Separable Ordinary Differential Equations

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1 Differential equations; what are they?

In general, a differential equation is an equation that relates the rate of change of a variable with another (such as the rate of change of y with respect to x *i.e.* *y'*).

A *separable* differential equation is a special type of first order differential equation (the *order* of a differential equation is just the highest power that a derivative is raised to.) where the 2 variables that we are dealing with can be *separated* in a way where we have a side expressed in terms of a variable, and the other in terms of a variable.

2 Solving such differential equations

Steps:

1. *Separate* your variables with their corresponding increment (dx, dy, dt etc)
2. *Integrate* both sides of the equation (don't forget the addition of a constant!)
3. *Apply* set conditions to find the constant (set conditions are just a list of rules that determine the constant; for example: when x=2, y=3 and so on)

2.1 Examples

a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6ye^{3x}}{2+e^{3x}} \quad \text{when x=0, y=1}$$

- Firstly we *separate* our variables

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6e^{3x}}{2+e^{3x}} \times y \rightarrow \frac{1}{y}\mathrm{dy} = \frac{6e^{3x}}{2+e^{3x}}\mathrm{dx}$$

- Secondly we *integrate*

  $\int \frac{1}{y}\mathrm{dy} = \int \frac{6e^{3x}}{2+e^{3x}}\mathrm{dx} \rightarrow \int \frac{1}{y}\mathrm{dy} = 2\int \frac{3e^{3x}}{2+e^{3x}}\mathrm{dx}$

  $\ln y = 2\ln(2+e^{3x}) + C$

- Finally, we apply the set conditions

$$\xrightarrow[\text{y=1}]{\text{x=0}} \ln 1 = 2\ln(2+e^{3\times 0}) + C$$

$$0 = \ln(2) + C \rightarrow C = -\ln 2$$

$$\boxed{\therefore \ln y = 2\ln(2+e^{3x}) - \ln 2}$$

3 Producing a Differential Equation from a Word Problem

Differential equations are a powerful tool in physics and engineering (generally, more complicated differential equations of higher order and with more variables arise) where most word problems will be built around scenarios that could face an engineer. Translating word problems into an actual differential equation may seem like a daunting task, ***however*** knowing the general order of procedure will provide immense aid

3.1 The General Idea

The general idea is that we want to express the rate of change of a variable (y) in terms of another variable (x). Easier said that done right? Not really. Let's take an example.

3.2 Example:

(1) The height, h metres, of a cherry tree is recorded every year for t years after it is planted. It is thought that the height of the tree is increasing at a rate proportional to 8-h. When the tree is planted it is 0.5 metres tall and after 5 years it is 2 metres tall.

Continued...

a) `Form and solve a differential equation to model this information. give your answer in the form` h`=f(`t`)`

b) `According to the model, what will the height of the cheery tree be when it is fully grown?`

Solution

(a)

Before starting the actual solution we can make an interesting observation; notice how the question used *proportional* ***NOT*** *equal* so we must take care to multiply by a constant k. This is also reinforced by the fact that the question gave us 2 condition which means that we have 2 unknowns to solve for;

`"When the tree is planted it is 0.5 metres tall and after 5 years it is 2 metres tall"` $\implies$ when t=0 h=0.5 AND when t=5, h=2.

Also note that the word "rate" implies a change per unit time; this is where the t variable will come from.

Now we can begin solving after analysing some things.

`"It is thought that the` ***height*** `of the tree is increasing at a rate` *proportional to 8-h*`"` $\implies \dfrac{\mathrm{d}h}{\mathrm{d}t} \propto 8-h \implies \dfrac{\mathrm{d}h}{\mathrm{d}t} = k(8-h)$ where k is a constant

We now go through our basic steps to solving a differential equation:

$$\frac{1}{8-h}\mathrm{dh} = k\ \mathrm{dt} \rightarrow \int -\frac{-1}{8-h}\mathrm{dh} = \int k\ \mathrm{dt}$$

$$-\ln|8-h| = kt + C$$

$$\xrightarrow[\text{h=0.5}]{\text{t=0}} -\ln(8-0.5) = k(0) + C$$

$$\boxed{C = -\ln\left(\frac{15}{2}\right)}$$

$$\xrightarrow[\text{h=2}]{\text{t=5}} -\ln(8-2) = 5k - \ln\left(\frac{15}{2}\right)$$

$$\ln\left(\frac{15}{2}\right) - \ln(6) = 5k$$

$$\ln\left(\frac{5}{4}\right) = 5k$$

$$\boxed{k = \frac{1}{5}\ln\left(\frac{5}{4}\right)}$$

$$\to \ln\left(\frac{15}{2}\right) - \ln(8-h) = \frac{1}{5}\ln\left(\frac{5}{4}\right)$$

$$\ln\left(\frac{15}{2(8-h)}\right) = \frac{1}{5}\ln\left(\frac{5}{4}\right)t$$

$$\frac{15}{2(8-h)} = e^{\frac{1}{5}\ln\left(\frac{5}{4}\right)t}$$

$$\xrightarrow[\text{the power of -1}]{\text{raise both sides to}} \frac{2(8-h)}{15} = e^{-\frac{1}{5}\ln\left(\frac{5}{4}\right)t}$$

$$(8-h) = \frac{15}{2}e^{\frac{1}{5}\ln\left[\left(\frac{5}{4}\right)^{-1}\right]}$$

$$\boxed{h = 8 - \frac{15}{2}e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}} \; \#$$

(b)

We know that every year the height of the tree is *increasing*
`"the height of the tree is increasing"` so as t increases, h also increases.

