Complex Numbers

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1 Introduction

This is, so far, my most extensive paper to be released. This paper is a staircase from the floor of real numbers into complex analysis (at the further math level that is). In this paper, the historical introduction of complex numbers is *crucial* for understanding what complex numbers actually are. Complex numbers are truly a fascinating subject in Mathematics and I hope this paper can portray this beauty. Note that for the Further Pure Math 2 part for Edexcel, only the parts that overlap with Camrbidge's Pure math 2 syllabus are covered.

2 The History of the Complex Numbers

2.1 Humans before complex numbers

From the dawn of time, what humans considered numbers was a continuously evolving subject. The universe (by the modern opinion) started with one set of numbers; the natural numbers ($\mathbb{N}$). A very good quote to portray this is by Leopold Kronecker (on the right): "God made natural numbers; all else is the work of man." Humans then developed (or found, a very philosophical question) fractions, decimals, zero, negative numbers and irrational numbers. The development of numbers was put on ice until 1545.

2.2 The birth of complex numbers

When it comes to recorded history, Gerolamo Cardano, an Italian polymath, was the first person to encounter complex numbers[1]explicitly when we considered the quadratic equation

$$x(10 - x) = 40$$

, or

$$x^2 - 10x + 40 = 0.$$

Using the quadratic formula one of the solutions is

$$x = 5 + \sqrt{-15}.$$

Plugging in this x values and going through the simplification process, he got that

$$(5 + \sqrt{-15})^2 - 10(5 + \sqrt{-15}) + 40$$

is indeed equal to 0. However, Cardano quickly dismissed this solution as it was deemed useless.

[1] They weren't known as complex numbers yet; however, the square roots of negative numbers were observed.

Complex numbers were once again perceived by an Italian mathematician called Rafael Bombelli during his work with cubic equations; the presence of complex numbers was confusing as it doesn't relate to tangible geometric interpretation. The negatives were a controversial topic, so you might imagine how "deviant" complex numbers were seen; can you think what $\sqrt{-1}$ might represents? [2] The appreciation for complex numbers grew in the 17$^{\text{th}}$ century when the "Fundamental Theorem of Algebra" (FTA for short) [3] was slowly taking shape. When the FTA was rigorously proved, the existence of square roots for negative numbers was accepted. However, most mathematicians, including Isaac Newton, Gottfried Leibniz and René Descartes, were reluctant about them. Physicists on the other hand were completely against this class of numbers. Erwin Schrödinger was initially uncomfortable about the existence of the imaginary unit i in his equation which practically describes our physical reality (governs the behaviour of quantum particles). He wrote, "What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function."

2.3 What came after complex numbers

To cut a long story short, Euler worked on complex numbers and infinite series to establish (arguably) the most famous equation using complex numbers:

$$e^{it} = \cos t + i \sin t$$

where $i = \sqrt{-1}$ (Euler was the first to denote $\sqrt{-1}$ as i. His work was later developed into Complex analysis which bloomed under the hands of Augustin Louis Cauchy and pushed further by Karl Weierstrass and Laurent Reinmann. In short, Complex Analysis is the study of complex numbers and functions; it can be used to aid in solving many problems in numerous field, including fluids, waves and even quantum mechanics, including functions with real values. For example, we will see how we can use complex numbers to solve integrals like

$$\int \sin^7 x \, \mathrm{d}x,$$

and finding the roots of

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0.$$

3 Cambridge P3 & Edexcel FP1

3.1 General definition of a complex number

Complex Numbers are all numbers of the form $a + ib$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $i = \sqrt{-1}$. We can easily see that the all real numbers are indeed complex numbers where $b = 0$. That is, $\mathbb{R} \subset \mathbb{C}$ We will now return to the days where the number line was the bulk of your problems.

We were taught that $\mathbb{R}$ (the set of real numbers) can be represented as a line known as the number line. All real numbers exist on that single line with no exception. This is known as the geometric representation of real numbers which really helps in understanding the nature of these numbers. To represent complex numbers geometrically however, we need an *extra dimension* to deal with the *imaginary* number ib. Hence, complex numbers can be represented geometrically as points, or vectors, on a 2D plane called the *complex plane* where the horizontal axis represents the real part of our complex number, and the vertical axis represents the imaginary part of our complex numbers.

Before jumping into the actual geometric representation (which is done on an argand diagram, named after Jean-Robert Argand), we will first introduce some notation:

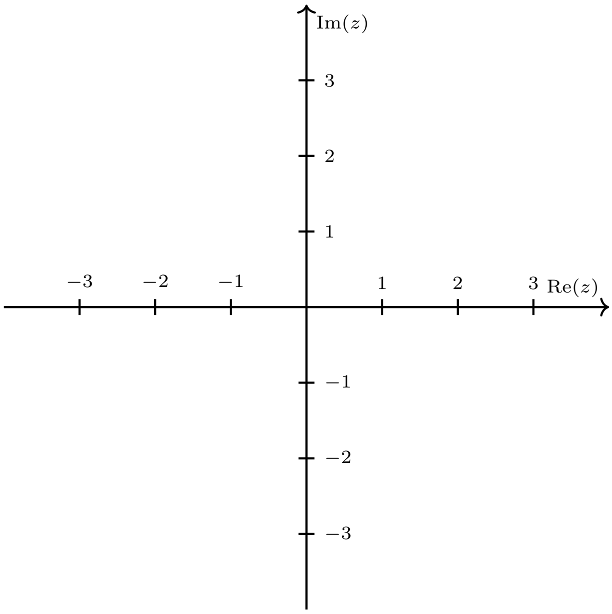
[2] It is important to note that mathematicians abandoned the Greek's method of axioms and deductive systems for geometric oriented approaches.

[3] an n^{th} degree polynomial, $p(x)$ with non zero coefficients has at least 1 solution if $x \in \mathbb{C}$.

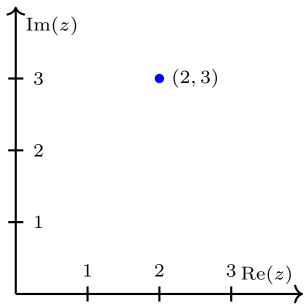
- z will be used to represent an arbitrary complex number, where $z = a + ib$.
- $\mathrm{Im}(z)$ will represent the imaginary part, or imaginary quantity/magnitude, of z-that is, $\mathrm{Im}(z) = b$ where $z = a + ib$ (as agreed before).
- $\mathrm{Re}(z)$ will represent the real part, or real quantity/magnitude, of z-that is, $\mathrm{Re}(z) = a$.

3.2 The Argand Diagram and vector nature of complex numbers

The following figure shows an argand diagram.



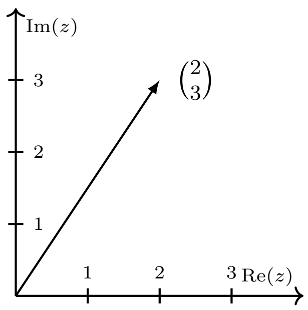
If we want to represent $2 + 3i$ on our Argand diagram, we can simply do so by representing our complex number as cartesian coordinates and plotting it on our Argand diagram.



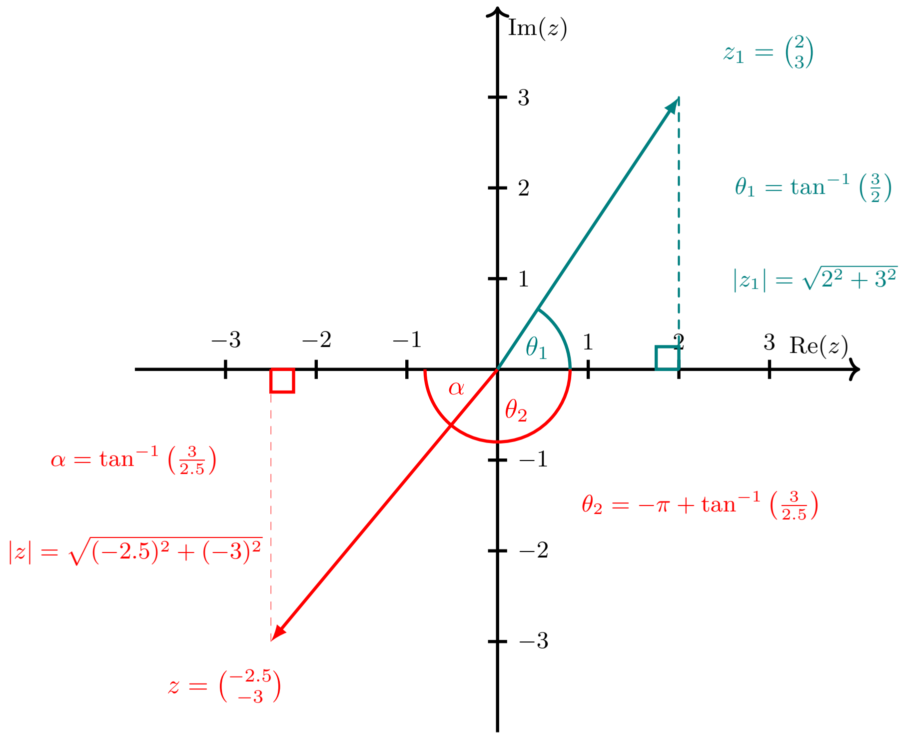
Alternatively, we can represent the complex number $2 + 3i$ as a position vector. It turns out that every complex number can be represented as a vector in the complex plane.[4] In facts, real numbers can also be represented as vectors.

