# Chapter 2

Probability

# Chapter Outline

2.1 Sample Space

2.2 Events

2.3 Counting Sample Points

2.4 Probability of an Event

2.5 Additive Rules

2.6 Conditional Probability, Independence, and the Product Rule

2.7 Bayes' Rule

# Sample Space

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol $S$.

Consider the experiment of rolling one die, the sample space is

$$S_1 = \{1, 2, 3, 4, 5, 6\}$$

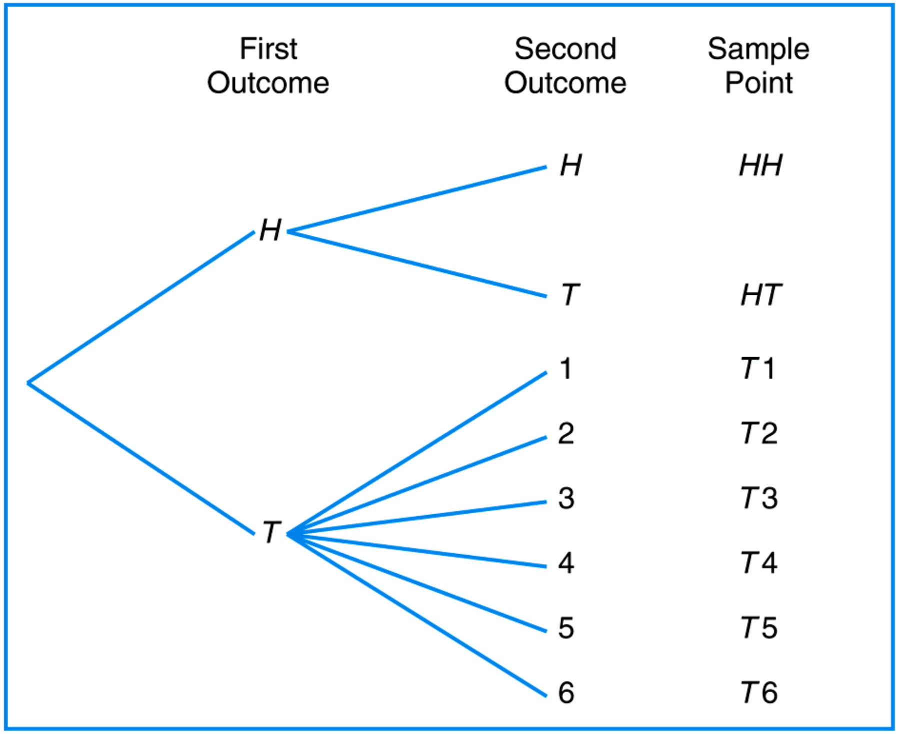
If we are interested only in whether the number is even or odd, the sample space is simply

$$S_2 = \{\text{even}, \text{odd}\}$$

# Tree Diagram

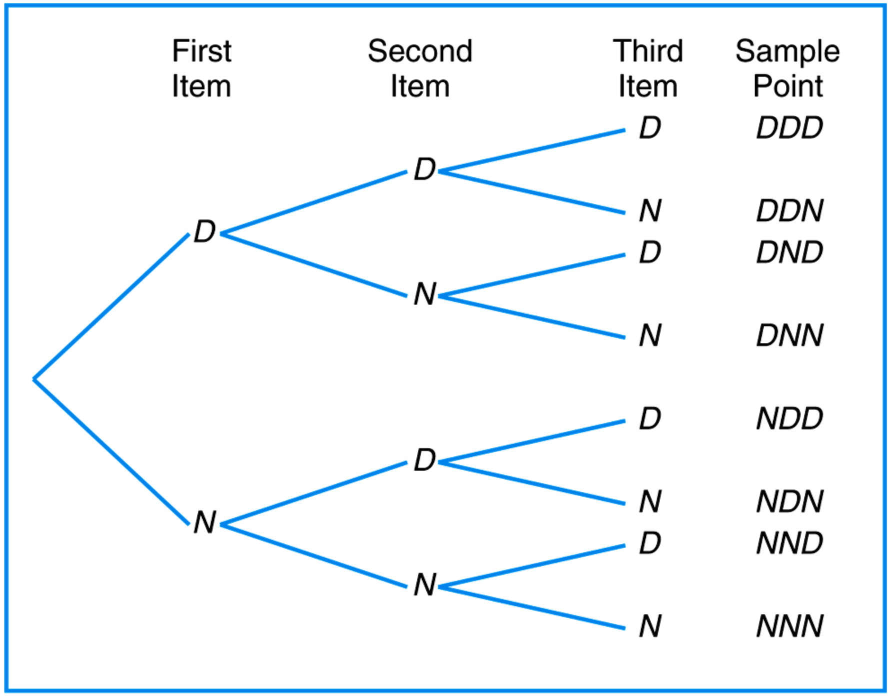
An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once, the sample space is

$$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$$



# Tree Diagram

Three items are selected at random from a manufacturing process. Each item is inspected and classified defective, *D,* or non-defective, *N.*



# Events

An **event** is a subset of a sample space.

