Proof by Contradiction

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1 Goal of the Paper

Proof by Contradiction is a powerful proof technique in a mathematicians arsenal. Establishing contradictions for the sake of proving *goals* is widely integrated in beginner, intermediate and advanced proofs. This paper tries to introduce this proof technique in a way that can be read informally and formally. It will gloss over the formal aspects of this technique that make it *logically and deductively valid* without introducing logic symbols ($\forall x$, $\exists k$, etc...)

2 Much Needed Definitions

- Conjecture: an **unproved** statement that has been established from mathematical intuition or testing– *a conjecture may or may not be true.*
- Theorem: a conjecture that has been proved rigorously.
- Givens: the set of information that we are given or have established.
- Goals: the set of information that we must deduce from the *givens*– the thing we must establish.
- Negation of P: the opposite of P– for example, the negation of "The weather is cold." is "The weather is *not* cold." but **not** "The weather is hot" as the weather may not be hot or cold.

3 Introduction

Do not focus on the examples as much as the general idea and logical reasoning.

What is it?

Proof by Contradiction is a proof technique that *assumes* the *truth* of the whole or some of the *goal*. You may not fully wrap your head around this paragraph, but that's okay and expected! Things will be cleared with examples; however, we must get formality aside.

But why?

That is a *very* good question that leads to a much deeper understanding. It's generally true that proving a positive statement– goal– is much easier than proving a negative one. For example, to prove that the earth is not flat– to logically/deductively prove that is– it would be easier to assume that it's flat and to deduce an absurdity which nullifies our previous assumptions. Since the earth is either flat or not flat, and assuming that it is flat leads to an absurdity, then the earth mustn't be flat![1] Well that's quite obvious, but let's zoom in

Is the earth round?

We *assumed* that the earth is flat and used this fact to deduce that it isn't. Let's break things down to givens and goals.

Theorem: If the earth has finite area, the earth is not flat.

Scratch work

Givens	Goals
Earth has finite area	Earth is not flat

Using *only* the fact that the Earth has finite area to prove our goal is surprisingly difficult! Let's try forcing a contradiction.

Assume that the Earth is indeed flat.

Givens	Goals
Earth has finite area	*Contradiction*
Earth is flat	—

Our new goal is to obtain a contradiction. Notice how we transformed some version of our goal– the opposite of our goal to be precise– into a given that we can use! Using our previous given and our assumption, we can say that if the Earth has finite area and it's flat, then earth must have edges. But the earth doesn't have edges, so the earth cannot be flat, which was what we wanted. For the sake of this non-mathematical example, we suppose that the earth not having any edges is a known, established truth.

Proof. Suppose that the Earth has finite area. Assume that the Earth is flat. If the earth has finite area and is flat, then it must have some edges. But the earth doesn't have any edges. Hence, our assumption is false, which implies that the earth is not flat.

But is it valid to deduce that the earth is round– or is any shape in general? No. It is always true that if some statement, P, is true, then the negation of P is false. The negation of "The earth is *not* is flat." is "The earth is flat." Since we proved that the statement, "The earth is flat." is false, we can *only* conclude that "The earth is *not* is flat." must be true. We cannot, however, establish that the earth is round, parabolic or anything as that would be an invalid deduction– we simply cannot infer anything about the shape of the earth, only if it is flat or not flat, the latter being true. But wait, why didn't we, for example, establish that the earth doesn't have a finite area? Why were we sure that our second, assumed, given was false? It's simple; the theorem reads "**If** the earth has finite area, **then** the earth is not flat." The theorem only deals with whether the earth is flat or not flat in the case where the earth has finite area. We are not concerned if the earth has or doesn't have finite area, as the theorem deals with the assumption that it does have finite area.

Before ending this interesting example, we must know that skepticism is the philosophy of proofs. You must question every step of your proofs: "Can I really make this jump in my proof?" is a question you should always be asking yourself. Let's get to actually doing mathematics.

[1] Another name for Proof by Contradiction is *Reductio ad absurdum*– reduction to absurdity in Latin– can you guess why?

4 The Math

By now, you should have an intuitive understanding of proving by contradiction already. It's fine if you feel that information is somewhat dispersed and disconnected; this section will chain everything together. We will do so via examples. The syllabus includes two core proofs– the irrationality of $\sqrt{2}$ and the infinity of primes– which we will begin with before proceeding to unfamiliar territory.

