Trigonometry

Yousef Ibrahim

October 2023

1 Introduction

This paper covers trigonometry for the P3 Edexcel syllabus. It is critical to mention that no matter what, you cannot cheat time and practice. The sole purpose of this paper is to provide an explanation for things only. You have to go through questions on your own to succeed.

2 Reciprocal trigonometric functions

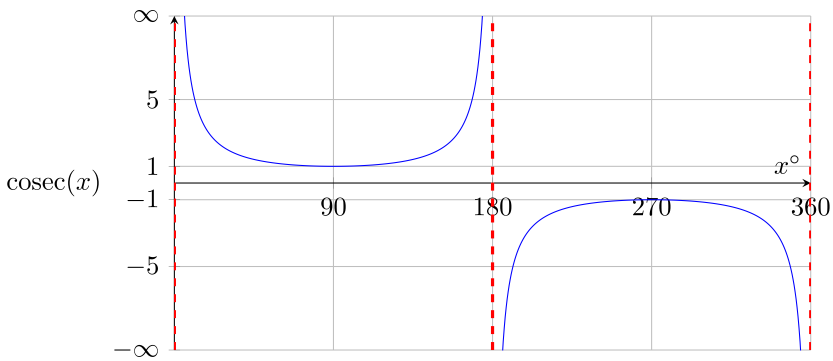
We will introduce a new set of trigonometric functions called reciprocal trigonometric functions. As the name suggested, they are the reciprocals the sine, cosine and tangent trigonometric functions you've been familiar with since secondary schools. They are called cosecant, secant and cotangent, respectively. They are defined as follows:

- $\mathrm{cosec} x = \dfrac{1}{\sin x}$
- $\sec x = \dfrac{1}{\cos x}$
- $\cot x = \dfrac{1}{\tan x} = \dfrac{\cos(x)}{\sin(x)}$

We will take a quick break to mention why these functions are so important that they received their own symbols. Reciprocal trigonometric functions have applications in fields like physics, engineering, and astronomy. For example, in physics, the periods of a simple harmonic oscillator is often expressed in terms of the reciprocal of the frequency. Also, on the pure side of things, they appear in many problems like in real analysis. Back to it.

We will now introduce the graphs of these functions and special properties of them.

2.1 cosec$x$



cosec$x$ will, predictably, have vertical asymptotes at all values of x such that $\sin(x) = 0$ as

