Lecture 5

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1 Cyclic Vectors

Suppose $n = \text{dim} V < \infty$. $v \in V$ is a cyclic vector for $T \in \mathcal{L}$, i.e.

$$v, Tv, T^2v, \dots$$

span V. Then

$$v_i = T^{i-1}v$$

is a basis of V, for i from 1 to n. In addition,

$$T^n v = -\sum_{i=1}^{n} a_{i-1} v_i$$

And T can be described by the composition matrix

$$A = \begin{pmatrix} 0 & 0 & \dots & -a_0 \\ 1 & 0 & \dots & -a_1 \\ 0 & 1 & \dots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -a_{n-1} \end{pmatrix}$$

Its characteristic polynomial is given by

$$q_T(z) = \det(zI_A)$$

Using the cofactor expansion along the last column for $C = zI - A$,

$$q_T(z) = \sum_{i=1}^{n} (-1)^{n+i} a_{i-1} \det(C^{[in]}) + z \det(C^{[in]})$$

Note that all of the terms involve the determinant of a matrix that can be represented as

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}$$

in block matrix form. Using this analogy, observe A' is lower triangular and A'' is upper triangular. The determinant is then given by

$$\det(C^{[in]}) = z^{i-1}(-1)^{n-i}$$

Summing the terms,

$$q_T(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

Note:
Given any monic polynomial $p(z)$ one can write down a matrix having $p(z)$ as the characteristic polynomial called the companion matrix.

Example 1.1.

$$p(z) = (z-1)^3 = z^3 - 3z^2 + 3z - 1$$

Then

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}$$

has this.

More on characteristic polynomials. Suppose $V = W_1 \oplus W_2$ with $T(W_i) \subseteq W_i$. Then

$$q_T(z) = q_{T|_{W_1}}(z) q_{T|_{W_2}}(z)$$

Reason: Choose basis for W_1, W_2 to form basis for V_1. Then $q_T(z)$ becomes the determinant of a block matrix.
In general, a T-invariant subspace $W_1 \subseteq V$ need not admit an invariant complement (i.e. there need not exist T-invariant W_2 such that $V = W_1 + W_2$)

Example 1.2. $V = \mathbb{R}^2, Te_1 = e_2, Te_2 = 0$. Then $W = ke_2$ is invariant, but there is no invariant complement.

If $W \subseteq V$ is T-invariant, then T induces quotient $\bar{T}$ on V/W where

$$\bar{T}(v + W) = Tv + W$$

Then

$$q_T(z) = q_{T|_W}(z) q_{\bar{T}}(z)$$

Proof:
T preserves W, so $zI - T$ also preserves W. Then

$$(zI_V - T)|_W = zI_W - T|_W$$

and

$$\overline{zI_V - T} = zI_{V/W} - \bar{T}$$

Thus from the homework problem,

$$\det(zI_V - T) = \det(zI_W - T|_W) \det(zI_{V/W} - \bar{T}) = q_{T|_W}(z) q_{\bar{T}}(z)$$

Theorem 1.1. *(Cayley-Hamilton) Let $T \in \mathcal{L}(V)$ with characteristic polynomial $q(z) = q_T(z)$. Then*

$$q(T) = 0$$

Proof:
Consider the first case that V has the cyclic vector v. The characteristic polynomial is hence

$$a_0 + a_1 z + a_2 z^2 + \dots a_{n-1} z^{n-1} + z^n$$

where the coefficients are defined as

$$T v_n = -a_0 v_1 - a_1 v_2 - \dots - a_{n-1} v_n$$

Rearranging yields

$$a_0 v + a_1 T v + \dots + a_{n-1} T^{n-1} v + T^n v = 0$$

Meaning $q(T)v = 0$. Then

$$q(T) v_2 = q(T) T v = T q(T) v = 0$$

where $q(T)$ and T commute as the former is a polynomial of T. Then this can be extended to v_n, which completes the proof if a cyclic vector exists. For the general case, we want to show $q(T)v = 0 \forall v \neq 0$. We know

