Lecture 1

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1 Traces and Determinants

1.1 Notation

- $\mathbb{F}$ is a field, usually $\mathbb{R}, \mathbb{C}$, sometimes $\mathbb{Z}_p$, where p is a prime
- $\mathcal{L}(V, W)$ is the set of linear maps from V to W
- Given bases of $V \cong \mathbb{F}^n, W \cong \mathbb{F}^m$, $\mathcal{L}(V, W) \cong \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = M_{m\times n}(F)$
- Special case: $W = V \Rightarrow \mathcal{L}(V) = \mathcal{L}(V, V)$

Definition 1.1. The trace of $A \in M_{n\times n}(F)$ is

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

Lemma 1.1.

$$A, B \in M_{n\times n}(F) \Rightarrow \mathrm{tr}(AB) = \mathrm{tr}(BA)$$

Proof:

$$\begin{aligned}\mathrm{tr}(AB) &= \sum_i \sum_k A_{ik} B_{ki} \\ &= \sum_k \sum_i B_{ki} A_{ki} \\ &= \mathrm{tr}(BA)\end{aligned}$$

Consequentially,

$$\mathrm{tr}(CAC^{-1}) = \mathrm{tr}(CC^{-1}A) = \mathrm{tr}(A)$$

Thus we can define

Definition 1.2. For $T \in \mathcal{L}(V)$, we define

$$\operatorname{tr}(T) = \operatorname{tr}(A)$$

where $A \in M_{n \times n}(F)$ is the matrix of T with basis V. (From above, the trace is invariant under change of basis)

Remember

If $\mathbb{F} = \mathbb{C}$, we can choose a basis such that

$$A = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix}$$

where $\lambda_i \in \mathbb{C}$, thus

$$\operatorname{tr}(T) = \operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i$$

Lemma 1.2. *The inverse of*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and it exists iff the determinant is nonzero.
Proof: trivial
In fact, define

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and if $AB = 0$, A cannot have an inverse, else

$$A^{-1}(AB) = 0 = B$$

implying $A = 0$, where counterexamples can easily be found.

Example 1.1.

$$
\begin{aligned}
2x_1 + 3x_2 &= 4 \\
2x_1 + x_2 &= 3
\end{aligned}
$$

Then

$$
\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}
$$

Plugging in the formula of inverse matrix yields

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \end{pmatrix}
$$

1.2 Area Calculations

Consider $\mathbb{F} = \mathbb{R}$. Given $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, we want to compute the volume of the parallelpiped, which is the area of the parallelogram for $n = 2$.
We denote this by $\mathrm{Vol}(v_1, v_2)$. The properties are as follows

- $\mathrm{Vol}(av_1, v_2) = a\mathrm{Vol}(v_1, v_2) = \mathrm{Vol}(v_1, av_2)$
- $\mathrm{Vol}(v_1 + av_2, v_2) = \mathrm{Vol}(v_1, v_2)$ (as the perpendicular height is the same)
- $\mathrm{Vol}(e_1, e_2) = 1$

The above properties are sufficient to derive a formula for the area.

Lemma 1.3. *Vol is bilinear.*
Proof: putting $a = 0$ in the first property, $\mathrm{Vol}(0, v_2) = 0 = \mathrm{Vol}(v_1, 0)$*. Then putting $v_1 = 0, a = 1, v_2 = v$ into the second property,* $\mathrm{Vol}(v, v) = 0$*. For linearity in the first argument, we want*

$$
\mathrm{Vol}(v_1 + v_1', v_2) = \mathrm{Vol}(v_1, v_2) + \mathrm{Vol}(v_1', v_2)
$$