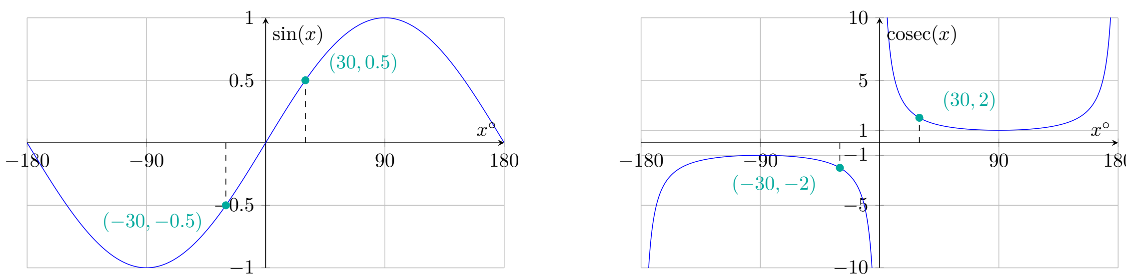
$$\text{cosec}x = \frac{1}{\sin x}$$

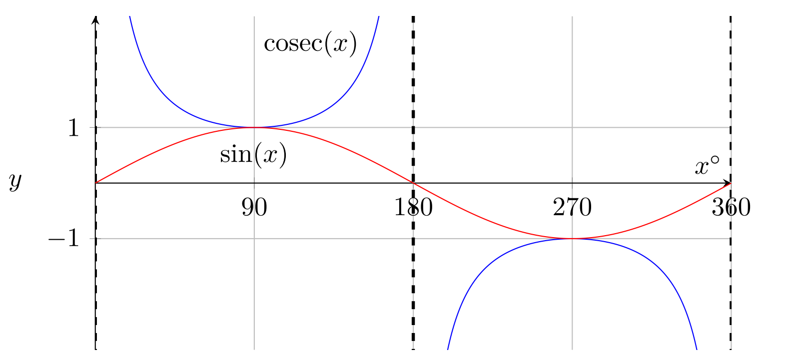
and $\frac{1}{0}$ is undefined. The turning points of cosec$x$ occur at the same x values as those for the turning points of $\sin x$, and as $\sin x$ approaches 0, cosec$x$ approaches $\pm\infty$ depending if we are approaching 0 from negative or positive values. The period of $\sin(x)$ and cosec(x) are the same, and they both are symmetric about the origin. This means that

$$\text{cosec}(-x) = -\text{cosec}(x)$$

and

$$\sin(-x) = -\sin(x)$$

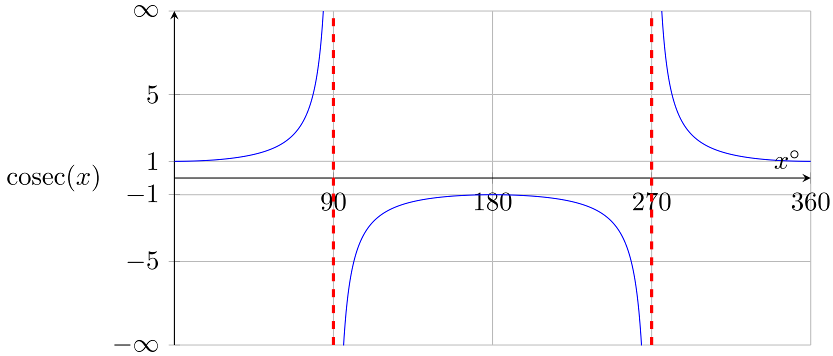




Other than that, there is not much to the graph.

For the geometric representation of cosec(x), it is just the ratio of hypotenuse to opposite side, in contrast to that of $\sin(x)$ which is the ratio of opposite side to hypotenuse.

2.2 $\sec x$



$\sec x$ too will have vertical asymptotes at all values of x such that $\cos(x) = 0$ as