Consider the experiment of rolling one die:

- $S = \{1, 2, 3, 4, 5, 6\}$
- $E_1 = \{1\}$; Event (simple) of occurring 1
- $E_2 = \{2, 4, 6\}$; Event (compound) of occurring even number

# Complements

The **complement** of an event $A$ with respect to $S$ is the subset of all elements of $S$ that are not in $A$. We denote the complement of $A$ by the symbol $A'$.

- Let *R* be the event that a red card is selected from an ordinary deck of 52 playing cards, and let *S* be the entire deck. Then *R′* is the event that the card selected from the deck is not a red card but a black card.

- Consider the sample space

  S = {book, cell phone, mp3, paper, stationery, laptop}

  Let

  *A* = {book, stationery, laptop, paper}

  Then the complement of *A* is

  *A′* = {cell phone, mp3}

# Intersection of Events

The **intersection** of two events $A$ and $B$, denoted by the symbol $A \cap B$, is the event containing all elements that are common to $A$ and $B$.

Let *E* be the event that a person selected at random in a classroom is majoring in ITC, and let *F* be the event that the person is female. Then $E \cap F$ is the event of selecting a female ITC student in the classroom.

# Disjoint Events

Two events $A$ and $B$ are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if $A$ and $B$ have no elements in common.

Let $V = \{a, e, i, o, u\}$ and $C = \{l, r, s, t\}$; then it follows that $V \cap C = \Phi$. That is, $V$ and $C$ have no elements in common and, therefore, cannot both simultaneously occur.

# Union of Events

The **union** of the two events $A$ and $B$, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to $A$ or $B$ or both.

In the die-tossing experiment, if

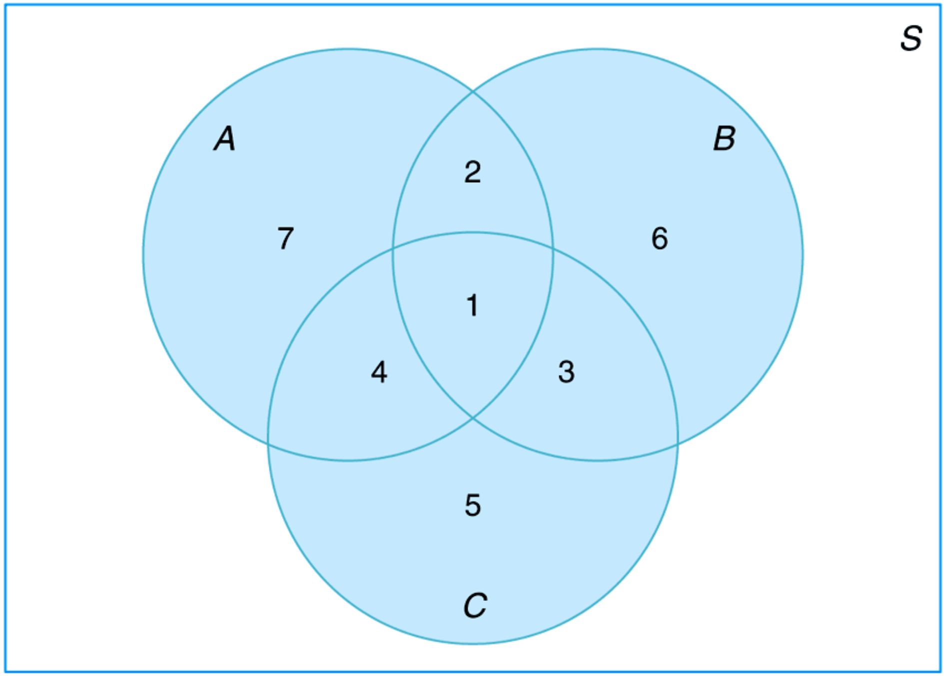
$$A = \{2, 4, 6\} \text{ and } B = \{4, 5, 6\},$$

we might be interested in either $A$ or $B$ occurring or both $A$ and $B$ occurring. Such an event, called the union of $A$ and $B$, will occur if the outcome is an element of the subset

