Name: Youssef Menacer

Hw#2: Stochastic Processes in Biology

Pb# 1	

Let
$$Y_1 \xrightarrow{c_1} 2Y_1 \quad R_1$$
$$Y_1 + Y_2 \xrightarrow{c_2} 2Y_2 \quad R_2$$
$$Y_2 \xrightarrow{c_3} \phi \quad R_3$$

be a stochastic Lotka-Volterra or "Prey - Predator" model

Where Y_1 denotes the concentration of preys And Y_2 denotes the concentration of predators

Where R_1: denotes the prey production
R_2: denotes the prey consumption to generate predator.
R_3: denotes the predator death.

$\boxed{1}$ As the scheme shows the existence of predators depends on the abundance (on the available preys), ~~and~~ ~~the~~ However, the absence of predators, helps preys grow exponentially. Thus, we call this model prey-predator model.

→ ODE associated,

$$\begin{cases} \dfrac{dY_1}{dt} = c_1 Y_1(t) - c_2 Y_1(t) Y_2(t) \\ \dfrac{dY_2}{dt} = -c_3 Y_2(t) + c_2 Y_1(t) Y_2(t) \end{cases}$$

Where: → $Y_1(t)$ denotes preys concentration @ time t. Similarly, $Y_2(t)$ denotes predators concentration @ time t.

→ c_1 denotes the prey reproduction rate

→ c_2 denotes the rate of predator consumption of prey.

→ c_3 denotes the predator death rate.

All $c_j > 0$ $j = 1, 2, 3$.

• If $Y_2(t) = 0$, i.e preys are isolated, we get from the ODE,

$$\begin{cases} \dfrac{dY_1(t)}{dt} = c_1 Y_1(t) \\ Y_1(0) = n_1 > 0 \end{cases} \Rightarrow Y_1(t) = n_1 e^{c_1 t}$$

$c_1 > 0$

$$\underset{t \to \infty}{\Longrightarrow} \boxed{Y_1(t) \longrightarrow \infty}$$

~~The~~ Preys grow exponentially in isolation.

• If $Y_1(t) = 0$, i.e predators are isolated, we get

$$\begin{cases} \dfrac{dY_2(t)}{dt} = -c_3 Y_2(t) \\ Y_2(0) = n_2 > 0 \end{cases} \rightarrow Y_2(t) = n_2 e^{-c_3 t} \underset{t \to \infty}{\longrightarrow} 0$$

$c_3 > 0$

Then predators goes to extinction in the absence of preys.

$\boxed{2}$ "Attached as .ipynb file (simulation_pb1)

$\boxed{3}$ If $c_2 > c_3$, i.e ~~the~~ consumption of preys is greater than their reproduction, preys concentration decreases and goes to extinction. As a result, and according to what I showed in 1, species 2 (predators) concentration goes to extinction as there is no food for them. (showed in the simulation).

Pb# 2	Consider the following chemical reaction

$$A \xrightarrow{k} X \xrightarrow{\alpha_1} \phi$$
$$B \xrightarrow{k} Y \xrightarrow{\alpha_2} \phi$$
$$X + Y \xrightarrow{k_a} C$$

And the associated ODE (deterministic case) is giving by

$$(*) \begin{cases} \dfrac{d[X]}{dt} = k - \alpha_1 [X] - k_a [X][Y] \\ \dfrac{d[Y]}{dt} = k - \alpha_2 [Y] - k_a [X][Y] \end{cases}$$

$\boxed{1}$ The fixed points:

