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HW # 2

Pb # 1

Prove that $\sum_n P_n(t) = 1 \iff \sum_n \frac{1}{\lambda_n} = \infty$

Such that $P_n(t) = Pr\{X(t) = n\}$

Let $S_n(t) = \sum_{j=0}^{n} P_j(t)$

Using the fact that $P_n(t)$ verifies

$$\begin{cases} P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & n \geq 1 \\ P_0'(t) = -\lambda_0 P_0(t) & \cancel{n \geq 1} \\ P_0(0) = 1, \quad P_n(0) = 0, & n \neq 0 \end{cases}$$

Then,

$$S_n'(t) = \sum_{j=1}^{n} P_j'(t) = \sum_{j=1}^{n} \left(-\lambda_j P_j(t) + \lambda_{j-1} P_{j-1}(t) \right)$$

$$= \cancel{\lambda_0 P_0'(t)} - \lambda_1 \cancel{P_1(t)} + \cancel{\lambda_0 P_0(t)} - \lambda_2 \cancel{P_2(t)} + \cancel{\lambda_1 P_1(t)}$$

$$+ \cdots + -\lambda_{k-1} \cancel{P_{k-1}(t)} + \lambda_{k-2} \cancel{P_{k-2}(t)} + \cdots$$

$$+ \cdots - \lambda_n P_n(t) + \cancel{\lambda_{n-1} P_{n-1}(t)}$$

$$\Rightarrow \boxed{S_n'(t) = -\lambda_n P_n(t) \qquad n \geq 1}$$

$$\Rightarrow \int_0^t S_n'(z)\,dz = -\int_0^t \lambda_n P_n(z)\,dz$$

$$\Rightarrow S_n(t) - S_n(0) = -\int_0^t \lambda_n P_n(z)\,dz$$

$$\Rightarrow S_n(0) - S_n(t) = \int_0^t \lambda_n P_n(z)\,dz$$

$$S_n(0) = \sum_{j=0}^{n} P_n(0) = \boxed{1}$$

Therefore,

$$\boxed{1 - S_n(t) = \int_0^t \lambda_n P_n(Z)\, dZ} \quad \ldots\ast$$

Since $P_j(t) = Pr\{x(t) = j\} \geq 0$

Then, $S_n(t) = \sum_{j=0}^{n} P_j(t)$ is an $\uparrow$ sequence with "n"

Therefore, $0 \leq 1 - S_n(t)$ is an $\downarrow$ sequence

Hence, $\exists\ \gamma(t) = \lim_{n \to \infty} 1 - S_n(t) = \min_n \{1 - S_n(t)\}$

Thus,

$$\lambda_n \int_0^t P_n(Z)\, dZ \geq \gamma(t)$$

$$\Rightarrow \int_0^t P_n(Z)\, dZ \geq \frac{\gamma(t)}{\lambda_n}, \forall\ n \geq 0$$

$$\Rightarrow \boxed{\int_0^t \sum_{j=0}^{n} P_j(Z) dZ \geq \gamma(t) \sum_{j=0}^{n} \frac{1}{\lambda_j}} \quad \ldots (\ast\ast)$$

Using $(\ast)$, $t \geq \int_0^t S_n(Z)\, dZ \geq \gamma(t) \sum_{j=0}^{n} \frac{1}{\lambda_j}$

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(i) Suppose $\sum_n \frac{1}{\lambda_n} = \infty$, then

~~the inequality (**)~~,

$$\gamma(t) \sum_n \frac{1}{\lambda_j} \leq \int_0^t S_\infty(\tau)\, d\tau \leq t$$

holds iff $\gamma(t) = 0$, for finite t. $(t < \infty)$

$$\Leftrightarrow \quad \gamma(t) = \lim_{n \to \infty} 1 - S_n(t) = 0$$

$$\Leftrightarrow \quad \boxed{\lim_{n \to \infty} S_n(t) = \sum_n P_n(t) = 1}$$

(ii) Suppose $\sum_n P_n(t) = 1 = \lim_{n \to \infty} S_n(t)$

Then, using (*)

$$\lambda_n = \frac{1 - S_n(t)}{\int_0^t P_n(\tau)\, d\tau} \Leftrightarrow \frac{1}{\lambda_n} = \frac{\int_0^t P_n(\tau)\, d\tau}{1 - S_n(t)}$$

