Problem Set 2

niceguy

September 24, 2022

1. The head on an electric toothbrush oscillates in simple harmonic motion with frequency $f = 300\text{Hz}$ and amplitude $A = 2.00\text{mm}$. What is the maximum speed of the toothbrush head?

> **Solution:**
> $$x = A\cos(\omega t + \phi)$$
> Taking the derivative,
> $$v = -A\omega\sin(\omega t + \phi)$$
> So the maximum velocity is given by $A\omega$. Substituting the values in,
> $$v = 2 \times 10^{-3} \times 2\pi \times 300 = 3.77 \text{ ms}^{-1}$$

2. A rotating cam in a car engine opens a valve once per rotation. The cam displacement can be written as
$$y(t) = 40 + 2\cos(\omega t + \phi_0)$$
and the valve is open whenever $y(t) > 41\text{mm}$. During what fraction of each cycle is the valve open?

> **Solution:** From the diagram, the valve is open between the two solutions where $y(t) = 41$, which can be simplified as $\cos(\omega t) = 0.5$. The two solutions are $\pm\frac{\pi}{3}$. Therefore the fraction is $\frac{2\pi}{3} \div 2\pi = \frac{1}{3}$.

3. At time $t = 0\text{s}$ a particle of mass $m = 0.21\text{kg}$ is moving through a position $x(0) = 0.110\text{m}$ with a velocity $v = 0.330\text{ms}^{-1}$. The particle passes through the same position with the same velocity $\Delta t = 1.25\text{s}$ later.
   (a) Determine the amplitude of the oscillation
   (b) Determine the phase constant of the motion

> **Solution:** Since the particle passes through the same position with the same velocity, a period (or several) must have passed, or
> $$1.25 = \frac{2k\pi}{\omega}, k \in \mathbb{N}$$
> We know that the position of the particle is given by
> $$x(t) = A\cos(\omega t + \phi)$$
> Substituting initial conditions,
> $$0.11 = A\cos\phi$$

$$0.33 = -A\omega \sin\phi$$

Dividing the second equation by the first equation,

$$-\omega \tan\phi = 3$$

The only way to solve for ϕ is by assuming $k = 1$, i.e. the particle does not pass through the same position with the same velocity for $t \in (0, 1.25)$. Substituting $\omega = \frac{2\pi}{T}$ yields

$$\phi = -0.538$$

Considering $v(0) > 0$, we have

$$\phi = 5.74 \text{ rad}$$

The amplitude can then be solved from the initial equations

$$A = 0.128 \text{ m}$$

4. A simple harmonic oscillator consists of a block of mass 2.00 kg attached to a spring of spring constant 50.0 N/m. The amplitude of the oscillation is measured to be $A = 0.12$m. At time $t = 0$s the position of the particle is measured to be $x(0) = 00.040$m with the particle moving to the right.

(a) Determine the angular frequency of the oscillation

Solution:

$$\begin{aligned}\omega &= \sqrt{\frac{k}{m}} \\ &= \sqrt{\frac{50}{2}} \\ &= 5 \text{ rad s}^{-1}\end{aligned}$$

(b) Determine the total mechanical energy of the system

Solution:

$$\begin{aligned}E &= \frac{1}{2}kA^2 \\ &= \frac{1}{2} \times 50 \times 0.12^2 \\ &= 0.36 \text{ J}\end{aligned}$$

(c) Determine the velocity of the particle at $t = 0$s.

Solution:

$$x(t) = 0.12\cos(5t + \phi)$$

Putting $x(0) = -0.04$ and considering $v(0) > 0$,

$$\phi = 4.37 \text{ rad}$$

Differentiation gives us

$$v(t) = -0.6 \sin(5t + \phi)$$

and substituting $t = 0$ yields

$$v(0) = 0.566 \text{ ms}^{-1}$$

(d) Determine the initial phase constant of this oscillation. Express your answer in radians, as a value between 0 and 2π.