Set $\begin{cases} \dfrac{d[X]}{dt} = 0 \\ d[Y]/dt = 0 \end{cases}$

$$\Longrightarrow \begin{cases} k - \alpha_1[X] - k_a[X][Y] = 0 & \dots (1) \\ k - \alpha_2[Y] - k_a[X][Y] = 0 & \dots (2) \end{cases}$$

$$(1) - (2) \Longrightarrow \boxed{[X] = \frac{\alpha_2}{\alpha_1}[Y]} \quad \dots (3)$$

$$\xrightarrow[\text{in } (1)]{\text{Plug } (3)} \alpha_2[Y] + k_a \frac{\alpha_2}{\alpha_2}[Y]^2 - k = 0$$

$$\Longrightarrow k_a \frac{\alpha_2}{\alpha_1}[Y]^2 + \alpha_2[Y] - k = 0$$

Set $y = [Y]$

$$\longrightarrow \boxed{k_a \frac{\alpha_2}{\alpha_1} y^2 + \alpha_2 y - k = 0}$$

$$\Delta = \alpha_2^2 + 4 k_a \frac{\alpha_2}{\alpha_1} k > 0$$

$$\Longrightarrow \boxed{y^*_{1,2} = \frac{-\alpha_2 \pm \sqrt{\Delta}}{2 k_a \frac{\alpha_2}{\alpha_1}}}$$

$$\Longrightarrow \begin{cases} (x_1^*, y_1^*) = \left(\frac{\alpha_2}{\alpha_1} y_1^*, y_1^*\right) \\ (x_2^*, y_2^*) = \left(\frac{\alpha_2}{\alpha_1} y_2^*, y_2^*\right) \end{cases}$$ The fixed points of (*)

Where $y_1^* = \frac{-\alpha_2 - \sqrt{\Delta}}{2 k_a \frac{\alpha_2}{\alpha_1}}$, and $y_2^* = \frac{-\alpha_2 + \sqrt{\Delta}}{2 k_a \frac{\alpha_2}{\alpha_1}}$

(i) For $h = 10,\ \alpha_1 = 10^{-6},\ \alpha_2 = 10^{-5},\ k_a = 10^{-5}$,

We get,

$$\frac{\alpha_2}{\alpha_1} = 10 \ , \ 2k_a\frac{\alpha_2}{\alpha_1} = 2\left(10^{-5}\right)\left(10\right) = \boxed{2.10^{-4}}$$

$$\Rightarrow \Delta = \alpha_2^2 + 4k_a\frac{\alpha_2}{\alpha_1}h$$

$$= 10^{-10}\left(1 + 4.10^{7}\right)$$

$$\Rightarrow y^*_{1,2} = \frac{-10^{-5} \pm 10^{-5}\sqrt{10^{-10}\left(1+4.10^{7}\right)}}{2.10^{-4}} = \boxed{\frac{-1 \pm \sqrt{1+4.10^{+7}}}{20}}$$

$$\Rightarrow \boxed{\left(x_i^*, y_i^*\right) = \left(\frac{\alpha_2}{\alpha_1}y_i^*, y_i^*\right) = \left(10y_i^*, y_i^*\right) \quad \text{for } i = 1, 2}$$

(ii) For $h = 10^{-3},\ \alpha_1 = 10^{-4},\ \alpha_2 = 10^{-3},\ k_a = 10^{-3}$, we get

$$\frac{\alpha_2}{\alpha_1} = 10 \Rightarrow 2k_a\frac{\alpha_2}{\alpha_1} = 2\,10^{-3}\,10 = \boxed{2.10^{-2}}$$

Then, $\Delta = 10^{-6} + 4\,\cancel{10^{-3}}\,10\,\cancel{10^{3}} = 10^{-6} + 4.10$

$$\boxed{= 10^{-6}\left(4.10^{+7} + 1\right)}$$

Therefore,

$$z^*_{1,2} = \frac{-10^{-3} \pm 10^{-3}\sqrt{1+4.10^{7}}}{2.10^{-2}}$$

$$= \frac{-1 \pm \sqrt{1+4.10^{7}}}{20} = y^*_{1,2}$$

$$\Rightarrow \left(w_i^*, z_i^*\right) = \left(\alpha_2/\alpha_1\, z_i^*, z_i^*\right) = \left(10z_i^*, z_i^*\right) = \left(x_i^*, y_i^*\right), i=1$$

Therefore for (i) and (ii), we get the same fixed points.

