Chapter 6

Some Continuous Probability Distributions

Chapter Outline

6.1 Continuous Uniform Distribution

6.2 Normal Distribution

6.3 Areas Under the Normal Curve

6.4 Applications of the Normal Distribution

6.5 Normal Approximation to the Binomial Distribution

Uniform Distribution

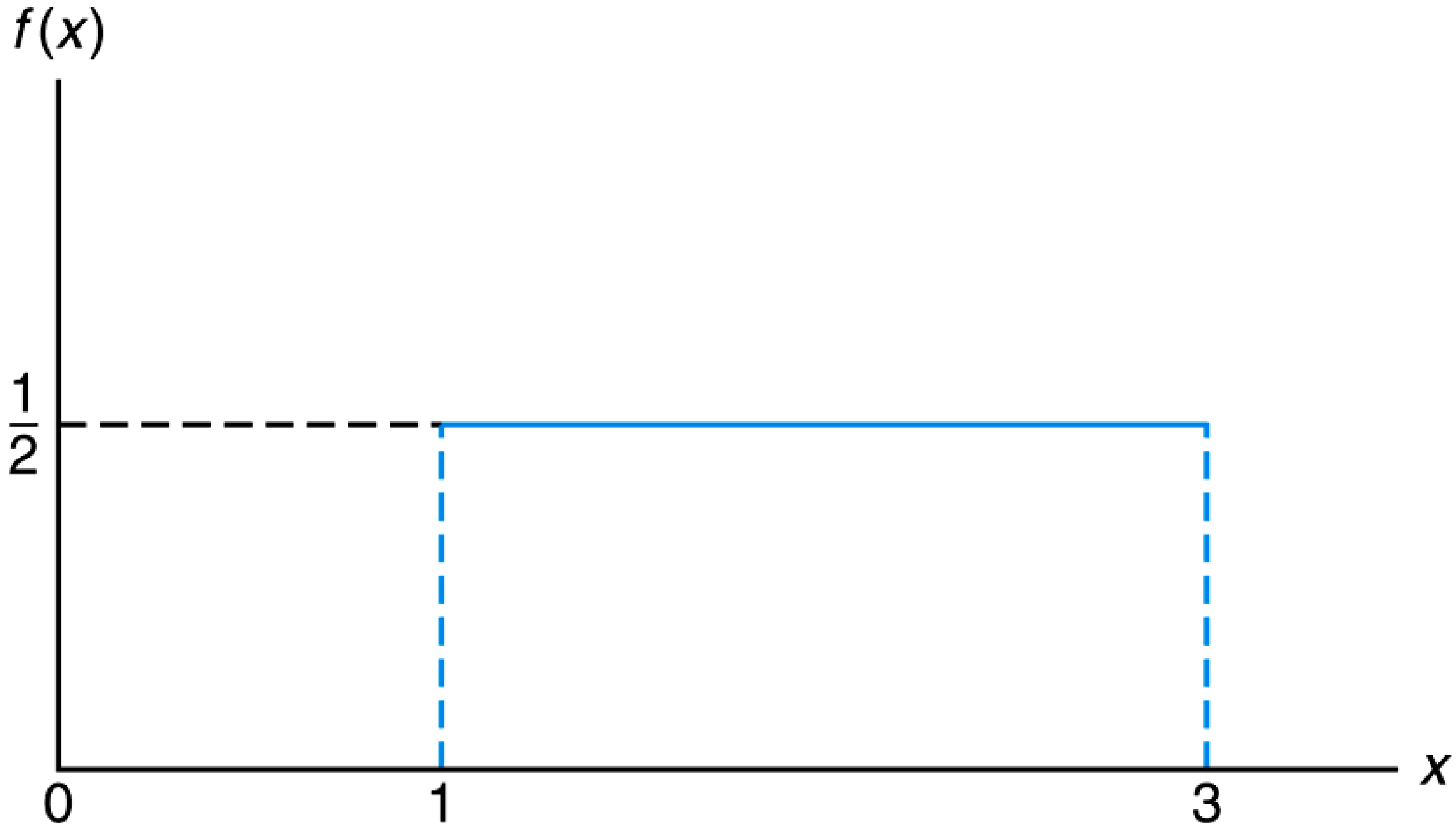
The density function of the continuous uniform random variable *X* on the interval [*A*, *B*] is

$$f(x; A, B) = \begin{cases} \dfrac{1}{B-A}, & A \le x \le B \\ 0, & \text{elsewhere} \end{cases}$$

The mean and variance of the uniform distribution are

$$\mu = \frac{A+B}{2} \text{ and } \sigma^2 = \frac{(B-A)^2}{12}.$$

The Density Function for a Random Variable on the Interval [1,3]



Example

Suppose that a large conference room at AOU can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length *X* of a conference has a uniform distribution on the interval [0, 4].

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

(c) What are the expected duration and the variance of a conference in this room?

Solution:

$$\text{(a) } f(x) = \begin{cases} \dfrac{1}{4}, & 0 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