The quantity $\alpha \int_0^t P_n(\tau)\, d\tau < \infty$

So

$$\lim_{n \to \infty} \frac{1}{\lambda_n} = \left(\int_0^t \lim_{n \to \infty} P_n(\tau)\, d\tau \right) \Big/ \underbrace{\lim_{n \to \infty} \left(1 - S_n(t)\right)}_{0^+}$$

$$= \infty$$

Therefore, $\sum_n \frac{1}{\lambda_n} = \infty$

Since the necessary condition of convergence is not verified (i.e $\lim_{n \to \infty} \frac{1}{\lambda_n} = 0$) in general for $\sum < \infty$

Pb #2

In the simple BD process $\begin{cases} \lambda_n = \lambda n & \text{(birth rate)} \\ \mu_n = \mu n & \text{(death rate)} \end{cases}$

$$\frac{dn}{dt} = (\lambda - \mu)\, n + \nu \quad \ldots (1)$$

Let $\{X(t),\ t \geq 0\}$ be the stochastic process of this model.

Denote $\bar{n}(t) = E[X(t)]$

$X(t)$: denote the population size @ time $= t$

for $X(0) = N$ (initial population size)

Then, $\bar{n}(t+h) = E[X(t+h)] = E[E[X(t+h) | X(t)]]$

The population @ time $t+h$ will either increase in size by 1 (for birth or immigration occurs) in $(t, t+h)$. However, it can decrease by 1 if a death happens in $(t, t+h)$.

i.e $X(t+h) = \begin{cases} X(t) + 1, & \text{with probn } [\nu + \lambda X(t)]h + o(h) \\ X(t) - 1, & \text{with probn } \mu X(t) h + o(h) \\ X(t) & \text{with probn } 1 - [\nu + \lambda X(t) + \mu X(t)]h + o(h) \end{cases}$

nothing happens

Therefore,

$$E[X(t+h) \mid X(t)] = (X(t)+1)[(\nu + \lambda X(t))h + o(h)]$$
$$+ [(X(t)-1)][\mu X(t) h + o(h)] +$$
$$+ X(t)[1 - [\nu + \lambda X(t) + \mu X(t)]h + o(h)]$$

$$\Rightarrow E[X(t+1) \mid X(t)] = X(t) + [\nu + \lambda X(t) - \mu X(t)]h + o(h)$$

$$\Rightarrow \bar{n}(t+h) = E[E[X(t+1) \mid X(t)]]$$
$$= \underbrace{E[X(t)]}_{\bar{n}(t)} + \nu h + (\lambda - \mu) h E[X(t)] + o(h)$$

$$\Rightarrow \frac{\bar{n}(t+h) - \bar{n}(t)}{h} = (\lambda - \mu)\bar{n}(t) + \nu + \frac{o(h)}{h}$$

$$\xrightarrow{h \to 0} \boxed{\begin{array}{l} \bar{n}'(t) = (\lambda - \mu)\bar{n}(t) + \nu \\ \bar{n}(0) = E[X(0)] = N \end{array}}$$

$\bar{n}(t)$ is a solution to (1)

- We need to prove the uniqueness

Assume $\bar{n}_1(t)$ and $\bar{n}_2(t)$ are two solutions to (1) i.e.

$$\begin{cases} n_1'(t) = (\lambda - \mu) n_1 + \nu \\ n_1(0) = N \end{cases} \quad \begin{cases} n_2'(t) = (\lambda - \mu) n_2 + \nu \\ n_2(0) = N \end{cases}$$

$w = n_1 - n_2$

$$\Rightarrow \begin{cases} w'(t) = (\lambda - \mu) w(t) \\ w(0) = 0 \end{cases} \Rightarrow w(t) = K e^{(\lambda-\mu)t}$$
$$w(t) = w(0) e^{(\lambda-\mu)t}$$
$$w(t) = 0 \quad \forall t$$

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Therefore $n_1(t) = n_2(t) \quad \forall t$

$\Longrightarrow$ the uniqueness of the solution of (1)

$\Rightarrow \bar{n}(t) = E[X(t)]$ is the solution of (1).