$$A \cup B = \{2, 4, 5, 6\}$$

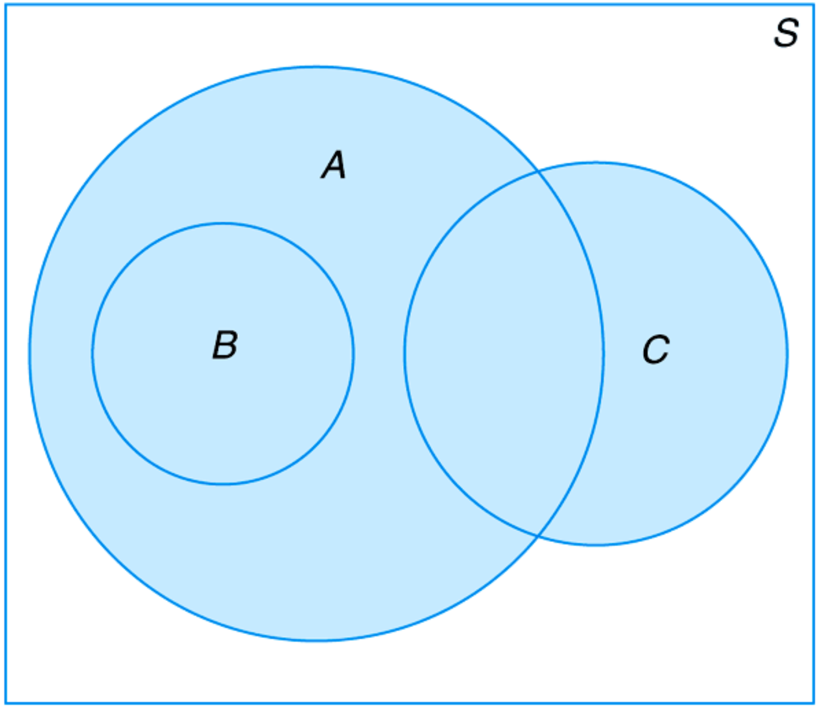
# Events Represented by Various Regions

- $A \cup C$ = regions 1, 2, 3, 4, 5, and 7
- $B \cap A$ = regions 4 and 7
- $A \cap B \cap C$ = region 1
- $(A \cup B) \cap C$ = regions 2, 6, and 7



# Events and Sample Space

- The events *A*, *B*, and *C* are all subsets of the sample space *S*
- The event *B* is a subset of event *A*
- Event $B \cap C$ has no elements and hence B and C are mutually exclusive
- Event $A \cap C$ has at least one element; and event $A \cup B = A$



# Multiplication Rule

If an operation can be performed in $n_1$ ways, and if for each of these ways a second operation can be performed in $n_2$ ways, then the two operations can be performed together in $n_1 n_2$ ways.

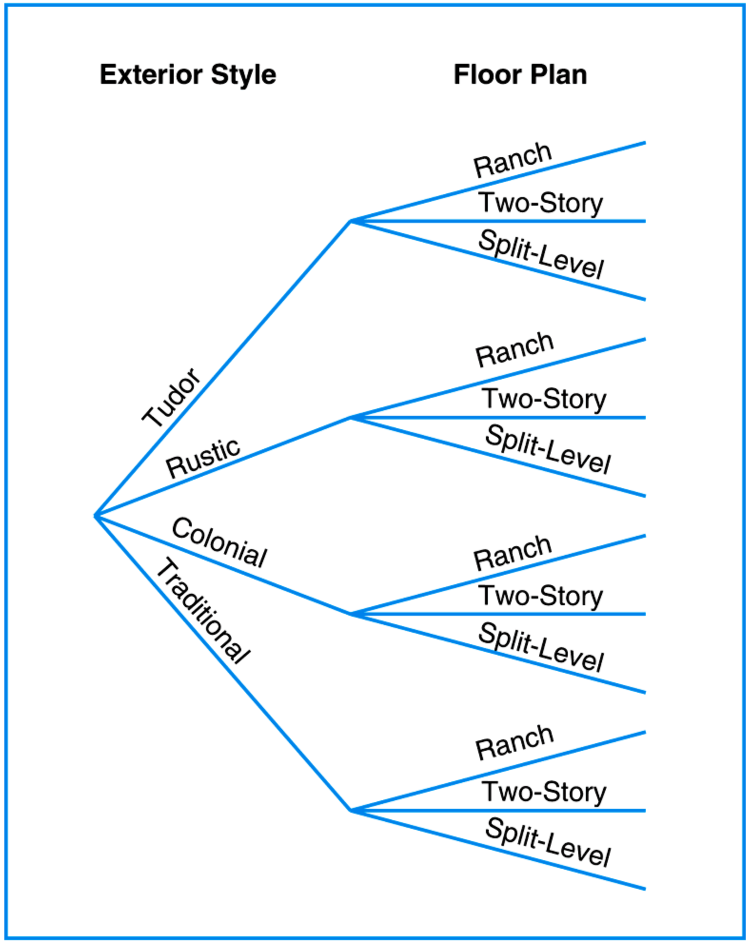
How many sample points are there in the sample space when a pair of dice is thrown once?

Solution:

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = 6 \times 6 = 36$ possible ways.

# Tree Diagram

A developer of a new subdivision offers prospective home buyers a choice of Tudor, rustic, colonial, and traditional exterior styling in ranch, two-story, and split-level floor plans. In how many different ways can a buyer order one of these homes?



# Multiplication Rule

If an operation can be performed in $n_1$ ways, and if for each of these a second operation can be performed in $n_2$ ways, and for each of the first two a third operation can be performed in $n_3$ ways, and so forth, then the sequence of $k$ operations can be performed in $n_1 n_2 \cdots n_k$ ways.