With the set of real numbers, we can identify any real number by knowing its magnitude and sign. For example, if we have some real number with a length of 7 and a negative sign, it can only be -7. With complex numbers, it's different; since we are dealing with a plane, we must have the magnitude of the complex number and the angle it makes (clockwise or counter-clockwise, will

[4] This is due to complex numbers satisfying the vector space axioms, though this is extra knowledge.



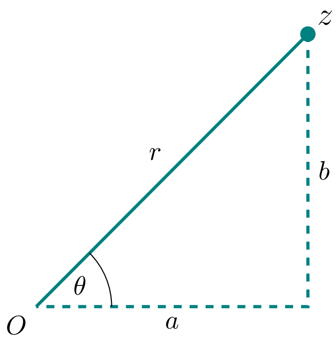
be discussed soon) with the positive or negative real axis. By convention, to identify a complex number, we take the angle made with the positive real axis and the counter-clockwise direction as positive with $-\pi < \theta \leq \pi$ unless specified otherwise.



This leads us to new notation:

- The angle a complex number (represented as a vector) makes with the positive real axis is called *the argument of* z and is denoted by $\arg(z)$
- The magnitude (length) of a complex number (represented as a vector) is called *the modulus of* z and is denoted by $|z|$

Other than the usual way of expressing a complex number ($z = a + ib$) there is another form known as the modulus argument form which, as the name suggests, expresses a complex number using its modulus and argument.



Suppose we have some complex number $z = a + ib$ such that $|z| = r$ and $\arg(z) = \theta$. Since $a = r\cos\theta$ and $b = r\sin\theta$, it is then true that

$$z = r\cos\theta + ri\sin\theta$$

or more simply

$$z = r\left(\cos\theta + i\sin\theta\right).$$

There is also another very important form to express some complex number z: the exponential form. In the early 1740's, Euler discovered a fundamental relationship in complex numbers:

$$z = re^{i\theta} = r\left(\cos\theta + i\sin\theta\right),^{5}$$

where $\theta = \arg(z)$ and $r = |z|$. Using the expenential form of complex numbers, we can establish some important results. *Let* $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$.

a)

$$z_1 \cdot z_2 = r_1e^{i\theta_1} \cdot r_2e^{i\theta_2}$$

$$z_1 \cdot z_2 = (r_1 \cdot r_2)e^{i(\theta_1+\theta_2)}$$

Hence,

i)

$$|z_1 \cdot z_2| = |z_1||z_2|$$
$$|z_1 \cdot z_2| = r_1 \cdot r_2$$

ii)

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$
$$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2$$

b)

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)e^{i(\theta_1-\theta_2)}$$

Hence,

[5] Euler's formula is a direct consequence of the Maclaurin series expansions of e^x, $\sin x$, and $\cos x$

i)

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 \cdot z_2| = r_1 \cdot r_2$$

ii)

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

We will define the reflection of z about the real axis as z^*. This would look like this

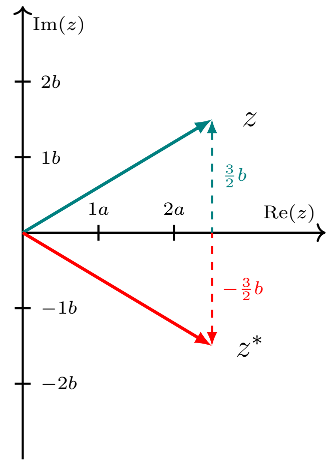


Figure 1: The Complex Numbers z and z^* plotted on an Argand Diagram.

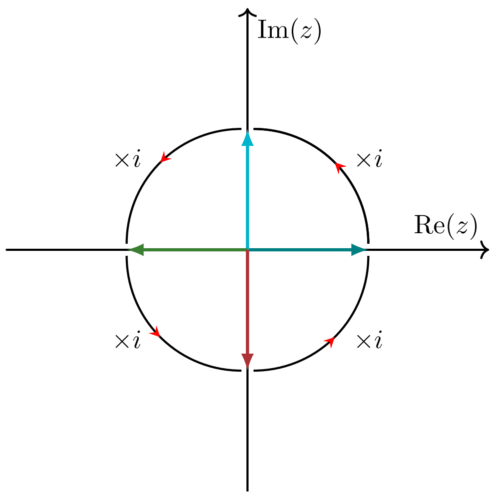
Algebraically, if $z = a + ib$ then $z^* = a - ib$. z^* is known as the complex conjugate of z. You might wonder why the complex conjugate of z is so important that it gets its own symbol: z^* is significant because the product of z and z^* happens to be a real number, which is an important fact in proofs (will not be discussed) and even computation (will be discussed). That is,

$$\begin{aligned}
z \cdot z^* &= (x + iy)(x - iy) \\
&= (x)^2 - (iy)^2 \\
&= x^2 - i^2 \cdot y^2 \\
&= x^2 - \left(\sqrt{-1}\right)^2 y^2 \\
&= x^2 - (-1)y^2 \\
z \cdot z^* &= x^2 + y^2.
\end{aligned}$$

Before jumping into operations with complex numbers, we want to discuss one more thing.

Multiplying by i has the consequence of rotating our vector (complex number) $\frac{\pi}{2}$ radians anti-clockwise centered at the origin.[6] Pretty cool no?

[6]Matrices actually related to complex numbers, especially in complex analysis, but that is way above A level Math and Further Math. You can actually relate i with the rotation matrix



3.3 Operations with complex numbers

We will not spend too much time on this since this is easy.

3.3.1 Addition and subtraction

Adding and subtracting complex number is straightforward: we add the real parts of the complex numbers together and do the same for the imaginary parts

Example 1:

$$(12 - 3i) + (6 + 9i) = (12 + 6) + (9i - 3i)$$
$$= 18 + 6i$$

Example 2:

$$(12 - 3i) - (-8 + 5i) = 12 - 3i + 8 - 5i$$
$$= 20 - 8i$$

3.3.2 Multiplication and division

Multiplication is also quite straightforward: the only thing extra which you must utilize is the fact that $i^2 = -1$ and that a fraction in its simplest form must not contain any imaginary numbers in the denominator (we make use of the fact that $z \cdot z^* = x^2 + y^2$).

Example 1:

$$\begin{aligned}(2 + 6i)(-6 - 5i) &= 2 \cdot -6 + 2 \cdot -5i + 6i \cdot -6 + 6i \cdot -5i \\ &= -12 - 10i - 36i - 30i^2\end{aligned}$$

since $i^2 = -1$

$$\begin{aligned}&= -12 - 30 \cdot -1 - 46i \\ &= 18 - 46i\end{aligned}$$

Example 2:

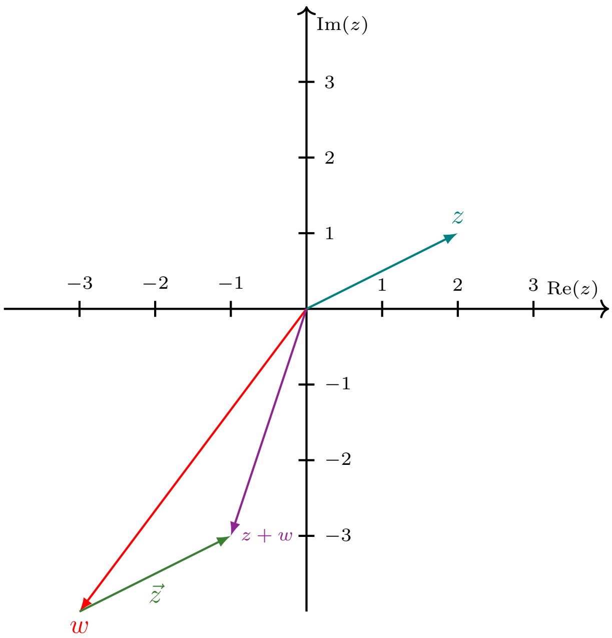
$$\begin{aligned}\frac{5-2i}{3+i} &= \frac{5-2i}{3+i}\times\frac{3-i}{3-i}\\ &= \frac{(5-2i)(3-i)}{(3+i)(3-i)}\\ &= \frac{13-11i}{3^2-i^2}\\ &= \frac{13-11i}{10}\end{aligned}$$

Complex numbers follow the rules of vector addition; we can easily see this using the following example.

