Chapter 9

Transportation and Assignment Models

Introduction

- In this chapter we will explore three special linear programming models:
  - The transportation problem.
  - The assignment problem.
  - The transshipment problem.
- These problems are members of a category of LP techniques called *network flow problems.*

The Transportation Problem

- The *transportation problem* deals with the distribution of goods from several points of supply (*sources*) to a number of points of demand (*destinations*).
- Usually we are given the capacity of goods at each source and the requirements at each destination.
- Typically the objective is to minimize total transportation and production costs.

The Transportation Problem

- The Executive Furniture Corporation manufactures office desks at three locations:
  - Cairo, Alex, and Portsaid.
- The firm distributes the desks through regional warehouses located in
  - Maadi, Nasr city, and Haram.

The Transportation Problem

Network Representation of a Transportation Problem, with Costs, Demands and Supplies

Executive Furniture Company

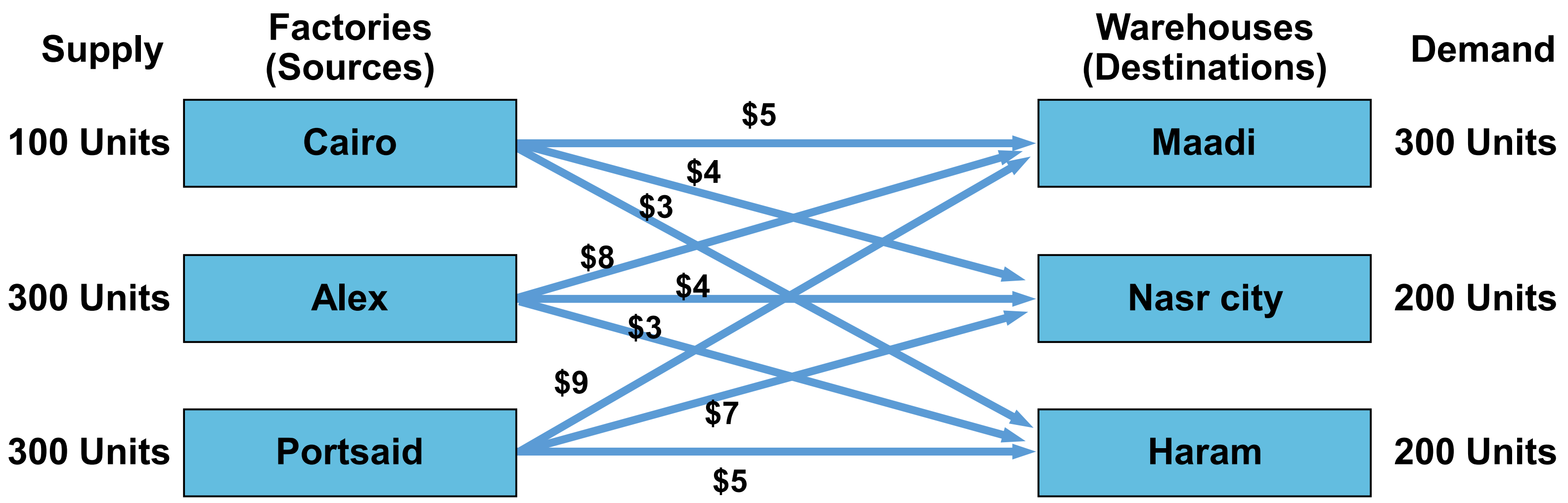


Figure 9.1

The Transportation Algorithm

- This is an iterative procedure in which a solution to a transportation problem is found and evaluated using a special procedure to determine whether the solution is optimal.
  - When the solution is optimal, the process stops.
  - If not, then a new solution is generated.