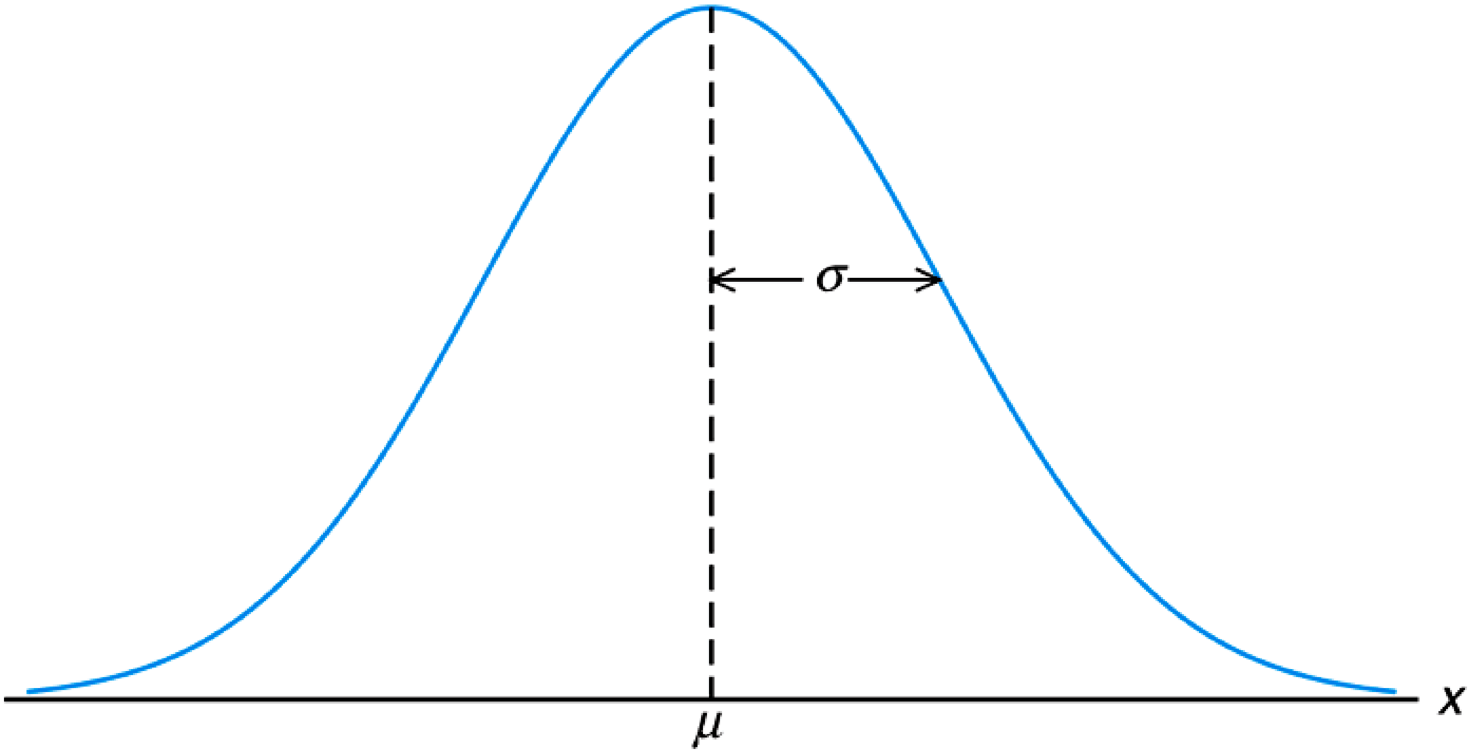
$$\text{(b) } P[X \ge 3] = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4} x \Big|_3^4 = 1 - \frac{3}{4} = 0.25$$

$$\text{(c) } \mu = \frac{0+4}{2} = 2, \ \ \sigma^2 = \frac{(4-0)^2}{12} = \frac{4}{3} = 1.33$$

Normal Distribution

- The most important continuous probability distribution in the entire field of statistics.
- Its graph, called the normal curve, is the bell-shaped curve which approximately describes many phenomena that occur in nature, industry, and research.
- For example, physical measurements in areas such as meteorological experiments, rainfall studies, and measurements of manufactured parts are often more than adequately explained with a normal distribution.
- In addition, errors in scientific measurements are extremely well approximated by a normal distribution.

The Normal Curve



Normal Distribution

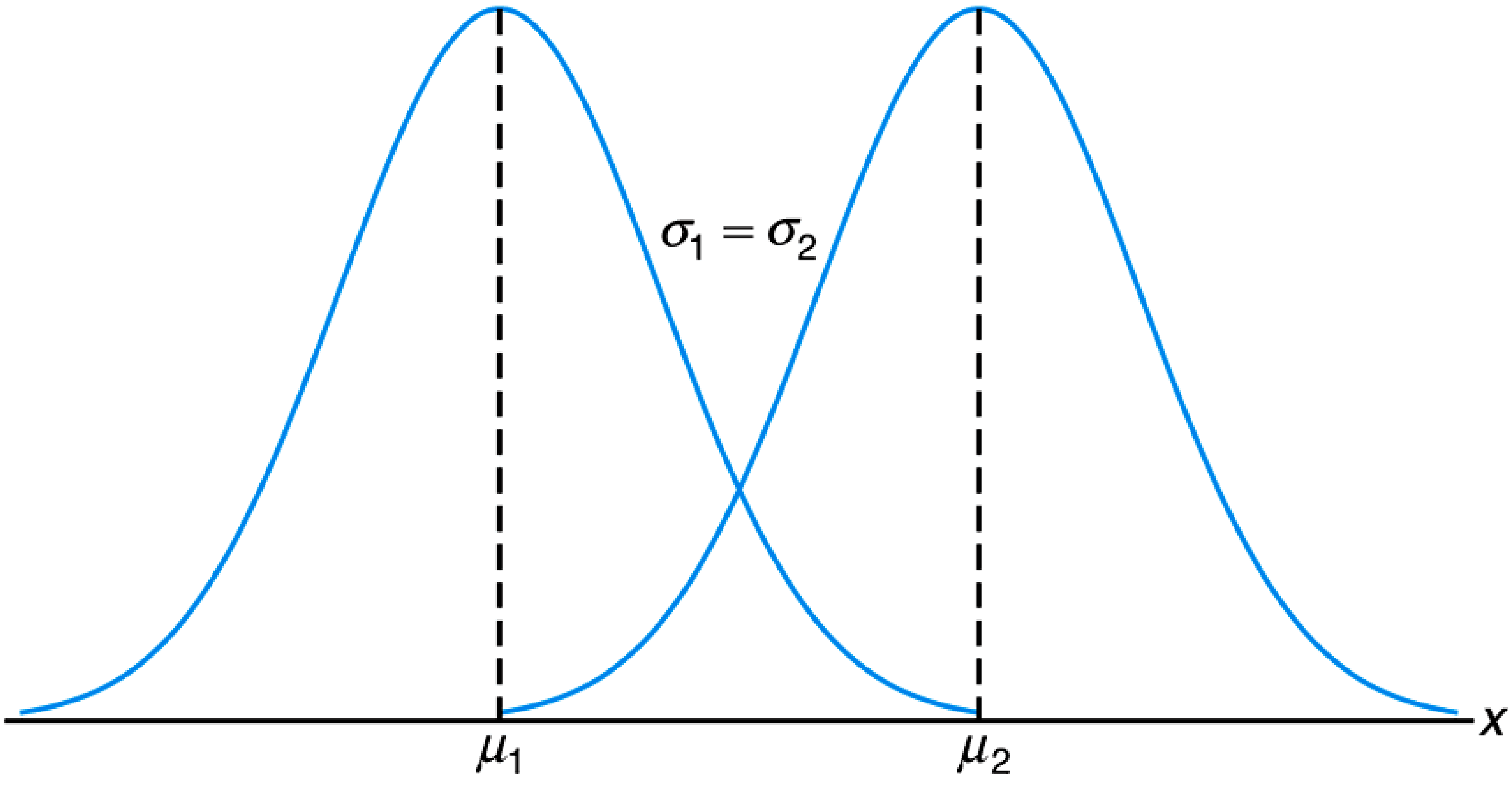
The density of the normal random variable *X*, with mean μ and variance σ^2, is