Let $z = 2 + i$ and $w = -3 - 4i$. Algebraically, $z + w = -1 - 3i$. Using vectors,

$$z + w = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-3\\-4\end{pmatrix} = \begin{pmatrix}-1\\-3\end{pmatrix} = -1 - 3i.$$

On our Argand diagram:



Realistically, you wont be dealing with complex numbers as vectors (explicitly) when it comes to operations. The geometric effect of multiplication and division are not needed.

3.4 Polynomials and complex numbers

By the Complex conjugate root theorem, if $p(x)$ is a polynomial with real coefficients and z is a root of the equation- that is, $p(z) = 0$- then z^* must also be a root of $p(z)$- that is, $p(z^*) = 0$. This is evidently true in quadratics: when solving the roots of a quadratic, the only possible way to get an imaginary number, consequentially a complex number, is if the discriminant is less than zero.

Example 1: Find the roots of the quadratic function $f(x) = 2x^2 + 6x + 9$

Solution:

For our roots x_1 and x_2, we will use the quadratic formula.

$$\begin{aligned}
x_1 &= \frac{-(6)}{2(2)} + \frac{\sqrt{(6)^2 - 4(2)(9)}}{2(2)} \\
&= -\frac{3}{2} + \frac{\sqrt{-9}}{2} \\
&= -\frac{3}{2} + \frac{\sqrt{-1 \times 9}}{2} \\
&-\frac{3}{2} + \frac{3}{2}\sqrt{-1} \\
x_1 &= -\frac{3}{2} + \frac{3}{2}i
\end{aligned}$$

Similarly,

$$\begin{aligned}
x_2 &= \frac{-(6)}{2(2)} - \frac{\sqrt{(6)^2 - 4(2)(9)}}{2(2)} \\
&= -\frac{3}{2} - \frac{\sqrt{-9}}{2} \\
&= -\frac{3}{2} - \frac{\sqrt{-1 \times 9}}{2} \\
&-\frac{3}{2} - \frac{3}{2}\sqrt{-1} \\
x_2 &= -\frac{3}{2} - \frac{3}{2}i
\end{aligned}$$

It is obvious from the example that the $\pm$ is the reason for our roots being a complex number and its conjugate. However, this is true for *all* polynomials with real coefficients! The fact that if z is a root to some polynomial with real coefficent implies that $z*$ is also a root. The usefulness of this fact will now be portrayed. Remember that an n^{th} degree polynomial with real coefficients has n solutions (both real and complex).

Example 2: Given than $p(x) = x^3 + 5x^2 + 31x + 75$ and $x = -1 + 2\sqrt{6}i$ is a solution for $p(x) = 0$, factorise $p(x)$ completely.

Solution:

If $x = -1 + 2\sqrt{6}i$ is a solution for $p(x) = 0$ then $x = -1 - 2\sqrt{6}i$ is another solution for $p(x) = 0$. Hence, $\left(x - (-1 + 2\sqrt{6}i)\right)$, $\left(x - (-1 - 2\sqrt{6}i)\right)$ are factors of $p(x)$. Therefore,

$$\left(x - \left(-1 + 2\sqrt{6}i\right)\right)\left(x - \left(-1 - 2\sqrt{6}i\right)\right)$$

is a factor of $p(x)$. Letting $z = -1 + 2\sqrt{6}i$

$$\begin{aligned}
\left(x - \left(-1 + 2\sqrt{6}i\right)\right)\left(x - \left(-1 - 2\sqrt{6}i\right)\right) &= (x - z)(x - z^*) \\
&= x^2 - xz^* - xz + zz^* \\
&= x^2 - x(z + z^*) + (1)^2 + (2\sqrt{6})^2 \\
&= x^2 - x(-2) + 1 + 4 \times 6 \\
\left(x - \left(-1 + 2\sqrt{6}i\right)\right)\left(x - \left(-1 - 2\sqrt{6}i\right)\right) &= x^2 + 2x + 25
\end{aligned}$$

Now, all that is left to find the linear factor of $p(x)$. You can do this by long division or comparing coefficients. We will do the latter.

$$p(x) = (ax + b)(x^2 + 2x + 25)$$

You don't need to fully factorise! Only find the relevant terms.

$$x^3 + 5x^2 + 31x + 75 = ax^3 + \cdots + 25b$$

Therefore,

$$\begin{aligned} ax^3 &= x^2 \\ \rightsquigarrow a &= 1. \end{aligned}$$

And,

$$\begin{aligned} 25b &= 75 \\ \rightsquigarrow b &= 3. \end{aligned}$$

Therefore,

$$p(x) = (x + 3)(x^2 + 2x + 25).$$

This leads us to an important case: finding the square roots of a complex number.

3.4.1 Square roots of complex numbers

Question: Find the square roots of $z = 10 - 4\sqrt{6}i$

Solution: *We will skip some steps regarding algebraic operations.* Let $w = a + bi$ some square root of z where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

$$\begin{aligned} (a + bi)^2 &= 10 - 4\sqrt{6}i \\ a^2 + 2abi - b^2 &= 10 - 4\sqrt{6}i \\ a^2 - b^2 &= 10 \qquad (1) \\ 2ab &= -4\sqrt{6} \qquad (2) \\ &\vdots \\ a_1 = \sqrt{2}i &\qquad b_1 = 2\sqrt{3}i \\ a_2 = -\sqrt{2}i &\qquad b_2 = -2\sqrt{3}i \\ a_3 = -2\sqrt{3} &\qquad b_3 = \sqrt{2} \\ a_4 = 2\sqrt{3} &\qquad b_4 = -\sqrt{2} \end{aligned}$$

But a and b are real numbers, so

$$w_1 = -2\sqrt{3} + \sqrt{2}i \qquad w_2 = 2\sqrt{3} - \sqrt{2}i$$

Note that with exponential form, just apply the usual rules of indices.

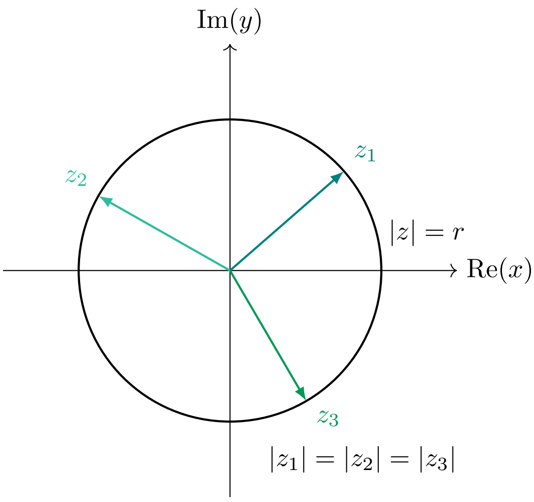
3.5 Loci

A ”locus” (plural: ”loci”) refers to a set of points that satisfy a particular geometric or algebraic condition. Our work with loci will be on the Argand diagram for complex numbers. We will deal with the following loci,

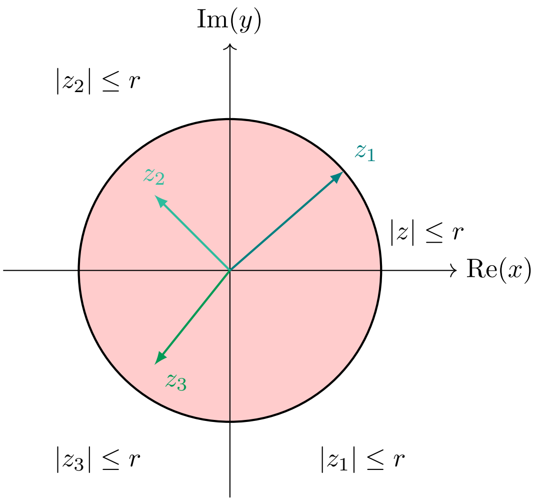
1. Circles with centre $(0, 0)$
2. Circles with centre (a, b)
3. Half-lines and part-lines
4. Perpendicular bisectors

