Chapter 13

Waiting Lines and Queuing Theory Models

Multichannel Queuing Model with Poisson Arrivals and Exponential Service Times (M/M/m)

Assumptions of the model:

- Arrivals are served on a FIFO basis.
- There is no balking or reneging.
- Arrivals are independent of each other but the arrival rate is constant over time.
- Arrivals follow a Poisson distribution.
- Service times are variable and independent but the average is known.
- Service times follow a negative exponential distribution.
- The average service rate is greater than the average arrival rate.

Multichannel Queuing Model with Poisson Arrivals and Exponential Service Times (M/M/m)

- Equations for the multichannel queuing model:
- Let

  m = number of channels open
  λ = average arrival rate
  μ = average service rate at each channel

1. The probability that there are zero customers in the system is:

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=m-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \frac{m\mu}{m\mu - \lambda}} \text{ for } m\mu > \lambda$$

Multichannel Model, Poisson Arrivals, Exponential Service Times (M/M/m)

2. The average number of customers or units in the system

$$L = \frac{\lambda\mu(\lambda/\mu)^m}{(m-1)!(m\mu-\lambda)^2}P_0 + \frac{\lambda}{\mu}$$

- The average time a unit spends in the waiting line or being serviced (namely, in the system):

$$W = \frac{\mu(\lambda/\mu)^m}{(m-1)!(m\mu-\lambda)^2}P_0 + \frac{1}{\mu} = \frac{L}{\lambda}$$

Multichannel Model, Poisson Arrivals, Exponential Service Times (M/M/m)

4. The average number of customers or units in line waiting for service:

$$L_q = L - \frac{\lambda}{\mu}$$

5. The average time a customer or unit spends in the queue waiting for service:

$$W_q = W - \frac{1}{\mu} = \frac{L_q}{\lambda}$$

6. Utilization rate:

$$\rho = \frac{\lambda}{m\mu}$$

Arnold's Muffler Shop Revisited

- **Arnold wants to investigate opening a second garage bay. He would hire a second worker who works at the same rate as his first worker.**
- **The customer arrival rate remains the same.**

λ = 2 cars arriving per hour, μ = 3 cars serviced per hour

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=m-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \frac{m\mu}{m\mu - \lambda}} \text{ for } m\mu > \lambda$$

$$P_0 = \frac{1}{\left[\sum_{n=0}^{1} \frac{1}{n!}\left(\frac{2}{3}\right)^n\right] + \frac{1}{2!}\left(\frac{2}{3}\right)^2\left(\frac{2(3)}{2(3) - 2}\right)}$$

$$= \frac{1}{1 + \frac{2}{3} + \frac{1}{2}\left(\frac{4}{9}\right)\left(\frac{6}{6 - 2}\right)} = \frac{1}{1 + \frac{2}{3} + \frac{1}{3}} = \frac{1}{2} = 0.5$$

= probability of 0 cars in the system

Arnold's Muffler Shop Revisited

- **Average number of cars in the system**

$$L = \frac{\lambda\mu(\lambda/\mu)^m}{(m-1)!(m\mu-\lambda)^2}P_0 + \frac{\lambda}{\mu}$$

$$L = \frac{2(3)(2/3)^2}{(1)![2(3)-2]^2}(\frac{1}{2}) + \frac{2}{3} = 0.75$$

- **Average time a car spends in the system**

$$W = \frac{\boxed{L}}{\lambda} = \frac{3}{8}\text{ hour} = 22\ {}^{1}\!/_{2}\text{ minutes}$$

Arnold's Muffler Shop Revisited

- Average number of cars in the queue

$$L_q = L - \frac{\lambda}{\mu} = \frac{3}{4} - \frac{2}{3} = \frac{1}{12} = 0.083$$

- Average time a car spends in the queue

$$W_q = W - \frac{1}{\mu} = \frac{L_q}{\lambda} = \frac{0.083}{2} = 0.0415 \text{ hour} = 2\,{}^1\!/_2 \text{ minutes}$$

