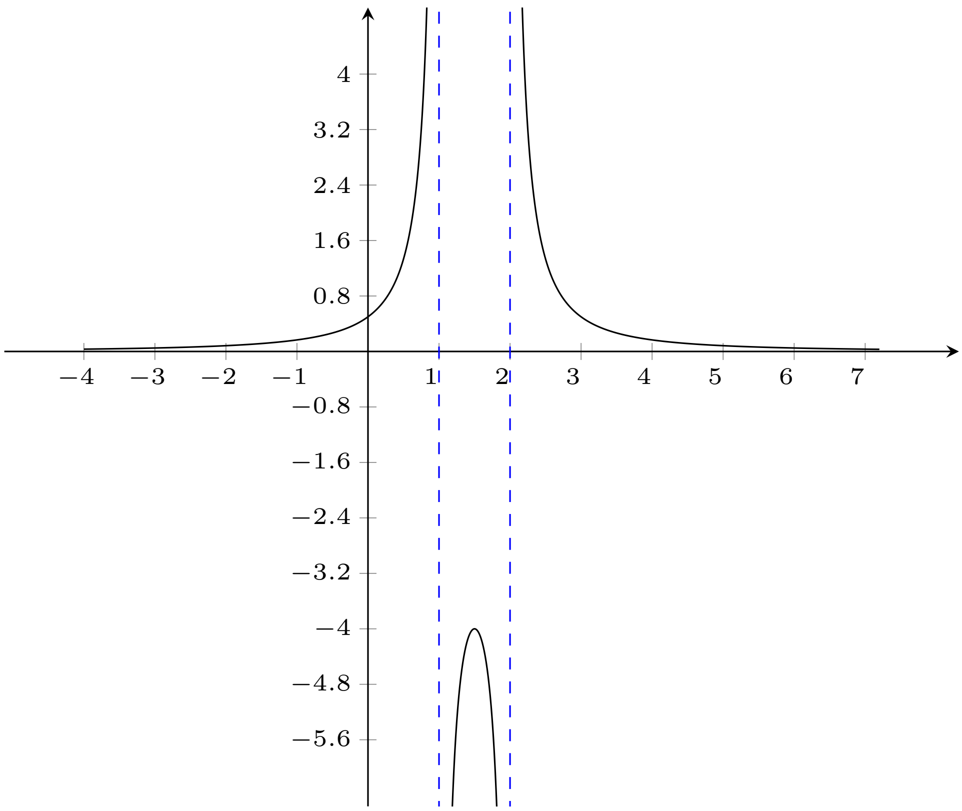
$$\therefore \frac{1}{f(x)} = \frac{1}{y}$$

So the turning point of $\dfrac{1}{x^2 - 3x + 2}$ is $(-1.5, -4)$

Investigating

$$\frac{1}{f(x)}$$

around the vertical asymptotes, we can deduce that the graph of the function will look like this



$f(x)$ and $|f(x)|$

Although you should know this already, we will briefly go through the modulus function

If x is positive, then

$$|x| = x$$

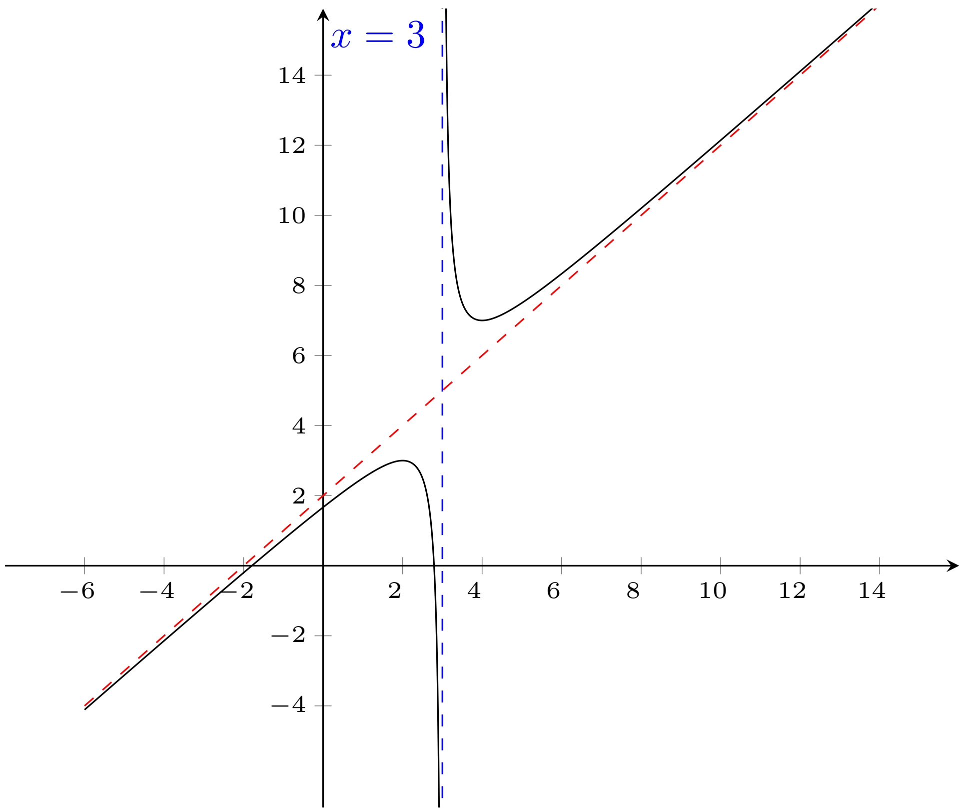
If x is negative, then

$$|x| = -x$$

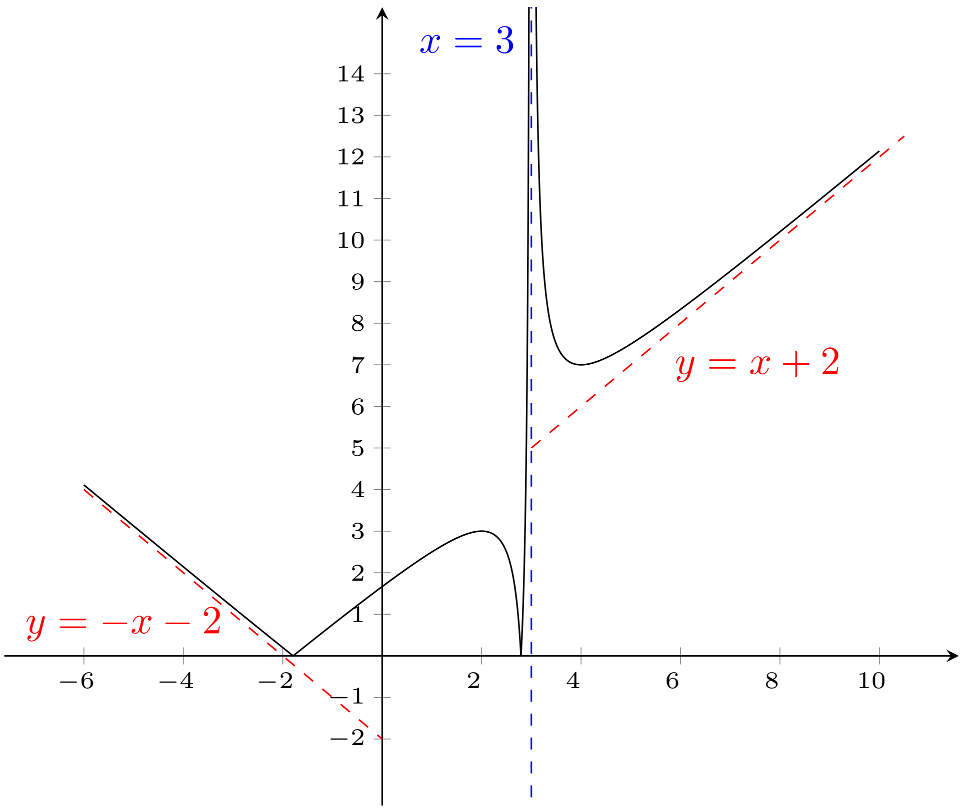
Since the modulus function is applied to $f(x)$ as a whole, our input is unchanged (x) while our output will flip signs if it is negative. Therefore, we "flip" the portion of our graph that is under the x-axis. A unique aspect of functions that include the modulus function is that they have "sharp" turns.

We will not go through how we graphed $f(x)$; we will shift our focus to the changes by the modulus function.

Consider $f(x) = \dfrac{x^2 - x - 5}{x - 3}$ and its corresponding graph



To sketch $|f(x)| = \left|\frac{x^2-x-5}{x-3}\right|$, all we have to do is flip the part of the curve that is below the x-axis upwards. Doing so, we will get something like this



Notice the following features and changes

There is a "sharp turn" at the roots since this is where we reflected the graph

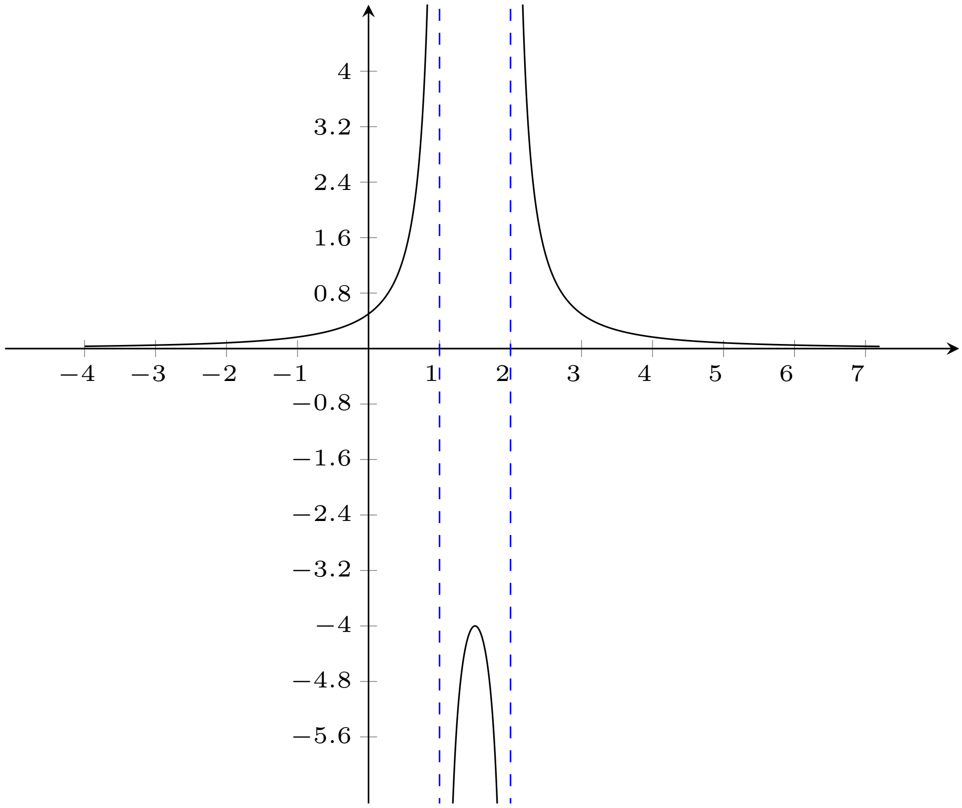
There was no change to the vertical asymptote as the "illegal" input is unchanged.

We will have two oblique asymptotes instead of one one; one belonging to the region of the curve that was no reflected in the x-axis ($y = x + 2$), and one belonging to the reflected region of the curve ($y = -x - 2$). The oblique asymptote will also be reflected to describe the end behaviour of the reflected portion of the curve.

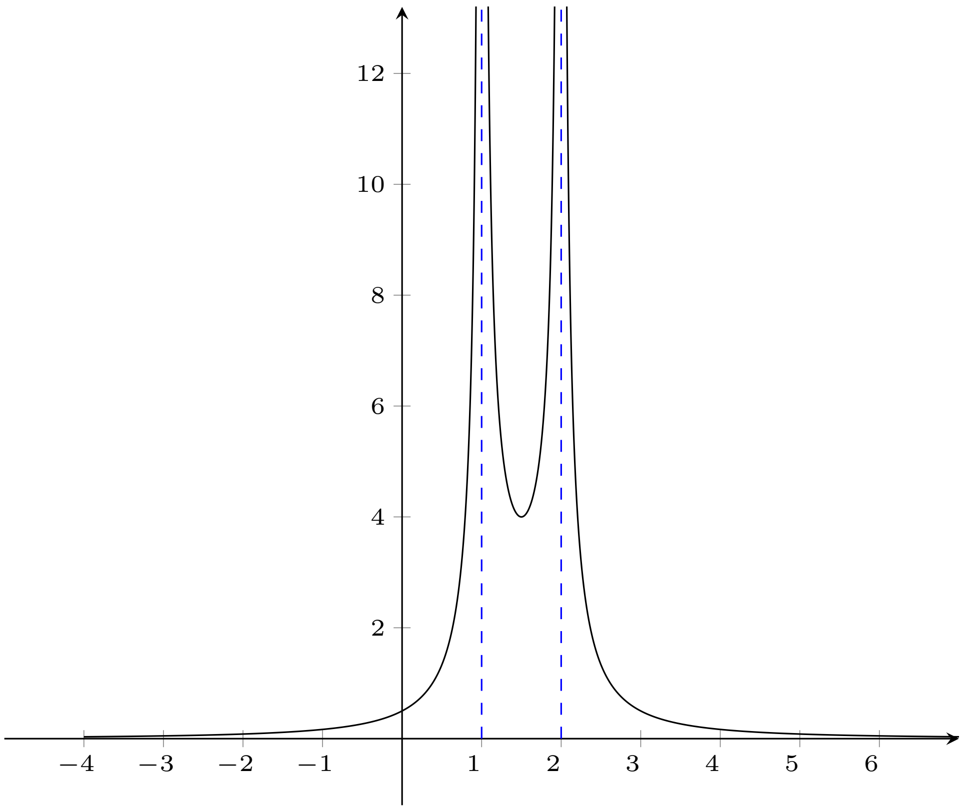
Remember that $-f(x)$ is a reflection of $f(x)$ in the x-axis

Trying an easier example, if we want to sketch $f(x) = \dfrac{1}{x^2 - 3x + 2}$ and $|f(x)| = \left|\dfrac{1}{x^2 - 3x + 2}\right|$

$f(x) = \dfrac{1}{x^2 - 3x + 2}$ will have the graph



$|f(x)| = \left|\dfrac{1}{x^2 - 3x + 2}\right|$ will differ by $f(x)$ by having the region between the vertical asymptotes reflected in the x-axis



Notice the the x-coordinate of the vertex is unchanged while the y-coordinate will have its sign flipped as it was below the x axis

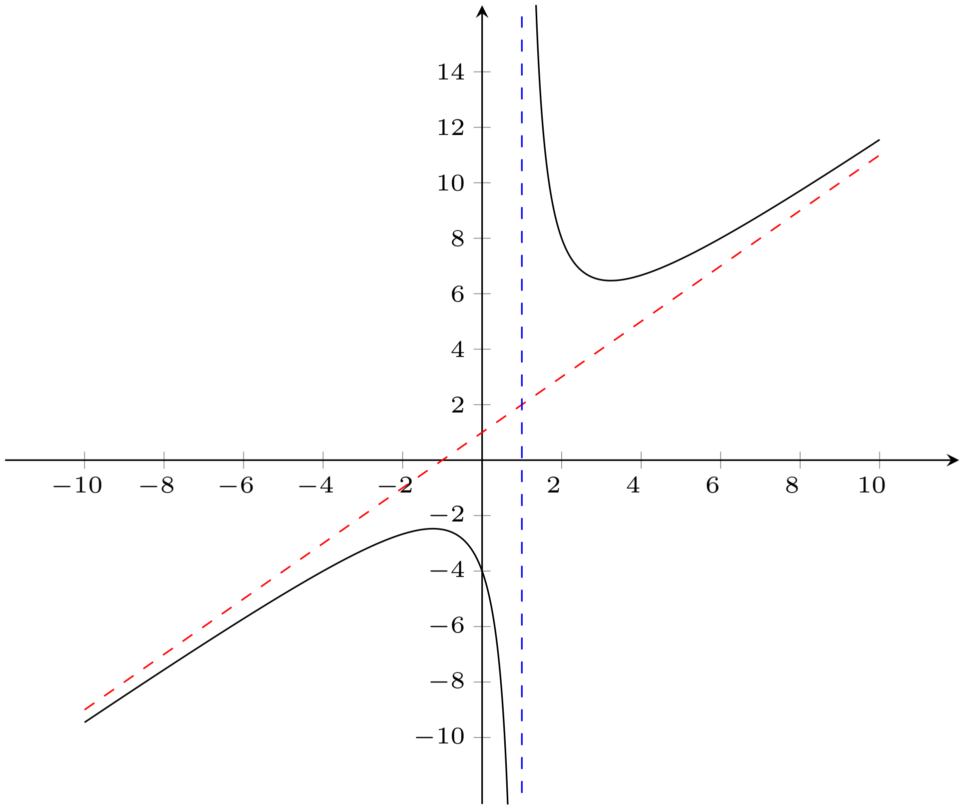
$f(x)$ and $f(|x|)$

In contrast to the previous subsection, the input is the one affected by the modulus function, while the output is unchanged here. This means that if the input x is positive, it wont be affected and will produce the same output as $f(x)$. However, if the input x is negative, it will multiplied by -1 and produce the output of $-x$ where x is negative here. Geometrically, this means that we will have symmetry about the y-axis; since each negative input is basically transformed into its positive counterpart, the negative x-axis, in someway, becomes another positive x-axis in the left direction.

In short, to sketch the graph of $f(|x|)$, we "delete" the part of $f(x)$ that is to the left of the y axis, and flip the part of $f(x)$ that is to the right of the y-axis.

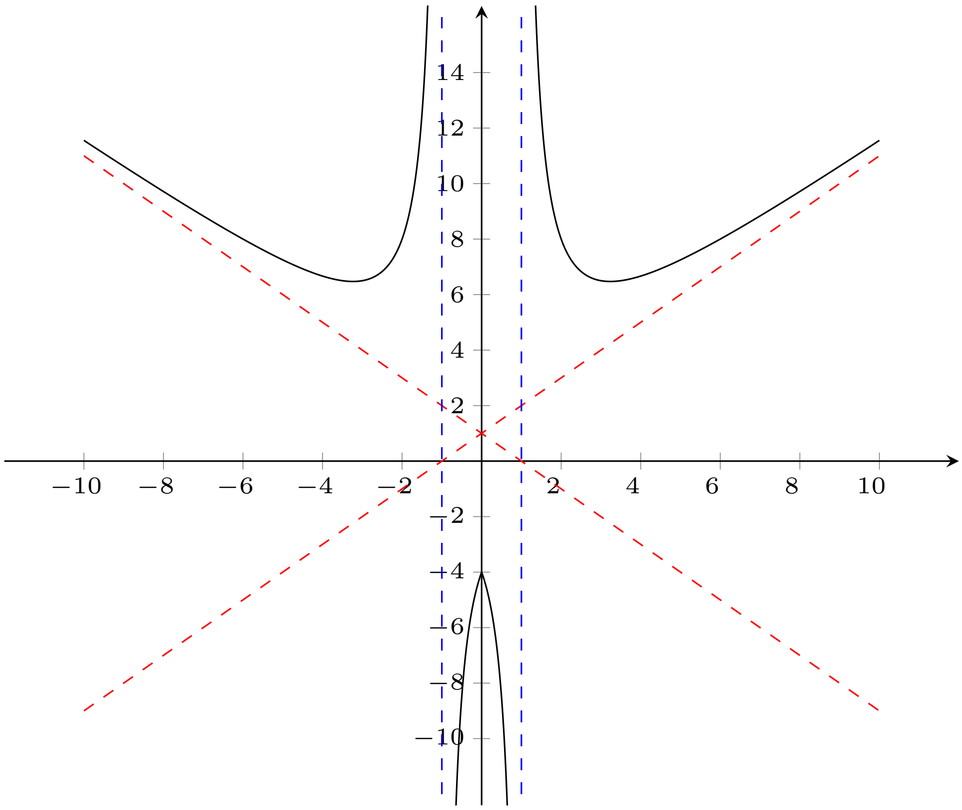
Consider $f(x) = \dfrac{x^2+4}{x-1}$

Sketching the function, we get



As for $f(|x|) = \dfrac{|x|^2+4}{|x|-1}$, we will ignore the part of $f(x)$ that is to the left of the y-axis, and we will reflect the part of $f(x)$ that is to the right of the y-axis about the y-axis.

Consequentially, we will obtain the following graph

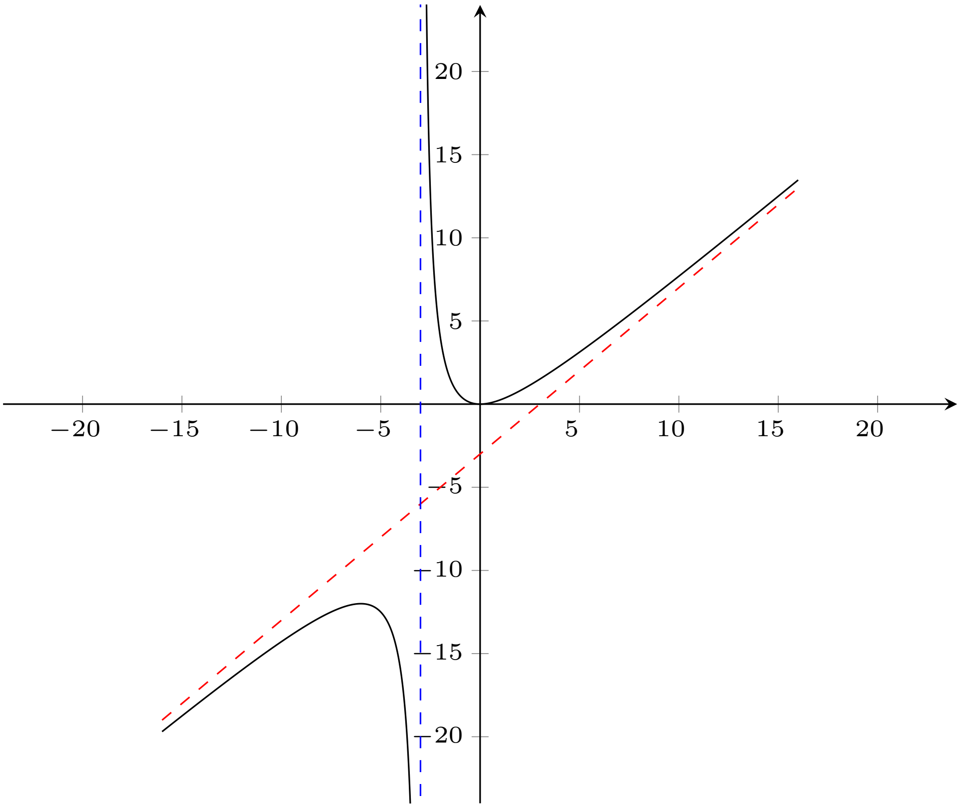


Notice the "sharp turn" at $x = 0$. Also notice how we will have two oblique asymptotes: our original asymptote ($y = x + 1$), and the version that has been reflected about the y-axis ($y = -x+1$). Remember that $f(-x)$ is a reflection of $f(x)$ about the y-axis. We can also see that we will have another vertical asymptote that is a reflection of the original one. We can now establish that any change done to the function by the modulus function also translates to our asymptote, whether it's $f(|x|)$ or $|f(x)|$. For the explanation, we have an issue when the input is $x = 1$, since $|-1| = 1$, we will also have an issue with $x = -1$; hence, we will have the vertical asymptote $x = -1$.

One last example.

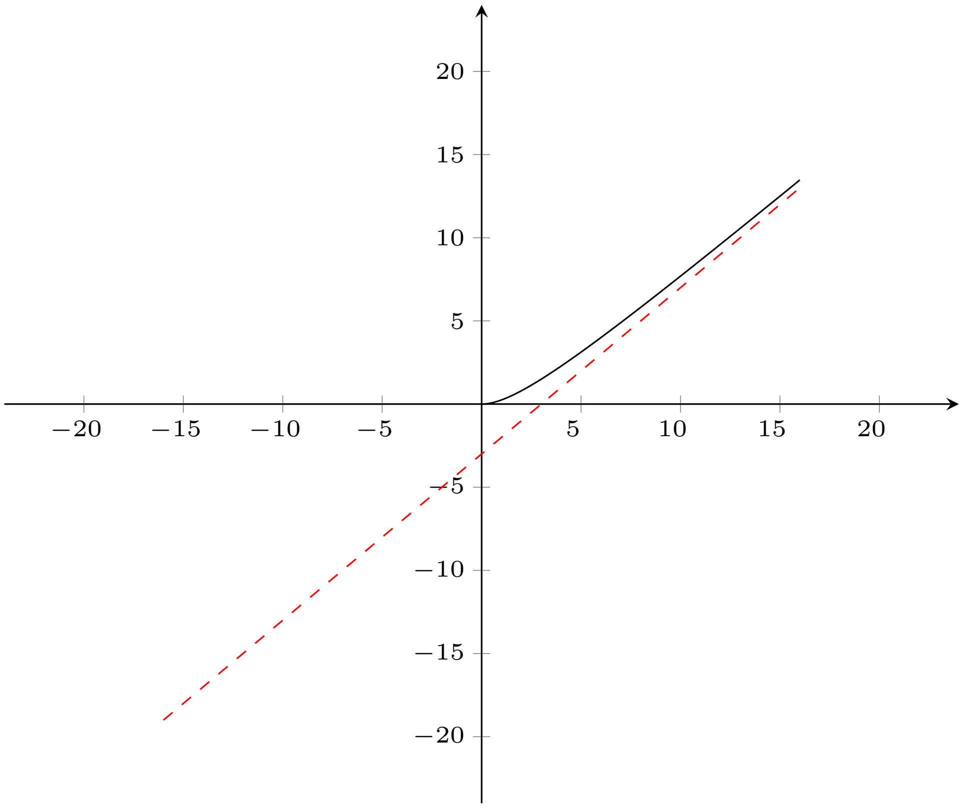
Let us sketch the graph of $f(x) = \dfrac{x^2}{x+3}$ and $f(|x|) = \dfrac{|x|^2}{|x|+3}$

$f(x) = \dfrac{x^2}{x+3}$:



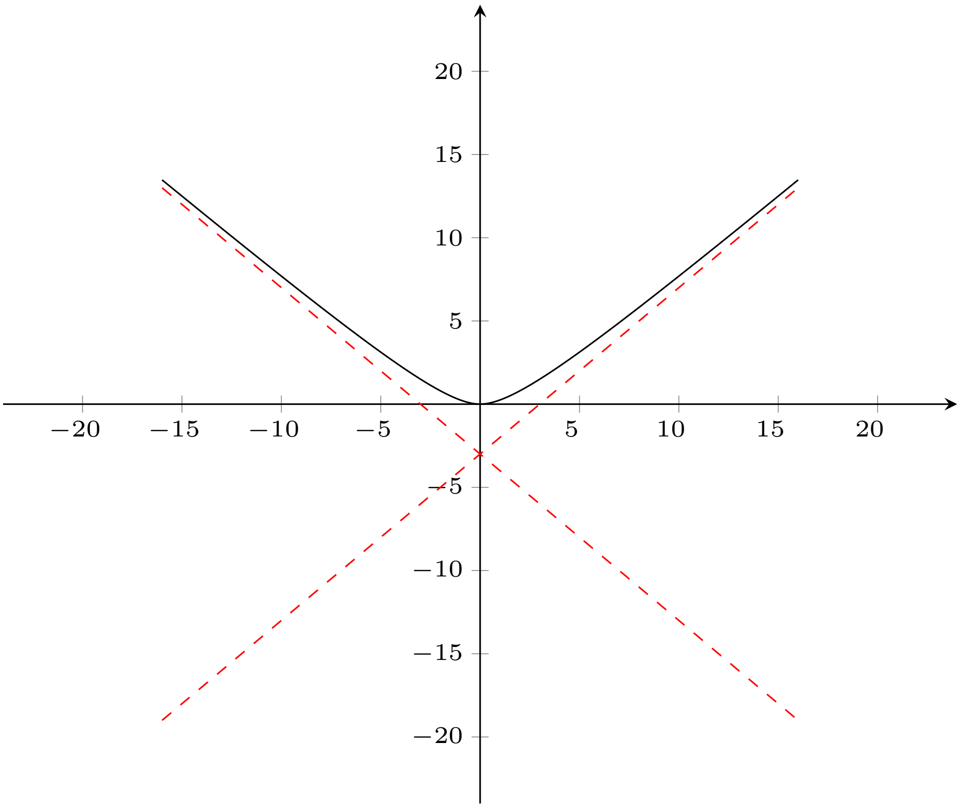
For $f(|x|)$, we have to work in the established steps

First, we "delete" the part of the curve that lies to the left of the y-axis. We get something like this



Notice how we no longer have a vertical asymptote

Next, we reflect the remaining part about the y-axis. We finally reach the desired end product.



Notice how we no longer have any discontinuity since there is no input that will cause a 0 denominator.

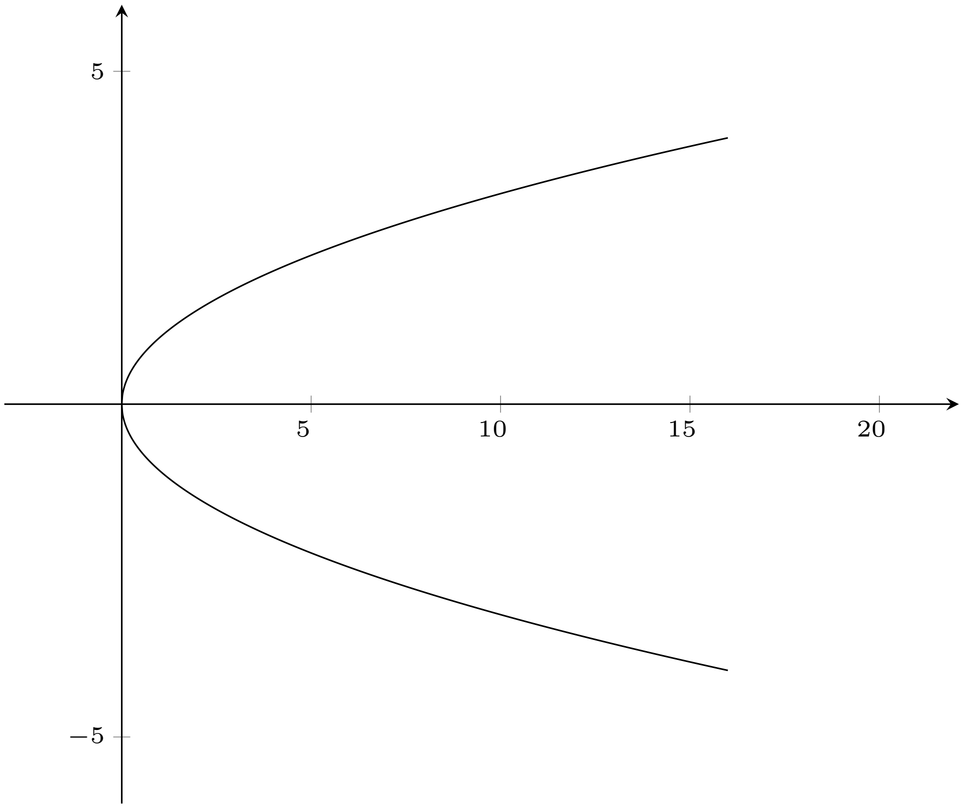
$y = f(x)$ and $y^2 = f(x)$

At last, we have the relationship between $y = f(x)$ and $y^2 = f(x)$.

We must first establish the symmetry of $y^2 = f(x)$ about the x-axis. We first note that $y = \pm\sqrt{f(x)}$; this means that for every input, x, we have two outputs, y and $-y$, of the same magnitude but different signs. Therefore, it would be sufficient to sketch the portion above the x-axis and then reflecting it about the x-axis.

