# Chapter 7

# Linear Programming Models: Graphical and Computer Methods

To accompany
*Quantitative Analysis for Management*, *Eleventh Edition*,
by Render, Stair, and Hanna
Power Point slides created by Brian Peterson

# *Learning Objectives*

**After completing this chapter, students will be able to:**

1. **Understand the basic assumptions and properties of linear programming (LP).**
2. **Graphically solve any LP problem that has only two variables by both the corner point and isoprofit line methods.**
3. **Understand special issues in LP such as infeasibility, unboundedness, redundancy, and alternative optimal solutions.**
4. **Understand the role of sensitivity analysis.**
5. **Use Excel spreadsheets to solve LP problems.**

# Chapter Outline

# *Introduction*

- **Many management decisions involve trying to make the most effective use of limited resources.**
- ***Linear programming*** **(*LP*) is a widely used mathematical modeling technique designed to help managers in planning and decision making relative to resource allocation.**
  - **This belongs to the broader field of *mathematical programming.***
  - **In this sense, *programming* refers to modeling and solving a problem mathematically.**

# *Requirements of a Linear Programming Problem*

- **All LP problems have 4 properties in common:**
  1. **All problems seek to *maximize* or *minimize* some quantity (the *objective function*).**
  2. **Restrictions or *constraints* that limit the degree to which we can pursue our objective are present.**
  3. **There must be alternative courses of action from which to choose.**
  4. **The objective and constraints in problems must be expressed in terms of *linear* equations or *inequalities.***

# *Basic Assumptions of LP*

- We assume conditions of ***certainty*** exist and numbers in the objective and constraints are known with certainty and do not change during the period being studied.
- We assume ***proportionality*** exists in the objective and constraints.
- We assume ***additivity*** in that the total of all activities equals the sum of the individual activities.
- We assume ***divisibility*** in that solutions need not be whole numbers.
- All answers or variables are ***nonnegative.***

# *LP Properties and Assumptions*

| PROPERTIES OF LINEAR PROGRAMS |
|---|
| 1. One objective function |
| 2. One or more constraints |
| 3. Alternative courses of action |
| 4. Objective function and constraints are linear – proportionality and divisibility |
| 5. Certainty |
| 6. Divisibility |
| 7. Nonnegative variables |

**Table 7.1**

# *Formulating LP Problems*

- **Formulating a linear program involves developing a mathematical model to represent the managerial problem.**
- **The steps in formulating a linear program are:**
  1. **Completely understand the managerial problem being faced.**
  2. **Identify the objective and the constraints.**
  3. **Define the decision variables.**
  4. **Use the decision variables to write mathematical expressions for the objective function and the constraints.**

# *Formulating LP Problems*

- One of the most common LP applications is the ***product mix problem.***
- Two or more products are produced using limited resources such as personnel, machines, and raw materials.
- The profit that the firm seeks to maximize is based on the profit contribution per unit of each product.
- The company would like to determine how many units of each product it should produce so as to maximize overall profit given its limited resources.

# *Flair Furniture Company*

- The Flair Furniture Company produces inexpensive tables and chairs.
- Processes are similar in that both require a certain amount of hours of carpentry work and in the painting and varnishing department.
- Each table takes 4 hours of carpentry and 2 hours of painting and varnishing.
- Each chair requires 3 of carpentry and 1 hour of painting and varnishing.
- There are 240 hours of carpentry time available and 100 hours of painting and varnishing.
- Each table yields a profit of $70 and each chair a profit of $50.

# *Flair Furniture Company Data*

**The company wants to determine the best combination of tables and chairs to produce to reach the maximum profit.**

| | HOURS REQUIRED TO PRODUCE 1 UNIT | | |
|---|---|---|---|
| DEPARTMENT | ($T$) TABLES | ($C$) CHAIRS | AVAILABLE HOURS THIS WEEK |
| Carpentry | 4 | 3 | 240 |
| Painting and varnishing | 2 | 1 | 100 |
| Profit per unit | $70 | $50 | |

Table 7.2

# Flair Furniture Company

- The objective is to:

  Maximize profit

- The constraints are:
  1. The hours of carpentry time used cannot exceed 240 hours per week.
  2. The hours of painting and varnishing time used cannot exceed 100 hours per week.
- The decision variables representing the actual decisions we will make are:

  $T$ = number of tables to be produced per week.

  $C$ = number of chairs to be produced per week.

# *Flair Furniture Company*

- **We create the LP objective function in terms of $T$ and $C$:**

**Maximize profit = \$70$T$ + \$50$C$**

- **Develop mathematical relationships for the two constraints:**
  - **For carpentry, total time used is:**

    **(4 hours per table)(Number of tables produced)**
    **+ (3 hours per chair)(Number of chairs produced).**
  - **We know that:**

**Carpentry time used ≤ Carpentry time available.**

**$4T + 3C \leq 240$ (hours of carpentry time)**

# *Flair Furniture Company*

- Similarly,

  Painting and varnishing time used
  ≤ Painting and varnishing time available.

  $2T + 1C \leq 100$ (hours of painting and varnishing time)

  This means that each table produced requires two hours of painting and varnishing time.

- Both of these constraints restrict production capacity and affect total profit.

# *Flair Furniture Company*

The values for $T$ and $C$ must be nonnegative.

$T \geq 0$ (number of tables produced is greater than or equal to 0)

$C \geq 0$ (number of chairs produced is greater than or equal to 0)

The complete problem stated mathematically:

$$\text{Maximize profit} = \$70T + \$50C$$

subject to

$4T + 3C \leq 240$ (carpentry constraint)

$2T + 1C \leq 100$ (painting and varnishing constraint)

$T, C \geq 0$ (nonnegativity constraint)

# *Graphical Solution to an LP Problem*

- **The easiest way to solve a small LP problems is graphically.**
- **The graphical method only works when there are just two decision variables.**
- **When there are more than two variables, a more complex approach is needed as it is not possible to plot the solution on a two-dimensional graph.**
- **The graphical method provides valuable insight into how other approaches work.**