Case when it doesn't work:

for $\lambda = \lambda(t)$ in general inhomogeneous
$\mu = \mu(t)$ BD with immigration
$\nu = \nu(t)$ process

$$\Rightarrow \frac{dn}{dt} = (\lambda(t) - \mu(t))\, n + \nu$$

Then for $\bar{n}(t) = E[X(t)]$

$\cancel{\bar{n}'(t)}$

$$\frac{\bar{n}(t+h) - \bar{n}(t)}{h} = (\lambda(t) - \mu(t))\, \bar{n}(t) + \nu(t) + \frac{o(h)}{h}$$

$$\Rightarrow \boxed{\bar{n}'(t) = (\lambda(t) - \mu(t))\, \bar{n}(t) + \nu(t)}$$

But we don't have uniqueness of the solution

i.e $\left(e^{-\int_0^t (\lambda(z) - \mu(z))\,dz}\, \bar{n}(t)\right)' =$

$$= -(\lambda(t) - \mu(t))\, e^{-\int_0^t (\lambda(z) - \mu(z))\,dz}\, \bar{n}(t) + e^{-\int_0^t (\lambda(z) - \mu(z))\,dz}\, \bar{n}'(t)$$

$$\Rightarrow \left(e^{-\int_0^t (\lambda(z)-\mu(z))\,dz}\, n(t) \right)' = \nu(t)\, e^{-\int_0^t (\lambda(z)-\mu(z))\,dz}$$

$$\Rightarrow n(t) = e^{\int_0^t (\lambda(z)-\mu(z))\,dz} \left[\int_0^t \nu(s)\, e^{-\int_0^s (\lambda(z)-\mu(z))\,dz}\, ds + \underbrace{N}_{n(0)} \right]$$

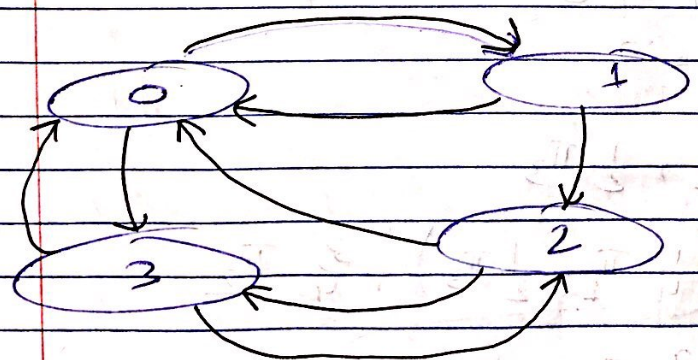
!!!

pb #3,

$$\text{Let } P = \begin{array}{c|cccc} & \multicolumn{4}{c}{\text{tomorrow } n+1} \\ & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \text{today } n \;\; 3 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array} \quad MC$$

$S = \{0, 1, 2, 3\}$

(a) The directed graph:



(b) (i) The MC is irreducible since we have one unique communicating class $C = \{0,1,2,3\}$. (every states $i \leftrightarrow j$ $\forall\, i \neq j$)

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(ii) Since all states communicate (irreducible) and this is a finite MC. Then it is a positive recurrent MC.

(iii) for any $i \in \{0, 1, 2, 3\}$, $d(i) = \text{period}(i)$

We see that $P_{ii} = 0 \quad \forall i \in \{0, 1, 2, 3\}$

Then, the period of i, $d(i) = 4 \quad \forall i \in \{0, 1, 2, 3\}$

(c) Find $\Pi = [\Pi_0, \Pi_1, \Pi_2, \Pi_3]$

$\Pi^T = P^T \Pi^T$

$$\Rightarrow \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix}$$

And $\sum_{j=0}^{3} \Pi_j = 1$

$$\Rightarrow \Pi_0 = \frac{1}{4}\Pi_1 + \frac{1}{2}\Pi_3$$

$$\Pi_1 = \frac{1}{2}\left(\frac{1}{4}\Pi_1 + \frac{1}{2}\Pi_3\right) + \frac{3}{4}\Pi_2$$