Sam is going to assemble a computer by himself. He has the choice of chips from 2 brands, a hard drive from 4, memory from 3, and an accessory bundle from 5 local stores. How many different ways can Sam order the parts?

Solution:

Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_1 \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

# Example

Suppose a computer-assisted test is to consist of 5 questions. A computer stores 5 comparable questions for the first test question, 8 for the second, 6 for the third, 5 for the forth, and 10 for the fifth. How many different 5-question tests can the computer select? (Two tests are considered different if they differ in one or more questions)

Solution:

Since $n_1 = 5$, $n_2 = 8$, $n_3 = 6$, $n_4 = 5$, and $n_5 = 10$, there are

$$n_1 \times n_2 \times n_3 \times n_4 \times n_5 = 5 \times 8 \times 6 \times 5 \times 10 = 12{,}000$$

different tests.

# Permutations

A **permutation** is an arrangement of all or part of a set of objects.

Consider the three letters *a, b,* and *c.* The possible permutations are

*abc, acb, bac, bca, cab, cba.*

Thus, we see that there are 6 distinct arrangements. Using the product rule, we could arrive at the answer 6 without actually listing the different orders by the following arguments:

There are $n_1 = 3$ choices for the first position. No matter which letter is chosen, there are always $n_2 = 2$ choices for the second position. No matter which two letters are chosen for the first two positions, there is only $n_3 = 1$ choice for the last position, giving a total of

$$n_1 n_2 n_3 = 3 \times 2 \times 1 = 6 \text{ permutations.}$$

In general, $n$ distinct objects can be arranged in

$$n(n-1)(n-2) \cdot \cdot \cdot (3)(2)(1) \text{ ways.}$$

# Factorials

For any non-negative integer $n$, $n!$, called "$n$ factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

The number of permutations of $n$ objects is $n!$.

e.g. $4! = 4 \times 3 \times 2 \times 1 = 24$.

e.g. The number of permutations of the four letters *a, b, c,* and *d* will *be* $4! = 24$.

# Permutations

The number of permutations of $n$ distinct objects taken $r$ at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

The number of permutations that are possible by taking two letters at a time from the four letters *a*, *b*, *c* and *d*. These would be

*ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc.*

Using the product rule, we have two positions to fill, with $n_1 = 4$ choices for the first and then $n_2 = 3$ choices for the second, for a total of

$$n_1 n_2 = 4 \times 3 = 12 \text{ permutations.}$$

In general, $n$ distinct objects taken $r$ at a time can be arranged in

$$n(n-1)(n-2) \cdot \cdot \cdot (n-r+1) \text{ ways.}$$

We represent this product by the symbol

$$_nP_r = \frac{n!}{(n-r)!}$$

# Example 1

In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a ITC department. If each student can receive at most one award, how many possible selections are there?

Solution:

Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$$_{25}P_3 = \frac{25!}{(25-22)!} = \frac{25\cdot 24\cdot 23\cdot 22!}{22!} = 25\cdot 24\cdot 23 = 13{,}800$$

# Example 2

Serial numbers for a product are to be made using 2 letters followed by 3 numbers. If the letters are to be taken from the first 8 letters of the alphabet with no repeats and the numbers are to be taken from the 10 digits (0 – 9) with no repeats, how many serial numbers are possible?

Solution:

The number of ways of selecting 2 letters out of 8 is

$$ {}_8P_2 = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56 $$

The number of ways of selecting 3 digits out of 10 is

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720 $$

Using the multiplication rule with $n_1 = 56$ and $n_2 = 720$, we have

56 × 720 = 40,320 serial numbers.

# Example 3

How many 3-letter code words are possible using the first 8 letters of the alphabet if

(a) No letter can be repeated?

(b) Letters can be repeated?

(c) Adjacent letters cannot be alike?

Solution:

(a) There are $8 \times 7 \times 6 = 336$ possible code words.

(b) There are $8 \times 8 \times 8 = 512$ possible code words.

(c) There are $8 \times 7 \times 7 = 392$ possible code words.

# Example 4

A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) There are no restrictions?

(b) *A* will serve only if he is president?

(c) *B* and *C* will serve together or not at all?

Solution:

(a) $${}_{50}P_2 = \frac{50!}{(50-2)!} = \frac{50!}{48!} = \frac{50 \cdot 49 \cdot 48!}{48!} = 50 \cdot 49 = 2450$$

(b) Since *A* will serve only if he is president, we have two situations here: *A* is selected as the president, which yields 49 possible outcomes for the treasurer's position, or officers are selected from the remaining 49 people without *A*, which has the number of choices ${}_{49}P_2 = 49 \times 48 = 2352$. Therefore, the total number of choices is 49 + 2352 = 2401.

(c) The number of selections when *B* and *C* serve together is 2. The number of selections when both *B* and *C* are not chosen is ${}_{48}P_2 = 2256$. Therefore, the total number of choices in this situation is 2 + 2256 = 2258.