# Graphical Representation of a Constraint

## Quadrant Containing All Positive Values

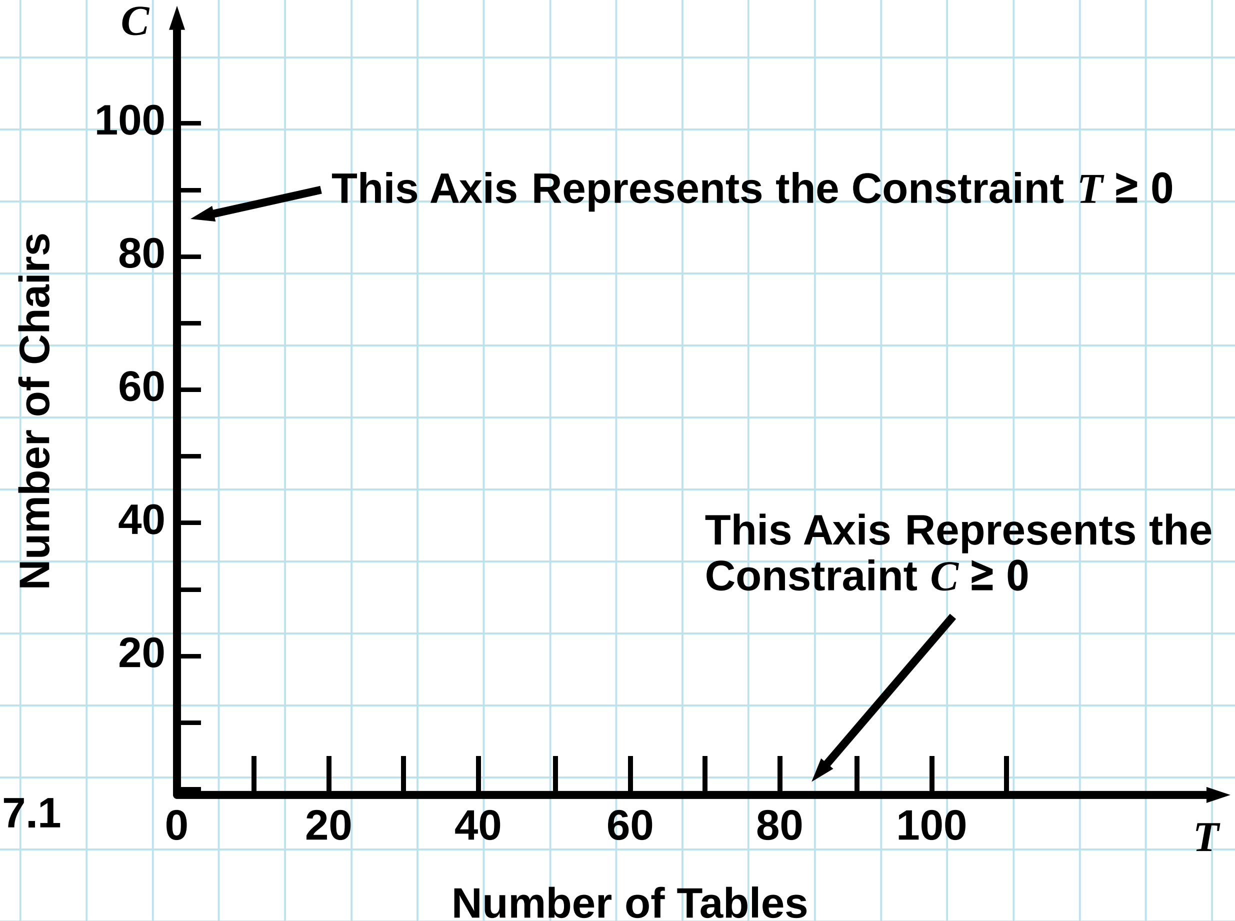



Figure 7.1

# *Graphical Representation of a Constraint*

- **The first step in solving the problem is to identify a set or region of feasible solutions.**
- **To do this we plot each constraint equation on a graph.**
- **We start by graphing the equality portion of the constraint equations:**

