Differentiation

Yousef Ibrahim

October 2023

1 Introduction

When we differentiate a function, we are essentially finding the rate of change of some variable with respect to another variable. When we find $\frac{\mathrm{d}y}{\mathrm{d}x}$, we are finding the rate of change of y with the respect to x; we are finding the change in$y$ due to a small change in x. Newton, *one* of the founders of calculus, used differentiation to find rates of change.[1] Calculus has a wide variety of adopted notations. Edexcel, Cambridge and most high school syllabuses, make use of Leibnitz's (the second father of calculus) and Lagrange's notation. The dot notation (Newton's notation; $\dot{x}$, $\dot{y}$, $\cdots$) only deals with *rates* of change (derivative of a variable with respect to time) and will not be used, though it is the most convenient when it can be used.

2 New standard derivatives

In P1 and P2, we only dealt with the derivatives of sums and differences of power functions ($x^n \pm x^k$) where $n, k \in \mathbb{R}$. In P3, we will deal with the derivatives of sums, differences, products and quotients of the exponential functions, trigonometric functions and the natural logarithmic function. We will start with the standard results.

1) $\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$

2) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x}$

3) $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$

4) $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$

a) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \sec^2(x)$

b) $\frac{\mathrm{d}}{\mathrm{d}x}(\sec(x)) = \sec(x)\tan(x)$

c) $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}(x)) = \mathrm{cosec}(x)\cot(x)$

d) $\frac{\mathrm{d}}{\mathrm{d}x}(\cot(x)) = -\mathrm{cosec}^2 x$

e) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

You should be able to independently derive the derivatives given by alphabet letters independently. This will be done by the product or quotients rule which we will discuss after the chain rule. For now, we will use the previous results as given. Moreover, you use the previous results listed by the alphabets as facts unless you are *explicitly* asked to do so.

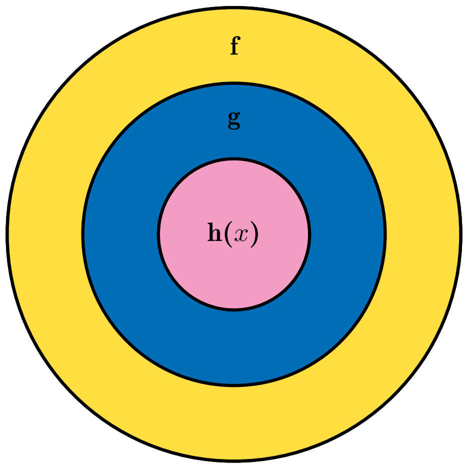
[1] interestingly, integration has roots that date back to the Greeks and Eudoxus of Cnidus' method of exhaustion.

3 Chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

This is the general rule which I don't expect you to instantly get. How I like to perceive this is as an onion; it has many layers, just like functions we use the chain rule on to differentiate.

To find the derivative of a function $f(g(h(x)))$, we will look at its layers:



To find the derivative of the whole function, we *peel* the **f**, then the **g** and finally the $\mathbf{h}(x)$ layer. This mean that we find the derivative of each layer in that order. We will first do an example with 2 layers.

Question Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = (2x+5)^{10}$

Solution:

Our $g(x)$ is $g(x) = 2x + 5$, and our $f(x)$ is $f(x) = x^{10}$; hence,

$$f(g(x)) = f(2x+5) = (2x+5)^{10}$$

. It is also true that

$$f(g(x)) = (g(x))^{10}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((g(x))^{10}\right) = 10(g(x))^9 \times \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)$$

is the derivative of the inner layer and the former term is the derivative of the outer layer. I hope how the chain rule now works is more clear. Of course, this is not some procedure that you will do when you find the derivative of some function. Quick note, the chain rule can be applied on *all* functions including the power functions we previously dealt with.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{10}\right) = 10 \times (x)^9 \times \frac{\mathrm{d}}{\mathrm{d}x}(x) = 10x^9$$

Let us try a 3 layer function.

Question Find the derivative of y with respect to x for $y = \sin^5\left(3x^4\right)$.

