CH.5 Integration

Lecture 2

Chapter Summary

5.1 Approximating Areas under Curves

5.2 Definite Integrals

- Properties of Definite Integrals

5.3 Fundamental Theorem of Calculus

- Area Functions
- Fundamental Theorem of Calculus **(PART 1)**
- Fundamental Theorem of Calculus **(PART 2)**

5.4 Working with Integrals

- Integrating Even and Odd Functions
- Average Value of a Function
- Mean Value Theorem for Integrals

5.5 Substitution Rule

Definite Integrals

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DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k.$$

- Properties of Definite Integrals

DEFINITION Reversing Limits and Identical Limits

Suppose f is integrable on $[a, b]$.

1. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$ **2.** $\int_a^a f(x)\, dx = 0$

QUICK CHECK 5 Evaluate $\int_a^b f(x)\, dx + \int_b^a f(x)\, dx$ if f is integrable on $[a, b]$.

Integral of a Sum

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$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

$$\int_a^b (f(x) + g(x))\, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} [f(x_k^*) + g(x_k^*)]\Delta x_k$$

Definition of definite integral

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k + \lim_{\Delta \to 0} \sum_{k=1}^{n} g(x_k^*)\Delta x_k$$

Split into two finite sums

Split into two limits.

$$= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Constants in Integrals

$$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx.$$

If f *is integrable on* $[a, b]$ and c *is a constant*

The justification is based on the fact that for finite sums,

$$\sum_{k=1}^{n} cf(x_k^*)\Delta x_k = c\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

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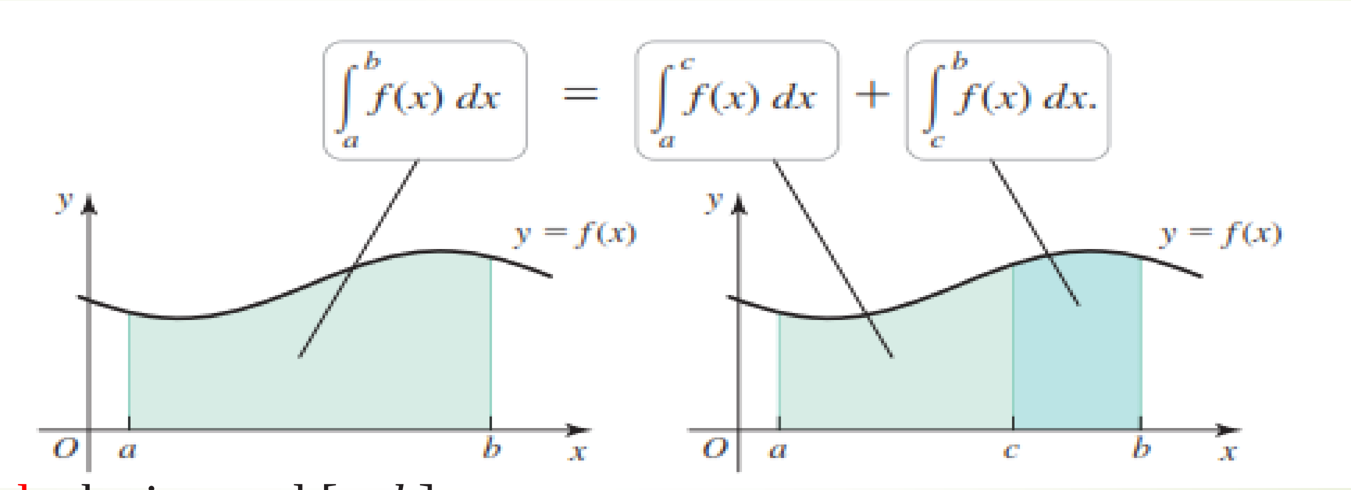
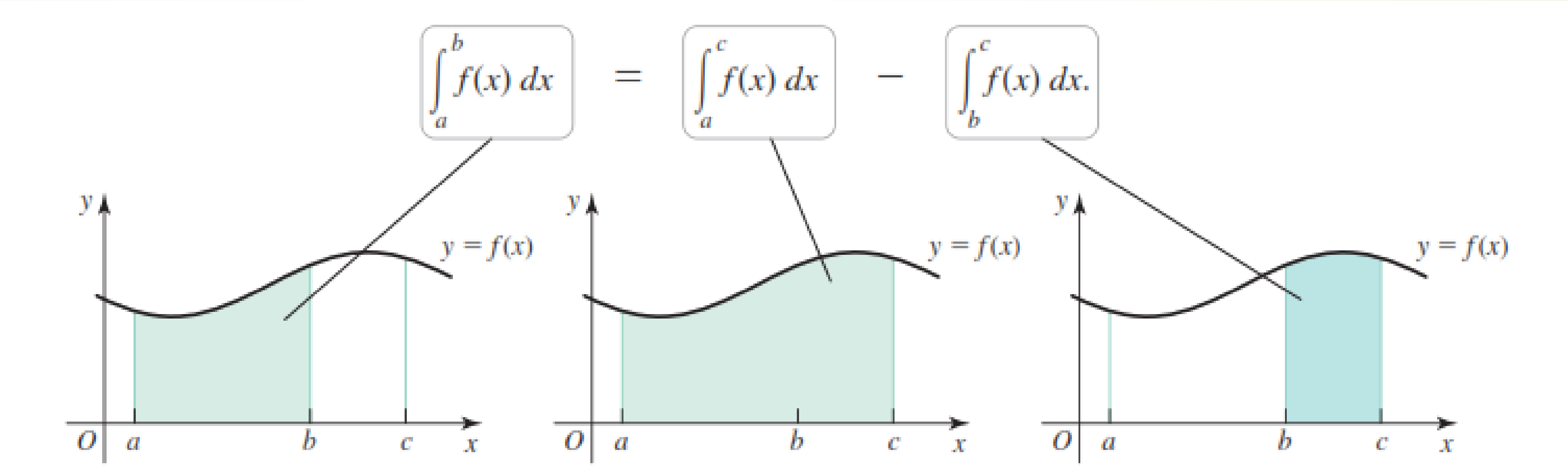
Integrals over Subintervals

If *c lies between a and b*, *then the integral on* $[a, b]$ may be split into two integrals.



when c lies outside the interval $[a, b]$

For example, if $a < b < c$ and f is integrable on $[a, c]$, then it follows that:



Integrals of Absolute Values

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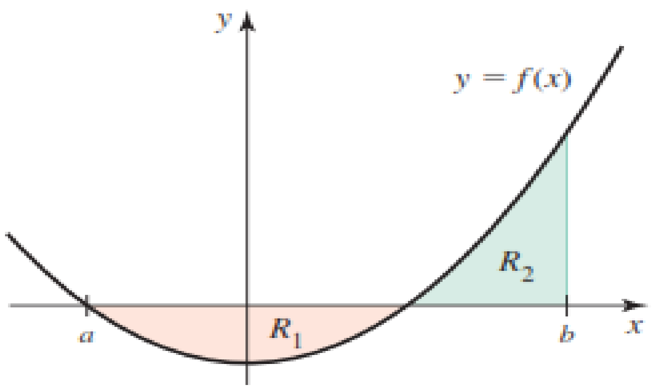
$$\int_a^b |f(x)|\,dx = \text{area of } R_1^* + \text{area of } R_2$$
$$= \text{area of } R_1 + \text{area of } R_2$$



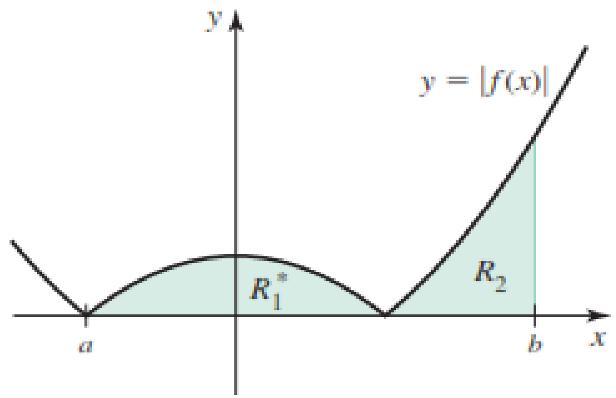


Table 5.4 Properties of definite integrals

Let f and g be integrable functions on an interval that contains a, b, and c.

1. $\int_a^a f(x)\,dx = 0$ Definition
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ Definition
3. $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ For any constant c
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. The function $|f|$ is integrable on $[a, b]$ and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

Example

Assume that $\int_0^5 f(x)\,dx = 3$ and $\int_0^7 f(x)\,dx = -10.$

Evaluate the following integrals, if possible.

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a. $\int_0^7 2f(x)\,dx$ **b.** $\int_5^7 f(x)\,dx$ **c.** $\int_5^0 f(x)\,dx$ **d.** $\int_7^0 6f(x)\,dx$ **e.** $\int_0^7 |f(x)|\,dx$

Solution

a. $\int_0^7 2\,f(x)\,dx = 2\int_0^7 f(x)\,dx = 2\cdot(-10) = -20.$ By Property 4 of Table 5.4,

b. $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx.$ Therefore, By Property 5 of Table 5.4,

$$\int_5^7 f(x)\,dx = \int_0^7 f(x)\,dx - \int_0^5 f(x)\,dx = -10 - 3 = -13.$$

c. $$\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -3.$$ By Property 2 of Table 5.4,

d. Using Properties 2 and 4 of Table 5.4, we have

$$\int_7^0 6f(x)\,dx = -\int_0^7 6f(x)\,dx = -6\int_0^7 f(x)\,dx = (-6)(-10) = 60.$$

e. This integral cannot be evaluated without knowing the intervals on which f *is positive* and negative. It could have any value greater than or equal to 10.

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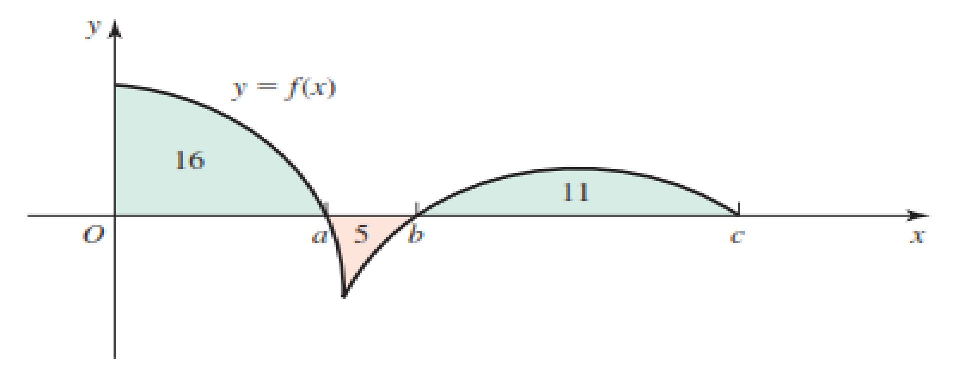
Quiz:

Evaluate $\int_{-1}^{2} x\,dx$ and $\int_{-1}^{2} |x|\,dx$ using geometry.

2. Evaluate the following integrals.

$$\int_0^a f(x)\,dx \qquad \int_0^b f(x)\,dx \qquad \int_a^c f(x)\,dx \qquad \int_0^c f(x)\,dx$$



3. Use the definition of the definite integral to evaluate the following definite integrals. Use right Riemann sums and Theorem 5.1.

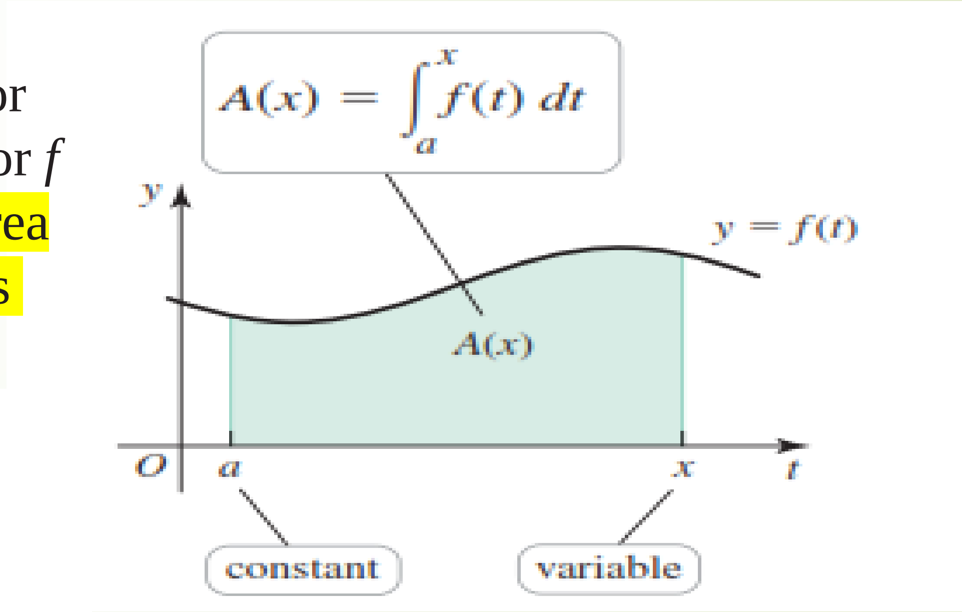
$$\int_0^2 (2x + 1)\,dx \qquad \int_1^4 (x^2 - 1)\,dx$$