$$4T + 3C = 240$$

- **We solve for the axis intercepts and draw the line.**

# *Graphical Representation of a Constraint*

- **When Flair produces no tables, the carpentry constraint is:**

$$4(0) + 3C = 240$$
$$3C = 240$$
$$C = 80$$

- **Similarly for no chairs:**

$$4T + 3(0) = 240$$
$$4T = 240$$
$$T = 60$$

- **This line is shown on the following graph:**

# Graphical Representation of a Constraint

## Graph of carpentry constraint equation

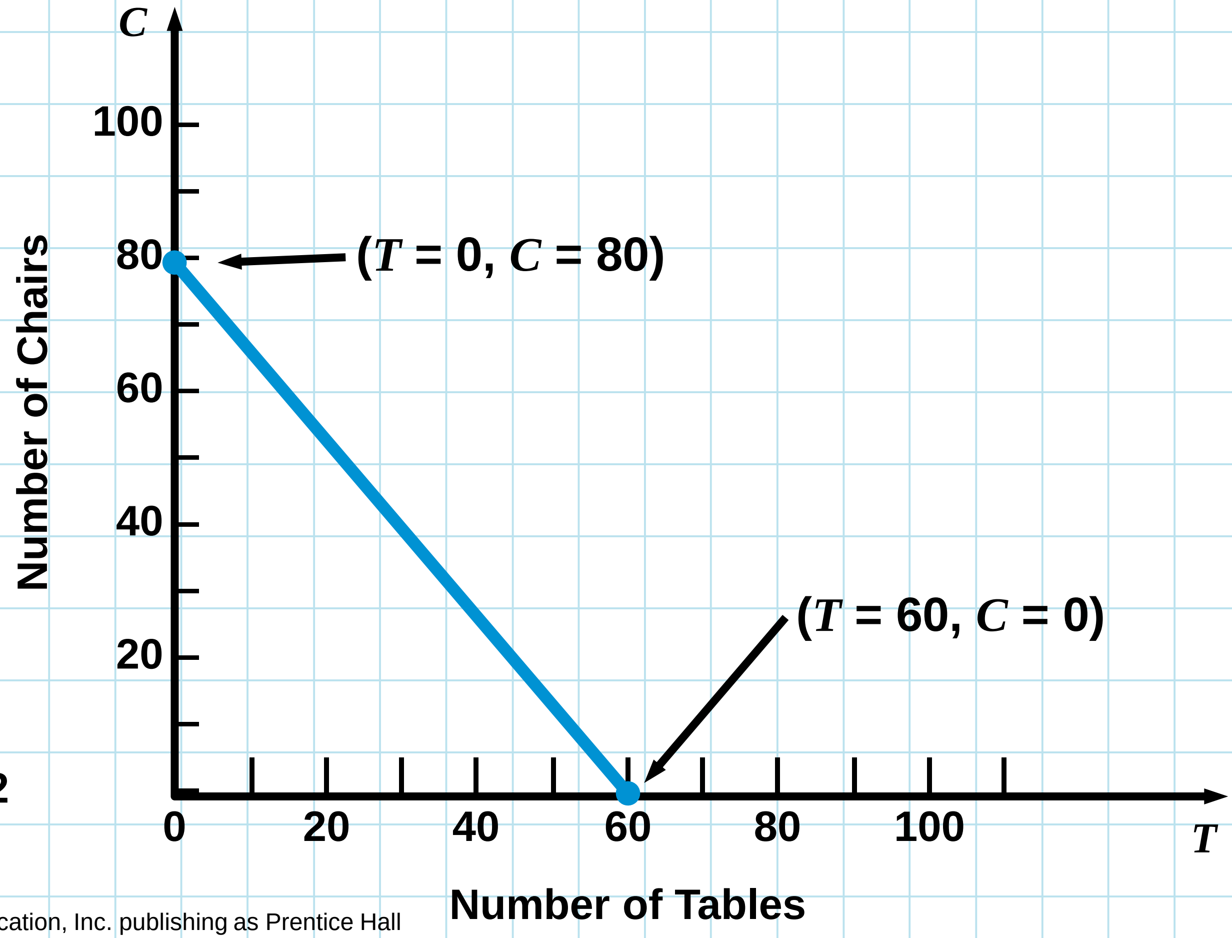



Figure 7.2

# Graphical Representation of a Constraint

## Region that Satisfies the Carpentry Constraint

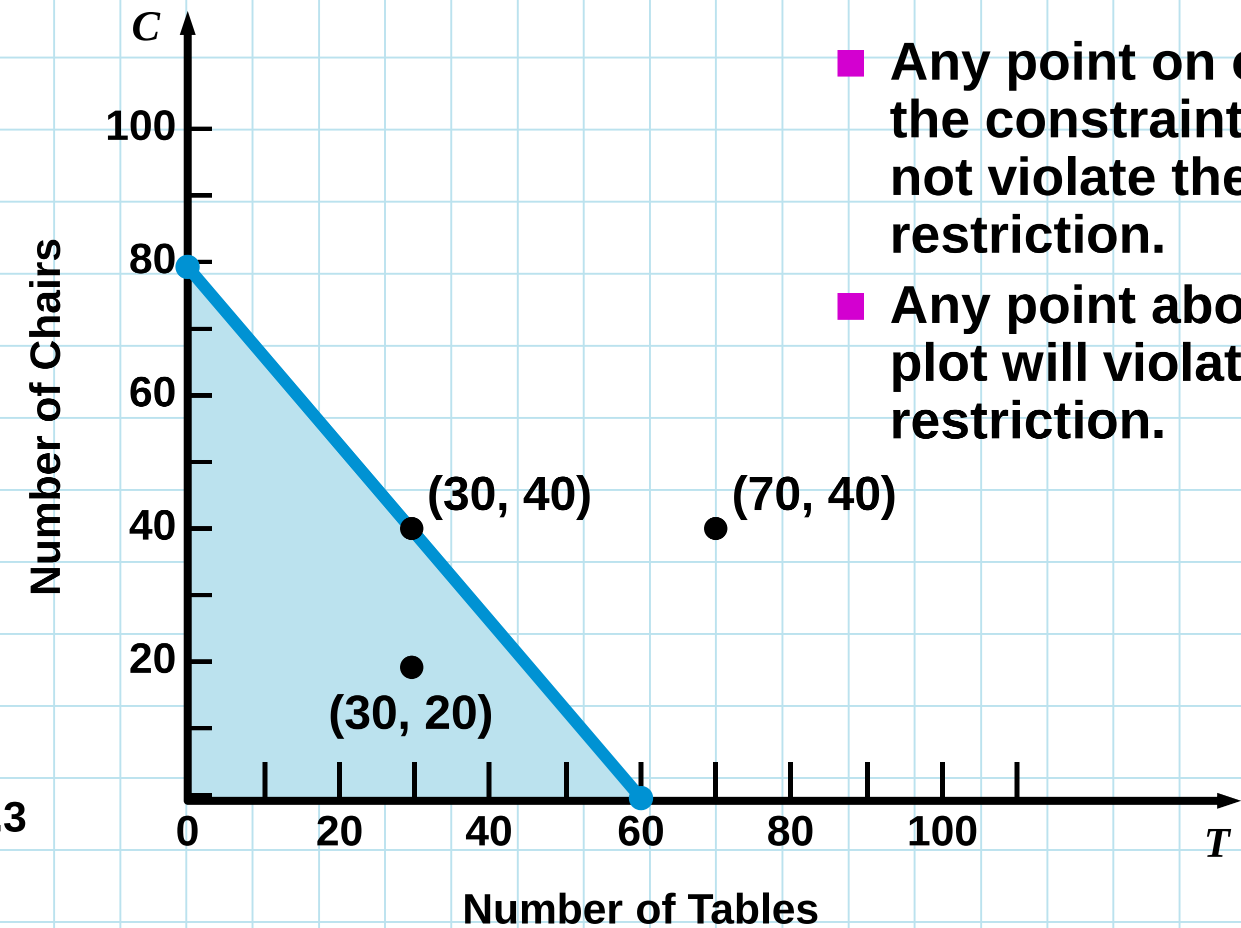



Figure 7.3

- Any point on or below the constraint plot will not violate the restriction.
- Any point above the plot will violate the restriction.

