Lecture 16

niceguy

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1 Recap

The Poission distribution is

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

The exponential distribution is

$$f(x;\beta) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$$

Where $\beta = \frac{1}{r}$, the time between events.

Example 1.1. In a factory, components fail every 4 days on average. What is the chance a component lasts longer than a week? Note that the exponential distribution is used.
We have $\beta = 4$. Then

$$P(X \geq 7) = \int_7^\infty \frac{1}{4}e^{-\frac{x}{4}}dx = e^{-\frac{7}{4}} \approx 0.17$$

Note that we expect $\frac{7}{4}$ failures in one week.

2 Memoryless Proposition

Given a random variable with exponential distribution,

$$P(X \geq s + t | X \geq s)$$

By definition of conditional probability, this is equal to

$$\frac{P(X \geq s+t)}{P(X \geq s)}$$

Now

$$P(X \geq s) = \int_s^\infty \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = e^{-\frac{s}{\beta}}$$

Then the quotient is $e^{-\frac{t}{\beta}}$, or $P(X \geq t)$.

3 Functions of Random Variables

For the discrete case, consider X with distribution $f(x)$. Let $Y = u(X)$, where u is bijective. Then we can write $X = u^{-1}(Y)$. Let $g(y)$ the the distribution of y.

$$\begin{aligned}
g(y) &= P(Y = y) \\
&= P(u^{-1}(Y) = u^{-1}y) \\
&= P(X = u^{-1}y) \\
&= P(X = u^{-1}y) \\
&= f(u^{-1}(y))
\end{aligned}$$

For the continuous case, let $G(y)$ be the cumulative distribution. Then

$$\begin{aligned}
G(y) &= P(Y \leq y) \\
&= P(u^{-1}(Y) \leq u^{-1}(y)) \\
&= P(X \leq u^{-1}(y))
\end{aligned}$$

Then

$$\begin{aligned}
g(y) &= \frac{d}{dy} G(y) \\
&= \frac{d}{dy} \int_{-\infty}^{u^{-1}(y)} f(t) dt \\
&= f(u^{-1}(y)) \frac{du^{-1}(y)}{dy}
\end{aligned}$$

u is either strictly increasing or strictly decreasing. In the latter case, we have an extra negative sign in the slope, giving us

$$g(y) = f(u^{-1}(y)) \left| \frac{du^{-1}(y)}{dy} \right|$$