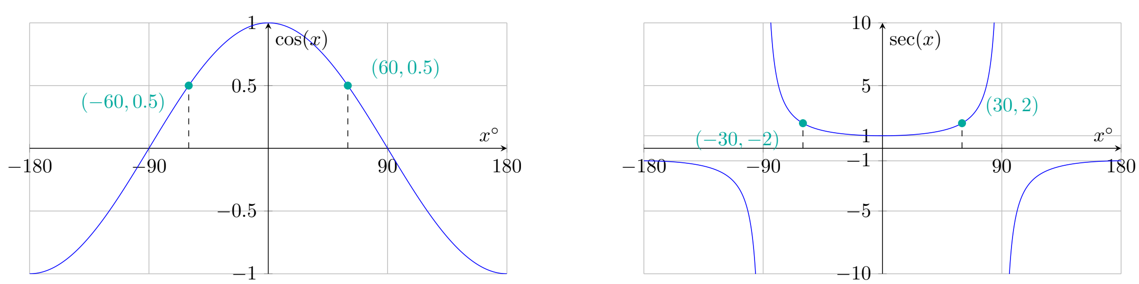
$$\sec = \frac{1}{\cos x}$$

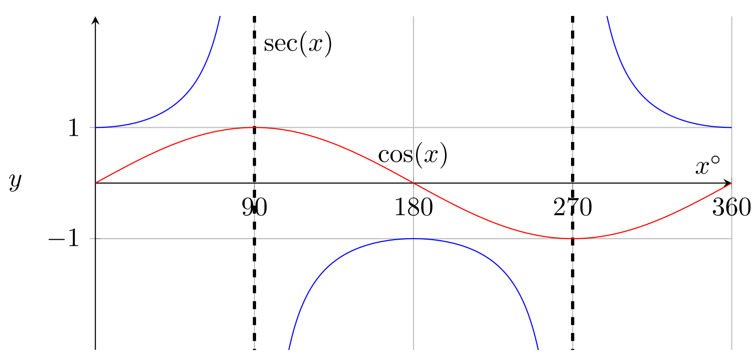
and $\frac{1}{0}$ is undefined. The turning points of $\sec x$ will also occur at the same x values as those for the turning points of $\cos x$, and as $\cos x$ approaches 0, cosec$x$ approaches $\pm\infty$ depending if we are approaching 0 from negative or positive values. The periods of $\cos(x)$ and $\sec(x)$ are the same, and they both are symmetric about the x-axis. This means that

$$\sec(x) = \sec(-x)$$

and

$$\cos(x) = \cos(-x)$$

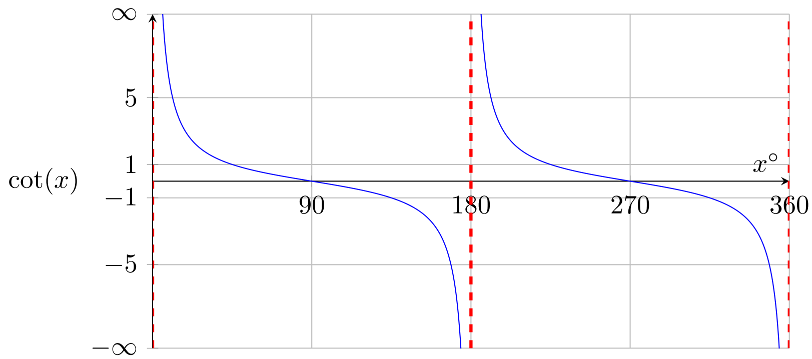




Other than that, there is not much to the graph.

For the geometric representation of $\sec(x)$, it is just the ratio of hypotenuse to adjacent side, in contrast to that of $\cos(x)$ which is the ratio of adjacent side to hypotenuse.

2.3 $\cot(x)$

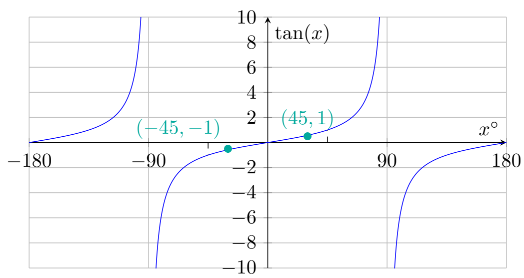


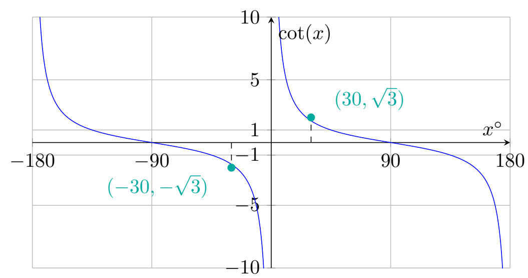
For the same reasons that you grew tired of reading, $\cot(x)$ will have vertical asymptotes at $x = 0°, 180°, 360°$. $\tan(x)$ and $\cot(X)$ have a period of $180°$ and both are symmetric about the origin. This means that

$$\cot(-x) = -\cot(x)$$

and

$$\tan(-x) = -\tan(x)$$





3 Trigonometric identities

A trigonometric identity is an equation involving trigonometric ratios that is true for all values of the variables for which both sides of the equality are defined. You may be asked to prove *any relevant* trigonometric identity. When it comes to using some trigonometric identity when solving some equation however, you will be asked to prove the relevant identity before using it. That is always the case *unless* you must use one of the vital identities (discussed in the following subsections); these identities are elementary and are the building blocks of P3 trigonometry.