To sketch the curve, identify the point where it crosses the x-axis.

We will first look at the graph of $y^2 = x$



You would be correct if you realised that this is a parabola that is rotated 90 degrees clockwise. It is important we know the general shape of this graph as all questions will have the right hand side as a linear equation in x, hence, we can apply simple transformations onto $y^2 = x$

Lets try to sketch $y^2 = 2x + 3$

First we will identify where it crosses the x-axis

$$2x + 3 = 0$$

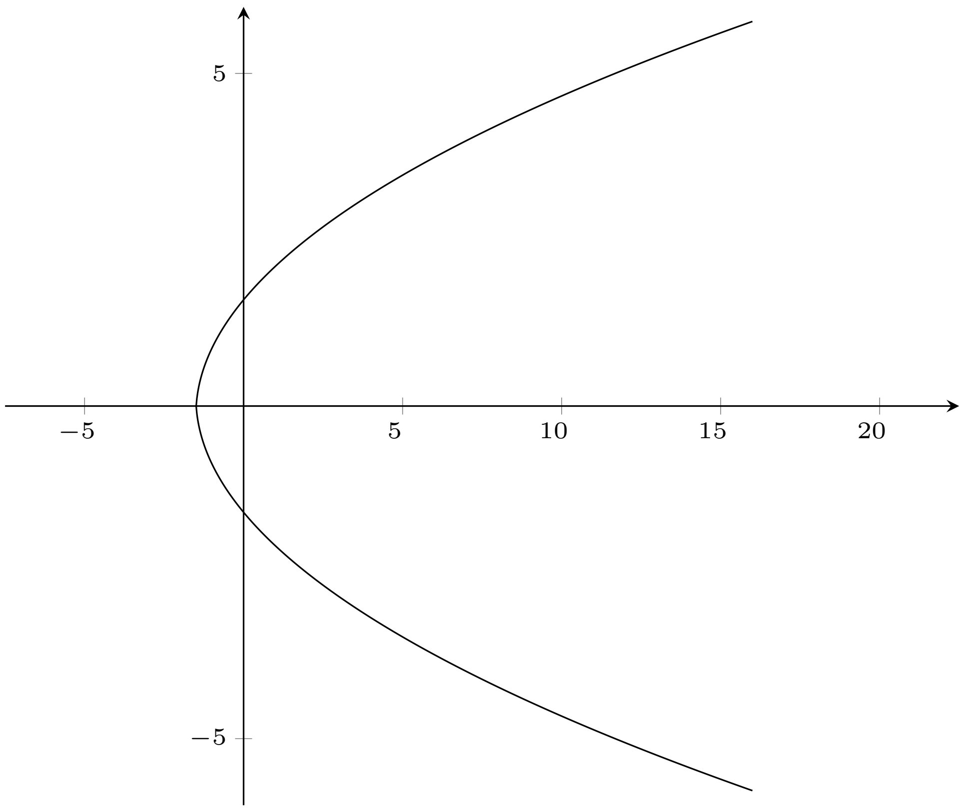
$$x = -\frac{3}{2}$$

Since the equation does not have $-x$ instead of x, it wont be reflected about the y-axis compared to $y^2 = x$.

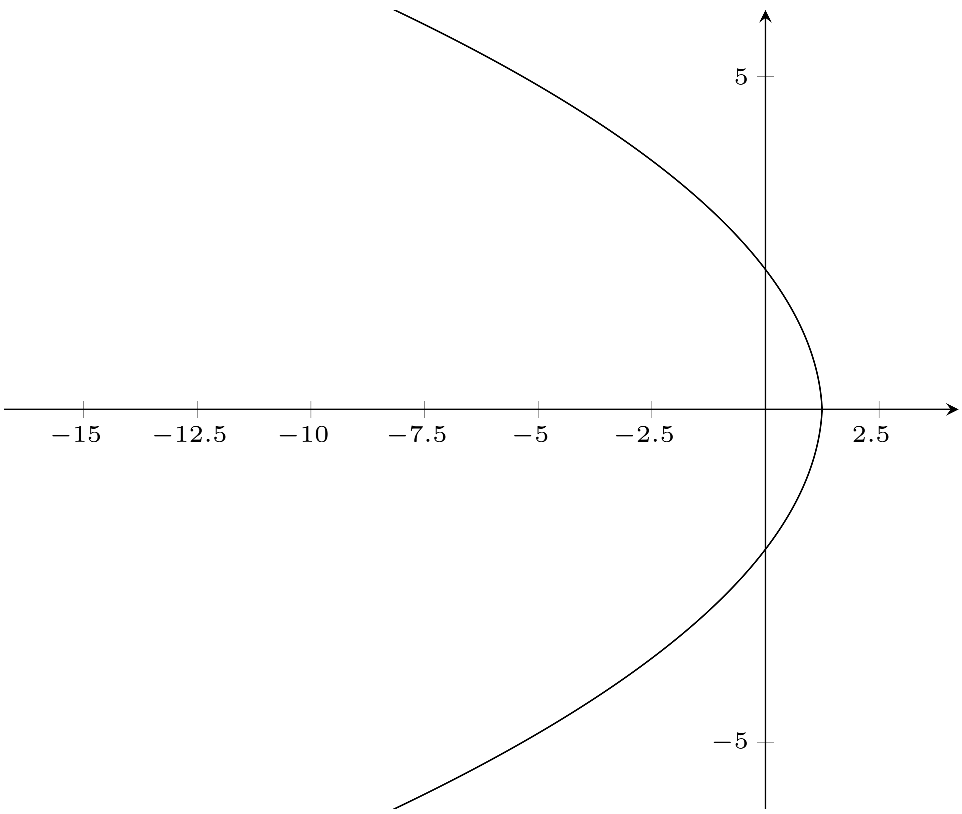
To find the y-intercepts, we let $x = 0$

$$y = \pm\sqrt{3}$$

Using the general shape of these types of graph, we can now sketch the curve to get something like this.



If we want to sketch $y^2 = 5 - 4x$, we first find the x-axis intercept, which is (1.25,0), and the y-axis intercepts, which are $(0, \pm\sqrt{5})$. We notice that the opening of the curve will be to the left instead of the right, and this is reinforced by the fact that we have $-4x$ in the equation. Sketching the curve we will get the follow



2.5 Important exercises

We will only go through one past paper question as they are all pretty vanilla in terms of difficulty once you fully master and understand the skills that we have discussed. You should test and hone your skills by doing past paper questions.

Question

Let a be a positive constant.

a) Sketch the equation $y = \dfrac{ax}{x+7}$

b) Sketch the curve with the equation $y = \left|\dfrac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\dfrac{ax}{x+7}\right| > \dfrac{a}{2}$

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Solution:

a) We first note that the numerator and denominator have the same degree of polynomial, so we will have a horizontal asymptote. To find the horizontal asymptote, you can do any of the discussed methods. However, i'll stick with the approach using limits

$$\lim_{x\to\infty} \frac{ax}{x+7}$$

$$= \lim_{x\to\infty} \frac{x}{x} \times \frac{x}{1+\frac{7}{x}}$$

$$= \frac{a}{1}$$

$\therefore$ our horizontal asymptote is $y = a$ where a is a positive constant

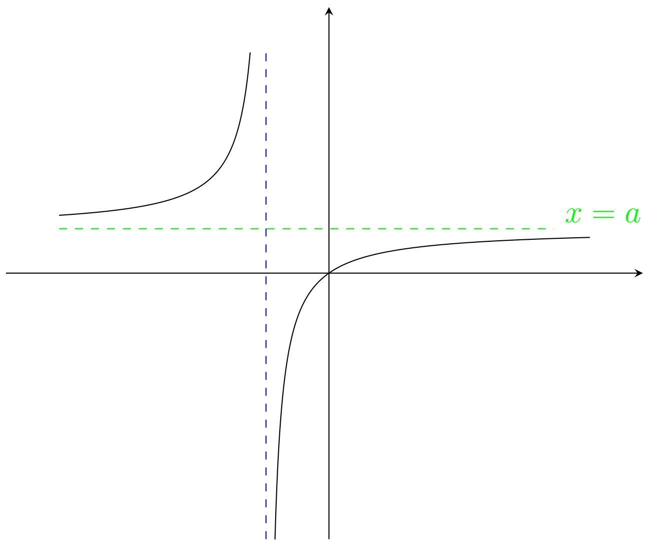
To find our horizontal asymptote, let $x+7 = 0 \rightsquigarrow x = -7$ is our vertical asymptote.

Finding the turning points is unnecessary as we know the general shape of the graph and we weren't asked to find them prior to being asked to sketch them. When turning points are critical, we will usually be asked to find them before being asked to sketch the graph.

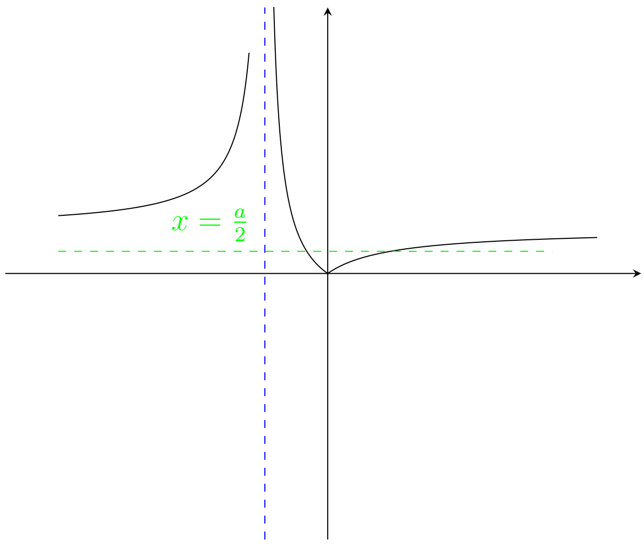
When $x = -7.01$, $y = 701a$ where a is a positive constant $\implies$ y approaches $+\infty$ as x approaches -7 from the right.

When $x = -6.99$, $y = -699a$ where a is a positive constant $\implies$ y approaches $-\infty$ as x approaches -7 from the left.

We can now sketch the graph of y accurately to get the following



b) Since we have $|f(x)|$ type of scenario, we only need to reflect the part of the curve that is below the x-axis about the x-axis, which is the region $-7 < x < 0$. We can now sketch $\left|\frac{ax}{x+7}\right|$



After plotting $x = \frac{a}{2}$, we can tell that we will have two inequalities. We first need to find the x-coordinates of the points of intersection between $x = \frac{a}{2}$ and $y = \left|\frac{ax}{x+7}\right|$

Considering the positive case:

$$\frac{ax}{x+7} = \frac{a}{2}$$

$$2x = x + 7$$

$$x = 7$$

Considering the negative case:

$$\frac{ax}{x+7} = -\frac{a}{2}$$

$$2x = -x - 7$$

$$3x = -7$$

$$x = -\frac{7}{3}$$

Since we want the region above $x = \frac{a}{2}$, the wanted inequalities are:

$x > 7 \quad ; \quad x < -\frac{7}{3} \quad ; \quad x \neq -7$

CHAPTER III
SUMMATION

Chapter III

Summation of series

What is the summation of series?

A *series* is the sum of terms in a mathematical *sequence*. You would be correct if you think the term 'Summation of sequence' is more appropriate as we are taking the sum of terms of a defined sequence. We will be using sigma notation to express our sums extensively in this chapter.

The idea of infinite summation is a very important concept. You may have heard of Riemann sums; Riemann sums are a method in calculus used to approximate the area under a curve or the definite integral of a function. We can define the definite integral in terms of Riemann sums as follows

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=0}^{n} f(c_i)\Delta x_i,$$

where c_i is a sample point between x_i and x_{i+1}, and Δx_i is equal to the width of the subinterval.

This is not required by the syllabus sadly.

Summation will also be used to produce a lower and upper bound of an integral in Further Pure Mathematics 2.

Sums also play a pivotal role in the approximation of function by Taylor and Maclaurin series (discussed in Further Pure Mathematics 2); this makes sums important in various fields, notably in physics as they are used to calculate the behavior of waves, fluids, and other physical systems. In engineering and computer science, series, such as Fourier series and power series, are employed to analyze and manipulate signals in applications like audio processing, image compression, and data compression. Summation of series is also used in analyzing and forecasting time-dependent data, such as stock prices, weather patterns, or population trends. They will also be used in subsequent work in Further Probability and statistics regarding probability generating functions specifically. We can probably make a small booklet if we want to discuss all of the applications of the summation of series, so we will cut things short here.

A historical introduction

The first known use of summation of series was by Archimedes in the 3rd century BC. He used the method of exhaustion to calculate the area under the arc of a parabola with the summation of an infinite series, and gave a remarkably accurate approximation of π. Fast forward to the 18th century, where calculus was largely developed and methods of working with series were formalized. Leonhard Euler made significant contributions to the study of infinite series. Euler introduced the notion of convergent and divergent series (discussed in a Calculus 2 course), developed methods for calculating sums of specific series, and explored the properties of power series expansions. Using his work, Leonhard Euler developed a more efficient method for calculating π using infinite series. In the 19th century, mathematicians like Augustin-Louis Cauchy and Karl Weierstrass took series theory to the next level. They gave calculus and analysis a more rigorous foundation, including a precise definition of convergence for infinite series. They also established criteria for determining whether a series converges or diverges, which paved the way for more systematic analysis of series summation. The 20th century saw developments in the study of Fourier series, which represent periodic functions as infinite sums of trigonometric functions, and the development of methods for numerical approximation of series using computers.

This will be a rather short chapter. However, though I see it as very fun and intuitive, many find questions regarding this topic rather difficult. So, practice past paper questions to fully master this topic.

3.1 Summation formula of $\sum r$, $\sum r^2$ and $\sum r^3$

We are required to make use of the following results:

- $\displaystyle\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$
- $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$
- $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$
- $\displaystyle\sum_{r=1}^{n} k = \underbrace{k+k+\cdots+k}_{n \text{ times}} = n \times k$

While we won't touch on how these results were established (not required), we will prove that these results are true by mathematical induction. We will focus more about techniques of solving and some underlying fundamental concepts.

Before proceeding, lets discuss the sigma notation

$$\sum_{r=1}^{n} u_r$$

Where

- $r = 1 \rightarrow$ the lower limit of the sum
- $n \rightarrow$ the upper limit of the sum which is variable
- $u_r \rightarrow$ the mathematical sequence that we want to add up

The best way to establish our techniques for solving questions regarding this topic is to actually take a crack at some questions. These questions will mostly cover all of the techniques you need to solve any question that comes up in the exam.

Question 1: Find $\sum_{r=1}^{n}(4r+1)$. Hence determine the value of $\sum_{17}^{48}(4r+1)$.

Answer:

$$\sum_{r=1}^{n}(4r+1) = 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 \qquad \text{Break the sum into smaller parts}$$

$$= 4 \times \frac{n}{2}(n+1) + n \qquad \text{Use the formula for the sum of } n \text{ natural numbers}$$

$$= 2n^2 + 3n \qquad \text{Simplify the result}$$

For the next part of the question we will introduce a way to easily deal with sums where the lower limit isn't 1. It may be unnecessary here, but when we are dealing with limits in terms of n or other constants it will be necessary.

Let $u_r = 4r + 1$

$$\overbrace{u_1 + u_2 + u_3 + \cdots + u_{15} + u_{16} + u_{17} + u_{18} + u_{19} + \cdots + u_{48}}^{\sum_{r=1}^{48} u_r}$$

$$\underbrace{u_1 + u_2 + u_3 + \cdots + u_{15} + u_{16}}_{\sum_{r=1}^{16} u_r} + \underbrace{u_{17} + u_{18} + u_{19} + \cdots + u_{48}}_{\sum_{r=17}^{48} u_r}$$

$$\therefore \sum_{r=17}^{48} u_r = \sum_{r=1}^{48} u_r - \sum_{r=1}^{16} u_r \qquad \text{Using the diagram}$$

$$= 2(48)^2 + 3(48) - 2(16)^2 - 3(16)$$

Use the previously established result as our lower limit is 1 and we substituted the value of n

$$= 4192$$

When the question is simple like this one, it is generally overkill to draw such a diagram.

Sometimes we *have* to deduce the sequence we want to sum

Question 2: Find an expression for 1+3+4+7+. . . , for the first n terms.

Solution:

This is the sum of the first n odd terms. The r^{th} term of the sequence can be expressed as $2r-1$. Therefore, our sum is equal to $\sum_{r=1}^{n} 2r-1$

$$\begin{aligned}\sum_{r=1}^{n} 2r-1 &= 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= 2\times\frac{n}{2}(n+1) - 1\times n \\ &= n^2\end{aligned}$$

Working with $\sum r^2$ is no different

Question 3: Find an expression in terms of n for $\sum_{r=1}^{n}(3r^2-4r+2)$.

Solution:

$$\begin{aligned}\sum_{r=1}^{n}(3r^2-4r+2) &= 3\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r + \sum_{r=1}^{n} 2 \\ &= \frac{3}{6}n(n+1)(2n+1) - \frac{4}{2}n(n+1) + 2n \\ &= \frac{1}{2}n(n+1)(2n+1) - 2n(n+1) + 2n \\ &= n^3 - \frac{1}{2}n^2 + \frac{1}{2}n\end{aligned}$$

Unsurprisingly, the difference in procedure when we are dealing with r^3 does not exist.

Question 4: Find $\sum_{r=1}^{n+1} r^2(r-1)$

Solution:

$$\begin{aligned}\sum_{r=1}^{n+1} r^2(r-1) &= \sum_{r=1}^{n+1} r^3 - \sum_{r=1}^{n+1} 1 \\ &= \frac{(n+1)^2}{4}((n+1)+1)^2 - (n+1) \\ &= \frac{(n+1)^2}{4}(n+2)^2 - n - 1\end{aligned}$$

notice how we have $n+1$ in place of n

We will go through select questions at the end of this chapter.

3.2 Converging series

We will be dealing with a new type of converging infinite series[1] that deals with the difference between two functions/sequences. These types of questions are considered safe marks for everyone. Actually, up and until now, we are dealing with easy concepts and it will remain like that for a while.

To clarify some terminology that will be used, a series is said to be convergent when it approaches or, more formally, *converges* to a specific value as we approach an infinite number of terms (upper limit $= \infty$). In contrast, a series is said to be divergent when it *does not converge* to a specific value as we approach an infinite number of terms (upper limit $= \infty$).

[1] The only one you know up and until now is the infinite geometric series with the condition of convergence being $|r| < 1$

The follow simple example portrays what we were talking about.

$$
\begin{aligned}
\sum_{r=1}^{N}\left(\frac{1}{r}-\frac{1}{r+1}\right) &= \left(\frac{1}{1}-\cancel{\frac{1}{2}}\right) \\
&+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right) \\
&+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right) \\
&\vdots \\
&+\left(\cancel{\frac{1}{N-1}}-\cancel{\frac{1}{N}}\right) \\
&+\left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right) \\
&= 1-\frac{1}{N+1}
\end{aligned}
$$

In words, we can observe that consecutive terms cancel each other. Since the first term is not preceded by its negative counterpart, it is not cancelled. Similarly, the last term is not followed by its positive counterpart so it also remains. The method we just used is called the method of differences.

The type of series that we have just dealt with is called a telescoping series because it deals with the telescoping property of the summation notation

$$\sum_{r=1}^{n}(a_r - a_{r+1}) = a_1 - a_n$$

Which can be easily proven.

<u>However</u>, it is *very* important to write out the first couple of terms (more than 4 preferably) and last couple of terms (2 at least), showing the cancellation pattern to obtain your method marks!

Consider $\displaystyle\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+1}\right)$

$$
\begin{aligned}
\sum_{r=1}^{\infty}\left(\frac{1}{r}-\frac{1}{r+1}\right) &= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right) \\
&= \lim_{n\to\infty} 1-\frac{1}{n+1} \\
&= 1
\end{aligned}
$$

Question 1: Find, in terms of n, the sum of $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$. Hence evaluate $\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)}$

Solution:

We are lucky that we only have two methods for evaluating sums: directly if we have $\sum r$, $\sum r^2$ and $\sum r^3$, or using the method of differences.

It is obvious we don't have/can't force a sum with only r and/or r^2 and/or r^3, so we must use the method of differences.

However, we can quickly tell that we are not dealing with the sum of a difference between two functions.

We conveniently have our quadratic denominator factored out; our fraction is ready for partial fraction decomposition.

We will show the steps for the decomposition this time but will be omitted for the rest of the chapter as you should already have this skill.

$$\frac{1}{(r+1)(r+2)} = \frac{A}{r+1} + \frac{B}{r+2}$$ Split into partial fractions

$$1 = A(r+2) + B(r+1)$$ Multiply both sides by $(r+1)(r+2)$

$$\xrightarrow[r=-2]{Let} 1 = -B \to B = -1$$ Choose convenient r values

$$\xrightarrow[r=-1]{Let} 1 = A$$

$$\therefore \frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}$$

$$\implies \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+2}$$

Where $\sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+2}$ is a sum of differences.

Lets now try to find $\displaystyle\sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+2}$

$$\begin{aligned}\sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+2} &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) \\ &+ \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \\ &+ \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) \\ &\vdots \\ &+ \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) \\ &+ \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) \\ &= \frac{1}{2} - \frac{1}{n+2}\end{aligned}$$

which lines up with the telescoping property.

Lets now explore cases where the series isn't telescoping.

We would like to remind you that there will be key exercises at the end of this chapter addressing most ideas and techniques.

Question 2: For the summation $\displaystyle\sum_{r=2}^{n} \frac{1}{(r-1)(r+1)}$, find an expression in terms of n.

Answer:

$$\frac{1}{(r-1)(r+1)} \equiv \frac{1}{2}\left(\frac{1}{r-1} - \frac{1}{r+1}\right) \qquad \text{Split into partial fractions}$$

$$\therefore \sum_{r=2}^{n} \frac{1}{(r-1)(r+1)} = \frac{1}{2}\sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1}$$

$$\rightarrow \text{consider } \sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1} = \boxed{\frac{1}{1}} + \boxed{-\frac{1}{3}}$$

$$+$$

$$\boxed{\frac{1}{2}} + \boxed{-\frac{1}{4}}$$

the negative terms cancel their positive counterpart that is 1 unit to the left and 2 units down, relative to the diagram.

$$+ \quad \boxed{\frac{1}{3}} + \boxed{-\frac{1}{5}}$$

$$+ \quad \boxed{\frac{1}{4}} + \boxed{-\frac{1}{6}}$$

these terms will be cancelled by following terms in the sequence

$$+$$

$$\vdots$$

$$+ \quad \boxed{\frac{1}{n-3}} + \boxed{\frac{-1}{n-1}}$$

these terms would have been cancelled by previous terms in the sequence

$$+ \quad \boxed{\frac{1}{n-2}} + \boxed{-\frac{1}{n}}$$

the positive terms are cancelled by their negative counterpart that is 1 unit to the right and 2 units up, relative to the diagram.

$$+ \quad \boxed{\frac{1}{n-1}} + \boxed{\frac{-1}{n+1}}$$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore \frac{1}{2} \sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

There would be countless ways to explain the pattern of how the terms cancel each other. I have omitted the cancelling of the terms to avoid cluttering the diagram. The diagram isn't supposed to be followed, it only serves the purpose of showing you how you should think of the pattern; we did that here in terms of "units up and down". With practice you will come up with your own way of thinking.

3.3 Important Exercises

Questions

1. **a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer.

   b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find
$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$

   c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$

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2. **a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}(1-r-r^2)$ in terms of n, simplifying your answer.

   b) Show that
$$\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)} = \frac{r+1}{(r+1)^2+1} - \frac{r}{r^2+1}$$
   and hence use the method of differences to find $\sum_{r=1}^{n} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$

   c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}.$

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3. **a)** Show that
$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}$$

   Let $u_r = \frac{1}{\cos(r+1)\cos}$

   b) Use the method of differences to find $\sum_{r=1}^{n} u_r$

   c) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge

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4. **a)** By simplifying $(x^n - \sqrt{x^{2n}+1})(x^n + \sqrt{x^{2n}+1})$, show that

$$\frac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$$

Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \dfrac{1}{x^n - \sqrt{x^{2n}+1}}$

b) Use the method of differences to find $\sum_{r=1}^{N} u_n$ in terms of N and x

c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \dots$$

is convergent and give the sum to infinity when this exists

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DETAILED SOLUTIONS ON THE NEXT PAGE. ATTEMPT BEFORE PROCEEDING

Solutions:

1. **a)** $\displaystyle\sum_{r=1}^{n} r(r+1)(r+2)$

$$= \sum_{r=1}^{n} r^3 + 3r^2 + 2r$$

Expand the brackets fully to use our standard sum results

$$= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

Expand the sum

$$= \frac{n^2}{4}(n+1)^2 + \frac{3}{6}n(n+1)(2n+1) + \frac{2}{2}n(n+1)$$

Substitute the standard results

$$= n(n+1)\left(\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right)$$

Factorise

$$= \frac{n(n+1)}{4}[n(n+1) + 2(2n+1) + 4]$$

Simplify

$$= \frac{n(n+1)}{4}(n^2 + n + 4n + 2 + 4)$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

b) When we are dealing with a three term sum scenario, we can take one of two approaches: we can lay out the terms in a way such that we see a cancellation pattern, or we try and force having two sums with each sum being for a difference between two functions.

Method 1

$$
\begin{aligned}
\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2\left(r+2\right)}\right) = {} & \frac{1}{2(1)} + -\frac{1}{2} + \cancel{\frac{1}{2(3)}} \\
& + \frac{1}{2(2)} + \cancel{-\frac{1}{3}} + \cancel{\frac{1}{2(4)}} \\
& + \cancel{\frac{1}{2(3)}} + \cancel{-\frac{1}{4}} + \cancel{\frac{1}{2(5)}} \\
& + \cancel{\frac{1}{2(4)}} + \cancel{-\frac{1}{5}} + \cancel{\frac{1}{2(6)}} \\
& + \cancel{\frac{1}{2(5)}} + \cancel{-\frac{1}{6}} + \cancel{\frac{1}{2(7)}} \\
& + \vdots \\
& + \cancel{\frac{1}{2(n-2)}} + \cancel{-\frac{1}{n-1}} + \cancel{\frac{1}{2(n)}} \\
& + \cancel{\frac{1}{2(n-1)}} + \cancel{-\frac{1}{n}} + \frac{1}{2(n+1)} \\
& + \cancel{\frac{1}{2(n)}} + -\frac{1}{n+1} + \frac{1}{2(n+2)}
\end{aligned}
$$

Notice how the terms cancel in a diagonal pattern; this means that the terms that are not a part of a diagonal will not fully cancel. Without writing the terms in the way like the diagram, it would've been impossible to notice this pattern.

Usually, the set up for the terms to notice the cancellation pattern must be as follows:
+ve → -ve → +ve

$$
=\frac{1}{2(2)}+\frac{1}{2(n+1)}+\frac{1}{2(n+2)}-\frac{1}{n+1}
$$

This method is fairly easy once you know how to set up the terms in the sum

Method 2

$$\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right)$$
$$=\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{2(r+1)}+\frac{1}{2(r+2)}-\frac{1}{2(r+1)}\right)$$
$$=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)+\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{(r+2)}-\frac{1}{(r+1)}\right)$$

Consider each sum separately

- $\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=\frac{1}{2}\left(1-\frac{1}{n+1}\right)$ [1]

- $\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{(r+2)}-\frac{1}{(r+1)}\right)=-\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{(r+1)}-\frac{1}{(r+2)}\right)$

  $=-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n+2}\right)$ [1]

$$\therefore\sum_{r=1}^{n}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right)=\frac{1}{2}\left(1-\frac{1}{n+1}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n+2}\right)$$
$$=\frac{1}{4}+\frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+1}\right)$$

c)

$$\sum_{r=1}^{\infty}\frac{1}{r(r+1)(r+2)}=\lim_{n\to\infty}\frac{1}{4}+\frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+1}\right)$$
$$=\frac{1}{4}$$

[1] I substituted the sum results directly as we had previously found them.

