CH.5 Integration

Lecture 3

Chapter Summary

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Substitution Rule for Indefinite Integrals

Substitution Rule for definite Integrals

CH.7 Integration Techniques

7.1 Basic Approaches

5.5 Substitution Rule

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Given just about any differentiable function, with enough know-how and persistence, you can compute its derivative. But the same cannot be said of antiderivatives.

Indefinite Integrals

One way to find new antiderivative rules is to start with familiar derivative rules and work backward.

Example

Antiderivatives by trial and error Find $\int cos\ (2x)dx$

Solution

The closest familiar indefinite integral related to this problem is

$$\int cos\ x\ dx = \sin x + C$$

which is true because

$$\frac{d}{dx}(\sin x + C) = \cos x$$

Therefore, we might incorrectly conclude that the indefinite integral of cos 2x is sin 2x + C. However, by the Chain Rule,

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$$\frac{d}{dx}(\sin 2x + C) = 2\cos 2x \neq \cos 2x.$$

Let's try $\frac{1}{2}\sin 2x$

$$\frac{d}{dx}\left(\frac{1}{2}\sin 2x\right) = \frac{1}{2}\cdot 2\cos 2x = \cos 2x.$$

It works! So we have

$$\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C.$$

The trial-and-error approach of this Example does not work for complicated integrals.

Consider a composite function $F(g(x))$ where F is an antiderivative of f; that is, $F'(x) = f(x)$. Using the Chain Rule to differentiate the composite function $F(g(x))$, we find that

$$\frac{d}{dx}[F(g(x))] = \underbrace{F'(g(x))}_{f(g(x))}g'(x) = f(g(x))g'(x).$$

This equation says that $F(g(x))$ *is an antiderivative of* $f(g(x))$ *which is written*

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C,$$

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where F is any antiderivative of f.

$$\int \underbrace{f(g(x))}_{f(u)}\underbrace{g'(x)dx}_{du} = \int f(u)\,du = F(u) + C.$$

called the Substitution Rule
(or Change of Variables Rule)

THEOREM 5.6 Substitution Rule for Indefinite Integrals
Let $u = g(x)$, where g' is continuous on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Consider the integral $\int 2x\cos(x^2)\,dx$. We can evaluate it if we remember the Chain Rule calculation

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$$

This tells us that $\sin(x^2)$ is an antiderivative of $2x\cos(x^2)$, and therefore,

$$\int \underbrace{2x}_{\substack{\text{Derivative of}\\ \text{inside function}}} \cos(\underbrace{x^2}_{\substack{\text{Inside}\\ \text{function}}})dx = \sin(x^2) + C$$

A similar Chain Rule calculation shows that

$$\int \underbrace{(1+3x^2)}_{\substack{\text{Derivative of}\\ \text{inside function}}} \cos(\underbrace{x+x^3}_{\substack{\text{Inside}\\ \text{function}}})dx = \sin(x+x^3)+C$$

In both cases, the integrand is the product of a composite function and the derivative of the inside function. The Chain Rule does not help if the derivative of the inside function is missing. For instance, we cannot use the Chain Rule to compute $\int \cos(x+x^3)\,dx$ because the factor $(1+3x^2)$ does not appear.

In general, if $F'(u) = f(u)$ then by the Chain Rule,

$$\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$$

This translates into the following integration formula.

The Substitution Method If $F'(x) = f(x)$, then

$$\int f(u(x))u'(x)dx = F(u(x)) + C$$

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Example 1.3.1. *Evaluate*

$$\int 3x^2 \sin\left(x^3\right) dx$$

Solution: *Let $u = x^3$, then $du = 3x^2 dx$, hence*

$$\begin{aligned}\int 3x^2 \sin\left(x^3\right) dx &= \int \sin(u)\, du \\ &= -\cos u + C \\ &= -\cos\left(x^3\right) + C\end{aligned}$$

PROCEDURE Substitution Rule (Change of Variables)

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)\, dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Disclaimer: Not all integrals yield to the Substitution Rule.