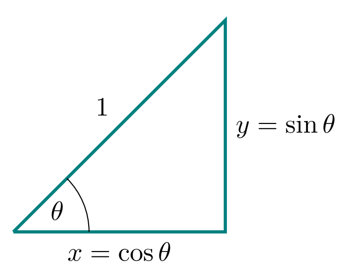
Identity 1.1: $\sin^2\theta + \cos^2\theta \equiv 1$.

Proof. Suppose we have a rectangle with hypotenuse 1, angle θ, opposite side y and adjacent side x. By the geometric definition of sine (ratio of opposite to hypotenuse) $y = \sin\theta \div 1 = \sin\theta$. By the geometric definition of cosine (ratio of adjacent to hypotenuse) $x = \cos\theta \div 1 = \cos\theta$. Hence, by Pythagoras theorem,

$$\sin^2\theta + \cos^2\theta = 1.$$

And since this is true for all real values of θ

$$\sin^2\theta + \cos^2\theta \equiv 1.$$



Identity 1.2: $1 + \tan^2\theta \equiv {}^{1} \sec^2\theta$

Proof. We know that

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

Dividing by $\cos^2\theta$, we get the required identity[2]

$$1 + \tan^2\theta \equiv \sec^2\theta$$

Identity 1.3: $1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$

Proof. We know that

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

Dividing by $\sin^2\theta$, we get the required identity[3]

$$1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$$

Identity 2.1: $\sin(A \pm B) \equiv \sin(A)\cos(B) \pm \sin(B)\cos(A)$

Proof is not required. However notice that

$$\sin(A)\cos(B) + \sin(B)\cos(B) = \sin(B)\cos(A) + \sin(A)\cos(B)$$

while

$$\sin(A)\cos(B) - \sin(B)\cos(B) \neq \sin(B)\cos(A) - \sin(A)\cos(B).$$

Identity 2.2: $\cos(A \pm B) \equiv \cos(A)\cos(B) \mp \sin(B)\sin(A)$

Proof is not required. However, notice how we flip the sign of addition/subtraction when going from the LHS (**L**eft **H**and **S**ide) to the RHS (**R**ight **H**and **S**ide). Hence

$$\cos(A + B) \equiv \cos(A)\cos(B) - \sin(B)\sin(A)$$

and vice-versa for the case of subtraction.

Identity 2.3: $\tan(A \pm B) \equiv \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$

Proof. Using $\tan(x) = \frac{\sin(x)}{\cos(x)}$,

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)}$$

$$= \frac{\sin(A)\cos(B) \pm \sin(B)\cos(A)}{\cos(A)\cos(B) \mp \sin(B)\sin(A)}$$

Dividing the numerator and denominator by $\cos(A)\cos(B)$

$$= \frac{\frac{\sin(A)\cancel{\cos(B)}}{\cos(A)\cancel{\cos(B)}} \pm \frac{\sin(B)\cancel{\cos(A)}}{\cancel{\cos(A)}\cos(B)}}{\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)} \mp \frac{\sin(B)\sin(A)}{\cos(A)\cos(B)}}$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

[1] Distinguishing between equivalence and equality is not really major.
[2] In the exam, you have to show *all* steps, including the step showing that $\tan\theta = {}^{\sin\theta}/_{\cos\theta}$
[3] In the exam, you have to show *all* steps, including the step showing that $\cot\theta = {}^{\cos\theta}/_{\sin\theta}$

Identity 3.1: $\sin(2A) \equiv 2\sin(A)\cos(A)$

We will skip the proof as it is very simple; just use the addition formula for sine, letting $B = A$.

Identity 3.2: $\cos(2A) \equiv \cos^2 A - \sin^2 A \equiv 1 - 2\sin^2 A \equiv 2\cos^2 A - 1$

For the first equivalence, use the addition formula with $B = A$. For the second and third equivalence, use the Pythagorean identity, where

$$\cos^2 A = 1 - \sin^2 A$$

and

$$\sin^2 A = 1 - \cos^2 A.$$

Identity 3.3: $\tan(2A) = \frac{2\tan(A)}{1-\tan^2 A}$

We will skip the proof as it is like the previous two proofs; just let $B = A$.