Solution

Seems pretty intimidating right? Not really.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\sin\left(3x^4\right)\right)^5\right) &= \overbrace{5\left(\sin\left(3x^4\right)\right)^4}^{\text{Derivative of layer 1}} \times \overbrace{\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(3x^4\right)\right)}_{\text{Houses the 1st and 2nd layers}}}^{\text{Derivative of layer 2 and 3}} \\
&= 5\left(\sin\left(3x^4\right)\right)^4 \times \overbrace{\cos\left(3x^4\right)}^{\text{Derivative of layer 2 \textit{only}}} \times \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(3x^4\right)\right)}_{\text{Derivative of layer 3}} \\
&= 5\left(\sin\left(3x^4\right)\right)^4 \times \cos\left(3x^4\right) \times \overbrace{12x^3}^{\text{Derivative of layer 3}} \\
&= 60x^3 \sin^4\left(3x^4\right)\cos\left(3x^4\right)
\end{aligned}
$$

There is another way to look at the chain rule. This look will help us down the road (P4).

Question Find the derivative of y with respect to x for $y = e^{5x}$.

Solution.

Let $u = 5x$. Then $y = e^u$.

$$\frac{\mathrm{d}y}{\mathrm{d}u} = e^u$$

That doesn't seem very helpful. We found the derivative of y with respect to u instead of x. But wait,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}.$$

We have $\frac{\mathrm{d}y}{\mathrm{d}u}$ and we only need $\frac{\mathrm{d}u}{\mathrm{d}x}$ to find the required derivative.

$$
\begin{aligned}
u &= 5x \\
\rightsquigarrow \frac{\mathrm{d}u}{\mathrm{d}x} &= 5 \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= e^u \times 5 \\
&= 5e^u
\end{aligned}
$$

Substituting $u = 5x$ back, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5e^{5x}$$

One more example.

Question Find the derivative of $y = \sin^2(2x)$ with respect to x.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\sin(2x)\right)^1 \times \cos(2x) \times 2 = 4\sin(2x)\cos(2x)$$

This leads to another important result.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

Question Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y where $x = 3\sec^2(2y)$. Hence, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values p and q.

Solution.

This is a 3 layer chain rule. The first layer is the power of 2 (which is a bracket), the second layer is the secant layer (the trigonometric function) and the third layer is the $2y$ layer (inner function of secant).

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 3\left(2 \cdot \sec(2y) \cdot \sec(2y)\tan(2y) \cdot 2\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 12\sec^2(2y)\tan(2y)$$

We know that the derivative of y with respect to x is the reciprocal of what we found.

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{12\sec^2(2y)\tan(2y)}$$

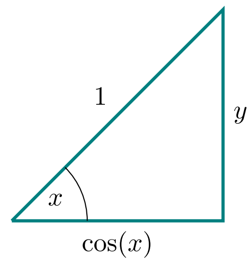
We know that we want the fraction in terms of x. We can start things off by using the substitution $x = 3\sec^2(2y)$

$$= \frac{1}{4 \times 3\sec^2(2y)\tan(2y)}$$

$$= \frac{1}{4x\tan(2y)}$$

All that's left is figuring out $\tan(2y)$ in terms of x. We will take a short break from the question now.

When we have $y = f(x)$ where $f(x)$ is any trigonometric function, we can find the values of all other trigonometric functions in terms of y. I am aware this does not make any sense now. We have to remember that *all* trigonometric functions are related, meaning I can express sine in terms of cosine, cotangent or any trigonometric function I desire. We can link the trigonometric functions geometrically using a right angle triangle! For example, let us express $\tan(x)$ in terms of y given $y = \sin(x)$. Before jumping into the right angle triangle, let us explain what the previous equation means; it means that there is some angle x where the sine of that angle is equal to y, so the ratio of opposite to hypotenuse is $y/1$. Let's now draw the right angle triangle with relevant sides and angles.



$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

We have $\sin(x)$ in terms of y and we only need $\cos(x)$ in terms of y to find $\tan(x)$ in terms of y. Using Pythagoras' theorem, we see that the adjacent side, $\cos(x)$, satisfies the equation

$$\begin{aligned}\cos(x) &= \sqrt{1^2 - \sin^2(x)} \\ &= \sqrt{1 - y^2}\end{aligned}$$

Hence,

$$\tan(x) = \frac{y}{\sqrt{1-y^2}}$$

You may have noticed that we used the fact that $\cos(x) = \sqrt{1^2 - \sin^2(x)}$. Hence, we can relate trigonometric ratios using right angle triangle and identities! Let us get back to our equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4x\tan(2y)}$$

We have to find $\tan(2y)$ in terms of x using the fact that $x = \sec^2(2y)$. We can use a right angle triangle (will be discussed) or we can more easily utilize the identity $1 + \tan^2\theta = \sec^2\theta$ where $\theta = 2y$ in our case