Example

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Use the Substitution Rule to find the following indefinite integrals. Check your work by differentiating.

a. $\int 2(2x+1)^3\,dx$ **b.** $\int 10e^{10x}\,dx$

Solution

$$\int \underbrace{(2x+1)^3}_{u^3} \cdot \underbrace{2\,dx}_{du} = \int u^3\,du$$ Substitute $u = 2x + 1$, $du = 2\,dx$.

$$= \frac{u^4}{4} + C$$ Antiderivative

$$= \frac{(2x+1)^4}{4} + C.$$ Replace u by $2x + 1$.

b. The composite function e^{10x} has the inner function $u = 10x$, which implies that $du = 10\,dx$. The change of variables appears as

$$\int \underbrace{e^{10x}}_{e^u} \underbrace{10\,dx}_{du} = \int e^u\,du$$ Substitute $u = 10x$, $du = 10\,dx$.

$$= e^u + C$$ Antiderivative

$$= e^{10x} + C.$$ Replace u by $10x$.

Example

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Introducing a constant

Find the following indefinite integrals.

a. $\int x^4(x^5+6)^9\,dx$ **b.** $\int \cos^3 x \sin x\,dx$

Solution

$$\int \underbrace{(x^5+6)^9}_{u^9}\underbrace{x^4\,dx}_{\frac{1}{5}du} = \int u^9 \frac{1}{5}\,du$$

Substitute $u = x^5 + 6$,
$du = 5x^4\,dx \Rightarrow x^4\,dx = \frac{1}{5}\,du$

$$= \frac{1}{5}\int u^9\,du$$

$\int c f(x)\,dx = c\int f(x)\,dx$

$$= \frac{1}{5}\cdot\frac{u^{10}}{10} + C$$

Antiderivative

$$= \frac{1}{50}(x^5+6)^{10} + C.$$

Replace u by $x^5 + 6$.

$$\int \underbrace{\cos^3 x}_{u^3}\,\underbrace{\sin x\,dx}_{-du} = -\int u^3\,du$$

$$= -\frac{u^4}{4} + C \quad = -\frac{\cos^4 x}{4} + C.$$

Example

Find $\int \frac{x}{\sqrt{x+1}}\,dx.$

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Solution

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du$$

Substitute $u = x + 1, du = dx.$

$$= \int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) du$$

Rewrite integrand.

$$= \int (u^{1/2} - u^{-1/2})\,du.$$

Fractional powers

$$= \frac{2}{3}u^{3/2} - 2u^{1/2} + C$$

Antiderivatives

$$= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C$$

Replace u by $x + 1.$

$$= \frac{2}{3}(x+1)^{1/2}(x-2) + C.$$

Simplify the integrand.

Substitution 2 Another possible substitution is $u = \sqrt{x+1}$,

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Rules for Integrating Common Functions

- The constant rule: $\int k\,dx = kx + C$ for constant k
- The power rule: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for all $n \neq -1$
- The logarithmic rule: $\int \frac{1}{x}\,dx = \ln|x| + C \quad$ for all $x \neq 0$
- The exponential rule: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \quad$ for constant $k \neq 0$

Example 1.1.2. *Evaluate*

$$\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx$$

Solution:

$$\begin{aligned}\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx &= 2\int x^5 dx + 8\int x^3 dx - 3\int x^2 dx + \int 5dx \\ &= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \\ &= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C\end{aligned}$$

Example 1.1.3. *Evaluate*

$$\int \left(\frac{x^3 + 2x - 7}{x} \right) dx$$

Solution:

$$\int \left(\frac{x^3 + 2x - 7}{x} \right) dx = \int \left(x^2 + 2 - \frac{7}{x} \right) dx$$
$$= \frac{1}{3}x^3 + 2x - 7\ln|x| + C$$

Example 1.1.4. *Evaluate*

$$\int \left(3e^{-5t} + \sqrt{t} \right) dt$$

Solution:

$$\int \left(3e^{-5t} + \sqrt{t} \right) dt = \int \left(3e^{-5t} + t^{1/2} \right) dt$$
$$= 3\left(\frac{1}{-5}e^{-5t} \right) + \frac{1}{3/2}t^{3/2} + C$$
$$= -\frac{3}{5}e^{-5t} + \frac{2}{3}t^{3/2} + C$$

Example 1.1.5. *Evaluate*

$$\int \frac{dx}{1+e^x}$$

Solution:

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{dx}{1+e^x}\frac{e^{-x}}{e^{-x}}\\ &= \int \frac{e^{-x}dx}{e^{-x}+1}\\ &= -\ln|1+e^{-x}|+C\end{aligned}$$

Example 1.1.6. *Evaluate*

$$\int \frac{xdx}{1+x^2}$$

Solution:

$$\begin{aligned}\int \frac{xdx}{1+x^2} &= \frac{1}{2}\int \frac{2xdx}{1+x^2}\\ &= \frac{1}{2}\ln|1+x^2|+C\end{aligned}$$