These previous 9 identities are fundamental for proving general identities. You must know how and when to employ them.

Regarding proving general identities like proving

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1-\cos\theta}{\cos\theta - \sin\theta} \equiv \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta},$$

we sadly cannot go over an infinite number of questions, and the purpose of this paper is not practice and doing problem as much as it is explaining. However, we will go through some tips for proving identities (and prove three random identity questions, including the previous one).

Tips:

- We start with the more complex side
- Look for the angles within trigonometric functions on both sides; if they are different, you must use double angle formulas (usually employed) or sum and difference identities (rarely employed relative to double angle formulas). We usually delay this as much as possible.
- If we have two fractions on one side and end up with one fraction on the other side, we must start by condensing the fraction.
- When it comes to the tangent and cotangent function, if they are present on one side and are not on the other side, we must use $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\cot(x) = \frac{\cos(x)}{\sin(x)}$.
- Sometimes, using some identities may complicate stuff. For example, if the LHS contains 2θ inside a trigonometric function and the RHS contains θ inside a trigonometric function, the trigonometric function with 2θ may cancel.
- As a last resort, if you can't go from one side to another, you can work on both sides and land somewhere in the middle.

We will do some questions together

Question 1 Prove that

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1-\cos\theta}{\cos\theta - \sin\theta} \equiv \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta}.$$

Proof.

We can see that the LHS is more complex relative to the RHS. Also, the LHS contains two fractions, while the RHS contains one fraction.

$$LHS = \frac{\sin\theta}{\cos\theta + \sin\theta} \times \frac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta} + \frac{1 - \cos\theta}{\cos\theta - \sin\theta} \times \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}$$

$$= \frac{\sin\theta(\cos\theta - \sin\theta) + (1 - \cos\theta)(\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta}$$

$$= \frac{\sin\theta\cos\theta - \sin^2\theta + \cos\theta + \sin\theta - \cos^2\theta - \sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

no cosine in the denominator, so we find a way to get rid of it.

$$= \frac{\cos\theta + \sin\theta - (\cos\theta + \sin\theta)}{1 - \sin^2\theta - \sin^2\theta}$$

$$= \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta} = RHS \quad \square$$

This might feel overwhelming at first because it is. However, with practice, you will soon realize how easy these marks are to bag.

Question 2 Prove that

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} \equiv 2$$

We first acknowledge that we begin with 2 fractions and end up with 1 fraction ($^2/_1$). We start with this and delay the problem of difference angles to a later step.

$$LHS = \frac{\sin 3\theta\cos\theta - \sin\theta\cos 3\theta}{\sin\theta\cos\theta}$$

The numerator looks oddly familiar to one identity! Checking back, we can see that the numerator is equal to $\sin(A - B)$ where $A = 3\theta$ and $B = \theta$.

$$= \frac{\sin(3\theta - \theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{\sin\theta\cos\theta}$$

Also the denominator is product of $\sin(A)$ and $\cos(A)$. Hence it is a scalar multiple of $\sin 2A$

$$= \frac{\sin 2\theta}{\frac{1}{2}(2\sin\theta\cos\theta)}$$

$$= \frac{2\sin 2\theta}{\sin 2\theta}$$

$$= 2 = RHS \quad \square$$

Question 3 Prove that

$$\cot^2 x - \tan^2 x \equiv 4\cot 2x \text{cosec} 2x$$

Proof.

