Chapter 5

Some Discrete Probability Distributions

Chapter Outline

5.1 Introduction and Motivation

5.2 Binomial Distribution

5.4 Geometric Distributions

5.5 Poisson Distribution and the Poisson Process

The Bernoulli Process

The Bernoulli process must possess the following properties:

- The experiment consists of repeated trials.
- Each trial results in an outcome that may be classified as a success or a failure.
- The probability of success, denoted by p, remains constant from trial to trial.
- The repeated trials are independent.

The Bernoulli Process

Consider the set of Bernoulli trials where three items are selected at random from a manufacturing process, inspected, and classified as defective or non-defective. A defective item is designated a success. The number of successes is a random variable *X* assuming integral values from 0 through 3. The eight possible outcomes and the corresponding values of *X are* Outcome

Outcomes	*NNN*	*NND*	*NDN*	*DNN*	*NDD*	*DND*	*DDN*	*DDD*
x	0	1	1	1	2	2	2	3

Since the items are selected independently and we assume that the process produces 25% defectives, we have

$$P(NDN) = P(N)P(D)P(N) = (3/4)(1/4)(3/4) = 9/46$$

The probability distribution of *X* is therefore

x	0	1	2	3
f (x)	27 / 64	27 / 64	9 / 64	1 / 64

Binomial Distribution

The number *X* of successes in *n* Bernoulli trials is called a binomial random variable. The probability distribution of this discrete random variable is called the **binomial distribution**, and its values will be denoted by $b(x; n, p)$ since they depend on the number of trials and the probability of a success on a given trial.

Thus, for the probability distribution of *X*, the number of defectives is

$$P(X = 2) = f(2) = b(2; 3, 1/4) = 9 / 64$$

Binomial Distribution

A **Bernoulli trial** can result in a success with probability *p* and a failure with probability $q = 1 - p$. Then the probability distribution of the binomial random variable *X*, the number of successes in *n* independent trials, is

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, \ldots$$

The mean and variance of the binomial distribution $b(x; n, p)$ are

$$\mu = np \text{ and } \sigma^2 = npq.$$

Example 1

The probability that a certain kind of component will survive a shock test is 3/4. Find the probability that exactly 2 of the next 4 components tested survive.

Solution:

Assuming that the tests are independent and p = 3/4 for each of the 4 tests, we obtain

$$b\left(2;4,\frac{3}{4}\right)=\binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2=\frac{4!}{2!\cdot 2!}\left(\frac{9}{16}\right)\left(\frac{1}{16}\right)=6\left(\frac{9}{266}\right)=\frac{27}{128}$$

Example 2

A large chain retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 3%. The inspector randomly picks 20 items from a shipment. What is the probability that there will be at least one defective item among these 20?

Solution:

Denote by *X* the number of defective devices among the 20. Then *X* follows a *b(x*; 20, 0.03*)* distribution. Hence,

$$
\begin{aligned}
P(X \geq 1) &= 1 - P(X = 0) \\
&= 1 - b(0; 20, 0.03) \\
&= 1 - \binom{20}{0}(0.03)^0(1-0.03)^{20-0} = 1-(0.97)^{20} = 0.4562
\end{aligned}
$$

Example 3

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) exactly 5 survive (b) at least 10 survive, and (c) from 3 to 8 survive?

Solution:

Let X be the number of people who survive.

$$\text{(a) } P(X=5)=b(5;15,0.4)=\binom{15}{5}(0.4)^5(0.6)^{10}=\frac{15!}{5!\cdot 10!}(0.4)^5(0.6)^{10}=0.1859$$

$$\text{(b) } P(X\geq 10)=1-P(X<10)=1-\sum_{x=0}^{9}b(x;15,0.4)=1-0.9662=0.0338$$

$$\text{(c) } P(3\leq X\leq 8)=\sum_{x=3}^{8}b(x;15,0.4)=\sum_{x=0}^{8}b(x;15,0.4)-\sum_{x=0}^{3}b(x;15,0.4)$$

$$=0.4032-0.2173=0.1859$$

Example 4

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease. Find the mean and variance of the binomial random variable.