We are asked for the *maximum* height.Since we know that the height increases with time, we should consider the case as t $\to \infty$

as $t \to \infty$ h $\to$ 8 because $h = 8 - \frac{15}{2}e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t}$ where $\ln\left(\frac{4}{5}\right)$ is negative so e will be raised to a **very large *negative* power** hence $-\frac{15}{2}e^{\frac{1}{5}\ln\left(\frac{4}{5}\right)t} \to 0 \; as \; t \to \infty$

$\boxed{\therefore h_{max} = 8}$ #

4 More Examples:

The following questions are from the June and October 2022 session for WMA14. A detailed solution will be attached to following pages.

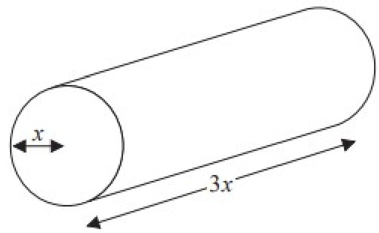

Figure 1

A tablet is dissolving in water.

The tablet is modelled as a cylinder, shown in Figure 1.

At t seconds after the tablet is dropped into the water, the radius of the tablet is x mm and the length of the tablet is $3x$ mm.

The cross-sectional area of the tablet is decreasing at a constant rate of $0.5\,\text{mm}^2\,\text{s}^{-1}$

(a) Find $\dfrac{dx}{dt}$ when $x = 7$

(4)

(b) Find, according to the model, the rate of decrease of the volume of the tablet when $x = 4$

(4)

Figure 1: EDEXCEL MAY JUNE 2022 WMA14

10. A spherical ball of ice of radius 12 cm is placed in a bucket of water.

In a model of the situation,

- the ball remains spherical as it melts
- t minutes after the ball of ice is placed in the bucket, its radius is r cm
- the rate of decrease of the radius of the ball of ice is inversely proportional to the square of the radius
- the radius of the ball of ice is 6 cm after 15 minutes

Using the model and the information given,

(a) find an equation linking r and t,

(5)

(b) find the time taken for the ball of ice to melt completely.

(2)

(c) On Diagram 1 on page 27, sketch a graph of r against t.

(1)

Figure 2: EDEXCEL OCTOBER NOVEMBER 2022 WMA14

SOLUTIONS ON NEXT PAGE

(IT IS STRONGLY ADVISED TO ATTEMPT THE QUESTIONS BEFORE PROCEEDING)

(Q1)

(a) We are given that $\dfrac{\mathrm{d}A}{\mathrm{d}t} = -0.5mm^2s^{-1}$ where A denotes the cross-sectional area of the cylinder.

Step 1: Finding $\dfrac{\mathrm{d}x}{\mathrm{d}t}$

By the chain rule,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}?} \times \frac{\mathrm{d}?}{\mathrm{d}t}$$

To choose the (?) we must consider what is given. Since we are given $\dfrac{\mathrm{d}A}{\mathrm{d}t}$, (?) must be A

$$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}?} \times -0.5$$

Step 2: Finding $\dfrac{\mathrm{d}x}{\mathrm{d}A}$

We must first relate A and x by an equation
$\hookrightarrow A = \pi x^2$ (*Remember that A is the* cross-sectional *area*)

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2\pi x \implies \frac{\mathrm{d}x}{\mathrm{d}A} = \frac{1}{2\pi x}$$

$$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2\pi x} \times -0.5 = -\frac{1}{4\pi x}$$

$$\xrightarrow[\text{at x=7}]{\text{Evaluate at}} \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=7} = -\frac{1}{4 \times \pi \times 7} \approx -0.0114(3\ s.f.)$$

(b) (`Same procedure, won't be discussed to the same detail level`)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$

(Using A instead of x would cause problems as it will be difficult to establish a relationship between A and V

$V = \pi x^2 \times 3x \implies \dfrac{\mathrm{d}V}{\mathrm{d}x} = 9\pi x^2$

$$\therefore \frac{\mathrm{d}V}{\mathrm{d}t} = 9\cancel{\pi} x^{\cancel{2}} \times -\frac{1}{4\cancel{\pi x}} = -\frac{9}{4}x$$

$$\xrightarrow[\text{at x=4}]{\text{Evaluate at}} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x=4} = -\frac{9}{4} \times 4 = -9$$

Note: this is NOT a differential equation question; however, it houses some important techniques which can be elementary to solving a differential equation question

(Q2)

(a)
We are given that

$$\frac{\mathrm{d}r}{\mathrm{d}t} \propto \frac{1}{r^2} \implies \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2}$$

We are also given the *initial* radius of the sphere and the radius of the sphere when 15 minutes have elapsed.

We will now solve the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2} \to r^2 \text{ dr} = k \text{ dt}$$

$$\int r^2 \text{ dr} = \int k \text{ dt}$$

$$\frac{r^3}{3} = kt + C$$

$$\xrightarrow[r=12]{\text{when t=0}} \frac{12^3}{3} = k(0) + C$$

$$\rightsquigarrow \boxed{C = 576}$$

$$\therefore \frac{r^3}{3} - 576 = kt$$

$$\xrightarrow[r=6]{\text{when t=15}} \frac{6^3}{3} - 576 = 15t$$

$$\rightsquigarrow \boxed{k = -\frac{168}{5}}$$

$$\therefore \frac{r^3}{3} - 576 = -\frac{-168}{5}t$$

(b)
The sphere melts completely when r=0

$$\therefore 576 - \frac{168}{5}t = 0 \rightsquigarrow t = \frac{120}{7}$$

Final remarks: Differential equations are a very important matter in engineering, physics and mathematics. Being fully capable of them is an important task.