2. **a)**

$$\begin{aligned}\sum_{r=1}^{n}(1-r-r^2) &= \sum_{r=1}^{n}1-\sum_{r=1}^{n}r-\sum_{r=1}^{n}r^2\\ &= n-\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)(2n+1)\\ &= n\left[1-\frac{n}{2}-\frac{1}{2}-\frac{1}{6}(2n^2+3n+1)\right]\\ &= -\frac{n}{6}\left(-6+3n+3+2n^2+3n+1\right)\\ &= -\frac{n}{6}\left(2n^2+6n-2\right)\\ &= -\frac{n}{3}\left(n^2+3n-1\right)\end{aligned}$$

b)

$$\begin{aligned}RHS &= \frac{r+1}{r^2+2r+2}-\frac{r}{r^2+1}\\ &= \frac{(r+1)(r^2+1)}{(r^2+2r+2)(r^2+1)}-\frac{r(r^2+2r+2)}{(r^2+1)(r^2+2r+2)}\\ &= \frac{r^3+r^2+r+1-r^3-2r^2-2r}{(r^2+1)(r^2+2r+2)}\\ &= \frac{1-r-r^2}{(r^2+1)(r^2+2r+2)} = LHS \quad \square\end{aligned}$$

$$\therefore \sum_{r=1}^{n} \frac{1-r-r^2}{(r^2+1)(r^2+2r+2)} = \sum_{r=1}^{n} \frac{r+1}{r^2+2r+2} - \frac{r}{r^2+1}$$

$$= \cancel{\frac{2}{5}} - \frac{1}{2}$$

$$+ \cancel{\frac{3}{10}} - \cancel{\frac{2}{5}}$$

$$+ \cancel{\frac{4}{17}} - \cancel{\frac{3}{10}}$$

The terms cancel in a diagonal with 2 terms

$$\vdots$$

$$+ \cancel{\frac{n}{(n-1)^2+2(n-1)+2}} - \cancel{\frac{n-1}{(n-1)^2+1}}$$

$$+ \frac{n+1}{n^2+2n+2} - \cancel{\frac{n}{n^2+1}}$$

$$= \frac{n+1}{n^2+2n+2} - \frac{1}{2}$$

c)

$$\sum_{r=1}^{\infty} \frac{1-r-r^2}{(r^2+1)(r^2+2r+2)} = \lim_{n\to\infty} \left(\frac{n+1}{n^2+2n+2} - \frac{1}{2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n}{n} \times \frac{1+\frac{1}{n}}{n+2+\frac{2}{n}} \right) - \frac{1}{2}$$

$$= -\frac{1}{2}$$

3. **a)** We begin with the simpler side (LHS). Notice how we only have sines and cosines in the RHS, so we wont use the double angle formula for $\tan(n+1)$. We must use the fact that

$$\tan x = \frac{\sin x}{\cos x}$$

While it will be messy at the beginning, always follow through your work if you are certain of your approach (which we are since there is no alternative).

$$
\begin{aligned}
LHS &= \tan(r+1) - \tan r \\
&= \frac{\sin(r+1)}{\cos(r+1)} - \frac{\sin r}{\cos r} \\
&= \frac{\cos r \sin(r+1)}{\cos r \cos(r+1)} - \frac{\sin r \cos(r+1)}{\cos r \cos(r+1)} \\
&= \frac{\cos r(\sin r \cos 1 + \sin 1 \cos r)}{\cos r \cos(r+1)} - \frac{\sin r(\cos r \cos 1 - \sin r \sin 1)}{\cos r \cos(r+1)} \\
&= \frac{\cancel{\cos r \sin r \cos 1} + \sin 1 \cos^2 r - \cancel{\cos r \sin r \cos 1} + \sin^2 r \sin 1}{\cos r \cos(r+1)} \\
&= \frac{\sin 1 \cos^2 r + \sin 1 \sin^2 r}{\cos r \cos(r+1)} \\
&= \frac{\sin 1(\sin^2 r + \cos^2 r)}{\cos r \cos(r+1)} \\
&= \frac{\sin 1}{\cos r \cos(r+1)} = RHS \quad \square
\end{aligned}
$$

b) Consider $\sum_{r=1}^{n} \sin 1 \times u_r = \sum_{r=1}^{n} \tan(r+1) - \tan r$

$$
\begin{aligned}
\sum_{r=1}^{n} \tan(r+1) - \tan r &= \cancel{\tan 2} - \tan 1 \\
&+ \cancel{\tan 3} - \cancel{\tan 2} \\
&+ \cancel{\tan 4} - \cancel{\tan 3} \\
&\vdots \\
&+ \cancel{\tan n} - \cancel{\tan(n-1)} \\
&+ \tan(n+1) - \cancel{\tan n} \\
&= \tan(n+1) - \tan 1
\end{aligned}
$$

$$\therefore \sum_{r=1}^{n} u_r = \frac{1}{\sin 1}\left[\tan(n+1) - \tan 1\right]$$

c)

$$\begin{aligned}
u_1 + u_2 + u_3 + \ldots &= \lim_{n\to\infty} \frac{1}{\sin 1}\left[\tan(n+1) - \tan 1\right] \\
&= \frac{1}{\sin 1}\lim_{n\to\infty}\tan(n+1) - \lim_{n\to\infty}\frac{\tan 1}{\sin 1} \\
&= \frac{1}{\sin 1}\lim_{n\to\infty}\tan(n+1) - \frac{\tan 1}{\sin 1}
\end{aligned}$$

Since $\tan(n+1)$ is an oscillating function, $\tan(n+1)$ does not converge to a specific value as it approaches infinity $\implies u_1 + u_2 + u_3 + \ldots$ does not converge to a value

4. **a)** This is a product that produces a difference between two squares. To make this more clear

$$\textit{Let} \quad a = x^n \quad \text{and} \quad b = \sqrt{x^{2n}+1}$$

$$\begin{aligned}
\therefore \left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right) &= (a-b)(a+b) \\
&= a^2 - b^2 \\
&= x^{2n} - x^{2n} - 1 \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\rightsquigarrow \frac{1}{x^n - \sqrt{x^{2n}+1}} \times \frac{x^n + \sqrt{x^{2n}+1}}{x^n + \sqrt{x^{2n}+1}} &= \frac{x^n + \sqrt{x^{2n}+1}}{\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right)} \\
&= \frac{x^n + \sqrt{x^{2n}+1}}{-1} \\
&= -x^n - \sqrt{x^{2n}+1}
\end{aligned}$$

b) We will obviously use the result that we found in part **(a)** to produce a sum of a difference of two function

$$\sum_{r=1}^{N} u_r = \sum_{r=1}^{N}\left(x^{n+1}+\sqrt{x^{2n+2}+1}+\frac{1}{x^n-\sqrt{x^{2n}+1}}\right)$$

$$= \sum_{r=1}^{N}\left[\left(x^{n+1}+\sqrt{x^{2n+2}+1}\right)-\left(x^{n}+\sqrt{x^{2n}+1}\right)\right]$$

$$= \cancel{\left(x^2+\sqrt{x^4+1}\right)}-\left(x+\sqrt{x^2+1}\right)$$

$$+\cancel{\left(x^3+\sqrt{x^6+1}\right)}-\cancel{\left(x^2+\sqrt{x^4+1}\right)}$$

$$+\cancel{\left(x^4+\sqrt{x^8+1}\right)}-\cancel{\left(x^3+\sqrt{x^6+1}\right)}$$

$$\vdots$$

$$+\left(x^{N+1}+\sqrt{x^{2N+2}+1}\right)-\cancel{\left(x^N+\sqrt{x^{2N}+1}\right)}$$

$$= \left(x^{N+1}+\sqrt{x^{2N+2}+1}\right)-\left(x+\sqrt{x^2+1}\right)$$

c) $u_1+u_2+u_3+\cdots = \sum_{r=1}^{\infty} u_r = \lim_{N\to\infty}\left(x^{N+1}+\sqrt{x^{2N+2}+1}\right)-\left(x+\sqrt{x^2+1}\right)$

For $u_1+u_2+u_3+\ldots$ to be convergent,

$$\lim_{N\to\infty}\left(x^{N+1}+\sqrt{x^{2N+2}+1}\right)-\left(x+\sqrt{x^2+1}\right)$$

must exist

For the limit to exist , $-1<x<1$ in this case $\therefore$ $u_1+u_2+u_3+\ldots$ is convergent when $-1<x<1$

When $-1<x<1$,

$$\lim_{N\to\infty}\left(x^{N+1}+\sqrt{x^{2N+2}+1}\right)-\left(x+\sqrt{x^2+1}\right) = 0+\sqrt{0+1}-x-\sqrt{x^2+1}$$

$$= 1-x-\sqrt{x^2+1}$$

DID YOU KNOW

MODERN CRYPTOLOGY ORIGINATED AMONG THE ARABS, THE FIRST PEOPLE TO SYSTEMATICALLY DOCUMENT CRYPTANALYTIC METHODS. AL-KHALIL WROTE THE BOOK OF CRYPTOGRAPHIC MESSAGES, WHICH CONTAINS THE FIRST USE OF PERMUTATIONS AND COMBINATIONS TO LIST ALL POSSIBLE ARABIC WORDS WITH AND WITHOUT VOWELS. THE INVENTION OF THE FREQUENCY ANALYSIS TECHNIQUE FOR BREAKING MONOALPHABETIC SUBSTITUTION CIPHERS, BY AL-KINDI, AN ARAB MATHEMATICIAN, PROVED TO BE THE SINGLE MOST SIGNIFICANT CRYPTANALYTIC ADVANCE UNTIL WORLD WAR II.

CHAPTER IV
MATRICES

$$\begin{pmatrix} 8 & 2 \\ 6 & 5 \end{pmatrix}$$

Chapter IV

Matrices 1

What are matrices?

Matrices (singular, matrix) are very interesting mathematical objects with various real life application. A matrices can be thought of as rectangular arrays of numbers, symbols, or expressions arranged in rows and columns. A matrix can also be thought of as a collection of vectors; this definition of a matrix is highly relevant in the context of linear transformations, something we will discuss in this chapter. Sadly, the pure modules of the Further Mathematics syllabus does not extensively cover Linear Algebra and only touches on matrices as a form of linear transformation, eigenvalues and eigenvectors and systems of equations.

The applications of matrices are extremely diverse and extensive, in addition to being incredibly useful; we will not do this mathematical object justice in listing some of its applications, but we will try nevertheless. Matrices are an essential tool in computer science as they are widely used for tasks such as 3D modeling, transformations (translation, rotation, scaling and we will actually dealing with these in this chapter), and rendering. Some modern encryption method make use of matrices; a string of numbers is converted into a new set of numbers my multiplying it with several non-singular square matrices. The message is decoded by the inverses. Matrices were also used to develop quantum mechanics in the early 20th century by Heisenberg. Cutting the list short, we will end with economics where they can be used to model and perform calculation on an economy; In 1973 the economist Wassily Leontief was awarded the Nobel prize for his work on economic modeling in which he used matrix methods to study the relationships among different sectors in an economy.

A historical introduction

Matrices as mathematical objects are not as old as those in the previous chapters. While they were technically used in ancient China and Babylon, they were only seen as an array of numbers used to organize and manipulate numerical data. The study of matrices picked up in the 17th centuary when Japanese mathematician Seki Kowa used matrices to solve systems of linear equations and to study determinants. While Seki's first manuscript was as early as Leibniz's first commentary on the subject, Seki dealt with more complex matters regarding matrices and dealt with bigger square matrices (up to 5x5 matrices). He also gave resultant and Laplace's formula of determinant for the nŒn case. The subject was forgotten in the West until Gabriel Cramer opened the closed seal in 1750. The term "matrix" was actually coined in 1850 by an English mathematician. In the following years, pioneers such as Leonhard Euler, Arthur Cayley, Werner Heisenberg and John von Neumann advanced the study of matrices extensively, and built the base for modern application.

4.1 Matrix operations

An $\mathbf{n} \times \mathbf{m}$ matrix is a matrix with $\mathbf{n}$ rows and $\mathbf{m}$ columns. We refer to matrices by their size.

For a general matrix of size $\mathbf{n} \times \mathbf{m}$ we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{pmatrix}$$

where a_{ij} represents the element in the $\hat{\imath}^{th}$ row and $\hat{\jmath}^{th}$ column.

We will be primarily using the definition of a matrix being a collection of vectors. Each column of a matrix represents a single vector. A specific row of a matrix corresponds to a specific component of the vectors.

Matrix addition and subtraction

The addition and subtraction of two matrices is only defined if both matrices are of the same size. For example, we can add and subtract two $\mathbf{m} \times \mathbf{n}$ matrices, but we cannot add and subtract an $\mathbf{m} \times \mathbf{n}$ matrix with an $\mathbf{n} \times \mathbf{n}$ matrix. In these operations, we add or subtract from corresponding elements.

Examples

i) If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5 & -8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1.5 & -6 \end{pmatrix}$

then $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 2 \\ -5 & -8 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 1.5 & -6 \end{pmatrix} = \begin{pmatrix} 1+4 & 2+1 \\ -5+1.5 & -8-6 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ -3.5 & -14 \end{pmatrix}$

ii) If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ -5 & -8 & 99 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1.5 & -6 \\ 6 & 3 \end{pmatrix}$

then $\pm\mathbf{A} \pm \mathbf{B}$ does not exist

In general, to add and subtract $\mathbf{2} \times \mathbf{2}$ matrices together:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$$

Scalar and matrix multiplication

If $\mathbf{A}$ is a $\mathbf{2} \times \mathbf{2}$ matrix, then

$$k\mathbf{A} = k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Similarly, If $\mathbf{B}$ is a 3×3 matrix, then

$$k\mathbf{B} = k\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{pmatrix}$$

For matrix multiplication, it gets a fair bit more complicated. We will first show how matrix multiplication is defined before attempting to explain why it is like that anyways.

However we will first discuss when matrix multiplication is valid.

If **A** and **B** are two matrices then **AB** is only defined if **A** is an $\mathbf{n} \times \mathbf{m}$ and **B** is an $\mathbf{m} \times \mathbf{v}$ matrix. The resulting size of **AB** is an $\mathbf{n} \times \mathbf{v}$ matrix.

For a $\mathbf{2} \times \mathbf{2}$ matrix, matrix multiplication is defined as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

To find the the entry of **AB** in the $\hat{\imath}^{th}$ row and $\hat{\jmath}^{th}$ column, with have to find the dot product of the $\hat{\imath}^{th}$ row of matrix **A** and the $\hat{\jmath}^{th}$ column of matrix **B**. There are other methods to find the matrix product **AB**, but this will do.

Similarly, for a $\mathbf{3} \times \mathbf{3}$ matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix} = \begin{pmatrix} aj+bm+cp & ak+bn+cq & al+bo+cr \\ dj+em+fp & dj+en+fq & dl+eo+fr \\ gj+hm+ip & gk+hn+iq & gl+ho+ir \end{pmatrix}$$

While this may seem complicated, it will be as intuitive as regular multiplication once you get ample practice in.

Now onto explaining why matrix multiplication is like this. It is important to know that what will be said doesn't explain fully how matrix multiplication came to be what it is to be. When looking at matrices as linear transformations, a matrix is applied to an array of vectors; this matrix transforms these vectors to new vectors. It is important to note that all linear transformations (matrix transformations) obey two rules: the origin (the zero vector) is unchanged by all linear transformations, and the fact that linearity is preserved. That is, **M** is a matrix transformation if and only if $\mathbf{Ma} + \mathbf{Mb} = \mathbf{M}(\mathbf{a} + \mathbf{b})$. Suppose we have the following defined matrix multiplication

$$\mathbf{A} \times (\mathbf{B})$$

If we want to explain this in terms of transformations, we say that **A** is a matrix transformation that is applied to **B** where **B** is an array of vectors.

Suppose that

$$\mathbf{A} \times (\mathbf{B}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

then we would say that the matrix **B** houses the two column vectors

$$\begin{pmatrix} e \\ g \end{pmatrix} \quad and \quad \begin{pmatrix} f \\ h \end{pmatrix}$$

The matrix **A** contains the linear transformation of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ which can be explained as follows:

$$\begin{pmatrix} a \\ c \end{pmatrix} \rightarrow \text{is the vector that } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is mapped to}$$

and

$$\begin{pmatrix} b \\ d \end{pmatrix} \rightarrow \text{is the vector that } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is mapped to}$$

Let's take an example. *Let* $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix}$ and $\mathbf{B}= \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ then the product **AB** is defined and is as such

$$\mathbf{AB} = \begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 19 & 13 \\ -24 & -20 \end{pmatrix}$$

since **B** is a matrix that houses the two column vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we will deal with each column vector and the result of the applied transformation alone.

$$\begin{aligned} \begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} &= \begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix} \left[3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= 3 \begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -9 \end{pmatrix} \\ &= \begin{pmatrix} 19 \\ -24 \end{pmatrix} \end{aligned}$$

$\therefore$ the column vector

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is mapped to

$$\begin{pmatrix} 19 \\ -24 \end{pmatrix}$$

under the matrix transformation

$$\begin{pmatrix} 3 & 5 \\ -2 & -9 \end{pmatrix}$$

Notice how this is the same as

$$\begin{pmatrix} 3 \times 3 + 2 \times 5 \\ 3 \times -2 + 2 \times -9 \end{pmatrix}$$

because we are taking the relevant result of applying the transformation to the $\hat{\imath}$ and $\hat{\jmath}$ components; in the top row we take the $\hat{\imath}$ result and in the bottom row we take the $\hat{\jmath}$ result.

The same goes for the other column vector. At last, we mush these results into a new matrix **AB**

Lets try some examples.

Examples

Let $\mathbf{A} = \begin{pmatrix} 3 & 4 & -1 \\ 5 & 2 & 1 \\ -2 & 1 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -2 & 3 \\ 2 & 0 & 4 \end{pmatrix}$

i)

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 & -1 \\ 5 & 2 & 1 \\ -2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ 0 & -2 & 3 \\ 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 23 \\ 7 & 6 & 35 \\ 6 & -6 & 9 \end{pmatrix}$$

ii)

$$\mathbf{BA} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & -2 & 3 \\ 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 & -1 \\ 5 & 2 & 1 \\ -2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 13 & 21 \\ -16 & -1 & 10 \\ -2 & 12 & 14 \end{pmatrix}$$

It is very interesting that $\mathbf{AB} \neq \mathbf{BA}$.

There is a special matrix called the identity matrix, I, where

$$\mathbf{I}_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \qquad \mathbf{I}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{I}_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The identity matrix is only defined for square matrices[1].

The identity matrix is the matrix equivalent of multiplying by 1; multiplying by the identity matrix does not change the other multiplied matrix. It is true that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

[1] a square matrix is an $\mathbf{n} \times \mathbf{n}$ matrix for any natural number n

Algebraic properties of matrices

Matrices do not abide by all algebraic properties of numbers. We will list some algebraic properties of matrices without proof as it is not needed and doesn't really give us a deeper insight for our purposes.

- $\mathbf{AB} \neq \mathbf{BA}$ unless [Matrix multiplication is not commutative]
  1. $\mathbf{A}$ or $\mathbf{B} = \mathbf{I}$
  2. $\mathbf{A} = \mathbf{B}^n$
  3. $\mathbf{A}$ or $\mathbf{B}$ is the zero matrix
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ [Matrix addition is commutative]
- $\mathbf{AB}(\mathbf{CD}) = \mathbf{A}(\mathbf{BC})\mathbf{D} = (\mathbf{AB})\mathbf{CD}$ [Matrix multiplication is associative]
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ [Matrix addition is associative]
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ [Left distributive law]
- $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ [Right distributive law]
- $\mathbf{B}(c\mathbf{A}) = c\mathbf{BA}$
- $(c - b)\mathbf{A} = c\mathbf{A} - b\mathbf{A}$
- $a(\mathbf{BC}) = (a\mathbf{A})\mathbf{C} = \mathbf{A}(a\mathbf{C})$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

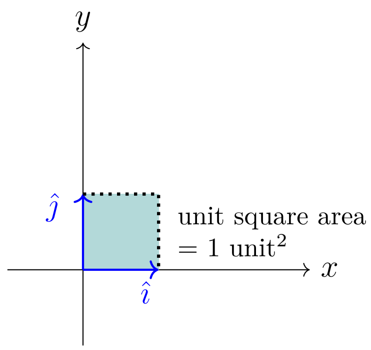
4.2 The matrix determinant

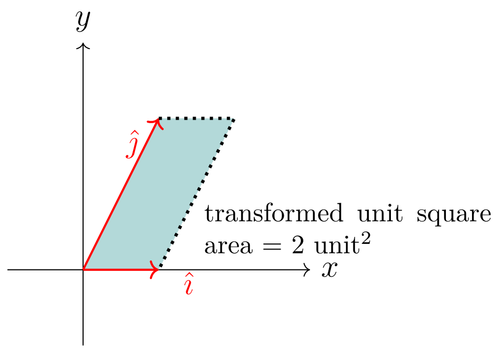
The matrix determinant is a scalar that represents the factor by which an area is scaled after the application of the matrix transformation. To visualize this, we will consider how the area of the unit square enclosed by $\hat{\imath}$ and $\hat{\jmath}$ changes. However, all areas will be scaled the same due to the linearity of matrix transformation

It is important to note that if we have a negative determinant, the area is scaled by the magnitude of the determinant. However, the negative sign causes the $\hat{\imath}$ and $\hat{\jmath}$ to flip positions.

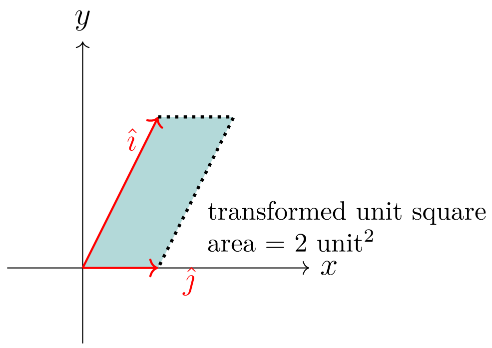
Before jumping into the diagrams, we will introduce determinant notation

$$\text{The determinant of } \mathbf{M} = \det(\mathbf{M}) = |\mathbf{M}|$$





The determinant of the matrix transformation $\mathbf{M}$ in the diagram above is 2 as area is scaled by a factor of 2 by the matrix transformation. We can tell that the determinant is 2 and not -2 because the relative position of $\hat{\imath}$ and $\hat{\jmath}$ is the same, that is, $\hat{\imath}$ is clockwise from $\hat{\jmath}$. If the determinant was -2 instead, the positions of $\hat{\imath}$ and $\hat{\jmath}$ will be as follows



Notice how $\hat{\imath}$ and $\hat{\jmath}$ flip positions.

Now, onto computing the determinant of $\mathbf{2} \times \mathbf{2}$ and $\mathbf{3} \times \mathbf{3}$ matrices. While we won't prove why the determinant is calculated the way it is, it does not provide a deeper understanding of the required material in the Further Mathematics syllabus.

Computing the determinant of a 2×2 matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of $\mathbf{A}$ is as follows

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c| = ad - bc$$

Minor and co factor entries

We will introduce the minor and co factor entries of a matrix; the determinant is calculated using them.

The minor of an entry is the determinant of the matrix resulting after covering the row and column of the entry. The co factor of an entry is the minor entry $\times\ (-1)^{m+n}$ where $m+n$ is the sum of the row and column number

Computing the determinant of a $\mathbf{3}\times\mathbf{3}$ matrix

$$\textit{Let } A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

$$\begin{pmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ \cancel{a_4} & a_5 & a_6 \\ \cancel{a_7} & a_8 & a_9 \end{pmatrix} \xrightarrow[\text{of } a_1]{\text{minor entry}} \begin{vmatrix} a_5 & a_6 \\ a_8 & a_9 \end{vmatrix} = a_5a_9 - a_6a_8$$

$$\begin{pmatrix} a_1 & \cancel{a_2} & a_3 \\ \cancel{a_4} & \cancel{a_5} & \cancel{a_6} \\ a_7 & \cancel{a_8} & a_9 \end{pmatrix} \xrightarrow[\text{of } a_5]{\text{minor entry}} \begin{vmatrix} a_1 & a_3 \\ a_7 & a_9 \end{vmatrix} = a_1a_9 - a_3a_7$$

$$\begin{pmatrix} a_1 & a_2 & \cancel{a_3} \\ a_4 & a_5 & \cancel{a_6} \\ \cancel{a_7} & \cancel{a_8} & \cancel{a_9} \end{pmatrix} \xrightarrow[\text{of } a_9]{\text{minor entry}} \begin{vmatrix} a_1 & a_2 \\ a_4 & a_5 \end{vmatrix} = a_1a_5 - a_2a_4$$

For the co factor entries, we have to multiply by $(-1)^{m+n}$ where $m+n$ is the sum of the row and column number of the corresponding entry of the matrix. Alternatively, we can just use the checker board patterns

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For example, the co factor entry of a in the matrix $\mathbf{A}$ is $(+1\ \times$ minor entry of $a)$. The co factor entry of b in the matrix $\mathbf{A}$ is $(-1\times$ minor entry of $b)$.

The same thing can be done with a $\mathbf{2}\times\mathbf{2}$ matrix, but that wont be necessary as applying the formula is easier.

Now, for all matrices of size $\mathbf{n} \times \mathbf{n}$, the determinant is equal to

$$\sum C_{ij} \times a_{ij}$$

across a row or column, where C_{ij} is the co factor entry for the matrix entry a_{ij}

Examples

Find the result of the following

i)

$$\begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix} = 3 \times 2 - (4 \times -2) = 14$$

ii)

$$\begin{vmatrix} 3 & -2 & 6 \\ 0 & -5 & 1 \\ 4 & -3 & 12 \end{vmatrix} = 3 \times \begin{vmatrix} -5 & 1 \\ -3 & 12 \end{vmatrix} - 0 \times \begin{vmatrix} -2 & 6 \\ -3 & 12 \end{vmatrix} + 4 \times \begin{vmatrix} -2 & 6 \\ -5 & 1 \end{vmatrix}$$

$$= 3(-5 \times 12 - 1 \times -3) + 4(-2 \times 1 - 6 \times -5)$$

$$= -59$$

Notice how we chose to take $\sum C_{ij} \times a_{ij}$ along the first column because there is a 0 which will simplify calculations

iii)

$$\begin{vmatrix} 5 & 9 & 3 \\ 5 & 6 & 15 \\ 4 & 1 & -2 \end{vmatrix} = 4 \times \begin{vmatrix} 9 & 3 \\ 6 & 15 \end{vmatrix} - 1 \times \begin{vmatrix} 5 & 3 \\ 5 & 15 \end{vmatrix} - 2 \times \begin{vmatrix} 5 & 9 \\ 5 & 6 \end{vmatrix}$$

$$= 4(9 \times 15 - 3 \times 6) - (5 \times 15 - 3 \times 5) - 2(5 \times 6 - 5 \times 9$$

$$= 438$$

That was a very long calculation. What if I told you we can simplify it?

Row operations and their effects on determinants

Matrix row operations are operations performed on the rows of a matrix to transform it into a different matrix. There are three types of row operations:

- row switching, where $r_i \leftrightarrow r_j$ [Determinant's sign changes]
- row multiplication, where $r_i \to kr_i$ [Scales the determinant by a factor of k]
- row addition, where $r_i \to r_i + kr_j$ [No effect on the determinant]

Lets try finding $\begin{vmatrix} 5 & 9 & 3 \\ 5 & 6 & 15 \\ 4 & 1 & -2 \end{vmatrix}$ now.

$$\begin{vmatrix} 5 & 9 & 3 \\ 5 & 6 & 15 \\ 4 & 1 & -2 \end{vmatrix} \xrightarrow{r_1 \to r_1 - r_2} \begin{vmatrix} 0 & 3 & -12 \\ 5 & 6 & 15 \\ 4 & 1 & -2 \end{vmatrix} \xrightarrow{r_2 \to r_2 - r_1} \begin{vmatrix} 0 & 3 & -12 \\ 1 & 5 & 17 \\ 4 & 1 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 3 & -12 \\ 1 & 5 & 17 \\ 4 & 1 & -2 \end{vmatrix} \xrightarrow{r_3 \to r_3 - 4r_2} \begin{vmatrix} 0 & 3 & -12 \\ 1 & 5 & 17 \\ 0 & -19 & -70 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 5 & 9 & 3 \\ 5 & 6 & 15 \\ 4 & 1 & -2 \end{vmatrix} = \underbrace{1 \times 1 \times 1}_{\substack{\text{effect of row} \\ \text{operations}}} \times \begin{vmatrix} 0 & 3 & -12 \\ 1 & 5 & 17 \\ 0 & -19 & -70 \end{vmatrix} = -1(3 \times -70 - (12 \times 19)) = 438$$

Suppose we do the following

$$\begin{vmatrix} 0 & 3 & -12 \\ 1 & 5 & 17 \\ 0 & -19 & -70 \end{vmatrix} \xrightarrow{r_2 \to 2r_2} \begin{vmatrix} 0 & 3 & -12 \\ 2 & 10 & 34 \\ 0 & -19 & -70 \end{vmatrix}$$

$$\begin{vmatrix} 0 & 3 & -12 \\ 2 & 10 & 34 \\ 0 & -19 & -70 \end{vmatrix} = -2(3 \times -70 - (12 \times 19)) = 2 \times 438$$

$$\therefore \begin{vmatrix} 5 & 9 & 3 \\ 5 & 6 & 15 \\ 4 & 1 & -2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 0 & 3 & -12 \\ 2 & 10 & 34 \\ 0 & -19 & -70 \end{vmatrix}$$

The determinant is multiplied by 2 due to our row operation. However, we would not do this in a real scenario as our goal is to establish as much zeroes as possible in a single row or column. We can perform row operations on $\mathbf{2} \times \mathbf{2}$ matrices but that isn't necessary.

4.3 The inverse matrix

We have already seen the matrix equivalent of multiplying by 1 (multiplying by the identity matrix). We will now discuss the matrix equivalent of multiplying by the reciprocal. Notice how we did not mention dividing as matrix division is undefined.

If $\mathbf{AB} = \mathbf{I}$ then $\mathbf{B} = \mathbf{A^{-1}}$ or $\mathbf{A} = \mathbf{B^{-1}}$

We call $\mathbf{A^{-1}}$ the inverse matrix of $\mathbf{A}$ and $\mathbf{B^{-1}}$ the inverse matrix of $\mathbf{B}$

Geometrically speaking, the inverse of a matrix *undoes* the transformation done by the original matrix. Remembering the geometric interpretation of the determinant, you may be able to establish a relationship between the determinant of a matrix and its inverse. If a matrix, $\mathbf{A}$, has a determinant d, then the inverse of that matrix, $\mathbf{A^{-1}}$ must have a determinant d^{-1} to undo the area scaling by the matrix $\mathbf{A}$. Since the inverse of a matrix undoes the effect of the matrix, the product of a matrix and its inverse is the identity matrix which is the matrix equivalent of multiplying by 1 and results in no transformation.

Before proceeding with computation, there are some things to note about the matrix inverse. A matrix can only have an inverse if it is a square matrix. However, a square matrix with a determinant of 0 does not have an inverse and the matrix is called a *singular* or a *non-invertible* matrix. In contrast, a square matrix with a non-zero determinant must have an inverse, and the matrix is called a *non-singular* or an *invertible* matrix.

We will now discuss how to find the inverse of a matrix

The inverse of a $\mathbf{2} \times \mathbf{2}$ square matrix

For a $\mathbf{2} \times \mathbf{2}$ square matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is given by

$$\mathbf{A^{-1}} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

given that $\det(A) \neq 0$

Things get a bit more complicated with a $\mathbf{3} \times \mathbf{3}$ square matrix.

The inverse of a 3×3 square matrix

There are many very good ways to find the inverse of a $\mathbf{3} \times \mathbf{3}$ square matrix. We will discuss two methods and will leave it up to you to decide which of them you prefer. There is a third very good way that will be discussed in Further Pure Mathematics 2 known as the Cayley-Hamilton theorem (feel free to check it out).

Using the adjoint matrix

The general formula for the inverse of an $\mathbf{n} \times \mathbf{n}$ square matrix $\mathbf{A}$ is given as

$$\mathbf{A}^{-\mathbf{1}} = \frac{\mathbf{1}}{\det(\mathbf{A})}\mathrm{Adj}(\mathbf{A})$$

where $\mathrm{Adj}(\mathbf{A})$ is the adjoint matrix of $\mathbf{A}$.

Before defining the adjoint of a matrix, we need to define the transpose of a matrix and the cofactor matrix.

The transpose of a matrix is a new matrix formed by interchanging its rows with columns; in other words, we reflect the entries about the main diagonal. We use a superscript T to denote the transpose of a matrix. For example $\mathbf{A}^T$ is the transpose matrix of $\mathbf{A}$.

$$\text{If } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \text{ then } \mathbf{A}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

The cofactor matrix is a matrix derived from a given square matrix by calculating the cofactors of each element. In general,

$$\text{If } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ then } \mathrm{Cof}(\mathbf{A}) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

Where C_{ij} is the cofactor entry of the element a_{ij}

$$\therefore \mathrm{Cof}(\mathbf{A}) = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ & & \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ & & \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

The computation of Cof($\mathbf{A}$) is annoying to say the least but not very difficult.

Finally, we reach the adjoint of a matrix. The adjoint of a matrix is the transpose of the cofacotor matrix.

$$\text{Adj}(\mathbf{A}) = [\text{Cof}(\mathbf{A})]^T$$

Notice that the determinant computation is now easy as we have found out all cofactor entries.

Example Find the inverse of the square matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

First we will find all 9 cofactor entries

$$C_{11} = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -7 \qquad C_{12} = -\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 4 \qquad C_{13} = \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 1 \qquad C_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \qquad C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \qquad \therefore$$

$$C_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5 \qquad C_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2 \qquad C_{33} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$$

$$\text{Cof}(\mathbf{A}) = \begin{pmatrix} -7 & 4 & 2 \\ 1 & -1 & 1 \\ 5 & -2 & -1 \end{pmatrix} \rightsquigarrow \text{Adj}(\mathbf{A}) = \begin{pmatrix} -7 & 4 & 2 \\ 1 & -1 & 1 \\ 5 & -2 & -1 \end{pmatrix}^T = \begin{pmatrix} -7 & 1 & 5 \\ 4 & -1 & -2 \\ 2 & 1 & -1 \end{pmatrix}$$

$\det(\mathbf{A}) = -7 \times 1 + 4 \times 2 + 2 \times 1 = 3$

Finally we have our result

$$\mathbf{A}^{-1} = \frac{1}{3} \begin{pmatrix} -7 & 1 & 5 \\ 4 & -1 & -2 \\ 2 & 1 & -1 \end{pmatrix}$$

Using an augmented matrix

This method is more intuitive; we *augment* the given matrix with the identity matrix. We then perform row operations on the augmented matrix to transform the given matrix to the identity matrix. The identity matrix after these row operations is the inverse of the given matrix.

But what is an augmented matrix and what is *augmenting*? An augmented matrix is a matrix produced by "mashing" two matricees together; the process of combining the two matrices is known as *augmenting*.

We will first start with a $\mathbf{2} \times \mathbf{2}$ matrix.

Example 1: Find the inverse matrix of $\mathbf{A}$ using an augmented matrix, where

$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 2 & 1 \end{pmatrix}$

Solution:

$$\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{r_2 \to r_2 - r_1} \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right) \xrightarrow[r_1 \to r_1 - 5r_2]{r_2 \to -\frac{1}{4}r_2} \left(\begin{array}{cc|cc} 2 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{4} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 2 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{4} \end{array}\right) \xrightarrow{r_1 \to \frac{1}{2}r_1} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{8} & \frac{5}{8} \\ 0 & 1 & \frac{1}{4} & -\frac{1}{4} \end{array}\right)$$

$$\therefore \mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

This is obviously much harder than using the formula directly, but consider this practice for using row operations and an augmented matrix before moving to the $\mathbf{3} \times \mathbf{3}$ case.