We will choose to start with the RHS. Try starting with the LHS; you will experience some trouble.

$$\begin{aligned}
RHS &= \frac{4}{\tan(2x)\sin(2x)} \\
&= \frac{4}{\frac{\sin(2x)}{\cos(2x)}\sin(2x)} \\
&= \frac{4\cos(2x)}{\sin^2(2x)} \\
&= \frac{4(\cos^2 x - \sin^2 x)}{(2\sin(x)\cos(x))^2} \\
&= \frac{4(\cos^2 x - \sin^2 x)}{4\sin^2(x)\cos^2(x)} \\
&= \frac{\cos^2 x - \sin^2 x}{\sin^2(x)\cos^2(x)}
\end{aligned}$$

We will now divide by $\cos^2 x$. As to why we chose to do this is simply intuition and some observation; usually, when we want to transform a fraction of sine and cosine to tangent, we divide by the highest power of cosine.

$$= \frac{1 - \tan^2 x}{\sin^2}$$

seperating into two form to achieve a similar form to the LHS

$$\begin{aligned}
&= \frac{1}{\sin^2 x} - \frac{\tan^2 x}{\sin^2 x} \\
&= \text{cosec}^2 x - \sec^2 x
\end{aligned}$$

You might think that we are far off, but substituting $\text{cosec}^2 = 1 + \cot^2 x$ and $\sec^2 x = 1 + \tan^2 x$

$$\begin{aligned}
&= \cot^2 x - \tan^2 x - 1 + 1 \\
&= \cot^2 x - \tan^2 x = RHS \quad \square
\end{aligned}$$

4 $a\cos\theta + b\sin\theta$ in the forms of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

This is just an application of the addition identities. We will make this clear with the aid of a question.

Question Express $\cos(x) + 2\sin(x)$ in the form $R\cos(x - \alpha)$ where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving the exact value of R and α in radians to 3 d.p.

Solution.

We know that

$$\begin{aligned}
\cos(x) + 2\sin(x) &\equiv R\cos(x - \alpha) \\
&\equiv R(\cos(x)\cos(\alpha) + \sin(x)\sin(\alpha)) \\
\cos(x) + 2\sin(x) &\equiv R\cos(\alpha)\cos(x) + R\sin(\alpha)\sin(x)
\end{aligned}$$

Hence,

$$R\cos(\alpha) = 1 \qquad , \qquad R\sin(\alpha) = 2$$

Dividing the second equation by the first equation

$$\tan(\alpha) = 2$$
$$\alpha = 1.107$$

Since,

$$\sin(\alpha) = \frac{2}{\sqrt{2^2 + 1^2}}$$[4]

then

$$R = \frac{2}{\frac{2}{\sqrt{5}}}$$

$$R = \sqrt{5}$$

So

$$\cos(x) + 2\sin(x) \equiv \frac{\sqrt{5}}{2}\cos(x - 1.107)$$

In general, it is always true that $R = \sqrt{a^2 + b^2}$. Trying this with our previous question

$$R = \sqrt{1^2 + 2^2} = \sqrt{5}$$

which is what we got.

For the form $R\sin(x - \alpha)$ it is the same series of steps but using the difference formula for sine instead of cosine.

Question Express $3\cos(x) + 2\cos(x - 60°$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° \leq \alpha \leq 90°$. State the exact value of R and give α correct to 2 decimal places.

Solution.

Note that to express in the R form, we must have our trigonometric expression to be in the form $a\cos\theta + b\sin\theta$. To do this, we first use the difference formula for cosine and then do our usual steps. We will only do the first part.

$$\begin{aligned} 3\cos(x) + 2\cos(x - 60) &= 3\cos(x) + 2\cos(x)\cos(60) + \sin(x)\sin(60) \\ &= 3\cos(x) + \cos(x) + \frac{\sqrt{3}}{2}\sin(x) \\ &= 4\cos(x) + \frac{\sqrt{3}}{2}\sin(x) \end{aligned}$$

The rest of the steps are standard.

Question Show that the equation $\sqrt{5}\sec(x) + \tan(x) = 4$ can be expressed as $Rcos(x + \alpha) = \sqrt{5}$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places.