Transportation Table for Executive Furniture Corporation

Cairo capacity constraint

FROM \ TO	WAREHOUSE AT Maadi	WAREHOUSE AT Nasr city	WAREHOUSE AT Haram	FACTORY CAPACITY
Cairo FACTORY	$5	$4	$3	100
Alex FACTORY	$8	$4	$3	300
Portsaid FACTORY	$9	$7	$5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.2

Cost of shipping 1 unit from portsaid factory to nasr city warehouse

Haram warehouse demand

Total supply and demand

Cell representing a source-to-destination (Alex to Haram) shipping assignment that could be made

Developing an Initial Solution: Northwest Corner Rule

- Once we have arranged the data in a table, we must establish an initial feasible solution.
- One systematic approach is known as the *northwest corner rule.*
- Start in the upper left-hand cell and allocate units to shipping routes as follows:
    1. Exhaust the supply (factory capacity) of each row before moving down to the next row.
    2. Exhaust the demand (warehouse) requirements of each column before moving to the right to the next column.
    3. Check that all supply and demand requirements are met.
- This problem takes five steps to make the initial shipping assignments.

Developing an Initial Solution: Northwest Corner Rule

1. Beginning in the upper left hand corner, we assign 100 units from Cairo to Maadi. This exhaust the supply from Cairo but leaves Maadi 200 desks short. We move to the second row in the same column.

FROM \ TO	Maadi (*A*)	NasrCity (*B*)	Haram (*C*)	FACTORY CAPACITY
Cairo (*D*)	100 | $5	$4	$3	100
Alex (*E*)	$8	$4	$3	300
PortSaid (*F*)	$9	$7	$5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Developing an Initial Solution: Northwest Corner Rule

2. Assign 200 units from Alex to Maadi. This meets Maadi's demand. Alex has 100 units remaining so we move to the right to the next column of the second row.

FROM \ TO	Maadi (*A*)		NasrCity (*B*)		Haram (*C*)		FACTORY CAPACITY
Cairo (*D*)	100	$5		$4		$3	100
Alex (*E*)	200	$8		$4		$3	300
PortSaid (*F*)		$9		$7		$5	300
WAREHOUSE REQUIREMENTS	300		200		200		700

Developing an Initial Solution: Northwest Corner Rule

3. Assign 100 units from Alex to NasrCity. The Alex supply has now been exhausted but NasrCity is still 100 units short. We move down vertically to the next row in the nasrcity column.

FROM \ TO	Maadi (*A*)	NasrCity (*B*)	Haram (*C*)	FACTORY CAPACITY
Cairo (*D*)	100 \$5	\$4	\$3	100
Alex (*E*)	200 \$8	100 \$4	\$3	300
PortSaid (*F*)	\$9	\$7	\$5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Developing an Initial Solution: Northwest Corner Rule

4. Assign 100 units from Fort Portsaid to NasrCity. This fulfills Nasrcity's demand and Portsaid still has 200 units available.

FROM \ TO	Maadi (*A*)	Nasr City (*B*)	Haram (*C*)	FACTORY CAPACITY
Cairo (*D*)	100 $5	$4	$3	100
Alex (*E*)	200 $8	100 $4	$3	300
PortSaid (*F*)	$9	100 $7	$5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Developing an Initial Solution: Northwest Corner Rule

5. Assign 200 units from Portsaid to Haram. This exhausts Portsaid's supply and Haram's demand. The initial shipment schedule is now complete.

FROM \ TO	Maadi (*A*)	Nasrcity (*B*)	Haram (*C*)	FACTORY CAPACITY
Cairo (*D*)	100 $5	$4	$3	100
Alex (*E*)	200 $8	100 $4	$3	300
Portsaid (*F*)	$9	100 $7	200 $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.3

Developing an Initial Solution: Northwest Corner Rule

The cost of this shipping assignment:

ROUTE FROM	ROUTE TO	UNITS SHIPPED	x	PER UNIT COST ($)	=	TOTAL COST ($)
D	*A*	100		5		500
E	*A*	200		8		1,600
E	*B*	100		4		400
F	*B*	100		7		700
F	*C*	200		5		1,000
						4,200

This solution is feasible but we need to check to see if it is optimal.