Essential information before beginning:

- All real numbers are either rational or irrational.
- All rational numbers **can** be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.
- All irrational numbers **cannot** be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.
- If a square number is even, then it must be the square of an even number.
- A prime number is a number that is is larger than 1 and cannot be written as a product of two smaller positive integers.

Question 1: Prove the following theorem: "The square root of 2 is irrational."

Scratch Work:

Givens	Goals
—	$\sqrt{2}$ is irrational

Note that the theorem doesn't give us anything in terms of givens; it only provides a goal. That doesn't mean, obviously, that we cannot use mathematical facts, such as $1 + 1 = 2$, $2 \times 0 = 0$, and the fact that all real numbers are either rational or irrational, all rational numbers can be expressed... while all irrational numbers cannot be expressed... Even then, it isn't clear what we should do. Let's trying proving with contradiction.

Givens	Goals
—	$\sqrt{2} \neq \frac{a}{b}$ where a and b are integers with no common factors, and $b \neq 0$

Assume that $\sqrt{2}$ is rational.

Givens	Goals
$\sqrt{2}$ is rational	*Contradiction*
$\sqrt{2} = \frac{a}{b}$ where a and b...	—
$a = 2k$ where $k \in \mathbb{Z}$	—

If $\sqrt{2}$ is rational, it follows that $\sqrt{2} = \frac{a}{b}$ where a and b are integers that do not have a common factor, except for 1.

Now the question of "*What* type of contradiction are we looking for?" is kind of unanswerable. However, we must deduce a contradicts between our givens, or a mathematical contradiction like 1=2. Squaring seems like a natural step to get rid of the square root. Since $\sqrt{2} = \frac{a}{b}$, then $2b^2 = a^2$. We can observe something interesting; a^2 is even and positive as it is a square number equal to an integer multiplied by 2. Hence, since if a^2 is even then a must also be even. Therefore, $a = 2k$ for some integer k.

Givens	Goals
$\sqrt{2}$ is rational	*Contradiction*
$\sqrt{2} = \frac{a}{b}$ where a and b...	—
$a = 2k$ where $k \in \mathbb{Z}$	—

Since $2b^2 = a^2$, then $2b^2 = (2k)^2$. It then follows that $2b^2 = 4k^2$ so $b^2 = 2k^2$. Since k^2 is an integer and $b^2 = 2k^2$, then b^2 must be even. Hence, b is also even. So $b = 2r$ for some integer value of r.

Givens	Goals
$\sqrt{2}$ is rational	*Contradiction*
$\sqrt{2} = \frac{a}{b}$ where a and b are integers that do not have a common factor, except for 1	—
$a = 2k$ where $k \in \mathbb{Z}$	—
$b = 2r$ where $r \in \mathbb{Z}$	—

So, we can say that $\sqrt{2} = {}^{2k}/_{2r}$. But wait! We said that a and b are integers that do not have a common factor, except for 1. Subbing in a and b in term of k and r, respectively, we see that they do have a common factor; 2. Hence, $\sqrt{2}$ is not rational. Since all real numbers are either rational or irrational, it must be true that $\sqrt{2}$ is irrational, which was what we wanted. Let's compile this into a proof.

Proof. Suppose $\sqrt{2}$ is rational. This means that, $\sqrt{2} = \frac{a}{b}$ where a and b are integers that do not have a common factor, except for 1. Squaring both sides of the previous equation, we get that

$$2 = \frac{a^2}{b^2}.$$

Hence, $2b^2 = a^2$. Since b^2 is an integer as b is an integer and a squared integer is also an integer, then a^2 must be even as it is equal to an integer multiplied by 2. Moreover, since a^2 is even, it follows that a is also even. Hence, $a = 2k$ where k is some integer. Substituting our result in the equation $2b^2 = a^2$, we get that $2b^2 = (2k)^2$. So, $2b^2 = 4k^2$. So, $b^2 = 2k^2$. Since k is integer, then k^2 is also an integer. Hence, b^2 is even as it is equal to an integer multiplied by 2. It follows that b is even, so $b = 2r$ where r is some integer. This means that

$$\sqrt{2} = \frac{2k}{2r}.$$

This contradicts that a and b are are integers that do not have a common factor, except for 1, as 2 is a common factor. Hence, it is not true that $\sqrt{2}$ is rational. Since all real numbers are either rational or irrational, $\sqrt{2}$ is irrational, which was what we wanted.

Note that our scratch work goes in depth to make sure the reader understands every step. It is **not** necessarily to do all of this work in the exam. The part after *Proof.* suffices. We will start reducing the amount of explanation in upcoming questions.