$$n(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2},-\infty<x<\infty,$$

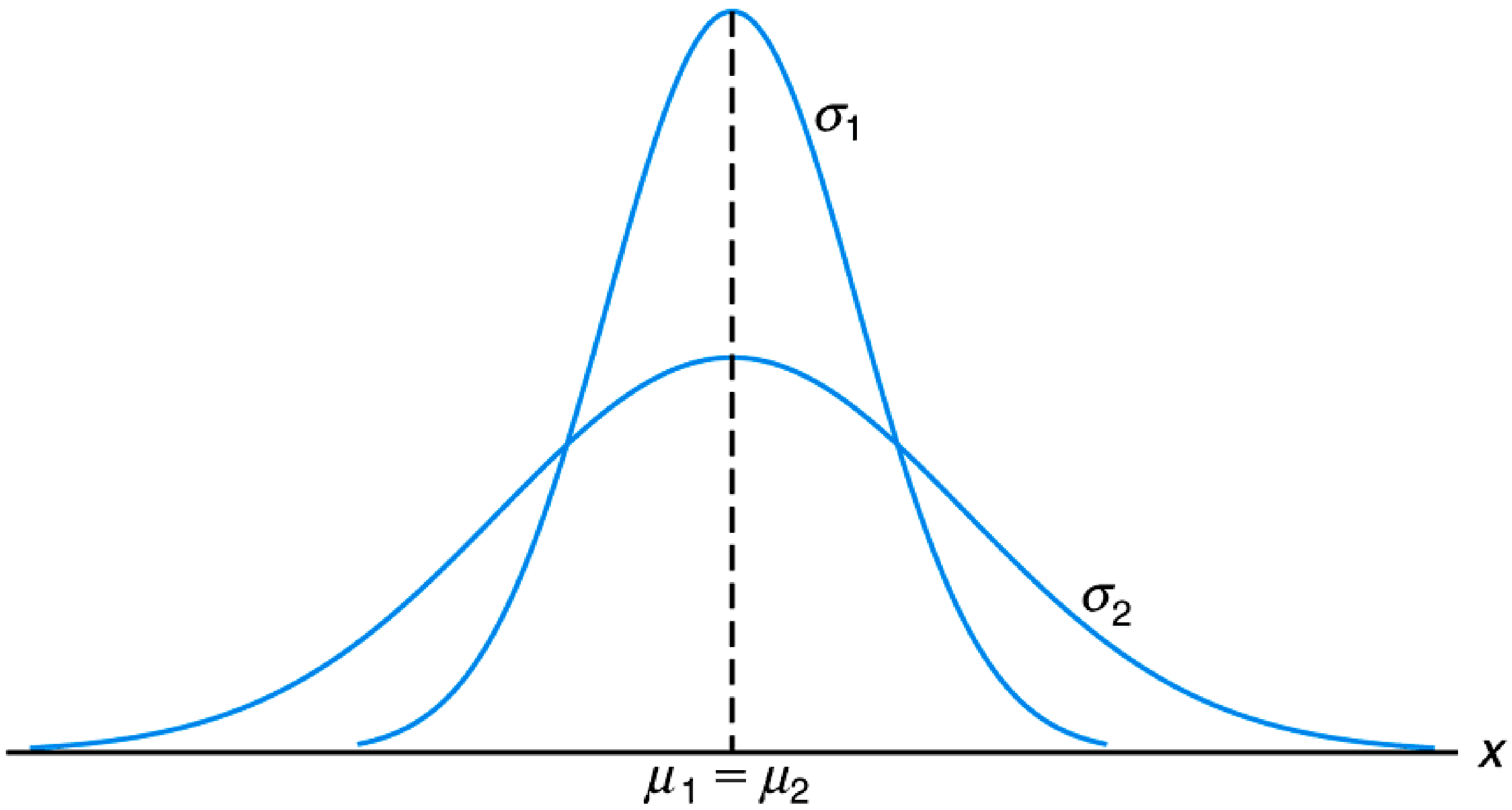
where $\pi = 3.141592654\ldots$ and $e = 2.718281828459\ldots.$

The mean and variance of $n(x;\mu,\sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

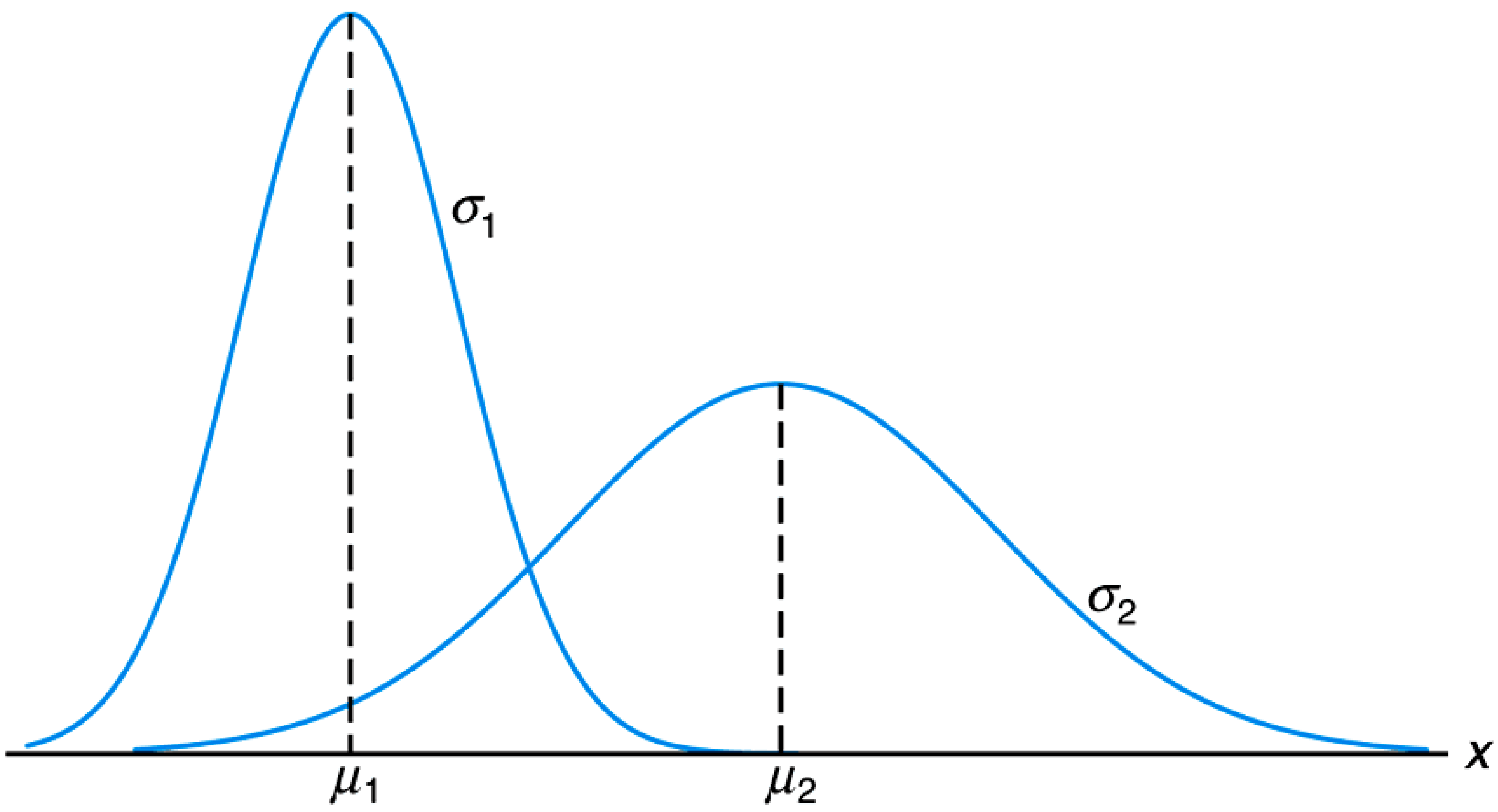
Normal Curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$



