Chapter 3

Random Variables and Probability Distributions

Chapter Outline

3.1 Concept of a Random Variable

3.2 Discrete Probability Distribution

3.3 Continuous Probability Distributions

Random Variables

A **random variable** is a function that associates a real number with each element in the sample space.

- The sample space giving each possible outcome when three electronic components are tested may be written

  $S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\}$,

  where N denotes non-defective and D denotes defective.

- One is naturally concerned with the number of defectives that occur. Thus, each point in the sample space will be assigned a numerical value of 0, 1, 2, or 3.

- These values are, of course, random quantities determined by the outcome of the experiment. They may be viewed as values assumed by the random variable X, the number of defective items when three electronic components are tested.

Example 1

Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values *y* of the random variable *Y*, where *Y* is the number of red balls, are

Sample Space	Y
RR	2
RB	1
BR	1
BB	0

Example 2

Suppose a sampling plan involves sampling items from a process until a defective is observed. The evaluation of the process will depend on how many consecutive items are observed. In that regard, let *X* be a random variable defined by the number of items observed before a defective is found.

Sample Space	***X***
D	0
ND	1
NND	2
NNND	3
.	.
.	.

Example 3

Let X be the random variable defined by the waiting time, in hours, between successive speeders spotted by a radar unit. The random variable X takes on all values x for which $x \geq 0$.

Discrete and Continuous Sample Spaces

If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a **discrete sample space**.

If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a **continuous sample space**.

Probability Mass Functions

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

Example 1

In the case of tossing a coin three times, the variable *X*, representing the number of heads, assumes the value 2 with probability 3/8, since 3 of the 8 equally likely sample points result in two heads and one tail.

x	0	1	2	3
$P(X = x) = f(x)$	1 / 8	3 / 8	3 / 8	1 / 8

Example 2

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution:

Let *X* be a random variable whose values *x* are the possible numbers of defective computers purchased by the school. Then *x* can only take the numbers 0, 1, and 2. Now

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95} \qquad f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190}$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}$$