Solution: This was solved for above.

$$\phi = 4.37 \text{ rad}$$

(e) Determine the times when particle is at $x = 0.040$m for the first and for the second time.

Solution:

$$0.04 = 0.12 \cos(5t + \phi)$$
$$\cos(5t + \phi) = \frac{1}{3}$$

The two smallest $\theta > \phi$ such that $\cos\theta = \frac{1}{3}$ are 5.05 and 7.51, giving the times

$$t = 0.136 \text{ s and } t = 0.628 \text{ s}$$

5. Figure 1 below shows the velocity as a function of time for first five oscillations of a mass undergoing simple harmonic motion. Determine the initial phase constant of this oscillation

Solution: From the figure,

$$-2 \sin\phi = -1$$

Considering initial velocity is negative,

$$\phi = \frac{5\pi}{6}$$

6. A mass $m = 2.0$kg is placed on a spring balance, displacing the balance by $\Delta l = 3.0$cm. the damping mechanism allows the balance to return to equilibrium in teh shortest possible time. What is the required coefficient b in a damping force $F = -bv$?

Solution:

$$k = \frac{F}{\Delta l} = \frac{mg}{\Delta l} = 654 \text{ Nm}^{-1}$$

For the balance to return to equilibrium in the shortest possible time, there must be critical damping, i.e.

$$\gamma = 2\omega_0 = 2\sqrt{\frac{k}{m}} = 36.2$$

We also know that $\gamma = \frac{b}{m}$, which gives us

$$b = 72.3 \text{ kgs}^{-1}$$

7. Figure 2 shows a graph of displacement x as a function of time t for a damped harmonic oscillator. Estimate the quality factor Q of the oscillator.

Solution: I'll do this later.

8. The energy of simple harmonic oscillator reduces by a factor of 3 after 20 complete cycles.
   (a) By what factor will it reduce after 100 complete cycles?

   Solution:

   $$3^5 = 243$$

   (b) How many cycles are required to reduce the amplitude of the oscillator by a factor of 3

   Solution:

   $$20 \times 2 = 40$$

9. Figure 3 shows three systems of mass m attached to the light springs that all oscillate with the same frequency ω. Show that the spring constant of the springs for the three systems, $k_a : k_b : k_c = 1 : \frac{1}{2} : 2$.

Solution:

$$k_a = \omega^2 m$$

In the second system,

$$F = -k_b x - k_b x$$

Letting $k_1 = 2k_b$, we have

$$k_b = \frac{1}{2}k_1 = \frac{1}{2}\omega^2 m = \frac{1}{2}k_a$$

In the third system, let x_a and x_b denote the deformations of both springs. Then

$$F = k_c x_a = k_c x_b$$

This gives us $x_a = x_b$. Since their sum is x, we have

$$F = \frac{k_c}{2}x$$

Letting $k_2 = \frac{k_c}{2}$,

$$k_c = 2k_2 = 2\omega^2 m = 2k_a$$

This gives us the required ratio.

10. A mass stands on a platform which executes SHM in the vertical direction at a frequency $f = 2.5Hz$. Show that teh mass loses contact with the platform when the amplitude of the displacement exceeds 4.0cm.

> **Solution:** This happens when the vertical acceleration is greater than gravity.
>
> $$x(t) = A\cos(\omega t + \phi)$$
> $$v(t) = -A\omega \sin(\omega t + \phi)$$
> $$a(t) = -A\omega^2 \cos(\omega t + \phi)$$
>
> So the maximum value of acceleration is when x is at a maximum, which causes the mass to lose contact. Solving for angular frequency,
>
> $$\omega = 2\pi f = 5\pi$$
>
> Hence
>
> $$A \times 25\pi^2 = 9.81 \Rightarrow A = 0.040 \text{ m} = 4.0 \text{ cm}$$

11. An electrical current has the form

$$I_1(t) = 10\cos(\omega t + 1)$$

To synchronize this current with an external power source, and additional current $I_2(t)$ is added to $I_1(t)$, so that the combined current has a phase constant of zero:

$$I(t) = I_1(t) + I_2(t) = I_0\cos(\omega t)$$

(a) Show that for the amplitude $|I|$ of the final current $I(t)$ to be also 10.0A, the magnitude of current $|I_2|$ has to be 9.52A.