Transportation Problem Cont.

Stepping-Stone Method
Finding a Least Cost Solution

Stepping-Stone Method: Finding a Least Cost Solution

- The *stepping-stone method* is an iterative technique for moving from an initial feasible solution to an optimal feasible solution.
- There are two distinct parts to the process:
  - Testing the current solution to determine if improvement is possible.
  - Making changes to the current solution to obtain an improved solution.
- This process continues until the optimal solution is reached.

Stepping-Stone Method: Finding a Least Cost Solution

- There is one very important rule: *The number of occupied routes (or squares) must always be equal to one less than the sum of the number of rows plus the number of columns*
  - In the Executive Furniture problem this means the initial solution must have 3 + 3 – 1 = 5 squares used.

Occupied shipping routes (squares) = Number of rows + Number of columns – 1

- **When the number of occupied rows is less than this, the solution is called *degenerate.***

Testing the Solution for Possible Improvement

- The stepping-stone method works by testing each unused square in the transportation table to see what would happen to total shipping costs if one unit of the product were tentatively shipped on an unused route.
- There are five steps in the process.

Five Steps to Test Unused Squares with the Stepping-Stone Method

1. Select an unused square to evaluate.
2. Beginning at this square, trace a closed path back to the original square via squares that are currently being used with only horizontal or vertical moves allowed.
3. Beginning with a plus (+) sign at the unused square, place alternate minus (–) signs and plus signs on each corner square of the closed path just traced.

Five Steps to Test Unused Squares with the Stepping-Stone Method

4. Calculate an *improvement index* by adding together the unit cost figures found in each square containing a plus sign and then subtracting the unit costs in each square containing a minus sign.

5. Repeat steps 1 to 4 until an improvement index has been calculated for all unused squares. If all indices computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease total shipping costs.

Five Steps to Test Unused Squares with the Stepping-Stone Method

For the Executive Furniture Corporation data:

***Steps 1 and 2*. Beginning with Cairo–NasrCity route we trace a closed path using only currently occupied squares, alternately placing plus and minus signs in the corners of the path.**

- **In a *closed path*, only squares currently used for shipping can be used in turning corners.**
- ***Only one* closed route is possible for each square we wish to test.**

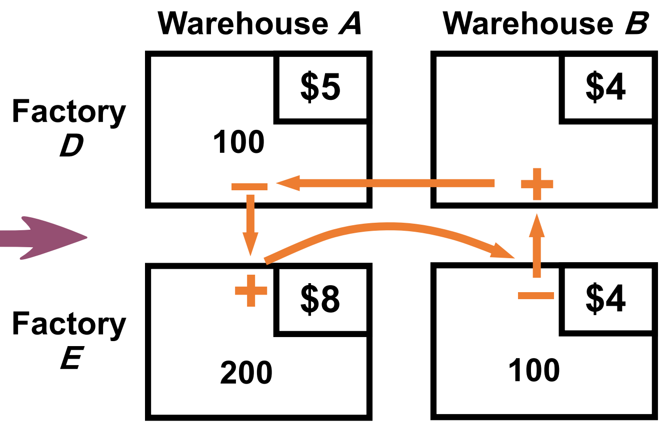
Five Steps to Test Unused Squares with the Stepping-Stone Method

Step 3. Test the cost-effectiveness of the Cario–Nasrcity shipping route by pretending that we are shipping one desk from Cairo to NasrCity. Put a plus in that box.