x	0	1	2
***P*(*X* = *x*)**	68 / 95	51 / 190	3 / 190

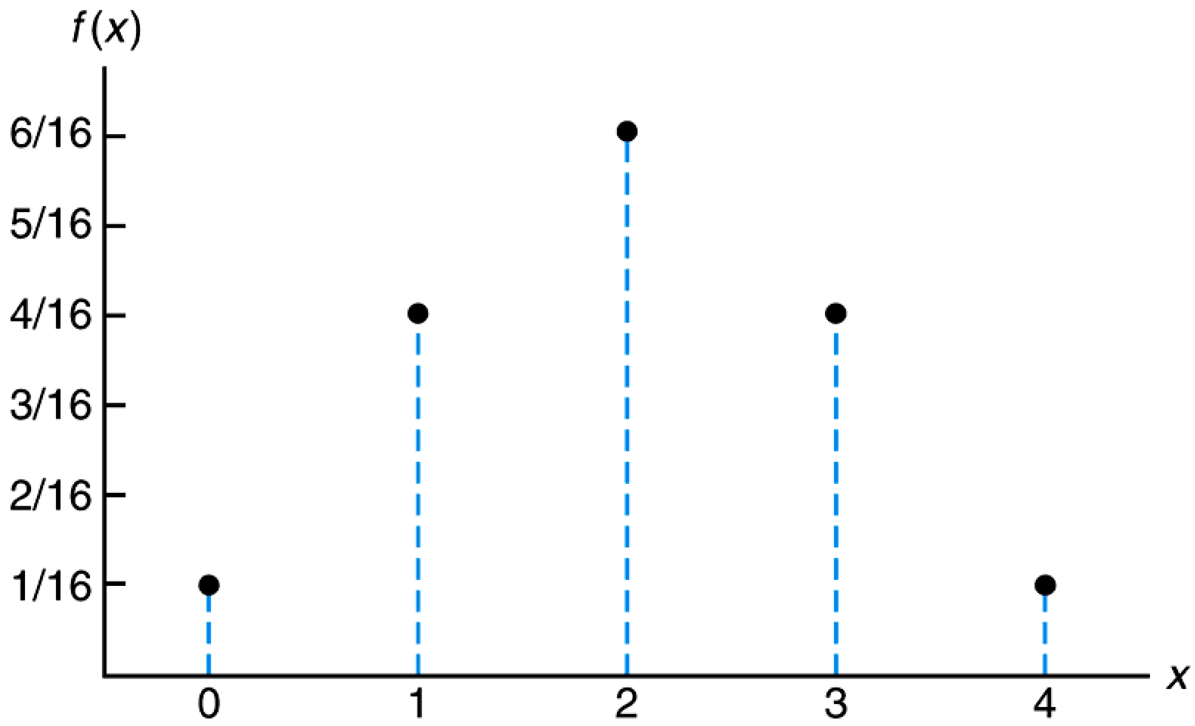
Cumulative Distribution Functions

The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

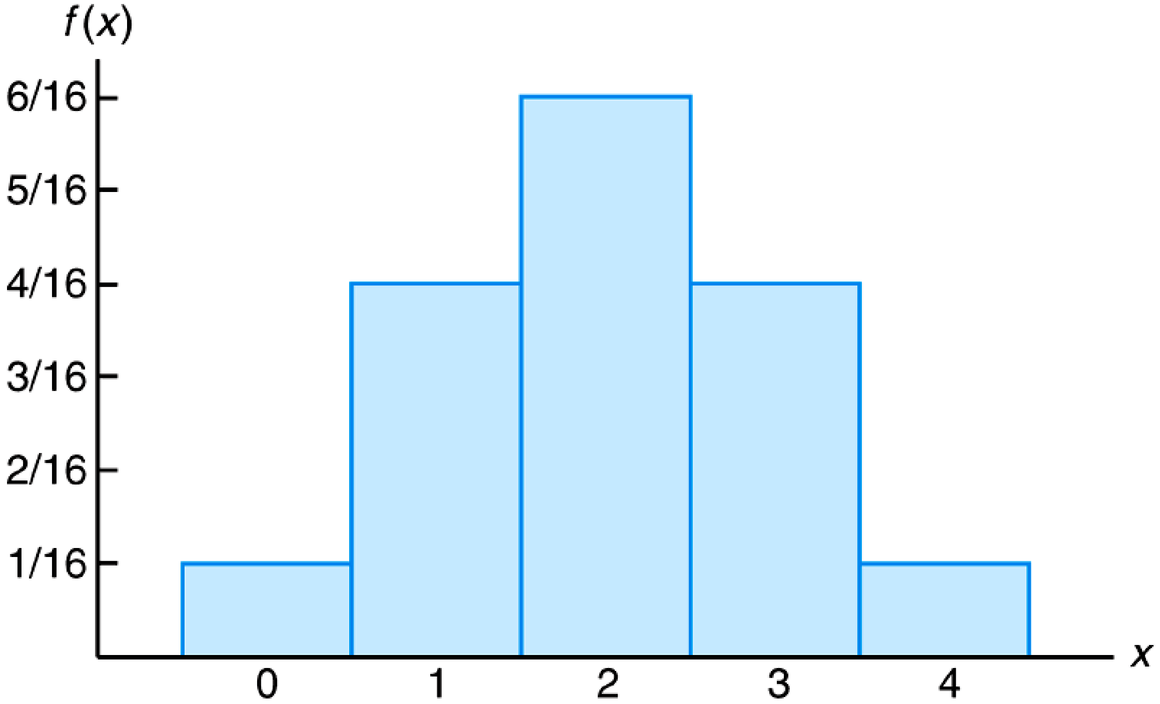
Probability Mass Function Plot

x	0	1	2	3	4
$P(X = x) = f(x)$	1 / 16	1 / 4	3 / 8	1 / 4	1 / 16



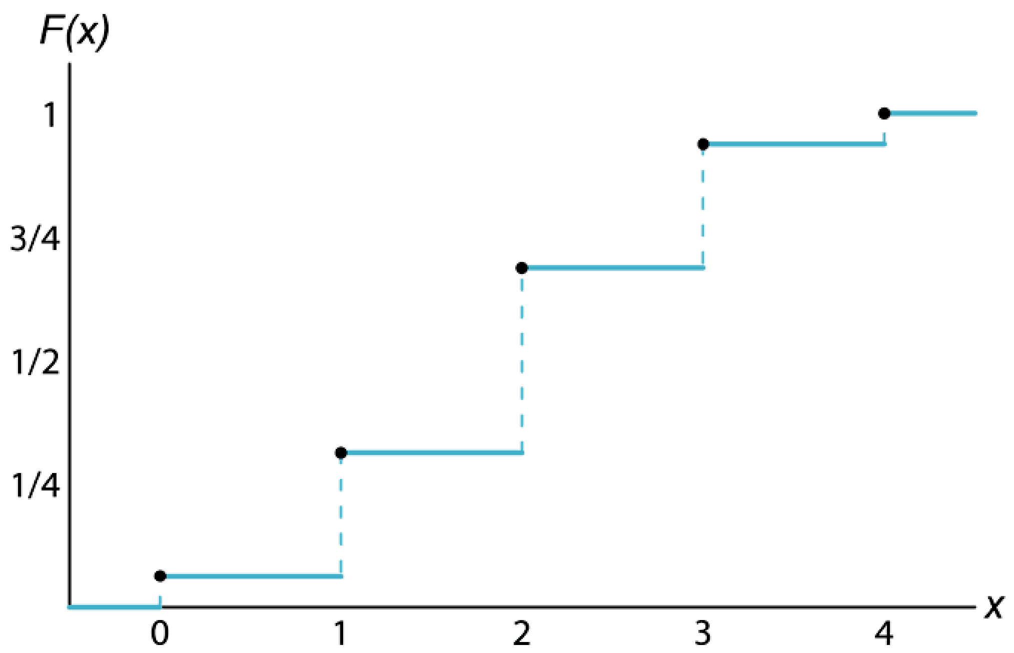
Probability Histogram

x	0	1	2	3	4
$P(X = x) = f(x)$	1 / 16	1 / 4	3 / 8	1 / 4	1 / 16



Discrete Cumulative Distribution Function

x	0	1	2	3	4
$P(X = x) = f(x)$	1 / 16	1 / 4	3 / 8	1 / 4	1 / 16
$P(X \leq x) = F(x)$	1 / 16	5 / 16	11 / 16	15 / 16	1