Arnold's Muffler Shop Revisited

- **Adding the second service bay reduces the waiting time in line but will increase the service cost as a second mechanic needs to be hired.**

$$\text{Total daily waiting cost} = (8 \text{ hours per day})\lambda W_q C_w$$
$$= (8)(2)(0.0415)(\$50) = \$33.20$$

$$\text{Total daily service cost} = (8 \text{ hours per day})mC_s$$
$$= (8)(2)(\$15) = \$240$$

- **So the total cost of the system is**

$$\text{Total system cost} = \$33.20 + \$240 = \$273.20$$

- **This is the cheapest option: open the second bay and hire a second worker at the same \$15 rate.**

Effect of Service Level on Arnold's Operating Characteristics

Table 13.2 OPERATING CHARACTERISTIC	LEVEL OF SERVICE ONE MECHANIC $\mu = 3$	TWO MECHANICS $\mu = 3$ FOR BOTH	ONE FAST MECHANIC $\mu = 4$
Probability that the system is empty (P_0)	0.33	0.50	0.50
Average number of cars in the system (L)	2 cars	0.75 cars	1 car
Average time spent in the system (W)	60 minutes	22.5 minutes	30 minutes
Average number of cars in the queue (L_q)	1.33 cars	0.083 car	0.50 car
Average time spent in the queue (W_q)	40 minutes	2.5 minutes	15 minutes
Total cost	$653.33	$273.20	$360

Constant Service Time Model (M/D/1)

- **Constant service times are used when customers or units are processed according to a fixed cycle.**
- **The values for L_q, W_q, L, and W are always less than they would be for models with variable service time.**
- **In fact both average queue length and average waiting time are *halved* in constant service rate models.**

Constant Service Time Model (M/D/1)

1. **Average length of the queue**

$$L_q = \frac{\lambda^2}{2\mu(\mu-\lambda)}$$

2. **Average waiting time in the queue**

$$W_q = \frac{\lambda}{2\mu(\mu-\lambda)}$$

Constant Service Time Model (M/D/1)

3. Average number of customers in the system

$$L = L_q + \frac{\lambda}{\mu}$$

4. Average time in the system

$$W = W_q + \frac{1}{\mu}$$

- **Garcia-Golding Recycling, Inc., collects and compacts aluminum cans and glass bottles in New York City. Its truck drivers, who arrive to unload these materials for recycling, currently wait an average of 15 minutes before emptying their loads. The cost of the driver and truck time wasted while in queue is valued at $60 per hour.**
- **A new automated compactor can be purchased that will process truck loads at a constant rate of 12 trucks per hour (i.e., 5 minutes per truck). Trucks arrive according to a Poisson distribution at an average rate of 8 per hour. If the new compactor is put in use, its cost will be amortized at a rate of $3 per truck unloaded.**
- **A summer intern from a local college did the following analysis to evaluate the costs versus benefits of the purchase:**

Garcia-Golding Recycling, Inc.

- The company collects and compacts aluminum cans and glass bottles. λ
- Trucks arrive at an average rate of 8 per hour (Poisson distribution).
- Truck drivers currently wait an average of 15 minutes before emptying their loads C_w $\frac{1}{4}$
- Drivers and trucks cost $60 per hour.
- A new automated machine can process truckloads at a constant rate of 12 per hour. $\mu =$
- A new compactor would be amortized at $3 per truck unloaded. C_s

Constant Service Time Model (M/D/1)

Analysis of cost versus benefit of the purchase

Current **waiting cost/trip**	**= ($^1/_4$ hour waiting time)($60/hour cost)** W_q
	= $15/trip
New **system:** λ	**= 8 trucks/hour arriving**
μ	**= 12 trucks/hour served**
Average waiting time in queue	**= W_q = $^1/_{12}$ hour**
Waiting cost/trip with new compactor	**= ($^1/_{12}$ hour wait)($60/hour cost) = $5/trip**
Savings with new equipment	**= $15 (current system) – $5 (new system)**
	= $10 per trip
Cost of new equipment amortized	**= $3/trip**
Net savings	**= $7/trip**