[4]You can find $\cos(\alpha)$ instead.

Solution.

This one looks hard, but it isnt. Let us first express the functions using our base trigonometric functions (sine and cosine).

$$\frac{\sqrt{5}}{\cos(x)} + \frac{\sin(x)}{\cos(x)} = 4$$

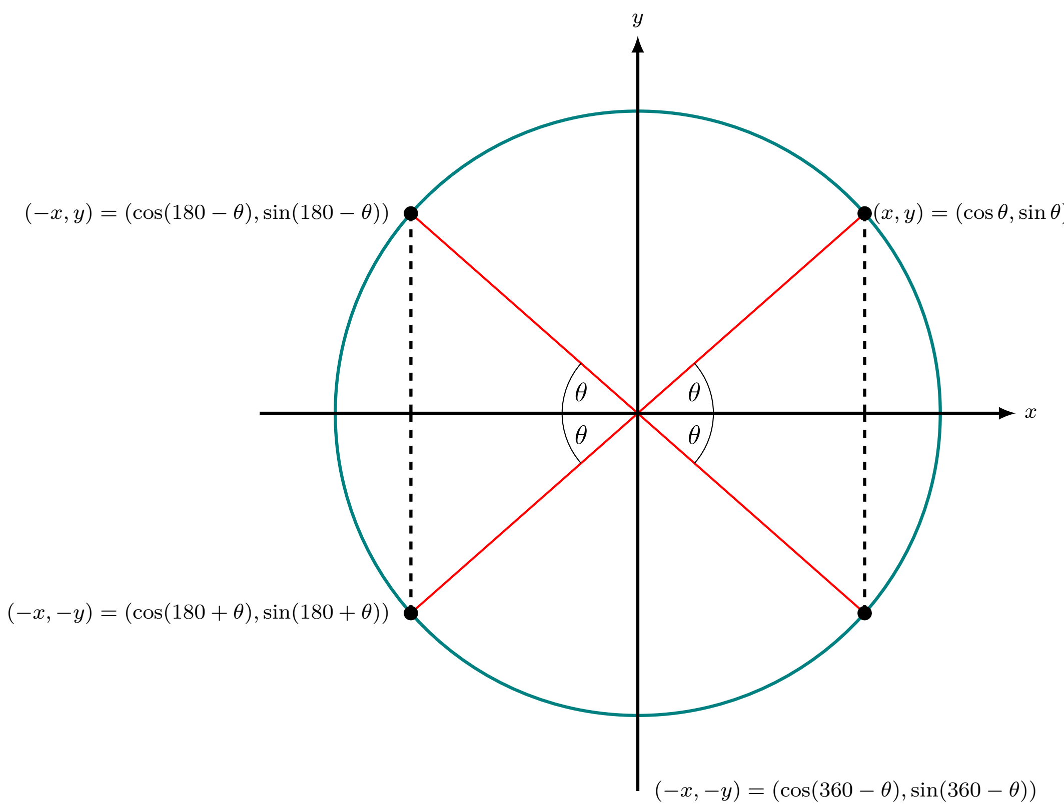
multiplying by $\cos(x)$

$$\sqrt{5} + \sin(x) = 4\cos(x)$$
$$4\cos(x) - \sin(x) = \sqrt{5}$$

You can now deal with this form according to the previous mentioned steps.

5 Solving trigonometric equations

It is very unfortunate to see a significant amount of students unable to solve trigonometric equations with ease. This is mainly a product of a weak base in trigonometry. We will deal with working with negative domains and multiples of the fundamental angles. We will use the unit circle.



