Forces and Moments

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Paper goal and structure

This paper develops the elementary ideas and characteristics of forces that are necessary to smoothly get into the idea of moments. Most– if not all– of Eedexcel's IAL moments question fall under those concerned with bridges on pivots, not ideas relevant to usage of lines of action. Hence, not much cares was given towards building up on that idea.

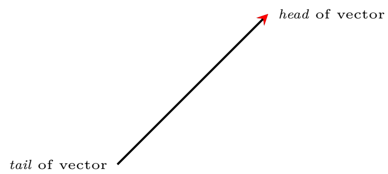
1 Forces

The building blocks of Newtonian Mechanics

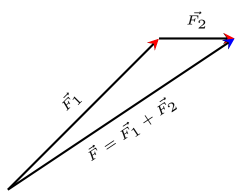
Dealing with forces is essentially dealing with the base structure of Newtonian Mechanics. For our needs, we can consider forces to be vectors with fixed points of application, i.e. we cannot freely move forces in 2D space while maintaining magnitude and direction. If a force acts on a ball then we cannot manipulate this force to act on another object. This seems trivial but acknowledging this explicitly will help in the formulation of bigger ideas!

Forces, by their nature act along a *line of action*– an important idea in moments.

Acknowledging the fact that forces are vectors, we can port many ideas from the realm of vectors to that of forces and mechanics. First, we will touch on the additive nature of vectors. Let's get some terminology in place first.

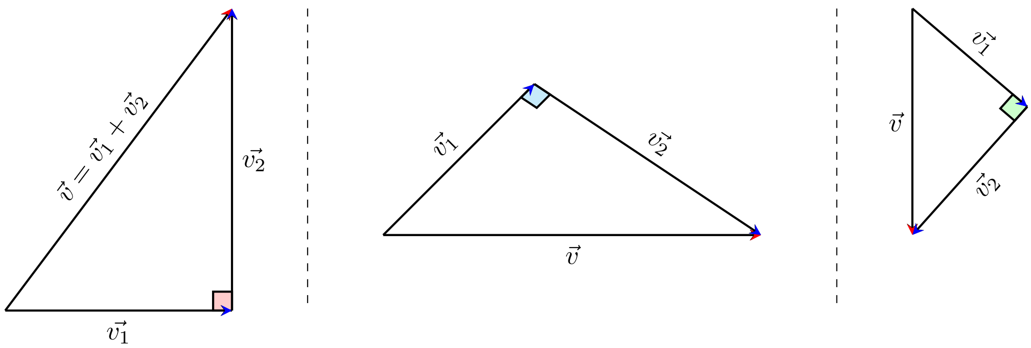


At the IGCSE level, we learned that we can add vectors from *tail* to *head*. This is better represented by the following diagram.



Notice that since the tail of $\vec{F_2}$ corresponds to the head of $\vec{F_1}$, their sum is just the vector with the tail at $(0,0)$– the tail of $\vec{F_1}$– and the head at $(3,2)$– the head of $\vec{F_2}$. This is the graphical way to represent vector addition. We are keeping this brief since this is elementary vectors. Note that the magnitude of $\vec{F}$ is greater than the magnitude of its parts (i.e. $\vec{F_1}$ and $\vec{F_2}$).

But why are we discussing vector addition in a mechanics? It turns out that representing forces as the sum of two perpendicular– hence smaller– forces is incredibly useful. Please note that there are infinite ways to break down a vector; however, in mechanics, we exclusively deal with the case where we represent a vector as the sum of two perpendicular vectors. We will refer to the procedure of expressing a vector as the sum of two smaller parts as *vector decomposition*. We are essentially decomposing a vector down to simpler, more useful parts. Let's see this in action.



Note that while we have been discussing force only, this is true for every other vector, including displacement and acceleration.

Going from the graphical representation of vectors, we meet an interesting individual known as Isaac Newton and his three laws. We are only concerned with his second law:

$$\vec{F}_{\text{net}} = m \times \vec{a},$$

noting that we are dealing with the resultant force. But wait, we just learned that we can break any force into two orthogonal[1], smaller vectors. Suppose we break our resultant force into two orthogonal vectors, $\vec{F_x}$ and $\vec{F_y}$, where x and y represent the directions of the vectors. Conventionally, x and y refer to the direction parallel to the x-axis and y-axis on a Cartesian plane, but in this case, we will use them to represent an arbitrary direction. It is then true that

$$\vec{F}_{net(x)} = m \times \vec{a}_x$$

and

$$\vec{F}_{net(y)} = m \times \vec{a}_y,$$

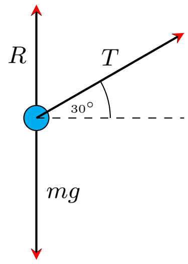
where $\vec{F}_{net(x)}$ and $\vec{F}_{net(y)}$ represent the resultant force in the x and y direction, respectively. This makes sense since the two orthogonal forces are, afterall, forces! It's interesting to note that an acceleration in some direction $\vec{d}$ is caused by a resultant force in the same direction $\vec{d}$.