Identity approach

Since

$$
\begin{aligned}
3+3\tan^2 2y &= 3\sec^2 2y \\
3\tan^2 2y &= 3\sec^2(2y)-3 \\
\sqrt{3}\tan(2y) &= \sqrt{3\sec^2(2y)-3} \\
\sqrt{3}\tan(2y) &= \sqrt{x-3} \\
\tan(2y) &= \frac{1}{\sqrt{3}}\sqrt{x-3} \\
\tan(2y) &= \frac{\sqrt{3}\sqrt{x-3}}{3} \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{4x\frac{\sqrt{3}\sqrt{x-3}}{3}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{4\sqrt{3}x\sqrt{x-3}}
\end{aligned}
$$

Suppose you used

$$
1+\tan^2 2y = \sec^2 2y
$$

Hence

$$
\begin{aligned}
\tan(2y) &= \sqrt{\sec^2(2y)-1} \\
\tan(2y) &= \sqrt{\sec^2(2y)-1} \\
&= \sqrt{\frac{x}{3}-1} \\
&= \sqrt{\frac{1}{3}(x-3)} \\
&= \frac{1}{\sqrt{3}}(x-3)
\end{aligned}
$$

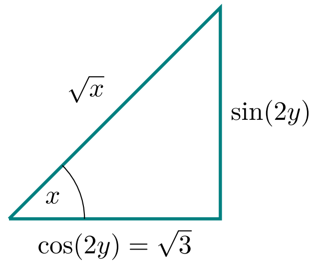
which would lead to the same thing.

Right angle triangle:

We know that

$$
\sec(2y) = \frac{\sqrt{x}}{\sqrt{3}}
$$

and secant is the ratio of hypotenuse to adjacent side



Hence, by Pythagoras

$$\sin(2y) = \sqrt{x-3}$$

$$\therefore \tan(2y) = \frac{\sqrt{x-3}}{\sqrt{3}}$$

and we continue with the solution.

Here is the general chain rule again.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(g(x))\right) = \overbrace{f'(g(x))}^{\substack{\text{Derivative of}\\ \text{layer 1}}} \cdot \overbrace{g'(x)}^{\substack{\text{Derivative of}\\ \text{layer 2}}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(g(h(x)))\right) = \overbrace{f'(g(h(x)))}^{\substack{\text{Derivative of}\\ \text{layer 1}}} \cdot \overbrace{g'(h(x))}^{\substack{\text{Derivative of}\\ \text{layer 2}}} \cdot \overbrace{h'(x)}^{\substack{\text{Derivative of}\\ \text{layer 3}}}$$

4 Product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u(x)\cdot v(x)\right) = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

where $v'(x)$ is the derivative of $v(x)$ with respect to x.

This rule is simpler to apply. However, it is usually mixed with the chain rule.

Question Find $f'(x)$ for $f(x) = e^{2x}\sin^3(2x)$.

Solution.

First, we apply the product rule

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{2x})\cdot\sin^3(2x) + \frac{\mathrm{d}}{\mathrm{d}x}\big(\sin^3(2x)\big)\cdot e^{2x} \\ &= 2e^{2x}\cdot\sin^3(2x) + 3\sin^2(2x)\cdot\cos(2x)\cdot 2\cdot e^{2x} \\ f'(x) &= 2e^{2x}\sin^3(2x) + 6e^{2x}\sin^2(2x)\cos(2x) \end{aligned}$$

We will spare ourselves from doing another example as the product rule itself is pretty straight forward. The problem is usually with the chain rule. Messing the chain rule is usually the most common mistake students make, so watch out for that. No matter how good or well honed you are, it is easy to make a mistake while differentiating.

