Coordinate Geometry in the (x,y) Plane.

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1 A General Look

What is coordinate geometry? Coordinate geometry is a branch of mathematics that deals with geometric figures on a coordinate plane, also known as the (x, y) plane or Cartesian plane. It combines the principles of algebra and geometry to study relationships between points, lines, curves, and shapes.

1.1 What Will be Covered

Generally speaking, we will be dealing with known mathematical concepts in the first half of the paper; however, we will apply them on circles. The remaining portion of the paper tackles circles on the Cartesian plane.

- Midpoints and perpendicular bisectors
- The locus definition of a circle
- The equation of a circle and its components
- Circle theorems and their application

1.2 Prerequisites

Some basics of coordinate geometry have been covered in Pure Mathematics (1). The paper assumes knowledge of these topics.

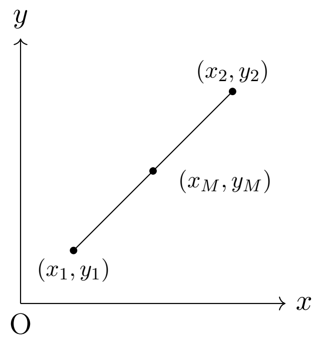
- Finding the equation of straight lines
- Parallel and perpendicular line, and their properties/relationships
- Find length and area on an (x, y) plane

2 Midpoints and perpendicular bisectors

This section will be short as it covers IGCSE and P(1)[1] level material.

[1] P(1)=Pure Mathematics (1)

2.1 Midpoint of a Line Segment

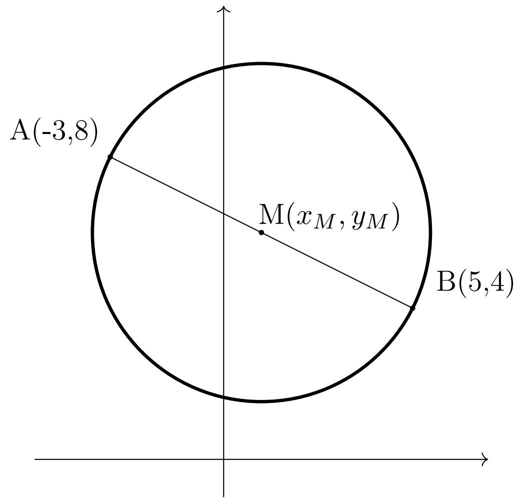


The midpoint of a line segment, M, is given by the following formulae:

$$M = (x_M, y_M) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Where x_1 and x_2 denote the x-coordinates of the endpoints of our line segment, and y_1 and y_2 denote the y-coordinates of the endpoints of our line segment.

If we consider how we can apply the midpoint of a line segment for circles, we have the following application:



Given that AB is a diameter, we can easily find the center of the circle in the diagram on the left. The center is the midpoint of the line segment AB:

$$M = \left(\frac{-3 + 5}{2}, \frac{8 + 4}{2}\right) = (1, 6)$$

2.2 Perpendicular lines

When two straight lines are perpendicular, it is true that

$$m_1 \times m_2 = -1$$

where m_1 is the gradient of ℓ_1, and m_2 is the gradient of ℓ_2

3 The Equation of a Circle

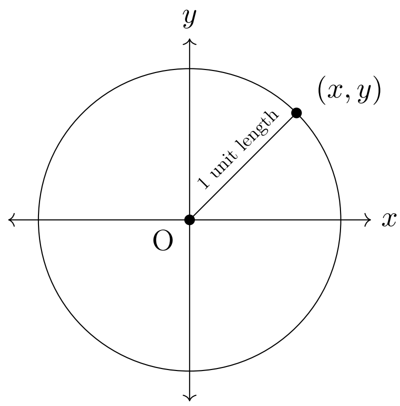
What is a circle? A circle is the collection of points that are equidistant from a point known as the center. In other terms, the collection of points such that the distance between the center and any point is equal to the radius.

3.1 Equation of a circle centred at the origin

Let the coordinates of a point lying on a circle of radius r with the origin as its center be (x, y). Since all points on the circle are a distance r from the center, the origin in this case, it is true that:

$$(x-0)^2 + (y-0)^2 = r^2$$
$$\boxed{x^2 + y^2 = r^2}$$

$x^2 + y^2 = r^2$ is the equation of a circle centered at the origin with radius r.



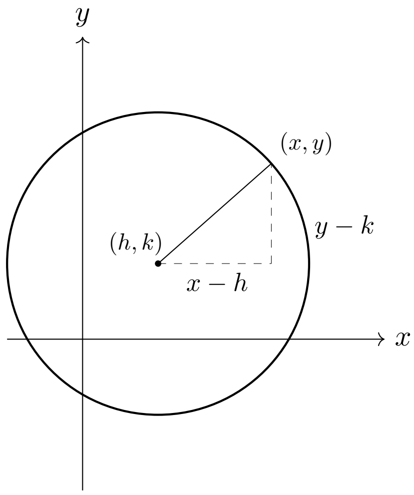
3.2 The General Equation of a Circle

What if our circle is not centred at the origin. How would we find the equation of the circle?

Suppose we have a circle centered at (h, k) with radius r. Let the coordinated of a point lying on the circle be (x, y). Since the distance between any point on the circle, (x, y), and the center, (h, k), is r, it is then true that

$$(x-h)^2 + (y-k)^2 = r^2$$

This is the equation of a circle centred (h, k) with a radius r i.e. The general equation of a circle.