Examples

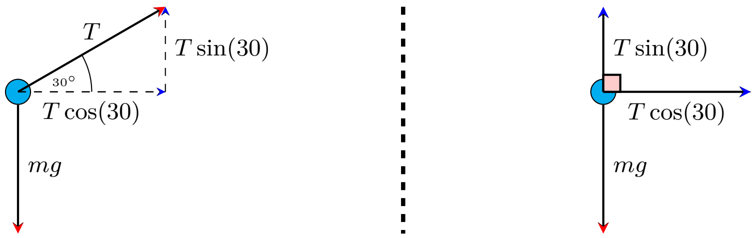
Question 1 A particle of mass 2.5 kilograms is connected to a taut string that makes an angle of 30° with the horizontal. The tension in the string is equal to 50 newtons. The particle is on a horizontal, smooth plane and is on the point of losing contact with the plane. Find the acceleration of the particle.

[1] Orthogonal: perpendicular; commonly used when talking about vectors.

Solution. The key to solving any mechanics question lies in your sketch. However, with more experience, you should be able to solve simple questions without sketching. We will draw a free body diagram of the particle.[2] We will use g to represent gravitational acceleration.



Where R is the reaction force on particle from plane. We will now decompose tension into the components parallel to the x-axis and y-axis. But why did we choose this direction? We can decompose it into different orthogonal vectors. The answer is simple; because our other force, weight, acts in the direction parallel to the y-axis, so breaking tension into those components is convenient as we can easily find the resultant vectors.



Since the particle is on the point of losing contact with the plane, i.e. moving up the plane, it is in a state of vertical equilibrium so the sum of forces on the particle in the vertical direction is equal to 0, and the floor doesn't exert any force on the particle so $R = 0$.[3] We have created two different diagrams, each displaying a valid way to think about vector decomposition. In the first diagram, we are representing the orthogonal vectors as "imaginary" parts of our "parent vector" T. The other diagram, transforms the original vector, T, into its parts, where the **vector** sum of $T\cos(30)$ and $T\sin(30)$ is equal to T, where the orthogonal vectors have the parents' point of application[4]. Note that $T \neq T\cos(30) + (T\sin(30)$ but $\vec{T} = \vec{T}\cos(3) + \vec{T}\sin(30)$; the first equation says that the magnitude of the parent vector is the sum of its part, which is not true as, by Pythagoras' theorem, $T^2 = (T\cos(30))^2 + (T\sin(30))^2$. The first representation is obviously more clean.

Now, we can start doing the math. Since the sum of forces in the vertical direction is zero,

$$T\sin(30) = 2.5g.$$

Although according to Edexcel's specification $g = 9.81$, we will use $g = 10$ for convenience. Hence,

$$T = 50\,N.$$

[2] A free body diagram is a diagram that isolates one object of our system and considers the forces directly acting on that single object only.

[3] Though we are yet to discuss equilibrium, it is the state when an object is stationary, and Σforces in any direction=0 and Σmoments at any point=0

[4] Point of Application: the point on which a force acts on, usually corresponding to the tail.

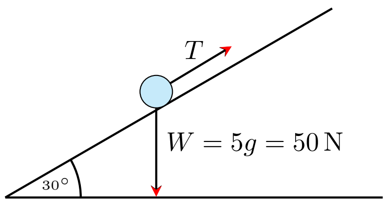
Resolving in the direction parallel to the x-axis, we get that

$$\begin{aligned}\vec{F}_{\text{net}(x)} &= m \times a_x \\ T\cos(30) &= 2.5 \times a_x \\ 50 \times \frac{\sqrt{3}}{2} &= 2.5a_x \\ a_x &= 10\sqrt{3} \approx 17.3.\end{aligned}$$

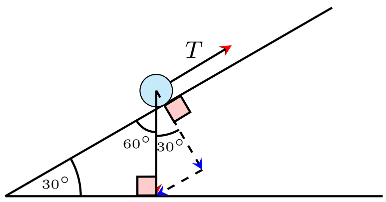
Since all of our forces either act in the direction parallel to x or y, and because the sum of forces– hence acceleration– in the direction parallel to y is zero, our resultant force– hence acceleration– is equal to the resultant force in the direction parallel to the x-axis. So $a = a_x = 17.3\,\text{ms}^{-2}$.