If we are trying to explain why this works logically, we can think of it as finding the row operations that cancel the effect of the matrix transformation and applying them to the identity matrix to get the inverse matrix. If you are interested behind the mechanism, look into *Elementary Matrices*.

Example 2: Find the inverse matrix of **B** using an augment matrix where
$\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \to r_3 - r_2]{r_3 \to r_3 - 2r_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right) \xrightarrow[r_1 \to r_1 + \frac{5}{3}r_3]{r_1 \to r_1 + 2r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right) \xrightarrow[r_2 \to -r_2]{r_2 \to r_2 + \frac{2}{3}r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right) \xrightarrow{r_3 \to -\frac{1}{3}r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$$\therefore \mathbf{B}^{-1} = \frac{1}{3} \begin{pmatrix} -7 & 1 & 5 \\ 4 & -1 & -2 \\ 2 & 1 & -1 \end{pmatrix}$$

4.4 Matrix transformations

We will now actually use matrices in the sense of transformations. We will mainly be dealing with the follow transformation matrices:

1. Stretch and enlargement matrices
2. Reflection matrix
3. Rotation matrix
4. Shear matrix

We will also learn about invariant points and lines.

Thankfully, we will only be dealing with the 2D cases; hence, we will be dealing with $\mathbf{2} \times \mathbf{2}$ matrices only.

Before beginning, it is very important to mention that matrices are applied like functions; inwards first and outwards last. For example, suppose that we have a vector $\mathbf{v}$ and two matrix transformations $\mathbf{A}$ and $\mathbf{B}$. For the operation $\mathbf{ABv}$. The matrix transformation $\mathbf{B}$ is applied to $\mathbf{v}$, followed by $\mathbf{A}$ applied to $\mathbf{Bv}$. The order by which matrix transformations are applied to a vector or another matrix is important as matrix multiplication is *not* commutative.

The stretch and enlargement matrices

Suppose the matrix $\mathbf{C}$ represents the position vectors of a rectangle's vertices on a Cartesian plane where

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

Suppose we want to stretch the shape by a factor of 2, parallel to the x-axis. The unit vector in the x direction, $\hat{\imath}$, must be transformed from the column vector

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

and the unit vector in the y-direction, $\hat{\jmath}$, must remain unchanged. Therefore, the transformation matrix must be M, where

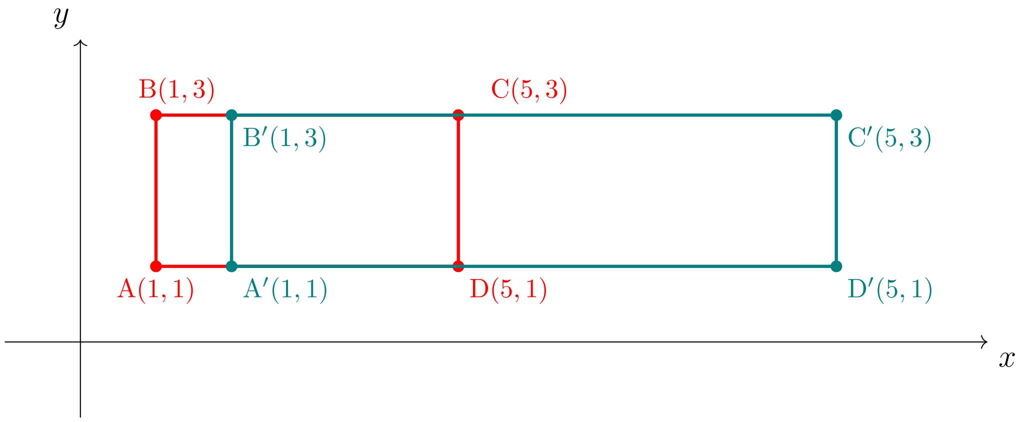
$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

We will now apply M onto C to find our new coordinates

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 & 5 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 10 & 10 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

Note: We will omit the step of finding the new coordinates; it's just matrix multiplication

The following diagram shows the change imposed by the transformation matrix $\mathbf{M}$



In general, the transformation matrix for a stretch in the x direction is

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

where k is the stretch factor

Now, for the same rectangle, suppose we want to stretch the shape by a factor of 2 parallel to the y-axis. The unit vector in the x direction, $\hat{\imath}$, must stay the same. The unit vector in the y direction however must be transformed from

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

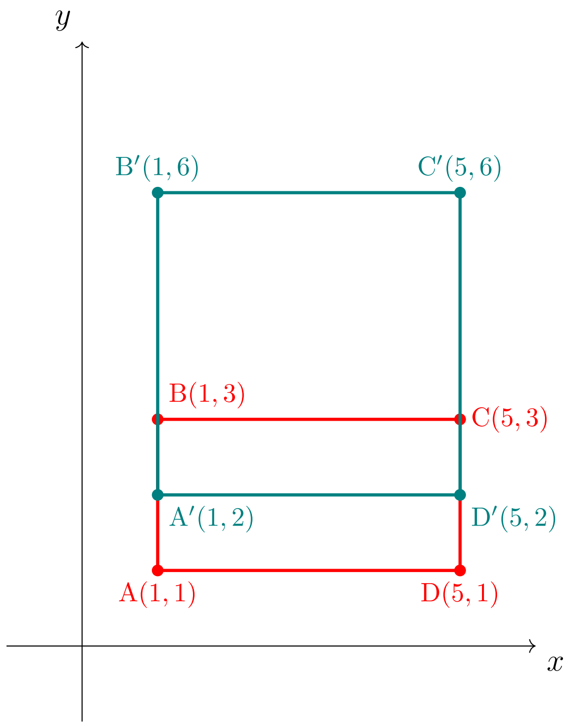
to

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Therefore, the transformation matrix must be $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The following diagram shows the change impose by the transformation matrix **M**



In general, the transformation matrix for a stretch in the x direction is

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$$

where k is the stretch factor

You may have noticed that the area of the rectangles double, which is expected as the transformation matrices have a determinant of 2.

If we combine the two matrices we will get the matrix **E**, where

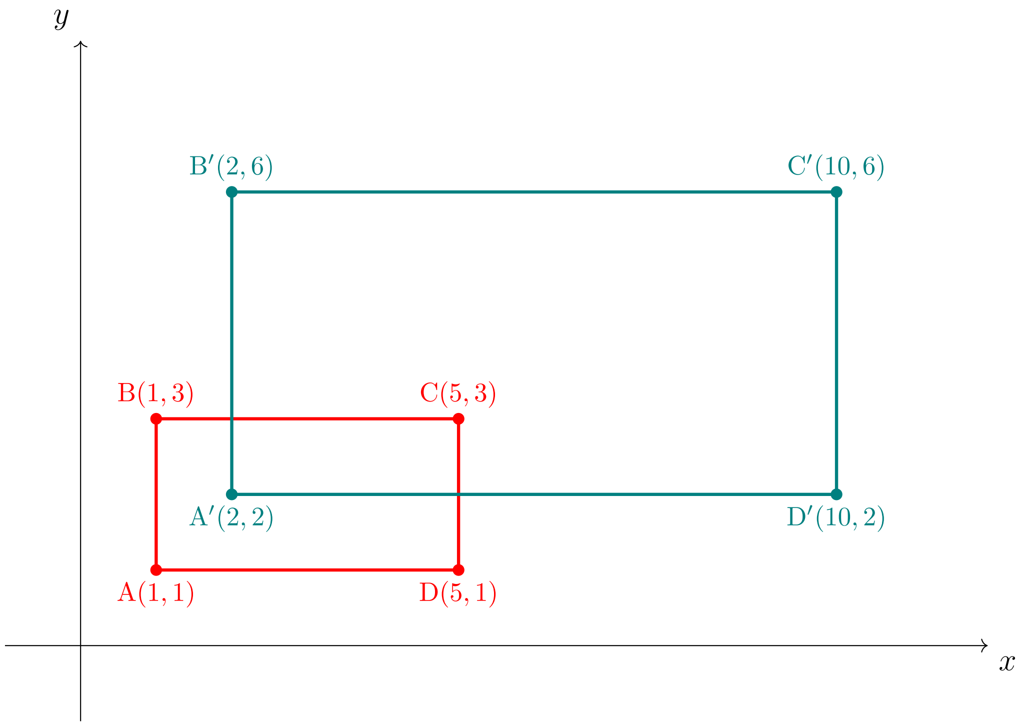
$$\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This enlarges the shape by a factor of 2. We say that the matrix **E** represents an enlargement centered at the origin with a scale factor of 2. In general, the transformation matrix for an enlargement centered at the origin is

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

where k is the scale factor of the enlargement

Applying the enlargement matrix with an enlargement factor of 2 onto the original rectangle, we achieve the following diagram



The area of the rectangle is increased by a factor of 4 after the matrix transformation **E** is applied. The determinant of **E** is 4, as expected.

The reflection matrix

For a reflection in the y-axis, the x-coordinates must change their signs. Therefore, the unit vector in the x-direction, $\hat{\imath}$, must be transformed from

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

to

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

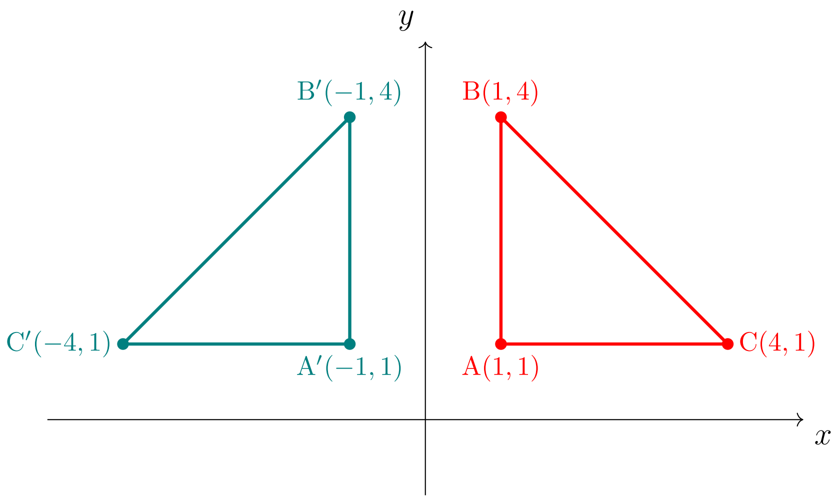
The unit vector in the y direction however must remain the same. Therefore, the reflection matrix, **R**, must be

$$\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let the matrix **T** represent the position vectors of some triangle, where

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 4 & 1 \end{pmatrix}$$

Applying **R** onto **T**, we get the transformed triangle in teal.



In contrast, for a reflection in the x-axis, the y-coordinates must change their signs. Therefore, the unit vector in the y direction, $\hat{\jmath}$, must be transformed from

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

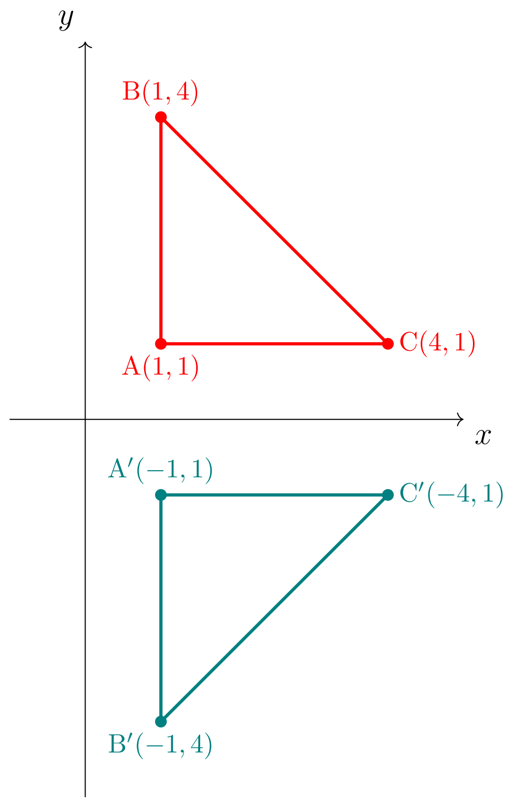
to

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The unit vector in the x direction however must remain the same. Therefore, the reflection matrix, **X**, must be

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

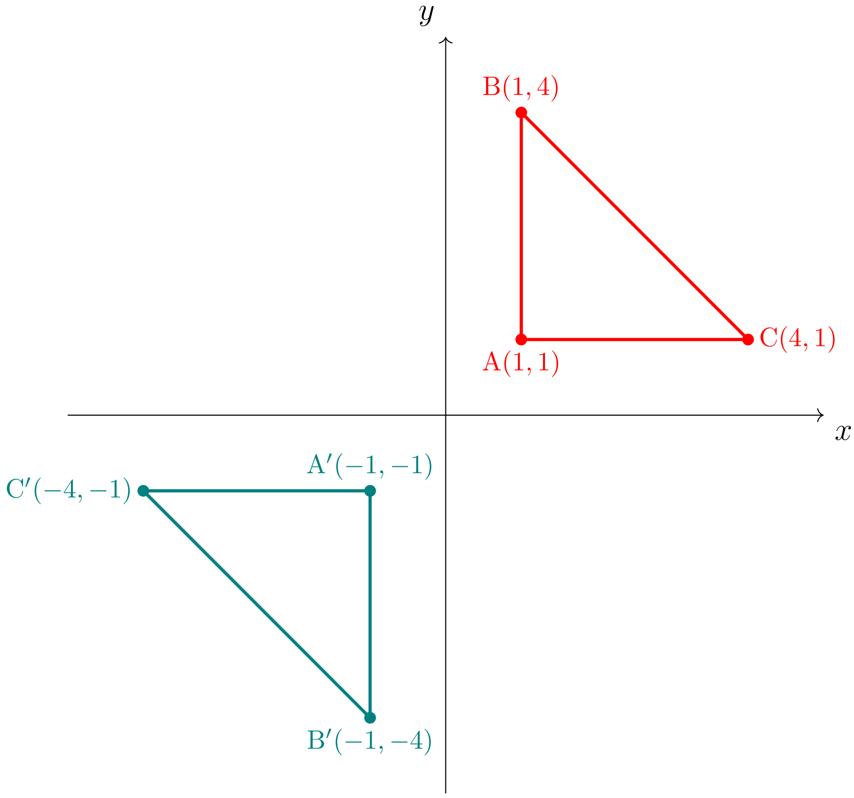
Applying **X** onto **T**, we get the transformed triangle in teal



Applying both a reflection in the x-axis follow by a reflection in the y-axis (or the other way around, order does not matter here), we get a 180° rotation about the origin. The corresponding matrix is **R** where

$$\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

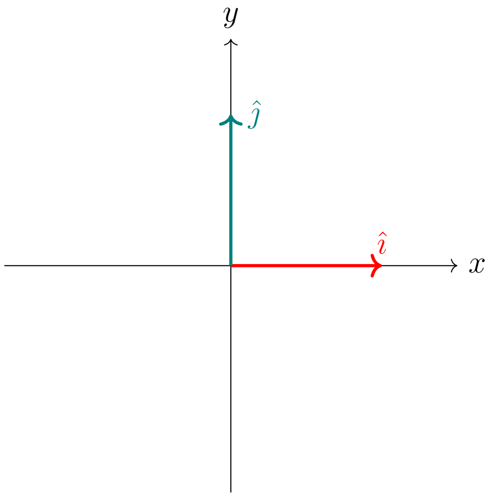
Applying the matrix **R** onto our triangle we get the following



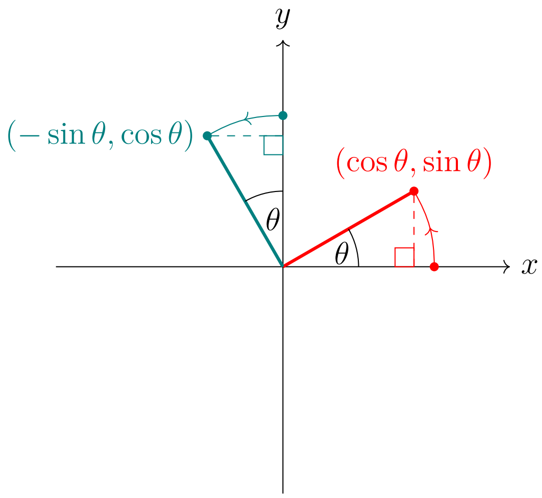
The rotation matrix

Establishing the rotation matrix is a bit more tricky.

Consider our unit vectors in the x and y directions.



Suppose we apply an anticlockwise rotation magnitude θ. The new positions of our unit vectors will be as follows



We can see the new position vector of $\hat{\imath}$ is now

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

instead of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the new position vector of $\hat{\jmath}$ is now

$$\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

instead of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore, the rotation matrix by an angle of θ anti-clockwise is $\mathbf{T}$, where

$$\mathbf{T} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The shear matrix

There is a special transformation known as a *shear*. It is a transformation that distorts the shape of an object by shifting its parts along a specific direction.

A shear in the x-direction is expressed by the matrix $\mathbf{T}_1$, where

$$\mathbf{T}_1 = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

and k is the shear factor.

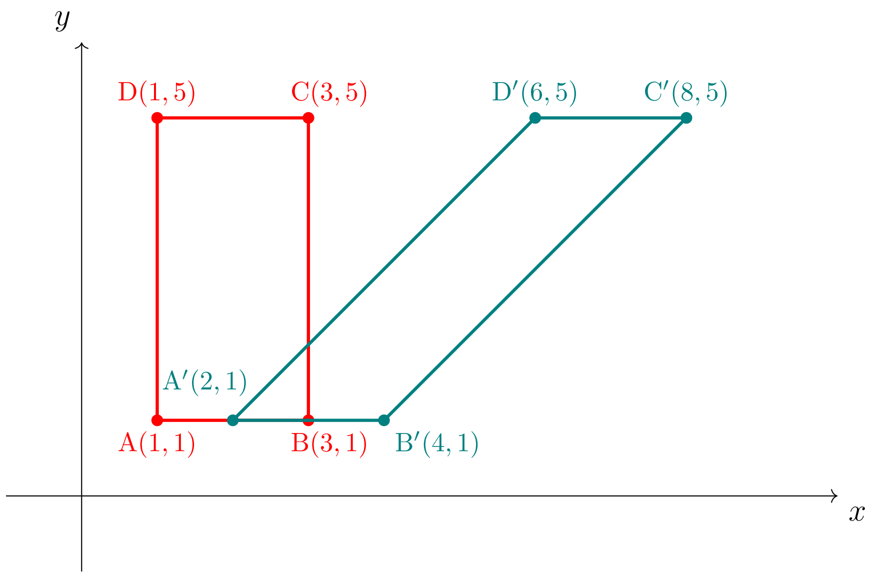
Consider the rectangle by which the position vectors of its vertices is expressed by the matrix $\begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 5 & 5 \end{pmatrix}$. Suppose we want to apply a shear to the rectangles parallel to the x-axis with a shear factor of 1; hence, we will apply the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

To find our new coordinates

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 5 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 8 & 6 \\ 1 & 1 & 5 & 5 \end{pmatrix}$$

On a diagram, our shear looks like this



A shear in the y-direction is expressed by the matrix $\mathbf{T}_2$, where

$$\mathbf{T}_2 = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$$

and k is the shear factor.

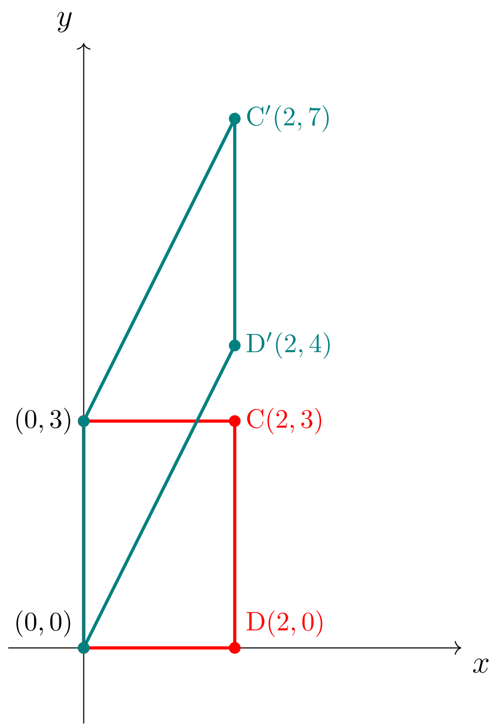
Consider the rectangle by which the position vectors of its vertices is expressed by the matrix $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 0 \end{pmatrix}$. Suppose we want to apply a shear to the rectangles parallel to the y-axis with a shear factor of 2; hence, we will apply the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

To find our new coordinates

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 7 & 4 \end{pmatrix}$$

On a diagram, our shear looks like this

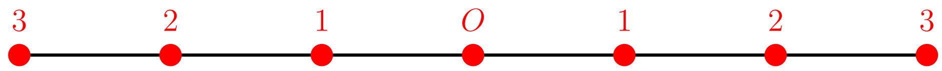


We will make some important observations regarding the previous diagram. Notice how the two points $(0, 0)$ and $(0, 3)$ are unaffected. We say that these points are *invariant* under the matrix transformation $\mathbf{T}_2$. While it is easy to draw the effect of a shear and deduce invariant points or lines, it may be difficult in other cases that we have not studied. The question arises, can we find invariant points and lines algebraically?

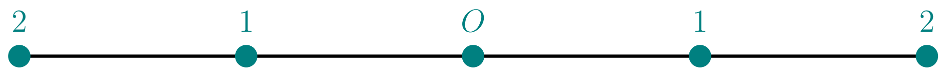
Invariant points and lines

Before discussing how we can find them, what are invariant points and lines? Invariant points are *points* that are unaffected by a matrix transformation, i.e. they remain in the same place. Invariant lines are *lines* that are unaffected by a matrix transformation. However, this *does not* mean that points on that line are not affected, it means that points on that line remain on that line but may move on it. To make it more clear, we will use a simple diagram to portray this.

Suppose the following line is an invariant line under the matrix transformation **T** (**T** is still not applied) and the red points are just some points on that line.



Applying **T** we see the following



Where O is the origin

Notice how the points 1 and 2 move further away from the origin (so does 3 but it does not fit on our diagram) but are still on the line.

It is important to notice that the origin is an invariant point under all linear and matrix transformations (all transformations that we will deal with here); it is one of the conditions for a transformation to be linear.

Onto tackling this algebraically. We will first deal with invariant points.

Suppose we have a matrix transformation $\mathbf{M}$ in 2D space and an invariant point $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $\mathbf{v}$ represents the position vector of the invariant point. It is then true that

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Sometimes, we will get an equation like y=3x when solving for x and y; this means that the invariant points join up to produce a *line of invariant lines*. Do not confuse this with an invariant line! Invariant lines have points that are mapped to another point on the same line.

Lets try this with an example

Example 1: Find the invariant points under the matrix transformation $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Commencing multiplication and comparing the components of the column vectors.

$$\begin{pmatrix} 4x + 3y \\ 2x + 3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \rightsquigarrow \quad 4x + 3y = x \quad (1) \quad , \quad 2x + 3y = y \quad (2)$$

$$(1) : \ 4x + 3y = x \xrightarrow{\text{simplify}} y = -x$$

$$(2) : \ 2x + 3y = y \xrightarrow{\text{simplify}} y = -x$$

$\therefore$ The invariant point is $y = -x \implies$ all points on the line $y = -x$ are invariant points.

Example 2: Find the invariant points under the matrix transformation $\mathbf{T}$, where

$$\mathbf{T} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Multiplying and comparing components

$$x - 2y = x \quad , \qquad -2x + y = y$$

$$y = 0 \qquad\qquad\qquad x = 0$$

This means that the origin is the only invariant point under the transformation T. However, there may be an invariant line; how can we investigate this?

Example 3: Find the invariant line(s) under the matrix transformation **T**, where

$$\mathbf{T} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

Solution:

Suppose the invariant line has the equation $y = mx$ (no $+C$ as the origin is an invariant point for all matrix/linear transformations, so all invariant lines must go through it). It is then true that

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

But $y = mx$ and $Y = mX$

$$\implies \mathbf{T}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

Plugging in $\mathbf{T} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

$$x - 2mx = X \quad , \quad -2x + mx = mX$$

Plugging in X from the first equation into the second equation

$$-2x + mx = m(x - 2mx)$$

$$-2 + m = m - 2m^2$$

$$2m^2 = 2$$

$$m = \pm 1$$

Therefore, the lines $y = \pm x$ are invariant under the transformation **T**

We will do one more example before wrapping this chapter up

Example 4: Find any invariant lines for the matrix **M**, where

$$\mathbf{M} = \begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

$$-3x + 2mx = X \quad , \quad -8x + 5mx = mX$$

Plugging in X from the first equation into the second equation

$$-8x + 5mx = m(-3x + 2mx)$$

$$-8 + 5m = -3m + 2m^2$$

$$2m^2 - 8m + 8 = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2$$

Therefore, there is only one invariant line which is $y = 2x$.

In general, questions regarding matrices are direct, plain and simple; so we will do two past paper questions. Usually, matrix questions are mixed with other concepts in mathematics.

4.5 Important Exercises

Questions

1. The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ where a and b are positive constants.

   a) The matrix $\mathbf{M}$ represents a sequence of two geometrical transformations.

   State the type of each transformation, and make clear the order in which they are applied.

   The unit square in the x-y plane is transformed by $\mathbf{M}$ onto parallelogram $OPQR$.

   b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square.

   It is given that the area of $OPQR$ is 2cm^2 and that the line $x + 3y = 0$ is invariant under the transformation represented by $\mathbf{M}$.

   c) Find the values of a and b.

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2. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

   Where k is a real constant.

   a) Show that $\mathbf{A}$ is non-singular.

   The matrices $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

   It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$

   b) Find the value of k.

   c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by $\mathbf{CAB}$.

Solutions

1. **a)** A stretch parallel to the x-axis with a stretch factor of a, followed by a shear parallel to the x-axis with a shear factor of b

   b) We want to find the matrix that reverses the action of $\mathbf{M}$; this is the inverse matrix of $\mathbf{M}$. Since this is a $\mathbf{2} \times \mathbf{2}$ matrix, we only have to multiply the matrices and find the determinant before directly applying the formula for the inverse of a $\mathbf{2} \times \mathbf{2}$ matrix.

$$\mathbf{M} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$$\det(\mathbf{M}) = a$$

$$\therefore \mathbf{M}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

   c) The unit square has an area of 1cm^2. Under $\mathbf{M}$, the unit square is transformed to a parallelogram with an area of $2\text{cm}^2 \implies \det(\mathbf{M}) = a = 2$

$$\therefore \mathbf{M} = \begin{pmatrix} 2 & b \\ 0 & 1 \end{pmatrix}$$

   Since the line $y = \frac{1}{3}x$ is invariant under $\mathbf{M}$ then

$$\begin{pmatrix} 2 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ -\frac{1}{3}x \end{pmatrix} = \begin{pmatrix} X \\ -\frac{1}{3}X \end{pmatrix}$$

$$2x - \frac{b}{3}x = X \quad , \quad -\frac{1}{3}x = -\frac{1}{3}X \to x = X$$

$$\rightsquigarrow \left[x\left(2 - \frac{b}{3}\right) = x\right] \times 3$$

$$6 - b = 3$$

$$b = 3$$

2. **a)**

$$\begin{vmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{vmatrix} = \begin{vmatrix} k & 0 & 2 \\ 1 & 0 & -1-k \\ 1 & 1 & -k \end{vmatrix}$$

   Taking $\sum (C_{ij} \times a_{ij})$ across the second column, we get

$$-\begin{vmatrix} k & 2 \\ 1 & -1-k \end{vmatrix} = -\left[k(-1-k) - 2\right]$$

$$= k^2 + k + 2$$

If the matrix is singular then $k^2+k+2=0$. But the discriminant of the quadratic equation is Δ, where

$$\Delta = (1)^2 - 4(1)(2) = -7 < 0 \therefore \text{equation does not have a solution.}$$

$\implies$ Determinant of $\mathbf{A} \neq 0 \iff \mathbf{A}$ is non-singular $\quad\square$

b) Since matrix multiplication is associate, we can find $\mathbf{AB}$ and then find $\mathbf{C}(\mathbf{AB})$

$$\mathbf{AB} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix} \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3k \\ 1 & -3 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{C}(\mathbf{AB}) = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -3k \\ 1 & -3 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 9k+3 \\ -1 & -3k-3 \end{pmatrix}$$

$$\therefore 9k+3 = -\frac{3}{2} \rightsquigarrow k = -\frac{1}{2}$$

c)

$$\begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

$$-2x - \frac{3}{2}mx = X \quad , \quad -x - \frac{3}{2}mx = mX$$

Plugging in X into the second equation

$$\left[-x - \frac{3}{2}mx = m\left(-2x - \frac{3}{2}mx\right) \right] \times -2$$

$$2x + 3mx = 4mx + 3m^2x$$

$$3m^2 + m - 2 = 0$$

$$(3m-2)(m+1) = 0$$

$$m = \frac{2}{3} \quad , \quad m = -1$$

Therefore, our invariant lines are $y=-x$ and $y = \frac{2}{3}x$

CHAPTER V
POLAR COORDINATES

Chapter V

Polar coordinates

What are polar coordinates?

The polar coordinate system is a two-dimensional coordinate system that represents points on a plane. Unlike the two dimensional Cartesian coordinate system, the polar coordinate system expresses the coordinates using two values: the radial distance from the pole (the origin in the Cartesian coordinate system), and the angle made with the polar axis ($y = 0$ for $x \geq 0$ in the Cartesian coordinate system). The radial distance from the pole is denoted by r, and the angle made with the polar axis is denoted by θ. Polar coordinates are expressed in the form (r,θ). By convention, the anti-clockwise direction is taken to be the positive angle opening, and the value of r must be greater than or equal to 0.

Polar coordinates are particularly useful when dealing with problems involving circular or rotational symmetry, such as in physics, engineering, and navigation. They provide a convenient way to describe points and vectors in terms of their distance and direction relative to a fixed point and reference direction. They can also simplify certain integrals. Moreover, it may be easier to sketch a curve using the polar equation instead of the rectangular equation.

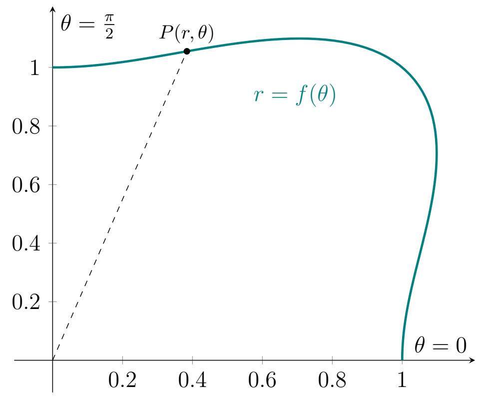
Going to more real life applications, polar coordinates are a convenient way to represent circular or rotational motion. They can be used to simplify the representation of angles and distances in systems with rotational symmetry, such as the motion of planets, gears, or rotating machinery. Furthermore, polar coordinates are a helpful way for engineers and physicists to simplify the analysis of circular or symmetric systems. For example, they're used in AC circuits to represent oscillations. Lastly, to end this non-exhaustive list, polar coordinates can be used in artistic and design applications to create visually appealing patterns, radial symmetry, and curved shapes.

A historical introduction

The concept of polar coordinates can be seen in the works of Greek mathematicians who used variations of polar coordinates to describe and study the motion of celestial bodies. During the Islamic Golden Age, Al-Biruni developed the concept of trigonometric functions and used them in the context of spherical trigonometry, which involved polar coordinates. During the Renaissance period, Johannes Kepler used polar coordinates extensively in his studies of planetary motion, leading to the formulation of his laws of planetary motion, which revolutionized our understanding of celestial bodies. Following the path of Johannes Kepler, renowned mathematician and physicist Isaac Newton used polar coordinates to describe the motion of objects under the influence of gravitational forces. In subsequent centuries, polar coordinates became more widely adopted and developed within the field of mathematics. Mathematicians and scientists like Leonhard Euler and Carl Friedrich Gauss made significant contributions to the theory and application of polar coordinates, further refining their understanding and expanding their applications.