# *Graphical Representation of a Constraint*

- **The point (30, 40) lies on the plot and exactly satisfies the constraint**

  **4(30) + 3(40) = 240.**

- **The point (30, 20) lies below the plot and satisfies the constraint**

  **4(30) + 3(20) = 180.**

- **The point (70, 40) lies above the plot and does not satisfy the constraint**

  **4(70) + 3(40) = 400.**

# Graphical Representation of a Constraint

## Region that Satisfies the Painting and Varnishing Constraint

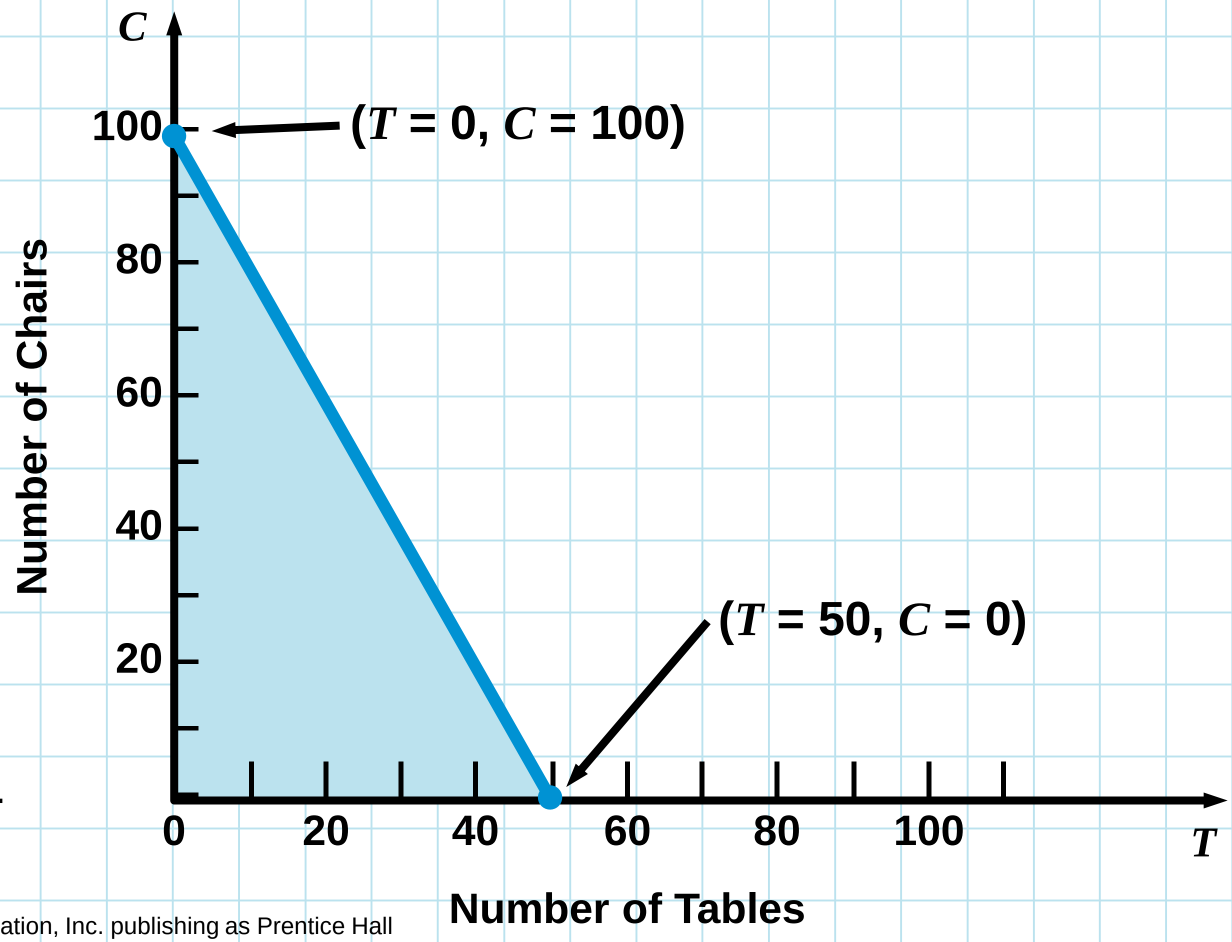



Figure 7.4

# Graphical Representation of a Constraint

- To produce tables and chairs, both departments must be used.
- We need to find a solution that satisfies both constraints ***simultaneously.***
- A new graph shows both constraint plots.
- The ***feasible region*** (or ***area of feasible solutions***) is where all constraints are satisfied.
- Any point inside this region is a ***feasible*** solution.
- Any point outside the region is an ***infeasible*** solution.