# Theorem

The number of distinct permutations of $n$ things of which $n_1$ are of one kind, $n_2$ of a second kind, $\ldots$, $n_k$ of a $k$th kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Solution:

Directly using the Theorem, we find that the total number of arrangements is

$$\frac{10!}{(1!)(2!)(4!)(3!)} = 12{,}600$$

# Theorem

The number of ways of partitioning a set of $n$ objects into $r$ cells with $n_1$ elements in the first cell, $n_2$ elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

Solution:

The total number of possible partitions would be

$$\binom{7}{3\ \ 2\ \ 2} = \frac{7!}{(3!)(2!)(2!)} = 210$$

# Combinations

The number of combinations of $n$ distinct objects taken $r$ at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Suppose that an art museum owns 8 paintings by a given artist and another art museum wishes to borrow 3 of these paintings for a special show. In how many ways the 3 painting can be selected for shipment?

Solution:

$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3!\,5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 1\cdot 5!} = 56$$

# Example 1

(a) In how many ways can we choose a chairperson, a vice-chairperson, and a secretary from 10 persons, assuming that one person cannot hold more than one position?

(b) In how many ways can we choose a subcommittee of 3 people?

Solution:

(a) $$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10\cdot 9\cdot 8 = 720$$

(b) $$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!\,7!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 120$$

# Example 2

A young boy asks his mother to get 5 Game-Boy cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother can get 3 arcade and 2 sports games?

Solution:
The number of ways of selecting 3 cartridges from 10 is

$$\binom{10}{3} = \frac{10!}{3! \cdot 7!} = 120$$

The number of ways of selecting 2 cartridges from 5 is

$$\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$$

Using the multiplication rule with $n_1$ = 120 and $n_2$ = 10, we have
120 × 10 = 1200 ways.

# Probability

The **probability** of an event $A$ is the sum of the weights of all sample points in $A$. Therefore,

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

# Example 1

A coin is tossed twice. What is the probability that at least 1 head occurs?

Solution:

The sample space for this experiment is

$$S = \{HH, HT, TH, TT\}$$

If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of $\omega$ to each sample point. Then $4\omega = 1$, or $\omega = 1/4$. If A represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \text{ and}$$

$$P(A) = (1/4) + (1/4) + (1/4) = 3/4$$

# Example 2

A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If $E$ is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

Solution:

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We assign a probability of $\omega$ to each odd number and a probability of $2\omega$ to each even number. Since the sum of the probabilities must be 1, we have $9\omega = 1$ or $\omega = 1/9$. Hence, probabilities of 1/9 and 2/9 are assigned to each odd and even number, respectively. Therefore,

$$E = \{1, 2, 3\} \text{ and}$$

$$P(E) = (1/9) + (2/9) + (1/9) = 4/9$$

# Example 3

A die is loaded in such a way that an even number is twice as likely to occur as an odd number. let $A$ be the event that an even number turns up and let $B$ be the event that a number divisible by 3 occurs. Find

$$P(A \cup B) \text{ and } P(A \cap B).$$

Solution:

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We assign a probability of $\omega$ to each odd number and a probability of $2\omega$ to each even number. Since the sum of the probabilities must be 1, we have $9\omega = 1$ or $\omega = 1/9$. Hence, probabilities of 1/9 and 2/9 are assigned to each odd and even number, respectively. For the events

$$A = \{2, 4, 6\} \text{ and } B = \{3, 6\}, \text{ we have}$$

$$A \cup B = \{2, 3, 4, 6\} \text{ and } A \cap B = \{6\}$$

$$P(A \cup B) = (2/9) + (1/9) + (2/9) + (2/9) = 7/9 \text{ and}$$

$$P(A \cap B) = 2/9$$

# Probability an Event

If an experiment can result in any one of $N$ different equally likely outcomes, and if exactly $n$ of these outcomes correspond to event $A$, then the probability of event $A$ is

$$P(A) = \frac{n}{N}.$$

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is

(a) An industrial engineering major.

(b) A civil engineering or an electrical engineering major.

Solution:

(a) $P(I) = 25 / 53$

(b) $P(C \cup E) = (8 + 10) / 53 = 18 / 53$

# Additive Rules of Probability

If $A$ and $B$ are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If $A$ and $B$ are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