[2] § Simulation_pb2

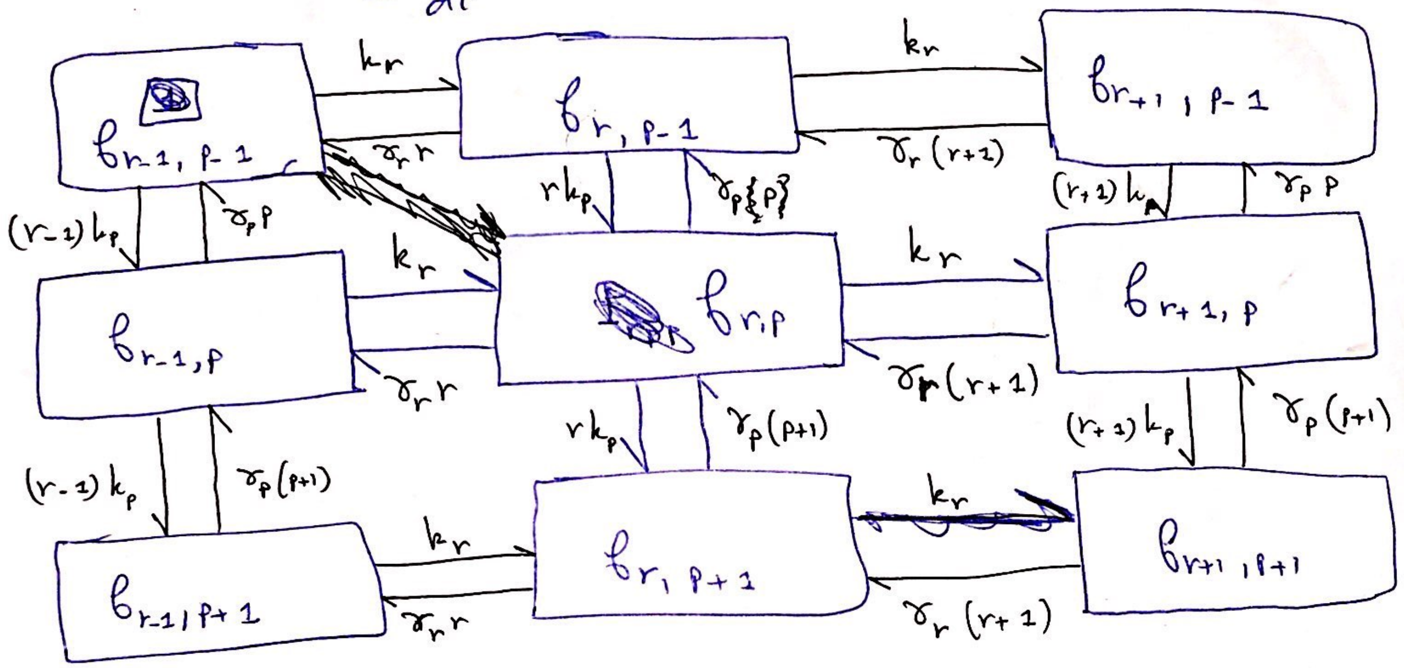
[3] The behavior in the two cases:

Pb #3. (1/)

For r : # mRNA and p : # protein, we get
/or [mRNA] or [protein]

$$\begin{cases} \dfrac{dr}{dt} = \overbrace{(k_\ell + \phi(p))}^{k_r} - \gamma_r r, \\ \dfrac{dp}{dt} = r k_p - \gamma_p p \end{cases}$$

$$\boxed{k_r = k_\ell + \phi(p)}$$



$$\Rightarrow \begin{cases} \dfrac{d b_r}{dt} = k_r \left[b_{r-1}(t) - b_r(t) \right] + \gamma_r \left[(r+1) b_{r+1}(t) - r b_r(t) \right] \\ \dfrac{d b_p}{dt} = r k_p \left[b_{p-1}(t) - b_p(t) \right] + \gamma_p \left[(p+1) b_{p+1}(t) - p b_p(t) \right] \end{cases}$$

Where $b_{i,j}$ denotes the proba being in state (i,j)

$\dfrac{db_r}{dt}$: denotes the variation of the proba of mRNA with the time t.