Example 1.3.10. *Evaluate*

$$\int x^3 e^{x^4+2} dx$$

Solution: *Let* $u = x^4 + 2$ *, then* $du = 4x^3 dx$*, hence*

$$\begin{aligned} \int x^3 e^{x^4+2} dx &= \int e^{x^4+2} \left(x^3 dx\right) \\ &= \int e^{u\left(\frac{1}{4} du\right)} \\ &= \frac{1}{4} e^u + C \\ &= \frac{1}{4} e^{x^4+2} + C \end{aligned}$$

Example 1.3.11. *Evaluate*

$$\int e^{5x+2} dx$$

Solution: *Let* $u = 5x + 2$ *, then* $du = 5dx$*, hence*

$$\begin{aligned} \int e^{5x+2} dx &= \int e^{5x} e^2 dx \\ &= e^2 \int e^{5x} dx \\ &= e^2 \left[\frac{e^{5x}}{5}\right] + c \\ &= \frac{1}{5} e^{5x+2} + c \end{aligned}$$

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Example 1.3.8. *Evaluate*

$$\int \frac{(\ln x)^2}{x} dx$$

Solution: *Let* $u = \ln x$ *, then* $du = \frac{1}{x}dx$*, hence*

$$\begin{aligned}\int \frac{(\ln x)^2}{x} dx &= \int (\ln x)^2 \left(\frac{1}{x} dx\right) \\ &= \int u^2 du = \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(\ln x)^3 + C\end{aligned}$$

Example 1.3.9. *Evaluate*

$$\int \frac{x}{x-1} dx$$

Solution: *Let* $u = x - 1$ *, then* $du = dx$*, hence*

$$\begin{aligned}\int \frac{x}{x-1} dx &= \int \frac{u+1}{u} du \\ &= \int \left[1 + \frac{1}{u}\right] du \\ &= u + \ln|u| + C \\ &= x - 1 + \ln|x-1| + C\end{aligned}$$

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Basic Trigonometric Integrals

- $\int \sin kx dx = \frac{-1}{k} \cos x + C$
- $\int \cos kx dx = \frac{1}{k} \sin x + C$
- $\int \sec^2 kx dx = \frac{1}{k} \tan x + C$
- $\int \csc^2 kx dx = \frac{-1}{k} \cot x + C$
- $\int \sec kx \tan kx dx = \frac{1}{k} \sec x + C$
- $\int \csc kx \cot kx dx = \frac{-1}{k} \csc x + C$

Example 1.2.5. *Evaluate*

$$\int (\sin 8t + 20 \cos 9t) dt$$

Solution:

$$\begin{aligned}\int (\sin 8t + 20 \cos 9t) dt &= \int \sin 8t dt + 20 \int \cos 9t dt \\ &= -\frac{1}{8} \cos 8t + \frac{20}{9} \sin 9t + C\end{aligned}$$

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Example 1.2.1. *Evaluate*

$$\int \tan x dx$$

Solution:

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln \cos x + C \\ &= \ln|\sec x| + C\end{aligned}$$

Example 1.2.3. *Evaluate*

$$\int \sec x dx$$

Solution:

$$\begin{aligned}\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

Definite Integrals

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THEOREM 5.7 Substitution Rule for Definite Integrals
Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

Evaluate the following integrals.

a. $\displaystyle\int_0^2 \frac{dx}{(x+3)^3}$ **b.** $\displaystyle\int_0^4 \frac{x}{x^2+1}\,dx$ **c.** $\displaystyle\int_0^{\pi/2} \sin^4 x \cos x\,dx$

Solution

a. Let the new variable be $u = x + 3$ and then $du = dx$. Because we have changed the variable of integration from x to u, the limits of integration must also be expressed in terms of u. In this case,

$x = 0$ implies u = 0 + 3 = 3, Lower limit
x = 2 implies u = 2 + 3 = 5. Upper limit

The entire integration is carried out as follows:

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$$\int_0^2 \frac{dx}{(x+3)^3} = \int_3^5 u^{-3}\,du$$ Substitute $u = x + 3$, $du = dx$.

$$= -\frac{u^{-2}}{2}\Bigg|_3^5$$ Fundamental Theorem

$$= -\frac{1}{2}(5^{-2} - 3^{-2}) = \frac{8}{225}.$$ Simplify.