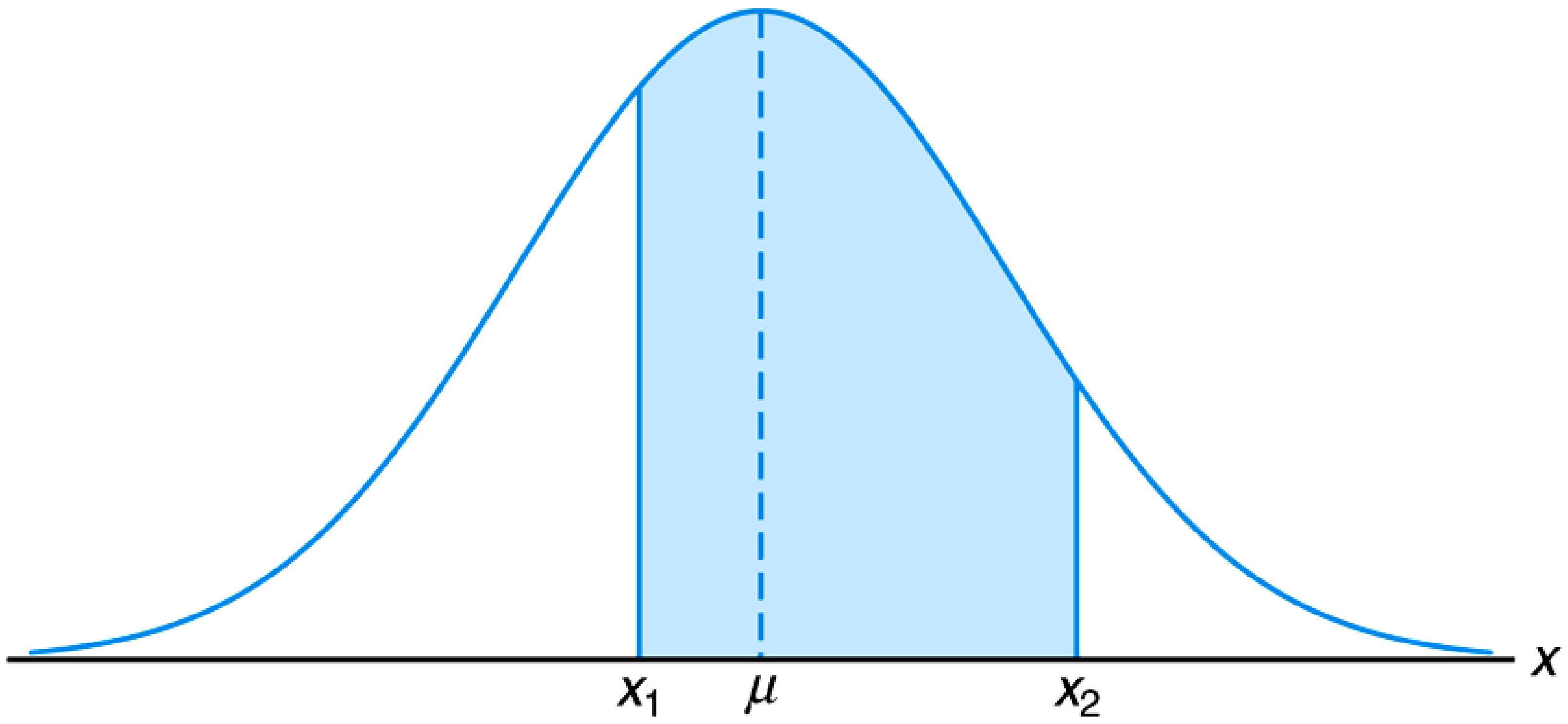
Normal Curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$



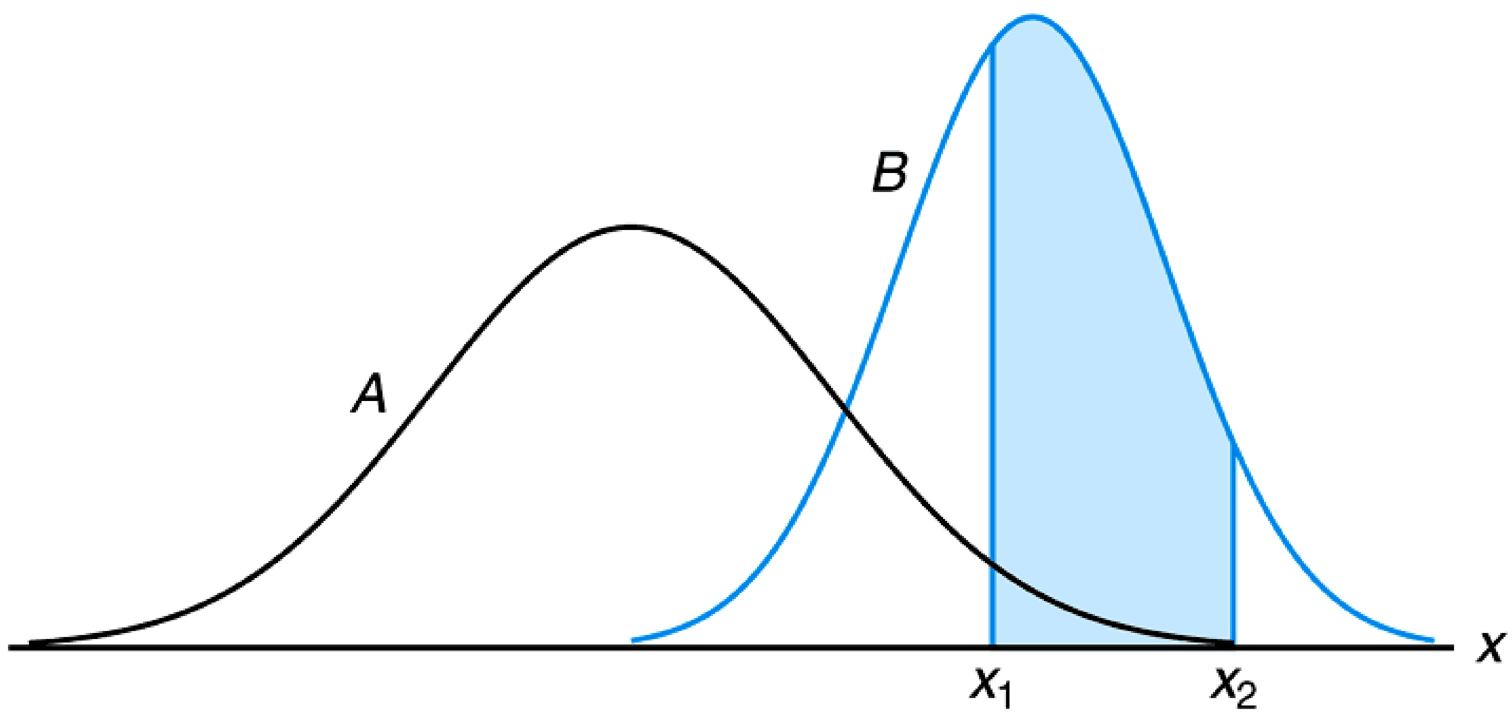
Normal Curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$