> **Solution:** $I_2(t)$ must also have the same frequency, as $I_2 = I - I_1$, where the left hand side is a periodic function with angular frequency ω. Hence we have
>
> $$10\cos(\omega t + 1) + A\cos(\omega t + \phi) = 10\cos\omega$$
>
> Expanding, we have
>
> $$10\cos\omega t\cos 1 - 10\sin\omega t\sin 1 + A\cos\omega t\cos\phi - A\sin\omega t\sin\phi = 10\cos\omega t$$
>
> We know that $\omega \neq 0$, so putting $t = 0$ tells us that the $\sin\omega t$ and $\cos\omega t$ terms are independent, which gives
>
> $$10\cos 1 + A\cos\phi = 10$$
>
> and
>
> $$-10\sin 1 - A\sin\phi = 0$$
>
> Rearranging and dividing the second equation by the first gives us
>
> $$\tan\phi = -1.83$$
>
> The second equation tells us $\sin\phi$ must be negative, meaning the only solution for ϕ is 5.21 rad. Substitution into the second equation again gives us $A = 9.59$.

(b) Show the smallest possible current amplitude $|I_2|$ that can be added to $I_1(t)$ to create a final current $I(t)$ with zero initial phase constant is $|I_2| = 8.41\text{A}$.

> **Solution:** We use the same approach as above, but substituting B for the unknown amplitude of I. We will still obtain
>
> $$-10\sin 1 - A\sin\phi = 0$$
>
> Considering A must be positive, and the maximum absolute value of $\sin\phi$ is 1, we have
>
> $$A = 10\sin 1 = 8.41$$

12. The potential energy $U(x)$ between two atoms in a diatomic molecule can be (approximately) expressed as

$$U(x) = -\frac{a}{x^6} + \frac{b}{x^{12}}$$

where x is the separation between the atoms and a and b are constants.

(a) Write an expression for the forces.

> **Solution:**
>
> $$\begin{aligned} F &= -\frac{dU}{dx} \\ &= -\frac{6a}{x^7} + \frac{12b}{x^{13}} \end{aligned}$$

(b) Show that the equilibrium separation x_0 of the atoms is $x_0 = \left(\frac{2b}{a}\right)^{\frac{1}{6}}$.

> **Solution:** At equilibrium, there is no net force, so $F = 0$, or
>
> $$\begin{aligned} \frac{6a}{x^7} &= \frac{12b}{x^{13}} \\ ax^6 &= 2b \\ x &= \left(\frac{2b}{a}\right)^{\frac{1}{6}} \end{aligned}$$

(c) Show that the system will oscillate with SHM when slightly displaced from equilibrium with angular frequency $\sqrt{\frac{k}{m}}$ where $k = 36a\left(\frac{a}{2b}\right)^{\frac{4}{3}}$.

> **Solution:**
>
> $$F'(x) = \frac{42a}{x^8} - \frac{156b}{x^{14}}$$

The first degree taylor approximation is then

$$
\begin{aligned}
F_{\text{approx}} &= F(x_0) + F'(x_0)(x - x_0) \\
&= 0 + \left(\frac{21a^2}{b} \times x_0^{-2} - \frac{39a^2}{b} \times x_0^{-2} \right) (x - x_0) \\
&= -18\frac{a^2}{b} \times \left(\frac{a}{2b} \right)^{\frac{1}{3}}
\end{aligned}
$$

In our SHM model, $F(x) = -kx$, so $F(0) = 0$. By substituting $x' = x - x_0$, the equation is in the form $F(x') = -kx'$, and we have

$$k = 18\frac{a^2}{b} \left(\frac{a}{2b} \right)^{\frac{1}{3}} = 36 \left(\frac{a}{2b} \right)^{\frac{4}{3}}$$