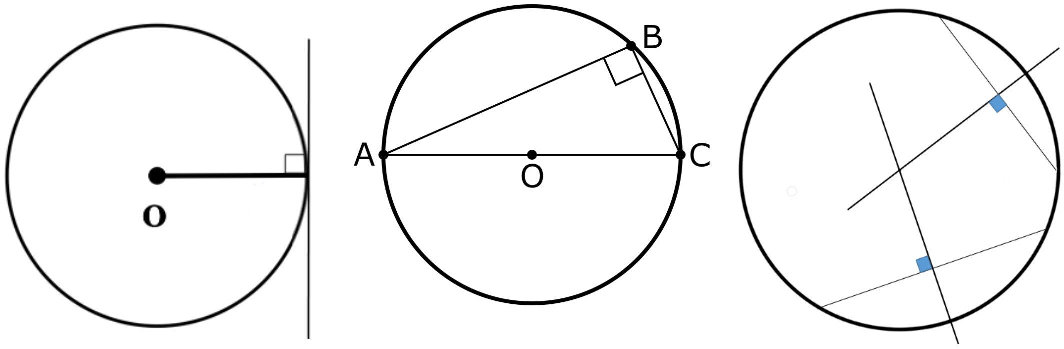


4 Some Important Circle Theorems

Although circle theorems as a topic is not required in the syllabus, there applications are of the upmost importance in certain questions. There are only 4 important circle theorems:

1. A tangent to a circle is perpendicular to the radius of the circle the the point of intersection
2. The perpendicular bisector of a chord will go through the centre of a circle
3. When the opposite ends of a diameter are subtended to a point, a right angle is formed

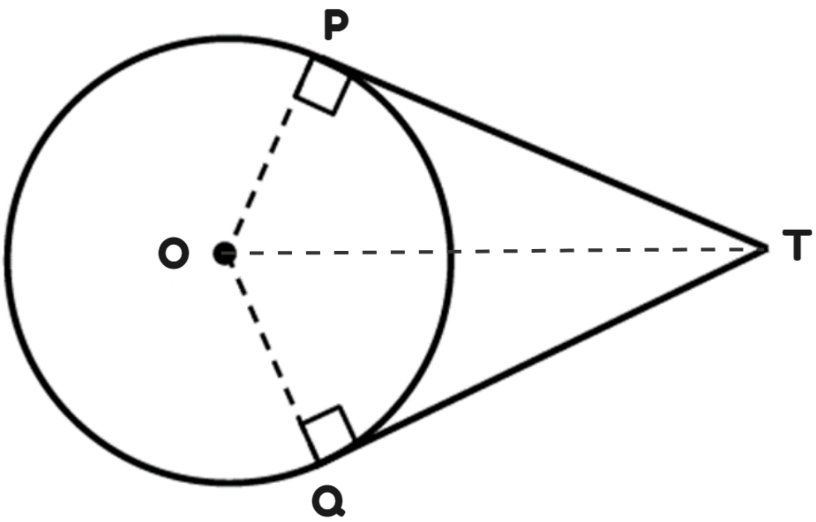
Diagrams representing each case (ordered by previous listing order):



You may have noticed that in Fig. 3 the perpendicular bisectors of each chord intersect at what looks like the center. You're correct; the intersection between two perpendicular bisectors of 2 chords is always at the center. This should make sense as we have stated previously that the perpendicular bisector of any chord passes through the center. Hence, the centre is a common point among all chord perpendicular bisectors.

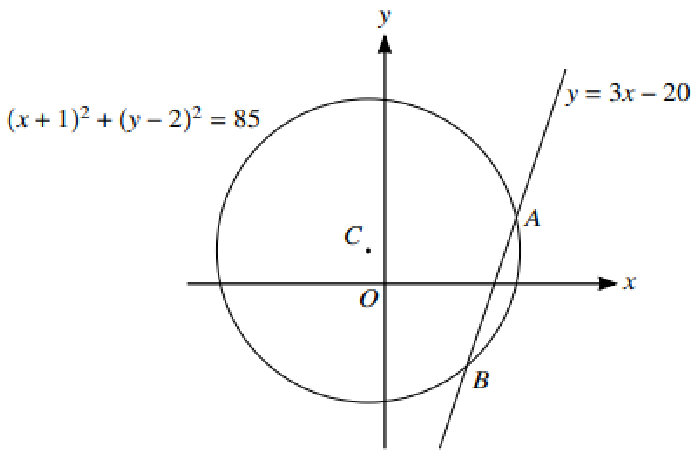
4.1 A Special Circle Theorem

There is however a very special circle theorem that simplifies a lot of work.



Notice how we can use trigonometry, properties of right angle triangles, properties of congruent triangles (notice that triangles TOP and TOQ are congruent) and much more to find certain sides, lengths, equations of tangents and much more! This is a very powerful circle theorem to have in your tool kit.

Unsolved Examples



The circle with equation $(x+1)^2+(y-2)^2=85$ and the straight line with equation $y = 3x - 20$ are shown in the diagram. The line intersects the circle at A and B, and the centre of the circle is at C.

(a) Find, by calculation, the coordinates of A and B. [4]

(b) Find an equation of the circle which has its centre at C and for which the line with equation $y = 3x - 20$ is a tangent to the circle. [4]

Figure 1: February/March 2022 9709/12

9 The equation of a circle is $x^2 + y^2 + 6x - 2y - 26 = 0$.

(a) Find the coordinates of the centre of the circle and the radius. Hence find the coordinates of the lowest point on the circle. [4]

(b) Find the set of values of the constant k for which the line with equation $y = kx - 5$ intersects the circle at two distinct points. [6]

Figure 2: May/June 2022 9709/11

Note: If a sketch isn't given, it is *very* important to make your own sketch

If you need further exercises, you can solve Cambridge A Level Mathematics 9709 circle questions from 2020–2023. They are identical to EDEXCEL's (maybe a little more difficult) and they test the same concepts.