We can easily see that

1. $(-x, y) = (-\cos\theta, \sin\theta)$
2. $(-x, -y) = (-\cos\theta, -\sin\theta)$
3. $(x, -y) = (\cos\theta, \sin\theta)$

because the vertical distance $y = \sin\theta$ is constant but we are only changing if it's y units above or below the x-axis. The same goes for the horizontal distance $x = \cos\theta$. This means that

1. $\sin\theta = \sin(180 - \theta)$
2. $\cos\theta = \cos(360 - \theta)$
3. $\tan\theta = \tan(180 + \theta)$

for all angles θ. It is also true that the sine, cosine or tangent for and angle θ is equal to $\theta \pm 360$ since adding or subtracting a full rotation lands us at the same place. Hence, we can deduce a set of steps for solving any trigonometric equation no matter what domain we have and what functions.

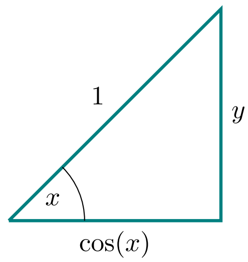
Let k be any real number such that $-1 \leq k \leq 1$.

- **Solving** $\sin\theta = k$:
  1. Take $\sin^{-1} k$ to find θ_1. We will call this our first major angle.
  2. Take $180 - \sin^{-1} k$ to find θ_2. We will call this our second major angle.
  3. Consider all angles $\theta_1 \pm 360n$ for integer values of n until you find all angles within the desired domain.
  4. Consider all angles $\theta_2 \pm 360n$ for integer values of n until you find all angles within the desired domain.
- **Solving** $\cos\theta = k$:
  1. Take $\cos^{-1} k$ to find θ_1. We will call this our first major angle.
  2. Take $360 - \cos^{-1} k$ to find θ_2. We will call this our second major angle.
  3. Consider all angles $\theta_1 \pm 360n$ for integer values of n until you find all angles within the desired domain.
  4. Consider all angles $\theta_2 \pm 360n$ for integer values of n until you find all angles within the desired domain.
- **Solving** $\tan\theta = k$:
  1. Take $\tan^{-1} k$ to find θ_1. We will call this our first major angle.
  2. Take $180 + \tan^{-1} k$ to find θ_2. We will call this our second major angle.
  3. Consider all angles $\theta_1 \pm 360n$ for integer values of n until you find all angles within the desired domain.
  4. Consider all angles $\theta_2 \pm 360n$ for integer values of n until you find all angles within the desired domain.

Note that for $\tan\theta$ we ± 180 not 360 because every 180 degrees, both sine and cosine change their sign for the corresponding coordinate, so the tangent function won't be affected as $\tan x = \frac{\sin x}{\cos x}$. In the context of graphs, the period of $\tan x$ is $180°$ (it repeats itself every $180°$. Note that if we have any equation with reciprocal trigonometric functions, we must first substitute in the standard trigonometric functions and the proceed with solving.

6 Binding trigonometric ratios together

When we have $y = f(x)$ where $f(x)$ is any trigonometric function, we can find the values of all other trigonometric functions in terms of y. I am aware this does not make any sense now. We have to remember that *all* trigonometric functions are related, meaning I can express sine in terms of cosine, cotangent or any trigonometric function I desire. We can link the trigonometric functions geometrically using a right angle triangle! For example, let us express $\tan(x)$ in terms of y given $y = \sin(x)$. Before jumping into the right angle triangle, let us explain what the previous equation means; it means that there is some angle x where the sine of that angle is equal to y, so the ratio of opposite to hypotenuse is $y/1$. Let's now draw the right angle triangle with relevant sides and angles.



$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

We have $\sin(x)$ in terms of y and we only need $\cos(x)$ in terms of y to find $\tan(x)$ in terms of y. Using Pythagoras' theorem, we see that the adjacent side, $\cos(x)$, satisfies the equation

$$\begin{aligned} \cos(x) &= \sqrt{1^2 - \sin^2(x)} \\ &= \sqrt{1 - y^2} \end{aligned}$$

Hence,

$$\tan(x) = \frac{y}{\sqrt{1 - y^2}}$$

You may have noticed that we used the fact that $\cos(x) = \sqrt{1^2 - \sin^2(x)}$. Hence, we can relate trigonometric ratios using right angle triangle and identities!