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b. Notice that a multiple of the derivative of the denominator appears in the numerator; therefore, we let $u = x^2 + 1$. Then $du = 2x\,dx$, or $x\,dx = \frac{1}{2}\,du$. *Changing limits of* integration,

$x = 0$ implies $u = 0 + 1 = 1$, *Lower limit*
$x = 4$ implies $u = 4^2 + 1 = 17$. Upper limit

Changing variables, we have:

$$\int_0^4 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{17} u^{-1}\,du$$ Substitute $u = x^2 + 1$, $du = 2x\,dx$.

$$= \frac{1}{2}\ln|u|\Bigg|_1^{17} = \frac{1}{2}(\ln 17 - \ln 1) = \frac{1}{2}\ln 17 \approx 1.417.$$

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c. Let $u = \sin x$, which implies that $du = \cos x\, dx$. The lower limit of integration becomes $u = 0$ and the upper limit becomes $u = 1$. Changing variables, we have

$$\int_0^{\pi/2} \sin^4 x \cos x\, dx = \int_0^1 u^4\, du \qquad u = \sin x,\ du = \cos x\, dx$$

$$= \left(\frac{u^5}{5}\right)\Big|_0^1 = \frac{1}{5}. \qquad \text{Fundamental Theorem}$$

Example

Integral of $\cos^2 x$. Evaluate $\int_0^{\pi/2} \cos^2 \theta\, d\theta.$

Solution

we use the identity for $\cos^2\theta$

$$\int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta.$$

The change of variables $u = 2\theta$ is now used for the second integral, and we have

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\cdot\frac{1}{2}\int \cos u\, du \qquad u = 2\theta,\ du = 2\, d\theta$$

Identities

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}.$$

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$$= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C. \qquad \text{Evaluate integrals; } u = 2\theta.$$

Using the Fundamental Theorem of Calculus, the value of the definite integral is

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi/2} = \left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right) - \left(0 + \frac{1}{4}\sin 0\right) = \frac{\pi}{4}.$$

Quiz:

Evaluate the following integrals.

$$\int (x^{3/2} + 8)^5 \sqrt{x}\, dx \qquad \int \sec 4w \tan 4w\, dw \qquad \int x\cos^2(x^2)\, dx$$

$$\int_1^{e^2} \frac{\ln x}{x}\, dx \qquad \int_2^3 \frac{x}{\sqrt[3]{x^2 - 1}}\, dx \qquad \int_0^{\pi/4} \cos^2 8\theta\, d\theta$$

$$\int \sec^2 10x\, dx \qquad \int_{-\pi}^0 \frac{\sin x}{2 + \cos x}\, dx \qquad \int_0^{\pi/2} \sin^4\theta\, d\theta$$

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CH.7 Integration Techniques

Chapter Summary

7.1 Basic Approaches

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Table 7.1 Basic Integration Formulas

1. $\int k\,dx = kx + C,\ k$ real
2. $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C,\ p \neq -1$ real
3. $\int \cos ax\,dx = \dfrac{1}{a}\sin ax + C$
4. $\int \sin ax\,dx = -\dfrac{1}{a}\cos ax + C$
5. $\int \sec^2 ax\,dx = \dfrac{1}{a}\tan ax + C$
6. $\int \csc^2 ax\,dx = -\dfrac{1}{a}\cot ax + C$
7. $\int \sec ax \tan ax\,dx = \dfrac{1}{a}\sec ax + C$
8. $\int \csc ax \cot ax\,dx = -\dfrac{1}{a}\csc ax + C$
9. $\int e^{ax}\,dx = \dfrac{1}{a}e^{ax} + C$
10. $\int \dfrac{dx}{x} = \ln|x| + C$
11. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\dfrac{x}{a} + C$
12. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$
13. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$

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Example **Substitution review**

Evaluate $\displaystyle\int_{-1}^{2} \frac{dx}{3+2x}$.

Solution

$$\int_{-1}^{2} \frac{dx}{3+2x} = \int_{1}^{7} \frac{1}{u}\underbrace{\frac{du}{2}}_{dx} = \frac{1}{2}\ln|u|\Big|_{1}^{7} = \frac{1}{2}\ln 7.$$

QUICK CHECK What change of variable would you use for the integral $\int (6+5x)^8\,dx$?

Example **Subtle substitution**

Evaluate $\displaystyle\int \frac{dx}{e^x + e^{-x}}$.