5.1 The polar system

Here, we can see what was mentioned in the first page of the chapter; we have the line $\theta = 0$, also known as the initial line, in place of the positive x-axis in the Cartesian coordinate system, and we have the line $\theta = \frac{\pi}{2}$ in place of the positive y-axis in the Cartesian coordinate system. Remember that, by convention, $r \geq 0$; if θ is measured anti-clockwise from the initial line θ is positive and θ is given in radians.



In the polar coordinate system, functions are in the form

$$r = f(\theta)$$

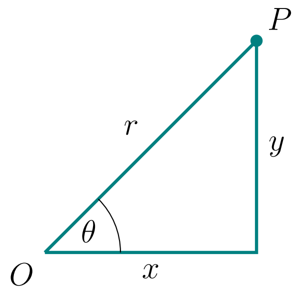
in contrast to the usual form of

$$y = f(x)$$

in the Cartesian coordinate system.

Since both coordinate systems are for the 2D plane, we can relate Cartesian coordinates to polar coordinates (remember that polar coordinates are just an alternative way to represent a point on a 2D plane).

If we consider the right angle triangle made when we drop a vertical line from the point P to the initial axis and join that to the pole, we can see how we can relate the two coordinate systems with Pythagoras' theorem and trigonometry. Using the diagram on the right, we deduce the following equations:



$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

These are the relevant equations when converting between Polar and Cartesian coordinates. Its pretty straight forward so we will skip examples of converting coordinates and jump straight into equations.

Example 1: Convert the polar equation $r = 2$ into its Cartesian form

Solution:

$$r^2 = 4$$

$$x^2 + y^2 = 4$$

which is the equation of a circle centered at the origin with a radius of 2

Example 2: Convert the polar equation $r^2 \cos 2\theta = 4$ into its Cartesian form

Solution:

$$r^2(\cos^2\theta - \sin^2\theta) = 4$$

$$r^2\cos^2\theta - r^2\sin^2\theta) = 4$$

$$x^2 - y^2 = 4$$

Example 3: Find the polar form of the Cartesian equation, $x^2 = \dfrac{y^4}{1-y^2}$, expressing r^2 in terms of θ

Solution:

$$r^2\cos^2\theta = \frac{r^4\sin^4\theta}{1-r^2\sin^2\theta}$$

$$\cancel{r^2}\cos^2\theta = \frac{r^{\cancel{4}2}\sin^4\theta}{1-r^2\sin^2\theta}$$

$$1-r^2\sin^2\theta = r^2\tan^2\theta\sin^2\theta \qquad \text{Multiply by } \frac{1-r^2\sin^2\theta}{\cos^2\theta}$$

$$1 = r^2\tan^2\theta\sin^2\theta + r^2\sin^2\theta$$

$$1 = r^2(\sin^2\theta\tan^2\theta + \sin^2\theta) \qquad \text{Factorise}$$

$$r^2 = \frac{1}{\sin^2\theta(1+\tan^2\theta)}$$

$$r^2 = \frac{1}{\sin^2\theta\sec^2\theta} \times \frac{\cos^2\theta}{\cos^2\theta} \qquad \text{Use } \sec^2\theta = 1+\tan^2\theta$$

$$r^2 = \frac{\cos^2\theta}{\sin^2\theta}$$

$$r^2 = \cot^2\theta$$

We will now tackle sketching curves using there polar equations. However, their are certain tests that can make sketching easier.

Standard graphs

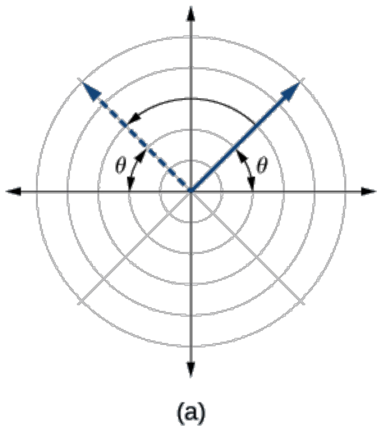
(a) Testing for symmetry about the initial line $\theta = 0$

If $f(-\theta) = f(\theta)$ then the curve is symmetrical about the x-axis

(b) Testing for symmetry about the half line $\theta = \frac{\pi}{2}$

If $f(180-\theta) = f(\theta)$ then the curve is symmetrical about the y-axis

If it is not obvious why these work, here is a diagram for aid:

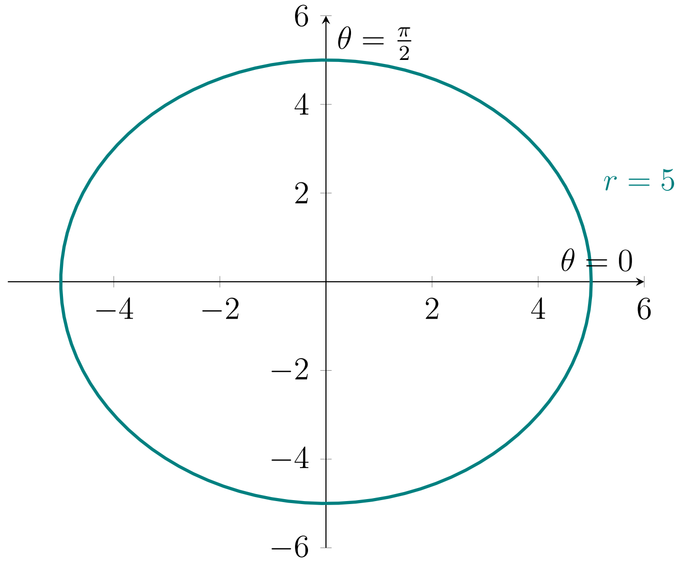


While there has never been a question where the value of r is negative for the given domain of θ, it is still important to remember that regions where $r < 0$ are omitted.

We will now introduce some major polar curves with their respective equations and some of their properties.

1) Circles (centered at the origin)

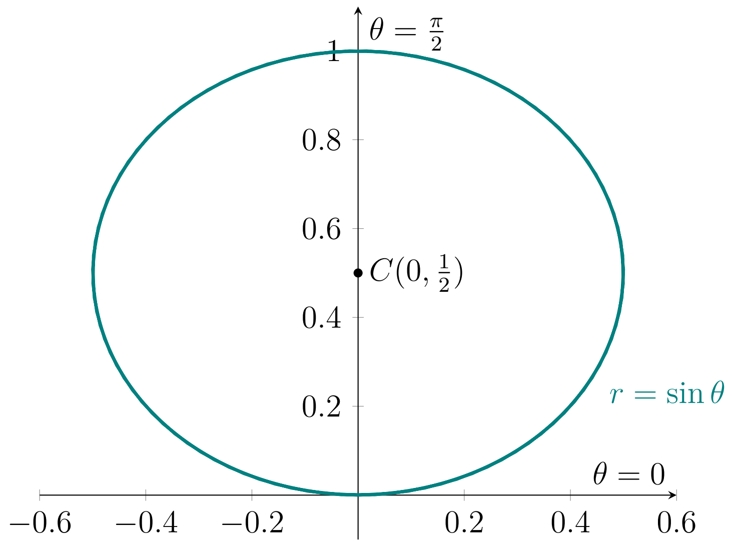
$r = a \quad , \quad a > 0$



It should be quite obvious why we get a circle; no matter the angle, the radial distance from the pole is a, which in turn creates a circle centered at the origin with a radius of a

2) Circles (not centered at the origin)

i) $r = a\sin\theta \quad , \quad a > 0$



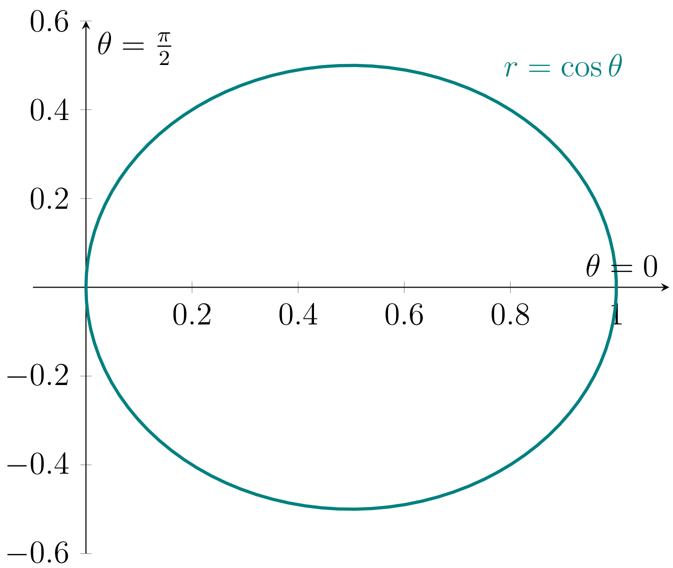
Properties of the graph:

a) the circle is tangential to the initial line and the line $\theta = 180°$

b) the radius of the circle is equal to a

c) circle intersects the half line $\theta = \frac{\pi}{2}$ perpendicularly

ii) $r = a\cos\theta \quad , \quad a > 0 \quad , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



Properties of the graph:

a) the radius of the circle is a

b) the circle is tangential to the half line $\theta = \pm\frac{\pi}{2}$

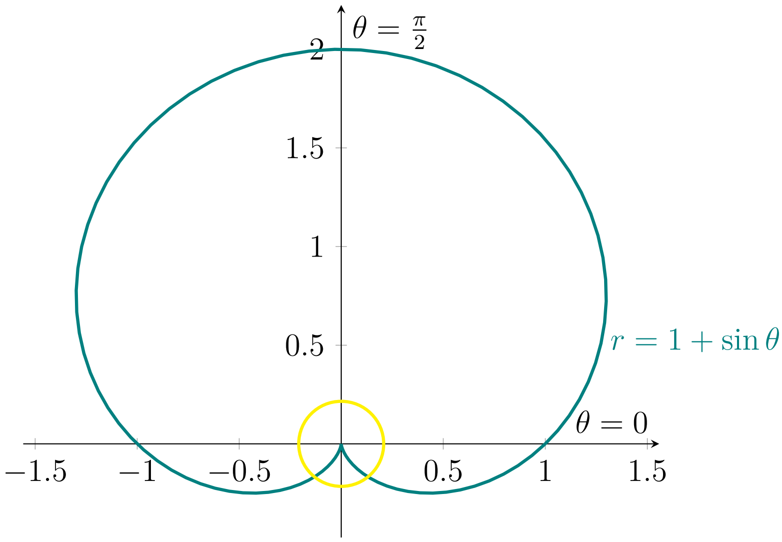
c) the circle intersects the initial line perpendicularly

If you notice, the cosine graph is the sine graph rotated 90 degrees clockwise about the pole.

3) Cardioids

As the name suggest, they have a shape that *resembles* a heart.

i) $a(1 + \sin\theta)$



The circle yellow part is called a *cusp*.

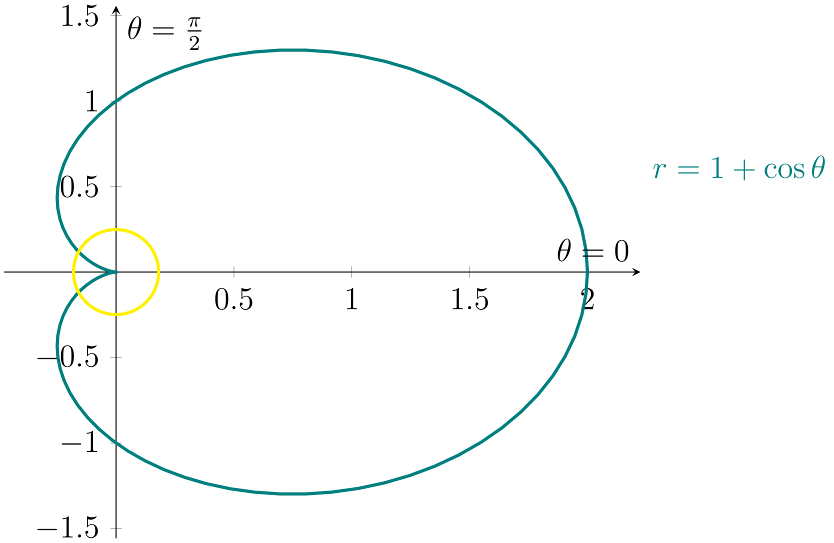
Properties of the graph:

a) lines emerging from the cusp are initially tangential to the half-line $\theta = -\frac{\pi}{2}$

b) the cardioid intersects the half-line $\theta = \frac{\pi}{2}$ perpendicularly

c) the cardioid intersects the initial line and $\theta = \pi$ at $\pm a$ and the half-line $\theta = \frac{\pi}{2}$ at 0 and $2a$

ii) $r = a(1 + \cos\theta)$



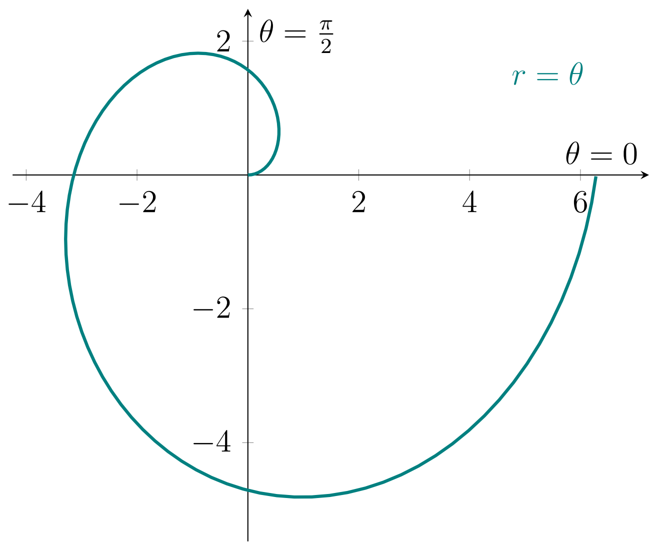
Properties of the graph:

i) lines emerging from the cusp are initially tangential to the initial axis
ii) cardioid intersects the initial line perpendicularly
iii) the cardioid intersects the initial line at 0 and $2a$, and the half-lines $\theta = \pm\frac{\pi}{2}$ at $\pm a$

4) Spirals

Although there are several equations for spirals, we will only discuss one; they all have one thing in common, which is that as θ increases, r increases i.e. the function is strictly increasing. We will look at the simplest spiral.

i) $r = \theta$

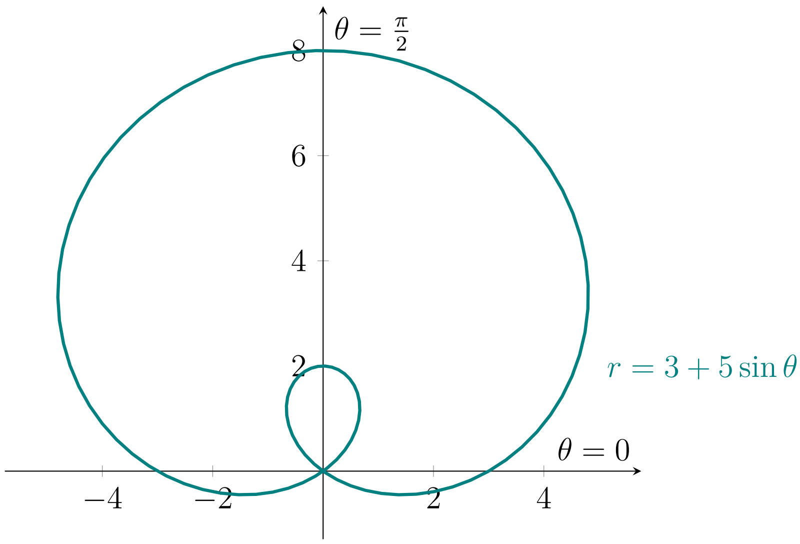


5) Inner-loop Limaçons

There is a special type of polar graphs known as *Limaçons*. Sketching them is very similar to cardioids so we will skip them, except for the *inner-loop Limaçons.* Remember that you are not supposed to memorize the graphs of all the polar equations that may come in the exam (there is no limited number); you are supposed to learn how to sketch them. However, these graphs are more or less standard graphs with properties you should know and illustrate when sketching. Techniques for sketching will be discussed later.

We will only look at the sine variant; the cosine variant is just the sine variant rotated 90 degrees clockwise about the origin

a) $a + b\sin\theta \quad , \quad a < b$



The point of showing the inner-loop Limaçon, which is not a standard graph and sketching it never came in an exam, is to show that polar graphs can be quirky, so we must learn how to properly sketch.

Techniques for sketching

In the exam, the domain of θ is usually from 0 to an angle less than $\frac{\pi}{2}$, so it wont be very hard to sketch the requested graphs. Usually, we would take points of reference with each step being the upper bound $\div$ 4. For example, if the interval of θ is $0 \leq \theta \leq \frac{\pi}{6}$, our step would be $\frac{\pi}{6\times4}$; therefore, our reference points are $\theta = 0$, $\frac{\pi}{24}$, $\frac{\pi}{12}$, $\frac{\pi}{8}$ and $\frac{\pi}{6}$ with their corresponding values of r.

If the interval for θ is $0 \leq \theta \leq \frac{3}{2}\pi$, then our step would be $\frac{3}{2\times4}\pi$; hence, our reference points are $\theta = 0$, $\frac{3}{8}\pi$, $\frac{3}{4}\pi$, $\frac{9}{8}\pi$ and $\frac{3}{2}\pi$ with their corresponding values of r.

It is highly recommended to make a table to keep track of your values. It is also highly recommended you use a protractor to plot your points; it may cost you an extra minute or two but you will be sure that your sketch is going to yield all marks. Taking into account all of the usual tips for sketching, such as having a sharp pencil and a good eraser, we are ready to start sketching.

Question 1 Sketch the the graph of $r = \sqrt{2}\sec\left(\theta - \frac{1}{4}\pi\right)$ for $-\frac{\pi}{4} < \theta \leq \frac{3}{4}\pi$, indicating clearly the polar coordinates of the intersection with the initial line.

Solution:

We will first pick our reference points which have a step of $\left(\frac{3}{4}\pi + \frac{\pi}{4}\right) \div 4 = \frac{\pi}{4}$

θ radians	$\theta°$	r
$-\dfrac{\pi}{4}$	-45	UNDEFINED
0	0	2
$\dfrac{\pi}{4}$	45	$\sqrt{2}$
$\dfrac{\pi}{2}$	90	2
$\dfrac{3}{4}\pi$	135	UNDEFINED

Lets take the limits at our undefined points

1. $\displaystyle\lim_{\theta \to -\frac{\pi}{4}} \sqrt{2}\sec\left(\theta - \frac{\pi}{4}\right) = \infty$

2. $\displaystyle\lim_{\theta \to -\frac{3}{4}\pi} \sqrt{2}\sec\left(\theta - \frac{\pi}{4}\right) = \infty$

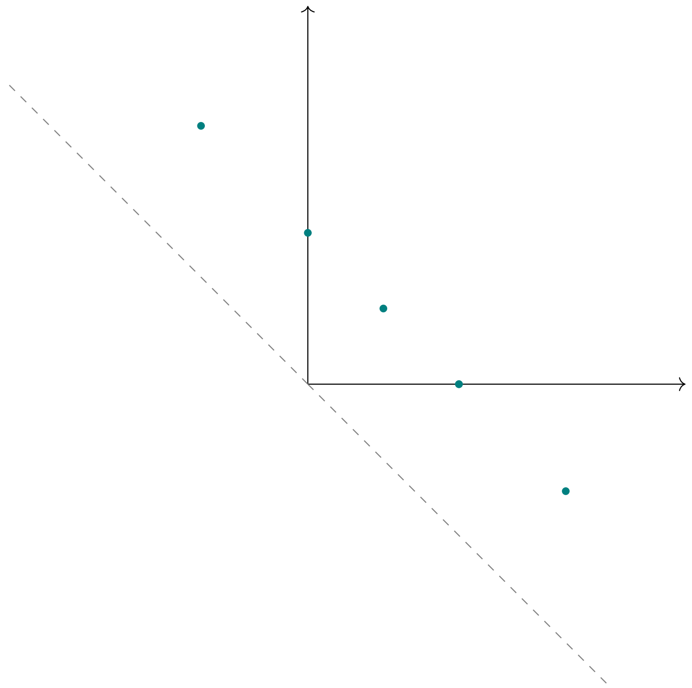
We will also take a reference point between $-\frac{\pi}{4}$ and 0, and $\frac{\pi}{2}$ and $\frac{3}{4}\pi$

When $\theta = \left(-\frac{\pi}{4} + 0\right) \div 2 = -\frac{\pi}{8}$, $r \approx 0.393$

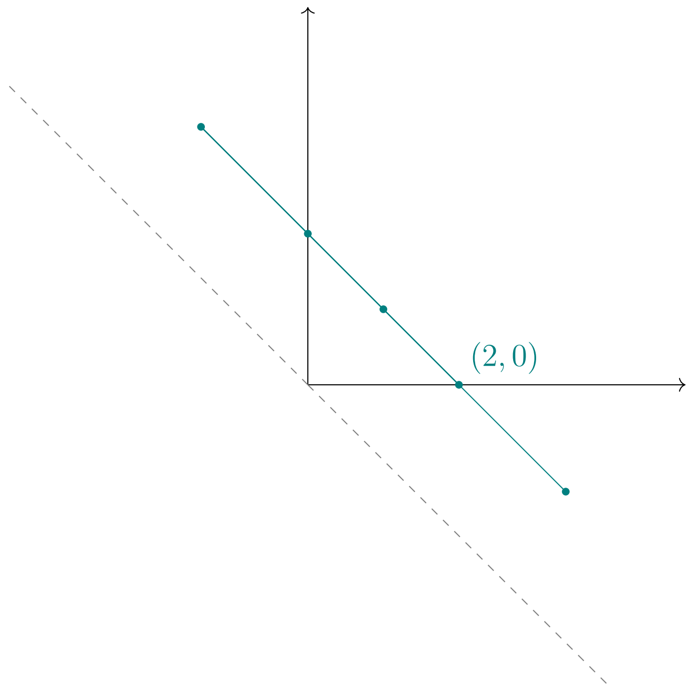
When $\theta = \left(\frac{\pi}{2} + \frac{3}{4}\pi\right) \div 2 = \frac{5}{8}\pi$, $r \approx 3.70$

We now can begin sketching.

First we will plot our points using a protractor and a ruler



It's quite obvious now that we have a straight line. Joining the points, we get the graph of $r = \sqrt{2}\sec\left(\theta - \frac{1}{4}\pi\right)$

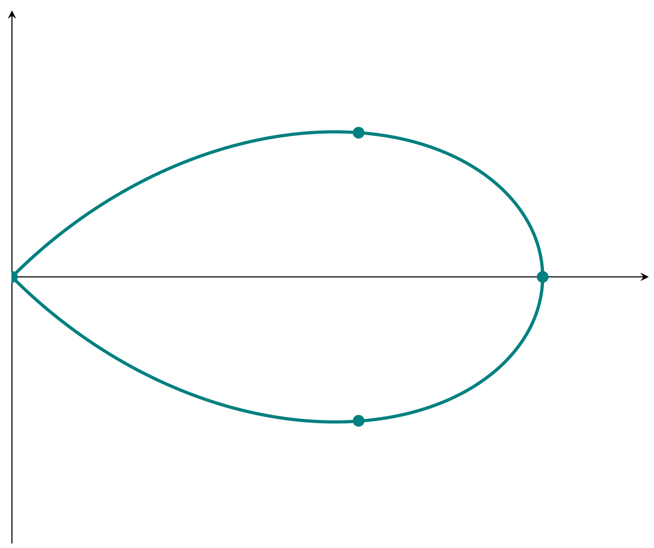


Question 2 The curve C has polar equation $r = \cos 2\theta$, for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Sketch C.

Solution:

θ radians	$\theta°$	r
$-\dfrac{\pi}{4}$	-45	0
$-\dfrac{\pi}{8}$	-22.5	$\dfrac{\sqrt{2}}{2}$
0	0	1
$\dfrac{\pi}{8}$	22.5	$\dfrac{\sqrt{2}}{2}$
$\dfrac{\pi}{4}$	45	0

Plotting our points and sketching:

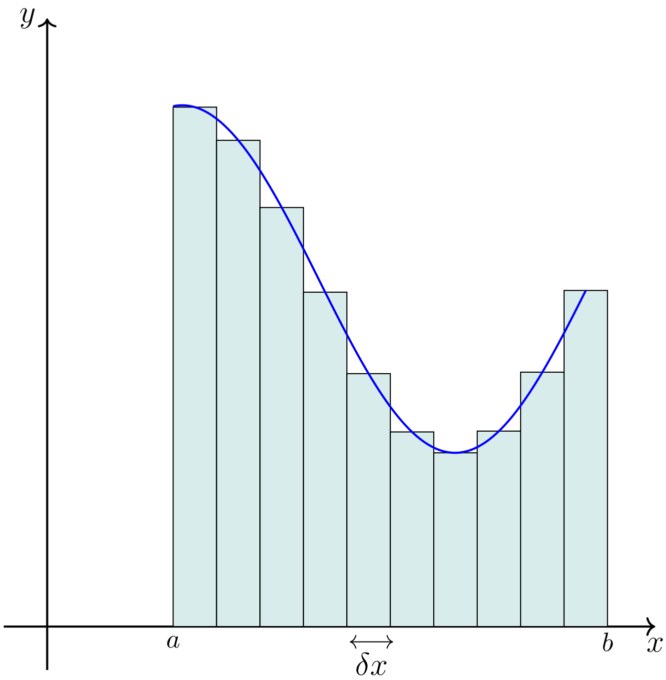


We never add a scale for our initial line and half-lines; we scale them as we wish by multiplying the values of r with a convenient constant.

5.2 Calculus with polar curves and equations

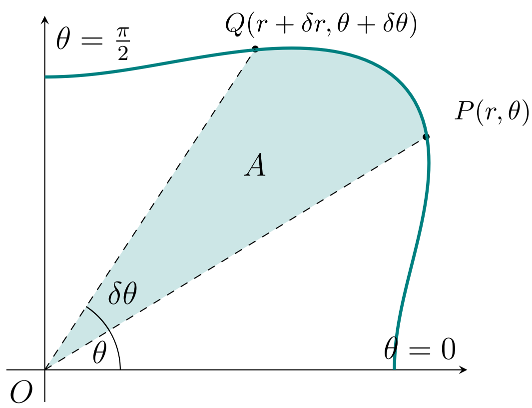
Integration

We already know that the geometric interpretation of integrating a function, $f(x)$, in the Cartesian form is equivalent to finding the areas of rectangles with height $f(x)$ and δx as $\delta x \to 0$.



Integrating functions in their polar form is a bit different. A small change in θ, call it $\delta\theta$, will cause a small change in r, call it δr.

We can see that as $\delta r \to 0$ and $\delta\theta \to 0$, the area of POQ approaches the area of a sector with radius r and angle $\delta\theta$, as $\delta\theta \to 0$. When integrating an equation in its Cartesian form, we are essentially finding the sum of areas of rectangles with width δx and height $f(x)$ from $x = a$ till $x = b$ as $\delta x \to 0$. Integrating a function in its polar form is essentially finding the sum of sectors with an angle $\delta\theta$ and radius r from $\theta = \alpha$ till $\theta = \beta$; this means that integrating functions in their polar forms gives out the area *enclosed* by the curve and some half-line (depends on how you set up the integral).

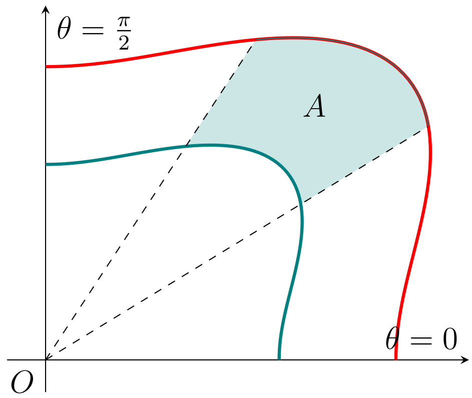


Finally, we can now establish the formula for the area enclosed by a polar curve:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta$$

Finding the area enclosed between two polar curves is very easy; it is the area enclosed by the bigger curve, r_2, minus the area enclosed by the smaller curve, r_1. To calculate it, we have the following formula:

$$\begin{aligned} A &= \frac{1}{2}\int_{\alpha}^{\beta} (r_2)^2 \, \mathrm{d}\theta - \frac{1}{2}\int_{\alpha}^{\beta} (r_1)^2 \, \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} \left({r_2}^2 - {r_1}^2\right) \, \mathrm{d}\theta \\ &\neq \frac{1}{2}\int_{\alpha}^{\beta} (r_2 - r_1)^2 \, \mathrm{d}\theta \end{aligned}$$

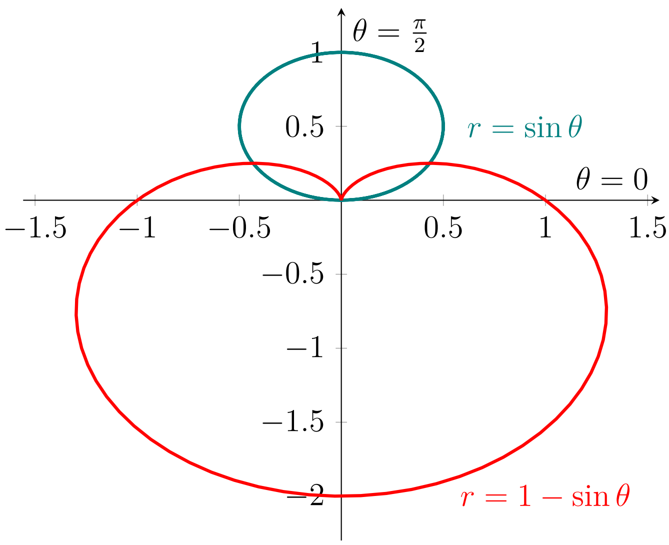


In the exam, the question regarding this chapter always starts with sketching the curves before anything else; the questions are designed in a linear manner, and won't ask you for the area enclosed by a curve and some half-line before sketching the curve in a previous part.

Example 1: Sketch the curves $r = \sin\theta$ and $r = 1 - \sin\theta$ for $0 \le \theta \le 2\pi$, and find the area enclosed between the two curves.

Solution:

Following our procedure for sketching curves, we get the following



We can see that we have two enclosed regions. By symmetry, they have an equal area; therefore, we can find the area of one enclosed region and multiply the result by 2. To find the area enclosed between the 2 curves, we have to find the points of intersection. The procedure is no different than with Cartesian equations; we equate both values of r.

$$r = \sin\theta \qquad\qquad r = 1 - \sin\theta$$

$$1 - \sin\theta = \sin\theta$$

$$\sin\theta = \frac{1}{2}$$

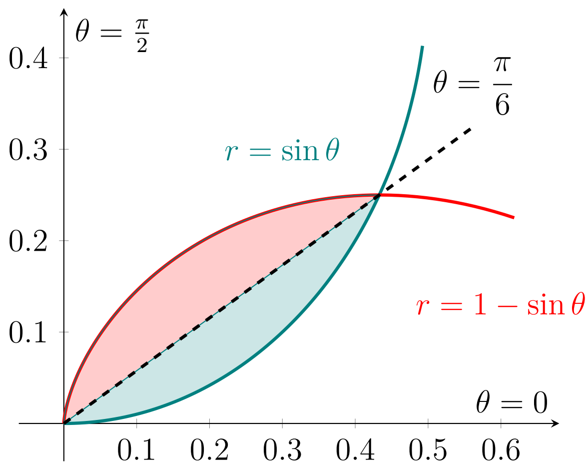
$$\theta = \frac{\pi}{6} \quad , \quad \frac{5}{6}\pi$$

We will focus on the enclosed region in the first quadrant where the θ value for the intersection is $\theta = \frac{\pi}{6}$

The total area enclosed on the right is equal to the blue area + red area.

