Lecture 11

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1 Ideal Gas

$$S(E,N,V) = k \ln \Omega(E,N,V)$$

and

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N}$$

We want to show this for the ideal gas.

$$U = \sum_i \frac{\vec{p_i}^2}{2m} \Rightarrow \sum_i \vec{p_i}^2 = 2m\vec{U}$$

Now for a sphere in n dimensions,

$$\sum_i x_i^2 = R^2$$

Both are similar, i.e. $2mU \propto R^2$, in the sense that they are the sum of squares. Using this analogy, we have a *radius* for $\vec{p_i}$. Now, the area of S^{3N-1} of radius R is

$$A = \frac{2\pi^{\frac{3N}{2}}}{\left(\frac{3N}{2}-1\right)!} R^{3N-1}$$

Now, it seems that

$$\Omega(N,U) \propto \frac{2\pi^{\frac{3N}{2}}}{\left(\frac{3N}{2}-1\right)!} (2mU)^{\frac{3N-1}{2}}$$

Note that $2mU$ shares the same units as $\vec{p}^2$, which is $\frac{\pi\hbar}{L}$. To nondimensionalise this,

$$\Omega(N,U) \propto \frac{2\pi^{\frac{3N}{2}}}{\left(\frac{3N}{2}-1\right)!}\left(\frac{\sqrt{2mU}}{\pi\hbar/L}\right)^{3N-1}$$

Secondly, since all $\vec{p_i}$ are positive, we need a factor of $\frac{1}{2^{3N}}$.

$$\Omega(N,U) \propto \frac{1}{2^{3N}}\frac{2\pi^{\frac{3N}{2}}}{\left(\frac{3N}{2}-1\right)!}\left(\frac{\sqrt{2mU}}{\pi\hbar/L}\right)^{3N-1}$$

There is also the indistinguishability of the particles. We only have access to the different $\vec{p_i}$, but not the corresponding particle. There are $N!$ permutations which we have to divide by. Then

$$\Omega(N,U) \propto \frac{1}{2^{3N}N!}\frac{2\pi^{\frac{3N}{2}}}{\left(\frac{3N}{2}-1\right)!}\left(\frac{\sqrt{2mU}}{\pi\hbar/L}\right)^{3N-1}$$

Using more approximations,

$$\begin{aligned}
\Omega(N,U) &\approx \left(\frac{e}{N}\right)^N \frac{1}{8^N}\frac{2(\pi^{1.5})^N}{\left(\frac{3N}{2e}\right)^{1.5N}}((2mU)^{1.5})^N\left(\frac{V}{(\pi\hbar)^3}\right)^N \\
&\approx \left[\frac{e^{1.5}}{N}\frac{1}{8}\frac{\pi^{1.5}2^{1.5}}{3^{1.5}}\frac{V}{(\pi\hbar)^3}\frac{2^{1.5}m^{1.5}}{N^{1.5}}U^{1.5}\right]^N \times 2 \\
&= 2\left[e^{1.5}\frac{V}{N}\frac{(4\pi mU)^{1.5}}{(3N(2\pi\hbar)^2)^{1.5}}\right]^N \\
\ln\Omega(N,U) &\approx kN\left[\ln\left(\frac{(4\pi mU)^{1.5}}{(3N(2\pi\hbar)^2)^{1.5}}\frac{V}{N}\right)+\frac{5}{2}\right]
\end{aligned}$$

Differentiating,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} = kN\frac{3}{2}\frac{1}{U} \Rightarrow U = \frac{3}{2}kNT$$

We conclude that

- Entropy is extensive (if V, N, U are all doubled, so is S)
- Entropy is huge, because of the factor of N
- U increases with S
- $C_V > 0$