Question 2 A particle of mass 5 kilograms lies on a smooth plane with a 30° inclination to the horizontal. The particle is attached to a string parallel to the inclined plane with a tension of T N. Given that the particle is moving up the plane with an acceleration of $1.5\,\text{ms}^{-2}$, find T.

Solution. We begin our solution with a sketch to better understand the given scenario.



We are faced with two option: we can break down T into two orthogonal vectors with their directions parallel to the x-axis and y-axis, or we can break down W into two orthogonal vectors with their directions parallel and perpendicular to the plane, noting that T is parallel to the plane. Notice that the question provided the acceleration up the plane, which is another term for the acceleration parallel to the plane. Hence, Σforces parallel to plane $= 5 \times 1.5 = 7.5$. This is our cue to break down W into two orthogonal vectors with their directions parallel and perpendicular to the plane.



To keep the diagram clear, we will note here that the component of W parallel to the plane is $W\sin(30) = \frac{W}{2}$, where W is the weight of the particle. Resolving parallel to the plane, we get

$$\begin{aligned}T - \frac{W}{2} &= 5 \times 1.5 \\ T &= 7.5 + \frac{5 \times 10}{2} \\ T &= 32.5\,\text{N}.\end{aligned}$$

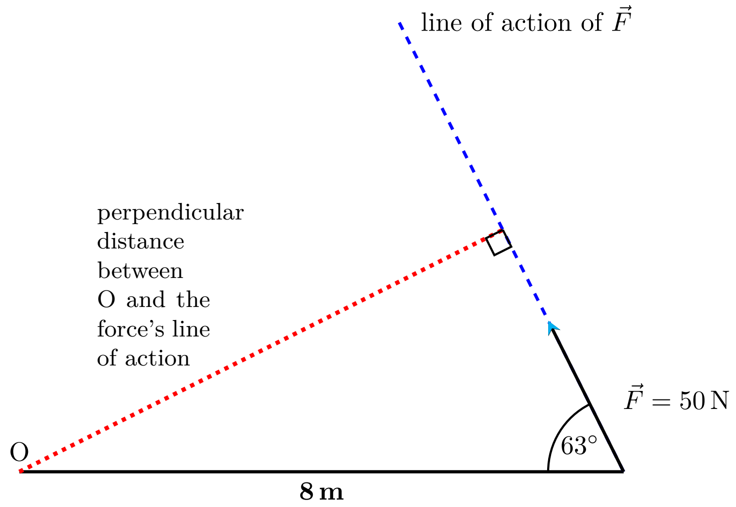
2 Moments

A moment or a torque is a vector quantity that exerts a tendency of rotation about a point on an object. The moment, $\vec{M}$, exerted by a force, $\vec{F}$, about a point O is given by

$$\vec{M} = \vec{F} \times d$$

where d is the perpendicular distance between the line of action of $\vec{F}$ and the point O. But what is the line of action for a force $\vec{F}$? The line of action of $\vec{F}$ is an imaginary line parallel to the direction of $\vec{F}$, going through its point of application. It is a geometric representation of the path along which the force acts. As the name suggests, it is the *line* along which the *action* of a force is sensed. Note that all of the objects that we deal with are *rigid*[5]. For example, we can find the moment, $\vec{M}$, by $\vec{F}$ on the weightless rod about O using the diagram below as follows.

$$\begin{aligned}\vec{M} &= \vec{F} \times d \\ &= 50 \times 8 \sin(63) \\ &\approx 356\,\text{Nm}\end{aligned}$$



An object is said to be in equilibrium if two conditions are met: Σforce on object=0 and Σmoments on object **about any point**=0. If the Σmoments on object **about any point**$\neq 0$, then the object will be rotating in one of two directions: clockwise or anti-clockwise rotation.

When it comes to questions regarding moments, we are usually dealing with rods, beams, planks, etc... If a rod is uniform, this means that it has equal density and cross-sectional area; hence, its mass is evenly distributed; thus, the rod's center of mass is in the center.

If a rod rests on supports, then we call the points where the rod rests on these supports the points of contacts. The supports exert a reaction force on the rod about the points of contacts. When we are dealing with an object in equilibrium and need to take moments about some point, we usually choose the points of contacts since the reaction forces will exert no moment since the line of action passes through the pivot; hence, the perpendicular force is zero! From this we can say that for an object in equilibrium, we almost always take the pivot in such a way that we cancel the moment of as many forces as possible.

[5]Rigid object: an object that experiences no deformation under any force.