Lecture 22

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1 Reynolds Transport Theorem

Derivation: Consider a one dimensional flow with one inlet and one outlet.
At time t,

$$B_{\text{sys}}(t) = B_{\text{cv}}(t)$$

As both the system and control volume are equivalent.
At time $t + \Delta t$,

$$B_{\text{sys}}(t + \Delta t) = B_{\text{cv}}(t + \Delta t) + \Delta B_{\text{out}} - \Delta B_{\text{in}}$$

As $B_{\text{sys}}(t + \Delta t)$ contains some B which has left the control volume, but it does not contain the new B that has entered it. Then

$$\begin{aligned}
\frac{dB_{\text{sys}}}{dt} &= \lim_{\Delta t \to 0} \frac{B_{\text{sys}}(t + \Delta t) - B_{\text{sys}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{B_{\text{cv}}(t + \Delta t) + \Delta B_{\text{out}} - \Delta B_{\text{in}} - B_{\text{cv}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{B_{\text{cv}}(t + \Delta t) - B_{\text{cv}}(t)}{\Delta t} + \dot{B}_{\text{out}} - \dot{B}_{\text{in}} \\
&= \frac{dB_{\text{cv}}}{dt} + \dot{B}_{\text{out}} - \dot{B}\text{in}
\end{aligned}$$

Derivation: General Case with multiple inlets and outlets, ρ, V, b may not be unform across the inlet and outlet parts, and $\vec{v}$ may not be normal to the inlet and outlet cross-sections.
We define

$$d\vec{A} := dA\hat{n}$$

Then

$$dm = \rho \vec{v} \cdot d\vec{A}$$

so

$$\dot{B}_{\text{net}} = \dot{B}_{\text{out}} - \dot{B}_{\text{in}} = \oiint_{cs} \rho b \vec{v} \cdot d\vec{A}$$

and finally

$$\frac{dB_{\text{sys}}}{dt} = \frac{dB_{\text{cv}}}{dt} + \oiint_{cs} \rho b \vec{v} \cdot d\vec{A}$$

Example 1.1. Derive the conservation of mass equation in Eulerian form using the RTT.
In Lagrangian form: $\frac{dm_{\text{sys}}}{dt} = 0$
In Eulerian form:

$$\frac{dm_{\text{cs}}}{dt} + \dot{m}_{\text{out}} - \dot{m}_{\text{in}} = 0$$

Which can be rewritten as

$$\frac{d}{dt} \iiint_{cv} \rho dV + \oiint_{cs} \rho \vec{v} \cdot d\vec{A} = 0$$

2 Momentum Equation

Derivation: In Lagrangian form,

$$\sum \vec{F}_{\text{sys}} = \frac{d}{dt}\vec{M} = \frac{d}{dt}\int_{\text{sys}} \vec{v} \rho dV$$

Substituting RTT,

$$\sum \vec{F}_{\text{sys}} = \frac{d}{dt}\vec{M}_{\text{cv}} + \dot{\vec{M}}_{\text{net}} = \frac{d}{dt}\iiint_{\text{cv}} \vec{v} \rho dV + \oiint_{\text{cv}} \vec{v} \left(\rho \vec{v} \cdot d\vec{A} \right)$$