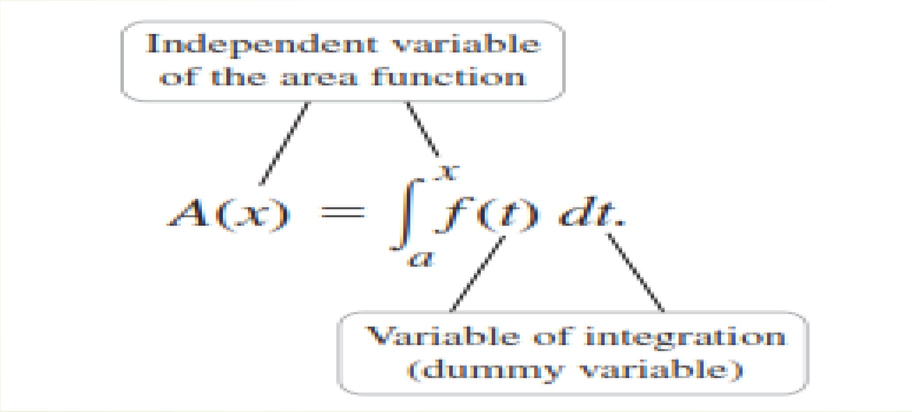
5.3 Fundamental Theorem of Calculus

Evaluating definite integrals using limits of Riemann sums, as described in Section 5.2, is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals, which is developed in this section.

Area Functions $A(x)$

We start with a continuous function $y = f(t)$ defined for $t \geq a$ where a is a fixed number. The *area function* for f with left endpoint a is denoted $\boldsymbol{A(x)}$ it gives the net area of the region bounded by the graph of f and the t-axis between $t = a$ and $t = x$.





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DEFINITION Area Function

Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint a** is

$$A(x) = \int_a^x f(t)\, dt,$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.

Example

The graph of f is shown in Figure 5.34 with areas of various regions marked.

Let $A(x) = \int_{-1}^{x} f(t)dt$ and $F(x) = \int_{3}^{x} f(t)dt$ be two area functions for f (note the different left endpoints).

Evaluate the following area functions.

a. A(3) and F(3) ***b. A(5) and F(5)***

c. A(9) and F(9)

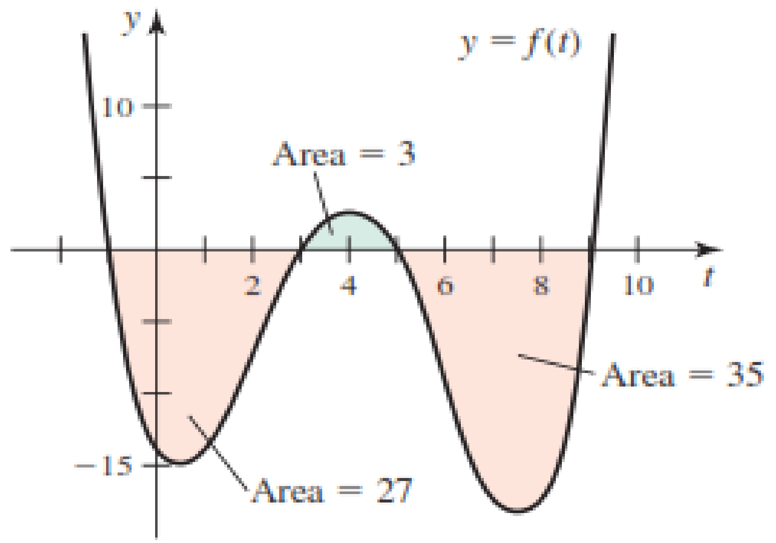

FIGURE 5.34

a. $A(3)$ **and** $F(3)$ **b.** $A(5)$ **and** $F(5)$
c. $A(9)$ **and** $F(9)$

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Solution

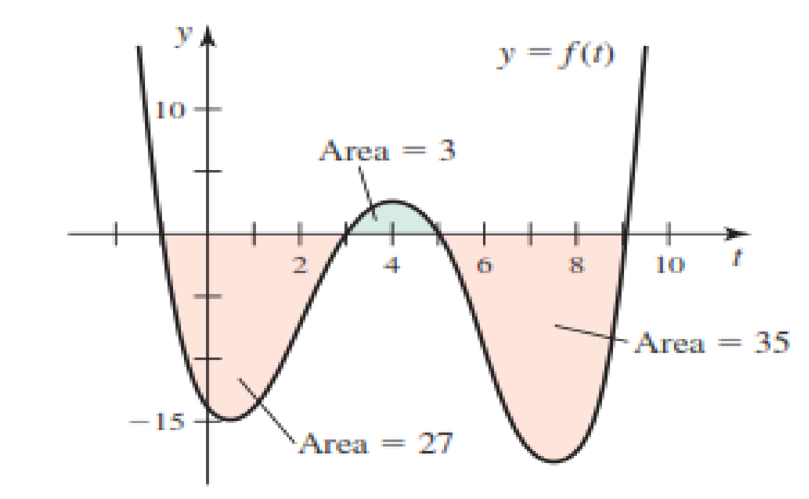

FIGURE 5.34

a. The value of $A(3)=\int_{-1}^{3} f(t)\,dt$ is the net area of the region bounded by the graph of f and the t-axis on the interval [-1, 3]. Using the graph of f, we see that $A(3) = -27$ (because this region has an area of 27 and lies below the t-axis).

On the other hand, $F(3)=\int_{3}^{3} f(t)\,dt = 0$.

b. $A(5)= \int_{-1}^{5} f(t)dt = \int_{-1}^{5} f(t)dt + \int_{-1}^{5} f(t)dt = 3 - 27 = -24.$

c. $A(9) = -27 + 3 - 35 = -59$ and $F(9) = 3 - 35 = -32$.