$P(x_1 < X < x_2)$ = Area of the Shaded Region



$P(x_1 < X < x_2)$ for Different Normal Curves

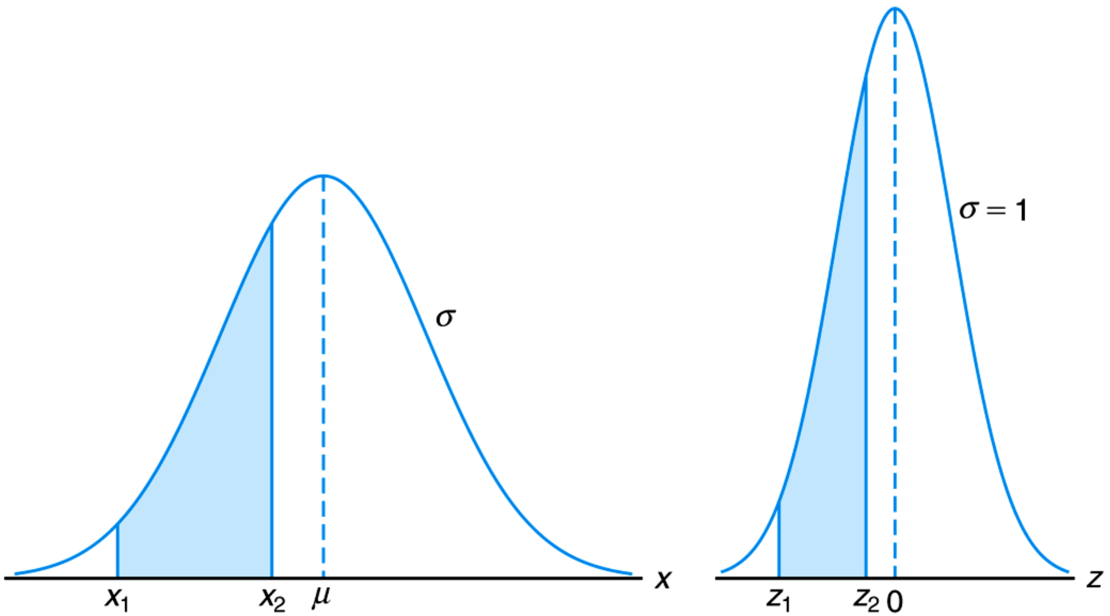


Standard Normal Distribution

The distribution of a normal random variable with mean 0 and variance 1 is called a standard normal distribution.

- We have now reduced the required number of tables of normal-curve areas to one, that of the standard normal distribution.
- The standard normal distribution table indicates the area under the standard normal curve corresponding to $P(Z < z)$ for values of z ranging from −3.49 to 3.49.
- **To illustrate the use of this table**, let us find the probability that Z is less than 1.74. First, we locate a value of z equal to 1.7 in the left column; then we move across the row to the column under 0.04, where we read 0.9591; therefore, $P(Z < 1.74) = 0.9591$.
- To find a z value corresponding to a given probability, the process is reversed; for example, the z value leaving an area of 0.2148 under the curve to the left of z is seen to be −0.79.

The Original and Transformed Normal Distributions



$$z = \frac{X - \mu}{\sigma}$$

Therefore, if X falls between the values $x = x_1$ and $x = x_2$, the random variable Z will fall between the corresponding values

$z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$.

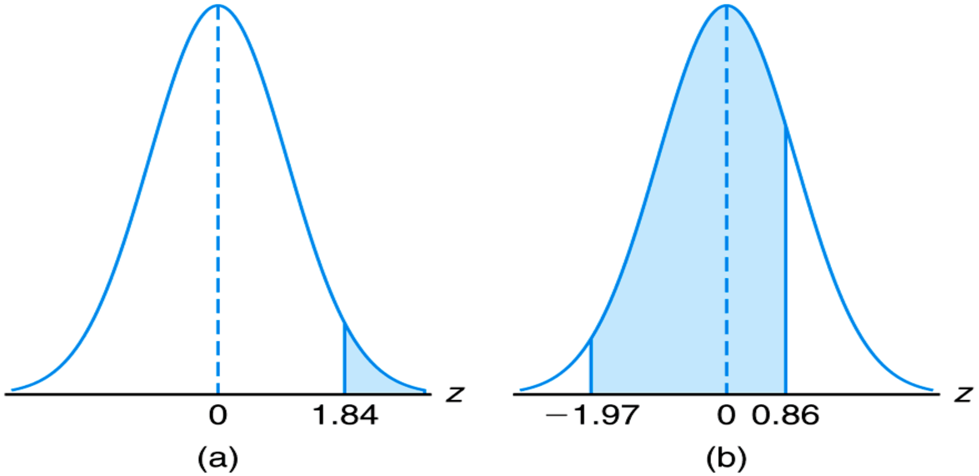
Example 1

Given a standard normal distribution, find the area under the curve that lies
(a) to the right of $z = 1.84$ and
(b) between $z = -1.97$ and $z = 0.86$.

Solution:

(a) The area in the right of $z = 1.84$ is equal to 1 minus the area to the left of $z = 1.84$, namely. From the table, the area is $1 - 0.9671 = 0.0329$.

(b) The area between $z = -1.97$ and $z = 0.86$ is equal to the area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$. From the table we find the desired area to be $0.8051 - 0.0244 = 0.7807$.