# Graphical Representation of a Constraint

**Feasible Solution Region for the Flair Furniture Company Problem**

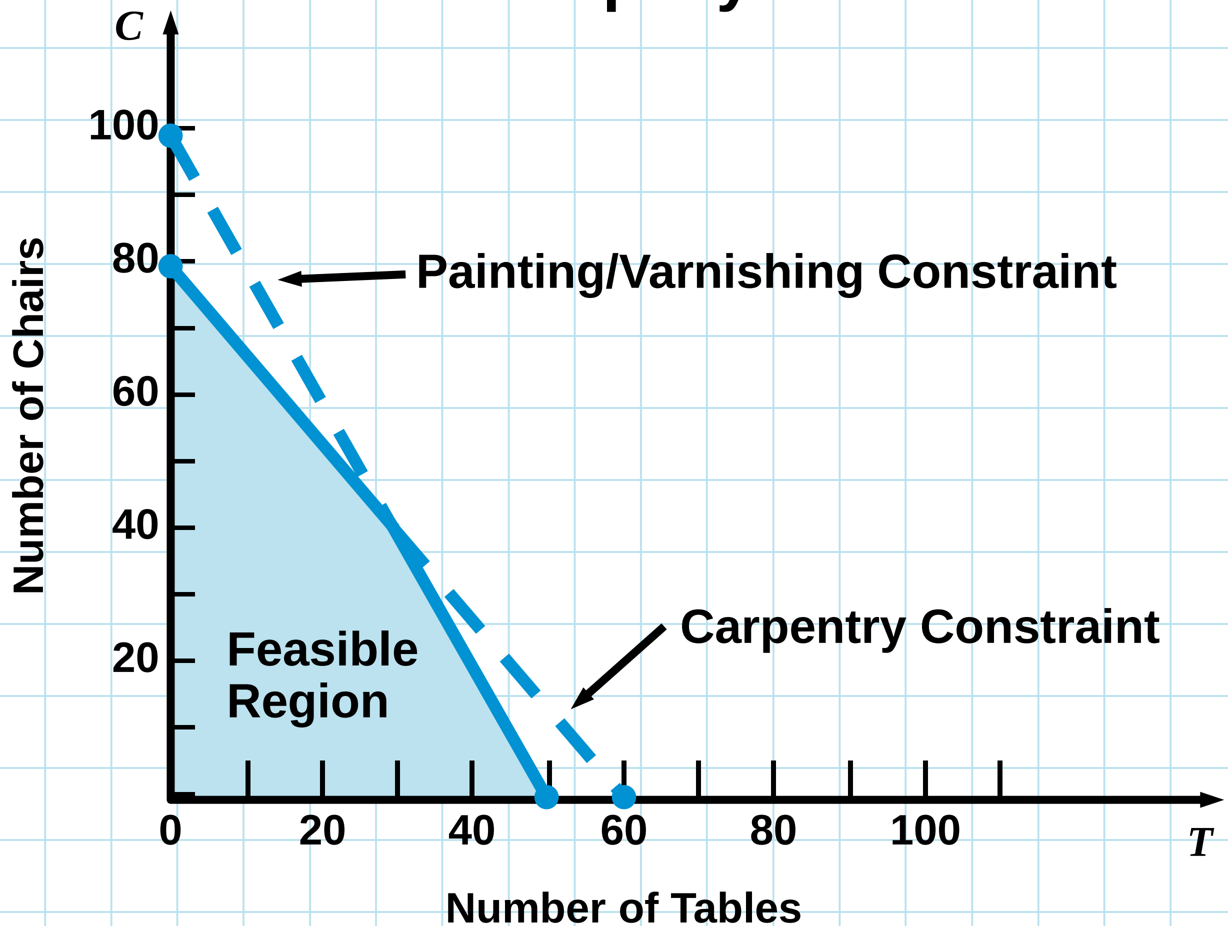



Figure 7.5

# Graphical Representation of a Constraint

- For the point (30, 20)

| | | |
|---|---|---|
| ***Carpentry constraint*** | $4T + 3C \leq 240$ hours available<br>(4)(30) + (3)(20) = 180 hours used | ✓ |
| ***Painting constraint*** | $2T + 1C \leq 100$ hours available<br>(2)(30) + (1)(20) = 80 hours used | ✓ |

- For the point (70, 40)

| | | |
|---|---|---|
| ***Carpentry constraint*** | $4T + 3C \leq 240$ hours available<br>(4)(70) + (3)(40) = 400 hours used | ✗ |
| ***Painting constraint*** | $2T + 1C \leq 100$ hours available<br>(2)(70) + (1)(40) = 180 hours used | ✗ |

# Graphical Representation of a Constraint

- **For the point (50, 5)**

| | | |
|---|---|---|
| ***Carpentry constraint*** | **$4T + 3C \leq 240$ hours available**<br>**$(4)(50) + (3)(5) = 215$ hours used** | ✓ |
| ***Painting constraint*** | **$2T + 1C \leq 100$ hours available**<br>**$(2)(50) + (1)(5) = 105$ hours used** | ✗ |

# *Isoprofit Line Solution Method*

- Once the feasible region has been graphed, we need to find the optimal solution from the many possible solutions.
- The speediest way to do this is to use the isoprofit line method.
- Starting with a small but possible profit value, we graph the objective function.
- We move the objective function line in the direction of increasing profit while maintaining the slope.
- The last point it touches in the feasible region is the optimal solution.

# *Isoprofit Line Solution Method*

- **For Flair Furniture, choose a profit of $2,100.**
- **The objective function is then**

$$\$2{,}100 = 70T + 50C$$

- **Solving for the axis intercepts, we can draw the graph.**
- **This is obviously not the best possible solution.**
- **Further graphs can be created using larger profits.**
- **The further we move from the origin, the larger the profit will be.**
- **The highest profit ($4,100) will be generated when the isoprofit line passes through the point (30, 40).**