Example

Consider the trapezoid bounded by the line *f(t)* = 2*t* + 3 *and the t-axis from t = 2 to t = x (Figure 5.35).The area function* $A(x)=\int_2^x f(t)dt$ gives the area of the trapezoid, for $x \geq 2$.

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a. Evaluate *A(2)*.
b. Evaluate *A(5)*.
c. Find and graph the area function $y = A(x)$ for $x \geq 2$.
d. Compare the derivative of *A* *to* *f* .

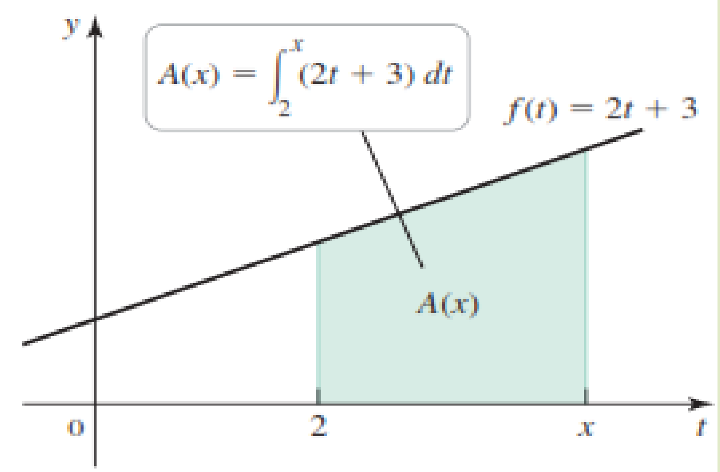

FIGURE 5.35

Solution

a. By Property 1 of Table 5.4,12 $A(2)=\int_2^2 f(t)dt = 0$

b. Notice that A(5) is the area of the trapezoid (Figure 5.35) bounded by the line $y = 2t + 3$ and the *t*-axis on the interval [2, 5]. Using the area formula for a trapezoid(Figure5.36),we find that

$$A(5) = \int_2^5 (2t+3)\,dt = \frac{1}{2}\underbrace{(5-2)}_{\text{distance between parallel sides}}\underbrace{(f(2)+f(5))}_{\text{sum of parallel side lengths}} = \frac{1}{2}\cdot 3(7+13) = 30.$$

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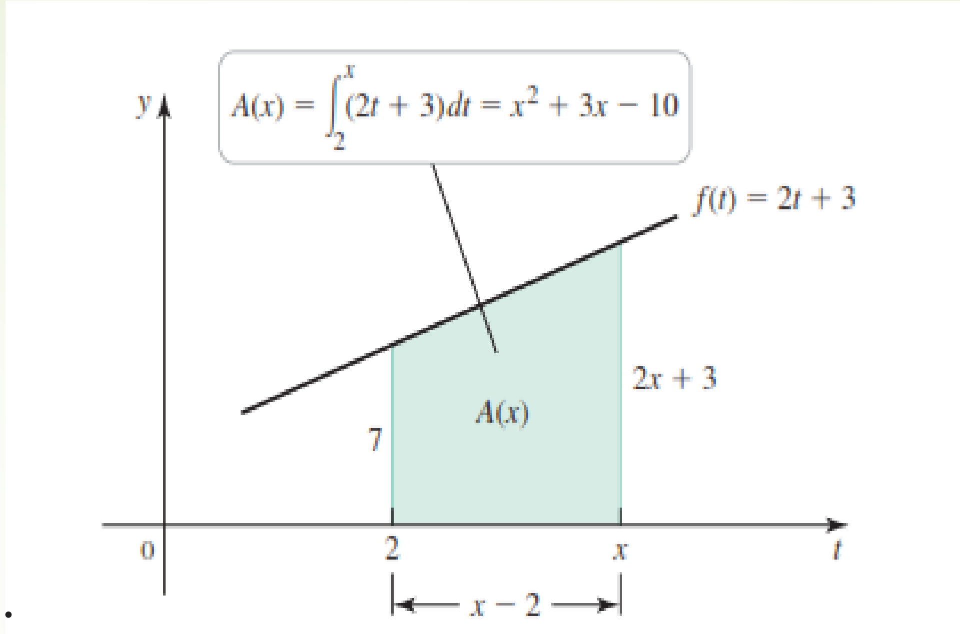
$$A(x) = \underbrace{\frac{1}{2}(x-2)}_{\text{distance between parallel sides}} \underbrace{(f(2)+f(x))}_{\text{sum of parallel side lengths}}$$

$$= \frac{1}{2}(x-2)(7+2x+3)$$

$$= (x-2)(x+5)$$

$$= x^2 + 3x - 10.$$

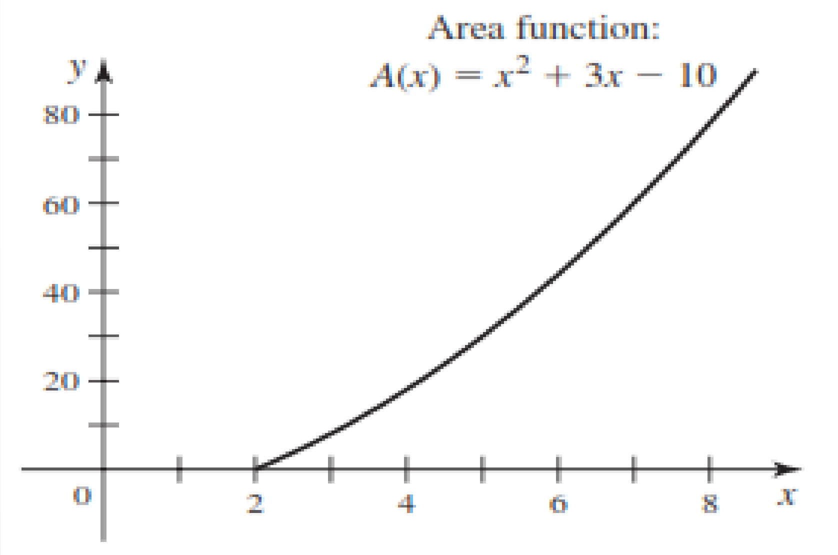


Because the line $f(t) = 2t + 3$ is above the t-axis, for $x \geq 2$. the area function $A(x) = x^2 + 3x - 10$, is an increasing function of x with $A(2) = 0$

d. Differentiating the area function, we find that:

$$A'(x) = \frac{d}{dx}(x^2 + 3x - 10) = 2x + 3 = f(x).$$

the area function A for a linear function f is an antiderivative of f; that is, $A'(x) = f(x)$.



