

Lecture 4

niceguy

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1 Review

1.1 Solving Separable Equations

Consider

$$\frac{du}{dt} = f(u)g(t)$$

This is solved by

$$\begin{aligned}\frac{1}{f(u)} \frac{du}{dt} &= g(t) \\ \int \frac{1}{f(u)} du &= \int g(t) dt + C, C \in \mathbb{R}\end{aligned}$$

Note: it might not be possible to find an explicit expression for u .

Example 1.1.

$$\begin{aligned}\frac{du}{dt} &= u^2, u(1) = -1 \\ \frac{1}{u^2} \frac{du}{dt} &= 1 \\ \int \frac{1}{u^2} du &= \int 1 dt \\ -\frac{1}{u} &= t + C \\ u &= -\frac{1}{t + C}, C \in \mathbb{R}\end{aligned}$$

where $u(1) = -1 \Rightarrow C = 0, u = -\frac{1}{t}$. This is a local solution, as the interval $(-\infty, 0)$ is not included. Consider

$$f(t) = \begin{cases} -\frac{1}{t} & t \in (0, \infty) \\ -\frac{1}{2+t} & t \in (-\infty, 0] \end{cases}$$

There is no unique solution for that interval.

2 First Order Linear Equations

$$a_0(t)u(t) + a_1(t)u'(t) = h(t)$$

Assuming $a_1(t) \neq 0$, it can be rearranged as

$$u'(t) + p(t)u(t) = q(t)$$

where $p(t)$ and $q(t)$ are the quotients.

Using the product rule, we have

$$\frac{d}{dt}\mu(t)u(t) = \mu'(t)u(t) + \mu(t)u'(t)$$

Factoring $/\mu(t)$ out, we have

$$\mu(t) \left(\frac{\mu'(t)}{\mu(t)}u(t) + u'(t) \right)$$

If $\frac{\mu'(t)}{\mu(t)} = p(t)$ and $\mu(t) \neq 0$, this can be solved conveniently. Assuming this is true,

$$\frac{d}{dt}(\mu(t)u(t)) = \mu(t)g(t)$$

Integrating both sides, we can solve for u by

$$u(t) = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}$$

What remains is to solve for $\mu(t)$.

$$\begin{aligned}\frac{1}{\mu} \frac{d\mu}{dt} &= p(t) \\ \int \frac{1}{\mu} d\mu &= \int p(t) dt \\ \ln |\mu| &= \int p(t) dt \\ \mu &= \pm e^{\int p(t) dt}\end{aligned}$$

Note that the \pm term and C terms are constants that are on both sides. Therefore, they can be ignored. In addition, it is an exponential, so it can never be 0.

Hence, defining $\mu(t) = e^{\int p(t) dt}$, the solution is given by

$$u(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^t \mu(s) q(s) ds + C \right)$$

$\mu(t)$ is called the **integrating factor**.

Example 2.1.

$$(t^2 - 1)u' + 2tu = t, u(0) = 1$$

Putting this into standard form,

$$u' + \frac{2t}{t^2 - 1}u = \frac{t}{t^2 - 1}$$

The integrating factor is given by

$$\mu(t) = t^2 - 1$$

Then

$$(t^2 - 1)u(t) = \frac{t^2}{2} + C$$

And rearranging gives us

$$u(t) = \frac{t^2}{2(t^2 - 1)} + \frac{C}{t^2 - 1}$$

Substituting the initial conditions,

$$u(t) = \frac{t^2}{2(t^2 - 1)} - \frac{1}{t^2 - 1}$$