Lecture 1

Calculus II

Textbook : Calculus for Scientists and Engineers, Early transcendentals

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Course Summary

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CH.5 Integration

Inverse Functions

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1.4 **Inverse Trigonometric Functions**

Derivatives and Integrals

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CH.5 Integration

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Chapter Summary

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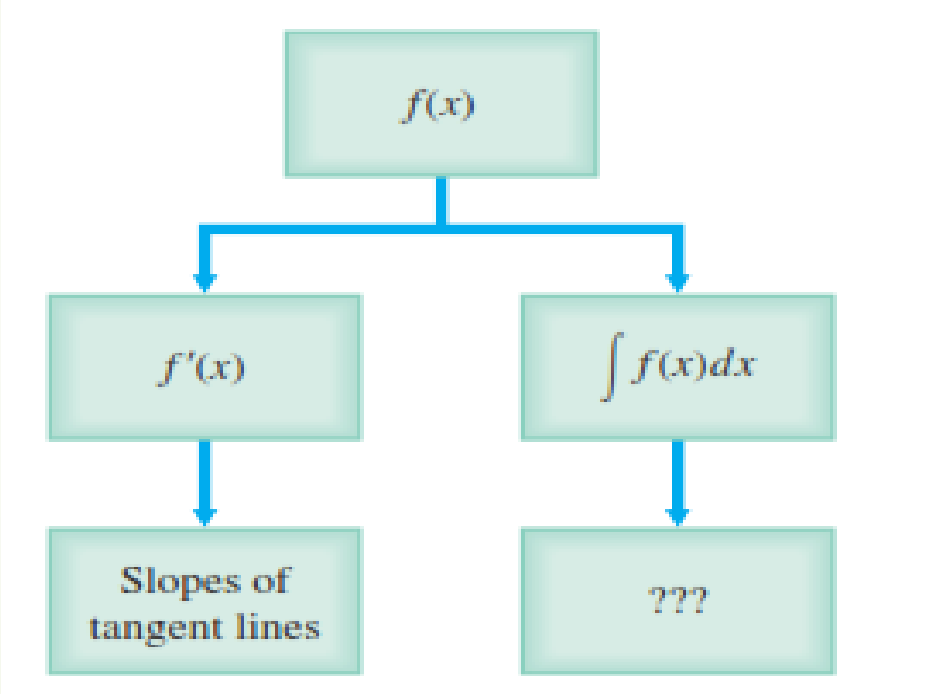
5.1 Approximating Areas under Curves

differentiation and integration.

The derivative of a function is associated with rates of change and slopes of tangent lines



antiderivatives (or indefinite integrals) reverse the derivative operation.

What is the geometric meaning of the integral?

The following example reveals a clue

Area under a Velocity Curve

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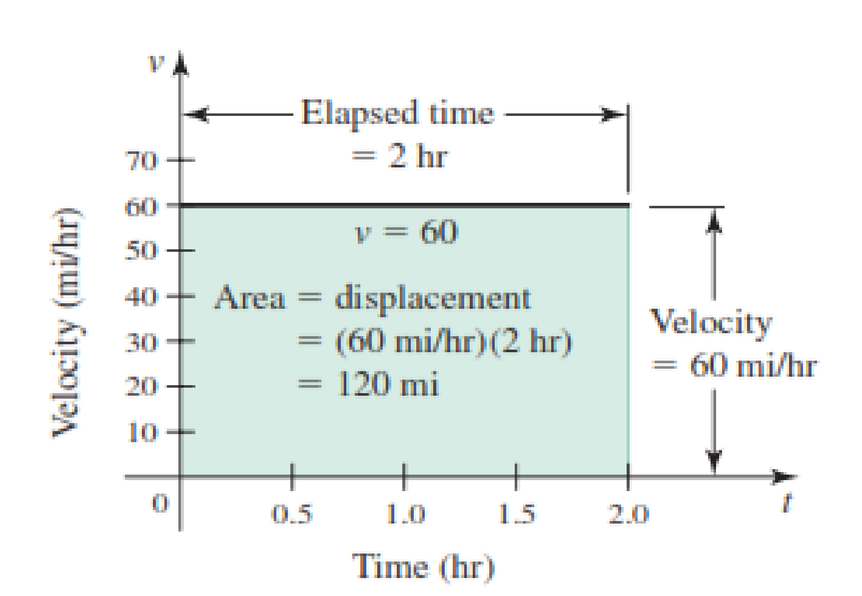
We begin by explaining why finding the area of regions bounded by the graphs of functions is such an important problem in calculus. Then you will see how antiderivatives lead to definite integrals,

Imagine a car traveling at a constant velocity of 60 mi / hr along a straight highway over a two-hour period. The graph of the velocity function $v = 60$ on the interval $0 \leq t \leq 2$ **is a horizontal line**

displacement = rate. Time
= 60 .2=120 mi

we see that the area between the velocity curve and the *t-axis* is *the displacement of the* moving object.

- we must extend these ideas to positive velocities that *change over an interval of time*



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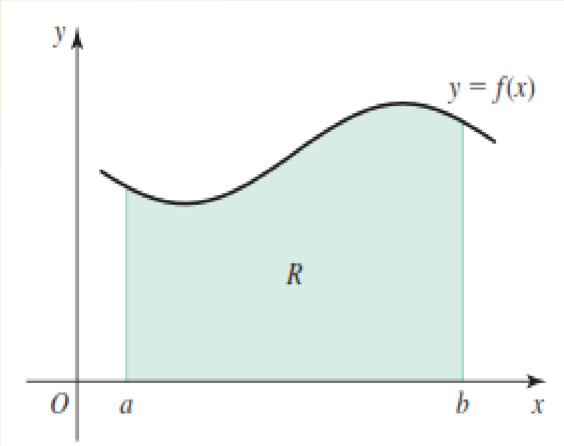
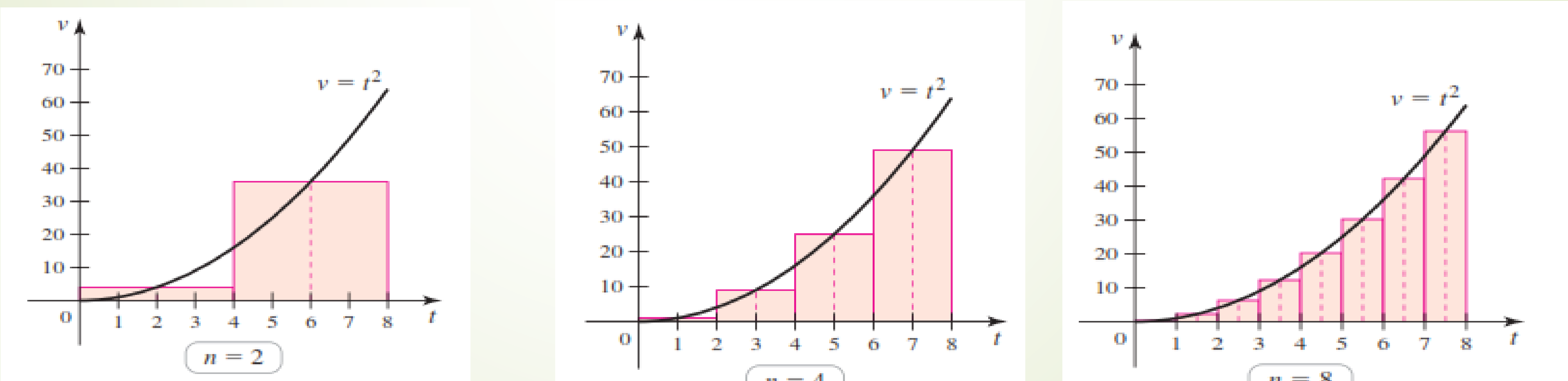
An approximation to the displacement:

- One strategy is to divide the time interval into many subintervals and approximate the velocity on each subinterval by a constant velocity. Then the displacements on each subinterval are calculated and summed.