Our goal is to show that this conjecture holds for more general functions

$$A'(x) = \frac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{A(x)} = f(x),$$

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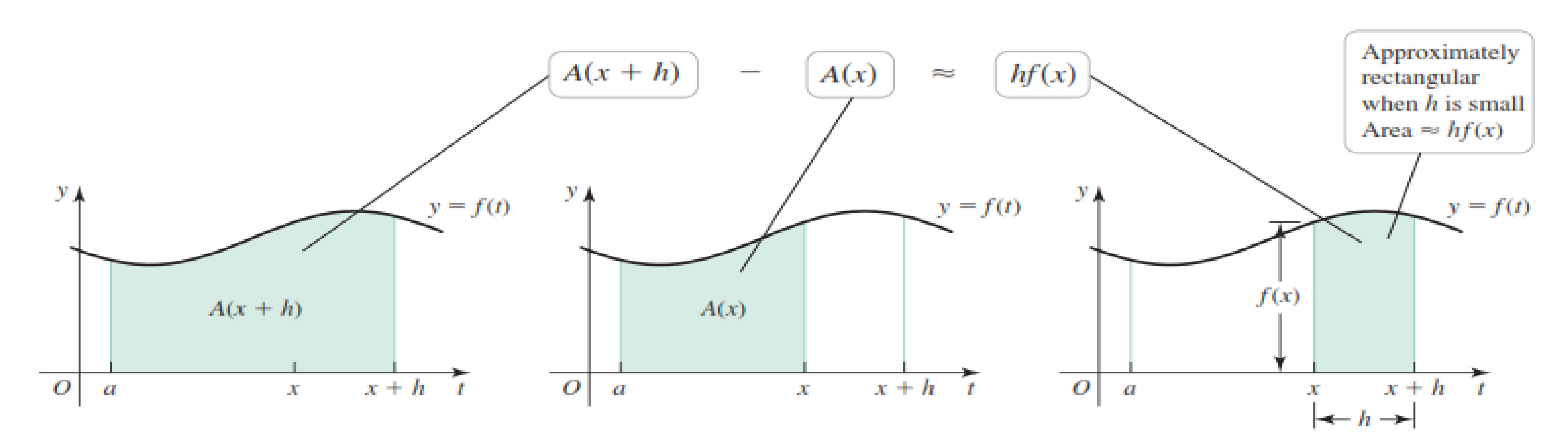
Assume that f is a continuous function defined on an interval $[a, b]$. As before, $A(x)$ it gives the net area of the region bounded by the graph of f and the t-axis between $t = a$ and $t = x$.



$$A(x+h) - A(x) \approx h f(x).$$

$$\frac{A(x+h) - A(x)}{h} \approx f(x). \quad \Rightarrow \quad \underbrace{\lim_{h\to 0}\frac{A(x+h) - A(x)}{h}}_{A'(x)} = \underbrace{\lim_{h\to 0} f(x)}_{f(x)}.$$

$$\Rightarrow \quad A'(x) = \frac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{A(x)} = f(x),$$

THEOREM 5.3 (PART 1) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for} \quad a \le x \le b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$; or, equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.

Example

"Derivatives of integrals"

Use Part 1 of the Fundamental Theorem to simplify the following expressions:

a. $\dfrac{d}{dx}\displaystyle\int_1^x \sin^2 t\,dt$ **b.** $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{t^2+1}\,dt$ **c.** $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \cos t^2\,dt$

Solution

a. Using Part 1 of the Fundamental Theorem, we see that

$$\frac{d}{dx}\int_1^x \sin^2 t\,dt = \sin^2 x.$$

b. To apply Part 1 of the Fundamental Theorem, the variable must appear in the upper limit. Therefore, we use the fact that:

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

and then apply the Fundamental Theorem:

$$\frac{d}{dx}\int_x^5 \sqrt{t^2+1}\,dt = -\frac{d}{dx}\int_5^x \sqrt{t^2+1}\,dt = -\sqrt{x^2+1}.$$

c. The upper limit of the integral is not x, but a function of x. Therefore, the function to be differentiated is a composite function, which requires the Chain Rule. We let $u = x^2$ to produce:

$$y = g(u) = \int_0^u \cos t^2\,dt.$$

By the Chain Rule,

$$\frac{d}{dx}\int_0^{x^2} \cos t^2\,dt = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left[\frac{d}{du}\int_0^u \cos t^2\,dt\right](2x) = (\cos u^2)(2x) = 2x\cos x^4.$$

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THEOREM 5.3 (PART 2) Fundamental Theorem of Calculus
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Given that *A is an antiderivative of f* on $[a, b]$, it is one short step to a powerful method for evaluating definite integrals. Remember (Section 4.9) that any two antiderivatives of f differ by a constant. Assuming that *F is any* other antiderivative of f *on* $[a, b]$, we have:

$$F(x) = A(x) + C, \text{ for } a \le x \le b.$$

Noting that $A(a) = 0$, it follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b).$$

Writing $A(b)$ in terms of a definite integral leads to the remarkable result:

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a).$$

We have shown that to evaluate a definite integral of f, *we*

- find any antiderivative of f, *which we call F;*
- compute $F(b) - F(a)$,

Example

Finding antiderivatives. Use what you know about derivatives to find all antiderivatives of the following functions.

a. $f(x) = 3x^2$ **b.** $f(x) = \dfrac{1}{1 + x^2}$ **c.** $f(x) = \sin x$

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Solution

a. Note that $\frac{d}{dx}(x^3) = 3x^2$. Therefore, an antiderivative of $f(x) = 3x^2$ is x^3 By Theorem 4.16, the complete family of antiderivatives is $F(x) = x^3 + C$, where C is an arbitrary constant.

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2},$$

$$\frac{d}{dx}(\cos x) = -\sin x.$$

THEOREM 4.17 Power Rule for Indefinite Integrals

$$\int x^p \, dx = \frac{x^{p+1}}{p + 1} + C,$$

where $p \neq -1$ is a real number and C is an arbitrary constant.