The blue area is the are enclosed between $\sin\theta$ and the half-line $\theta = \frac{\pi}{6}$ (think of the sectors subtended from the origin to the curve). Let the blue area $= A_B$.
Therefore,

$$A_B = \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin^2\theta \,\mathrm{d}\theta$$



The red area is enclosed between the half-line $\theta = \frac{\pi}{6}$ and the curve $r = 1 - \sin\theta$. Let the red area $= A_R$. Therefore,

$$A_R = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \sin\theta)^2 \,\mathrm{d}\theta$$

So the total enclosed area, A, is equal to $A_B + A_R$. Therefore,

$$A = \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin^2\theta \,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 - \sin\theta)^2 \,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin^2\theta \,\mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin^2\theta - 2\sin\theta + 1\right) \mathrm{d}\theta$$

Since $2\sin^2\theta = 1 - \cos 2\theta$, then

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(\frac{1-\cos 2\theta}{2}\right) \mathrm{d}\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2} - 2\sin\theta + 1\right) \mathrm{d}\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\ \mathrm{d}\theta + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1-\cos 2\theta - 4\sin\theta + 2)\ \mathrm{d}\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}} + \frac{1}{4}\left[3\theta + 4\cos\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\to \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\left[\left(\frac{\pi}{6} - \frac{\sin\left(2\times\frac{\pi}{6}\right)}{2}\right) - \left(0 - \frac{\sin 2\times 0}{2}\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$\to \frac{1}{4}\left[3\theta + 4\cos\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{4}\left(3\times\frac{\pi}{2} + 4\cos\frac{\pi}{2} - \frac{\sin\left(2\times\frac{\pi}{2}\right)}{2}\right)$$

$$- \frac{1}{4}\left(3\times\frac{\pi}{6} + 4\cos\frac{\pi}{6} - \frac{\sin\left(2\times\frac{\pi}{6}\right)}{2}\right)$$

$$= \frac{1}{4}\left(\pi - \frac{7\sqrt{3}}{4}\right)$$

$$\therefore A = \frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \pi - \frac{7\sqrt{3}}{4}\right)$$

$$= \frac{7}{24}\pi - \frac{\sqrt{3}}{2}$$

Therefore, the area of *both* enclosed regions is $\frac{7}{12}\pi - \sqrt{3}$

Differentiation

We know that differentiation is mainly used to find the rate of change at a point and certain maximum and minimum values. This will also be true here. However, we must take care what we are differentiating and with respect to what; for example, we can express r in terms of x, x in therms of θ and so on.

The maximum distance from the pole

To find the maximum distance from the pole, we must find the value of θ that gives us r_{max}. To do this, we must differentiate r with respect to θ, and find the value of θ when

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$$

Lets look at some examples:

Example 1: Find the value of θ and the corresponding value of r, when the value of r is at its maximum for $r = \sin 3\theta$ where $0 \leq \theta \leq \frac{\pi}{3}$

Solution:

We first have to find $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ and find the value of θ where $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 3\cos 3\theta$$

Equating to 0

$$3\cos 3\theta = 0$$

$$\cos 3\theta = 0$$

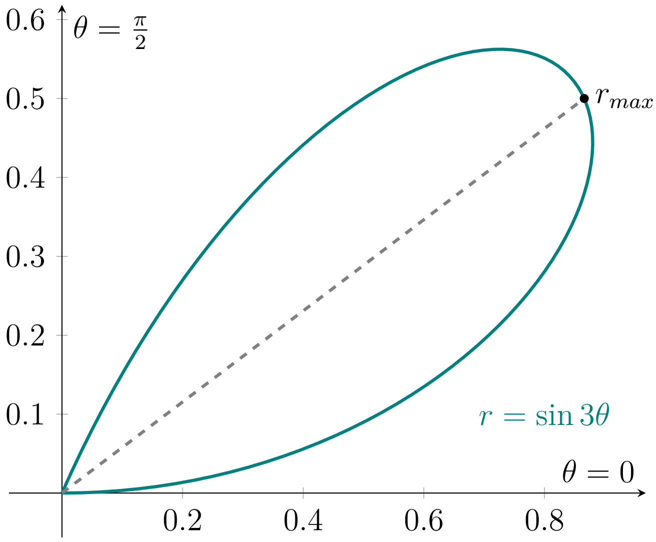
$$3\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

Therefore, when $\theta = \frac{\pi}{6}$, r is at its maximum value on the interval $0 \leq \theta \leq \frac{\pi}{3}$

$$\therefore r_{max} = \sin\left(3 \times \frac{\pi}{6}\right) = 1$$

We can look at the graph of $r = \sin 3\theta$ for $0 \leq \theta \leq \frac{\pi}{3}$ to visualise how that would look



Example 2: Find the value of θ and the corresponding value of r, when the value of r is at its maximum for $r = e^{\theta} \cos\theta$ where $0 \leq \theta \leq \frac{\pi}{2}$

Solution:

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta$$

Equating to 0

$$e^{\theta}\cos\theta - e^{\theta}\sin\theta = 0$$

$$e^{\theta}(\cos\theta - \sin\theta) = 0$$

$$e^{\theta} = 0 \qquad \text{or} \qquad \cos\theta - \sin\theta = 0$$

$$e^{\theta} \neq 0 \qquad\qquad \tan\theta = 1$$

$$\therefore \theta = \frac{\pi}{4}$$

$$\rightarrow r_{max} = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}$$

We will omit the sketch as it isnt needed. Nevertheless, you can explore the sketch on geogebra or desmos.

The maximum distance from the initial axis

For the maximum distance from the initial axis, we have to find to find the value of θ for which $|y|$ is maximum. Therefore, we have to express y in terms of θ, differentiate y with respect to θ and equate to 0. To express y in terms of θ, we have to use the fact that

$$y = r\sin\theta$$

where r is a function of θ.

Lets take some examples.

Example 1: Find the minimum and maximum distance from the initial axis for $r = 1 - \sin\theta$ on $0 \leq \theta \leq 2\pi$

Solution:

Since $r = 1 - \sin\theta$, then

$$y = (1 - \sin\theta)\sin\theta$$

$$y = \sin\theta - \sin^2\theta$$

$$\rightsquigarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta - 2\sin\theta\cos\theta$$

Equating the derivative to 0

$$\cos\theta - 2\sin\theta\cos\theta = 0$$

$$\cos\theta\,(1 - 2\sin\theta) = 0$$

$$\cos\theta = 0 \qquad , \qquad 1 - 2\sin\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3}{2}\pi \qquad\qquad \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5}{6}\pi$$

The minimum distance from the initial axis is 0, occurring at $\theta = \frac{\pi}{2}$

The maximum distance from the initial axis is $|-2| = 2$,[1]occurring at $\theta = \frac{3}{2}\pi$.

At $\theta = \frac{\pi}{6}$ and $\frac{5}{6}\pi$ we have local maximas (not required; differentiating between local maximas and maximums is not required). The curve has been sketched before.

[1]Notice how we differentiate between maximum *distance* from the initial axis and maximum *y value* for a point

Example 2: Find the minimum y value, y_{min}, for the curve $r = 1+\sin\theta$ on the interval $0 \leq \theta \leq 2\pi$.

Solution:

Since $r = 1 + \sin\theta$, then

$$y = (1+\sin\theta)\sin\theta$$

$$y = \sin\theta + \sin^2\theta$$

$$\rightsquigarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = \cos\theta + 2\sin\theta\cos\theta$$

Equating the derivative to 0

$$\cos\theta + 2\sin\theta\cos\theta = 0$$

$$\cos\theta\,(1+2\sin\theta) = 0$$

$$\cos\theta = 0 \qquad , \qquad 1 + 2\sin\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3}{2}\pi \qquad\qquad \sin\theta = -\frac{1}{2}$$

$$\theta = \frac{7}{6}\pi, \frac{11}{6}\pi$$

By testing the θ values, $y_{min} = -\frac{1}{4}$ for $\theta = \frac{7}{6}\pi$ and $\frac{11}{6}\pi$.

The maximum distance from the half-line $\theta = \frac{\pi}{2}$

For the maximum distance from the half-line $\theta = \frac{\pi}{2}$, we have to find the value of θ for which $|x|$ is maximum. Therefore, we have to express x in terms of θ, differentiate x with respect to θ and equate to 0. To express x in terms of θ, we have to use the fact that

$$x = r\cos\theta$$

where r is a function of θ.

We will only take one example since its essentially the same thing as the previous ideas.

Example: Find the maximum distance from the half-line $\theta = \frac{\pi}{2}$ for $r = \sin 2\theta$ where $0 \leq \theta \leq \frac{\pi}{2}$

Solution:

Since $r = \sin 2\theta$, then

$$x = \sin 2\theta \cos\theta$$

$$\rightsquigarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos 2\theta\cos\theta - \sin 2\theta \sin\theta$$

Equating to 0

$$2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0$$

Using double angle formulaes

$$2\left(2\cos^2\theta - 1\right)\cos\theta - 2\sin^2\theta\cos\theta = 0$$

$$4\cos^3\theta - 2\cos\theta - 2\cos\theta\left(1-\cos^2\theta\right) = 0$$

$$4\cos^3\theta - 2\cos\theta - 2\cos\theta + 2\cos^3\theta = 0$$

$$6\cos^3\theta - 4\cos\theta = 0$$

$$2\cos\theta\left(3\cos^2\theta - 2\right) = 0$$

$$\cos\theta = 0 \qquad , \qquad 3\cos^2\theta - 2 = 0$$

$$\theta = \frac{\pi}{2} \qquad\qquad \cos\theta = \pm\sqrt{\frac{2}{3}}$$

$$\cos\theta = \pm\sqrt{\frac{2}{3}}$$

$$\cos\theta = \sqrt{\frac{2}{3}} \qquad , \qquad \cos\theta = -\sqrt{\frac{2}{3}}$$

Since $0 \leq \theta \leq \frac{\pi}{2}$, then

$$\cos\theta = \sqrt{\frac{2}{3}} \text{ ONLY}$$

Since we cannot find the exact value of θ, we have to use the values of $\sin\theta$ and $\cos\theta$

$$\rightarrow \sin\theta = \frac{\sqrt{\left(\sqrt{3}\right)^2 - \left(\sqrt{2}\right)^2}}{\sqrt{3}}$$

$$\sin\theta = \frac{1}{\sqrt{3}}$$

$$\therefore x = \sin 2\theta \cos\theta = 2\sin\theta\cos^2\theta$$

$$x = 2 \times \frac{1}{\sqrt{3}} \times \left(\sqrt{\frac{2}{3}}\right)^2$$

$$= \frac{4}{9}\sqrt{3}$$

5.3 Important exercises

Questions:

1. The curve C has polar equation $r = a\cot\left(\frac{\pi}{3} - \theta\right)$ where a is a positive constant and $0 \le \theta \le \frac{\pi}{6}$.
   It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$

   a) Sketch C and show that $a = 2$.

   b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{\pi}{6}$.

   c) Show that C has Cartesian equation $2\left(x + y\sqrt{3}\right) = \left(x\sqrt{3} - y\right)\sqrt{x^2 + y^2}$

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2. The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \le \theta \le \frac{\pi}{2}$

   a) The point on C_1 furthest from the line $\theta = \frac{\pi}{2}$ is denoted by P. Show that, at P,

   $$2\theta\tan\theta - 1 = 0$$

   and verify that this equation has a root between 0.6 and 0.7.

   The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \le \theta \le \frac{\pi}{2}$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

b) Find the polar coordinates of Q, giving your answers in exact form.

c) Sketch C_1 and C_2 on the same diagram.

d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2.

3. The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leq \theta \leq \frac{\pi}{4}$.

a) Sketch C and state the greatest distance of a point on C from the pole.

b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{\pi}{4}$.

c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$.

d) Using your answer to part **b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$

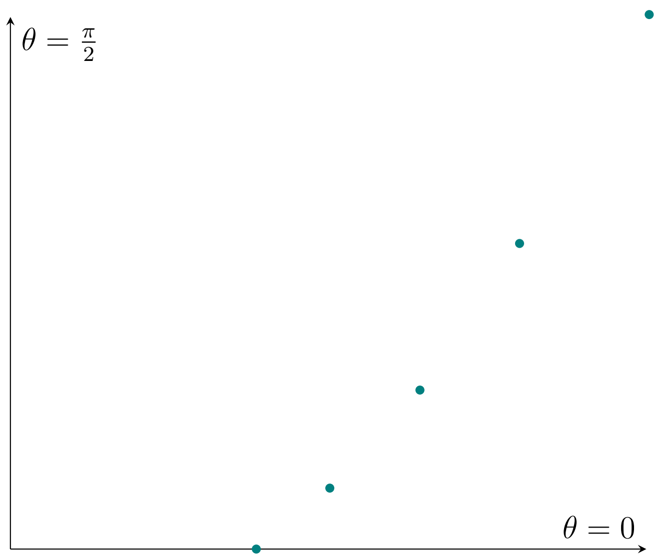
DETAILED SOLUTIONS ON THE NEXT PAGE. ATTEMPT BEFORE PROCEEDING

Solutions:

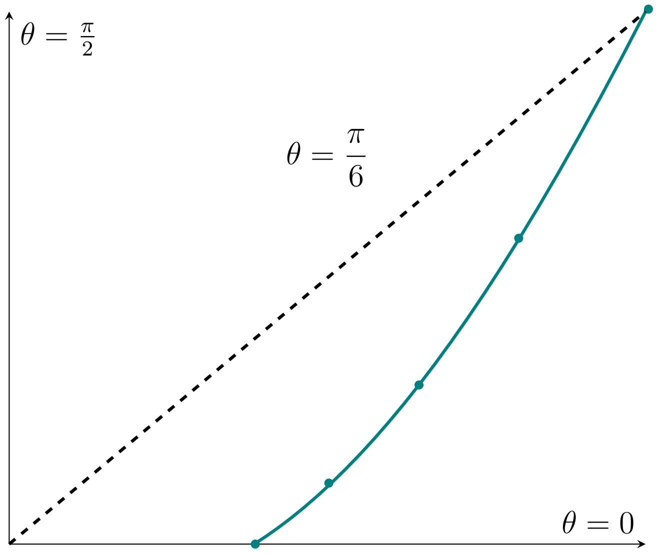
1. **a)** For sketching purposes, we assume that $a = 5$
   Our step will be equal to $\frac{\pi}{6\times 4} = \frac{\pi}{24}$

θ radians	$\theta°$	r
0	0	2.9
$\frac{\pi}{24}$	7.5	3.8
$\frac{\pi}{12}$	15	5
$\frac{\pi}{8}$	22.5	6.5
$\frac{\pi}{6}$	30	8.7

We took the value of r to one decimal place since this is what you can measure with your rule.
Plotting our points we get



Joining the points as best as we can, we get the following curve:



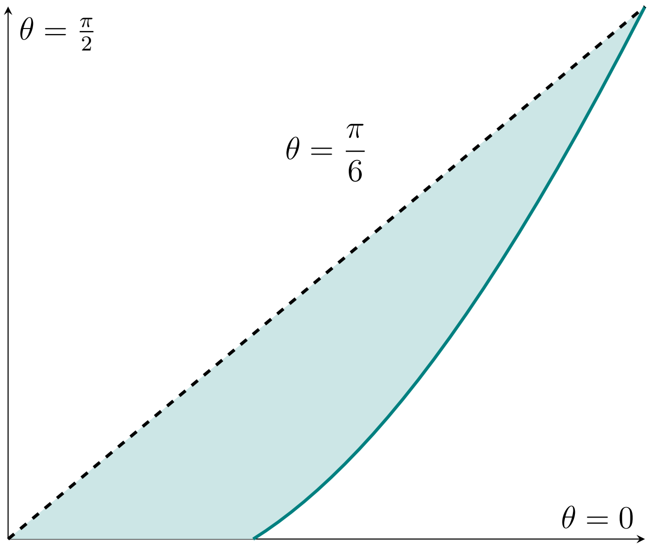
The distance from the pole is at its maximum when $\theta = \frac{\pi}{6}$. Therefore

$$a \cot\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 2\sqrt{3}$$

$$a\sqrt{3} = 2\sqrt{3}$$

$$a = 2 \quad \square$$

b) The region bounded by C and the half-line $\theta = \frac{\pi}{6}$, A, is the one shaded below.



$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\frac{\pi}{6}} 4\cot^2\left(\frac{\pi}{3}-\theta\right)\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{6}} \cot^2\left(\frac{\pi}{3}-\theta\right)\,\mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{6}} \left(\csc^2\left(\frac{\pi}{3}-\theta\right)-1\right)\,\mathrm{d}\theta
\end{aligned}
$$

Since $\dfrac{d}{d\theta}(\cot\theta) = -\csc^2\theta$ then $\displaystyle\int \csc^2\theta\,\mathrm{d}\theta = -\cot\theta$

$$\therefore A = 2\left[\cot\left(\frac{\pi}{3}-\theta\right)-\theta\right]_0^{\frac{\pi}{6}}$$

Skipping past the substiution and calculations, we get

$$A = \frac{4}{3}\sqrt{3}-\frac{\pi}{3}$$

c) Our goal is $r = 2\cot\left(\frac{\pi}{3}-\theta\right) \to 2\left(x+y\sqrt{3}\right) = \left(x\sqrt{3}-y\right)\sqrt{x^2+y^2}$

$$r = \frac{2\cos\left(\frac{\pi}{3}-\theta\right)}{\sin\left(\frac{\pi}{3}-\theta\right)} \qquad \text{Use } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$r\sin\left(\frac{\pi}{3}-\theta\right) = 2\cos\left(\frac{\pi}{3}-\theta\right)$$

$$r\left(\sin\frac{\pi}{3}\cos\theta - \sin\theta\cos\frac{\pi}{3}\right) = 2\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\cos\theta\right)$$ Using the double angle formulaes

$$\left[r\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) = 2\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\cos\theta\right)\right]\times 2$$

$$r\left(\sqrt{3}\cos\theta - \sin\theta\right) = 2\left(\cos\theta + \sqrt{3}\cos\theta\right)$$

Since $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$, then

$$\left[r\left(\frac{x}{r}\sqrt{3}-\frac{y}{r}\right) = 2\left(\frac{y}{r}+\frac{x}{r}\sqrt{3}\right)\right]\times r$$

$$\sqrt{x^2+y^2}\left(x\sqrt{3}-y\right) = 2\left(y+x\sqrt{3}\right) \quad \square \qquad \text{as } r = \sqrt{x^2+y^2}$$

2. **a)**

$$x = (\theta \cos\theta)\cos\theta$$

$$x = \theta \cos^2\theta$$

$$\rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos^2\theta - 2\sin\theta\cos\theta$$

Equating to 0

$$\left[\cos^2\theta - 2\sin\theta\cos\theta = 0\right] \times -\frac{1}{\cos^2\theta}$$

$$2\tan\theta - 1 = 0 \quad \square$$

Let $f(\theta) = 2\theta\tan\theta - 1$

When $\theta = 0.6$:

$$f(0.6) = 2 \times 0.6\tan 0.6 - 1 = -0.18 < 0$$

When $\theta = 0.7$:

$$f(0.7) = 2 \times 0.7\tan 0.7 - 1 = 0.18 > 0$$

Since $f(\theta)$ is continuous on $[0.6, 0.7]$ and there is a sign change between 0.6 and 0.7, then there must be a root between 0.6 and 0.7.

b) For our intersection, $C_1 = C_2$

$$\theta\sin\theta = \theta\cos\theta$$

$$\theta\sin\theta - \theta\cos\theta = 0$$

$$\theta(\sin\theta - \cos\theta) = 0$$

$$\theta = 0 \qquad , \qquad \sin\theta - \cos\theta = 0$$

$$\tan\theta = 1$$

$$\theta = \frac{\pi}{4}$$

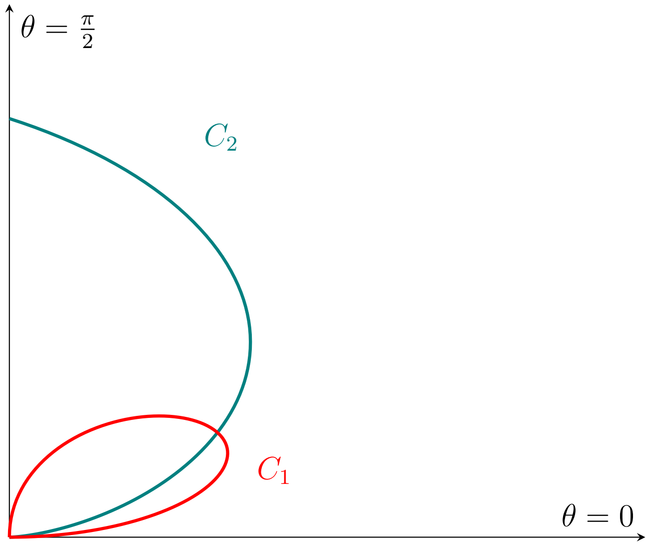
$$\therefore \theta_Q = \frac{\pi}{4}$$

$$r_Q = \frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)$$

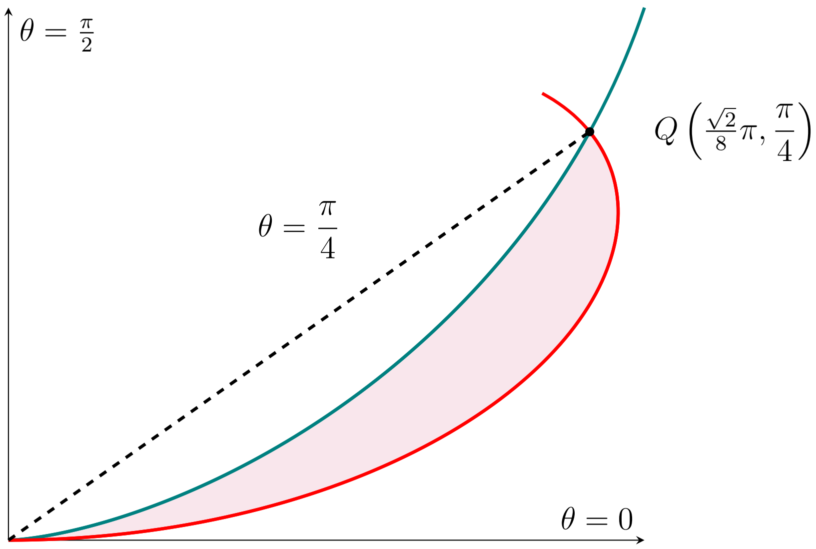
$$r_Q = \frac{\sqrt{2}}{8}\pi$$

$$\therefore Q\left(\frac{\sqrt{2}}{8}\pi, \frac{\pi}{4}\right)$$

c) We will skip the procedure on how to come up with the sketch as we have done this enough already.



d) The enclosed area between OQ of C_1 and OQ of C_2 is the area *bordered* by OQ of C_1 and OQ of C_2 and is shaded below.



The shaded region, A, is the area enclosed by C_1 and the half-line $\theta = \frac{\pi}{4}$ - the area enclosed by C_2 and the half-line $\theta = \frac{\pi}{4}$

Therefore,

$$\begin{aligned}
A &= \int_0^{\frac{\pi}{4}} \left((\theta\cos\theta)^2 - (\theta\sin\theta)^2\right)\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\theta^2\cos^2\theta - \theta^2\sin^2\theta\right)\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\theta^2\left(\cos^2\theta - \sin^2\theta\right)\right)\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \theta^2\cos 2\theta\,\mathrm{d}\theta
\end{aligned}$$

Letting $u = \theta^2$ and $\mathrm{d}v = \cos 2\theta$, and integrating by parts,

$$A = \left[\frac{\theta^2\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \theta\sin 2\theta\,\mathrm{d}\theta$$

Consider $\int_0^{\frac{\pi}{4}} \theta\sin 2\theta\,\mathrm{d}\theta$. Letting $u = \theta$ and $\mathrm{d}v = \sin 2\theta$, and integrating by parts,

$$\begin{aligned}
\to \int_0^{\frac{\pi}{4}} \theta\sin 2\theta\,\mathrm{d}\theta &= \left[-\frac{\theta\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2\theta\,\mathrm{d}\theta \\
&= \left[-\frac{\theta\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}} + \left[\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}
\end{aligned}$$

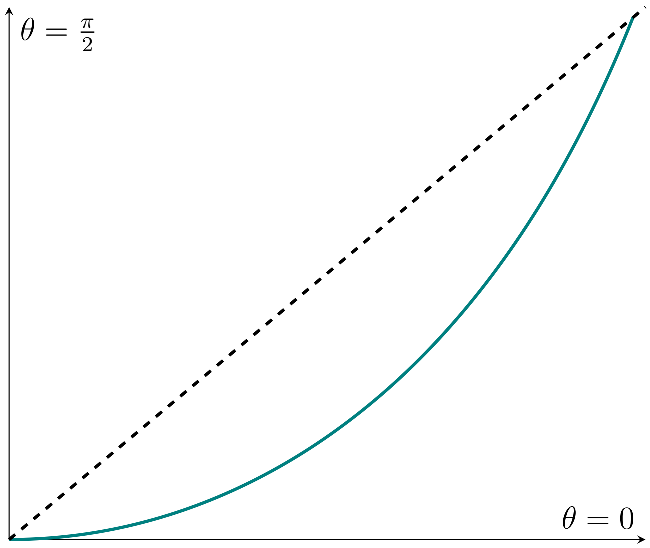
Substituting our result back into A, we get the following:

$$\begin{aligned}
A &= \left[\frac{\theta^2\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} - \left[-\frac{\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}} \\
&= \left[\frac{\theta^2\sin 2\theta}{2} + \frac{\theta\cos 2\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}
\end{aligned}$$

Substituting and simplifying, we get

$$A = \frac{\pi^2}{64} - \frac{1}{8}$$

3. **a)** We will assume that $a = 5$ and proceed with the standard procedure of sketching.



The distance of the curve from the pole is at its maximum when $\theta = \frac{\pi}{4}$; therefore, $r_{max} = a \tan \frac{\pi}{4} = a$. [2]

b) Let the area bordered (enclosed) by the curve and the half-line $\theta = \frac{\pi}{4}$.

$$A = \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \tan^2 \theta \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{4}} \left(\sec^2 \theta - 1\right) \, d\theta$$

$$= \frac{a^2}{2} \left[\tan \theta - \theta\right]_0^{\frac{\pi}{4}}$$

$$A = \frac{1}{2} a^2 \left(1 - \frac{1}{4}\pi\right)$$

[2]Remember that we only used a=5 for sketching purposes!

c) Our goal is $r = a\tan\theta \to y = \frac{x^2}{\sqrt{a^2-x^2}}$

$$r = a\frac{\sin\theta}{\cos\theta}$$

$$\sqrt{x^2+y^2} = a\frac{y}{x}$$

$$x\sqrt{x^2+y^2} = ay$$

$$\left[x\sqrt{x^2+y^2}\right]^2 = [ay]^2$$

$$x^2\left(x^2+y^2\right) = a^2y^2$$

$$x^4 + x^2y^2 = a^2y^2$$

$$x^4 = a^2y^2 - x^2y^2$$

$$x^4 = y^2\left(a^2-x^2\right)$$

$$\frac{x^4}{a^2-x^2} = y^2$$

$$y = \pm\sqrt{\frac{x^4}{a^2-x^2}}$$

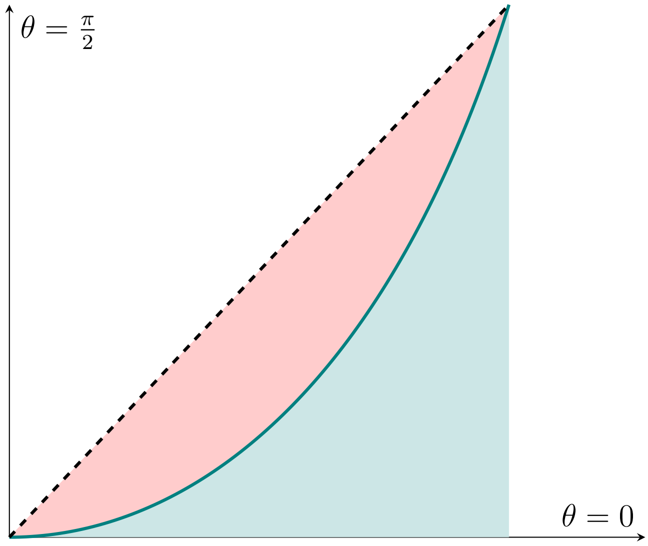
Since y is always positive, we only take the positive square root

$$y = \frac{x^2}{\sqrt{a^2-x^2}} \quad \square$$

We know that we have to square both sides at some point since we have a^2 some where. Generally speaking, if you ever get stuck, it is a good idea to convert all θ's and r's to x's and y's and then attempt to land at the wanted equation.

d) Notice that our integrand is the Cartesian form of our curve. This means that we are finding the area *under* the curve, or the area enclosed between the positive x-axis and the curve. We have previously found the area enclosed between the curve and the half-line $\theta = \frac{\pi}{4}$.

We can see this more clearly with diagrams



This is the area that we have found in part **b)** is the *red* area. The are that we are asked to find is the *blue* area. The blue area can be easily found by subtracting the red area from the area of the triangle that is made up from both regions.
The hypotenuse of the triangle is a, and the base of the triangle is

$$a\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}a = \text{upper limit of the integral}$$

$$\begin{aligned}\therefore \int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x &= \frac{1}{2} \times a \times \frac{\sqrt{2}}{2}a \times \sin\frac{\pi}{4} - \frac{1}{2}a^2\left(1 - \frac{1}{4}\pi\right) \\ &= \frac{1}{4}a^2\left(\frac{\pi}{2} - 1\right)\end{aligned}$$

CHAPTER VI
VECTORS

$\overleftrightarrow{AB}$

Chapter VI

Vectors

What are vectors?

Vectors are mathematical objects used to represent quantities that have both magnitude and direction. A vector can be thought of as an arrow in space, with the length of the arrow representing the magnitude of the vector and the direction indicating its direction. You should already have considerable experience dealing with vectors in both mathematics and physics.

As you might expect, vectors have a wide range of applications, most being elementary. For example, in physics, vectors are used to represent the forces acting on a body. An interesting application is the use of vectors to calculate the amount of light that is reflected or refracted by an object. The amount of light that is reflected or refracted by an object depends on the object's material properties and the angle of incidence of the light ray. These properties can be represented by vectors, and the calculations can be done using vector operations.

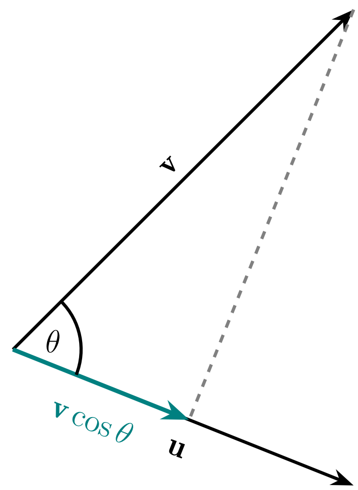
A historical introduction

It is surprising to know that the formalization of vector algebra began in the late 19th and early 20th centuries. Mathematicians such as Josiah Willard Gibbs, Oliver Heaviside, and Hermann Grassmann made significant contributions to the development of vector algebra and vector analysis. They developed the algebraic rules and notation for manipulating vectors and introduced concepts like dot product and cross product. However, it is obvious that the *idea* of vectors can be traced back to older times. While there is some evidence to suggest that the ancient Egyptians may have had an intuitive understanding of vectors, Newton used the *concept* of vectors through out his development of classical mechanics. You would've probably came to the conclusion when studying physics at the IGCSE level, or equivalent, that Newton created the notion of vectors; however, Sir Isaac Newton did not use the term *vectors* in any of his work! It would be frightening to imagine what Newton could have done with the modern understanding of vectors and matrices. The field of vector calculus emerged in the late 19th century with the work of James Clerk Maxwell and others (you may have heard of Maxwell through the famous *Maxwell equation*.

