CH.7 Integration Techniques

Lecture 4

Chapter Summary

7.1 Basic Approaches
7.2 Integration by Parts
7.3 Trigonometric Integrals
7.4 Trigonometric Substitutions
7.5 Partial Fractions
7.6 Other Integration Strategies

7.2 Integration by Parts

Lecture 4

Consider the indefinite integrals

$$\int e^x\,dx = e^x + C \quad \text{and} \quad \int xe^x\,dx = ?$$

The first integral is an elementary integral that we have already encountered.

The second integral is only slightly different—and yet, the appearance of the product xe^x in the integrand makes this integral (at the moment) impossible to evaluate. Integration by parts is ideally suited for evaluating integrals of products of functions.

Integration by Parts for Indefinite Integrals

Given two differentiable functions u *and* v, *the Product Rule states that*

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x).$$

By integrating both sides, we can write this rule in terms of an indefinite integral:

$$\int u(x)\underbrace{v'(x)\,dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)\,dx}_{du} \;\longrightarrow\; \int u\,dv = uv - \int v\,du.$$

Lecture 4

Integration by Parts

Suppose that u and v are differentiable functions. Then

$$\int u\,dv = uv - \int v\,du.$$

The integral $\int u\,dv$ *is viewed as the given integral, and we use integration by parts to express* it in terms of a new integral $\int v\,du$. *The technique is successful if the new integral* can be evaluated.

Example

"Integration by parts" Evaluate $\int xe^x\,dx.$

Solution

Functions in original integral	$u = x$	$dv = e^x\,dx$
Functions in new integral	$du = dx$	$v = e^x$

The integration by parts rule is now applied:

$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du}.$$

$$= xe^x - e^x + C.$$ Evaluate the new integral.

Lecture 4

Example **"Integration by parts"** **Evaluate** $\int x \sin x \, dx.$

Solution Remembering that powers of x are often a good choice for u, we form the following table.

$u = x$	$dv = \sin x \, dx$
$du = dx$	$v = -\cos x$

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv} = \underbrace{x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{dx}_{du}$$

$$= -x \cos x + \sin x + C.$$

Integration by parts

Evaluate $\int \cos x \, dx = \sin x.$

In general, integration by parts works when we can easily integrate the choice for and when the new integral is easier to evaluate than the original.

Example **"Repeated use of integration by parts"** **Evaluate** $\int x^2 e^x \, dx.$

Solution The factor x^2 is a good choice for u, leaving $dv = e^x \, dx$. We then have

$$\int \underbrace{x^2}_{u} \underbrace{e^x \, dx}_{dv} = \underbrace{x^2}_{u} \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{2x \, dx}_{du}.$$

Notice that the new integral on the right side is simpler than the original integral because the power of x has been reduced by one

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx$$

$$= x^2 e^x - 2(xe^x - e^x) + C$$

$$= e^x(x^2 - 2x + 2) + C.$$

Example **“Repeated use of integration by parts”** **Evaluate** $\int e^{2x}\sin x\,dx.$

Solution

the integrand consists of a product, which suggests integration by parts. In this case there is no obvious choice for u and dv, so let’s try the following choices:

$u = e^{2x}$	$dv = \sin x\,dx$
$du = 2e^{2x}\,dx$	$v = -\cos x$

The integral then becomes

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx.$$

Suppose we evaluate $\int e^{2x}\cos x\,dx$ sing integration by parts with the following choices:

$u = e^{2x}$	$dv = \cos x\,dx$
$du = 2e^{2x}\,dx$	$v = \sin x$

Integrating by parts, we have:

$$\int e^{2x}\cos x\,dx = e^{2x}\sin x - 2\int e^{2x}\sin x\,dx.$$

Now observe that equation (2) contains the original Integral, $\int e^{2x}\sin x\,dx.$ *Substituting* the result of equation (2) into equation (1), we find that

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$

$$= -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right)$$ Substitute for $\int e^{2x}\cos x\,dx.$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx.$$ Simplify.

Now it is a matter of solving for $\int e^{2x}\sin x\,dx$ *and including the constant of integration:*

$$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C.$$

Evaluate

$\int e^{3x}\cos 2x\,dx$ $\int e^{-x}\sin 4x\,dx$ $\int x^2 e^{4x}\,dx$

$\int e^{x}\cos x\,dx$ $\int x^2\ln^2 x\,dx$ $\int x^2\sin 2x\,dx$

Integration by Parts for Definite Integrals

Lecture 4

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

Example "A definite integral" **Evaluate** $\int_1^2 \ln x\,dx.$

$u = \ln x$	$dv = dx$
$du = \frac{1}{x}\,dx$	$v = x$

Solution

$$\int_1^2 \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = \big(\underbrace{(\ln x)}_{u}\,\underbrace{x}_{v}\big)\Big|_1^2 - \int_1^2 \underbrace{x}_{v}\,\underbrace{\frac{1}{x}\,dx}_{du} \qquad \text{Integration by parts}$$

$$= x\ln x\Big|_1^2 - \int_1^2 dx \qquad \text{Evaluate.}$$

$$= (2\ln 2 - 0) - (2 - 1)$$

$$= 2\ln 2 - 1 \approx 0.386.$$

Evaluate $\displaystyle\int_1^{e^2} x^2 \ln x\,dx$

Quiz:

Lecture 4

Integral of ln x

$$\int \ln x \, dx = x \ln x - x + C$$

Evaluate

13. $\int x^2 \ln x^3 \, dx$

14. $\int \theta \sec^2 \theta \, d\theta$

15. $\int x^2 \ln x \, dx$

16. $\int x \ln x \, dx$

17. $\int \frac{\ln x}{x^{10}} \, dx$

18. $\int \sin^{-1} x \, dx$

19. $\int \tan^{-1} x \, dx$

20. $\int x \sec^{-1} x \, dx, \ x \geq 1$

21. $\int x \sin x \cos x \, dx$

22. $\int x \tan^{-1} x^2 \, dx$

7.3 Trigonometric Integrals

Lecture 4

1. Integrating Powers of *sin x* or *cos x*

$\int \sin^m x\,dx$ $\int \cos^n x\,dx,$ where *m and n are positive integers.*

Example

"Powers of sine or cosine" Evaluate the following integrals.

a. $\int \cos^5 x\,dx$

b. $\int \sin^4 x\,dx$

Solution

a. Integrals involving odd powers of *cos x* (or *sin x)* are most easily evaluated by splitting off a single factor of *cos x* (or *sin x*) In this case, we rewrite $cos^5(x)$ as $cos^4(x)$. Cos(*x*). Now, $cos^4(x)$ can be written in terms of *sin x* using the identity $cos^2 x$ = 1 - $sin^2 x$. *The* result is an integrand that readily yields to the substitution *u = sin x:*

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx$$

$$= \int (1 - \sin^2 x)^2 \cdot \cos x\,dx = \int (1 - u^2)^2\,du \qquad \text{Let } u = \sin x;\ du = \cos x\,dx.$$

$$= \int (1 - 2u^2 + u^4)\,du \qquad \text{Expand.}$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

Replace u with $\sin x$.

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

b. With even powers of *sin x* or *cos x*, we use the half-angle formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the powers in the integrand:

$$\int \sin^4 x\,dx = \int \Big(\underbrace{\frac{1 - \cos 2x}{2}}_{\sin^2 x}\Big)^2 dx$$

Half-angle formula

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx.$$

Using the half-angle formula again for $cos^2 2x$, the evaluation may be completed:

$$\int \sin^4 x\,dx = \frac{1}{4}\int \Big(1 - 2\cos 2x + \underbrace{\frac{1 + \cos 4x}{2}}_{\cos^2 2x}\Big) dx \quad = \frac{1}{4}\int \Big(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\Big) dx$$

$$= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$$

Integrating Products of Powers of sin *x* and cos *x*

$$\int \sin^m x \cos^n x \, dx.$$

Lecture 4

1. If *m is an odd*, positive integer, we split off a factor of *sin x* and write the remaining even power of sin *x* in terms of cosine Functions

   "This step prepares the integrand for the substitution *u = cos x, and the* resulting integral is readily evaluated."

A similar strategy is used when *n* is an odd, positive integer.

2. If both *m* and *n* are even positive integers, the half-angle formulas are used to transform the integrand into a polynomial in *cos 2x*, each of whose terms can be integrated.

Example

"Products of sine and cosine" Evaluate the following integrals.

a. $\int \sin^4 x \cos^2 x \, dx$ **b.** $\int \sin^3 x \cos^{-2} x \, dx$

Solution

a. When both powers are even, the half-angle formulas are used:

$$\int \sin^4 x \cos^2 x \, dx = \int \Bigg(\underbrace{\frac{1-\cos 2x}{2}}_{\sin^2 x}\Bigg)^2 \Bigg(\underbrace{\frac{1+\cos 2x}{2}}_{\cos^2 x}\Bigg) dx$$

Half-angle formulas

Dr. Mohamed Abdel-Aal
Calculus II

$$= \frac{1}{8}\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\, dx.$$

The third term in the integrand is rewritten with a half-angle formula. For the last term, a factor of *cos 2x* is split off, and the resulting even power of *cos 2x* is written in terms of *sin 2x* to prepare for a *u*-substitution:

$$\int \sin^4 x \cos^2 x\, dx = \frac{1}{8}\int \left[1 - \cos 2x - \overbrace{\left(\frac{1 + \cos 4x}{2}\right)}\right] dx + \frac{1}{8}\int \overbrace{(1 - \sin^2 2x)}^{\cos^2 2x} \cdot \cos 2x\, dx.$$