Solution:

$$\mu = (15)(0.4) = 6 \text{ and}$$

$$\sigma^2 = (15)(0.4)(0.6) = 3.6$$

Geometric Distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is

$$g(x; p) = pq^{x-1}, \; x = 1, 2, 3, \ldots.$$

The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}.$$

Example 1

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Solution:

Using the geometric distribution with $x = 5$ and $p = 0.01$, we have

$$g(5; 0.01) = (0.01)(0.99)^4 = 0.0096.$$

Example 2

A family decided to continue having children until they have a boy.

(a) What is the probability that they will have 4 children?

(b) What is the probability that they have at most 4 children.

Solution:

Let *X* be the number of children in the family.

$$\text{(a) } P(X = 4) = g(4,0.5) = (0.5)(0.5)^3 = \frac{1}{16}$$

$$\text{(b) } P(X \le 4) = \sum_{x=1}^{4} g(x;0.5) = (0.5)(0.5)^3 + (0.5)(0.5)^2 + (0.5)(0.5)^1 + (0.5)(0.5)^0 = \frac{15}{16}$$

Example 3

At a "busy time," a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection. let $p = 0.05$ be the probability of a connection during a busy time.

(a) What is the probability that 5 attempts are necessary for a successful call?

(b) What is the expected number of calls necessary to make a connection?

Solution:

(a) Using the geometric distribution with $x = 5$ and $p = 0.05$ yields

$$P(X = x) = g(5; 0.05) = (0.05)(0.95)^4 = 0.041.$$

(b) $\mu = 1 / 0.05 = 20$.

Poisson Process

- Experiments yielding numerical values of a random variable X, the number of outcomes occurring during a given time interval or in a specified region, are called Poisson experiments.

- The given time interval may be of any length, such as a minute, a day, a week, a month, or even a year.

- A Poisson experiment can generate observations for the random variable X representing the number of telephone calls received per hour by an office, the number of days school is closed due to snow during the winter, or the number of games postponed due to rain during a baseball season.

- The specified region could be a line segment, an area, a volume, or perhaps a piece of material. In such instances, X might represent the number of field mice per acre, the number of bacteria in a given culture, or the number of typing errors per page.

Poisson Process

- The number of outcomes occurring in one time interval or specified region of space is independent of the number that occur in any other disjoint time interval or region. In this sense we say that the Poisson process has no memory.

- The probability that a single outcome will occur during a very short time interval or in a small region is proportional to the length of the time interval or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.

- The probability that more than one outcome will occur in such a short time interval or fall in such a small region is negligible.

The number X of outcomes occurring during a Poisson experiment is called a Poisson random variable, and its probability distribution is called the **Poisson distribution.**

Poisson Distribution

The probability distribution of the Poisson random variable *X*, representing the number of outcomes occurring in a given time interval or specified region denoted by *t*, is

$$p(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

where λ is the average number of outcomes per unit time, distance, area, or volume and *e* = 2.718281828459 . . . .

Both the mean and the variance of the Poisson distribution $p(x;\lambda t)$ are λt.

Example 1

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Solution:

Using the Poisson distribution with $x = 6$ and $\lambda t = 4$, we have

$$p(6;4) = \frac{e^{-4} 4^{6}}{6!} = 0.1042$$

Example 2

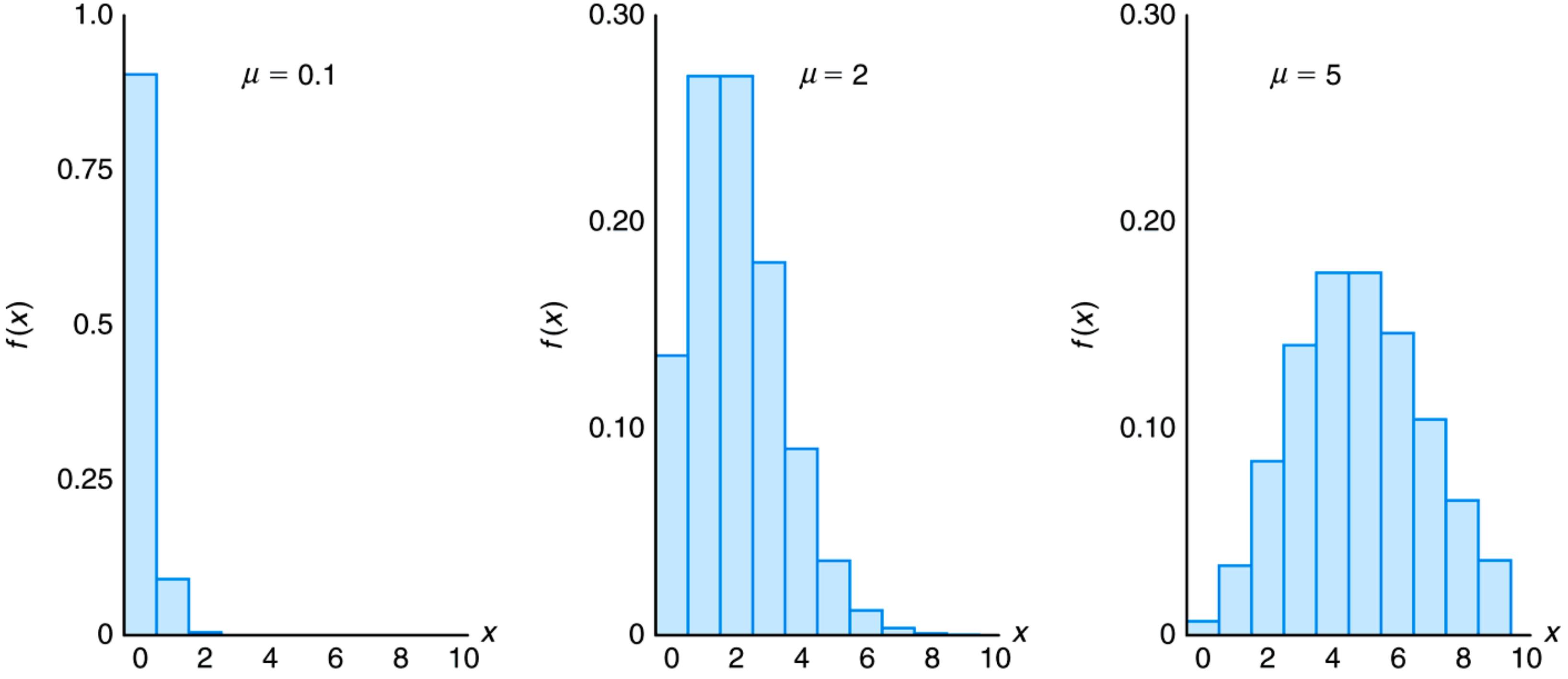
Ten is the average number of oil tankers arriving each day at a certain port. What is the probability that on a given day the port can handle at most 15 tankers per day?

Solution:

Let *X* be the number of tankers arriving each day, $\lambda t = 10$:

$$P(X \le 15) = \sum_{x=0}^{15} p(x;10) = \sum_{x=0}^{15} \frac{e^{-10}(10)^x}{x!} = 0.9513$$

Poisson Density Functions for Different Means



Theorem

Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$, and $np \overset{n\to\infty}{\longrightarrow} \mu$ remains constant,

$$b(x; n, p) \overset{n\to\infty}{\longrightarrow} p(x; \mu).$$

Example 1

In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

(a) What is the probability that in any given period of 400 days there will be an accident on one day?

(b) What is the probability that there are at most three days with an accident?

Solution:

Let X be a binomial random variable with n = 400 and p = 0.005. Thus, np = 2. Using the Poisson approximation,

(a) $P(X = 1) = e^{-2}\, 2^{1} = 0.271$

(b) $$P(X \le 3) = \sum_{x=0}^{3} \frac{e^{-2}\,2^{x}}{x!} = \frac{e^{-2}\,2^{0}}{0!} + \frac{e^{-2}\,2^{1}}{1!} + \frac{e^{-2}\,2^{2}}{2!} + \frac{e^{-2}\,2^{3}}{3!} = 0.857$$

Example 2

In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Solution:

This is a binomial experiment with n = 8000 and p = 0.001. Since p is very close to 0 and n is quite large, we shall approximate with the Poisson distribution using

$$\mu = (8000)(0.001) = 8.$$

Hence, if X represents the number of bubbles, we have

$$P(X < 7) = \sum_{x=0}^{6} b(x;800,0.001) \cong \sum_{x=0}^{6} p(x;8) = 0.3134$$