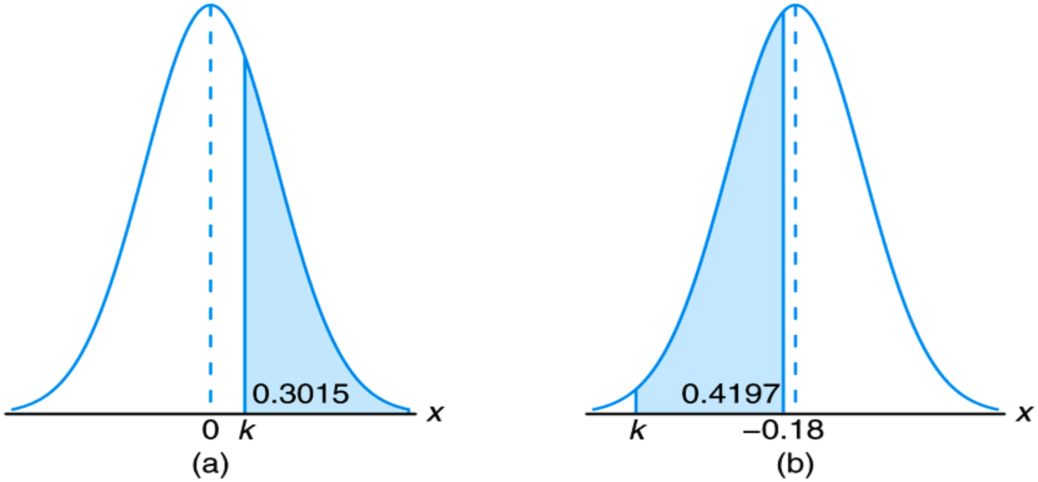
Example 2

Given a standard normal distribution, find the value of k such that
(a) $P(Z > k) = 0.3015$ and
(b) $P(k < Z < -0.18) = 0.4197$.

Solution:

(a) We see that the k value leaving an area of 0.3015 to the right must then leave an area of 0.6985 to the left. From the table it follows that $k = 0.52$.

(b) From the Table we note that the total area to the left of −0.18 is equal to 0.4286. We see that the area between k and −0.18 is 0.4197, so the area to the left of k must be $0.4286 - 0.4197 = 0.0089$. Hence, $k = -2.37$.



Example 3

Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

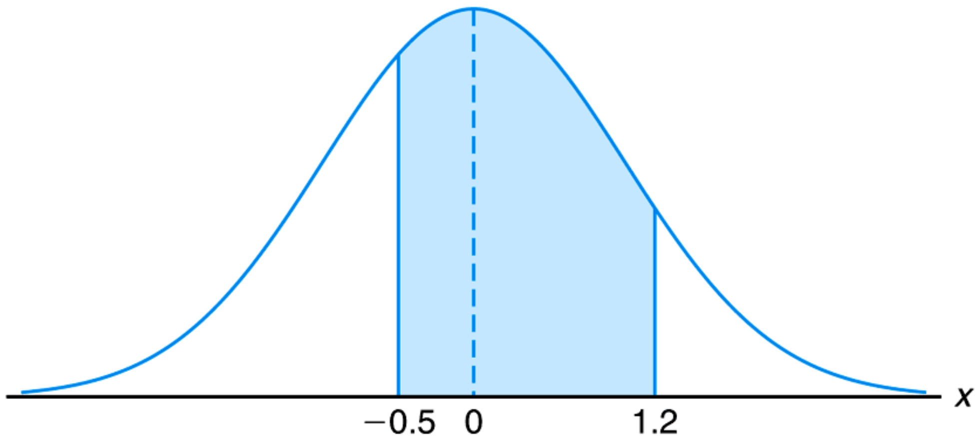
Solution:

The z values corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$z_1 = (45 - 50)/10 = -0.5 \quad \text{and } z_2 = (62 - 50)/10 = 1.2$$

Therefore,

$$P(45 < X < 62) = P(-0.5 < Z < 1.2) = P(Z < 1.2) - P(Z < -0.5)$$
$$= 0.8849 - 0.3085 = 0.5764.$$

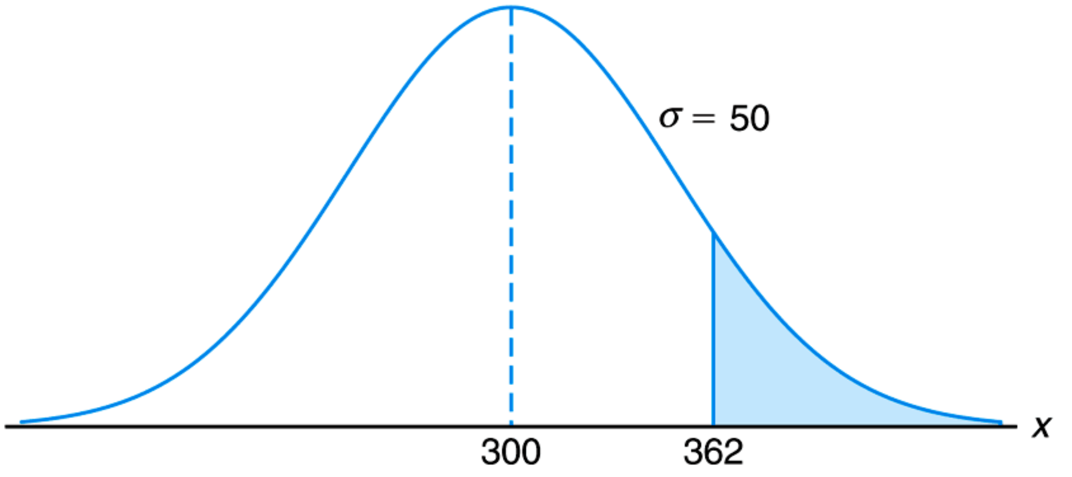


Example 4

Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

Solution:

$$P(X > 362) = P((X - 300)/50 > (362 - 300)/50) = P(Z > 1.24)$$
$$= 1 - P(Z < 1.24) = 1 - 0.8925 = 0.1075.$$



Example 5

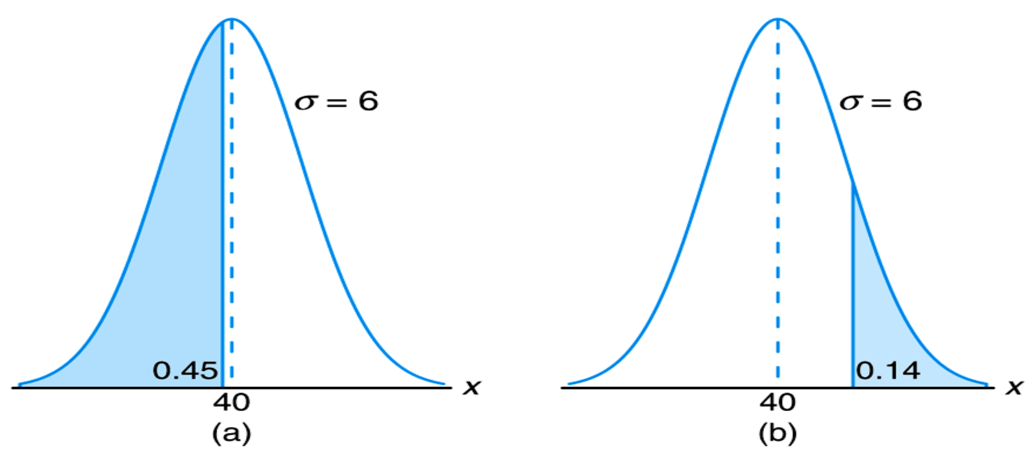
Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of x that has
(a) 45% of the area to the left and
(b) 14% of the area to the right.