Example

An object moving along a line is given by the function $v = t^2$, where $0 \le t \le 8$.

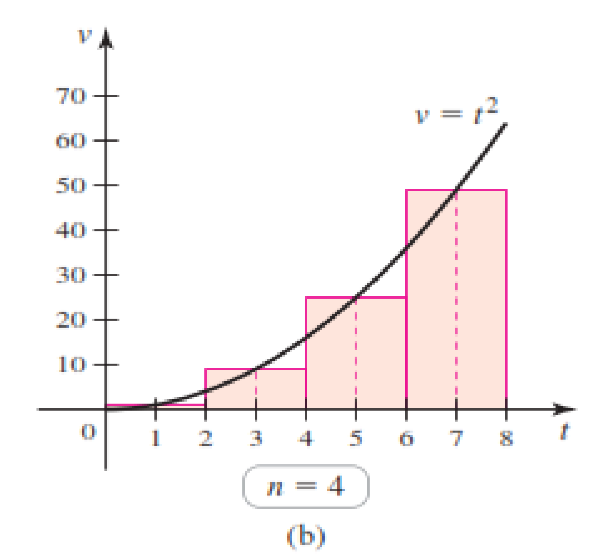


(a) (b) (c)

Approximate the displacement of the object by dividing the time interval [0,8] into *n subintervals* of equal length. On each subinterval, approximate the velocity by a constant equal to the value of v evaluated at the midpoint of the subinterval.

Solution

Divide [0, 8] into $n = 4$ *subintervals*:[0 2], [2, 4},[4, 6] , and [6,8]

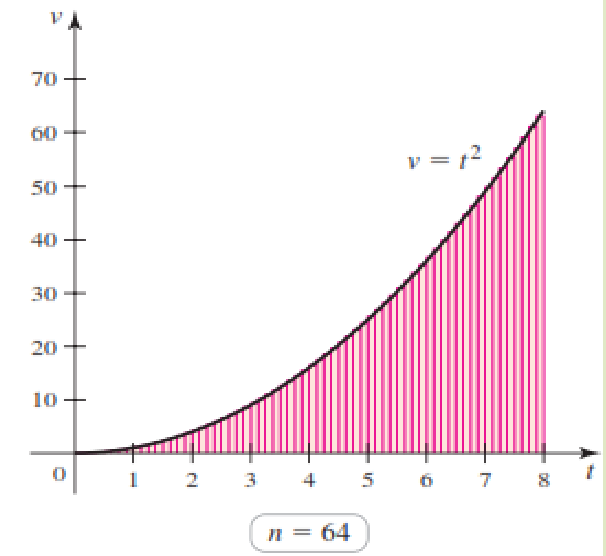


(b)

- With n = 4 (Figure 5.4b), each subinterval has length 2. The approximate displacement over the entire interval is

$$\underbrace{(1\,\text{m/s}\cdot 2\,\text{s})}_{v(1)} + \underbrace{(9\,\text{m/s}\cdot 2\,\text{s})}_{v(3)} + \underbrace{(25\,\text{m/s}\cdot 2\,\text{s})}_{v(5)} + \underbrace{(49\,\text{m/s}\cdot 2\,\text{s})}_{v(7)} = 168\,\text{m}.$$

- With $n = 8$ subintervals (Figure 5.4c), the approximation to the displacement is 170 m.



- ❑ this approximation generally improves as the number of subintervals increases
- ❑ The limit is the exact displacement, which is represented by the area of the region under the velocity curve

Approximating Areas by Riemann Sums:

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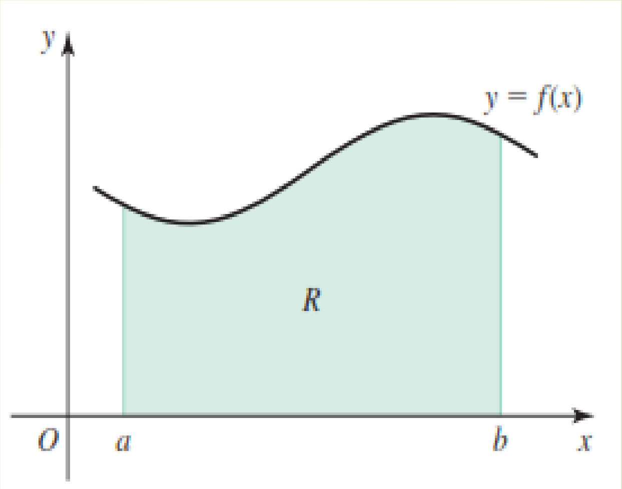
We now develop a method for approximating areas under curves.

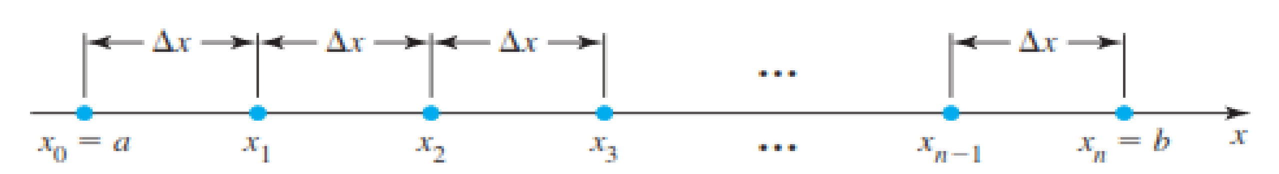
- We begin by dividing the interval $[a, b]$ into n subintervals of equal length,

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$



- The length of each subinterval, denoted Δx, $\Delta x = \dfrac{b-a}{n}.$

grid points

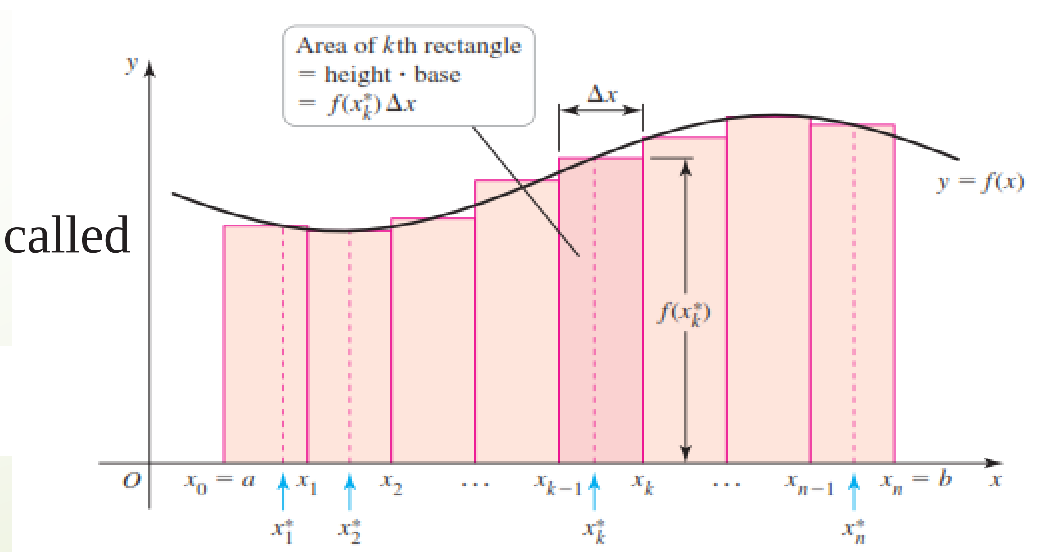


- In general, the *kth grid point is* $x_k = a + k\Delta x$, for $k = 0, 1, 2, \ldots, n.$
- The area of the rectangle on the *kth subinterval* is:

$$\text{height} \cdot \text{base} = f(x_k^*)\Delta x,$$

- we obtain an approximation to the area of R, which is called a **Riemann sum:**

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$



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DEFINITION Riemann Sum

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (Figure 5.9);
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (Figure 5.10); and
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (Figure 5.11), for $k = 1, 2, \ldots, n$.