$$\Rightarrow \boxed{\Pi_1 = \frac{2}{7}\Pi_3 + \frac{6}{7}\Pi_2}$$

And $\Pi_2 = \frac{3}{4}\left(\frac{2}{7}\Pi_3 + \frac{6}{7}\Pi_2\right) + \frac{1}{2}\Pi_3$

$$= \frac{10}{14}\Pi_3 + \frac{18}{28}\Pi_2 \Rightarrow \boxed{\Pi_2 = 2\Pi_3}$$

$$\Rightarrow \boxed{\Pi_1 = 2\Pi_3}$$

For $\boxed{\pi_0 = \frac{1}{2}\pi_3 + \frac{1}{2}\pi_3 = \pi_3}$

Then pluggin

$$2\pi_3 + 4\pi_3 = 6\pi_3 = 1 \Rightarrow \begin{cases} \pi_3 = \frac{1}{6} = \pi_0 \\ \pi_2 = \pi_1 = \frac{1}{3} \end{cases}$$

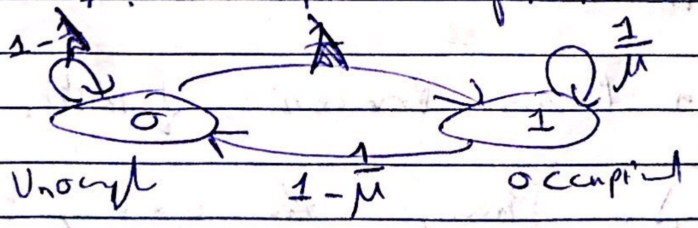
$$\Rightarrow \boxed{\pi^T = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)^T}$$

Pb #4

We get two states here

0: unoccupied promoter

1: occupied promoter



$$\Rightarrow P = \begin{pmatrix} 1-\lambda & \lambda \\ 1-\frac{1}{\mu} & \frac{1}{\mu} \end{pmatrix}$$

$$\pi^T = \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1-\frac{1}{\mu} \\ \lambda & \frac{1}{\mu} \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \end{pmatrix}$$

$\Rightarrow$

$$\begin{cases} (1-\lambda)\Pi_0 + \frac{1}{\mu}\Pi_1 = \Pi_0 \\ \lambda\Pi_0 + (1-\frac{1}{\mu})\Pi_1 = \Pi_1 \end{cases}$$

$$\Rightarrow \lambda \Pi_0 = \frac{\Pi_1}{\mu} \rightarrow \boxed{\Pi_1 = \lambda\mu\Pi_0}$$

$$\Pi_0 + \Pi_1 = 1 \rightarrow \Pi_0(1+\lambda\mu) = 1$$

$$\rightarrow \boxed{\Pi_0 = \frac{1}{1+\lambda\mu}}$$: the long fraction of time that the promoter is unoccupied.

Pb # 5

λ: the rate where (APs) fired by a neuron.

We have a Poisson Process $\{N(t), t \geq 0\}$ with rate λ. Then, the waiting time $T_1 \hookrightarrow \mathcal{E}(\lambda)$ exponential distribut with rate λ.

$$P(T_{(t)} > t) = e^{-\lambda t}$$ in the interval $(t, t+h)$

$\Rightarrow Pr\{ AP_s \text{ fixed after time } t \} = e^{-\lambda t}$

$\forall t \in (t, t+h)$

$\Rightarrow Pr\{ AP_s \text{ fixed before time } t \} = 1 - e^{-\lambda t}$

$= F_{T(t)}(t)$ ⟸ the distribution of $T(t)$

We know that $T_i \hookrightarrow \mathcal{E}(\lambda)$ has a pdf

$$f(t) = \lambda e^{-\lambda t}, \quad \forall t > 0$$

i.e $$E[T(t)] = \int_0^\infty t(\lambda e^{-\lambda t})\, dt = \frac{1}{\lambda}\underbrace{\int_0^\infty y^{2-1} e^{-y}\, dy}_{= \Gamma(2) = 1!}$$

$$y = \lambda t \Rightarrow t = \frac{y}{\lambda} \Rightarrow dt = \frac{dy}{\lambda}$$

$$\Rightarrow \boxed{E[T(t)] = \frac{1}{\lambda}} \quad \text{for } \lambda > 0$$

The mean of the time to the nearest AP in time $= \frac{1}{\lambda}$.