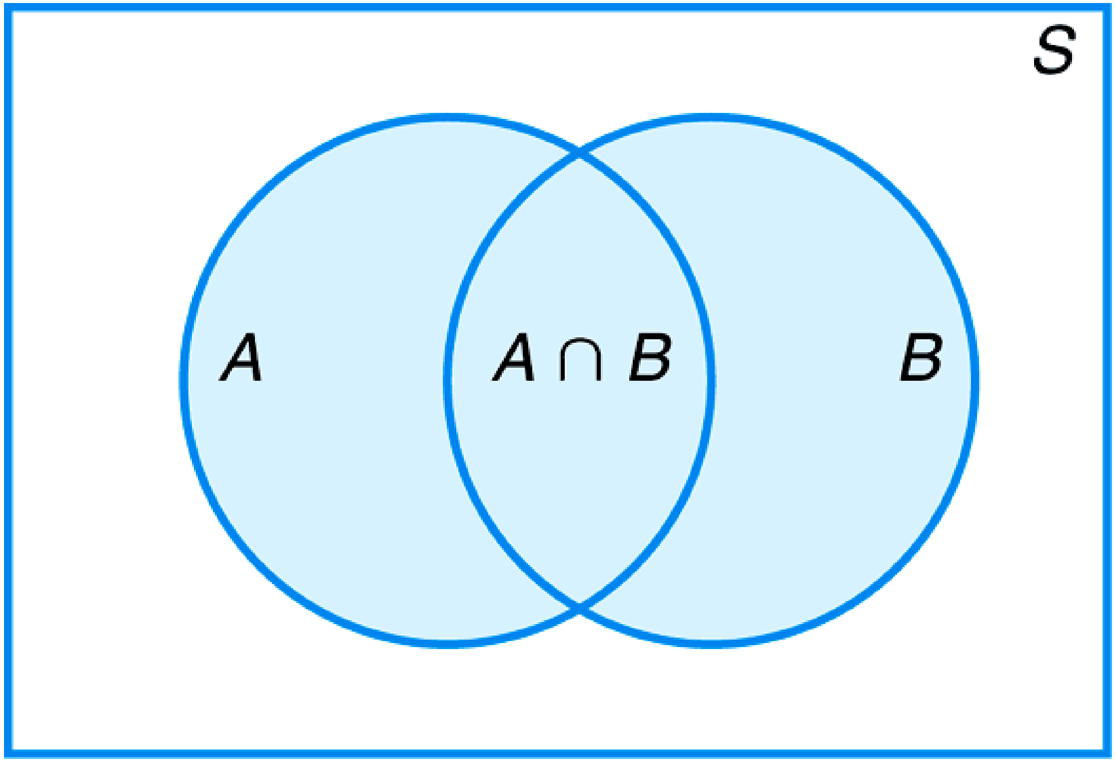
If $A_1,\ A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

If $A_1,\ A_2, \ldots, A_n$ is a partition of sample space $S$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

# Additive Rules of Probability



# Additive Rules of Probability

For three events $A$, $B$, and $C$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

If $A$ and $A'$ are complementary events, then

$$P(A) + P(A') = 1.$$

$$P(A \cap B') = P(A) - P(A \cap B)$$

# Example 1

What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

Solution:

Let $A$ be the event that 7 occurs and $B$ the event that 11 comes up. Now, a total of 7 occurs for 6 of the 36 sample points, and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$. The events $A$ and $B$ are mutually exclusive, since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = (1/6) + (1/18) = 2/9$$

This result could also have been obtained by counting the total number of points for the event $A \cup B$, namely 8, and writing

$$P(A \cup B) = 8/36 = 2/9$$

# Example 2

John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company *A* is 0.8, and his probability of getting an offer from company *B* is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that

a) he will get at least one offer from these two companies;

b) he will get a offer from only one company.

Solution:

Using the additive rule, we have

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.6 - 0.5 = 0.9$

b) $P(A \cap B') + P(B \cap A') = (0.8 - 0.5) + (0.6 - 0.5) = 0.4$

# Example 3

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that it will service at least 5 cars on his next day at work?

Solution:

Let $E$ be the event that at least 5 cars are serviced. Now, $P(E) = 1 - P(E')$, where $E'$ is the event that fewer than 5 cars are serviced. Since

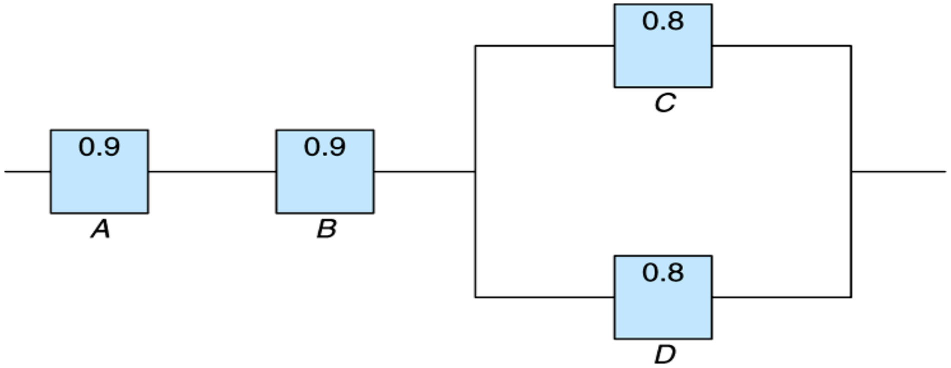
$$P(E') = 0.12 + 0.19 = 0.31,$$

it follows that

$$P(E) = 1 - 0.31 = 0.69$$

# Example 4

An electrical system consists of four components as illustrated in Figure. The system works if components *A* and *B* work and either of the components *C* or *D* works. The reliability (probability of working) of each component is also shown in Figure. Find the probability that the entire system works.