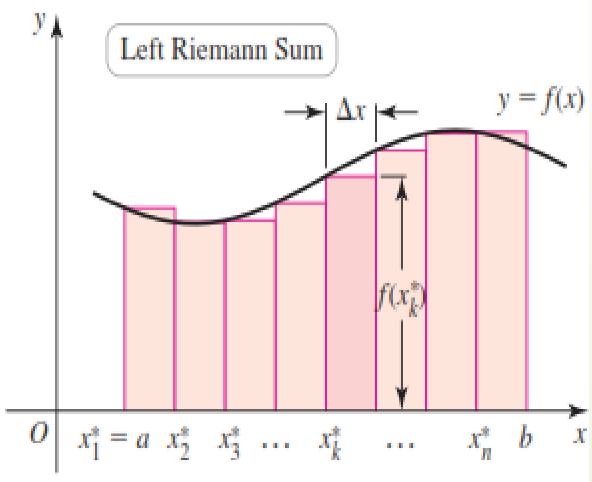

FIGURE 5.9

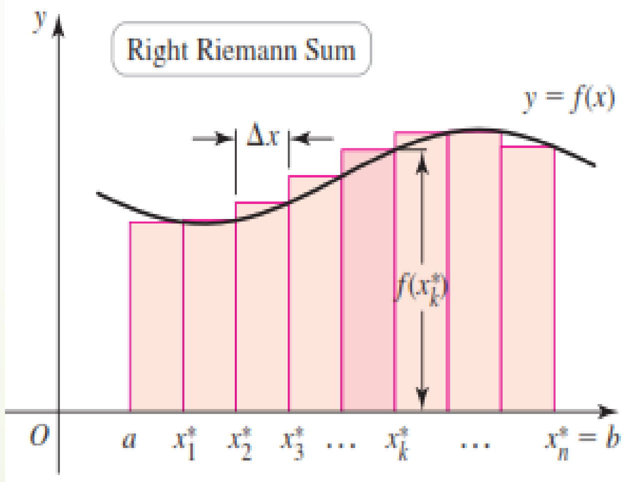

FIGURE 5.10

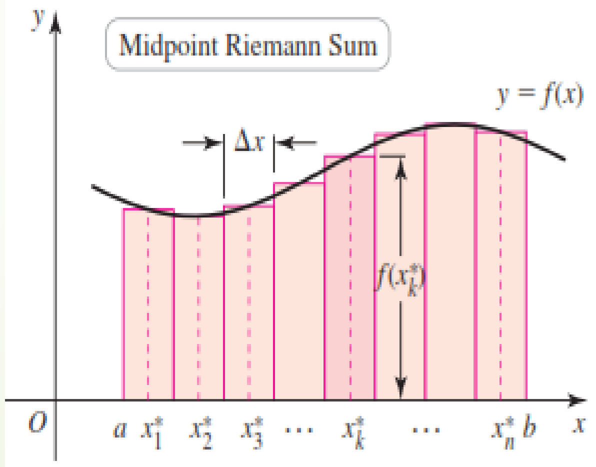

FIGURE 5.11

Example

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Area under the sine curve Let R be the region bounded by the graph of $f(x) = \sin x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$.

a) *Approximate the area of R using a left Riemann sum with* $n = 6$ *subintervals.* Illustrate the sum with the appropriate rectangles.

b) *Approximate the area of R* using a *right Riemann sum* with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

c) How do the area approximations in parts (a) and (b) compare to the actual area under the curve?

d) *Approximate the area of R* using a *midpoint Riemann sum* with $n = 6$ subintervals. Illustrate the sum with the appropriate rectangles.

Solution

Dividing the interval $[a, b] = [0, \frac{\pi}{2}]$ into $n = 6$ subintervals means the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}.$$

To find the left Riemann sum, we set $x_1^*, x_2^*, x_3^*, \ldots x_6^*$ equal to the left endpoints of the six subintervals. The heights of the rectangles are $f(x_k^*)$, for $k = 1, 2, \ldots 6$.

a) The resulting left Riemann sum (Figure 5.12) is:

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$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_6^*)\Delta x$$

$$= \left[\sin(0)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{12}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{5\pi}{12}\right)\cdot\frac{\pi}{12}\right]$$

$$\approx 0.863.$$

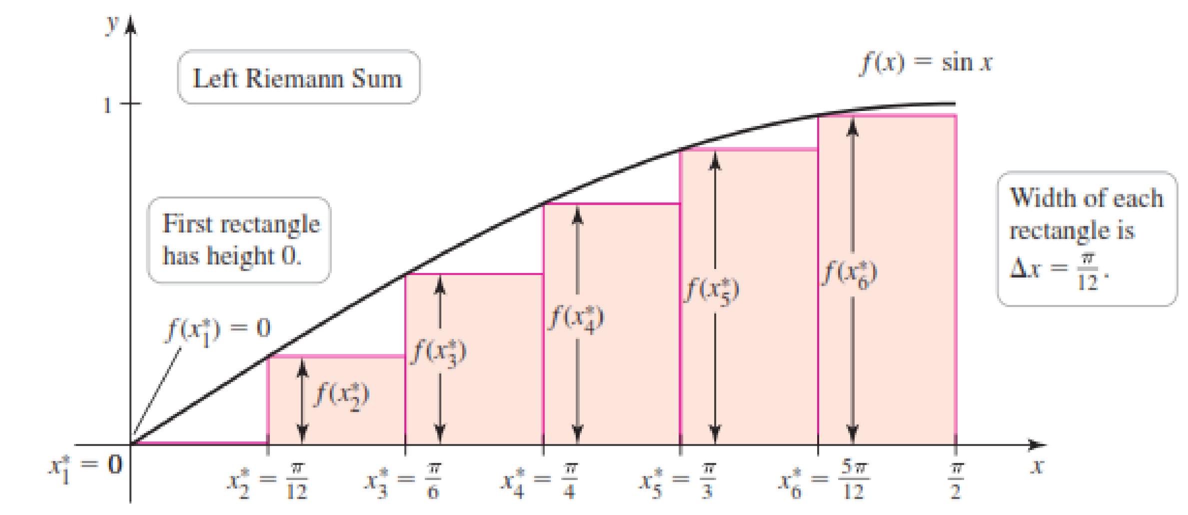

FIGURE 5.12

b) The resulting right Riemann sum (Figure 5.13) is:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_6^*)\Delta x$$

$$= \left[\sin\left(\frac{\pi}{12}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{5\pi}{12}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{12}\right]$$

$$\approx 1.125.$$

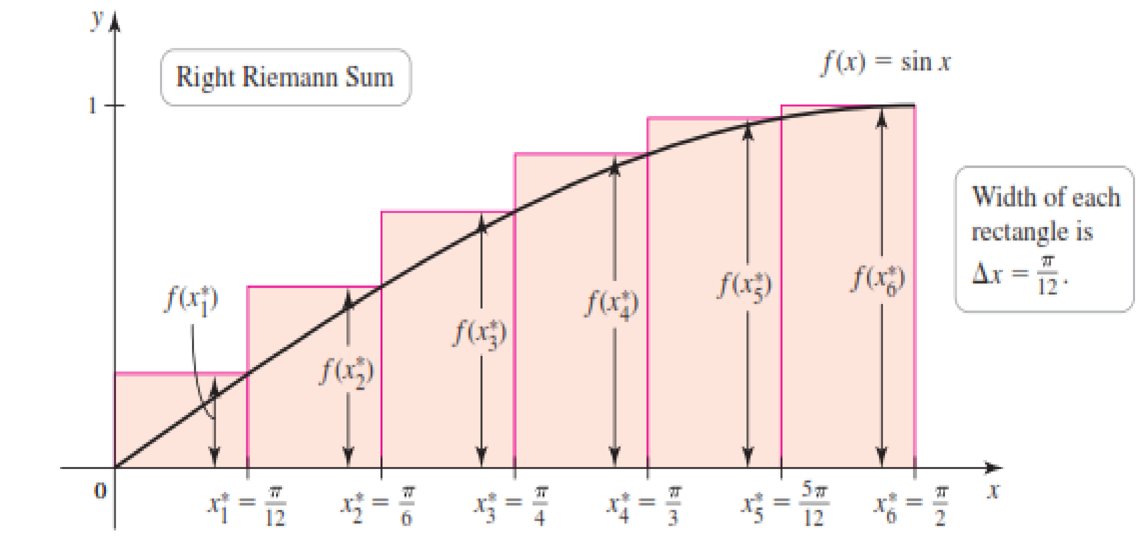

FIGURE 5.13

c) Looking at the graphs, we see that *the left Riemann sum* in part (a) underestimates the actual area of *R*, *whereas the right Riemann sum in part (b) overestimates the area of R*.
Therefore, the area of R is between 0.863 *and* 1.125.
As *the number of rectangles* increases, these approximations improve.