- **But if we ship one *more* unit out of Cairo we will be sending out 101 units.**
- **Since the Cairo factory capacity is only 100, we must ship *fewer* desks from Cairo to Maadi so place a minus sign in that box.**
- **But that leaves Maadi one unit short so increase the shipment from Alex to Maadi by one unit and so on until the entire closed path is completed.**

Five Steps to Test Unused Squares with the Stepping-Stone Method

Evaluating the unused cairo–Nasrcity shipping route

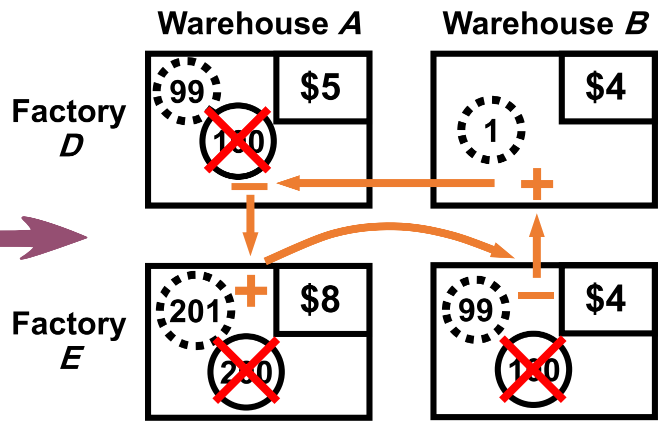


TO / FROM	Maadi		Nasrcity		Haram		FACTORY CAPACITY
Cairo	100	$5		$4		$3	100
Alex	200	$8	100	$4		$3	300
Portsaid		$9	100	$7	200	$5	300
WAREHOUSE REQUIREMENTS	300		200		200		700

Table 9.3

Five Steps to Test Unused Squares with the Stepping-Stone Method

Evaluating the unused cairo–nasrcity shipping route

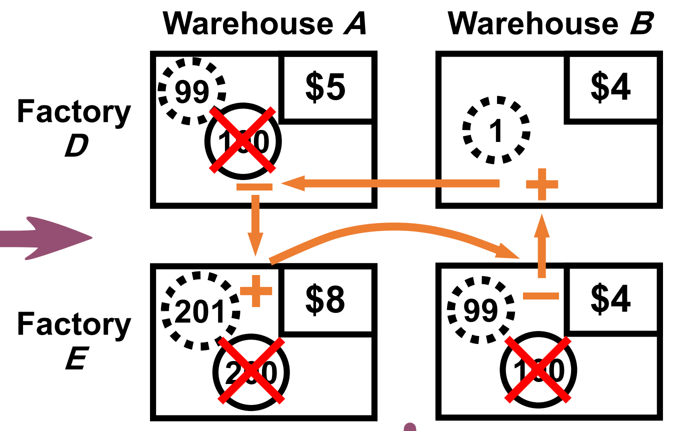


FROM \ TO	Maadi	Nasrcity	Haram	FACTORY CAPACITY
Cairo	100 $5	$4	$3	100
Alex	200 $8	100 $4	$3	300
Portsaid	$9	100 $7	200 $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.4

Five Steps to Test Unused Squares with the Stepping-Stone Method

Evaluating the unused Cairo–nasrcity shipping route



FROM \ TO	Maadi	Nasrcity	Haram	
Cairo	100 $5	$4	$3	
Alex	200 $8	100 $4	$3	
Portsaid	$9	100 $7	200 $5	
WAREHOUSE REQUIREMENTS	300	200	200	700

Result of Proposed Shift in Allocation

= 1 x $4
– 1 x $5
+ 1 x $8
– 1 x $4 = +$3

Table 9.4

Five Steps to Test Unused Squares with the Stepping-Stone Method

Step 4. Now compute an *improvement index* (I_{ij}) for the Cairo–Nasrcity route.

Add the costs in the squares with plus signs and subtract the costs in the squares with minus signs:

Cairo-nasrcity index $= I_{DB} = +\$4 - \$5 + \$5 - \$4 = +\$3$

This means for every desk shipped via the Cairo–Nasrcity route, total transportation cost will *increase* by \$3 over their current level.

Five Steps to Test Unused Squares with the Stepping-Stone Method

Step 5. Now examine the Cairo–Haram unused route which is slightly more difficult to draw.