# *Isoprofit Line Solution Method*

## Profit line of $2,100 Plotted for the Flair Furniture Company

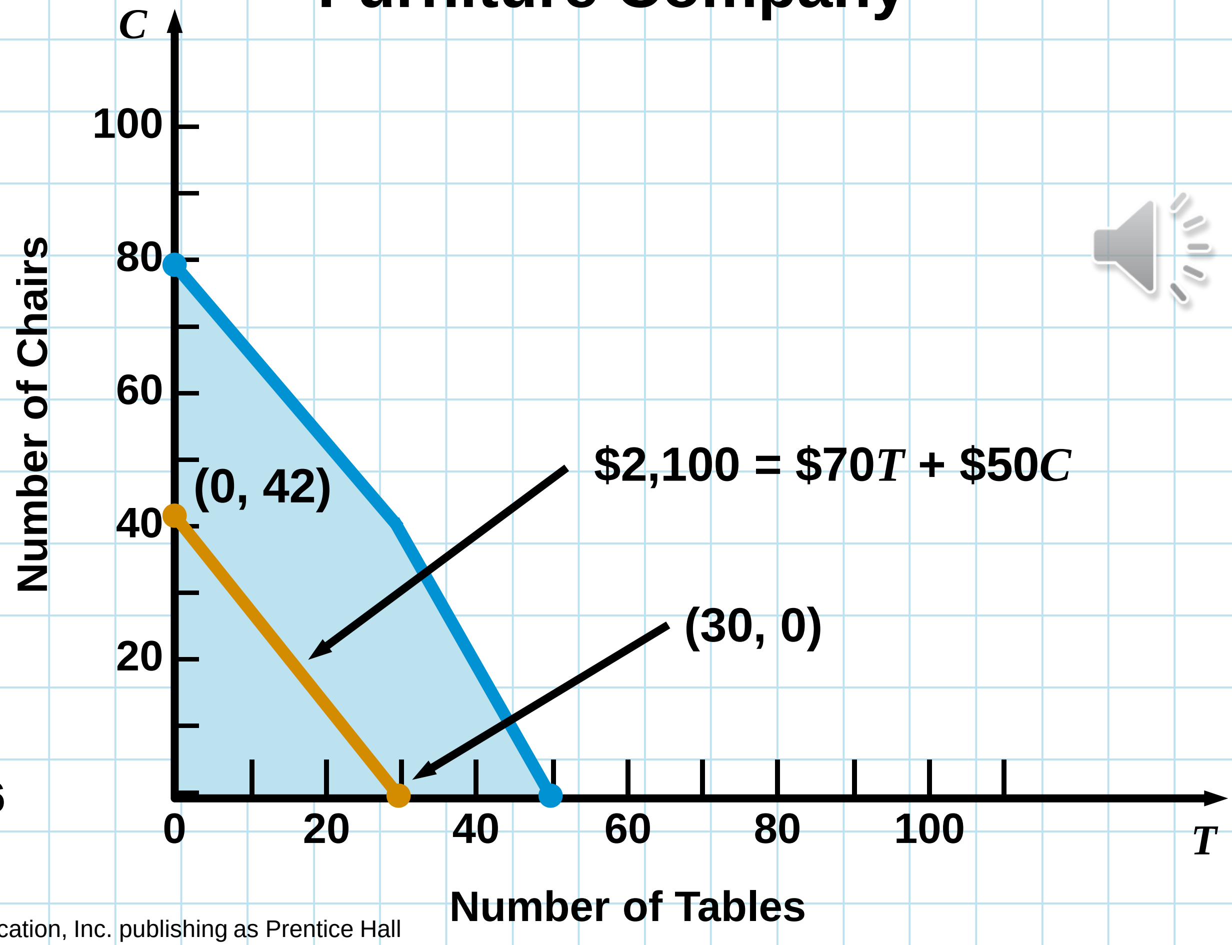



Figure 7.6

# *Isoprofit Line Solution Method*

**Four Isoprofit Lines Plotted for the Flair Furniture Company**

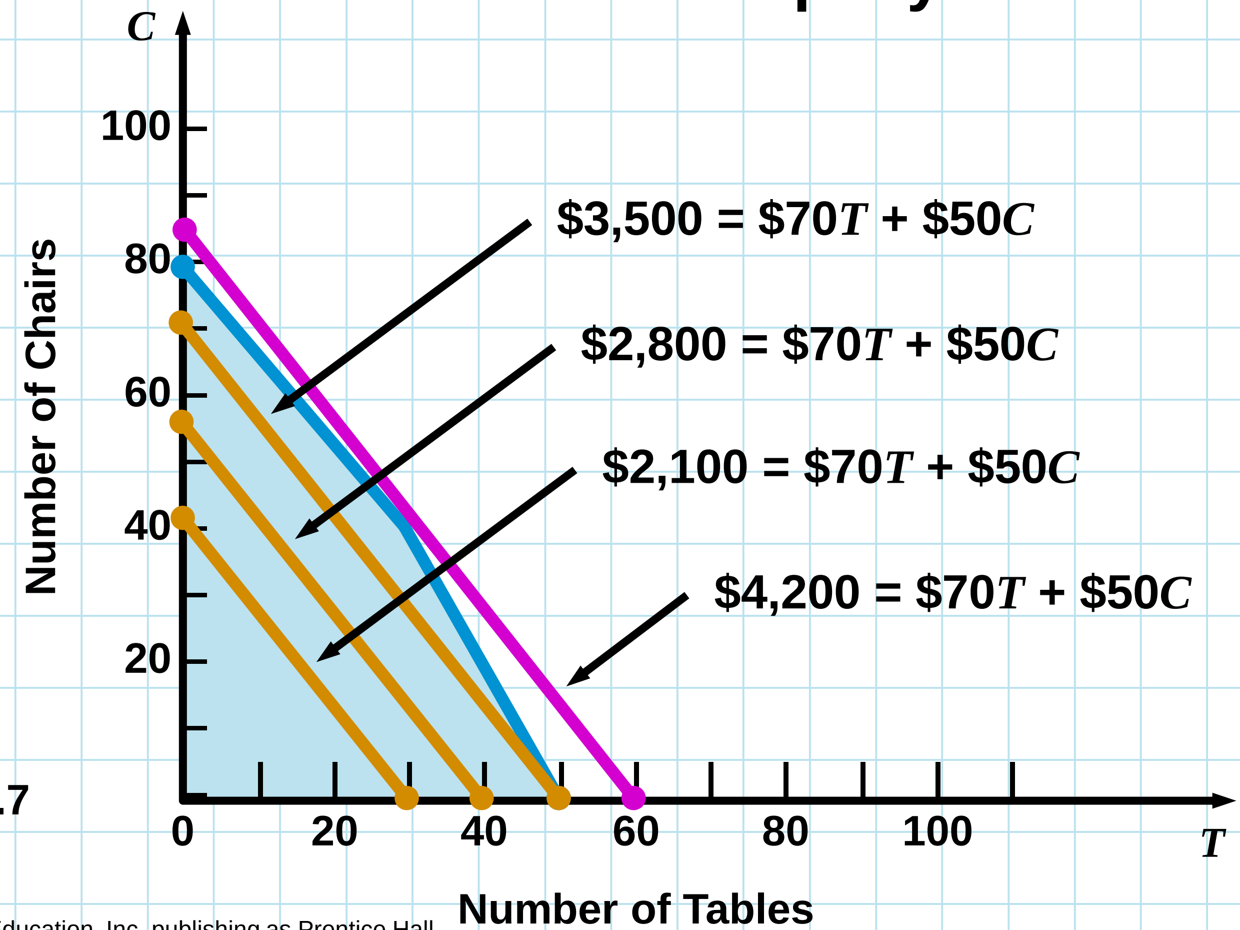



Figure 7.7

# *Isoprofit Line Solution Method*

## Optimal Solution to the Flair Furniture problem

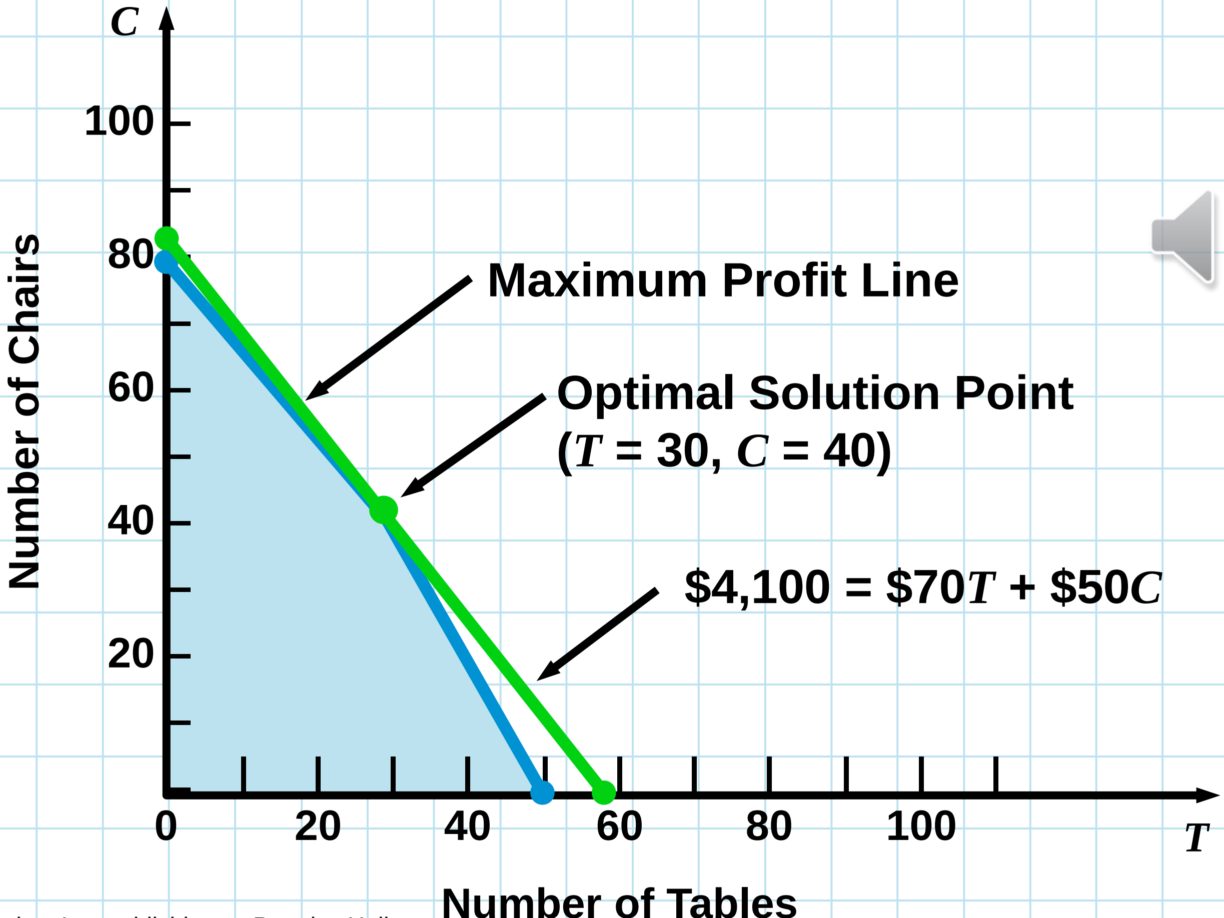



Figure 7.8

# *Corner Point Solution Method*

- A second approach to solving LP problems employs the ***corner point method.***
- It involves looking at the profit at every corner point of the feasible region.
- The mathematical theory behind LP is that the optimal solution must lie at one of the ***corner points***, or ***extreme point***, in the feasible region.
