Sigma Notation and Periodic Sequences

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Introduction

The following paper will address the sigma notation and the sum of periodic sequences. It is assumed that you know everything else excluding these topics regarding arithmetic and geometric sums.

1 Sigma notation

$$\sum_{\substack{r=k \\ \downarrow \\ \text{lower limit}}}^{\substack{\text{upper limit} \\ \uparrow \\ n}} u_r \longrightarrow \text{sequence}$$

- Sequence: This is the sequence that we are summing.
- Lower limit: This is the number, k, that we substitute in our sequence. Essentially, the lower limit, k, is the term that we start at. In the given sum, we start at the k^{th} term of the sequence u_r.
- Upper limit: This is the term we *terminate* our sum at. The last term we add in our sum is the n^{th} term.

It is especially important you understand the sigma notation for when we find the sum of periodic sequences.

Example: $\displaystyle\sum_{r=5}^{7} 2^r = 2^5 + 2^6 + 2^7 = 224$

There is a simple trick for finding the number of terms that we are adding using the sigma notation: we take the upper limit, n, subtract from it the lower limit, k and then add 1. So using the previous sum as an example, we have $7 - 5 + 1 = 3$ terms. But why did we add 1?

When we subtract the lower limit from the upper limit, we are essentially finding *how many jumps* we are making between terms; hence, we are not considering the fact that the first term exists (since we are not "jumping" to the first term)!

2 Periodic sums

We will first discuss what a periodic sequence is.

A periodic sequence is a sequence that repeats itself every n terms. This is almost always the case when we have trigonometric functions involved; trigonometric functions are periodic function, so the output values repeat themselves for an infinite number of input values.

Example: If $u_r = \cos\dfrac{\pi}{2}r$, then $u_1 = 0$, $u_2 = -1$, $u_3 = 0$, $u_4 = 1$, $u_5 = 0$, $u_6 = -1$
We can see that the sequence repeats itself every 4 terms.

We can also look at this in another way, which will be more suitable for the sum scenario. We can see that we can bundle every 4 terms together to create identical bundles, since every 4 terms are the same.

Suppose we want to find the result of

$$\sum_{r=1}^{12}\left(2\cos\left(\frac{\pi}{2}r\right) + \sin\left(\frac{\pi}{2}r\right)\right)$$

Listing all the terms, we have

$$\sum_{r=1}^{12}\left(2\cos\left(\frac{\pi}{2}r\right) + \sin\left(\frac{\pi}{2}r\right)\right) = (1-2-1+2) + (1-2-1+2) + (1-2-1+2) = 0$$

While this sum is easy to list out, and the sum of every bundle is 0, other sums may be more difficult. What if we had the upper limit as 90? If we were to list out terms, we would have $90 - 1 + 1 = 90$ terms. We will instead find the number of *bundles*. Since every 4 terms make up a "bundle", we will have $\frac{90}{4} = 22.5$ bundles. So we have 22 bundles and 2 terms. Since each bundle is equal to 0, the result of the 22 bundles is 0.

We will now investigate the 2 terms. The 2 terms come at the beginning of the bundle. The first 2 terms of each bundles are 1 and -2. So the result of the sum is (0)+(1-2)=-1.

Let's check one more question.

Question A sequence $a_1,\ a_2,\ a_3,\ \cdots$ is defined by

$$a_n = \cos^2\left(\frac{n\pi}{3}\right)$$

Find the exact values of

a) **i)** a_1

ii) a_2

iii) a_3

b) Hence find the exact value of

$$\sum_{n=1}^{50}\left[n + \cos^2\left(\frac{n\pi}{3}\right)\right]$$

You must make your method clear.

Solution

a) **i)** $a_1 = \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$

ii) $a_1 = \cos^2\left(\frac{2\pi}{3}\right) = \frac{1}{4}$

iii) $a_1 = \cos^2\left(\frac{\pi}{3}\right) = 1$

b)

$$\begin{aligned}\sum_{n=1}^{50}\left[n + \cos^2\left(\frac{n\pi}{3}\right)\right] &= \sum_{n=1}^{50} n + \sum_{n=1}^{50}\cos^2\left(\frac{n\pi}{3}\right) \\ &= 1275 + \sum_{n=1}^{50}\cos^2\left(\frac{n\pi}{3}\right)\end{aligned}$$

Let us deal with $\sum_{n=1}^{50}\cos^2\left(\frac{n\pi}{3}\right)$

If we list more terms, we can quickly realise that the sequence we are summing repeats itself every 3 terms. So each "bundle" of ours is equal to

$$0.25 + 0.25 + 1 = 1.5.$$

We have $50 - 1 + 1 = 50$ terms, so we have $50 \div 3 = 16\frac{2}{3}$; hence, we have 16 bundles and 2 terms.

$$\begin{aligned}\therefore \sum_{n=1}^{50}\left[n + \cos^2\left(\frac{n\pi}{3}\right)\right] &= 1275 + 16 \times 1.5 + \frac{1}{4} + \frac{1}{4} \\ &= 1299.5\end{aligned}$$