$\dfrac{db_p}{dt}$: denotes the variation of the proba of the protein over time t.

(2/

$$\underline{P} = \begin{array}{c|c|c|c} & p-1 & p & p+1 \\ \hline r-1 & -k_r - (r-1)k_p + \gamma_p p + \gamma_r r & -\gamma_p p - (r-1)k_p + (r-1)k_p + \gamma_p(p+1) - k_r + \gamma_r r & -\gamma_p(p+1) - k_r + (r-1)k_p + \gamma_r r \\ \hline r & -\gamma_r r - k_r - r k_p + k_r + \gamma_r(r+1) + \gamma_p p & -\gamma_p p - r k_p - k_r - \gamma_r r + r k_p + \gamma_p(p+1) + k_r + \gamma_r(r+1) & -k_r - \gamma_p(p+1) - \gamma_r r + k_r + r k_p + \gamma_r(r+1) \\ \hline r+1 & -\gamma_r(r+1) - (r+1)k_p + k_r + \gamma_p p & -\gamma_p p - \gamma_r(r+1) - (r+1)k_p + (r+1)k_p + k_r + \gamma_p(p+1) & -\gamma_p(p+1) - \gamma_r(r+1) + k_r + (r+1)k_p \end{array}$$

(3)

$$\Rightarrow \underline{P} = \begin{array}{cc} & \begin{array}{ccc} p-1 & p & p+1 \end{array} \\ \begin{array}{c} r-1 \\ \\ r \\ \\ r+1 \end{array} & \begin{pmatrix} -k_r-(r-1)k_p+\gamma_p p+\gamma_r r & \gamma_p-k_r+\gamma_r r & -\gamma_p(p+1)-k_r+(r-1)k_p+\gamma_r r \\ -r k_p+\gamma_r+\gamma_p p & \gamma_p+\gamma_r & \gamma_r+r k_p-\gamma_p(p+1) \\ -\gamma_r(r+1)-(r+1)k_p+k_r+\gamma_p p & \gamma_p+k_r-\gamma_r(r+1) & -\gamma_p(p+1)-\gamma_r(r+1)+k_r+(r+1)k_p \end{pmatrix} \end{array}$$

Pb #3

For $\phi(p) = \dfrac{h_o\left(\frac{p}{k}\right)^n}{1+\left(\frac{p}{k}\right)^n}$

$k_L = 0,\ \gamma_r = \gamma_r = k_p = h_o = 1,\ K = 0.5$

$$\Rightarrow \begin{cases} \dfrac{dr}{dt} = \dfrac{(2p)^n}{1+(2p)^n} - r \\ \dfrac{dp}{dt} = r - p \end{cases}$$

i) $n = 1$

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{2p}{1+2p} - r = f(r,p) \\ \dfrac{dp}{dt} = r - p = g(r,p) \end{cases}$$

Find the fixed pts.

$$\begin{cases} \dot{r} = 0 \\ \dot{p} = 0 \end{cases} \Rightarrow \dfrac{2p}{1+2p} = r \quad \dots (1)$$

$$\boxed{r = p}$$

$$\Rightarrow \frac{2r}{1+2r} = r \Leftrightarrow r\left(1 - \frac{2}{1+2r}\right) = 0$$

$$\Rightarrow r = 0 \text{ or } \frac{2}{1+2r} = 1$$

$$\Rightarrow 1 + 2r = 2$$

$$\Rightarrow \boxed{r = \frac{1}{2}}$$

$$\Rightarrow \begin{cases} (r_1^*, p_1^*) = (0,0) \\ (r_2^*, p_2^*) = \left(\frac{1}{2}, \frac{1}{2}\right) \end{cases}$$