Finally, the integrals are evaluated, using the substitution $u = \sin 2x$ *for the second* integral. After simplification, we find that

$$\int \sin^4 x \cos^2 x\, dx = \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C.$$

b. When at least one power is odd, the following approach works:

$$\int \sin^3 x \cos^{-2} x\, dx = \int \sin^2 x \cos^{-2} x \cdot \sin x\, dx$$ Split off $\sin x$.

$$= \int (1 - \cos^2 x) \cos^{-2} x \cdot \sin x\, dx$$ Pythagorean identity

$$= -\int (1 - u^2)\, u^{-2}\, du$$ $u = \cos x$; $du = -\sin x\, dx$

$$= \int (1 - u^{-2})\, du = u + \frac{1}{u} + C$$ Evaluate the integral.

$$= \cos x + \sec x + C.$$ Replace u with $\cos x$.

QUICK CHECK 2 What strategy would you use to evaluate $\int \sin^3 x \cos^3 x\, dx$?

Lecture 4

summarizes the techniques used to evaluate integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

Table 7.2

$\int \sin^m x \cos^n x \, dx$	**Strategy**
m odd, n real	Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, and then use $u = \cos x$.
n odd, m real	Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, and then use $u = \sin x$.
m and n both even, nonnegative integers	Use half-angle identities to transform the integrand into a polynomial in $\cos 2x$, and apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formulas

Lecture 4

A reduction formula equates an integral involving a power of a function with another integral in which the power is reduced;

Reduction Formulas

Assume n is a positive integer.

1. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$
2. $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$
3. $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,\ n \neq 1$
4. $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx,\ n \neq 1$

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

2. Integrating Powers of *tan x* or *sec x*

Lecture 4

Example

"Powers of tan x " Evaluate $\int \tan^4 x\,dx.$

Solution

Reduction formula 3 gives

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \underbrace{\int \tan^2 x\,dx}_{\text{use (3) again}}$$

$$= \frac{1}{3}\tan^3 x - \left(\tan x - \int \underbrace{\tan^0 x}_{=1}\,dx\right)$$

$$\boxed{= \frac{1}{3}\tan^3 x - \tan x + x + C.}$$

An alternative solution uses the identity $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^4 x\,dx = \int \tan^2 x\,\underbrace{(\sec^2 x - 1)}_{\tan^2 x}\,dx$$

$$= \int \underbrace{\tan^2 x}_{u^2}\,\underbrace{\sec^2 x\,dx}_{du} - \int \tan^2 x\,dx$$ Substitution and identity

$$= \int u^2\,du - \int (\sec^2 x - 1)\,dx$$ Evaluate integrals.

$$= \frac{u^3}{3} - \tan x + x + C$$ $u = \tan x$

$$\boxed{= \frac{1}{3}\tan^3 x - \tan x + x + C.}$$

Lecture 4

Note that for odd powers of *tan x and sec x*, the use of reduction formula 3 or 4 will eventually lead to $\int \tan x\, dx$ or $\int \sec x\, dx$. Theorem 7.1 gives these integrals, along with the integrals of $\cot x$ and $\csc x$.

THEOREM 7.1 Integrals of tan x, cot x, sec x, and csc x

$$\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C \qquad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

$u = \cos x;\ du = -\sin x\, dx$

$$= -\int \frac{1}{u}\, du = -\ln|u| + C$$

$$= -\ln|\cos x| + C.$$

$$\boxed{= \ln|(\cos x)^{-1}| + C = \ln|\sec x| + C.}$$

$$\int \sec x\,dx = \int \sec x \cdot \underbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}_{1}\,dx$$

Multiply integrand by 1.

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

Expand numerator.

$$= \int \frac{du}{u}$$

$u = \sec x + \tan x;\ du = (\sec^2 x + \sec x \tan x)\,dx$

$$= \ln|u| + C$$

Integrate.

$$= \ln|\sec x + \tan x| + C.$$

$u = \sec x + \tan x$

Derivations of the remaining integrals are left to Exercises .

Lecture 4

Integrating Products of Powers of *tan x and sec x*

$$\int \tan^m x \sec^n x \, dx$$

Integrals of the form $\int \tan^m x \sec^n x \, dx$

- if n is even, we split off a factor of $\sec^2 x$ and write the remaining even power of *sec x* in terms of *tan x*.
  "This step prepares the integral for the substitution $u = \tan x$". $\int \tan^3 x \sec^4 x \, dx$

- . If m is odd, we split off a factor of *sec x tan x* (the derivative of *sec x*),
  "which prepares the integral for the substitution $u = \sec x$". $\int \tan^3 x \sec^4 x \, dx$

- If m is even and n is odd, the integrand is expressed as a polynomial in *sec x*, each of whose terms is handled by a reduction formula.

$$\int \tan^2 x \sec x \, dx$$

Example

"Products of tan x and sec x" Evaluate the following integrals.