6.1 The cross product

6.1.1 The geometric interpretation

We know that the geometric interpretation of the dot product of some vector **u** and some vector **v** (i.e. $\mathbf{u}\cdot\mathbf{v}$) is equal to the length of **u** projected onto **v** multiplied by the length of **v**.



As a refresher,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta \\ &= |\mathbf{v}||\mathbf{u}|\cos\theta \\ &= \mathbf{v}\cdot\mathbf{u}\end{aligned}$$

this shows that the dot product between two vectors is commutative.

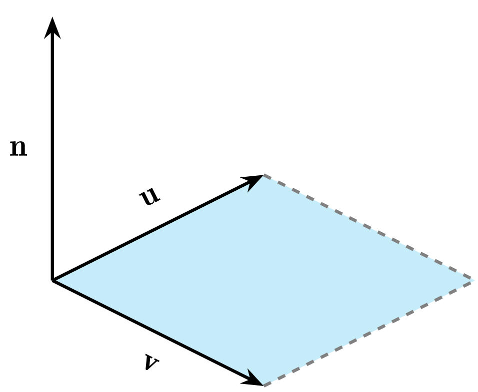
There is also the fact that dot product multiplication is distributive.

Another name for the *dot* product is the *scalar* product: this is due to the *scalar* nature of the output.

We will now look at the cross product.

There is another name for the *cross* product, the *vector* product; you would be correct if you deduce the fact that the output of the cross product between two vectors will be a vector.

Geometrically, the cross product between two vectors **u** and **v** is the vector **n**, where **n** has a direction vector that is perpendicular to both **u** and **v** (i.e. it's the common perpendicular vector to both multiplied vectors), and a magnitude equal to the *area* of the parallelogram enclosed by both vectors (shaded in blue).



The notation for the cross product is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v}$$

You would be wrong to assume that cross product multiplication is commutative; however, it is *anti-commutative*. This means that

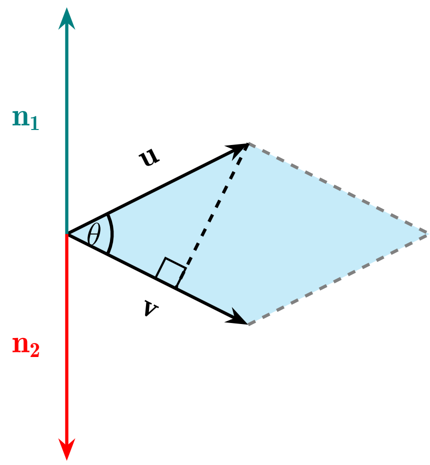
$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

Now what does this mean geometrically? This means that the vector equal to $\mathbf{v} \times \mathbf{u}$ has a direction that is *opposite* to $\mathbf{u} \times \mathbf{v}$. This is due to the fact that

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

This result should make sense! There are two vectors that are perpendicular to both $\mathbf{u}$ and $\mathbf{v}$; the vector $\mathbf{n}$ (as seen in the diagram), and the vector that is opposite to $\mathbf{n}$ (the one pointing down from $\mathbf{n}$).

We now come up with a new visualization of the cross product in both scenarios where $\mathbf{u}$ and $\mathbf{v}$ are in front or behind the $\times$.



We can see that

$$\mathbf{n_1} = \mathbf{u} \times \mathbf{v}$$

and

$$\mathbf{n_2} = \mathbf{v} \times \mathbf{u}$$

We also get the following results:

<u>Number one:</u>

$|\mathbf{n_1}| = |\mathbf{n_2}| = |\mathbf{v} \times \mathbf{u}| = |\mathbf{u} \times \mathbf{v}| = \texttt{Blue Area}$

Where the `Blue Area` is the area of the parallelogram enclosed by $\mathbf{u}$ and $\mathbf{v}$. Since the area of the parallelogram, A is

$$A = |\mathbf{u}||\mathbf{v}|\sin\theta$$

then

$$|\mathbf{n_1}| = |\mathbf{n_2}| = |\mathbf{v} \times \mathbf{u}| = |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = \texttt{Blue Area}.$$

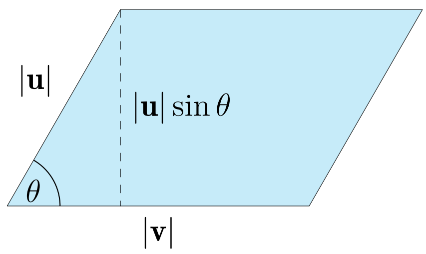
To make things shorter

$$|\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

Number two:

$\mathbf{n} = |\mathbf{u}||\mathbf{v}|\hat{\mathbf{n}} \sin\theta$

Where $\hat{\mathbf{n}}$ is the unit vector of $\mathbf{n}$
Now this result may need some explaining.
$|\mathbf{u}||\mathbf{v}|\sin\theta$ takes care of the *magnitude* of $\mathbf{n}$. $\hat{\mathbf{n}}$ takes care of the *direction* of $\mathbf{n}$. Notice that $\hat{\mathbf{n}}$ does *not* contribute anything to the magnitude of $\mathbf{n}$ as it is a unit vector and thus has a magnitude of 1.



6.1.2 Computing the cross product

We won't go into the explanation of why the cross product is computed as it is, due to it being rather complex. It is obviously unrequired knowledge.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

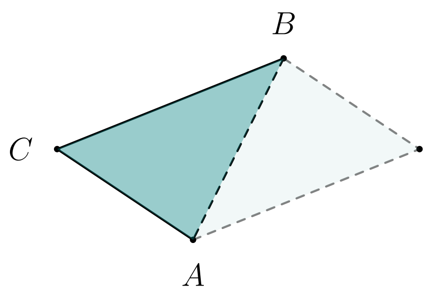
By now, computing a 3×3 determinant should be no problem.

We will now solve an example that utilizes the cross product.

Example: Find the area of the triangle ABC, given that the vertices are $A(4, 1, -2)$, $B(5, 5, 6)$ and $C(0, 3, 7)$.

Solution:

Notice that the area of the triangle ABC is *half* the area of the parallelogram with the three vertices $ABCD$ where D is unknown. This is made clear in the diagram where we have two congruent triangles, each of which has an area equal to half of the area of the parallelogram.



To find the area of the parallelogram using the cross product, we need to find $\overrightarrow{CB} = \mathbf{b}$ and $\overrightarrow{CA} = \mathbf{a}$

$$\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -9 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -9 \\ 5 & 2 & -1 \end{vmatrix} = 20\mathbf{i} - 41\mathbf{j} + 18\mathbf{k}$$

Therefore,

$$\begin{aligned} \text{Area} &= |\mathbf{a} \times \mathbf{b}| \\ &= |20\mathbf{i} - 41\mathbf{j} + 18\mathbf{k}| \\ &= \sqrt{(20)^2 + (-41)^2 + (18)^2} \\ &= \sqrt{2405} \end{aligned}$$

Since the area of triangle ABC is $\frac{1}{2}A$

$$\text{A}_{ABC} = \frac{1}{2}\sqrt{2405}$$

Note that this question can be solved using the dot product; however, it will take much longer. You can attempt solving this question with the dot product. You will see that it will take longer.

6.2 The vector equation of a line

You should have already met and dealt with the vector equation of a line. As a refresher

$$\ell : \mathbf{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \mu \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where

$$(x_1)\mathbf{i} + (y_1)\mathbf{j} + (z_1)\mathbf{k}$$

is the position vector of a point on the line,

$$(a)\mathbf{i} + (b)\mathbf{j} + (c)\mathbf{k}$$

is the direction vector of the line, and

$$\mu$$

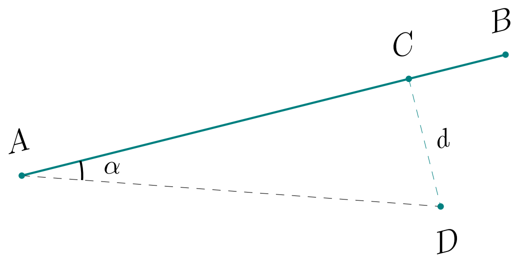
is a parameter.

We will learn how to use the cross product to solve some problems concerned with the vector equation of lines. Although it will take much longer, most questions and problems here can be solved using the dot product. However, it is *strongly* discouraged to use the dot product in the cases where you can use the cross product for familiarity purposes.

The shortest distance between a point and a line

Suppose we want to find the shortest distance between some point, D, and a line with A on it and goes through the points A and B. By the diagram,

$$d = |AD| \sin\theta$$



Now, we know that the magnitude of any unit vector is 1, so multiplying $|AD| \sin\theta$ by the magnitude of any unit vector remains unchanged.

Let $\mathbf{u}$ represent the direction vector $\overrightarrow{AB}$. Then,

$$d = |AD||\hat{\mathbf{u}}| \sin\theta$$

where $|\hat{\mathbf{u}}|$=1

But

$$\left|\overrightarrow{AD} \times \hat{\mathbf{u}}\right| = |AD||\hat{\mathbf{u}}| \sin\theta.$$

So in order to find d, we have to find $\overrightarrow{AD}$ and the unit vector $\mathbf{u}$.

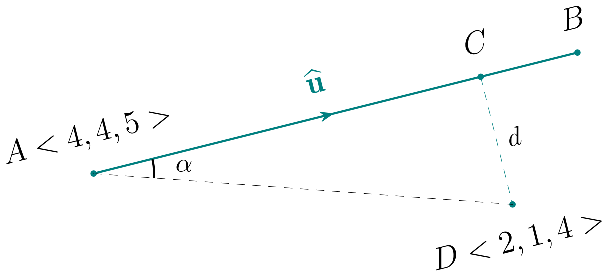
Example : Find the shortest distance between the point $P(2, 1, 4)$ and the line

$$\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Solution:

First, it is highly recommended you make a sketch that is just a copy of the one we have previously; however, we will now add coordinates.

Now that we have a diagram, things are easier. We will begin with finding the relevant vectors.



$$\overrightarrow{DA} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}\hat{\mathbf{u}} &= \frac{1}{\sqrt{(1)^2 + (1)^2 + (1)^2}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\end{aligned}$$

We now have all the relevant vectors, we can find d via the cross product.

$$\begin{aligned}d &= \left|\overrightarrow{DA} \times \hat{\mathbf{u}}\right| \\ &= \left|\frac{1}{\sqrt{3}}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & 1 & 1\end{vmatrix}\right| \\ &= \left|\frac{1}{\sqrt{3}}(2\mathbf{i} - \mathbf{j} - \mathbf{k})\right| \\ &= \frac{1}{\sqrt{3}} \times |2\mathbf{i} - \mathbf{j} - \mathbf{k}| \\ &= \frac{1}{\sqrt{3}} \times \sqrt{(2)^2 + (-1)^2 + (-1)^2} \\ d &= \sqrt{2}\end{aligned}$$

This is a standard procedure that doesn't deviate from question to question, so no further examples about this specific topic will be given.

The shortest distance between two skew lines

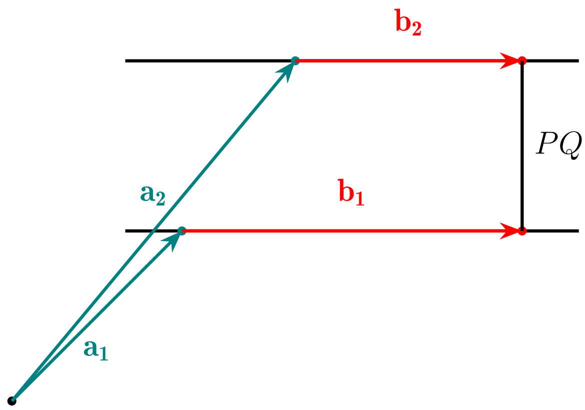
We will omit the visual derivation for the rule which we will give shortly. However, I strongly encourage you to look the derivation up on YouTube (search for the "Shortest distance between two skew lines").

Suppose the shortest distance between two skew lines ℓ_1 and ℓ_2 is PQ. It is true that

$$PQ = \frac{(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2})}{(\mathbf{d_1} \times \mathbf{d_2})}$$

Where:

- $\mathbf{a_1}$ is the position vector of a point on ℓ_1
- $\mathbf{a_2}$ is the position vector of a point on ℓ_2
- $\mathbf{b_1}$ is the direction vector of ℓ_1
- $\mathbf{b_2}$ is the direction vector of ℓ_2



This is just a matter of memorizing the formula and applying it. There is no way to add a trick to such question, so they are considered free marks.

Example 1: Find the shortest distance between the lines $\mathbf{r_1} = 2\mathbf{i}+\mathbf{j}+\mathbf{k}+(\mathbf{i}+2\mathbf{j}+3\mathbf{k})s$ and $\mathbf{r_2} = 3\mathbf{i}+4\mathbf{k}+(-2\mathbf{i}-\mathbf{j}+5\mathbf{k})t$

Solution:

We will find the relevant results first

1. $\mathbf{a_2} = 3\mathbf{i} + 4\mathbf{k}$, $\mathbf{a_1} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. $\mathbf{b_1} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b_2} = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$
3. $\mathbf{a_2} - \mathbf{a_1} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$
4. $\mathbf{b_1} \times \mathbf{b_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & -1 & 5 \end{vmatrix} = 13\mathbf{i} - 11\mathbf{j} + 3\mathbf{k}$
5. $|\mathbf{b_1} \times \mathbf{b_2}| = \sqrt{(13)^2 + (-11)^2 + (3)^2} = \sqrt{299}$
6. $(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2}) = (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (13\mathbf{i} - 11\mathbf{j} + 3\mathbf{k}) = 1 \times 13 - 1 \times -11 + 3 \times 3 = 33$

Let the shortest distance between the two lines be d.

$$d = \frac{33}{\sqrt{299}}$$

You might encounter one more type of question regarding two skew lines; in this type, we completely ditch the cross product and resort to the dot product (this is the *only* type of question where the dot product is favoured).

Example 2: The points P and Q lie on the lines $\mathbf{r_1} = 2\mathbf{i} + 3\mathbf{j} + s(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r_2} = 3\mathbf{i} + \mathbf{j} + t(-\mathbf{i} + \mathbf{j})$, respectively, such that $\overrightarrow{PQ}$ is perpendicular to both lines. Find the coordinates of P and Q, and find the distance between them.

Solution:

Since P is on $\mathbf{r_1}$, then

$$\begin{aligned}\overrightarrow{OP} &= 2\mathbf{i} + 3\mathbf{j} - s\mathbf{i} + 2s\mathbf{j} + s\mathbf{k} \\ &= (2 - s)\mathbf{i} + (3 + 2s)\mathbf{j} + s\mathbf{k}.\end{aligned}$$

for some value of s.

Since Q is on $\mathbf{r_2}$, then

$$\overrightarrow{OQ} = (3 - t)\mathbf{i} + (1 + t)\mathbf{j}$$

for some value of t.

Therefore,

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= (3 - t)\mathbf{i} + (1 + t)\mathbf{j} - ((2 - s)\mathbf{i} + (3 + 2s)\mathbf{j} + s\mathbf{k}) \\ &= (3 - t)\mathbf{i} + (1 + t)\mathbf{j} - (2 - s)\mathbf{i} - (3 + 2s)\mathbf{j} - s\mathbf{k} \\ \overrightarrow{PQ} &= (1 - t + s)\mathbf{i} + (-2 + t - 2s)\mathbf{j} - s\mathbf{k}\end{aligned}$$

Looking at the diagram again, we can see that

$$\overrightarrow{PQ}$$

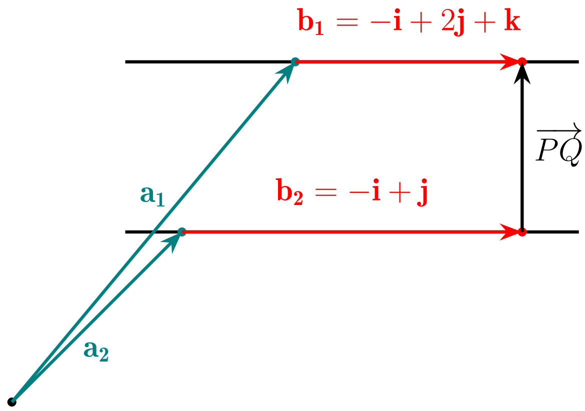
is perpendicular to the direction vectors of both lines; this means that

$$\overrightarrow{PQ} \cdot \mathbf{b_1} = 0$$

and

$$\overrightarrow{PQ} \cdot \mathbf{b_2} = 0$$

Notice that we have two variables with two equations that are not multiples of each other; hence, we can solve for s and t.



Equation 1

$$((1-t+s)\mathbf{i} + (-2+t-2s)\mathbf{j} - s\mathbf{k}) \cdot (-\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 0$$

$$(-1+t-s) + (-4+2t-4s) - s = 0$$

$$-5 + 3t - 6s = 0$$

$$3t - 6s = 5 \qquad (1)$$

Equation 2

$$((1-t+s)\mathbf{i} + (-2+t-2s)\mathbf{j} - s\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j}) = 0$$

$$(-1+t-s) + (-2+t-2s) = 0$$

$$-3 + 2t - 3s = 0$$

$$2t - 3s = 3 \qquad (2)$$

Solving the system of equations, we get that

$$s = -\frac{1}{3}$$

and

$$t = 1$$

Therefore,

$$\overrightarrow{OP} = \left(2 - \left(-\frac{1}{3}\right)\right)\mathbf{i} + \left(3 + 2 \times -\frac{1}{3}\right)\mathbf{j} - \frac{1}{3}\mathbf{k}$$

$$= \frac{7}{3}\mathbf{i} + \frac{7}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

$$\implies P\left(\frac{7}{3}, \ \frac{7}{3}, -\frac{1}{3}\right)$$

and

$$\begin{aligned}\overrightarrow{OQ} &= (3-1)\mathbf{i} + (1+1)\mathbf{j} \\ &= 2\mathbf{i} + 2\mathbf{j} \\ \Longrightarrow &Q(2,2,0)\end{aligned}$$

For the length of PQ,

$$\begin{aligned}PQ &= |\overrightarrow{PQ}| \\ &= \left|-\frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right| \\ &= \sqrt{\left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

6.3 Planes

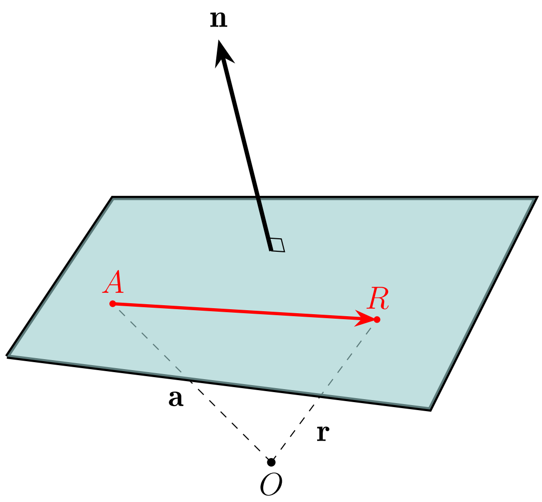
This is a new concept that we have not seen yet. A plane is an infinite two-dimensional surface. The plane is the collection of all points satisfying a linear equation with 3 variables (x, y and z). Touching on linear algebra, the linear equation has infinitely many solutions (infinitely many points satisfying the equation) that are free to vary in two directions (which maps out a plane).

It is understandable if you don't fully understand the previous paragraph right now.

When we are dealing with planes, there is a very important direction vector that we always deal with: the vector that is normal/orthogonal to the plane. The normal vector to a plane is very important because it can be used to define the equation of the plane. The normal vector is a vector that is perpendicular to the plane at any point. As we will soon see, this normal vector can be found given 3 non-collinear points.

Aside from defining planes, we will mainly deal with the interaction between two planes or a plane and a line.

Suppose we have some plane Π with a normal vector $\mathbf{n}$, where $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. *Let* A be some known point on the plane Π with the coordinates $A(p, q, r)$, and let R be any point on the plane Π with the coordinates $R(x, y, z)$. Since $\overrightarrow{AR}$ is a vector in the plane, and $\mathbf{n}$ is perpendicular to the plane, then



$$\mathbf{n} \cdot \overrightarrow{AR} = 0$$

but

$$\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$$

therefore

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$$

since dot product multiplication is distributive, then

$$\mathbf{n} \cdot \mathbf{r} - \mathbf{n} \cdot \mathbf{a} = 0$$

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{n} \cdot \mathbf{a}$$

$$ax + by + cz = \mathbf{n} \cdot \mathbf{a}$$

Let $\mathbf{n} \cdot \mathbf{a} = d$ where d is some real number. Therefore,

$$ax + by + cz = d$$

We now have the Cartesian equation of a plane:

$$\Pi : ax + by + cz = d,$$

where

- a, b and c are the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components, respectively, of the vector normal to the plane Π, $\mathbf{n}$.
- d is the result of $\mathbf{n} \cdot \mathbf{a}$, where $\mathbf{a}$ is the position vector of some point on the plane, and $\mathbf{n}$ is the vector normal to the plane Π.

Example 1: Find the Cartesian equation of the plane with the normal vector,$\mathbf{n}$, where $\mathbf{n} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, and goes through the point $(3, -5, 4)$

Solution:

Given n, we know that

$$\Pi : (1)x + (-3)y + (1)z = d$$

$$x - 3y + z = d$$

but d is the dot product between the normal and any point on the plane. Hence,

$$d = (\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k})$$

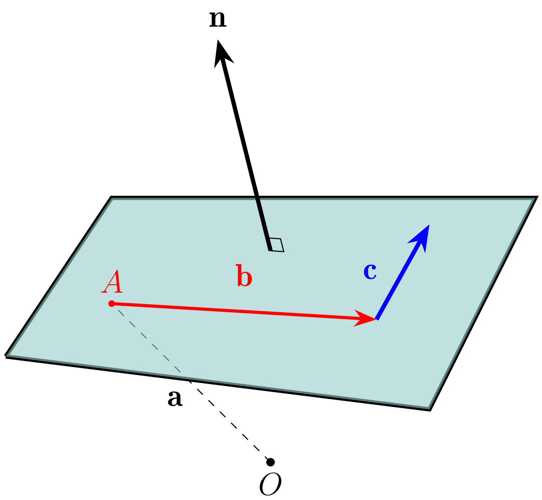
$$= 1 \times 3 + -3 \times -5 + 1 \times 4$$

$$d = 22$$

$$\therefore \ \Pi : x - 3y + z = 22$$

We also have the vector equation of a plane.

The main point is to produce an equation that can give an output of any point on the plane. In the vector equation of a plane, we first go from the origin to the plane ($\mathbf{a}$), and then navigate our way in the plane to other points using a combination of two direction vectors that are in the plane ($s\mathbf{b} + t\mathbf{c}$).



The vector equation of a plane Π is

$$\Pi : \mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$$

Notice that $\mathbf{n} = \mathbf{b} \times \mathbf{c}$, since $\mathbf{n}$ is a common perpendicular vector to both vectors in the plane.

Example 2: Write down the vector and Cartesian equation of the plane Π which contains the points $A(2, -5, 1)$, $B(0, 3, 2)$ and $C(-4, 1, 1)$.

Solution:

$$\overrightarrow{AB} = -2\mathbf{i} + 8\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{AC} = -6\mathbf{i} + 6\mathbf{j}$$

Therefore, the vector equation of the plane is

$$\mathbf{r} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix} + t\begin{pmatrix} -6 \\ 6 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} \\ &= (-2\mathbf{i} + 8\mathbf{j} + \mathbf{k}) \times (-6\mathbf{i} + 6\mathbf{j}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 8 & 1 \\ -6 & 6 & 0 \end{vmatrix} \\ \mathbf{n} &= -6\mathbf{i} - 6\mathbf{j} + 36\mathbf{k}. \end{aligned}$$

All that is now left is to find d in the Cartesian equation of a plane.

$$\begin{aligned} d &= (-6\mathbf{i} - 6\mathbf{j} + 36\mathbf{k}) \cdot (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \\ &= -6 \times 2 - 6 \times -5 + 36 \times 1 \\ &= 54 \end{aligned}$$

Therefore, the Cartesian equation of a plane is

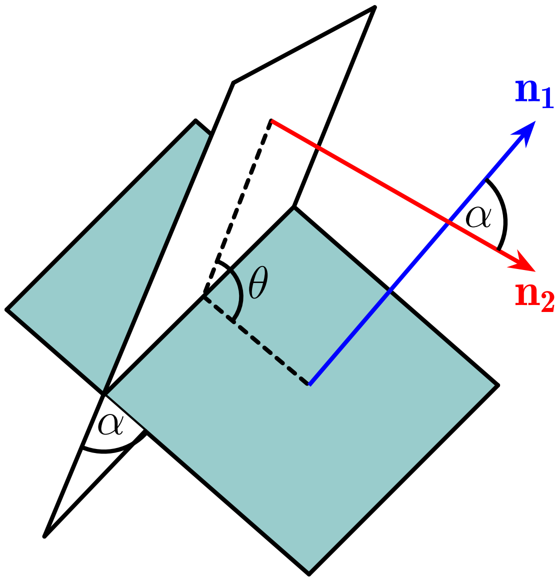
$$\Pi : \; -6x - 6y + 36z = 54$$

Simplifying,

$$\Pi : \; x + y - 4z = -9$$

Angle between two planes

When two planes intersect, two angles form between both planes: one acute (α) and one obtuse (θ), where $\theta = 180 - \alpha$ (as seen on diagram).

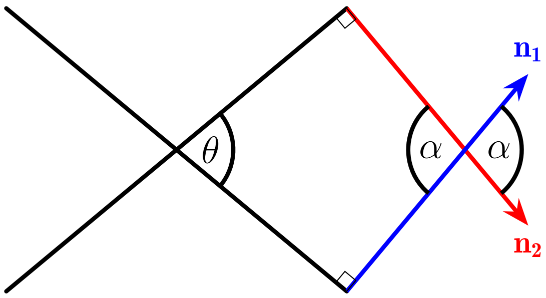


We can also find the angle α between the two normal vectors, $\mathbf{n_1}$ and $\mathbf{n_2}$, using the dot product:

$$\cos\alpha = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|}.$$

If you are asked to find the acute angle and the result from using the dot product ends up being an obtuse angle, simply subtract the result from 180°. The same is true the other way around.

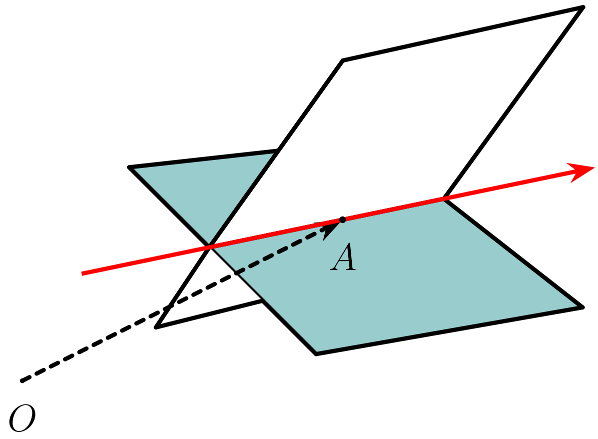
Here is a simplified diagram to help you make heads or tails of the angles.



Line of intersection between two intersecting planes

When two planes intersect, a line of intersection is formed. The line of intersection is parallel to both planes as it's contained in both planes. Consequently, the line of intersection is perpendicular to both normal vectors of each plane. Therefore, the direction vector of the line of intersection is $\mathbf{v}$, where

$$\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$$



The next task to tackle when finding the vector equation for the line of intersection is finding some point, A, which lies on it. We will solve the system of equations including both planes; however, we have 3 variables, x, y and z, with 2 equations, so we must have an infinite number of solutions (which is obviously true). To obtain a point, we

1. cancel out one variable,
2. get one of the remaining 2 variables be a convenient value,
3. find the value of the other remaining variable,
4. plug in the value of the 2 variables to find the last variable.

This may seem completed, but you will see with an example how easy it is.

Example 1: Find the vector equation of the line of intersection formed by the planes $\Pi_1 : \ 3x + y - z = 2$ and $\Pi_2 : \ x - y + 5z = 2$

Solution:

From the Cartesian equations of both planes, we deduce that

$$\mathbf{n_1} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$$

and

$$\mathbf{n_2} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

$$\therefore \mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & -1 & 5 \end{vmatrix}$$
$$\mathbf{v} = 4\mathbf{i} - 16\mathbf{j} - 4\mathbf{k}$$

since the magnitude of the direction vector doesn't matter,

$$\mathbf{v} = \begin{pmatrix} 4 \\ -16 \\ -4 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$$

To find some point on the line,

$$3x + y - z = 2 \tag{1}$$
$$x - y + 5z = 2 \tag{2}$$

adding (1) to (2), we get

$$4x + 4z = 4$$

dividing by 4 and making x the subject of the formula, we get

$$x = 1 - z$$

Let $z = 0$

$$x = 1 - 0$$
$$= 1$$

plugging our results into (2), we get

$$(1) - y + 5(0) = 2$$
$$1 - y = 2$$
$$y = -1$$
$$\therefore (1, -1, 0) \text{ is on the line.}$$

Now, we can construct the vector equation for the line of intersection.

$$\mathbf{r} = \mathbf{i} - \mathbf{j} + s(\mathbf{i} - 4\mathbf{j} - \mathbf{k})$$

Example 2: Find the vector equation of the line of intersection of the planes $x + 3y - z = 1$ and $2x + 4y + z = 6$.

Solution:

For the direction vector:

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 4 & 1 \end{vmatrix}$$

$$= 7\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

For a point on the line

$$x + 3y - z = 1 \tag{1}$$

$$2x + 4y + z = 6 \tag{2}$$

adding (1) to (2)

$$3x + 7y = 7$$

Let $x = 0$

$$7y = 7$$

$$y = 1$$

Plugging our results into (1)

$$0 + 3(1) - z = 1$$

$$z = 2$$

Therefore,

$$(0, 1, 2)$$

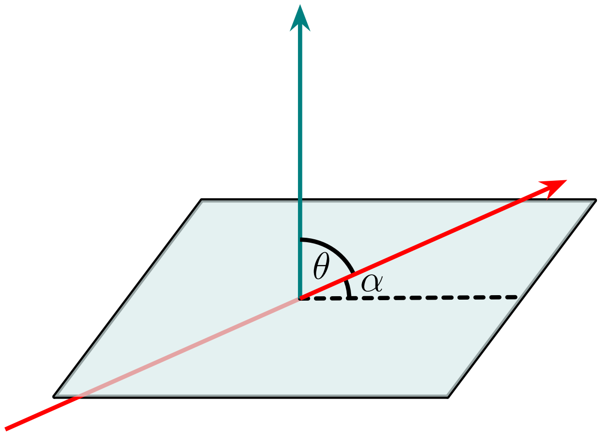
is on the line.

$$\rightsquigarrow \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 7 \\ -3 \\ -2 \end{pmatrix}$$

Angle between a line intersecting a plane

When a plane and a line intersect, an angle between the line and the plane forms. While we can't find α directly, we can find θ: the angle between the direction of the line, $\mathbf{v}$, and the normal vector, $\mathbf{n}$. We can then easily find α as $\alpha = 90° - \theta$ if θ is acute, or $\alpha = \theta - 90°$ if θ is obtuse.

$$\cos\theta = \frac{\mathbf{n}\cdot\mathbf{v}}{|\mathbf{n}||\mathbf{v}|}.$$



The reason why θ can both be acute or obtuse is due to the different ways the line can intersect the plane. If the line intersect the plane from the bottom, θ will be acute. If the line intersects the plane from the top however, θ will be obtuse.

Example: Find the angle made between the line $\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - 6\mathbf{k})$ and the plane $x + 2y + z = 5$.