- For Flair Furniture, the feasible region is a four-sided polygon with four corner points labeled 1, 2, 3, and 4 on the graph.

# Corner Point Solution Method

## Four Corner Points of the Feasible Region

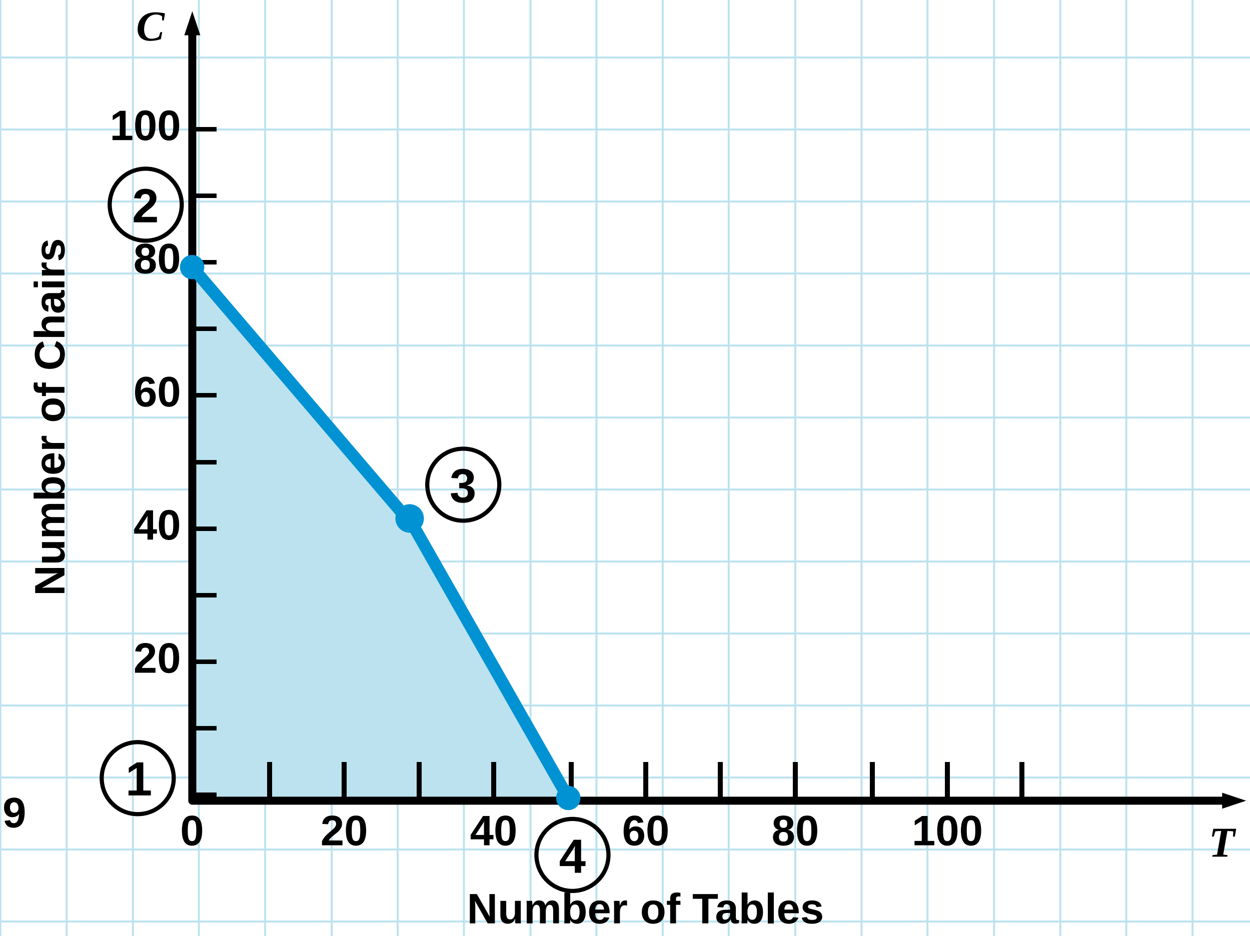



Figure 7.9

# Corner Point Solution Method

- To find the coordinates for Point 3 accurately we have to solve for the intersection of the two constraint lines.
- Using the ***simultaneous equations method***, we multiply the painting equation by –2 and add it to the carpentry equation

$$\begin{aligned} 4T + 3C &= 240 && \text{(carpentry line)} \\ -4T - 2C &= -200 && \text{(painting line)} \\ C &= 40 \end{aligned}$$