Discrete Cumulative Distribution Function

x	0	1	2	3	4
***P*(*X* = *x*) = *f*(*x*)**	1 / 16	1 / 4	3 / 8	1 / 4	1 / 16
***P*(*X* ≤ *x*) = *F*(*x*)**	1 / 16	5 / 16	11 / 16	15 / 16	1

$$\sum_{x=0}^{4} f(x) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} + \frac{1}{4} + \frac{1}{16} = 1$$

$$P(X \le 2) = \sum_{x=0}^{2} f(x) = F(2) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} = \frac{11}{16}$$

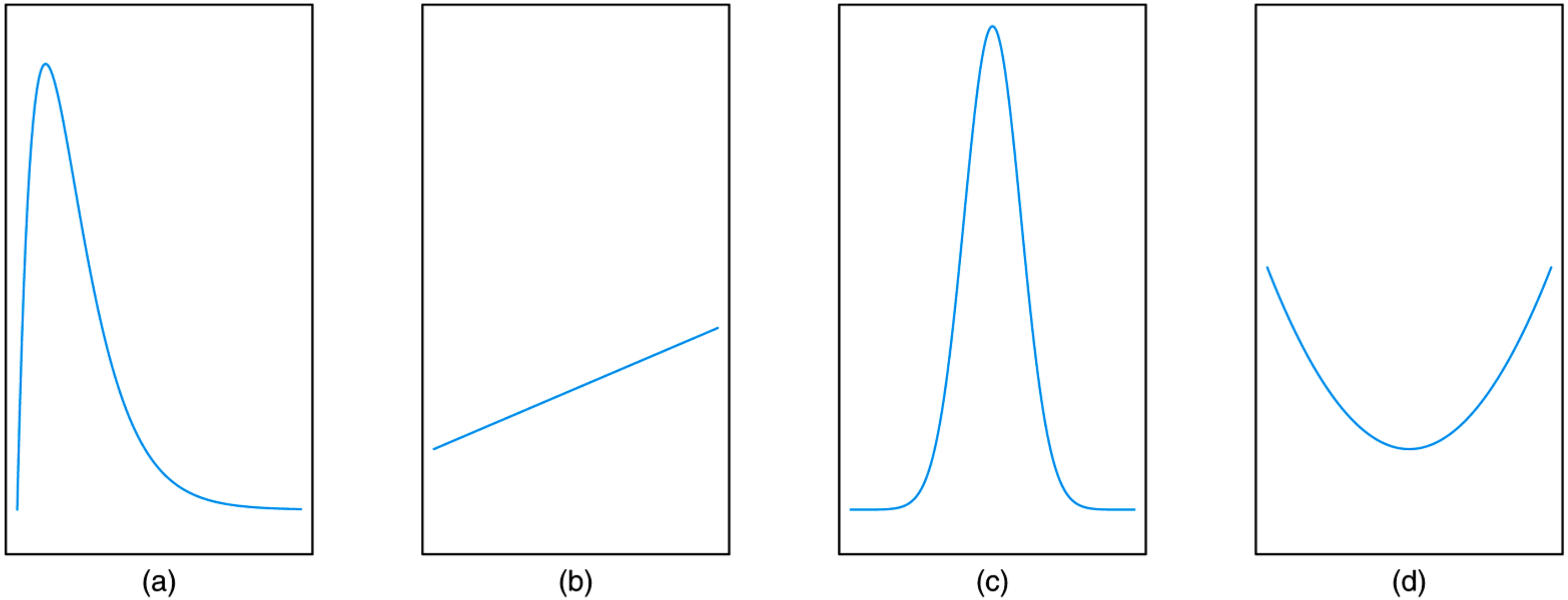
$$P(1 \le X \le 3) = \sum_{x=1}^{3} f(x) = F(3) - F(0) = \frac{1}{4} + \frac{3}{8} + \frac{1}{4} = \frac{14}{16}$$