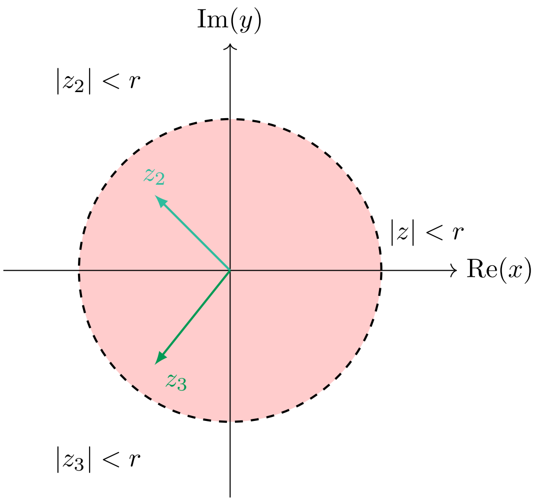
3.5.1 Circles with centre $(0, 0)$

A circle is a trace of all the points r units from some point which we call its center. Let z be some complex number. If we consider all the complex numbers z such that $|z| = r$, a trace of a circle with radius r will be mapped out.



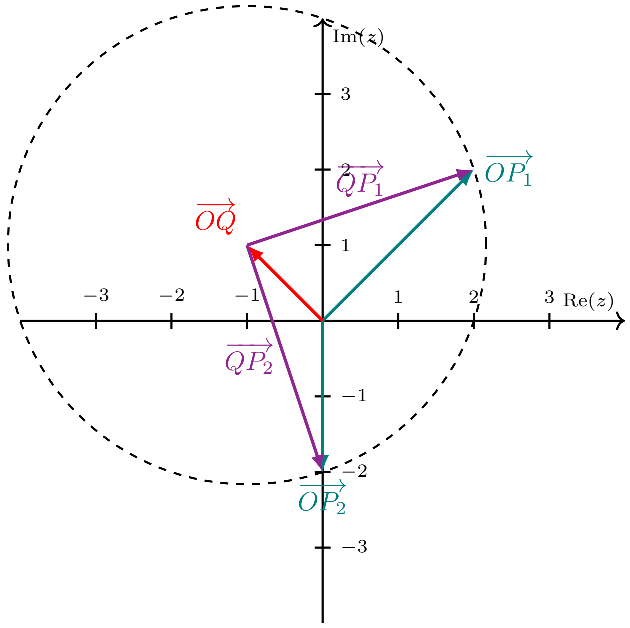
We can also extend this to regions instead of points by the use of inequality signs. Note that we do not consider z_1 in the strict inequality.





3.5.2 Circles with centre (a,b)

We will make heavy use of vectors here



We know from our previous work that $\overrightarrow{QP_n} = \overrightarrow{OP_n} - \overrightarrow{OQ}$. Now we will be a little bit laid back when it comes to alternating between vectors and complex numbers since it would be difficult to prove our transitions from vectors to complex numbers. The vector $\overrightarrow{OQ}$ is known and constant; however, the vector $\overrightarrow{OP_n}$ is an arbitrary vector on a circle with a radius r and center (a, b). $|\overrightarrow{QP_n}| = r$ and is constant. Therefore,

$$|\overrightarrow{QP_n}| = r$$
$$\therefore |\overrightarrow{OP_n} - \overrightarrow{OQ}| = r$$

Let $\overrightarrow{OP_n} = z$ where z is an arbitrary complex number on our circle, and $\overrightarrow{OQ} = z_1$.

$$|z - z_1| = r$$

Where z_1 is the complex number representing our circle's center.

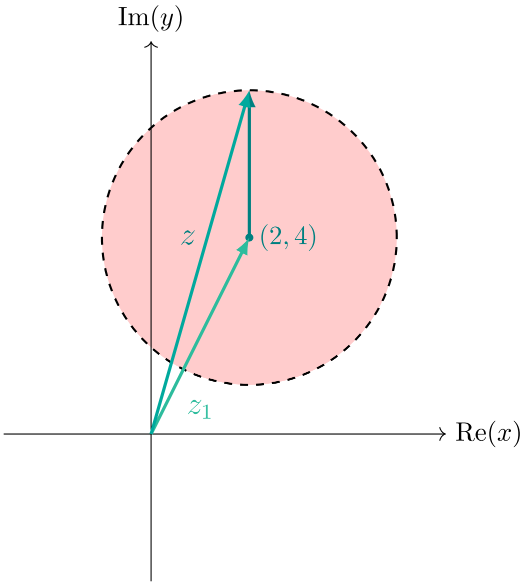
To sum things up, $|z - z_1| = r$ is the trace of all complex numbers z that are a distance r from z_1. The points, once traced form a circle. Again, we can employ inequalities to outlines regions in the same matter as a circle with a center $(0, 0)$

Question 1: Sketch the locus $|z - 2 - 4i| < 3$.

Solution: Let us first set up the inequality in the form $|z - z_1| < r$.

$$|z - 2 - 4i| < 3$$
$$|z - (2 + 4i)| < 3$$

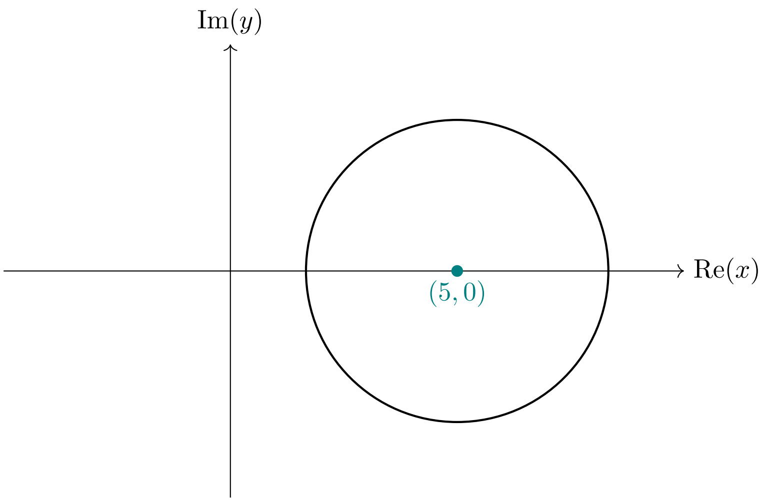
Hence, we have the region enclosed within a circle centered at $(2, 4)$ with a radius of 3



We only added vectors to clear somethings up but they are not needed at all.

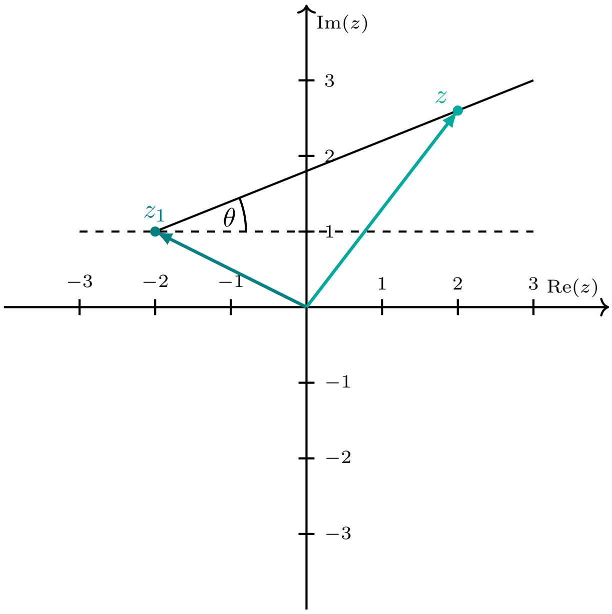
Question 2: Sketch the locus $|z - 5| = 2$

Solution: We have a circle centered at $(5, 0)$ with a radius of 2. Sketching it should be fairly straightforward.



