Lecture 11

niceguy

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1 Fundamental Theorem for Line Integrals

If $\vec{F}$ is a gradient of a scalar function f, then

$$\int_a^b \vec{F} \cdot d\vec{r} = f(\vec{r_b}) - f(\vec{r_a})$$

. If $C = \bigcup_i C_i$ and $C' = \bigcup_i C_i'$ where both share the same endpoints, we can see that

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{f} \cdot d\vec{r}$$

In fact, if the curve starts and ends at the same point, we call this a *closed* integral where

$$\oint \vec{F} \cdot d\vec{r} \equiv 0$$

Conversely, if this is true for any closed curve C, we know that it must be path independent. (Consider two paths C_1 and C_2 with the same endpoints. Then their difference is a closed curve which is zero, implying they are equal). To sum up, the following statements are equal

1. $\vec{F}$ is conservative
2. the integral of $\vec{F} \cdot d\vec{r}$ is path independent
3. $\oint \vec{F} \cdot d\vec{r} \equiv 0$

Consider $\vec{F} = P(x,y)\hat{i} + Q(x,y)\hat{j}$. If we assume $P(x,y) = \frac{\partial f}{\partial x}$ and $Q(x,y) = \frac{\partial f}{\partial y}$, we have

$$\frac{\partial P}{y} = \frac{\partial Q}{x}$$

assuming mixed partials commute. Hence the above equation can be used to verify if $\vec{F}$ is conservative or not (proof not given).

Example 1.1.

$$\vec{F}(x,y) = y\hat{i} - x\hat{j}$$

$\vec{F}$ is not conservative, as the mixed partials are not equal.

Example 1.2.

$$\vec{f}(x,y) = y\hat{i} + x\hat{j}$$

$\vec{F}$ is conservative, as the mixed partials are equal. Integrating P and Q and using $g(y)$ and $g(x)$ as "constant" functions, we get $f = xy + C$ where $g(x) = g(y) = C$ by comparison.

2 Green's Theorem

2.1 Terminology

- Simple curve: a curve that does not intersect itself except at endpoints

- Orientation: orientation is positive when curve goes counterclockwise, vice versa

2.2 Theorem and Proof

Theorem 2.1. *Let be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let R be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ are continuous and have continuous first partial derivatives throughout the region R, then*

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Proof:
We can express the region as $C = C_1 \cup C_2$ or $C = C_3 \cup C_4$ where

$$R = \{(x, y) | a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\}$$

and

$$R = \{(x, y) | x_1(y) \leq x \leq x_2(y), c \leq y \leq d\}$$

where a, b are the left and right endpoints, and c, d are the top and bottom endpoints. We can write

$$\begin{aligned}
\oint_C P(x, y)dx &= \int_{C_1} P(x, y)dx + \int_{C_2} P(x, y)dx \\
&= \int_a^b P(x, y_1(x))dx + \int_b^a P(x, y_2(x))dx \\
&= -\int_a^b P(x, y_2(x)) - P(x, y_1(x))dx \\
&= -\int_a^b P(x, y)\Big|_{y=y_1(x)}^{y=y_2(x)} dx \\
&= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy dx
\end{aligned}$$

Similarly,

$$\begin{aligned}
\oint_C Q(x, y)dy &= \int_{C_3} Q(x, y)dy + \int_{C_4} Q(x, y)dy \\
&= \int_d^c Q(x_1(y), y)dy + \int_c^d Q(x_2(y), y)dy \\
&= \int_c^d Q(x_2(y), y) - Q(x_1(y), y)dy \\
&= \int_c^d Q(x, y)\Big|_{x=x_1(y)}^{x=x_2(y)} dy \\
&= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x} dx dy
\end{aligned}$$

Summing both gives us

$$\oint Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

2.3 Examples

Example 2.1. Verify Green's Theorem for the integral $\oint_C ydx - xdy$ where C is the circle $C : x^2 + y^2 = 1$ traversed counterclockwise.
The left hand side gives

$$\begin{aligned}
I &= \oint \vec{F} \cdot d\vec{r} \\
&= \int_0^{2\pi} (\sin t\hat{i} - \cos t\hat{j}) \cdot (-\sin t\hat{i} + \cos t\hat{j}) dt \\
&= \int_0^{2\pi} -\sin^2 t - \cos^2 t dt \\
&= -2\pi
\end{aligned}$$

The right had side gives

$$\begin{aligned}
I &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \iint_R -2dA \\
&= -2\pi
\end{aligned}$$

as the area of a unit circle is π.

Example 2.2.

$$I = \oint_C \left(4 - e^{\sqrt{x}}\right) dx + \left(\sin y + 3x^2\right) dy$$

where C goes counter clockwise around the disk bounded by $a \leq r \leq b$ in the first quadrant.

$$\begin{aligned}\oint_C \left(4 - e^{\sqrt{x}}\right) dx + \left(\sin y + 3x^2\right) dy &= \iint_R 6x dA \\ &= \int_0^{\frac{\pi}{2}} \int_a^b 6r\cos\theta r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} 2(b^3 - a^3)\cos\theta d\theta \\ &= 2(b^3 - a^3)\end{aligned}$$

3 Parametric Surfaces and their Surface Areas

Just as how a curve can be parametrised by one variable, a surface can be parametrised by two variables, where

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

The simplest way to do this is by $S : z = f(x,y)$ where $u = x$ and $v = y$.

Example 3.1. Parametrise an upper hemisphere given by the equation $x^2 + y^2 + z^2 = a^2$.
Rearranging, we have

$$z = \sqrt{a^2 - x^2 - y^2}$$

so we have

$$\vec{r}(u,v) = u\hat{i} + v\hat{j} + \sqrt{a^2 - u^2 - v^2}\hat{k}$$

We can also use spherical coordinates which gives

$$\vec{r}(u,v) = a\vec{r} + u\hat{\theta} + v\hat{\phi}$$

where $u \in [0, 2\pi], v \in \left[0, \frac{\pi}{2}\right]$.

4 Tangent Planes

Let S be a surface parametrised by the differentiable vector function

$$\vec{r}(u,v) = (u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

where $(u, v) \in D$.
Then the tangent plane at (u_0, v_0) is spanned by

$$\vec{r_v}(u) = \frac{\partial \vec{r}}{\partial v}\Big|_{u_0, v_0}$$

and

$$\vec{r_u}(v) = \frac{\partial \vec{r}}{\partial u}\Big|_{u_0, v_0}$$

Definition 4.1. A surface is *smooth* if for every point,

$$\vec{r_v}(u) \times \vec{r_u}(v) \neq 0$$