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Limits of integration

x is variable of integration

Antiderivative of f evaluated at a and b

Integrand

Shorthand notation

The Inverse Relationship between Differentiation and Integration

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

Example

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Evaluate the following definite integrals using the Fundamental Theorem of Calculus, Part 2. Interpret each result geometrically.

a. $\int_0^{10} (60x - 6x^2)\,dx$ **b.** $\int_0^{2\pi} 3 \sin x\,dx$ **c.** $\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t}\,dt$

Solution

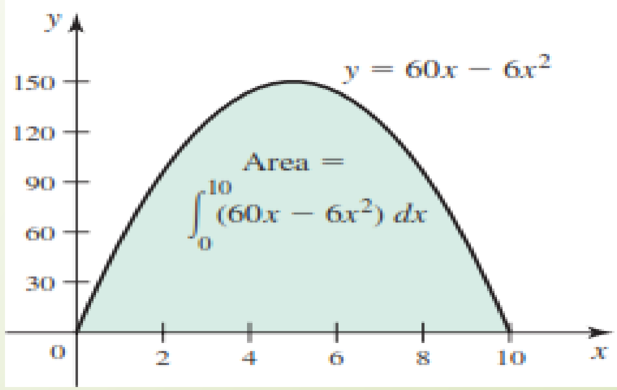
a. Using the antiderivative rules of **Section 4.9**, an antiderivative of $60x - 6x^2$ is 30 x^2 - 2 x^3. By the Fundamental Theorem, the value of the definite integral is:

$$\int_0^{10} (60x - 6x^2)\,dx = (30x^2 - 2x^3)\Big|_0^{10}$$ Evaluate at $x = 10$ and $x = 0$.

$$= (30 \cdot 10^2 - 2 \cdot 10^3) - (30 \cdot 0^2 - 2 \cdot 0^3)$$ Simplify.

$$= (3000 - 2000) - 0$$

$$= 1000.$$



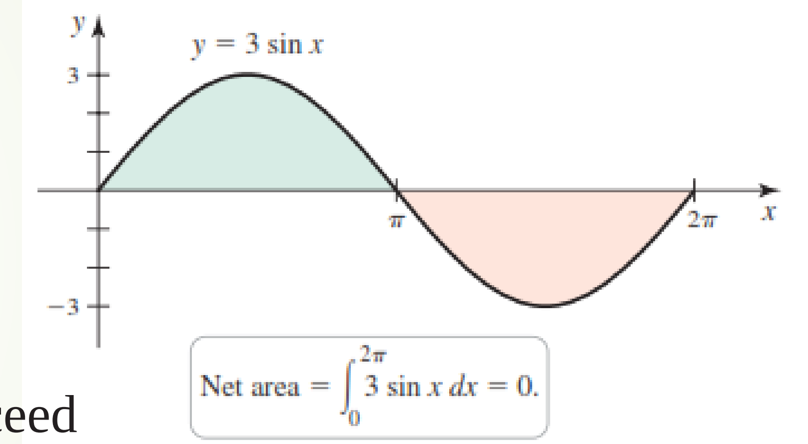
b. As shown in Figure 5.42, the region bounded by the graph of $f(x) = 3 \sin x$ and the x-axis on $[0, 2\pi]$ consists of two parts, one above the x-axis and one below the x-axis. By the symmetry of f, these two regions have the same area, so the definite integral over $[0, 2\pi]$ is zero.

Let's confirm this fact. An antiderivative of $f(x) = 3 \sin x$ is $-3 \cos x$. Therefore, the value of the definite integral is

$$\int_0^{2\pi} 3 \sin x \, dx = -3 \cos x \Big|_0^{2\pi}$$

$$= (-3 \cos(2\pi)) - (-3 \cos(0))$$

$$= -3 - (-3) = 0.$$



c. Although the variable of integration is t, rather than x, we proceed as in parts (a) and (b) after simplifying the integrand:

$$\int_{1/16}^{1/4} \frac{\sqrt{t} - 1}{t} dt = \int_{1/16}^{1/4} \left(t^{-1/2} - \frac{1}{t}\right) dt = 2t^{1/2} - \ln|t| \Big|_{1/16}^{1/4}$$

$$= \left[2\left(\frac{1}{4}\right)^{1/2} - \ln\frac{1}{4}\right] - \left[2\left(\frac{1}{16}\right)^{1/2} - \ln\frac{1}{16}\right]$$

$$= 1 - \ln\frac{1}{4} - \frac{1}{2} + \ln\frac{1}{16} = \frac{1}{2} - \ln 4 \approx -0.8863.$$

We know that $\frac{d}{dt}(t^{1/2}) = \frac{1}{2}t^{-1/2}.$

Therefore, $\int \frac{1}{2} t^{-1/2} dt = t^{1/2} + C$

Example **Net areas and definite integrals**

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The graph of $f(x) = 6x(x+1)(x-2)$ is shown in Figure 5.44. The region R_1 is bounded by the curve and the x-axis on the interval [-1, 0], and R_2 is bounded by the curve and the x-axis on the interval [0, 2]

a. Find the net area of the region between the curve and the x-axis on [-1, 2].

b. Find the area of the region between the curve and the x-axis on [-1, 2].

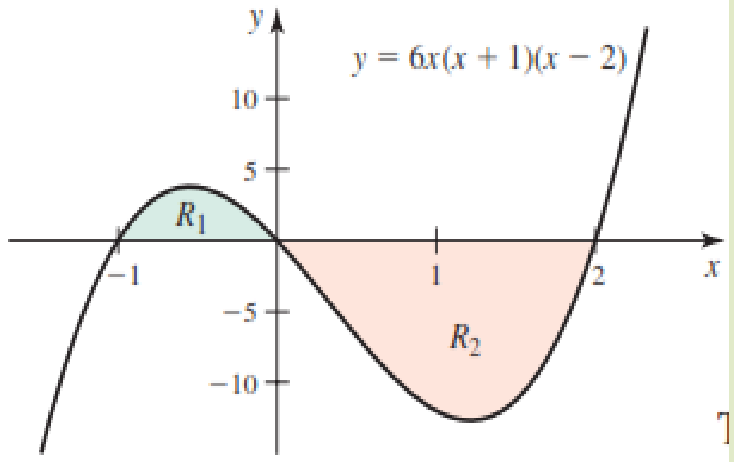

FIGURE 5.44

Solution

a. The net area of the region is given by a definite integral. The integrand f is first expanded in order to find an antiderivative:

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} (6x^3 - 6x^2 - 12x)\,dx. \quad \text{Expanding } f$$

$$= \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{-1}^{2} \quad \text{Fundamental Theorem}$$

$$= -\frac{27}{2}. \quad \text{Simplify.}$$

b. The region R_1 lies above the x-axis, so its area is

$$\int_{-1}^{0}(6x^3 - 6x^2 - 12x)\,dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{-1}^{0} = \frac{5}{2}.$$

The region R_2 lies below the x-axis, so its net area is negative:

$$\int_{0}^{2}(6x^3 - 6x^2 - 12x)\,dx = \left(\frac{3}{2}x^4 - 2x^3 - 6x^2\right)\Big|_{0}^{2} = -16$$

Therefore, the *area of* R_2 is – (-16)=16

The combined area of R_1 and R_2 is $\frac{5}{2} + 16 = \frac{37}{2}$

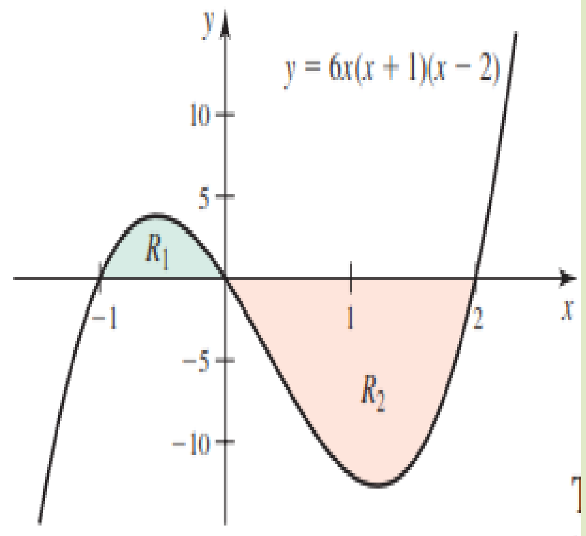

FIGURE 5.44

We could also find the area of this region directly by evaluating $\int_{-1}^{2}|f(x)|dx$

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Quiz:

1. Simplify the following expressions.

$$\frac{d}{dx}\int_3^x (t^2+t+1)\,dt$$

$$\frac{d}{dx}\int_{x^2}^{10} \frac{dz}{z^2+1}$$

$$\frac{d}{dx}\int_{-x}^{x} \sqrt{1+t^2}\,dt$$

$$\frac{d}{dx}\int_{x}^{0} \frac{dp}{p^2+1}$$

2. Evaluate the following integrals using the Fundamental Theorem of Calculus.

$$\int_0^2 4x^3\,dx$$

$$\int_0^2 (3x^2+2x)dx$$

$$\int_0^4 x(x-2)(x-4)\,dx$$

$$\int_0^{\pi/4} \sec^2\theta\,d\theta$$

5.4 Working with Integrals

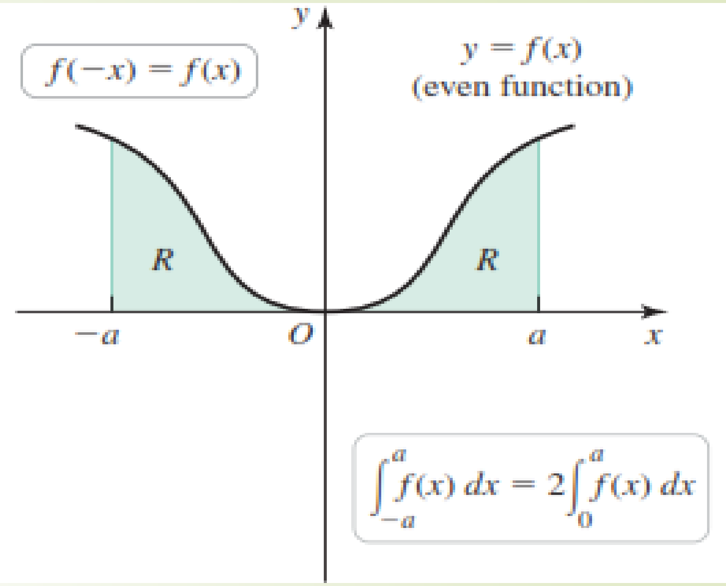
Lecture 3

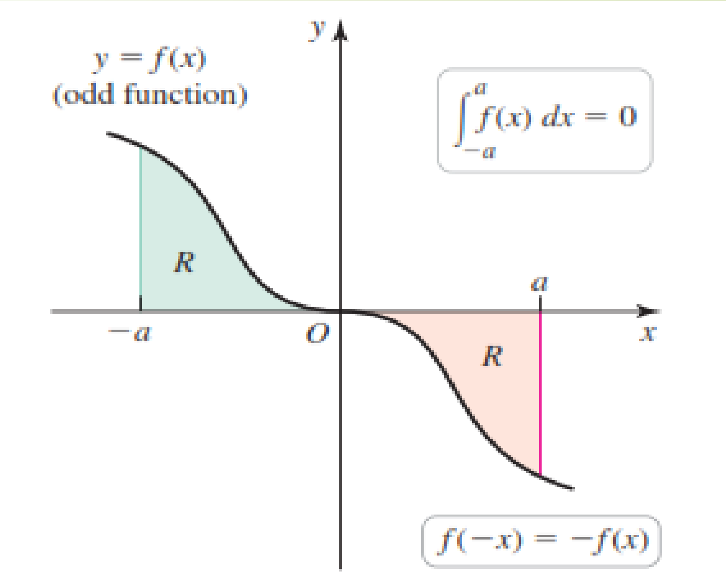
Integrating Even and Odd Functions

THEOREM 5.4 Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.