3.5.3 Half-lines and part-lines

In geometry, a *half-line* is a one-dimensional geometric object that starts at a specific point and extends infinitely in one direction. A half-line is also referred to as a *ray*.[7] Note that the direction



vector between the point z_1 and z (which is some arbitrary complex number) is the same for all z; therefore, the argument of z_1 will be equal to the direction vector of $\overrightarrow{z_1 z}$ and hence constant, where z_1 is some fixed complex number and z is some arbitrary constant number.

Therefore,

$$\arg(z - z_1) = \theta$$

is some half-line starting at z_1 and extending to infinity θ degrees anti-clockwise from the positive horizontal subtended from z_1.

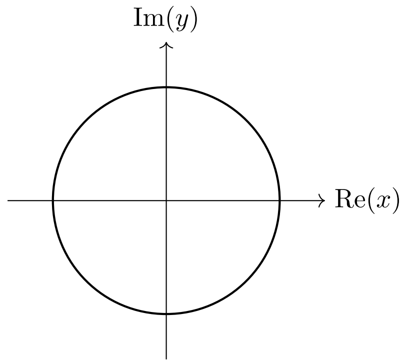
Example 1: On a single Argand diagram, sketch the loci $|z| = 5$ and $\arg(z + 2 + 2i) = \frac{\pi}{4}$.

Show that there is only one complex number, z, that satisfies both loci. Label this point as P on your diagram.

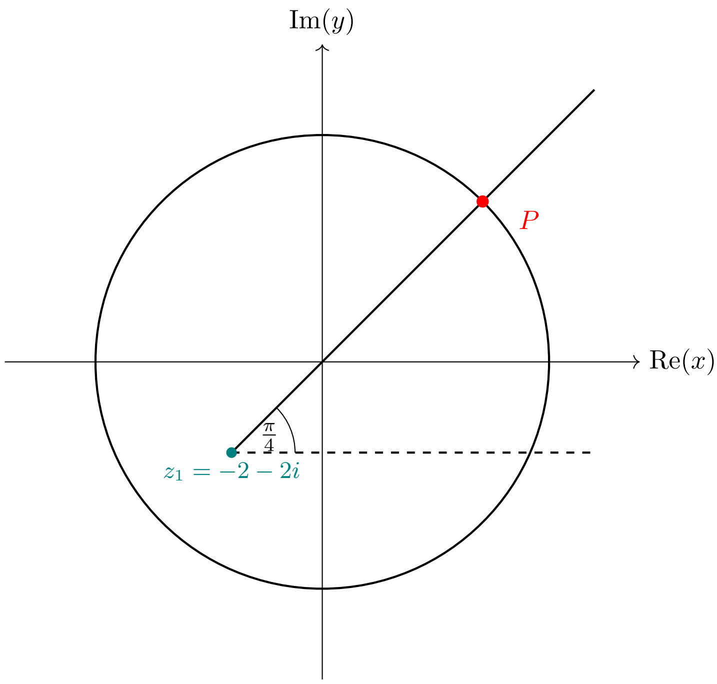
Solution:

Beginning with $|z| = 5$, we have a circle centered at the origin with a radius of 5.

[7]Note that a half-line (or a ray) is simply an infinite collection of points too!



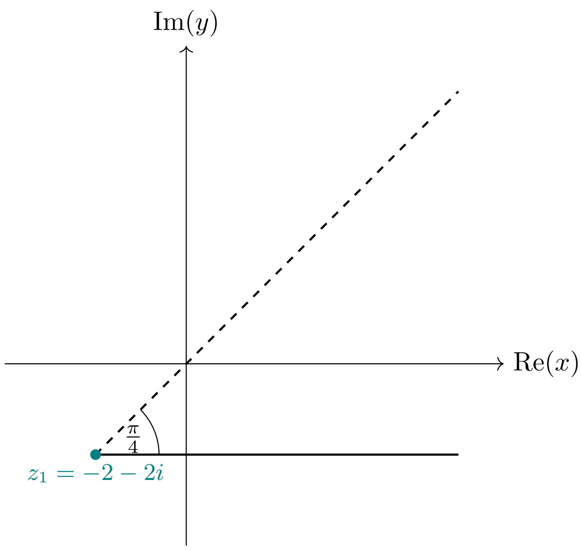
For $\arg(z+2+2i) = \dfrac{\pi}{4}$, let us rewrite this in the optimum form, which is $\arg(z-(-2-2i)) = \dfrac{\pi}{4}$. This is a half line starting at $z = -2-2i$ and extending to infinity, making an angle of $\frac{\pi}{4}$ radians with the positive horizontal extending from $z = -2-2i$. Plotting this on the same diagram, we get



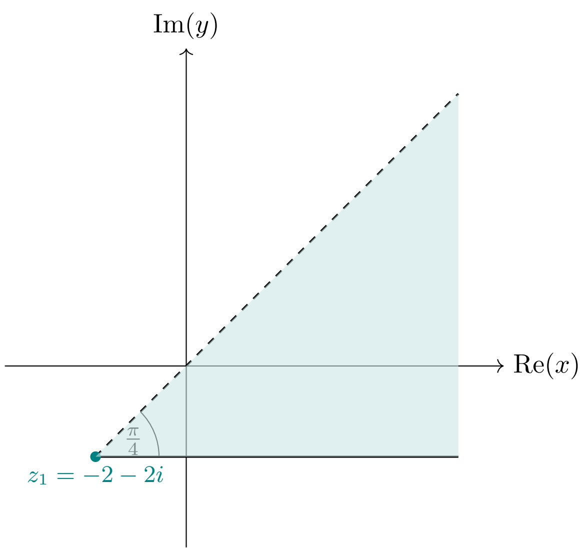
Example 2: Shade the region satisfying the inequality $0 \leq \arg(z + 2 + 2i) < \dfrac{\pi}{4}$

Solution:

First, we sketch the half line, which has been done in the previous part.



The inequality translates to all vectors starting at $-2-2i$ such that their direction vector makes an angle greater than or equal to 0 but less that $\frac{\pi}{4}$ radians with the positive horizontal. If we were to shade this, we will get the following.

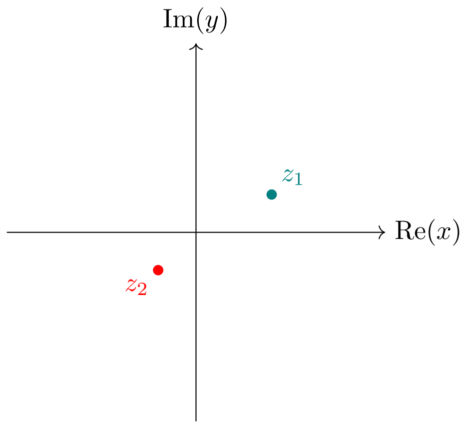


3.5.4 Perpendicular bisectors

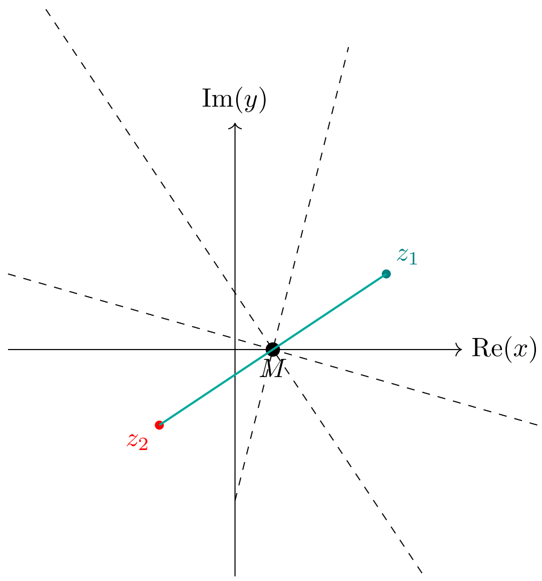
We have previously established that $z - z_1$ is the collection of all vectors on the complex plane starting at z_1 where z is *any* complex number.[8] Similarly, $z - z_2$ is the collection of all vectors on the complex plane starting at z_2 where z follows the same previous definition.

[8]We previously restricted $z - z_1$ to one direction vector; hence the collection of z too by mentioning the argument.