Probability Density Functions

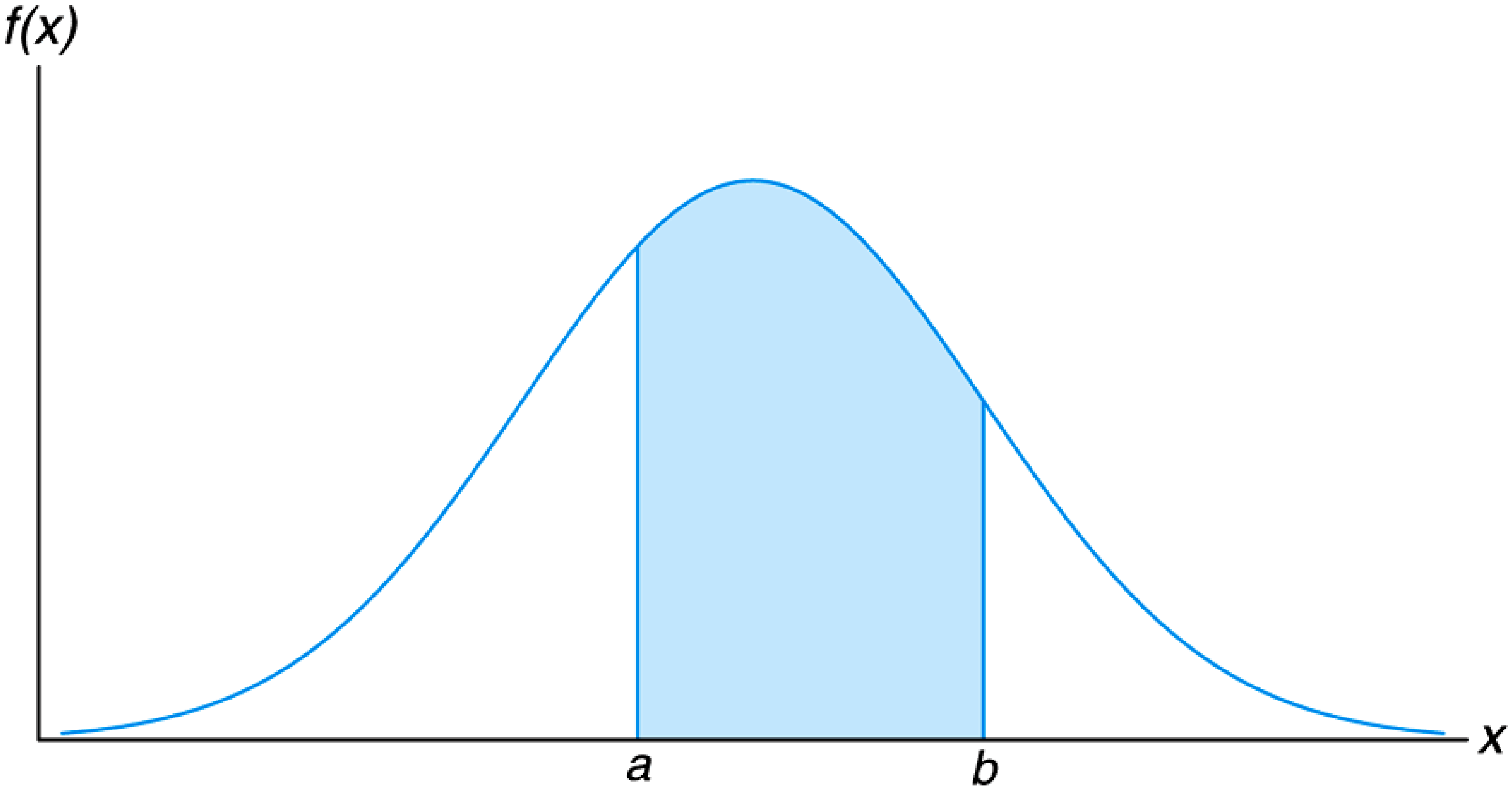
The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\, dx$.