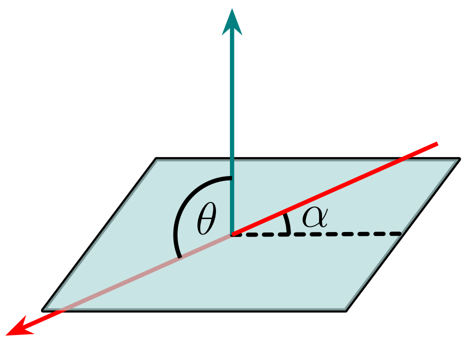
Solution:

The normal vector, $\mathbf{n}$, is

$$\mathbf{n} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

the angle between the normal and the line is θ, where

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-6\mathbf{k})}{|\mathbf{i}+2\mathbf{j}+\mathbf{k}|\,|\mathbf{i}+2\mathbf{j}-6\mathbf{k}|}\right)\\ &= \cos^{-1}\left(\frac{1\times 1 + 2\times 2 + 1\times -6}{\sqrt{1^2+2^2+1^2}\sqrt{1^2+2^2+(-6)^2}}\right)\\ &= \cos^{-1}\left(\frac{-1}{\sqrt{6}\sqrt{41}}\right)\\ &= \cos^{-1}\left(\frac{-1}{\sqrt{246}}\right)\\ \theta &= 93.7°\end{aligned}$$



Therefore, for α:

$\alpha = 93.66 - 90$

$\alpha = 3.66°$

Coordinates of intersection between a plane and a line

When we are finding the coordinates where a plane and a line intersect, we are finding the point where the **i**, **j** and **k** components of both the plane and the line are equal. Remember that the Cartesian equation of a plane, $ax + by + cz = d$, is the dot product between a known point, (a, b, c), and some arbitrary point on the plane, (x, y, z). We will substitute in the respective components of the line into the arbitrary point, (x, y, z), to find the coordinates of intersection. As usual, this will be made much clearer with an example.

Example: Find the coordinates of intersection between the line $\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - 6\mathbf{k})$ and the plane $x + 2y + z = 5$.

Solution:

We first write down the parametric form of the vector line equation

$$\mathbf{r} = (1 + s)\mathbf{i} + (4 + 2s)\mathbf{j} + (-3 - 6s)\mathbf{k}$$

We want to see which point lies on the line, **r**, and satisfies the Cartesian plane equation, and hence lies on the plane.

$$(1 + s) + 2(4 + 2s) + (-3 - 6s) = 5$$

$$1 + s + 8 + 4s - 3 - 6s = 5$$

$$-s + 6 = 5$$

$$-s = -1$$

$$s = 1$$

Hence, when $s = 1$, we have a point that lies on both the line $\mathbf{r}$, and the plane. To get the coordinates of the point, plug in $s = 1$ into the line equation.

$$(1+1)\mathbf{i} + (4 + 2\times 1)\mathbf{j} + (-3 - 6\times 1)\mathbf{k} = 2\mathbf{i} + 6\mathbf{j} - 9\mathbf{k}$$

$$\therefore (2, 6, -9)$$

is the point of intersection between the lines and the plane.

There are two more cases to consider: the line is parallel to the plane and doesn't intersect it, or the line is contained within the plane.

If the line is contained in the plane (parallel to the plane and intersects it), plugging in the parametric equation of the line into the Cartesian equation of the plane yields

$$k = k$$

where k is a real number.

Consider the plane $3x + y + 2z = 6$ and the line $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + s(4\mathbf{i} - 2\mathbf{j} - 5\mathbf{k})$. The parametric equation of the line is $\mathbf{r} = (1+4s)\mathbf{i} + (-1-2s)\mathbf{j} + (2-5s)\mathbf{k}$. Plugging this into the plane equation, we get

$$3(1+4s) + (-1-2s) + 2(2-5s) \stackrel{?}{=} 6$$

$$3 - 1 + 4 + 12s - 2s - 10s \stackrel{?}{=} 6$$

$$6 = 6$$

Therefore, the line is contained in the plane.

If the line is parallel to the plane and does not intersect it, plugging in the parametric equation of the line into the Cartesian equation of the plane will yield

$$k = c$$

where k and c are two distinct real numbers.

Consider the plane $3x + y + 2z = 6$ and the line $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + t(4\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$. The parametric equation of the line is $\mathbf{r} = (2+4t)\mathbf{i} + (1-2t)\mathbf{j} + (3-5t)\mathbf{k}$. Plugging this into the the plane equation, we get

$$3(2+4t) + (1-2t) + 2(3-5t) \stackrel{?}{=} 6$$

$$6 + 12t + 1 - 2t + 6 - 10t \stackrel{?}{=} 6$$

$$13 \neq 6$$

Therefore, the line is parallel to the plane and does not intersect it.

Shortest distance between a point and a plane

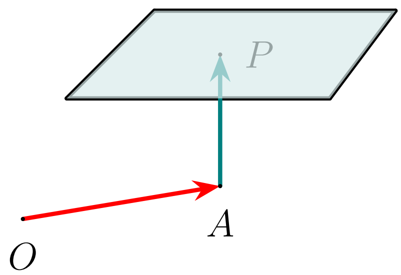
There are two possible methods for finding the shortest distance between a point and a plane.

Method 1

We know that the shortest distance between a point and a plane is the perpendicular distance between the point and that plane. We have the direction of the vector that is perpendicular to the plane (the normal plane vector $\mathbf{n}$) but we do not know its magnitude (the shortest distance between the point and the plane).

To find the shortest distance between the point A and the plane (i.e. the distance AP) we need to find the point P, which is the intersection of the line

$$\begin{aligned}\mathbf{r} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \overrightarrow{OA} + s\mathbf{n}\end{aligned}$$



with the plane. After finding s, the shortest distance is $\left|\overrightarrow{AP}\right| = |s\mathbf{n}|$.

Example: Consider the plane $2x - 3y + 6z = -14$ and the point $P(4, -5, 2)$. Find the shortest distance between the point and the plane.

Solution:

We will skip the explanation as it was previously discussed and go straight into the steps.

$$\begin{aligned}\mathbf{r} &= 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} + s(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) \\ &= (4 + 2s)\mathbf{i} + (-5 - 3s)\mathbf{j} + (2 + 6s)\mathbf{k}\end{aligned}$$

Plugging into the plane equation

$$2(4 + 2s) - 3(-5 - 3s) + 6(2 + 6s) = -14$$

$$8 + 4s + 15 + 9s + 12 + 36s = -14$$

$$s = -1$$

Therefore,

$$\begin{aligned}\left|\overrightarrow{AP}\right| &= |-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}| \\ &= \sqrt{(-2)^2 + (3)^2 + (-6)^2} \\ &= \sqrt{4 + 9 + 36} \\ &= \sqrt{49} \\ &= 7\end{aligned}$$

Therefore, the distance is 7.

Method 2

This method involves the direct application of a formula. The shortest distance between a point (x, y, z) and some plane $ax + by + cz = d$ is D, where

$$D = \frac{ax + by + cz - d}{\sqrt{a^2 + b^2 + c^2}}$$

We will not discuss the derivation of the formula; however, you can easily find it on YouTube (search for "Shortest distance between a point and a plane.")

Example: Consider the plane $2x - 3y + 6z = -14$ and the point P(4,-5,2). Find the shortest distance between the point and the plane.

Solution:

The shortest distance between the point and the plane,D, is

$$\begin{aligned}D &= \frac{2 \times 4 + -3 \times -5 + 6 \times 2 + 14}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} \\ &= 7\end{aligned}$$

Hence, the shortest distance between the point and the plane is 7.

Note:

All vector questions more or less follow the same scheme.

We will opt out from solving questions here, as there are legitimately no tricks to these questions. You should just go through past paper questions to practice these questions and become fast doing them.

CHAPTER VII
PROOF BY INDUCTION

Chapter VII

Proof by induction

What is Mathematical Induction?

Mathematical induction is a method of proof usually used to prove that some statement is true for the set of natural numbers or a subset of it.

Mathematical induction is used in areas of mathematics that deal with whole numbers: it is used in combinatorics, number theory and computer science for instance.

A historical introduction

While the earliest known use of mathematical induction is in the work of the sixteenth-century mathematician Francesco Maurolico where he used it to prove that the sum of the first n odd integers is n^2, similar ideas can be traced back to the ancient times. The Greeks developed a similar method known as *Proof by Exhausation* or *Proof by cases.* The formalization of mathematical induction as a proof technique is credited to the German mathematician Carl Friedrich Gauss; he emphasized the need for explicit statements of the base case and inductive step and provided a more rigorous framework for applying induction in mathematical proofs. It is worthy of mentioning that Fermat and Bernoulli did have their fair share of contributions in the formalization and development of mathematical induction.

7.1 Principal of mathematical induction

Mathematical induction works like toppling over a domino in a line of dominoes standing upright. Suppose that the first domino is pushed such that it topples.

Suppose we *assume* that some domino in the line (any domino in the line, it doesn't matter) has toppled.

If it is true that the domino in front of the domino that is assumed to topple topples, then the whole line of dominoes must topple. Remember that the domino assumed to have toppling can be *any* domino; consequently, we have established that *all* dominoes topple.

Steps for proving by mathematical induction:

1. Prove that the base case is true. The base case is the smallest value in the set of numbers that the statement is being considered for. This is usually $n = 1$. This is the step that knocks the first domino.
2. Assume that the statement is true for some value k where k is any number from the set of numbers that is being evaluated. You can deduce which part this is in our domino analogy.
3. Showing that the statement is true for the value $k+1$ and finishing with a concluding statement.

We have essentially proved that the statement is true for the base case, and that it is true for any number and the one following it in the set of numbers being tested.

There are no fancy tricks or techniques here, so a couple of examples on each main idea will do the trick.

7.1.1 Induction for results of summation

We will now prove that a given result for some sum is true by induction.

Example 1: prove that

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

for all positive integers n.

Solution:

Let the statement P_n be that for some positive integer value n,

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

For $n = 1$:

By computation:

$$\sum_{r=1}^{1} r = 1$$

By statement:

$$\begin{aligned}\sum_{r=1}^{1} r &= \frac{1}{2}(1+1) \\ &= \frac{2}{2} \\ &= 1\end{aligned}$$

hence, P_1 is true

Assume that P_k is true for some positive integer value k, such that

$$\sum_{r=1}^{k} r = \frac{k}{2}(k+1)$$

is true for some positive integer k.

Note: In our $n = k + 1$ case, we *always* use the assumption that we have made in the previous step. In our case here, we must use the assumption that

$$\sum_{r=1}^{k} r = \frac{k}{2}(k+1)$$

is true for some positive integer k. This should be more clear as we work through the $n = k + 1$ case.

For $n = k + 1$:

By computation:

$$\begin{aligned}\sum_{r=1}^{k+1} r &= \sum_{r=1}^{k} r + k + 1 \\ &= \frac{k}{2}(k+1) + k + 1 \\ &= (k+1)\left(\frac{k}{2} + 1\right) \\ &= \frac{k+1}{2}(k+2) \\ &= \frac{k+1}{2}((k+1)+1)\end{aligned}$$

$$\sum_{r=1}^{k+1} r = \frac{k+1}{2}((k+1)+1)$$

By statement:

$$\sum_{r=1}^{k+1} r = \frac{k+1}{2}((k+1)+1)$$

hence, P_{k+1} is true.

Therefore, by the principals of mathematical induction, P_n is true $\forall n \in \mathbb{Z}^+$. That is,

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

for all positive integer n, as P_1 is true and $P_k \implies P_{k+1}$

Example 2: Prove, by mathematical induction, that

$$\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$$

is true for all positive integers n.

Solution:

Let P_n be the statement that for some positive integer value n,

$$\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$$

<u>For $n = 1$:</u>

By computation:

$$\sum_{r=1}^{1} r\ln\left(\frac{r+1}{r}\right) = \ln(2)$$

By statement:

$$\begin{aligned}\sum_{r=1}^{1} r\ln\left(\frac{r+1}{r}\right) &= \ln\left(\frac{(1+1)^1}{1!}\right)\\ &= \ln(2)\end{aligned}$$

hence, P_1 is true.

Assume that P_k is true for some positive integer value k, such that

$$\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(k+1)^k}{k!}\right)$$

For $n = k + 1$:

By computation:

$$\begin{aligned}
\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) &= \sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) + (k+1)\ln\left(\frac{k+1+1}{k+1}\right) \\
&= \ln\left(\frac{(k+1)^k}{k!}\right) + (k+1)\ln\left(\frac{k+2}{k+1}\right) \\
&= \ln\left(\frac{(k+1)^k}{k!}\right) + \ln\left(\frac{(k+2)^{k+1}}{(k+1)^{k+1}}\right) \\
&= \ln\left(\frac{\cancel{(k+1)^k}(k+2)^{k+1}}{k!(k+1)^{\cancel{k+1}}}\right) \\
\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) &= \ln\left(\frac{((k+1)+1)^{k+1}}{(k+1)!}\right)
\end{aligned}$$

By statement:

$$\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{((k+1)+1)^{k+1}}{(k+1)!}\right)$$

hence P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true for all positive integers n, that is,

$$\sum_{r=1}^{n} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$$

is true for all positive integers n, as P_1 is true and $P_k \implies P_{k+1}$.

7.1.2 Induction for recurrence relations

Example 1: The sequence $u_1, u_2, u_2, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geq 1$. Prove by induction that $u_n = 2^n - 1$ for all positive integers n.

Solution:

Let P_n be the statement that for some positive integer n,

$$u_n = 2^n - 1$$

For $n = 1$:

By computation (given in question), we have that

$$u_1 = 1$$

By statement, we have that P_1:

$$u_1 = 2^1 - 1 = 1$$

hence, P_1 is true.

Assume that P_k is true for some positive integer value k, such that

$$u_k = 2^k - 1.$$

For $n = k + 1$:

By computation:

$$\begin{aligned} u_{k+1} &= 2u_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

By statement:

$$u_{k+1} = 2^{k+1} - 1$$

hence, P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true, that is,

$$u_n = 2^n - 1$$

for all positive integer values of n, as P_1 is true and $P_k \implies P_{k+1}$.

Example 2: Given that

$$u_{n+1} = \frac{4u_n + 3}{u_n + 2}$$

and $u_1 = 1$, prove by mathematical induction that $u_n < 3$ for all positive integer values of n.

Solution:

<u>For $n = 1$:</u>

$$u_1 = 1 < 3$$

hence, $u_n < 3$ for $n = 1$.

Assume that $u_k < 3$ for some positive integer value k.

<u>For $n = k + 1$:</u>

$$u_{k+1} = \frac{4u_k + 3}{u_k + 2}$$

Here we have a little trick; we subtract 3 from both sides of the equation. Subtracting 3 from both sides makes subsequent work easier and more apparent. Subtracting n from both sides is a standard procedure when proving that $u_k < n$, $u_k \leq n$, $u_k > n$ and $u_k \geq n$.

$$\begin{aligned} u_{k+1} - 3 &= \frac{4u_k + 3}{u_k + 2} - 3 \\ &= \frac{4u_k + 3}{u_k + 2} - \frac{3(u_k + 2)}{u_k + 2} \\ &= \frac{4u_k + 3 - 3u_k - 6}{u_k + 2} \\ u_{k+1} - 3 &= \frac{u_k - 3}{u_k + 2} \\ u_{k+1} &= 3 + \frac{u_k - 3}{u_k + 2} \end{aligned}$$

Since u_k is assumed to be less than 3, that is

$$u_k < 3$$

then

$$u_k - 3 < 0.$$

Consequently,

$$\frac{u_k - 3}{u_k + 2} < 0$$

and

$$3 + \frac{u_k - 3}{u_k + 2} < 3$$

but

$$u_{k+1} = 3 + \frac{u_k - 3}{u_k + 2}.$$

Therefore,

$$u_{k+1} < 3$$

Hence, by principals of mathematical induction, $u_n < 3$ for all positive integer values of n, since $u_1 < 3$, and $u_k < 3 \implies u_{k+1} < 3$.

7.1.3 Induction for matrix multiplication

Possibly the easiest type of induction question that can come up.

Example 1: *Let*

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$

Solution:

Let the P_n be the statement that for some positive integer value n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$$

<u>For $n = 1$:</u>

By definition:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

By statement:

$$\mathbf{A}^1 = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

hence, P_1 is true.

Assume that P_k is true for some positive integer value k, such that

$$\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}.$$

For $n = k + 1$:

By computation:

$$\begin{aligned}
\mathbf{A}^{k+1} &= \mathbf{A}^k \times \mathbf{A} \\
&= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 2 + 1 & 1 \end{pmatrix} \\
\mathbf{A}^{k+1} &= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}
\end{aligned}$$

By statement:

$$\mathbf{A}^{k+1} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$$

hence, P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true for all positive integer values of n, that is

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix},$$

$\forall n \in \mathbb{Z}^+$, as P_1 is true, and $P_k \implies P_{k+1}$

Example 2: *Let*

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

Prove by mathematical induction that, for every positive integer n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3\left(2^n - 1\right) \\ 0 & 1 \end{pmatrix}$$

Solution:

Let the P_n be the statement that for some positive integer value n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 3\left(2^n - 1\right) \\ 0 & 1 \end{pmatrix}$$

For $n = 1$:

By definition,

$$\mathbf{A}^1 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

By statement,

$$\begin{aligned} \mathbf{A^1} &= \begin{pmatrix} 2^1 & 3\left(2^1 - 1\right) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3\left(1\right) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

hence, P_1 is true.

Assume that P_k is true for some positive integer k, such that

$$\mathbf{A}^k = \begin{pmatrix} 2^k & 3\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}$$

For $n = k + 1$:

By computation:

$$\begin{aligned} \mathbf{A}^{k+1} &= \mathbf{A}^k \times \mathbf{A} \\ &= \begin{pmatrix} 2^k & 3\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 3 \times 2^k + 3 \times 2^k - 3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 6 \times 2^k - 3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 3\left(2 \times 2^k - 1\right) \\ 0 & 1 \end{pmatrix} \\ \mathbf{A}^{k+1} &= \begin{pmatrix} 2^{k+1} & 3\left(2^{k+1} - 1\right) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

By statement,

$$\mathbf{A}^{k+1} = \begin{pmatrix} 2^{k+1} & 3\left(2^{k+1}-1\right) \\ 0 & 1 \end{pmatrix}$$

Hence, P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true for all positive integer values of n, such that

$$\mathbf{A}^{n} = \begin{pmatrix} 2^{n} & 3\left(2^{n}-1\right) \\ 0 & 1 \end{pmatrix},$$

as P_1 is true, and $P_k \implies P_{k+1}$.

7.1.4 Induction for divisibility

This is a common type of induction question that is always repeated.

Before proceeding with the examples, we have to outline a specific property of numbers. If a number, k, is divisible by some number, c, then c must be a factor of k. For example, since 10 is divisible by 5, 5 must be a factor of 10; this is the case as $10 = 5 \times 2$.

Example 1: Prove by mathematical induction that $5^{2n} - 1$ is divisible by 8 for all positive integer values of n.

Solution:

Let P_n be the statement that for some positive integer value n, the function $f(n)$ is divisible by 8, where $f(n) = 5^{2n} - 1$.

For $n = 1$:

$$\begin{aligned} f(1) &= 5^{2(1)} - 1 \\ &= 5^2 - 1 \\ &= 25 - 1 \\ &= 24 \\ &= 8 \times 3 \end{aligned}$$

which is divisible by 8

hence, P_1 is true.

Assume that P_k is true for some positive integer k, such that

$$f(k) = 5^{2k} - 1$$

is divisible by 8.

For $n = k + 1$:

$$f(k+1) = 5^{2(k+1)} - 1$$

Here we employ the handy trick of subtracting $f(k)$ from both sides; this will make proving divisibility for $n = k + 1$ easier. This trick is used for all questions of this type.

$$\begin{aligned}
f(k+1) - f(k) &= 5^{2(k+1)} - 1 - \left(5^{2k} - 1\right) \\
&= 5^{2k+2} - 1 - 5^{2k} + 1 \\
&= 5^2 \times 5^{2k} - 5^{2k} \\
&= 25 \times 5^{2k} - 5^{2k} \\
f(k+1) - f(k) &= 24 \times 5^{2k} \\
f(k+1) &= 8 \times 3 \times 5^{2k} + f(k)
\end{aligned}$$

$8 \times 3 \times 5^{2k}$ is divisible by 8. $f(k)$ is divisible by 8 via assumption. Therefore, f(k+1) is divisible by 8.

Hence, P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true for all positive integer values of n, that is, $5^{2n} - 1$ is divisible by 8 for all positive integer values of n, as P_1 is true and$P_k \implies P_{k+1}$.

Example 2: Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n.

Solution:

Let P_n be the statement that for some positive integer value n, the function $f(n)$ is divisible by 13, where $f(n) = 3^{3n} - 1$.

For $n = 1$:

$$\begin{aligned} f(1) &= 3^{3(1)} - 1 \\ &= 3^3 - 1 \\ &= 27 - 1 \\ &= 26 \\ &= 13 \times 2 \end{aligned}$$

which is divisible by 13.

hence, P_1 is true.

Assume that P_k is true, such that

$$f(k) = 3^{3k} - 1$$

is divisible by 13

For $n = k + 1$:

$$\begin{aligned} f(k+1) &= 3^{3(k+1)} - 1 \\ f(k+1) - f(k) &= 3^{3k+3} - 1 - \left(3^{3k} - 1\right) \\ &= 3^3 \times 3^{3k} - 1 - 3^{3k} + 1 \\ &= 27 \times 3^{3k} - 3^{3k} \\ &= 26 \times 3^{3k} \\ f(k+1) &= 13 \times 2 \times 3^{3k} + f(k) \end{aligned}$$

$13 \times 2 \times 3^{3k}$ is divisible by 13. $f(k)$ is divisible by 13 via assumption.

Hence, P_{k+1}.

Therefore, by principals of mathematical induction, P_n is true for all positive integer values of n, that is, $3^{3n} - 1$ is divisible by 13 for all positive integer values of n, as P_1 is true and $P_k \implies P_{k+1}$.

7.1.5 Induction for derivatives

This is the last standard type of induction problem. It should be noted that you are expected to be well versed in all differentiation techniques from the Pure Mathematics 3 syllabus.

Example 1: Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^x \sin x\right) = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{1}{4}n\pi\right)$$

Solution:

Let P_n be the statement that for some positive integer value n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^x \sin x\right) = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{1}{4}n\pi\right)$$

<u>For $n = 1$:</u>

By computation,

$$\begin{aligned}\frac{\mathrm{d}^1}{\mathrm{d}x^1}\left(e^x \sin x\right) &= e^x \sin x + e^x \cos x \\ &= e^x \left(\sin x + \cos x\right)\end{aligned}$$

Consider $\sin x + \cos x \equiv R \sin\left(x + \alpha\right)$

$$\sin x + \cos x \equiv R \sin\left(x + \alpha\right)$$

$$\sin x + \cos x \equiv R \sin x \cos \alpha + R \sin \alpha \cos x$$

$$R \cos \alpha = 1 \tag{1}$$

$$R \sin \alpha = 1 \tag{2}$$

Solving, we get the following:

$$\sin x + \cos x \equiv \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

$$\therefore \frac{\mathrm{d}^1}{\mathrm{d}x^1}\left(e^x \sin x\right) = \sqrt{2} e^x \sin\left(x + \frac{\pi}{4}\right)$$

By statement,

$$\begin{aligned}\frac{\mathrm{d}^1}{\mathrm{d}x^1}\left(e^x \sin x\right) &= \left(\sqrt{2}\right)^1 e^x \sin\left(x + \frac{1}{4} \times 1 \times \pi\right) \\ &= \sqrt{2} e^x \sin\left(x + \frac{\pi}{4}\right)\end{aligned}$$

hence, P_1 is true.

Assume that P_k is true for some positive integer value k, such that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(e^x \sin x\right) = \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{1}{4}k\pi\right)$$

<u>For $n = k + 1$:</u>

By computation,

$$\begin{aligned}
\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(e^x \sin x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(e^x \sin x\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{1}{4}k\pi\right)\right) \\
&= \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{1}{4}k\pi\right) + \left(\sqrt{2}\right)^k e^x \cos\left(x + \frac{1}{4}k\pi\right) \\
&= \left(\sqrt{2}\right)^k e^x \left(\sin\left(x + \frac{1}{4}k\pi\right) + \cos\left(x + \frac{1}{4}k\pi\right)\right)
\end{aligned}$$

Let $u = x + \frac{1}{4}k\pi$

$$\begin{aligned}
&= \left(\sqrt{2}\right)^k e^x \left(\sin u + \cos u\right) \\
&= \left(\sqrt{2}\right)^k e^x \left(\sqrt{2}\sin\left(u + \frac{\pi}{4}\right)\right) \\
&= \left(\sqrt{2}\right)^{k+1} e^x \sin\left(x + \frac{1}{4}k\pi + \frac{\pi}{4}\right) \\
&= \left(\sqrt{2}\right)^{k+1} e^x \sin\left(x + (k+1)\frac{1}{4}\pi\right)
\end{aligned}$$

By statement,

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(e^x \sin x\right) = \left(\sqrt{2}\right)^{k+1} e^x \sin\left(x + (k+1)\frac{1}{4}\pi\right)$$

hence, P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true for all positive integer values of n, that is

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^x \sin x\right) = \left(\sqrt{2}\right)^n e^x \sin\left(x + \frac{1}{4}n\pi\right)$$

for every positive integer value of n, as P_1 is true and $P_k \implies P_{k+1}$.

Example 2: Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}\left(x\sin x\right) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right)$$

Before proceeding with the solution, notice how we are essentially finding the odd numbered derivatives of $x\sin x$; so between $n = k$ and $n = k+1$, we have to differentiate twice.

Solution:

Let P_n be the statement that for some positive integer value n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}\left(x\sin x\right) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right)$$

For $n = 1$:

By computation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sin x\right) = \sin x + x\cos x$$

By statement,

$$\begin{aligned}\frac{\mathrm{d}^{2(1)-1}}{\mathrm{d}x^{2(1)-1}}\left(x\sin x\right) &= \frac{\mathrm{d}^{1}}{\mathrm{d}x^{1}}\left(x\sin x\right)\\ &= (-1)^{1-1}\left(x\cos x + (2(1)-1)\sin x\right)\\ &= \sin x + x\cos x\end{aligned}$$

hence, P_1 is true.

Assume that P_k is true for some positive integer value k, that is

$$\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}\left(x\sin x\right) = (-1)^{k-1}\left(x\cos x + (2k-1)\sin x\right)$$

<u>For $n = k + 1$:</u>

By computation,

$$\begin{aligned}
\frac{\mathrm{d}^{2(k+1)-1}}{\mathrm{d}x^{2(k+1)-1}}\left(x\sin x\right) &= \frac{\mathrm{d}^{2k+2-1}}{\mathrm{d}x^{2k+2-1}}\left(x\sin x\right) \\
&= \frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}\left(x\sin x\right) \\
&= \frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}\left(\frac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}\left(x\sin x\right)\right) \\
&= \frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}\left((-1)^{k-1}\left(x\cos x + (2k-1)\sin x\right)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^{k-1}\left(\cos x - x\sin x + (2k-1)\cos x\right)\right) \\
&= (-1)^{k-1}\left(-\sin x - \sin x - x\cos x - (2k-1)\sin x\right) \\
&= (-1)^{k-1}\left(-(2k-1)\sin x - 2\sin x - x\cos x\right) \\
&= (-1)^{k-1}\times(-1)\left(x\cos x(2k+1)\sin x\right) \\
\frac{\mathrm{d}^{2(k+1)-1}}{\mathrm{d}x^{2(k+1)-1}}\left(x\sin x\right) &= (-1)^{(k+1)-1}\left(x\cos x + (2(k+1)-1)\sin x\right)
\end{aligned}$$

By statement,

$$\frac{\mathrm{d}^{2(k+1)-1}}{\mathrm{d}x^{2(k+1)-1}}\left(x\sin x\right) = (-1)^{(k+1)-1}\left(x\cos x + (2(k+1)-1)\sin x\right)$$

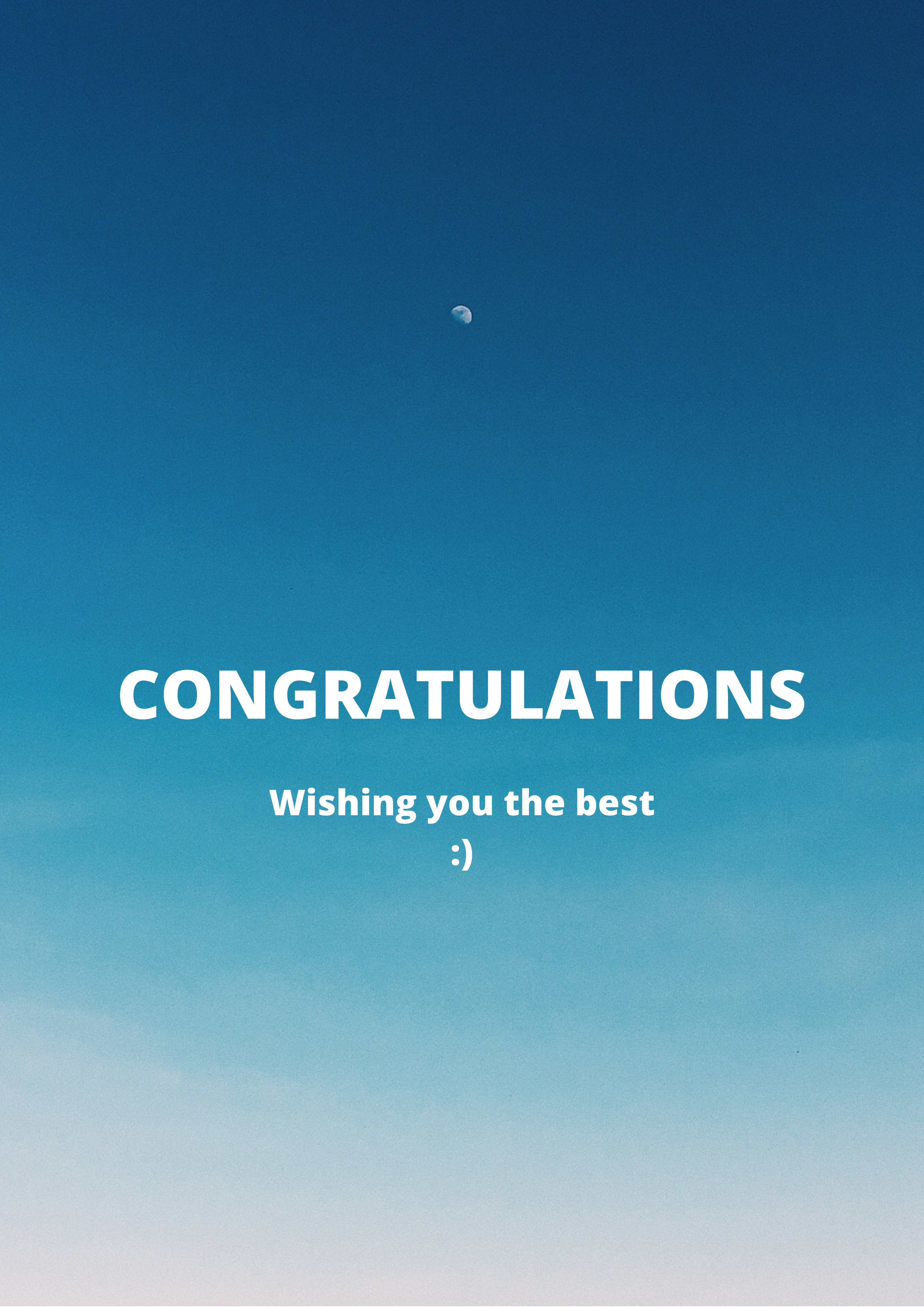
hence, P_{k+1} is true.

Therefore, by principals of mathematical induction, P_n is true for all positive integer values of n, that is

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}\left(x\sin x\right) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right)$$

for every positive integer n, as P_1 is true and $P_k \implies P_{k+1}$.

Note: Sometimes, it can be useful to find some results of the given statement ($n = 1$, $n = 2$, ...)


CONGRATULATIONS
Wishing you the best
:)