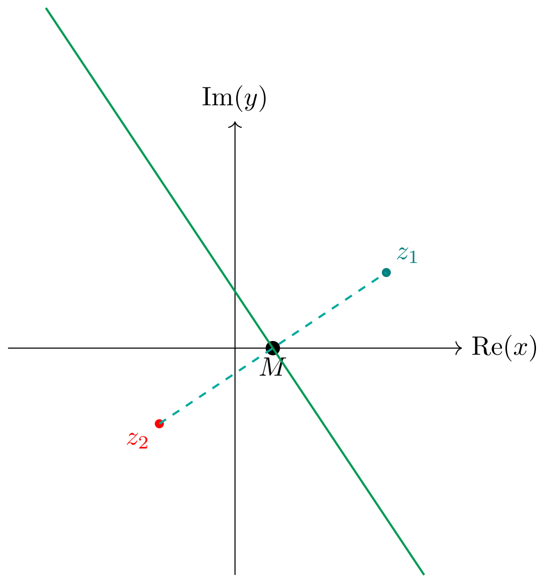
If we were to take $|z - z_1| = |z - z_2|$, we are looking at the collection of complex numbers (points) z such that the distance between them and each starting point (z_1 and z_2) is equal. Before drawing the loci, let us plot down z_1 and z_2 (we will take some random complex number).



We know that one point on the loci is the midpoint between z_1 and z_2. Let us plot it down and call it M



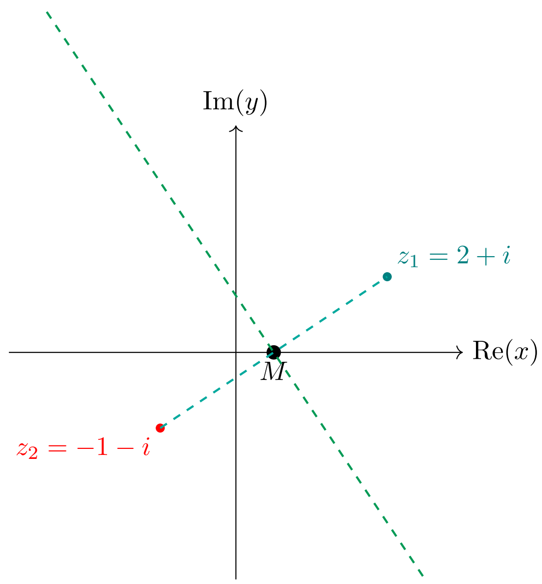
It is easy to see that the line (collection of points) that satisfies $|z - z_1| = |z - z_2|$ is the lines perpendicular to the one passing through z_1 and z_2. Any other line will have some "tilt" towards one of the points. Hence, our loci is the line perpendicular to the one passing through z_1 and z_2.



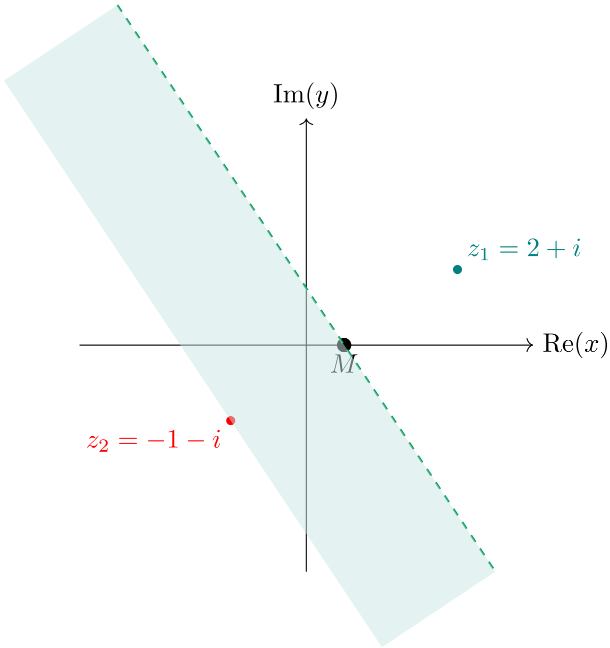
Example: Sketch the locus $|z + 1 + i| < |z - 2 - i|$

Solution:

Rewriting in the proper form, we get $|z - (-1 - i)| < |z - (2 + i)|$. Sketching we get



we want the distance between any point and $2 + i$ ($|z - (2 + i)|$) to be greater than the distance between any point and $-1 - i$ ($|z - (-1 - i)|$); this area would be to the left of the perpendicular line. Shading we get



Of course we should cover the area behind z_2 but we don't want to clutter the diagram more.

4 Cambridge FP2 & Edexcel FP2

The following content will *all* be consequences of De Moivre's Theorem.

4.1 Defining De Moivre's Theorem

De Moivre's Theorem: De Moivre's Theorem states that for any real number r and any integer n, the n^{th} power of a complex number z can be calculated as:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

where r is the magnitude of the complex number, θ is the argument of the complex number, and n is the exponent to which the complex number is raised. Proving De Moivre's theorem for positive integers is required by the Cambridge syllabus. Edexcel syllabus requires both positive and negative integers.

Proof of De Moivre's Theorem for Positive Integer Exponents by Mathematical Induction

We want to prove that for any real number r and any positive integer n, the n^{th} power of a complex number z can be written as follows:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Base Case ($n = 1$): For $n = 1$, we have:

$$\begin{aligned}(r(\cos\theta + i\sin\theta))^1 &= r(\cos\theta + i\sin\theta)\\ &= r^1(\cos(1\times\theta) + i\sin(1\times\theta))\end{aligned}$$

Since the $LHS = RHS$ when $n = 1$, the given statement is true for $n = 1$.

Inductive Hypothesis (Assume true for $n = k$): Assume that for some positive integer k, De Moivre's Theorem holds:

$$(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta)$$

Inductive Step (Prove true for $n = k+1$): Now, we want to show that De Moivre's Theorem holds for $n = k + 1$. We have:

$$\begin{aligned}
(rcis)^{k+1} &= (rcis)^k \cdot (rcis) \\
&= \left(r^k(\cos(k\theta) + i\sin(k\theta))\right) \cdot (rcis) \\
&= r^{k+1}(\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + ir^{k+1}(\cos(k\theta)\sin(\theta) + \sin(k\theta)\cos(\theta)) \\
&= r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))
\end{aligned}$$

Where $rcis = r(\cos\theta + i\sin\theta)$

The last step follows from the angle addition formulas for sine and cosine.

So, by mathematical induction, De Moivre's Theorem holds for all positive integers n.

Negative integers For negative integers, let n represent a positive integer and z some complex number.

$$\begin{aligned}
z^{-n} &= \frac{1}{z^n} \\
&= \frac{1}{r^n} \times \frac{1}{\cos(n\theta) + i\sin(n\theta)} \\
&= r^{-n} \times \frac{1}{\cos(n\theta) + i\sin(n\theta)} \times \overbrace{\frac{\cos(n\theta) - i\sin(n\theta)}{\cos(n\theta) - i\sin(n\theta)}}^{\text{the complex conjugate}} \\
&= r^{-n} \times \frac{\cos(n\theta) - i\sin(n\theta)}{\cos^2(n\theta) + \sin^2(n\theta)}
\end{aligned}$$

Since $\cos\theta = \cos -\theta$, $-\sin\theta = \sin -\theta$ and $\cos^2 x + \sin^2 x = 1$,

$$z^{-n} = r^{-n}(\cos(-n\theta) + i\sin(-n\theta)) \quad \square$$

4.2 Applying De Moivre's theorem

We will almost always be using some, or most of the following results.

- $\mathrm{Re}(\cos\theta + i\sin\theta) = \mathrm{Re}(e^{i\theta}) = \cos\theta$.
- $\mathrm{Im}(\cos\theta + i\sin\theta) = \mathrm{Im}(e^{i\theta}) = \sin\theta$.
- $e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$.
- $e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2\sin\theta$.

Since we will be talking about applications of De Moivre's theorem, we will explain things with examples.

4.2.1 Ratios of trigonometric functions and solving polynomials

Expressing $\cos n\theta$ in terms of $\cos\theta$: We will do this when $n = 4$; however, the procedure is the same for all integer values of n i.e. where De Moivre's theorem applies.