Solution

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x}+1}\,dx.$$

multiply numerator and denominator of the integrand by e^x

$$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{du}{u^2+1}$$

Substitute $u = e^x$, $du = e^x\,dx$.

$$= \tan^{-1} u + C$$

$u = e^x$

$$= \tan^{-1} e^x + C.$$

Example Split up fractions

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Solution

Evaluate $\displaystyle\int \frac{\cos x + \sin^3 x}{\sec x}\,dx.$

$$\int \frac{\cos x + \sin^3 x}{\sec x}\,dx = \int \frac{\cos x}{\sec x}\,dx + \int \frac{\sin^3 x}{\sec x}\,dx$$

Split fraction.

$$= \int \cos^2 x\,dx + \int \sin^3 x \cos x\,dx.$$

$\sec x = \dfrac{1}{\cos x}$

$$= \int \frac{1 + \cos 2x}{2}\,dx + \int \sin^3 x \cos x\,dx$$

Half-angle formula

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx + \int u^3\,du$$

$u = \sin x,\ du = \cos x\,dx$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + \frac{1}{4}\sin^4 x +$$

- Half-angle formulas

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

QUICK CHECK Explain how to simplify the integrand of $\displaystyle\int \frac{x^3 + \sqrt{x}}{x^{3/2}}\,dx$ Before integrating.

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Example **Division with rational functions**

Evaluate $$\int \frac{x^2 + 2x - 1}{x + 4}\, dx.$$

Solution

When integrating rational functions (polynomials in the numerator and denominator), check to see if the function is *improper* (*the degree of the numerator is* greater than or equal to the degree of the denominator)

$$\int \frac{x^2 + 2x - 1}{x + 4}\, dx = \int (x - 2)\, dx + \int \frac{7}{x + 4}\, dx \quad \text{Long division}$$

$$= \frac{x^2}{2} - 2x + 7 \ln |x + 4| + C. \quad \text{Evaluate integrals.}$$

$$\begin{array}{r}
x - 2 \\
x + 4 \overline{\big) x^2 + 2x - 1} \\
\underline{x^2 + 4x} \\
-2x - 1 \\
\underline{-2x - 8} \\
7
\end{array}$$

QUICK CHECK Explain how to simplify the integrand of $\int \frac{x + 1}{x - 1}\, dx$ Before integrating.

Example **Multiply by 1**

Evaluate $\int \frac{dx}{1+\cos x}$.

The idea is to multiply the integrand by 1, but the challenge is finding the appropriate representation of 1. In this case, we use

$$1 = \frac{1-\cos x}{1-\cos x}.$$

The integral is evaluated as follows:

$$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}\,dx \qquad \text{Multiply by 1.}$$

$$= \int \frac{1-\cos x}{1-\cos^2 x}\,dx \qquad \text{Simplify.}$$

$$= \int \frac{1-\cos x}{\sin^2 x}\,dx \qquad 1-\cos^2 x = \sin^2 x$$

$$= \int \frac{1}{\sin^2 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx \qquad \text{Split up the fraction.}$$

$$= \int \csc^2 x\,dx - \int \csc x \cot x\,dx \qquad \csc x = \frac{1}{\sin x},\ \cot x = \frac{\cos x}{\sin x}$$

$$= -\cot x + \csc x + C.$$

Example **Complete the square**

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Evaluate $\displaystyle\int \frac{dx}{\sqrt{-7-8x-x^2}}$.

the key is to complete the square on the polynomial in the denominator. We find that

$$\begin{aligned}
-7-8x-x^2 &= -(x^2+8x+7) \\
&= -(x^2+8x+\underbrace{16-16}_{\text{add and subtract 16}}+7) && \text{Complete the square.} \\
&= -((x+4)^2-9) && \text{Factor and combine terms.} \\
&= 9-(x+4)^2. && \text{Rearrange terms.}
\end{aligned}$$

After a change of variables, the integral is recognizable:

$$\begin{aligned}
\int \frac{dx}{\sqrt{-7-8x-x^2}} &= \int \frac{dx}{\sqrt{9-(x+4)^2}} && \text{Complete the square.} \\
&= \int \frac{du}{\sqrt{9-u^2}} && u = x+4,\ du = dx \\
&= \sin^{-1}\frac{u}{3} + C && \text{Table 7.1} \\
&= \sin^{-1}\left(\frac{x+4}{3}\right) + C. && \text{Replace } u \text{ by } x+4.
\end{aligned}$$

Thank You!