5 Quotient rule

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)\cdot v(x) - u(x)\cdot v'(x)}{(v(x))^2}$$

An easy way to memorize it is by the following phrase:

LOW DEE-HIGH minus HIGH DEE-LOW over LOW LOW

- LOW $\rightarrow$ denominator
- DEE-LOW $\rightarrow$ derivative of the denominator
- HIGH $\rightarrow$ numerator
- DEE-HIGH $\rightarrow$ derivative of the numerator

Question Find $f'(x)$ for $f(x) = \dfrac{\ln(2x^2+3x-5)}{e^{2x}}$

Solution.

$$f'(x) = \frac{\overbrace{(e^x)}^{\text{LOW}} \times \overbrace{\left(\frac{1}{2x^2+3x-5} \times \underbrace{(4x+3)}_{\text{Derivate of inner ln function}}\right)}^{\text{DEE-HIGH}} - \overbrace{\ln(2x^2+3x-5)}^{\text{HIGH}} \times \overbrace{\underbrace{(2)}_{\frac{\mathrm{d}}{\mathrm{d}x}(2x)} \quad e^{2x}}^{\text{DEE-LOW}}}{\underbrace{\left(e^{2x} \times e^{2x}\right)}_{\text{LOW LOW}}}$$

Again, you can see that the difficulty of this question lies in the chain rule application. However, I strongly recommend breaking down the *layers* of the layered functions ($\ln(2x^2+3x-5)$ and e^{2x}): the first layers are the actual logarithmic and exponential functions and the derivative of these layers are

$$\frac{1}{2x^2+3x-5}$$

and

$$e^{2x},$$

respectively.

The second layers are the inner functions ($2x^2+3x-5$ and $2x$) with the derivatives

$$4x+3,$$

and

$$2$$

respectively.

You can see that we can actually just use the product rule if we shift the denominator upwards by raising it to the power of negative one (will be clarified in a moment). This is true, and sometimes even easier (in the previous example, a product rule application could've been easier.

Alternative solution.

$$
\begin{aligned}
f(x) &= \frac{\ln(2x^2+3x-5)}{e^{2x}} \\
&= \ln(2x^2+3x-5) \times \frac{1}{e^{2x}} \\
f(x) &= \ln(2x^2+3x-5) \times e^{-2x} \\
\therefore f'(x) &= \left(\ln(2x^2+3x-5) \cdot e^{-2x} \cdot -2\right) + \left(\frac{1}{2x^2+3x-5} \cdot (4x+3) \cdot e^{-2x}\right) \\
&= -2e^{-2x}\ln(2x^2+3x-5) - \frac{e^{-2x}(4x+3)}{2x^2+3x-5}
\end{aligned}
$$

6 Exponential growth and decay

We will now talk about the derivative of a^{kx}. We discussed this when $a = e = 2.71828\cdots$. It is true that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{kx}\right) = ke^{kx}$$

We will use this fact to find the derivative of a^{kx}

$$
\begin{aligned}
a^{kx} &= e^{\left(\ln\left(a^{kx}\right)\right)} \\
a^{kx} &= e^{(xk\ln(a))}
\end{aligned}
$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{kx}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(xk\ln(a))}\right)$$

By the chain rule

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(xk\ln(a)\right) \cdot e^{(xk\ln(a))} \\
&= k\ln(a) \cdot e^{(xk\ln(a))} \\
&= k\ln(a) \cdot e^{\left(\ln\left(a^{xk}\right)\right)}
\end{aligned}
$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{kx}\right) = k\ln(a) \cdot a^{kx}$$

You can go through these steps in a shorter manner or memorize the general derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = k \cdot a^{kx} \cdot \ln(a)$$

The questions regarding exponential growth and decay including modelling ones usually just incorporate differentiating some exponential function and evaluating the derivative at some point in context of the question itself.

7 Inverse trigonometric function

Differentiating inverse trigonometric functions is just applying previous things and nothing new; just a matter of technique.

Question Differentiate y with respect to x for $y = \arcsin(x)$

Solution.

$$\begin{aligned}
y &= \arcsin(x) \\
x &= \sin(y) \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= \cos(y) \\
&= \sqrt{1 - \sin^2(y)} \\
&= \sqrt{1 - (\sin(y))^2} \\
&= \sqrt{1 - x^2} \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

Question Differentiate y with respect to x for $y = \arccos(x)$

Solution.

$$\begin{aligned}
y &= \arccos(x) \\
x &= \cos(y) \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= -\sin(y) \\
&= -\sqrt{1 - \cos^2(y)} \\
&= -\sqrt{1 - (\cos(y))^2} \\
&= -\sqrt{1 - x^2} \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

Question Differentiate y with respect to x for $y = \arctan(x)$

Solution.

$$\begin{aligned}
x &= \tan(y) \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= \sec^2(y) \\
&= 1 + \tan^2(y) \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= 1 + x^2 \\
\therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1 + x^2}
\end{aligned}$$

test of a box at the end of a 2. 5 inch line'