If $v_2 = 0$, both sides are 0, so we assume $v_2 \neq 0$. If $v_1 = av_2$, we use the second property. Now knowing that v_1 and v_2 are linearly independent, we can rewrite v_1' as

$$
v_1' = \lambda v_1 + \mu v_2
$$

Thus

$$\begin{aligned}
\mathrm{Vol}(v_1 + v_1', v_2) &= \mathrm{Vol}(v_1 + \lambda v_1 + \mu v_2, v_2) \\
&= \mathrm{Vol}((1+\lambda)v_1, v_2) \\
&= (1+\lambda)\mathrm{Vol}(v_1, v_2) \\
&= \mathrm{Vol}(v_1, v_2) + \mathrm{Vol}(\lambda v_1, v_2) \\
&= \mathrm{Vol}(v_1, v_2) + \mathrm{Vol}(\lambda v_1 + \mu v_2, v_2) \\
&= \mathrm{Vol}(v_1, v_2) + \mathrm{Vol}(v_1', v_2)
\end{aligned}$$

Using the above, we know Vol is skew symmetric, as

$$\mathrm{Vol}(v, v) = 0$$

where expanding under the substitution $v = v_1 + v_2$ yields the desired result. Conversely, this implies $\mathrm{Vol}(v, v) = 0$.

Proposition 1.1. *Let $v_1, v_2 \in \mathbb{R}^2$ be the columns of $A \in M_{2\times 2}(\mathbb{R})$. Then $\mathrm{Vol}(v_1, v_2) = \det(A)$.*
Proof: write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$\begin{aligned}
\mathrm{Vol}(v_1, v_2) &= \mathrm{Vol}(ae_1 + ce_2, be_1 + de_2) \\
&= a\mathrm{Vol}(e_1, be_1 + de_2) + c\mathrm{Vol}(e_2, be_1 + de_2) \\
&= -ad\mathrm{Vol}(e_2, e_1) - bc\mathrm{Vol}(e_1, e_2) \\
&= ad - bc \\
&= \det(A)
\end{aligned}$$

Theorem 1.1. *There eixsts a unique multi-linear functional*

$$\det : (\mathbb{F}^n)^n \to F$$

such that

- $\det(v_1, v_2, \ldots, v_n) = 0$ *if* $v_i = v_j$ *for* $i \neq j$
- $\det(e_1, e_2, \ldots, e_n) = 1$

We know that to be a multilinear map,

$$\det(v_1, \ldots, \lambda v_i, \ldots, v_n) = \lambda \det(v_1, \ldots, v_i, \ldots, v_n)$$

and

$$\det(v_1, \ldots, v_i + v_i', \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$$

We have proved this for $n = 2$.

Definition 1.3. A permutation of a finite set X is a bijection $\sigma : X \to X$. We will just consider $X = \{1, \ldots, n\}$, so permutations go by $\sigma(1), \sigma(2), \ldots, \sigma(n)$. Then there are $n!$ permutations, and other permutations are $(\sigma(1), \ldots, \sigma(n))$. A permutation is even if the number of pairs in wrong order is even ,i.e. the cardinality of

$$\{(\sigma(i), \sigma(j)) | i < j, \sigma(i) > \sigma(j)\}$$

In this case we write $\text{sign}(\sigma) = 1$. Odd permutations are defined similarly, with $\text{sign}(\sigma) = -1$. Thus for $\sigma = \{4, 3, 1, 2\}$, $\text{sign}(\sigma) = -1$ (trust me bro).

Observation:
If σ' is obtained from σ by interchanging two adjacent elements then $\text{sign}(\sigma') = -\text{sign}(\sigma)$. (Reason: one pair is corrected/wronged, and the other pairs do not change). Extending this, this holds for switching any pair (induction probably works). Hence, one can compute $\text{sign}(\sigma)$ by putting everything in right order through transpositions.

Example 1.2.

$$(4312) \to (1342) \to (1243) \to (1234)$$

There are 3 switches, so its sign is -1.

Another remark on permutation:
The set S_n of permutations is a group with group multiplication $\sigma \cdot \sigma' = \sigma \times \sigma'$. We define sign: $S_n \to \{1, -1\}$, and we have $\text{sign}(\sigma \times \sigma') = \text{sign}(\sigma) \times \text{sign}(\sigma')$. That is, sign is a group homomorphism.