Solution:

(a) We require a z value that leaves an area of 0.45 to the left. From the table we find $P(Z < -0.13) = 0.45$, so the desired z value is -0.13. Hence,

$$(x - 40)/6 = -0.13 \rightarrow x = (6)(-0.13) + 40 = 39.22.$$

(b) We require a z value that leaves 0.14 of the area to the right and hence an area of 0.86 to the left. Again, from the Table, we find $P(Z < 1.08) = 0.86$, so the desired z value is 1.08 and $x = (6)(1.08) + 40 = 46.48$.



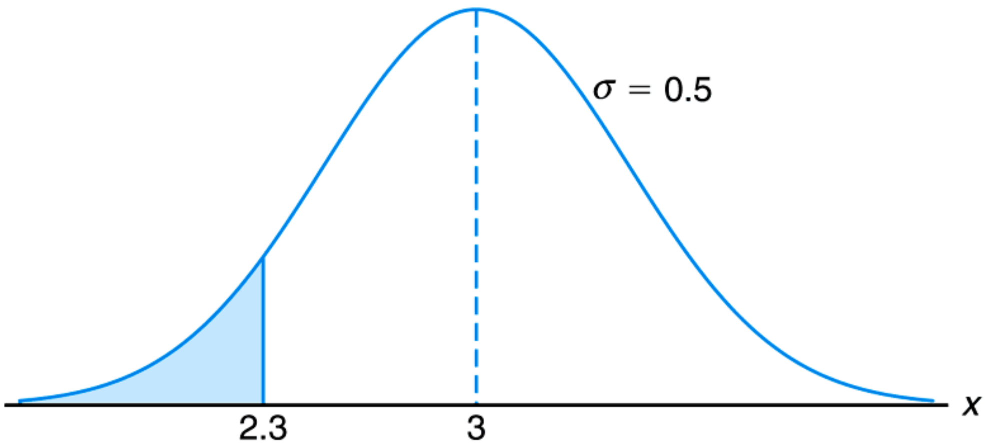
Application 1

A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

Solution :

To find $P(X < 2.3)$, we need to evaluate the area under the normal curve to the left of 2.3. This is accomplished by finding the area to the left of the corresponding z value. Hence, we find that

$$P(X < 2.3) = P((X - 3)/0.5 < (2.3 - 3)/0.5) = P(Z < -1.4) = 0.0808$$

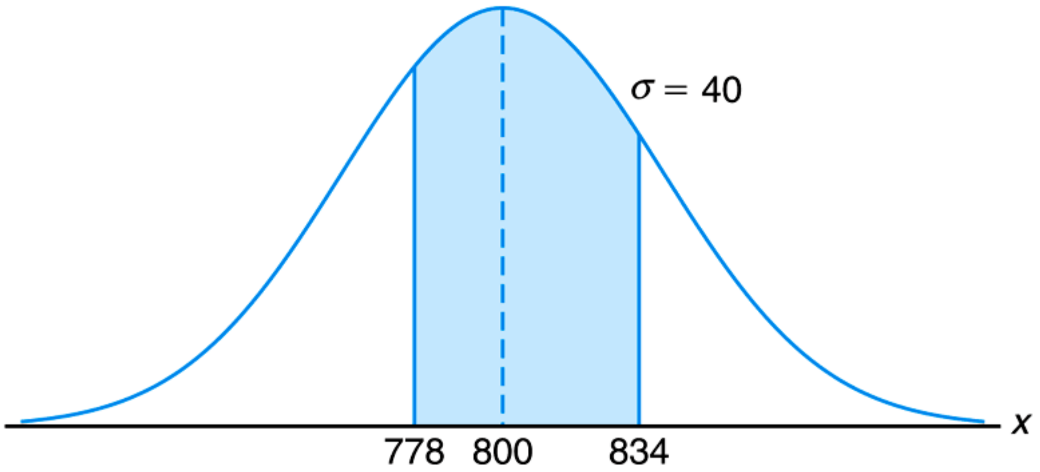


Application 2

An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

Solution :

$$P(778 < X < 834) = P(-0.55 < Z < 0.85) = P(Z < 0.85) - P(Z < -0.55)$$
$$= 0.8023 - 0.2912 = 0.5111$$



Approximating Binomial Distribution by Normal Distribution

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Normal approximation of $b(4; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$

To illustrate the normal approximation to the binomial distribution, we first draw the histogram for $b(x; 15, 0.4)$ and then superimpose the particular normal curve having the same mean and variance as the binomial variable X. Hence, we draw a normal curve with

$$\mu = np = (15)(0.4) = 6 \quad \text{and} \quad \sigma^2 = npq = (15)(0.4)(0.6) = 3.6\ ,\ \sigma = 1.897$$