Solution:

$$P[A \cap B \cap (C \cup D)] = P(A)P(B)P(C \cup D) = P(A)P(B)[1 - P(C' \cap D')]$$
$$= P(A)P(B)[1 - P(C')P(D')]$$
$$= (0.9)(0.9)[1 - (1 - 0.8)(1 - 0.8)] = 0.7776$$

# Conditional Probability

The conditional probability of $B$, given $A$, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

If in an experiment the events $A$ and $B$ can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

# Example 1: Categorization of the Adults in a Small Town

| | Employed | Unemployed | Total |
|---|---|---|---|
| Male | 460 | 40 | 500 |
| Female | 140 | 260 | 400 |
| Total | 600 | 300 | 900 |

*M*: a man is chosen,
*E*: the one chosen is employed.
Using the reduced sample space *E,* we find that

$$P(M|E) = 460 / 600 = 23 / 30$$

Or,

$$P(E) = 600 / 900 = 2 / 3$$
$$P(E \cap M) = 460 / 900 = 23 / 45$$
$$P(M|E) = P(E \cap M) / P(E) = (23 / 45) / ( 2 / 3) = 23 / 30$$

# Example 2

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane
(a) Arrives on time, given that it departed on time.
(b) Departed on time, given that it has arrived on time.

Solution:

(a) $$P(A \mid D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83} = 0.94$$

(b) $$P(D \mid A) = \frac{P(A \cap D)}{P(A)} = \frac{0.78}{0.82} = 0.95$$

# Example 3

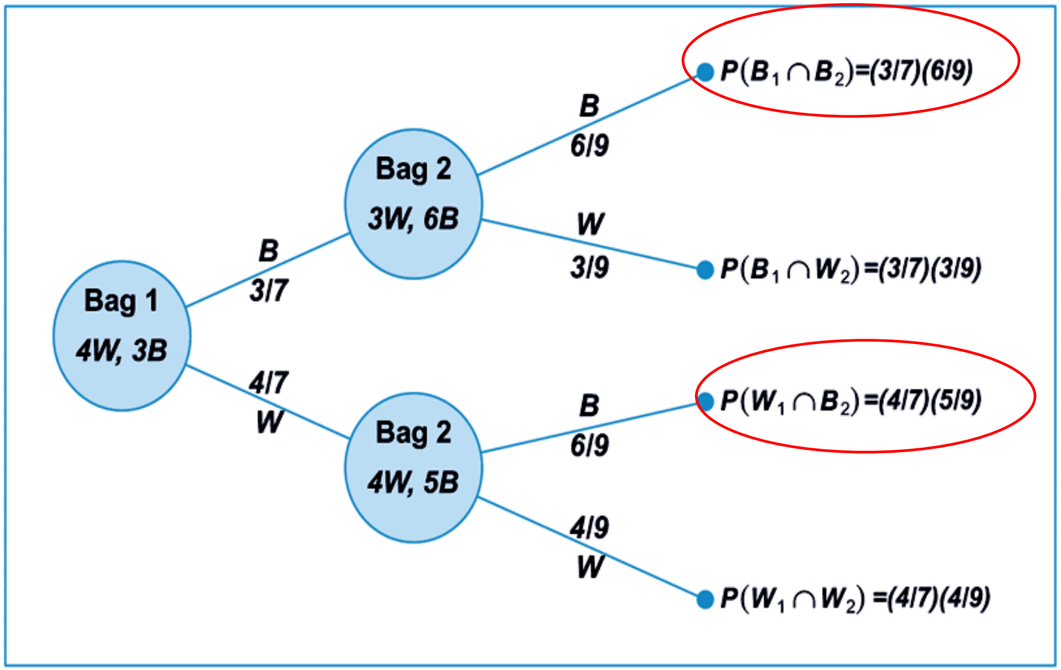
One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag.
(a) What is the probability that a ball drawn from the second bag is black?
(b) What is the probability that the first ball is black given a ball drawn from the second bag is black?

Solution:

$$\text{(a) } P(B_2) = \frac{3}{7}\cdot\frac{6}{9} + \frac{4}{7}\cdot\frac{5}{9} = \frac{38}{63}$$

$$\text{(b) } P(B_1 \mid B_2) = \frac{P(B_2 \cap B_1)}{P(B_2)}$$

$$= \frac{\frac{3}{7}\cdot\frac{6}{9}}{\frac{38}{63}} = \frac{18}{38}$$



# Independent Events

Two events $A$ and $B$ are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, $A$ and $B$ are **dependent**.