To express $\cos 4\theta$ in terms of $\cos\theta$:

We know that

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$

but also,

$$(\cos\theta + i\sin\theta)^4 = c^4 + 4ic^3\mathrm{s} - 6\mathrm{c}^2\mathrm{s}^2 - 4\mathrm{cs}^3 + \mathrm{s}^4$$

Therefore,

$$\cos 4\theta + i\sin 4\theta = c^4 + 4ic^3\mathrm{s} - 6\mathrm{c}^2\mathrm{s}^2 - 4\mathrm{cs}^3 + i\mathrm{s}^4$$

From this, we can say that

$$\begin{aligned}
\mathrm{Re}(\cos 4\theta + i\sin 4\theta) &= \mathrm{Re}(c^4 + 4ic^3\mathrm{s} - 6\mathrm{c}^2\mathrm{s}^2 - 4i\mathrm{cs}^3 + \mathrm{s}^4) \\
\cos 4\theta &= \mathrm{c}^4 - 6\mathrm{c}^2\mathrm{s}^2 + \mathrm{s}^4 \\
&= \mathrm{c}^4 - 6\mathrm{c}^2(1 - \mathrm{c}^2) + (1 - \mathrm{c}^2)^2 \\
&= \mathrm{c}^4 - 6\mathrm{c}^2 + 6\mathrm{c}^4 + \mathrm{c}^4 - 2\mathrm{c}^2 + 1 \\
\cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

Expressing $\sin n\theta$ in terms of $\sin\theta$: The procedure is identical to the cosine variant, but we take the imaginary part of the resulting complex number instead of the real one.

Expressing $\tan n\theta$ in terms of $\tan\theta$: This procedure is a little bit different and will extend to another concept. We will explain this using $n = 5$.

We know that

$$\begin{aligned}
\tan 5\theta &= \frac{\sin 5\theta}{\cos 5\theta} \\
&= \frac{\mathrm{Im}(\cos 5\theta + i\sin 5\theta)}{\mathrm{Re}(\cos 5\theta + i\sin 5\theta)}
\end{aligned}$$

Using our previous technique,

$$\begin{aligned}
\tan 5\theta &= \frac{5\mathrm{c}^4\mathrm{s} - 10\mathrm{s}^3\mathrm{c}^2 + \mathrm{s}^5}{\mathrm{c}^5 - 10\mathrm{c}^3\mathrm{s}^2 + 5\mathrm{s}^4\mathrm{c}} \\
&= \frac{5\mathrm{c}^4\mathrm{s} - 10\mathrm{s}^3\mathrm{c}^2 + \mathrm{s}^5}{\mathrm{c}^5 - 10\mathrm{c}^3\mathrm{s}^2 + 5\mathrm{s}^4\mathrm{c}} \div \frac{\mathrm{c}^5}{\mathrm{c}^5} \\
&= \frac{5\cancel{\mathrm{c}^4}\frac{\mathrm{s}}{\mathrm{c}} - 10\frac{\mathrm{s}^3}{\mathrm{c}^3}\cancel{\mathrm{c}^2} + \frac{\mathrm{s}^5}{\mathrm{c}^5}}{1\cancel{\mathrm{c}^5} - 10\cancel{\mathrm{c}^3}\frac{\mathrm{s}^2}{\mathrm{c}^2} + 5\frac{\mathrm{s}^4}{\mathrm{c}^4}\cancel{\mathrm{c}}}
\end{aligned}$$

using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

This leads us to a new type of question: solving polynomials.

Solving polynomials using trigonometric ratios and De Moivre's theorem :

Question: Express $\tan 4\theta$ in terms of $\tan\theta$. Hence, find the solutions of the equation

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0.$$

Solution:

We will skip the part of expressing $\tan 4\theta$ in terms of $\tan\theta$ as the procedure is identical to what we did in the previous question. We will focus on the second part of the question. After working it out,

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Hence implies that we have to use the previous part. Since we are dealing with two equations, we have to find a way such that they *correspond* to each other. Therefore, we have to transform the second equation into something *similar* to the first equation.

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

We notice that we have some common coefficients with the numerator

$$\begin{aligned} x^4 - 6x^2 + 1 &= 4x - 4x^3 \\ \frac{4x - 4x^3}{x^4 - 6x^2 + 1} &= 1 \end{aligned}$$

let $x = \tan\theta$

$$\rightsquigarrow \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$$

but $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$; so,

$$\begin{aligned} \tan 4\theta &= 1 \\ 4\theta &= \frac{\pi}{4}, \frac{5}{4}\pi, \frac{9}{4}\pi, \frac{13}{4}\pi \\ \theta &= \frac{\pi}{16}, \frac{5}{16}\pi, \frac{9}{16}\pi, \frac{13}{16}\pi \end{aligned}$$

plugging back to find x in terms of $\tan q\pi$

$$x = \tan q\pi$$

where

$$q = \frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16}$$

In the exam, you will first be asked to find the relevant trigonometric relationship and then solve some related polynomial equation.

4.2.2 From powers of sine and cosine to multiples of the fundamental angle

In this section, we will assume that $z = \cos\theta + i\sin\theta$, taking $r = 1$ always. We will use these results.

- $z + z^{-1} = 2\cos\theta \rightsquigarrow \left(z + z^{-1}\right)^n = (2\cos\theta)^n$
- $z - z^{-1} = 2i\sin\theta \rightsquigarrow \left(z - z^{-1}\right)^n = (2i\sin\theta)^n$
- $z^n + z^{-n} = 2\cos n\theta$
- $z - z^{-1} = 2i\sin n\theta$

For the 3$^{\text{rd}}$ and 4$^{\text{th}}$ results, they are a direct consequence of De Moivre's theorem:

$$\begin{aligned} z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ z^n + z^{-n} &= 2\cos n\theta \quad \square \end{aligned}$$

We will leave the 4$^{\text{th}}$ result for you to prove.

How can we use this?

Question: Express $\sin^4\theta$ in terms of $\cos n\theta$ for integer values of n.

Solution:

We know that

$$2i\sin\theta = z - z^{-1}$$

Hence,

$$\begin{aligned} (2i\sin\theta)^4 &= \left(z - z^{-1}\right)^4 \\ &= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4} \\ &= (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6 \\ 16\sin^4\theta &= 2\cos 4\theta - 8\cos 2\theta + 6 \\ \sin^4\theta &= \frac{1}{8}\left(\cos 4\theta - 4\cos 2\theta + 3\right) \end{aligned}$$

Let us try this with $\cos^4\theta$ with more brevity.

Question: Express $\sin^4\theta$ in terms of $\cos n\theta$ for integer values of n.

Solution:

$$\begin{aligned} (2\cos\theta)^4 &= (z + z^{-1})^4 \\ 16\cos^4\theta &= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} \\ &= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6 \\ 16\cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6 \\ \cos^4\theta &= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) \end{aligned}$$

Nearly identical!

But what can we do with this?

Question: Using your previous result, find $\int \cos^4\theta \, \mathrm{d}\theta$

Technically, you *can* use integration by parts and you technically *can* drink cement; you don't want to try doing that. You can also use the reduction formula of $\cos^n\theta$ but you probably don't have that on hand. Anyways, the question did ask us to use our previously established result, which happens to be the easiest way.

$$\begin{aligned}
\int \cos^4\theta \, \mathrm{d}\theta &= \int \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right) \mathrm{d}\theta \\
&= \frac{1}{8}\int \left(\cos 4\theta + 4\cos 2\theta + 3\right) \mathrm{d}\theta \\
&= \frac{1}{8}\left(\frac{\sin 4\theta}{4} + \frac{4\sin 2\theta}{2} + 3\theta\right) + C \\
\int \cos^4\theta \, \mathrm{d}\theta &= \frac{1}{32}(\sin 4\theta + 8\sin 2\theta + 12\theta) + C
\end{aligned}$$

This is one application of expressing powers of trigonometric functions in terms of multiples of the fundamental angle.

We will do one more problem involving the exponential form of a z:

- $z + z^{-1} = e^{i\theta} + e^{-i\theta}$
- $z - z^{-1} = e^{i\theta} - e^{-i\theta}$
- $z^n + z^{-n} = e^{in\theta} + e^{-in\theta}$
- $z^n - z^{-n} = e^{in\theta} - e^{-in\theta}$

Question: Express $\cos^4\theta - \sin^4\theta$ in terms of exponents.

Solution:

For $\cos^4\theta$

$$\begin{aligned}
2\cos\theta &= e^{i\theta} + e^{-i\theta} \\
(2\cos\theta)^4 &= \left(e^{i\theta} + e^{-i\theta}\right)^4 \\
&= e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta} \\
&= \left(e^{4i\theta} + e^{-4i\theta}\right) + 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6 \\
\cos^4\theta &= \frac{1}{16}\left(\left(e^{4i\theta} + e^{-4i\theta}\right) + 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6\right)
\end{aligned}$$

We will stop here since we require exponents not trigonometric functions.