d) To find the *midpoint Riemann sum*, we set *x* equal to the midpoints of the subintervals. The midpoint of the first subinterval is the average of x_0 and x_1, which is

$$x_1^* = \frac{x_1 + x_0}{2} = \frac{\pi/12 + 0}{2} = \frac{\pi}{24}.$$

The remaining midpoints are also computed by averaging the two nearest grid points.

The resulting midpoint Riemann sum (Figure 5.14) is:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_6^*)\Delta x$$
$$= \left[\sin\left(\frac{\pi}{24}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{3\pi}{24}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{5\pi}{24}\right)\cdot\frac{\pi}{12}\right]$$
$$+ \left[\sin\left(\frac{7\pi}{24}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{9\pi}{24}\right)\cdot\frac{\pi}{12}\right] + \left[\sin\left(\frac{11\pi}{24}\right)\cdot\frac{\pi}{12}\right]$$
$$\approx 1.003.$$

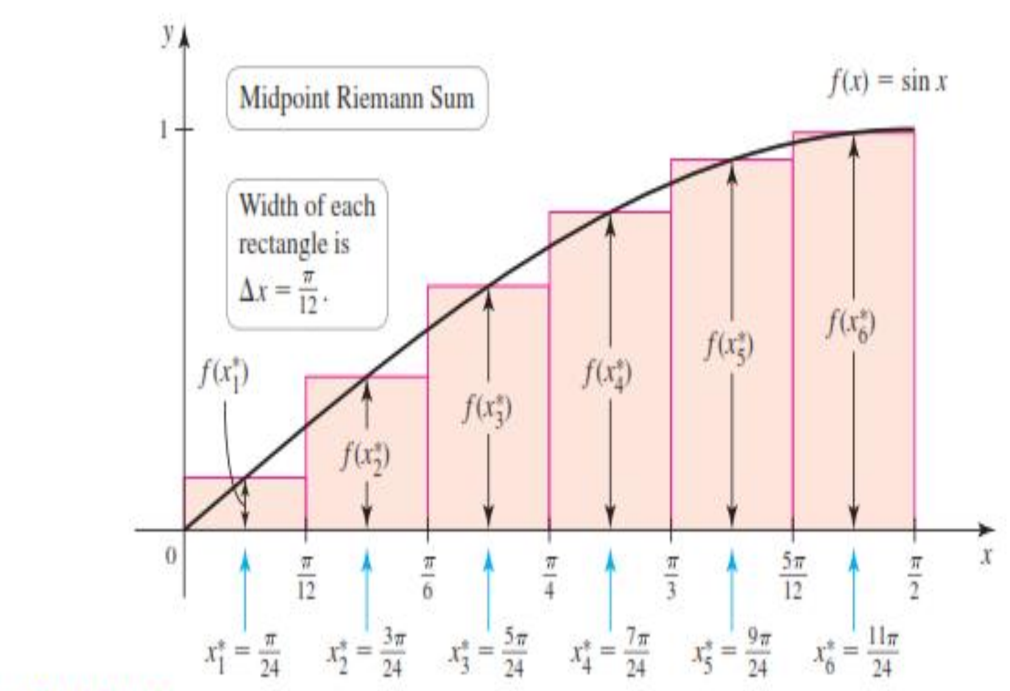

FIGURE 5.14

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Comparing the midpoint Riemann sum (Figure 5.14) with the left (Figure 5.12) and right (Figure 5.13) Riemann sums suggests that the midpoint sum is a more accurate estimate of the area under the curve.

Sigma (Summation) Notation

Working with Riemann sums is cumbersome with large numbers of subintervals. Therefore, we pause for a moment to introduce some notation that simplifies our work.

Sigma (or summation) notation is used to express sums in a compact way.

the upper limit

$$\sum_{k=1}^{10} k$$

The **index**

the lower limit

For example,

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$$

$$\sum_{k=0}^{3} k^2 = 0^2 + 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p=1}^{99} p.$$

Two properties of sums

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- *Constant Multiple Rule* $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k.$$

- *Addition Rule* $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

THEOREM 5.1 Sums of Positive Integers

Let n be a positive integer.

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

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Riemann Sums Using Sigma Notation

With sigma notation, a Riemann sum has the convenient compact form:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

- To express left, right, and midpoint Riemann sums in sigma notation, we must identify the points x_k^*.
  - For left Riemann sums, the left endpoints of the subintervals are

    $$x_k^* = a + (k-1)\Delta x, \text{ for } k = 1, \ldots, n.$$

  - For right Riemann sums, the right endpoints of the subintervals are

    $$x_k^* = a + k\Delta x, \quad \text{for } k = 1, \ldots, n.$$

  - For midpoint Riemann sums, the midpoints of the subintervals are

    $$x_k^* = a + (k - \tfrac{1}{2})\Delta x, \text{ for } k = 1, \ldots, n.$$

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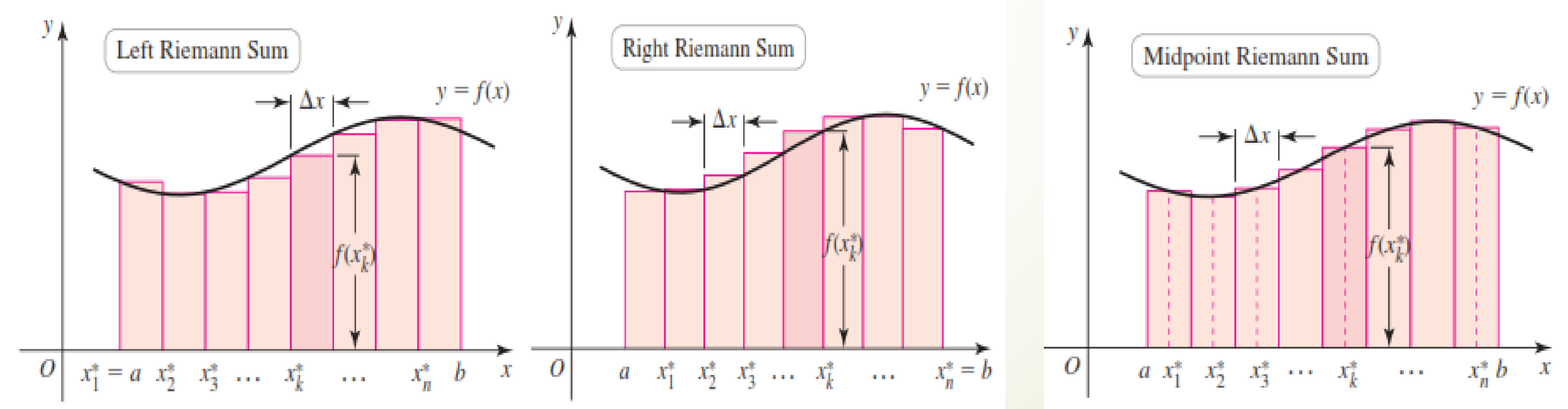
DEFINITION Left, Right, and Midpoint Riemann Sums in Sigma Notation

Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the **Riemann sum** of f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k^*)\Delta x$. Three cases arise in practice.