Two events $A$ and $B$ are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

# Example

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Solution:

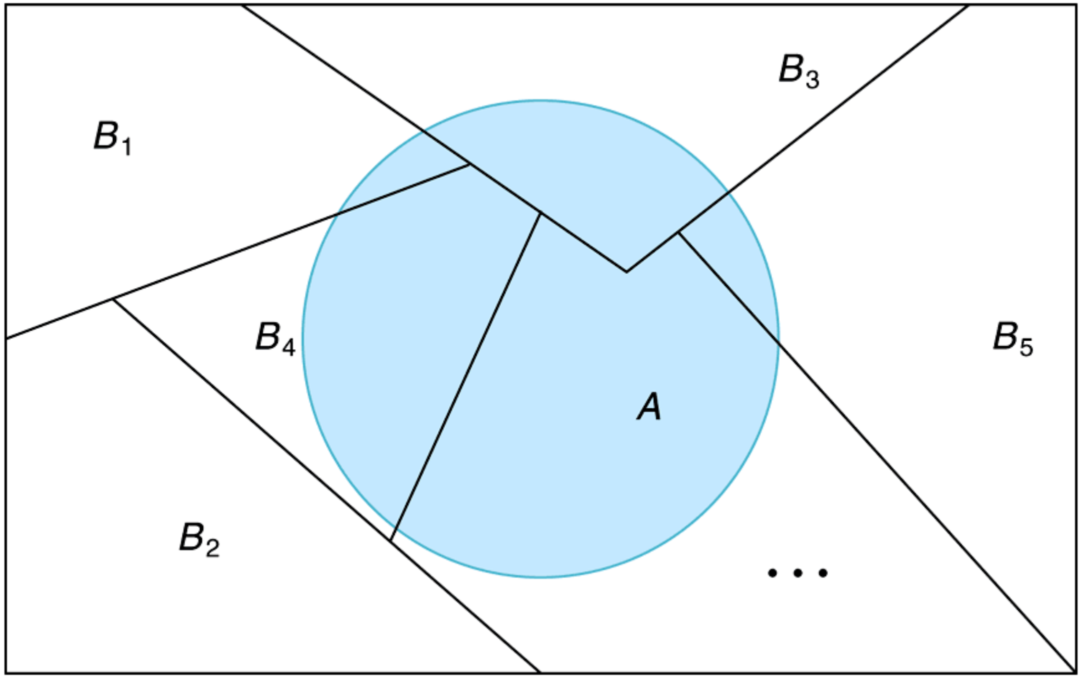
Let $A$ and $B$ represent the respective events that the fire engine and the ambulance are available. Then

$$P(A \cap B) = P(A)P(B) = (0.98)(0.92) = 0.9016.$$

# Theorem

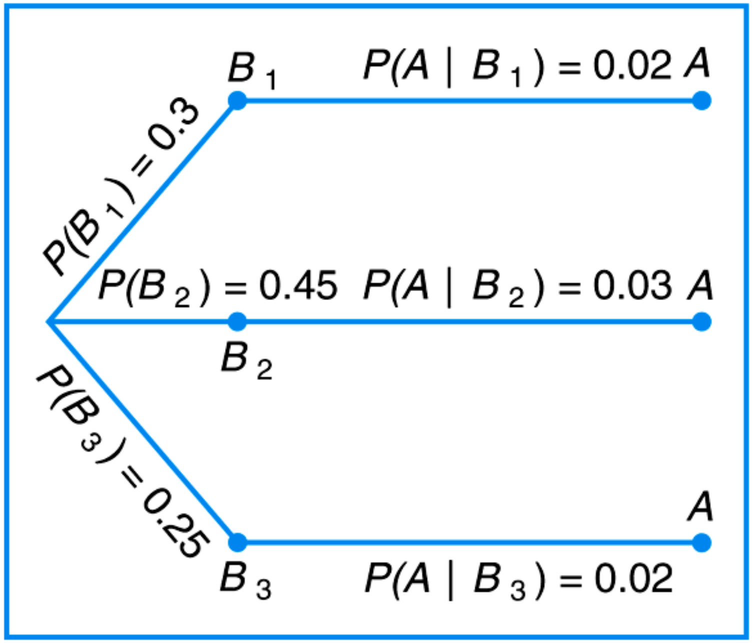
If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space $S$ such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event $A$ of $S$,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$



# Example

In a certain assembly plant, three machines, $B_1$, $B_2$, and $B_3$, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?




Solution:

Consider the following events:

$A$: the product is defective

$B_1$: the product is made by machine $B_1$

$B_2$: the product is made by machine $B_2$

$B_3$: the product is made by machine $B_3$

$P(B_1)P(A|B_1) = (0.3)(0.02) = 0.006$

$P(B_2)P(A|B_2) = (0.45)(0.03) = 0.0135$

$P(B_3)P(A|B_3) = (0.25)(0.02) = 0.005$

and hence

$P(A) = 0.006 + 0.0135 + 0.005 = 0.0245$

# Bayes' Rule

**(Bayes' Rule)** If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space $S$ such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event $A$ in $S$ such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

# Example

In a certain assembly plant, three machines, $B_1$, $B_2$, and $B_3$, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. If a product was chosen randomly and found to be defective, what is the probability that it was made by machine $B_3$?

Solution:

$$P(B_3 \mid A) = \frac{P(B_3)P(A \mid B_3)}{P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3)}$$

$$= \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{10}{49}$$