$$v, Tv, T^2 v, \dots$$

gives a T-invariant subspace $W \subseteq V$ and v is a cyclic vector for $T|_W$. Then

$$q_T(z) = q_{T|_W}(z) q_{\bar{T}}(z)$$

so

$$q_T(T)v = q_{\bar{T}}(z) q_{T|_W}(T)v = q_{\bar{T}}(z) q_{T|_W}(T|_W)v = 0$$

There are other polynomials $p(z)$ with $p(T) = 0$. There's a unique one of lowest degree

$$I_V, T, T^2, T^3, \ldots, \in \mathcal{L}(V)$$

Let k be the smallest natural number such that $I_V, T, \ldots, T^k$ is linearly dependent. Then

$$T^k = -b_0 - b_1 T - \cdots - T^{k-1} \Rightarrow b_0 + b_1 T + \cdots + b_{k-1} T^{k-1} + T^k = 0$$

Hence

$$p(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} + z^n$$

satisfies $p(T) = 0$

Definition 1.1. The polynomial $p(z) = p_T(z)$ defined in this way is the minimal polynomial of T.

Example 1.3. If $T^2 = T$, then the minimal polynomial $p_T(z) = z^2 - z$ satisfies unless $T = I_V$ or $T = 0$.

Example 1.4. $T = I_V$ has the minimal polynomial

$$p(z) = z - 1$$

$T = 0_V$ has the minimal polynomal

$$p(z) = z$$

Example 1.5. $T \in \mathcal{L}(V)$ is called *involution* if $T^2 = I$. Its minimal polynomial is

$$p(z) = z^2 - 1$$

unless $T = I_V$ or $T = -I_V$

Example 1.6.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in M_{3\times 3}(\mathbb{C})$$

Then $A^2 = -I$ (trust me bro) and its minimal polynomial is $z^2 + 1$.

Theorem 1.2. *Let $q(z)$ be any monic polynomial with $q(T) = 0$. Then q is divisible by the minimal polynomial $p_T(z)$.*

Proof:
By long division, since the degree of q_T is lower than that of q

$$q(z) = p_T(z)u(z) + r(z)$$

where u, r are polynomials with the degree of r lower than that of p_T

$$0 = q(T) = p_T(T)u(T) + r(T) = r(T)$$

so $r = 0$, else p_T is not minimal. Then

$$q(t) = p_T(T)u(T)$$

which completes the proof.

Theorem 1.3. *Let $T \in \mathcal{L}(V)$, where the vector space V with dimension $n < \infty$ over $F = \mathbb{C}$. Then there exists a basis of V in which the matrix A of T has block diagonal form here each block A_i is of form*

$$A_i = \begin{pmatrix} \lambda_i & 1 & 0 & \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \\ 0 & 0 & \dots & \lambda_i \end{pmatrix}$$

The characteristic polynomial of such A is

$$q_T(z) = q_A(z) = q_{A_1}(z) \dots q_{A_k}(z) = (z - \lambda_1)^{r_1} \dots (z - \lambda_k)^{r_k}$$

For the minimal polynomial, consider first case of single Jordan block N. If $\lambda = 0$, then the diagonal "1"s in N "shift" to the top right to form a linearly independent matrix. Then obviously

$$p_N(z) = z^n$$

For the general Jordan block $\mathcal{J}$, note

$$\mathcal{J} - \lambda I = N$$

Thus

$$(\mathcal{J} - \lambda I)^n = 0$$

and

$$p_{\mathcal{J}}(z) = (z - \lambda)^n$$

Note that it is the minimal polynomial. Or else, the substitution $\mathcal{J} = N + \lambda I$ gives a lower degree polynomial that vanishes for N, which does not exist. Hence we have that the minimal polynomial agrees with the characteristic polynomial.
In general for $T \in \mathcal{L}(V)$ as above, the characteristic polynomial

$$q(T)(z) = (z - \lambda_1)^{r_1} \dots (z - \lambda_k)^{r_K}$$

where the eigenvalues may repeat. Then we can eliminate all terms with repeated eigenvalues with a smaller r_i.