Lecture 4

a. $\int \tan^3 x \sec^4 x\, dx$ **b.** $\int \tan^2 x \sec x\, dx$

Solution

a. With an even power of *sec x*, we split off a factor of $\sec^2 x$, and prepare the integral for the substitution $u = \tan x$:

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x \sec^2 x \cdot \sec^2 x\, dx$$

$\sec^2 x = \tan^2 x + 1$

$$= \int \tan^3 x\,(\tan^2 x + 1) \cdot \sec^2 x\, dx$$

$u = \tan x;\ du = \sec^2 x\, dx$

$$= \int u^3 (u^2 + 1)\, du$$

$$\boxed{= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C.}$$

Evaluate; $u = \tan x$.

Because the integrand also has an odd power of *tan x*, an alternative solution is to split off a factor of *sec x tan x*, and prepare the integral for the substitution $u = \sec x$:

$$\int \tan^3 x \sec^4 x\, dx = \int \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec^3 x \cdot \sec x \tan x\, dx$$

$$= \int (\sec^2 x - 1) \sec^3 x \cdot \sec x \tan x\, dx$$

$u = \sec x;$
$du = \sec x \tan x\, dx$

$$= \int (u^2 - 1)\, u^3\, du \quad \boxed{= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C.}$$

Evaluate; $u = \sec x$.

b. In this case, we write the even power of *tan x* in terms of *sec x*:

$$\int \tan^2 x \sec x\, dx = \int (\sec^2 x - 1) \sec x\, dx$$

$\tan^2 x = \sec^2 x - 1$

$$= \int \sec^3 x\, dx - \int \sec x\, dx$$

$$= \overbrace{\frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\, dx}^{\text{reduction formula 4}} - \int \sec x\, dx$$

Add secant integrals;
use Theorem 7.1.

$$\boxed{= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C.}$$

Lecture 4

Table 7.3 summarizes the methods used to integrate $\int \tan^m x \sec^n x\,dx$. Analogous techniques are used for $\int \cot^m x \csc^n x\,dx$.

Table 7.3

$\int \tan^m x \sec^n x\,dx$	**Strategy**
n even	Split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$, and use $u = \tan x$.
m odd	Split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$, and use $u = \sec x$.
m even and n odd	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial in $\sec x$; apply reduction formula 4 to each term.

Integrals of the form $\int \sin mx \cos nx \, dx$ Use the following three identities to evaluate the given integrals.

$$\sin mx \sin nx = \frac{1}{2}\left[\cos\left((m-n)x\right) - \cos\left((m+n)x\right)\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin\left((m-n)x\right) + \sin\left((m+n)x\right)\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos\left((m-n)x\right) + \cos\left((m+n)x\right)\right]$$

Example

$$\int \cos 3x \cos 4x \, dx = \tfrac{1}{2}\int \left(\cos(-x) + \cos 7x\right) dx =$$

$$= \tfrac{1}{2}\int \left(\cos x + \cos 7x\right) dx = \tfrac{1}{2}\sin x + \tfrac{1}{14}\sin 7x + C$$

$$\int \sin 3x \cos 2x \, dx = \tfrac{1}{2}\int (\sin x + \sin 5x) \, dx = -\tfrac{1}{2}\cos x - \tfrac{1}{10}\cos 5x + C$$

Evaluate the following integrals.

15. $\int \sin^2 x \cos^2 x \, dx$

16. $\int \sin^3 x \cos^5 x \, dx$

19. $\int \cos^3 x \sqrt{\sin x} \, dx$

21. $\int \sin^5 x \cos^{-2} x \, dx$

23. $\int \sin^2 x \cos^4 x \, dx$

24. $\int \sin^3 x \cos^{3/2} x \, dx$

25. $\int \tan^2 x \, dx$

26. $\int 6 \sec^4 x \, dx$

28. $\int \tan^3 \theta \, d\theta$

31. $\int 10 \tan^9 x \sec^2 x \, dx$

36. $\int \dfrac{\sec^2 x}{\tan^5 x} \, dx$

39. $\int \dfrac{\csc^4 x}{\cot^2 x} \, dx$

38. $\int \sec^{-2} x \tan^3 x \, dx$

44. $\int_0^{\pi/4} \tan^3 \theta \sec^2 \theta \, d\theta$

34. $\int \sqrt{\tan x} \sec^4 x \, dx$

Lecture 4