

Lecture 7

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1 Taylor Series and Approximations for Two-Variable Functions

A first approximation for a function at somewhere near x_0 can simply be

$$f(x_0 + \Delta x) \approx f(x_0)$$

To make this a better approximation, we could consider the *tangent* at x_0 , which gives us the slightly more "accurate" approximation

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x$$

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We can then have a quadratic approximation

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2$$

and so on. In general, the n^{th} degree Taylor polynomial is given by

$$f(x_0 + \Delta x) = \sum_{i=1}^n \frac{1}{i!} f^{(i)}(x_0) \Delta x^i$$

Let us consider the two dimensional case. Let P be the known point, and Q be the point we wish to approximate. Then the parametric equations of line PQ is

$$\begin{aligned}x(t) &= x_0 + \Delta x \\ y(t) &= y_0 + \Delta y\end{aligned}$$

where $t \in [0, 1]$.

We then define

$$F(t) = f(x_0 + \Delta x, y_0 + \Delta y)$$

We now have a single variable function $F(t)$. Note that $F(0) = f(x_0, y_0)$ and $F(1) = f(x_0 + \Delta x, y_0 + \Delta y)$. We want to estimate $F(1)$.

$$F'(t) = \frac{d}{dt}F(t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

The second derivative is then

$$F''(t) = \frac{d}{dt} \left(\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right) = \frac{\partial^2 f}{\partial x^2} \frac{dx}{dt} \Delta x + \frac{\partial^2 f}{\partial y \partial x} \frac{dy}{dt} \Delta x + \frac{\partial^2 f}{\partial x \partial y} \frac{dx}{dt} \Delta y + \frac{\partial^2 f}{\partial y^2} \frac{dy}{dt} \Delta y$$

If Clairaut's Theorem holds,

$$F''(t) = \frac{\partial^2 f}{\partial x^2} \Delta x^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f}{\partial y^2} \Delta y^2$$

Applying Taylor's approximation on $F(t)$, we have

$$F(1) = \sum_{i=1}^n \frac{1}{i!} F^{(i)}(0)$$

where

$$F^{(n)}(0) = \sum_{i=1}^n \binom{n}{i} \frac{\partial^n f}{\partial x^i \partial y^{n-i}} \Delta x^i \Delta y^{n-i}$$

which expands to an ugly sum left to the reader as an exercise. The reader can also combine both equations into one (solving for $F(1)$) or touch grass. Note that the first order approximation gives us the tangent plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1.1. Find the 2nd degree polynomial approximation to the function $f(x, y) = \sqrt{x^2 + y^3}$ near $(1, 2)$.

$$f(1, 2) = 3$$

$$f_x = \frac{x}{\sqrt{x^2 + y^3}} \rightarrow f_x(1, 2) = \frac{1}{3}$$

$$f_y = \frac{3y^2}{2\sqrt{x^2 + y^3}} \rightarrow f_y(1, 2) = 2$$

$$f_{xx} = \frac{y^3}{(x^2 + y^3)^{\frac{3}{2}}} \rightarrow f_{xx}(1, 2) = \frac{8}{27}$$

$$f_{xy} = -\frac{3xy^2}{2(x^2 + y^3)^{\frac{3}{2}}} \rightarrow f_{xy}(1, 2) = -\frac{2}{9}$$

$$f_{yy} = \frac{12x^2y + 3y^4}{4(x^2 + y^3)^{\frac{3}{2}}} \rightarrow f_{yy}(1, 2) = \frac{2}{3}$$

Substituting into the formula, we have

$$f(x, y) \approx 3 + \frac{1}{3}(x - 1) + 2(y - 2) + \frac{4}{27}(x - 1)^2 - \frac{2}{9}(x - 1)(y - 2) + \frac{1}{3}(y - 2)^2$$

Example 1.2. Find the third order Taylor Expansion of $f(x, y) = e^{x-2y}$ about $(0, 0)$.

The formula gives us

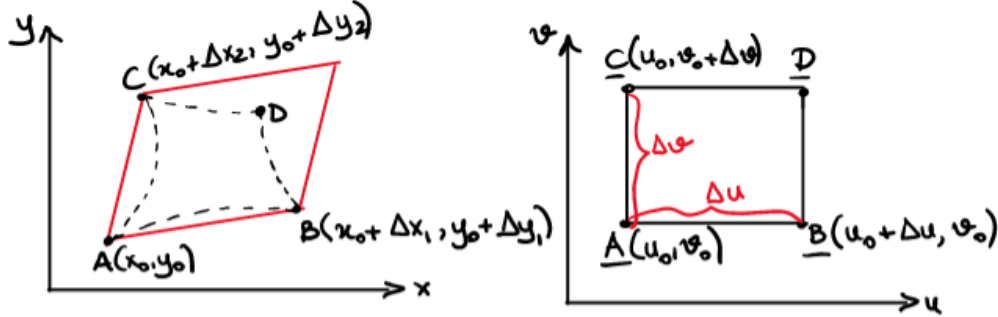
$$f(x, y) = 1 + x - 2y + \frac{1}{2}(x^2 - 4xy + 4y^2) + \frac{1}{6}(x^3 - 6x^2y + 12xy^2 - 8y^3)$$

2 Change of Variables

In u substitution, we let u be a function of x to simplify integrations.

Example 2.1.

$$\int_1^3 2x\sqrt{x^2 + 1}dx = \int_2^{10} \sqrt{u}du$$



Loosely speaking, we need a $\frac{dx}{dt}$ term to "scale" the integral. Consider 2 different partitions of the same region, one in squares (x and y) and the other in parallelograms (p and q). If we simply convert between $dx dy$ and $dp dq$, we will be off by a scale determined by the ratio between the areas of $||\Delta x \times \Delta y||$ and $||\Delta p \times \Delta q||$ where \times denotes the cross product.

From Fig 2, we let

$$x = g(u, v)$$

and

$$y = h(u, v)$$

We have

$$\Delta x_1 = x_0 + x_1 - x_0 = g(u_0 + \Delta u, v_0) - g(u_0, v_0) = g_u(u_0, v_0) \Delta u$$

Similarly,

$$\Delta x_2 = g_v(u_0, v_0) \Delta v$$

$$\Delta y_1 = h_u(u_0, v_0) \Delta u$$

$$\Delta y_2 = h_v(u_0, v_0) \Delta v$$

The area in the xy plane is then

$$\begin{aligned} ||AB \times AC|| &= ||(\Delta x_1 \hat{i} + \Delta y_1 \hat{j}) \times (\Delta x_2 \hat{i} + \Delta y_2 \hat{j})|| \\ &= |\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1| \\ &= |g_u \Delta u h_v \Delta v - g_v \Delta v h_u \Delta u| \\ &= \left| \det \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix} \Delta u \Delta v \right| \end{aligned}$$

Definition 2.1. We define the **Jacobian** as

$$J = \left| \det \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix} \right| = \left| \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \right| = \frac{\partial(x, y)}{\partial(u, v)}$$

We can then use the Jacobian to change the bases of integrals.