- **Again, only turn corners at squares that represent existing routes.**
- **Pass through the Alex–Haram square but we can not turn there or put a + or – sign.**
- **The closed path we will use is:**

$$+\ DC - DA + EA - EB + FB - FC$$

Five Steps to Test Unused Squares with the Stepping-Stone Method

Evaluating the Cairo–Haram Shipping Route

FROM \ TO	Maadi	Nasrcity	Haram	FACTORY CAPACITY
Cairo	100 – $5	$4	Start + $3	100
Alex	200 + $8	100 – $4	$3	300
Portsaid	$9	100 + $7	200 – $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.5

Cairo–Haram improvement index $= I_{DC} = + \$3 - \$5 + \$8 - \$4 + \$7 - \$5 = + \$4$

Five Steps to Test Unused Squares with the Stepping-Stone Method

Opening the Cairo–Haram route will not lower our total shipping costs.

Evaluating the other two routes we find:

Alex-Haram index $= I_{EC} = + \$3 - \$4 + \$7 - \$5 = + \$1$

The closed path is

$+ EC - EB + FB - FC$

Portsaid–Maadi index $= I_{FA} = + \$9 - \$7 + \$4 - \$8 = - \$2$

The closed path is

$+ FA - FB + EB - EA$

Opening the portsaid-Maadi route *will* lower our total transportation costs.

Five Steps to Test Unused Squares with the Stepping-Stone Method

Evaluating the Alex-Haram Shipping Route

FROM \ TO	Maadi	Nasrcity	Haram	FACTORY CAPACITY
Cairo	100 $5	$4	$3	100
Alex	200 $8	100 $4	Start $3	300
Portsaid	$9	100 $7	200 $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.5

Five Steps to Test Unused Squares with the Stepping-Stone Method

Evaluating the Portsaid–Maadi Shipping Route

FROM \ TO	Maadi	Nasrcity	Haram	FACTORY CAPACITY
Cairo	100 $5	$4	$3	100
Alex	200 $8	100 $4	$3	300
Portsaid	**Start** $9	100 $7	200 $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.5

Obtaining an Improved Solution

- In the Executive Furniture problem there is only one unused route with a negative index (Portsaid-Maadi).
  - If there was more than one route with a negative index, we would choose the one with the largest improvement
- We now want to ship the maximum allowable number of units on the new route
- The quantity to ship is found by referring to the closed path of plus and minus signs for the new route and selecting the *smallest number* found in those squares containing minus signs.

Obtaining an Improved Solution

- To obtain a new solution, that number is added to all squares on the closed path with plus signs and subtracted from all squares the closed path with minus signs.
- All other squares are unchanged.
- In this case, the maximum number that can be shipped is 100 desks as this is the smallest value in a box with a negative sign (*FB* route).
- We add 100 units to the *FA* and *EB* routes and subtract 100 from *FB* and *EA* routes.
- This leaves balanced rows and columns and an improved solution.

Obtaining an Improved Solution

Stepping-Stone Path Used to Evaluate Route *F-A*

FROM \ TO	*A*	*B*	*C*	FACTORY CAPACITY
D	100 \$5	\$4	\$3	100
E	200 \$8 −	100 \$4 +	\$3	300
F	\$9 +	100 \$7 −	200 \$5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.6

Obtaining an Improved Solution

Second Solution to the Executive Furniture Problem

FROM \ TO	A	B	C	FACTORY CAPACITY
D	100 \$5	\$4	\$3	100
E	100 \$8	200 \$4	\$3	300
F	100 \$9	\$7	200 \$5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.7

Total shipping costs have been reduced by (100 units) x (\$2 saved per unit) and now equals \$4,000.