- Substituting 40 for $C$ in either of the original equations allows us to determine the value of $T$.

$$\begin{aligned} 4T + (3)(40) &= 240 && \text{(carpentry line)} \\ 4T + 120 &= 240 \\ T &= 30 \end{aligned}$$

# *Corner Point Solution Method*

**Point 1: ($T$ = 0, $C$ = 0)** **Profit = \$70(0) + \$50(0) = \$0**

**Point 2: ($T$ = 0, $C$ = 80)** **Profit = \$70(0) + \$50(80) = \$4,000**

**Point 4: ($T$ = 50, $C$ = 0)** **Profit = \$70(50) + \$50(0) = \$3,500**

**Point 3: ($T$ = 30, $C$ = 40)** **Profit = \$70(30) + \$50(40) = \$4,100**

**Because Point 3 returns the highest profit, this is the optimal solution.**

# *Slack and Surplus*

- **Slack is the amount of a resource that is not used. For a less-than-or-equal constraint:**
  - **Slack = Amount of resource available – amount of resource used.**
- **Surplus is used with a greater-than-or-equal constraint to indicate the amount by which the right hand side of the constraint is exceeded.**
  - **Surplus = Actual amount – minimum amount.**

# Summary of Graphical Solution Methods

| ISOPROFIT METHOD |
| --- |
| 1. Graph all constraints and find the feasible region. |
| 2. Select a specific profit (or cost) line and graph it to find the slope. |
| 3. Move the objective function line in the direction of increasing profit (or decreasing cost) while maintaining the slope. The last point it touches in the feasible region is the optimal solution. |
| 4. Find the values of the decision variables at this last point and compute the profit (or cost). |
| **CORNER POINT METHOD** |
| 1. Graph all constraints and find the feasible region. |
| 2. Find the corner points of the feasible reason. |
| 3. Compute the profit (or cost) at each of the feasible corner points. |
| 4. Select the corner point with the best value of the objective function found in Step 3. This is the optimal solution. |

Table 7.4

# *Example:*

- **A manufacturing company makes two types of television sets; one is black and white and the other is colour. The company has resources to make at most 300 sets a week. It takes \$ 1800 to make a black and white set and \$ 2700 to make a coloured set.**
- **The company can spend not more than \$ 648000 a week to make television sets.**
- **If it makes a profit of \$510 per black and white set and \$ 675 per coloured set, how many sets of each type should be produced so that the company has maximum profit? Formulate this problem as a LPP given that the objective is to maximise the profit.**

# Decision variables

- **Solution** Let $x$ and $y$ denote, respectively, the number of black and white sets and coloured sets made each week. Thus

- $\boldsymbol{x \geq 0, y \geq 0}$

# Constraints

- C1: Since the company can make at most 300 sets a week, therefore,

$$C1: x + y \leq 300$$

- C2: Weekly cost (in Rs) of manufacturing the set is

$$1800x + 2700y$$

and the company can spend upto Rs. 648000. Therefore,

- 1800x + 2700y £ 648000, i.e., or

$$C2: 2x + 3y \leq 720$$

# Objective Function

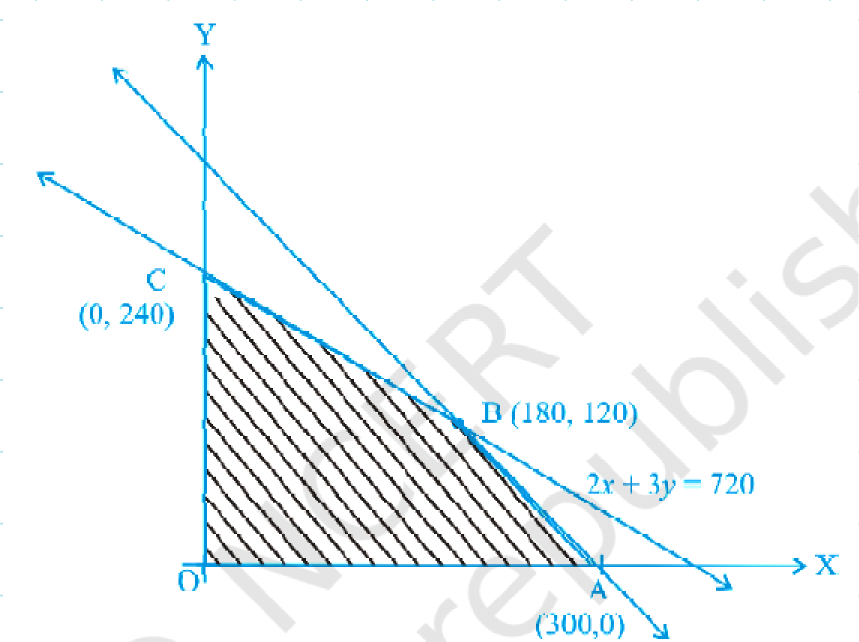
- The total profit on x black and white sets and y colour sets is Rs (510x + 675y). Let
- This is the **objective function**.

$$\text{Maximize} = 510x + 675y \; .$$

# Mathematical Formulation

- Thus, the mathematical formulation of the problem is
- Maximise Z = 510$x$ + 675$y$
- subject to the constraints :

$$\left.\begin{aligned} x + y &\le 300 \\ 2x + 3y &\le 720 \\ x \ge 0,\ y &\ge 0 \end{aligned}\right\}$$



# Corner points solution

| Corner Point | Value of Z | |
|---|---|---|
| O (0, 0) | 510 (0) + 675 (0) = 0 | |
| A (300, 0) | 510 (300) + 675 (0) = 153000 | |
| B (180, 120) | 510 (180) + 675 (120) = **172800** | ← Maximum |
| C (0, 240) | 510 (0) + 675 (240) = 162000 | |

Thus, maximum Z is 172800 at the point (180, 120), i.e., the company should produce 180 black and white television sets and 120 coloured television sets to get maximum profit.