- **left Riemann sum** if $x_k^* = a + (k - 1)\Delta x$
- **right Riemann sum** if $x_k^* = a + k\Delta x$
- **midpoint Riemann sum** if $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$, for $k = 1, 2, \ldots, n$



Example

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Calculating Riemann sums Evaluate the left, right, and midpoint Riemann sums of $f(x)=x^3+1$ between $a = 0$ *and* $b = 2$ *using* $n = 50$ *subintervals.* Make a conjecture about the exact area of the region under the curve (Figure 5.15).

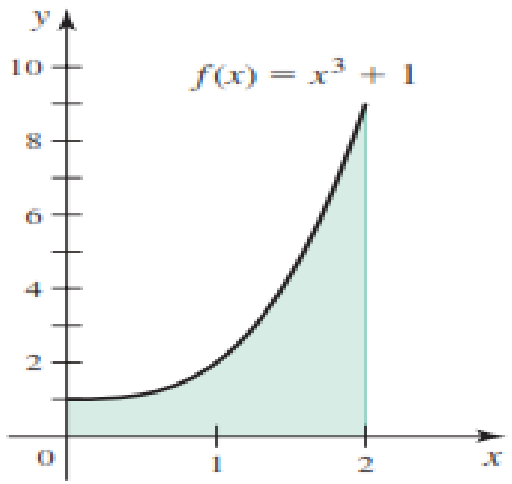

FIGURE 5.15

Solution

With $n = 50$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{50} = \frac{1}{25} = 0.04.$$

The value of x_k^* for the left Riemann sum is

$$x_k^* = a + (k-1)\Delta x = 0 + 0.04(k-1) = 0.04k - 0.04, \text{ for } k = 1, 2, 3, \ldots, 50.$$

Therefore, the left Riemann sum, evaluated with a **calculator,** is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{50} f(0.04k - 0.04)0.04 = 5.8416.$$

To evaluate the right Riemann sum, we let $x_k^* = a + k\Delta x = 0.04k$ and find that:

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{50} f(0.04k)0.04 = 6.1616.$$

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Rather than evaluating this sum with a calculator, we note that $f(0.04k) = (0.04k)^3 + 1$

and then use the properties of sums:

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{50} \underbrace{((0.04k)^3 + 1)}_{f(x_k^*)}\underbrace{0.04}_{\Delta x}$$

$$\sum (a_k + b_k) = \sum a_k + \sum b_k$$

$$= \sum_{k=1}^{50} (0.04k)^3\, 0.04 + \sum_{k=1}^{50} 1 \cdot 0.04$$

$$\sum c a_k = c \sum a_k$$

$$= (0.04)^4 \sum_{k=1}^{50} k^3 + 0.04 \sum_{k=1}^{50} 1.$$

Using the summation formulas for powers of integers in Theorem 5.1, we find that

$$\sum_{k=1}^{50} 1 = 50 \quad \text{and} \quad \sum_{k=1}^{50} k^3 = \frac{50^2 \cdot 51^2}{4}.$$

Substituting the values of these sums into the right Riemann sum, its value is

$$\sum_{k=1}^{50} f(x_k^*)\Delta x = \frac{3851}{625} = 6.1616,$$

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For the midpoint Riemann sum, we let

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = 0 + 0.04\left(k - \frac{1}{2}\right) = 0.04k - 0.02.$$

The value of the sum is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{50} f(0.04k - 0.02)0.04 \approx 5.9992.$$

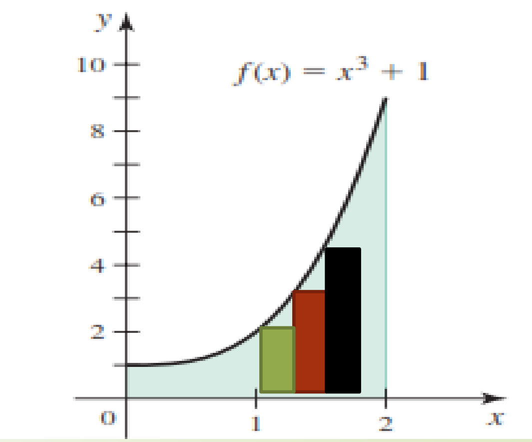
Note :

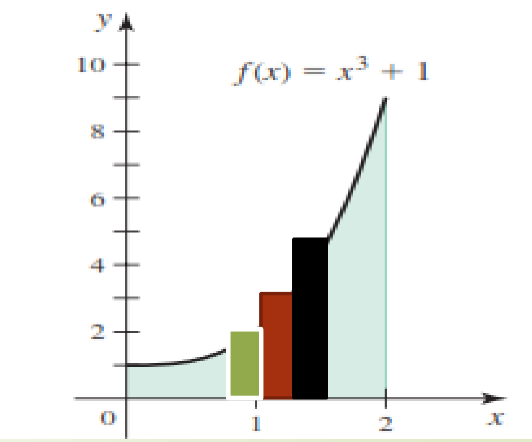
Because f *is increasing on* [0, 2],

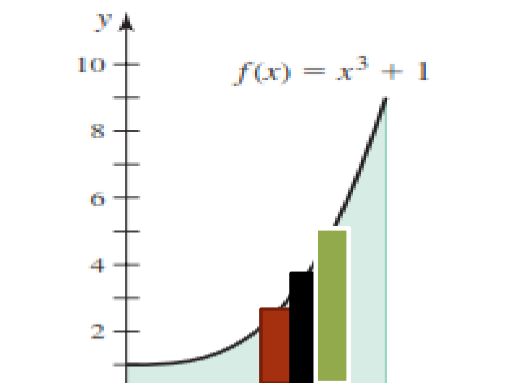
- the left Riemann sum underestimates the area of the shaded region in Figure 5.15, while
- the right Riemann sum overestimates the area. Therefore,

the exact area lies between 5.8416 and 6.1616.

- The midpoint Riemann sum usually gives the best estimate for increasing or decreasing functions.







Quiz:

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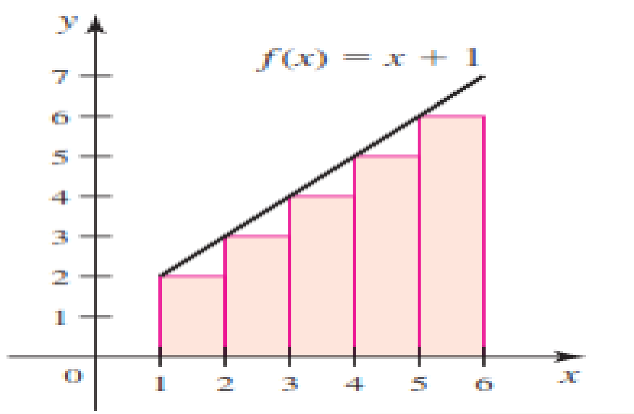
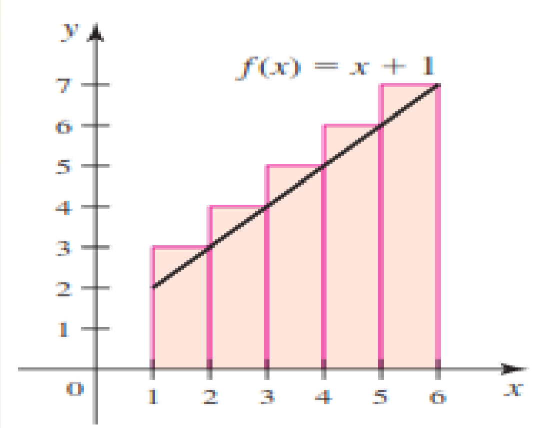
1. Use the figures to calculate the left and right Riemann sums for f on the given interval and for the given value of *n*:

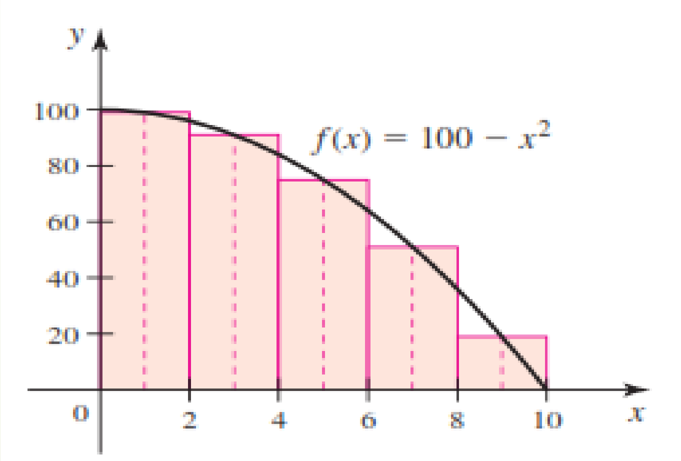
$f(x) = x + 1$ on $[1, 6]$; $n = 5$





2. Use the figures to calculate the left and right Riemann sums for *f* on the given interval and for the given value of *n*:

$$f(x) = 100 - x^2 \; on \; [0,10]; n = 5$$



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3. Evaluate the following expressions:

a. $\sum_{k=1}^{10} k$ **b.** $\sum_{k=1}^{6} (2k+1)$

c. $\sum_{k=1}^{4} k^2$ **d.** $\sum_{n=1}^{5} (1+n^2)$

4. Use sigma notation to write the following Riemann sums. Then evaluate each Riemann sum using Theorem 5.1

The right Riemann sum for $f(x) = x + 1$ *on* $[0,4]; n = 50$

5.2 Definite Integrals

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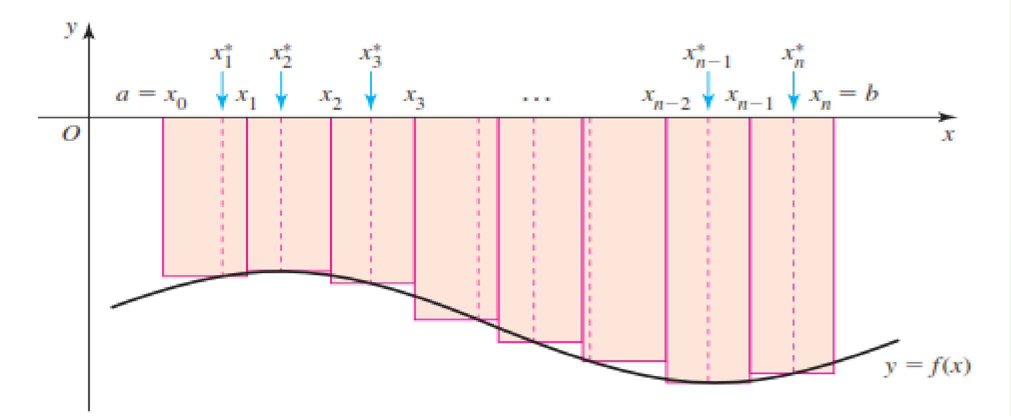
We introduced Riemann sums in Section 5.1 as a way to approximate the area of a region bounded by a curve $y = f(x)$ and the x-axis on an interval $[a, b]$. In that discussion, we assumed f to be nonnegative on the interval.

How do we interpret Riemann sums when f is **negative** on some or all of [a, b]?

- On intervals where $f(x) < 0$,

The Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$ approximates the negative of the area of the region bounded between the x-axis and the curve.



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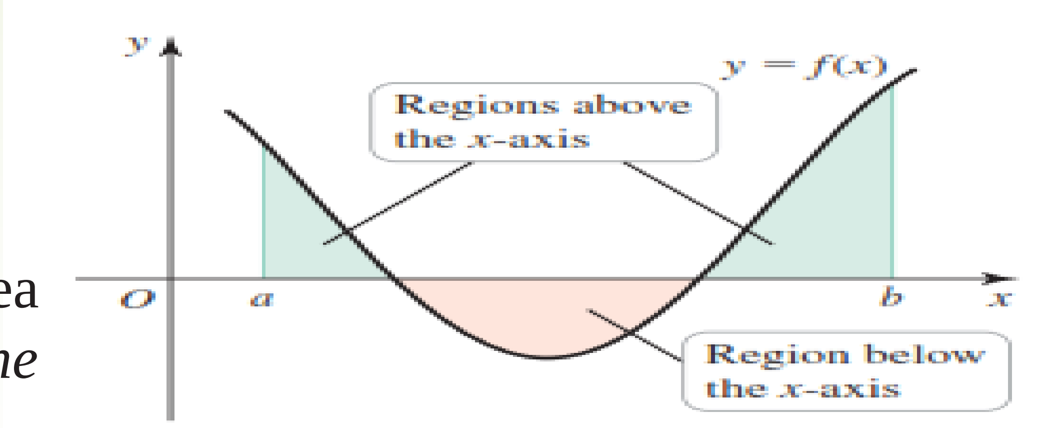
- In the more general case that f is positive on only part of $[a, b]$.



In this case, Riemann sums approximate the area of the regions that lie above the *x-axis* **minus** *the* area of the regions that lie *below the x-axis*

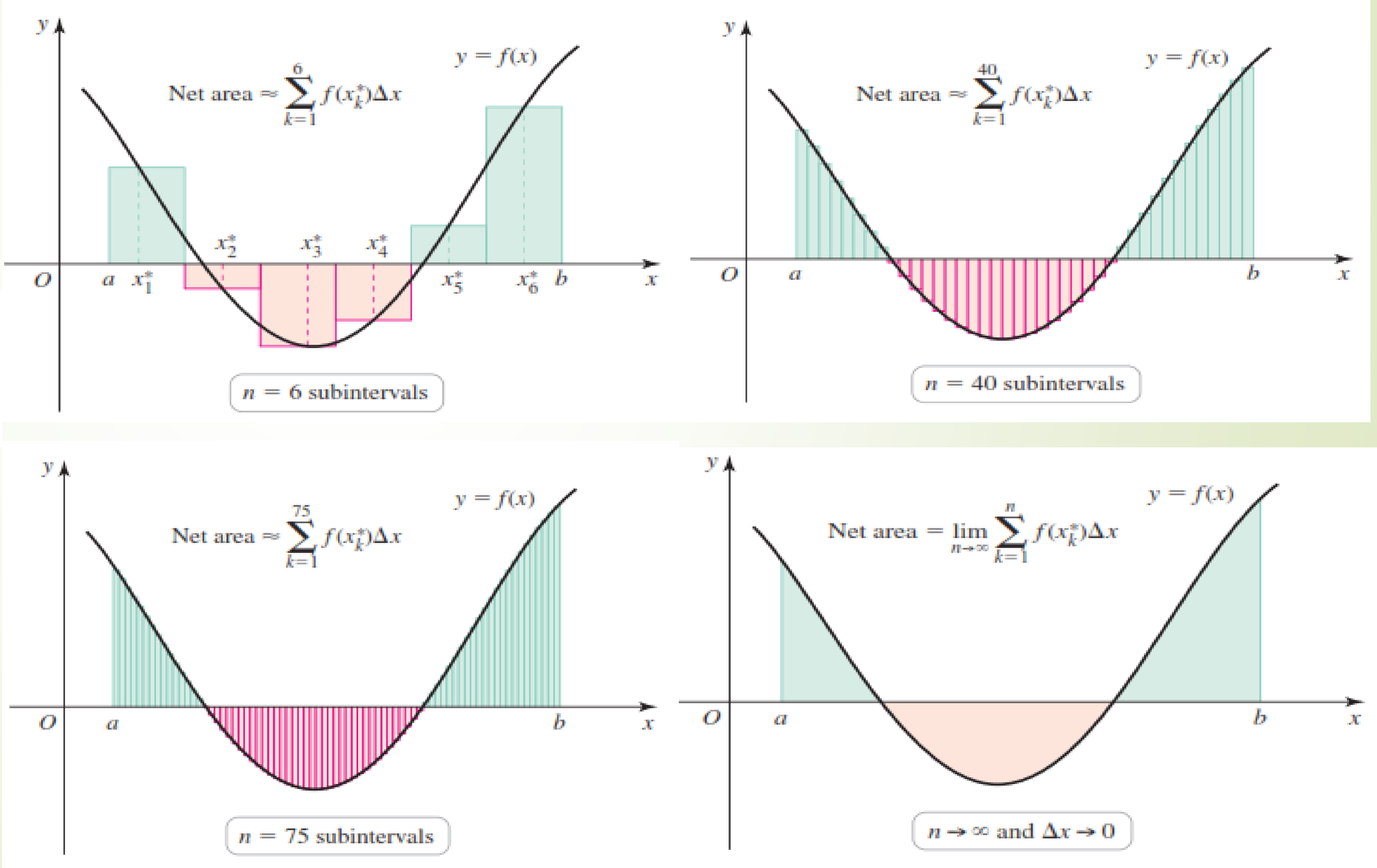
This difference between the positive and negative contributions is called the **net area**;

