Lecture 9

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1 Examples on the Change of Variables in Multiple Integrals

Example 1.1. Evaluate $\int_R e^{(x+y)/(x-y)} dA$ where R is the trapezoidal region with vertices $(1,0), (2,0), (0,-2)$, and $(0,-1)$.
If we let

$$u = x + y$$
$$v = x - y$$

The boundaries will be v from 1 to 2, and u from $-v$ to v. The Jacobian is then

$$
\begin{aligned}
J &= \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \\
&= \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}^{-1} \\
&= \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\
&= (-2)^{-1} \\
&= -\frac{1}{2}
\end{aligned}
$$

Then the integral is given by

$$I = \iint_S e^{\frac{u}{v}} \times \frac{1}{2} du dv$$

$$= \int_1^2 \int_{-v}^v \frac{1}{2} e^{\frac{u}{v}} du dv$$

$$= \frac{1}{2} \int_1^2 v(e - e^{-1}) dv$$

$$= \frac{1}{2}(e - e^{-1})$$

Example 1.2. Evaluate $\iint_R (x^2 - y^2) e^{xy} dx dy$, where the region R is the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 2$ and the lines $y = x$ and $y = x + 2$.

Note that substitution for $x^2 - y^2$ and xy doesn't work (try it!). Instead, we use

$$u = xy$$

$$v = y - x$$

The Jacobian is given by

$$J = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}^{-1}$$

$$= \det \begin{bmatrix} y & x \\ -1 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{x + y}$$

The integral is then

$$I = \int_0^2 \int_1^4 (x^2 - y^2) e^{xy} \times \frac{1}{x+y} du dv$$

$$= \int_0^2 \int_1^4 (x - y) e^{xy} du dv$$

$$= \int_0^2 \int_1^4 -v e^u du dv$$

$$= \int_0^2 v(e - e^4) dv$$

$$= 2(e - e^4)$$

Example 1.3. Find the volume of the region bounded by the hyperbolic cylinders $xy = 1, xy = 9, xz = 4, xz = 36, yz = 25, yz = 49$. We then let $u = xy, v = xz, w = yz$. The Jacobian is then

$$
J = \det \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}^{-1}
$$

$$
= \det \begin{bmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}^{-1}
$$

$$
= -\frac{1}{2xyz}
$$

$$
= -\frac{1}{2\sqrt{uvw}}
$$

The integral is then given by

$$
I = \int_{25}^{49} \int_{4}^{36} \int_{1}^{9} \frac{du\,dv\,dw}{2\sqrt{uvw}}
$$

$$
= 4(7 - 5)(6 - 2)(3 - 1)
$$

$$
= 64
$$

2   Line Integrals

We can integrate along lines (not necessarily straight lines) by

$$
\int_C f(x, y)\,ds
$$

This can be evaluated by parametrisation

$$
\begin{cases} x = x(t) & t \in [a, b] \\ y = y(t) & t \in [a, b] \end{cases}
$$

We assume that $f(x, y)$ is continuous over C and that C is smooth, or $\vec{r}'(t)$ is continuous and $\vec{r}(t) \neq \vec{0}$ except at the endpoints. We can then express ds as

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt
$$

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Note that the direction of integration does not matter if f is a scalar function, unlike single integration.

Example 2.1. Find the centre of mass of a semi-circular length of wire $y = \sqrt{a^2 - x^2}, a > 0$. Length density is constant.
By symmetry, $\bar{x} = 0$.
Paramatrisation gives us $x(t) = a \cos t$ and $y(t) = a \sin t$. Then

$$ds = \sqrt{(-a \sin t)^2 + (a \cos t)^2}dt = a\,dt$$

$$m\bar{y} = \int_C y\rho\,ds$$
$$a\pi\bar{y} = \int_0^\pi a \sin t \times a\,dt$$
$$\bar{y} = \frac{a}{\pi} \times 2$$
$$= \frac{2a}{\pi}$$

Example 2.2. In the special case where C is parallel to the x axis, e.g. from $(a, 0)$ to $(b, 0)$, this can be integrated normally, as $ds = dx$.

2.1   3 Dimensional Case

As expected, ds is now expressed as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Example 2.3. Find the mass of a spring in the shape of the circular helix defined parametrically by $x2 \cos t$, $y = t$, $z = 2 \sin t$ for $t \in [0, 6\pi]$ with density of $\rho(x, y, z) = 2y$.

$$ds = \sqrt{(-2 \sin t)^2 + 1^2 + (2 \cos t)^2}\,dt = \sqrt{5}\,dt$$

The mass is then given by

$$m = \int_C \rho ds$$
$$= \int_C 2y ds$$
$$= \int_0^{6\pi} 2t \times \sqrt{5} dt$$
$$= 36\sqrt{5}\pi^2$$

2.2   Piecewise smooth curves

Our curve is now

$$C = \bigcup_i C_i$$

where C may not be smooth but C_i is always smooth. Then

$$\int_C f ds = \sum_i \int_{C_i} f ds$$

2.3   Line Integrals of Vector Fields

$$\vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

This can be rewritten as

$$\vec{F}(x, y, z) = \vec{F}(\vec{r})$$

Example 2.4. Physical Examples

$$W = \vec{F} \cdot \vec{d}$$

where W stands for work done. Then

$$W = \int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{r}' dt$$

This can also be written as

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b (P\hat{i} + Q\hat{j} + R\hat{k}) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) dt$$
$$= \int_C P dx + Q dy + R dz$$

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