Stability

$$\text{Jacobian} = J = \begin{pmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial p} \end{pmatrix}$$

$$\Rightarrow J(r,p) = \begin{pmatrix} -1 & \frac{2(1+2p)-4p}{(1+2p)^2} \\ 1 & -1 \end{pmatrix}$$

ⓐ $A = J(0,0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

$\det(A) = 1 - 2 = -1 < 0 \Rightarrow \boxed{(0,0) \text{ saddle point}}$

ⓑ $B = J\left(\frac{1}{2},\frac{1}{2}\right) = \begin{pmatrix} -1 & \frac{4-2}{4} \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & -1 \end{pmatrix}$

$\det(B) = 1 - \frac{1}{2} = \frac{1}{2} > 0$

$\operatorname{tr}(B) = -1-1 = -2 < 0$ $\Rightarrow \boxed{\left(\frac{1}{2},\frac{1}{2}\right) \text{ is stable}}$

(ii) $n = 10$ $\begin{cases} \dot{r} = \frac{(2p)^{10}}{1+(2p)^{10}} - r \\ \dot{p} = r - p \end{cases}$

F pts.

$\begin{cases} \dot{r} = 0 \\ \dot{p} = 0 \end{cases} \Rightarrow r = p \Rightarrow p\left(\frac{2^{10}p^{10}}{1+2^{10}p^{10}} - 1\right) = 0$

$\Rightarrow p = 0$ or $2^{10}p^{10} = 1 + 2^{10}p^{10}$

$$\frac{(2p)^n}{1+(2p)^n} - r = 0$$

$r = p \Rightarrow p\left(\frac{2^n p^{n-1}}{1+2^n p^n} - 1\right) = 0$

$\Rightarrow p = 0$ or $2^n p^{n-1} = 1 + 2^n p^n$

$\Rightarrow 2^n (p^n - p^{n-1}) = -1$

$\Rightarrow p^{n-1}(p-1) = -\frac{1}{2^n}$

$\underline{n=10}$ $\quad p^9 (p-1) = -\frac{1}{2^{10}}$

Using Some app

$\Longrightarrow \boxed{p = 0.5, \quad p = 0.999015}$

$$\Rightarrow \begin{cases} (r_1^*, p_1^*) = (0, 0) \\ (r_2^*, p_2^*) = (0.5, 0.5) \\ (r_3^*, p_3^*) = (0.999015, 0.999015) \end{cases}$$

<u>Stability</u>

$$J = \begin{pmatrix} -1 & \frac{10\, 2^{10} p^9 (1 + 2^{10} p^{10}) - 2^{10}\, 10\, p^9\, 2^{10} p^{10}}{(1+(2p)^{10})^2} \\ 1 & -1 \end{pmatrix}$$

(a) $A = J(0,0) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$

$\det(A) = 1 > 0 \quad tr(A) = -2 < 0 \Rightarrow (0,0)$ is stable

(b) $B = J\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} -1 & 10\, 2^{10} \frac{1}{2^9}\left(1 + \frac{2^{10}}{2^{10}}\right) - 2^{10} \dfrac{\frac{10}{2^9}\frac{2^{10}}{4^{10}}}{(1+1)^2} \\ 1 & -1 \end{pmatrix}$

$\Rightarrow B = \begin{pmatrix} -1 & \frac{40-2}{4} \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{19}{2} \\ 1 & -1 \end{pmatrix}$

$\det(B) = 1 - \frac{19}{2} < 0 \Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ is unstable

(c) $0.999015 \simeq 1$

$C = J(0.999015, 0.999015) \simeq J(1,1)$

$= \begin{pmatrix} -1 & 2^{10}\cancel{(10)}(1 + \cancel{2^{10}}) - 2^{10}\cancel{(10)}\, 2^{10} \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 2^{10} \\ 1 & -1 \end{pmatrix}$

$\det(C) = 1 - 2^{10} < 0$

$\Rightarrow (0.999015, 0.999015)$ is unstable

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