DEFINITION Net Area

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

Riemann sums for f *on*$[a, b]$ give *approximations to the* ***net area*** of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$, where $a < b$.

$$\text{net area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x.$$



➢ General Riemann Sum

The Riemann sums we have used so far involve **regular partitions** in which the subintervals have the same length Δx.

We now introduce partitions of $[a, b]$ in which the lengths of the subintervals are not necessarily equal. A **general partition** of $[a, b]$ consists of the n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

Where, $x_0 = a, x_n = b$ The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \ldots n$

We let x_k^* be any point in the subinterval $[x_k, x_{k-1}]$.

This general partition is used to define the **general Riemann sum**

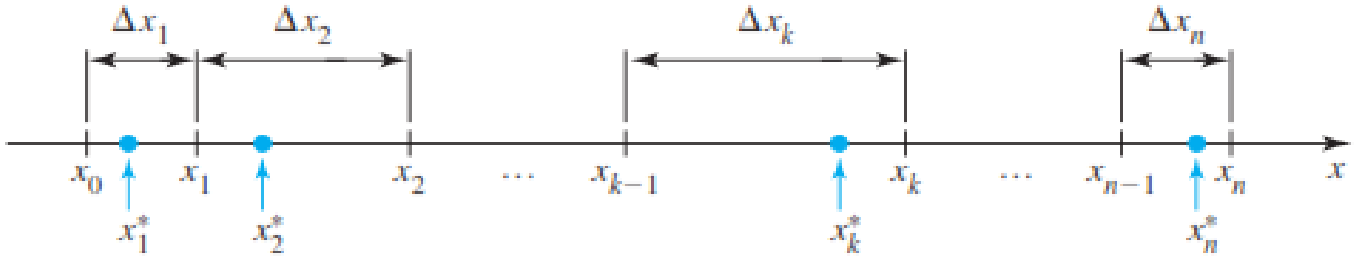
Lecture 1

DEFINITION General Riemann Sum

Suppose $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Let Δx_k be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.



If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum for f on $[a, b]$**.

$$\boxed{\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.}$$

to exist, it must have the same value over all general partitions of $[a, b]$ and for all choices of x_k^* on a partition.

We let Δ denote the largest value of Δx_k for $k = 1, 2, \ldots n$

5.2 Definite Integrals

DEFINITION Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim\limits_{\Delta \to 0} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. This limit is the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Upper limit of integration

Upper limit of summation

$$\int_a^b \underbrace{f(x)}_{\text{Integrand}}\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Lower limit of integration

Lower limit of summation

x is the variable of integration.

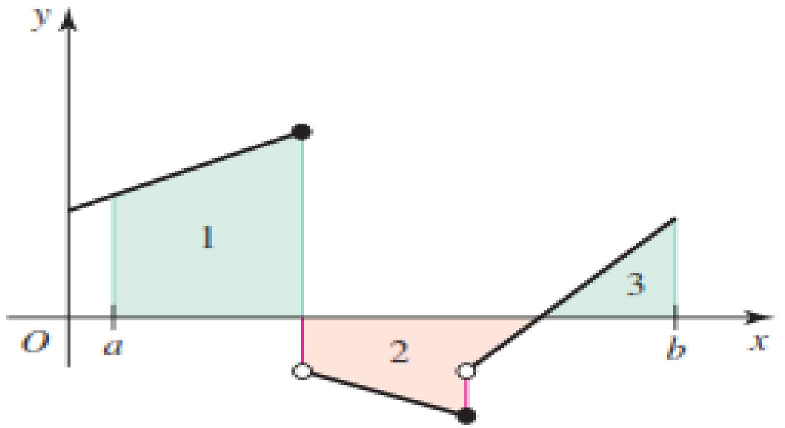
Lecture 1

THEOREM 5.2 Integrable Functions
If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

A function f is bounded on an interval I if there is a number M such that $|f(x)| < M$ for all x in I.

Figure 5.23 illustrates how the idea of net area carries over to piecewise continuous functions.

$$\text{Net area} = \int_a^b f(x)\,dx$$
$$= \text{area above } x\text{-axis (Regions 1 and 3)}$$
$$- \text{ area below } x\text{-axis (Region 2)}$$



A bounded piecewise continuous function is integrable.

FIGURE 5.23

Example **"Using geometry"**

Use familiar area formulas to evaluate the following definite integrals

a. $\displaystyle\int_2^4 (2x+3)\,dx$ **b.** $\displaystyle\int_1^6 (2x-6)\,dx$ **c.** $\displaystyle\int_3^4 \sqrt{1-(x-3)^2}\,dx$

Solution

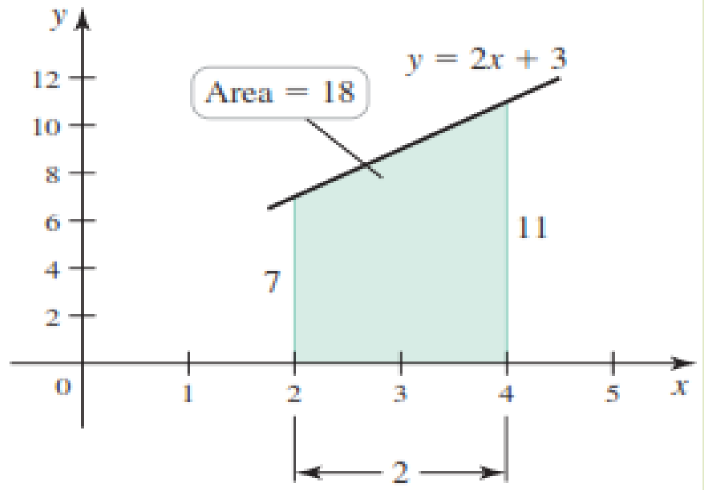
To evaluate these definite integrals geometrically, a sketch of the corresponding region is essential.

a. The definite integral $\displaystyle\int_2^4 (2x+3)\,dx$

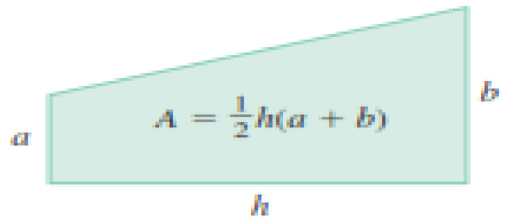
is the area of the trapezoid bounded by the *x-axis* and the line $y = 2x + 3$ from $x = 2$ to $x = 4$

The width of its base is 2 and the lengths of its two parallel sides are $f(2) = 7$ and $f(4) = 11$. Using the area formula for a trapezoid we have

$$\int_2^4 (2x+3)\,dx = \frac{1}{2}\cdot 2(11+7) = 18.$$

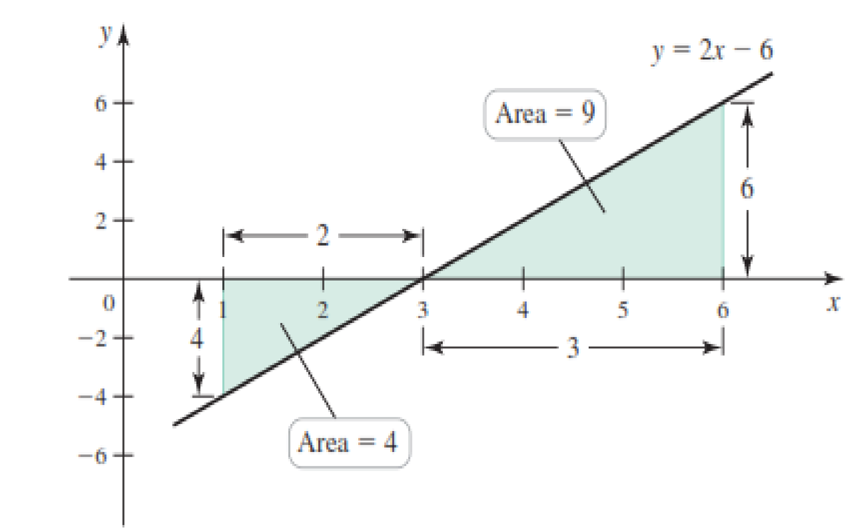


A trapezoid and its area When $a = 0$, we get the area of a triangle. When $a = b$, we get the area of a rectangle.