For $\sin^4 \theta$

$$\begin{aligned}
2\sin\theta &= e^{i\theta} - e^{-i\theta} \\
(2\sin\theta)^4 &= \left(e^{i\theta} - e^{-i\theta}\right)^4 \\
&= e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta} \\
&= \left(e^{4i\theta} + e^{-4i\theta}\right) - 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6 \\
\sin^4\theta &= \frac{1}{16}\left(\left(e^{4i\theta} + e^{-4i\theta}\right) - 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6\right)
\end{aligned}$$

Subtracting $\sin^4 \theta$ from $\cos^4 \theta$

$$\begin{aligned}
\cos^4\theta - \sin^4\theta &= \frac{1}{16}\left(\left(e^{4i\theta} + e^{-4i\theta}\right) + 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6\right) \\
&- \frac{1}{16}\left(\left(e^{4i\theta} + e^{-4i\theta}\right) - 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6\right) \\
&= \frac{1}{16}\left[8\left(e^{2i\theta} + e^{-2i\theta}\right)\right] \\
\cos^4\theta - \sin^2\theta &= \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)
\end{aligned}$$

4.2.3 Complex summation

When it comes to infinite series (and series in general), you have dealt with telescoping series, arithmetic series and geometric series. Today we will deal with sums of trigonometric functions.

We will link geometric series with sums of trigonometric functions using complex numbers.

A kind reminder of the following:

- For a geometric series:
  1. $u_n = ar^{n-1}$
  2. $S_n = \frac{a(1-r^n)}{1-r}$
  3. $S_\infty = \frac{a}{1-r}$ $(|r| < 1)$

The main idea is to relate the trigonometric function we are summing with z and the geometric series.

Let's see this with a question.

Question: Find

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \cos\left(\frac{n\pi}{3}\right)$$

Solution:

Let the infinite sum equal S. Laying out the terms we get

$$S = \frac{2}{3}\cos\left(\frac{\pi}{3}\right) + \cdots$$

This doesn't simplify it in any way.

Consider the following. Let $z = \frac{2}{3}\left(\cos n\frac{\pi}{3} + i\sin n\frac{\pi}{3}\right)$. It is then true that $\mathrm{Re}(z) = \frac{2}{3}\cos n\frac{\pi}{3}$, and $\mathrm{Re}(z^n) = \left(\frac{2}{3}\right)^n \cos n\frac{\pi}{3}$. Hence,

$$\mathrm{Re}\left(\sum_{n=1}^{\infty} z\right) = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n \cos\left(\frac{n\pi}{3}\right)$$

Now, let's work with $\displaystyle\sum_{n=1}^{\infty} z$

$$\begin{aligned}\sum_{n=1}^{\infty} z &= z + z^2 + \cdots \\ &= \frac{z}{1-z}\end{aligned}$$

To find $\mathrm{Re}\left(\sum_{n=1}^{\infty} z^n\right)$ we have to simplify things a bit further.

$$\sum_{n=1}^{\infty} z = \frac{1}{1-z} - 1$$

Multiplying by the complex conjugate to obtain a real denominator and a complex numerator that we can split into real and imaginary parts,

$$\begin{aligned}&= \frac{1}{1-\frac{2}{3}e^{i\frac{\pi}{3}}} \times \frac{1-\frac{2}{3}e^{-i\frac{\pi}{3}}}{1-\frac{2}{3}e^{-i\frac{\pi}{3}}} - 1 \\ &= \frac{1-\frac{2}{3}e^{-i\frac{\pi}{3}}}{1+\frac{4}{9}-\frac{2}{3}\left(e^{i\frac{\pi}{3}}+e^{-i\frac{\pi}{3}}\right)} - 1\end{aligned}$$

does $e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{3}}$ remind you of something?

$$= \frac{1-\frac{2}{3}\cos\frac{\pi}{3}+\frac{2}{3}i\sin\frac{\pi}{3}}{\frac{13}{9}-\frac{4}{3}\cos\frac{\pi}{3}} - 1$$

Taking the real part.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\cos\left(\frac{n\pi}{3}\right) &= \frac{9-6\cos\frac{\pi}{3}}{13-12\cos\frac{\pi}{3}} - 1 \\ \therefore \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\cos\left(\frac{n\pi}{3}\right) &= -\frac{1}{7}\end{aligned}$$

Quite simple. Let us try another question which is aimed to just show a small trick.

Question: Find $\displaystyle\sum_{n=0}^{N-1}\cos n\theta$

Solution:

Consider $z = \cos n\theta + i\sin\theta$. It is true that $\displaystyle\sum_{n=0}^{N-1}\cos n\theta = \mathrm{Re}\left(\sum_{n=0}^{N-1} z^n\right)$.

$$
\begin{aligned}
\sum_{n=0}^{N-1} z^n &= 1 + z + z^2 + \cdots \\
&= \frac{1 - z^N}{1 - z} \\
&= \frac{1 - e^{iN\theta}}{1 - e^{i\theta}}
\end{aligned}
$$

instead of multiplying by the complex conjugate, we can do something simpler

$$
\begin{aligned}
&= \frac{1 - e^{iN\theta}}{1 - e^{i\theta}} \times \frac{-e^{-\frac{1}{2}i\theta}}{-e^{-\frac{1}{2}i\theta}} \\
&= \frac{e^{i\theta\left(N-\frac{1}{2}\right)} - e^{-\frac{1}{2}i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}} \\
&= \frac{e^{i\theta\left(N-\frac{1}{2}\right)} - e^{-\frac{1}{2}i\theta}}{2i\sin\frac{1}{2}\theta}
\end{aligned}
$$

we cannot have i in the denominator to choose what part is real

$$
\begin{aligned}
&= \frac{\cos\theta\left(N-\frac{1}{2}\right) + i\sin\theta\left(N-\frac{1}{2}\right) - \cos-\frac{1}{2}\theta - i\sin-\frac{1}{2}\theta}{2i\sin\frac{1}{2}\theta} \times \frac{i}{i} \\
&= \frac{i\cos\theta\left(N-\frac{1}{2}\right) - \sin\theta\left(N-\frac{1}{2}\right) - i\cos\frac{1}{2}\theta - \sin\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta} \\
\therefore \sum_{n=0}^{N-1} \cos n\theta &= \frac{\sin\theta\left(N-\frac{1}{2}\right) + \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}
\end{aligned}
$$

4.3 The roots of unity and the n^{th} roots of any complex number

A root of unity is a complex number that, when raised to a positive integer power, results in 1. Finding these is pretty simple. Like we did previously, we will explain with an example.

Question: Find the 5^{th} roots of unity.

Solution:

$$z^5 = 1$$

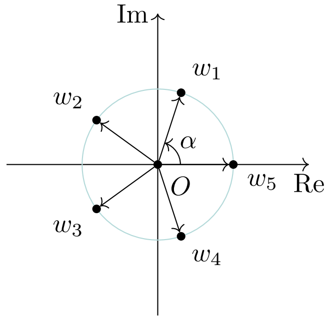
$|1| = 1$ and $\arg(1) = 0$ or $\arg(1) = 0 + 2\pi k$ for integer values of k

$$
\begin{aligned}
z^5 &= e^{i(2\pi k)} \\
z &= e^{i\left(\frac{2}{5}\pi k\right)}
\end{aligned}
$$

Hence, for $k = 0, 1, 2, 3$

$$z = e^{i(\frac{2}{5}\pi k)}$$

Discussing the geometric picture of this, we can see that successive roots are separated by $\frac{2}{5}\pi$ radians. Letting w_n denote the n^{th} root of z and plotting this, we get



Due to the even spacing of unity roots and equal magnitude, the sum of the n^{th} roots of unity is always equal to 0.

For other complex numbers, the procedure is similar

Question: find all solutions for $z^6 = 4 + 4\sqrt{3}i$

Solution:

$$z^6 = 4 + 4\sqrt{3}i$$
$$|4 + 4\sqrt{3}i| = \sqrt{4^2 + (4\sqrt{3})^2} = 8$$
$$\arg(4 + 4\sqrt{3}i) = \arctan\left(\frac{4\sqrt{3}}{4}\right) = \frac{\pi}{3}$$
$$z_k = \sqrt[6]{8}e^{i\left(\frac{\frac{\pi}{3}+2\pi k}{6}\right)} \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5$$

The sum of roots here is also 0, where the angle between two successive roots is $\frac{\pi}{3}$. In general the sum of all n^{th} unity roots must equal 0 for some complex number.