Question 2: Prove the following theorem: ”There are infinitely many primes.”

Scratch Work

Suppose there is a finite number of primes. Let $p_1, p_2, p_3, \ldots, p_n$ be the list of all primes, where n can be any positive integer. Let $N = p_1 \times p_2 \times p_3 \times \ldots \times p_n$. Since every single prime number is equal to itself multiplied by 1 only, we can say that N is only divisible by the prime numbers, 1, and nothing else. $N+1 = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$. For $N+1$ to be divisible by some number a, it must be a factor of $p_1 \times p_2 \times p_3 \times \ldots \times p_n$ and 1. If we attempt to divide by some prime number from our finite prime numbers, that prime number will be a factor of $p_1 \times p_2 \times p_3 \times \ldots \times p_n$ but not 1, leaving a remainder of 1. If we attempt to divide by a non-prime number, that number will not divide by $p_1 \times p_2 \times p_3 \times \ldots \times p_n$ or 1, **unless** our number is 1. All integers greater than 1 are either prime or can be expressed as a product of primes. Since $N + 1$ is only divisible by itself and 1 it must be a prime number. But $N + 1$ cannot be in our finite list of prime numbers as it is the product of every number in that last plus 1, so $N + 1$ is larger than every number in our list of primes; hence, since n can be any positive integer, we must have infinitely many primes.

Proof.

Assume there is a finite number of primes. Let $p_1, p_2, p_3, \ldots, p_n$ be the list of all primes, where n can be any positive integer. Let $N = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$. Dividing N by any prime number will always leave a remainder of 1[2]. Dividing N by any non-prime number, except for N and 1, will leave a remainder of $p_1 \times p_2 \times p_3 \times \ldots \times p_n$[3]. Hence, N is only divisible by N and 1, which is the definition of a prime number. Therefore, N is a prime number that is not in the list. Since n can be any positive integer, our assumption that there is a finite number of primes is nullified/contradicted. Hence there is not a finite number of primes, which only leaves the possibility of an infinite number of primes, which was what we wanted.

Our next proof will move faster

Question 3: Prove that if ab is irrational then at least one of a or b is irrational

Scratch Work

Note that we are given the ab is irrational so we don’t need to worry about that part. Note that the opposite of ”at least one of a or b is irrational” is ”none of a or b is irrational” i.e. ”a and b are rational” then $a = {}^{p}/_{q}$ where p and q are integers and $q \neq 0$, except for 1; and $b = {}^{k}/_{r}$ where k and r are integers and $k \neq 0$. $ab = \frac{kp}{rq}$ where both the numerator and denominator are integers. But we just expressed ab as a fraction with an integer numerator and denominator; this implies that ab is rational which is a contradiction. Hence, at least one of a or b must be irrational.

We will leave the tidied up *Proof.* part to the reader.

Question 3: Three **consecutive** terms in a sequence of real numbers are given by

$$k,\ 1 + 2k \ \text{ and } \ 3 + 3k$$

Scratch Work

Let’s make a table— **Table 1** next page— that shows our givens and goals.

We know assume that the negation of our goal is true. That is, we assume that the given sequence is indeed a geometric sequence. We will now look for a contradiction. Note that one of our givens can be broken down into simpler givens. That is we can make an additional table— **Table 2** next page.

[2]this is because 1 is only divisible by itself

[3]This is because $p_1 \times p_2 \times p_3 \times \ldots \times p_n$ has no common factors with any non-primes

Givens	Goals
k, $1 + 2k$ and $3 + 3k$ are three consecutive terms in some sequence.	Prove that the given sequence is *not* a geometric sequence.

Table 1

Givens	Goals
k, $1 + 2k$ and $3 + 3k$ are three consecutive terms in some sequence.	*Contradiction.*
Ratio between two consecutive terms is constant.	—

Table 2

Since the ratio between any two consecutive terms is constant, then there exists some value of k such that

$$\frac{3+3k}{1+2k} = \frac{1+2k}{k}.$$

Simplifying the equation into a quadratic, we get that

$$k^2 + k + 1 = 0.$$

Calculating the discriminant, we get

$$\Delta = 1^2 - 4(1)(1) = -3 < 0.$$

Since the discriminant is less than 0, then there doesn't exist any k such that the ratio between two consecutive terms in the given sequence is constant; hence, our assumption is false and the sequence cannot be a geometric sequence.

Again, the tidied up *Proof.* is left to the reader.

The paper covers all the fundamental ideas and standard techniques; however, practice is key. Good luck.