Obtaining an Improved Solution

- This second solution may or may not be optimal.
- To determine whether further improvement is possible, we return to the first five steps to test each square that is *now* unused.
- The four new improvement indices are:

***D* to *B* = I_{DB} = + \$4 – \$5 + \$8 – \$4 = + \$3**

(closed path: + *DB* – *DA* + *EA* – *EB*)

***D* to *C* = I_{DC} = + \$3 – \$5 + \$9 – \$5 = + \$2**

(closed path: + *DC* – *DA* + *FA* – *FC*)

***E* to *C* = I_{EC} = + \$3 – \$8 + \$9 – \$5 = – \$1**

(closed path: + *EC* – *EA* + *FA* – *FC*)

***F* to *B* = I_{FB} = + \$7 – \$4 + \$8 – \$9 = + \$2**

(closed path: + *FB* – *EB* + *EA* – *FA*)

Obtaining an Improved Solution

Second Solution to the Executive Furniture Problem

FROM \ TO	A	B	C	FACTORY CAPACITY
D	100 ($5)	($4)	($3)	100
E	100 ($8)	200 ($4)	($3)	300
F	100 ($9)	($7)	200 ($5)	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.7

Total shipping costs have been reduced by (100 units) x ($2 saved per unit) and now equals $4,000.

Obtaining an Improved Solution

Path to Evaluate the *E-C* Route

FROM \ TO	*A*	*B*	*C*	FACTORY CAPACITY
D	100 $5	$4	$3	100
E	100 $8	200 $4	Start $3	300
F	100 $9	$7	200 $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.8

An improvement can be made by shipping the maximum allowable number of units from *E* to *C.*

Obtaining an Improved Solution

Path to Evaluate the *E-C* Route

FROM \ TO	*A*	*B*	*C*	FACTORY CAPACITY
D	100 \$5	\$4	\$3	100
E	100 \$8 −	200 \$4	Start \$3 +	300
F	100 \$9 +	\$7	200 \$5 −	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.8

An improvement can be made by shipping the maximum allowable number of units from *E* to *C.*

Obtaining an Improved Solution

Total cost of third solution:

ROUTE		DESKS SHIPPED	x	PER UNIT COST ($)	=	TOTAL COST ($)
FROM	TO					
D	*A*	100		5		500
E	*B*	200		4		800
E	*C*	100		3		300
F	*A*	200		9		1,800
F	*C*	100		5		500
						3,900

Obtaining an Improved Solution

Third and optimal solution:

FROM \ TO	*A*	*B*	*C*	FACTORY CAPACITY
D	100 | $5	| $4	| $3	100
E	| $8	200 | $4	100 | $3	300
F	200 | $9	| $7	100 | $5	300
WAREHOUSE REQUIREMENTS	300	200	200	700

Table 9.9

Obtaining an Improved Solution

This solution is optimal as the improvement indices that can be computed are all greater than or equal to zero.

D to *B* = I_{DB} = + \$4 – \$5 + \$9 – \$5 + \$3 – \$4 = + \$2
(closed path: + *DB* – *DA* + *FA* – *FC* + *EC* – *EB*)

D to *C* = I_{DC} = + \$3 – \$5 + \$9 – \$5 = + \$2
(closed path: + *DC* – *DA* + *FA* – *FC*)

E to *A* = I_{EA} = + \$8 – \$9 + \$5 – \$3 = + \$1
(closed path: + *EA* – *FA* + *FC* – *EC*)

F to *B* = I_{FB} = + \$7 – \$5 + \$3 – \$4 = + \$1
(closed path: + *FB* – *FC* + *EC* – *EB*)

Summary of Steps in Transportation Algorithm (Minimization)

1. Set up a balanced transportation table.
2. Develop initial solution using the northwest corner method method.
3. Calculate an improvement index for each empty cell using the stepping-stone method. If improvement indices are all nonnegative, stop as the optimal solution has been found. If any index is negative, continue to step 4.
4. Select the cell with the improvement index indicating the greatest decrease in cost. Fill this cell using the stepping-stone path and go to step 3.