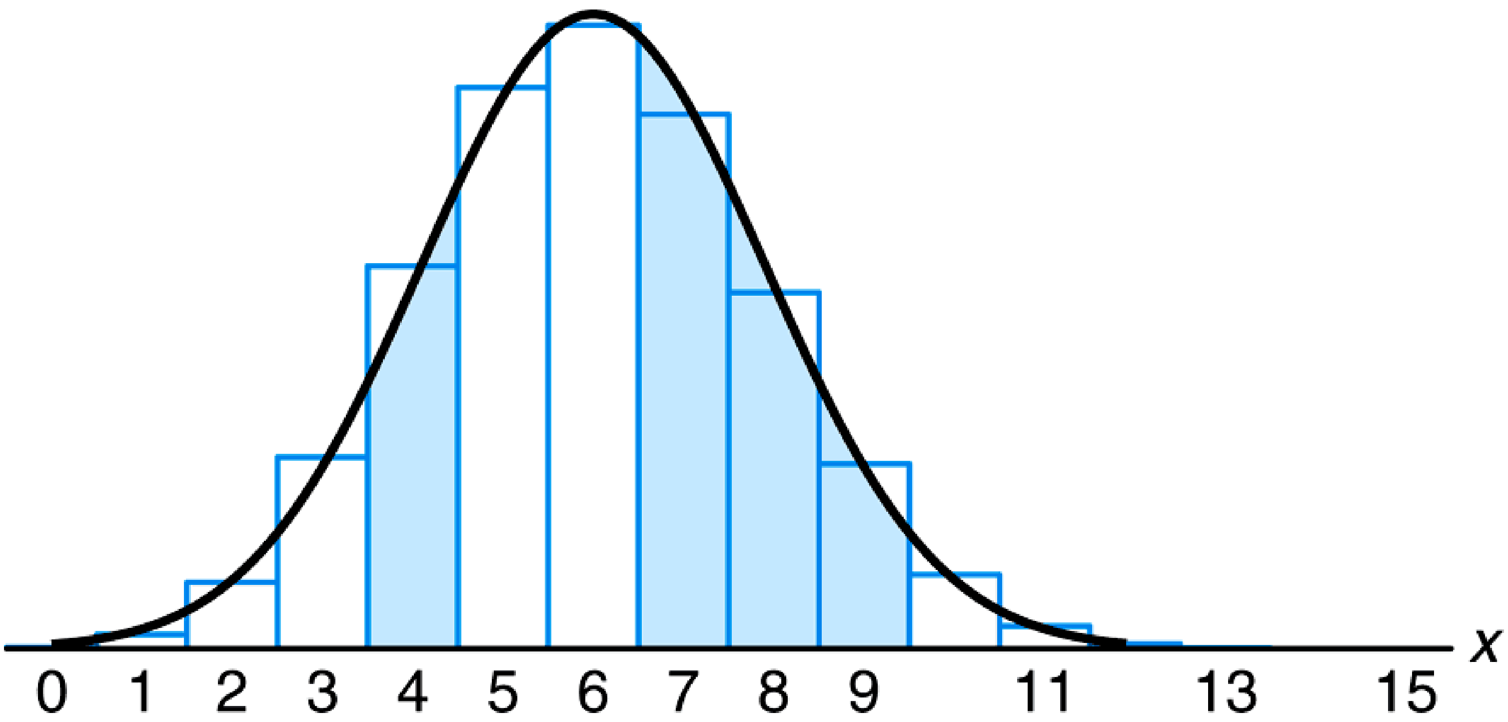
$$P(X = 4) = b(4; 15, 0.4) = 0.1268$$

$$P(X = 4) \approx P(3.5 < X < 4.5) = P(-1.32 < Z < -0.79)$$
$$= P(Z < -0.79) - P(Z < -1.32) = 0.2148 - 0.0934 = 0.1214$$

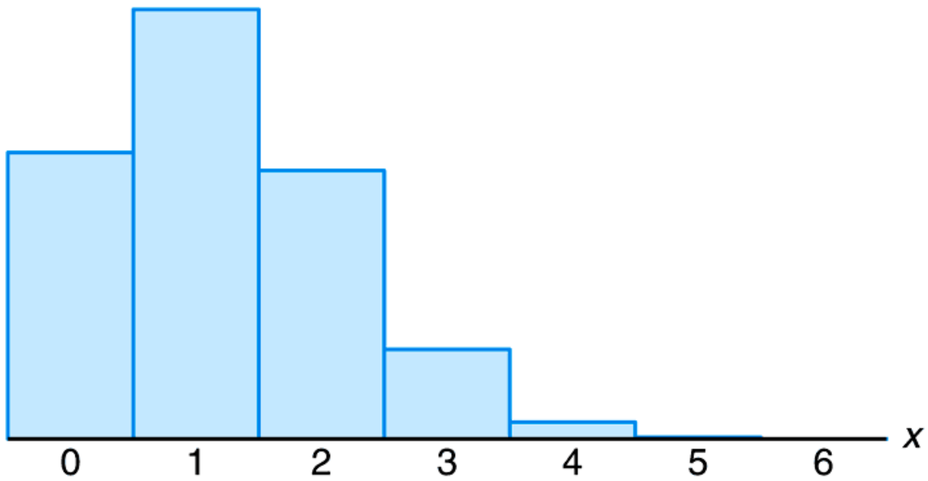
$$P(7 \le X \le 9) = \sum_{x=0}^{9} b(x;15,0.4) - \sum_{x=0}^{7} b(x;15,0.4) = 0.9662 - 0.6098 = 0.3564$$

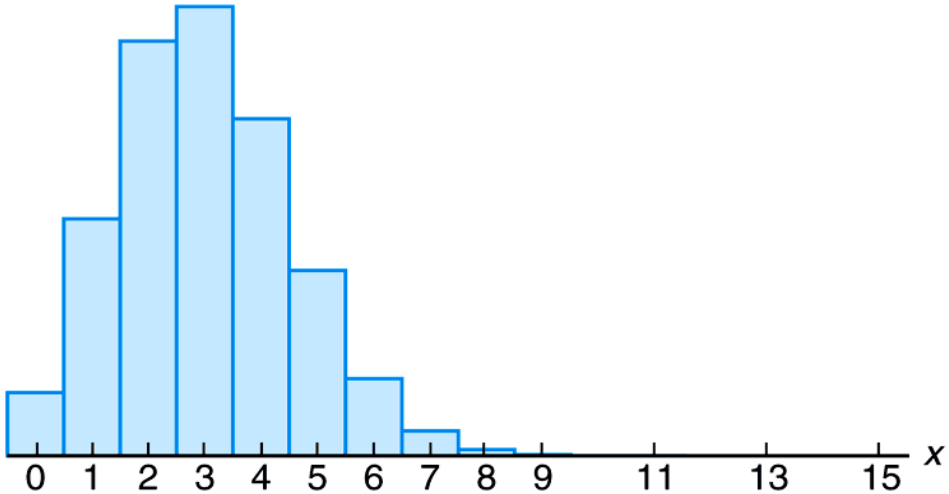
$$P(7 \le X \le 9) \approx P(0.26 < Z < 1.85) = P(Z < 1.85) - P(Z < 0.26)$$
$$= 0.9678 - 0.6026 = 0.3652$$

Normal approximation of $b(4; 15, 0.4)$ and $\sum_{x=7}^{9} b(x; 15, 0.4)$



Histograms for *b*(*x*; 6, 0.2) and *b*(*x*; 15, 0.2)





Example

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

Solution:

Let the binomial variable X represent the number of patients who survive. Since n = 100, we should obtain fairly accurate results using the normal-curve approximation with

$$\mu = np = (100)(0.4) = 40 \text{ and } \sigma^2 = npq = (100)(0.4)(0.6) = 24,\ \sigma = 4.899$$

$$P(X < 30) \approx P((X - 40)/4.899 < (30 - 40)/4.899)) = P(Z < -2.14) = 0.0162$$