Example

Evaluate the following integrals using symmetry arguments.

a. $\int_{-2}^{2} (x^4 - 3x^3)\,dx$ **b.** $\int_{-\pi/2}^{\pi/2} (\cos x - 4\sin^3 x)\,dx$

Solution

a. Using Properties 3 and 4 of Table 5.4, we split the integral into two integrals and use symmetry:

$$\int_{-2}^{2}(x^4 - 3x^3)\,dx = \int_{-2}^{2} x^4\,dx - 3\underbrace{\int_{-2}^{2} x^3\,dx}_{0}$$

x^4 is even, x^3 is odd.

$$= 2\int_{0}^{2} x^4\,dx - 0 \qquad = 2\left(\frac{x^5}{5}\right)\Bigg|_0^2$$

$$= 2\left(\frac{32}{5}\right) = \frac{64}{5}.$$

$$\int_{-\pi/2}^{\pi/2}(\cos x - 4\sin^3 x)\,dx = 2\int_{0}^{\pi/2} \cos x\,dx - 0$$

$$= 2\sin x\Big|_0^{\pi/2}$$

$$= 2(1 - 0) = 2.$$

Average Value of a Function

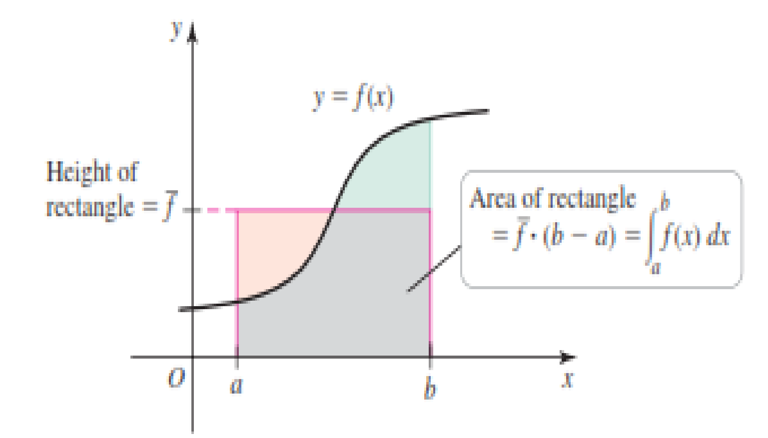
Lecture 3

DEFINITION Average Value of a Function
The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The average value of a function f on an interval $[a, b]$ has a clear geometrical interpretation. Multiplying both sides of the definition of average value by $(b - a)$, *we have*

$$\underbrace{(b-a)\bar{f}}_{\text{net area of rectangle}} = \underbrace{\int_a^b f(x)\,dx.}_{\text{net area of region bounded by curve}}$$



We see that the average value is the height of the rectangle with base $[a, b]$ that has the same net area as the region bounded by the graph of f on the interval $[a, b]$. (We need to use net area incase f is negative on part of $[a, b]$, which could make $\bar{f}$ negative.)

Lecture 3

Example

Average elevation A hiking trail has an elevation given by

$$f(x) = 60x^3 - 650x^2 + 1200x + 4500,$$

where f is measured in feet above sea level and x represents horizontal distance along the trail in miles, with $0 \le x \le 5$. What is the average elevation of the trail?

Solution

The trail ranges between elevations of about 2000 and 5000 ft (Figure 5.52). If we let the endpoints of the trail correspond to the horizontal distances $a = 0$ and $b = 5$, the average elevation of the trail in feet is

$$\bar{f} = \frac{1}{5}\int_0^5 (60x^3 - 650x^2 + 1200x + 4500)\,dx$$

$$= \frac{1}{5}\left(60\frac{x^4}{4} - 650\frac{x^3}{3} + 1200\frac{x^2}{2} + 4500x\right)\Big|_0^5 \quad \text{Fundamental Theorem}$$

$$= 3958\tfrac{1}{3}. \quad \text{Simplify.}$$

The average elevation of the trail is slightly less than 3960 ft.

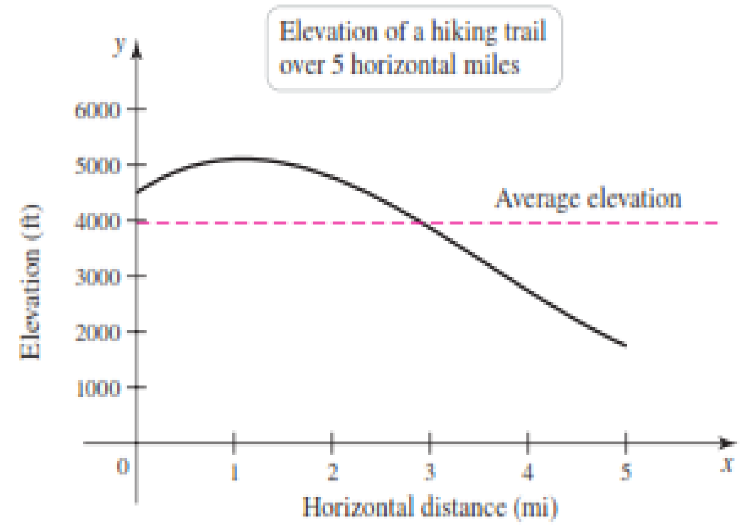


FIGURE 5.52

Mean Value Theorem for Integrals

Lecture 3

The average value of a function brings us close to an important theoretical result. The Mean Value Theorem for Integrals says that if f *is continuous on* $[a, b]$ then there is at least one point c in the interval $[a, b]$ such that f (c) equals the average value of f on $[a, b]$.

THEOREM 5.5 Mean Value Theorem for Integrals

Let f be continuous on the interval $[a, b]$. There exists a point c in $[a, b]$ such that

$$f(c) = \bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Proof: We begin by letting $F(x) = \int_a^x f(t)\,dt$ and noting that F is continuous on $[a, b]$ and differentiable on (a, b) (by Theorem 5.3, Part 1). We now apply the Mean Value Theorem for derivatives (Theorem 4.9) to F and conclude that there exists at least one point c in (a, b) such that

$$\underbrace{F'(c)}_{f(c)} = \frac{F(b) - F(a)}{b - a}.$$

By Theorem 5.3, Part 1, we know that $F'(c) = f(c)$ and by Theorem 5.3, Part 2, we know that

$$F(b) - F(a) = \int_a^b f(t)dt.$$

Combining these observations, we have

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt,$$ where c is a point in (a, b).

Example

Average value equals function value Find the point(s) on the interval $[0, 1]$ at which $f(x) = 2x(1 - x)$ equals its average value on $[0, 1]$.

Solution

The average value of f *on* $[0, 1]$ is

$$\bar{f} = \frac{1}{1-0}\int_0^1 2x(1-x)\,dx = \left(x^2 - \frac{2}{3}x^3\right)\Big|_0^1 = \frac{1}{3}.$$

We must find the points on $[0, 1]$ at which $f(x) = \frac{1}{3}$. Using the quadratic formula, the two solutions of $f(x) = 2x(1-x) = \frac{1}{3}$ are

$$\frac{1-\sqrt{1/3}}{2} \approx 0.211 \quad \text{and} \quad \frac{1+\sqrt{1/3}}{2} \approx 0.789.$$


Thank
You!