Lecture 1

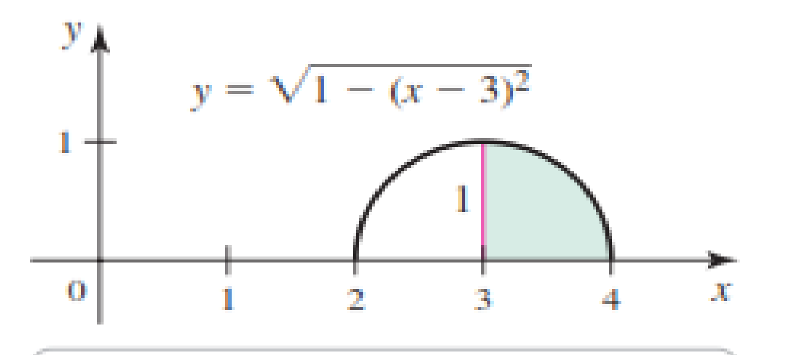
$$\int_1^6 (2x-6)\,dx = \text{net area} = 9 - 4 = 5.$$



We first let $y = \sqrt{1-(x-3)^2}$ and observe that $y \geq 0$ when $2 \leq x \leq 4$.

Squaring both sides leads to the equation $(x-3)^2 + y^2 = 1$

$$\int_3^4 \sqrt{1-(x-3)^2}\,dx = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}.$$



QUICK CHECK 4 Let $f(x) = 5$ and use geometry to evaluate $\int_1^3 f(x)\,dx$. What is the value of $\int_a^b c\,dx$ where c is a real number?

Example

Lecture 1

Definite integrals from graphs Figure 5.28 shows the graph of a function f with the areas of the regions bounded by its graph and the x-axis given. Find the values of the following definite integrals

a. $\int_a^b f(x)\,dx$ **b.** $\int_b^c f(x)\,dx$ **c.** $\int_a^c f(x)\,dx$ **d.** $\int_b^d f(x)\,dx$

Solution

a) Because f *is positive on* [a, b], the value of the definite integral is the area of the region between the graph and the *x-axis on* [a, b]; that is $\int_a^b f(x)\,dx = 12.$

b) Because f is negative on [b, c], the value of the definite integral is the negative of the area of the corresponding region; that is, $\int_b^c f(x)\,dx = -10.$

$$\int_a^c f(x)\,dx = 12 - 10 = 2.$$

$$\int_b^d f(x)\,dx = -10 + 8 = -2.$$

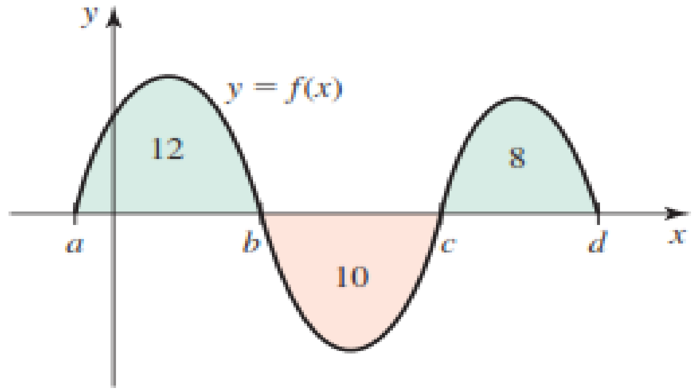


FIGURE 5.28

Evaluating Definite Integrals Using Limits

Lecture 1

We know that if f is integrable on $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k,$

for any partition of $[a, b]$ and any points x_k^* .

To simplify these calculations, we use equally spaced grid points and right Riemann sums. That is, for each value of n we let

$$\Delta x_k = \Delta x = \frac{b-a}{n} \text{ and } x_k^* = a + k\,\Delta x, \text{ for } k = 1, 2, \ldots, n.$$

Then, as $n \to \infty$ and $\Delta \to 0$,

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x.$$

Example

Lecture 1

Find the value of $\int_0^2 (x^3 + 1)\, dx$

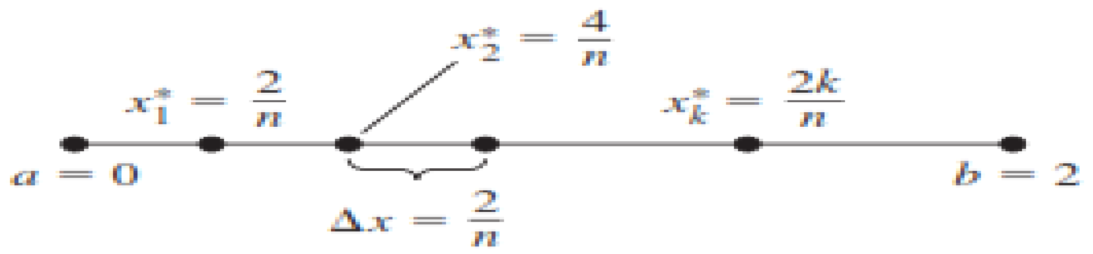
By evaluating a right Riemann sum and letting $n \longrightarrow \infty$

Solution

we conjectured that the value of this integral is **6**. To verify this conjecture, we now evaluate the integral exactly. The interval $[a, b]=[0, 2]$ is divided into n subintervals of length $\Delta x = \frac{b-a}{n} = \frac{2}{n}$,

which produces the grid points

$x_k^* = a + k\,\Delta x = 0 + k \cdot \frac{2}{n} = \frac{2k}{n}$, for $k = 1, 2, \ldots, n$.



Letting $f(x) = x^3 + 1$, the right Riemann sum is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[\left(\frac{2k}{n}\right)^3 + 1\right]\frac{2}{n}$$

$$= \frac{2}{n}\sum_{k=1}^{n}\left(\frac{8k^3}{n^3} + 1\right)$$

$$\sum_{k=1}^{n} c a_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

Lecture 1

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{2}{n}\left(\frac{8}{n^3}\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 1\right)$$

$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ and $\sum_{k=1}^{n} 1 = n$; Theorem 5.1

$$= \frac{2}{n}\left[\frac{8}{n^3}\left(\frac{n^2(n+1)^2}{4}\right) + n\right]$$

$$= \frac{4(n^2+2n+1)}{n^2} + 2. \quad \text{Simplify.}$$

Now we evaluate $\int_0^2 (x^3+1)\,dx$ by letting $n \longrightarrow \infty$ in the Riemann sum:

$$\int_0^2 (x^3+1)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to\infty}\left[\frac{4(n^2+2n+1)}{n^2} + 2\right]$$

$$= 4\lim_{n\to\infty}\left(\frac{n^2+2n+1}{n^2}\right) + \lim_{n\to\infty} 2$$

$$= 4(1) + 2 = 6.$$

Therefore, $\int_0^2 (x^3+1)\,dx = 6$,


Thank You!