Typical Density Functions



$P(a < X < b)$



Example

Suppose that the error in the reaction temperature, in ◦*C*, for a controlled laboratory experiment is a continuous random variable *X* having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Verify that *f(x)* is a density function.
(b) Find $P(0 < X \leq 1)$.

Solution:

(a) $f(x) \geq 0. \int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3}\, dx = \left.\frac{x^3}{9}\right|_{-1}^{2} = \left[\frac{8}{9}\right] - \left[\frac{-1}{9}\right] = 1$

(b) $P(0 < X \leq 1) = \int_{0}^{1} \frac{x^2}{3}\, dx = \left.\frac{x^3}{9}\right|_{0}^{1} = \frac{1}{9}$

Cumulative Distribution Functions

The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad \text{for } -\infty < x < \infty.$$

$$P(a \leq x \leq b) = P(a < x < b) = \int_{a}^{b} f(x)dx = F(b) - F(a)$$

$$P(x \leq b) = \int_{-\infty}^{b} f(x)dx = F(b)$$

$$P(x \geq b) = 1 - P(x \leq b) = 1 - \int_{-\infty}^{b} f(x)dx = 1 - F(b)$$

Continuous Cumulative Distribution Function

For the probability density function

$$f(x)=\begin{cases}\dfrac{x^2}{3}, & -1<x<2\\ 0, & \text{otherwise}\end{cases}$$

find *F(x),* and use it to evaluate *P(*0 < *X* ≤ 1*).*

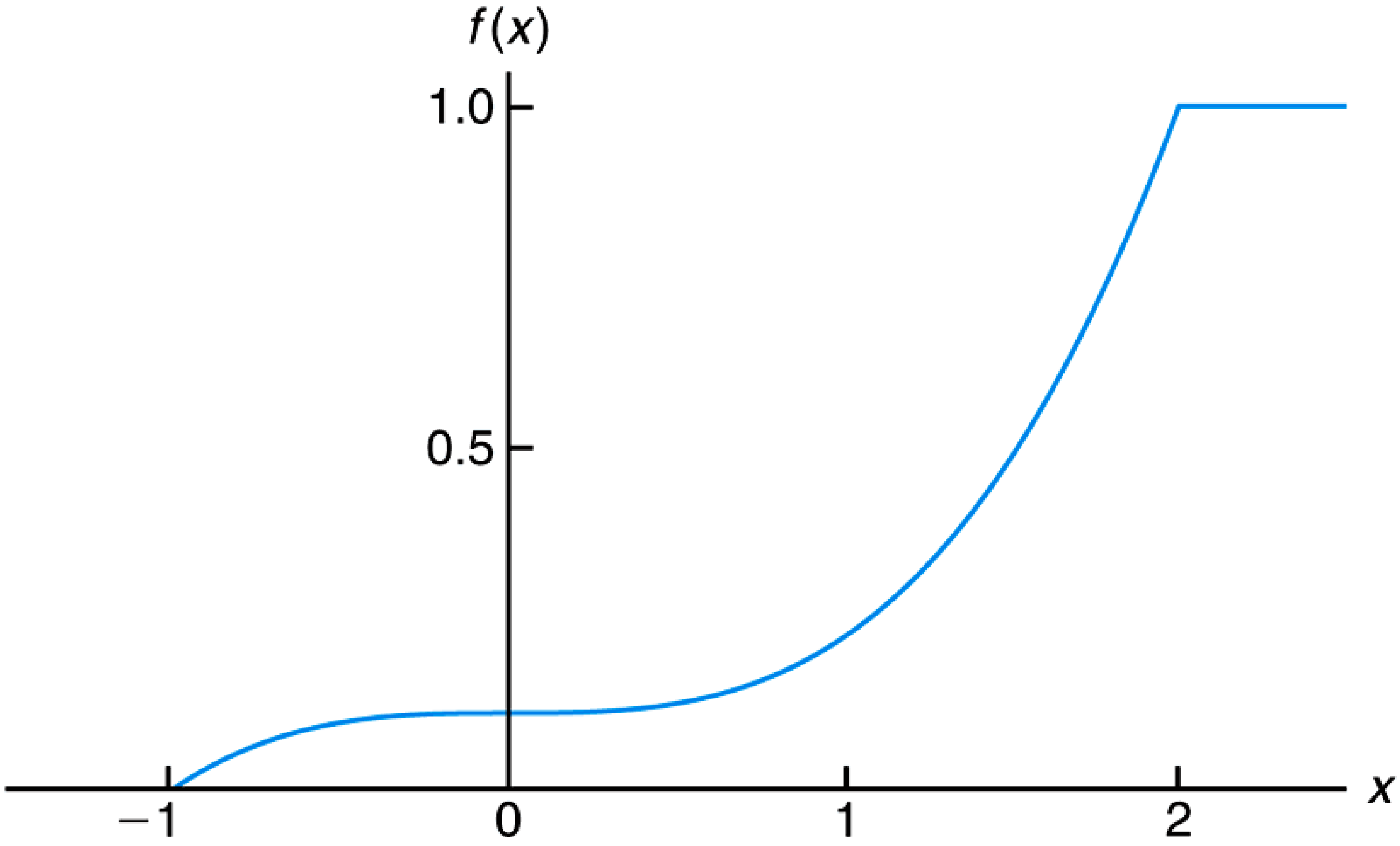
Solution:

$$F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-1}^{x}\frac{t^2}{3}dt=\frac{t^3}{9}\bigg|_{-1}^{x}=\left[\frac{x^3}{9}\right]-\left[\frac{-1}{9}\right]=\frac{x^3+1}{9}$$

$$F(x)=\begin{cases}0 & x<-1\\ \dfrac{x^3+1}{9} & -1\le x<2\\ 1 & x\ge 2\end{cases}$$

$$P(0<X\le 1)=F(1)-F(0)=\frac{2}{9}-\frac{1}{9}=\frac{1}{9}$$

Continuous Cumulative Distribution Function Graph



Example 1

Find the constant a such that $f(x)=\begin{cases} ax & \text{if } 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$ is a density function.

Find $P(0.5\le x\le 1)$.

Solution:

$$\int_{-\infty}^{\infty} f(x)dx=1 \Rightarrow \int_0^1 ax\ dx=1 \Rightarrow \left.\frac{ax^2}{2}\right|_0^1=1 \Rightarrow \frac{a}{2}=1 \Rightarrow a=2$$

$$f(x)=\begin{cases} 2x & \text{if } 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(0.5\le x\le 1)=\int_{0.5}^{1} 2x\ dx=\left. x^2\right|_{0.5}^{1}=1-0.25=0.75$$

Example 2

Let $f(x) = e^{-x}, x \geq 0.$

(a) Show that $f(x)$ is a density function.

(b) Find $P(2 \leq x \leq 3)$.

(c) Find $P(x \leq 4)$.

(d) Find $P(x > 4)$.

Solution:

$$\text{(a)} \int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} e^{-x}dx = -e^{-x}\Big|_0^{\infty} = -[0-1] = 1$$

$$\text{(b)}\, P(2 \leq x \leq 3) = \int_{2}^{3} f(x)dx = \int_{2}^{3} e^{-x}dx = -e^{-x}\Big|_2^{3} = -[e^{-3} - e^{-2}] = 0.018$$

$$\text{(c)}\, P(x \leq 4) = \int_{0}^{4} f(x)dx = \int_{0}^{4} e^{-x}dx = -e^{-x}\Big|_0^{4} = -[e^{-4} - 1] = 1 - e^{-4}$$

$$\text{(d)}\, P(x > 4) = 1 - P(x \leq 4) = e^{-4}$$

Example 3

Let $f(x)=a(1+x^2), 0\le x\le 2.$ Find the value of a such that $f(x)$ is a density function.

Solution:

$$\int_0^2 f(x)dx=1 \Rightarrow \int_0^2 a(1+x^2)dx=1 \Rightarrow a\left[x+\frac{1}{3}x^3\right]\Bigg|_0^2=1$$

$$\Rightarrow a\left\{\left[2+\frac{8}{3}\right]-[0]\right\}=1 \Rightarrow \frac{14a}{3}=1 \Rightarrow a=\frac{3}{14}.$$

Thus, $f(x)=\dfrac{3}{14}(1+x^2), 0\le x\le 2.$