Lecture 7

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1 Random Variables

Example 1.1. Consider 3 coin flips, where X denotes the number of heads. Then

$$P(X=0)=\frac{1}{8}, P(X=1)=\frac{3}{8}, P(X=2)=\frac{3}{8}, P(X=3)=\frac{1}{8}$$

Definition 1.1. $f(x)$ is a probability mass function (PMF) of the discrete random variable X if

- $f(x) \geq 0 \forall x \in S$
- $\sum_{x \in S} f(x) = 1$
- $P(X=x) = f(x)$

Example 1.2. From 1.1, we can write

$$f(0) = f(3) = \frac{1}{8}$$

and

$$f(1) = f(2) = \frac{3}{8}$$

Definition 1.2. $F(x)$ is a cumulative distribution function (CDF) of discrete random variables if

$$F(x) = \sum_{t \leq x} f(t)$$

hence

$$F(x) = P(X \leq x)$$

Example 1.3. From 1.1,

$$F(-1) = 0, F(0) = \frac{1}{8}, F(1) = \frac{1}{2}, F(2) = \frac{7}{8}, F(3) = 1$$

Abusing notation we have (the limit is implied)

$$F(-\infty) = 0, F(\infty) = 1$$

Consider if $X \in \mathbb{R}$ is a continuous random variable, then

$$P(X = 5) = 0$$

but

$$P(4 \leq X \leq 6) > 0$$

in general.

Definition 1.3. $f(x)$ is a probability distribution/density function (PDF) of the continuous random variable X if

- $f(x) \geq 0 \forall x$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = \int_a^b f(x)dx$

Definition 1.4. $F(x)$ is the cumulative density function of X if

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

Observe that these identities still hold

$$F(x) = P(X \leq x)$$

$$F(\infty) = 1$$

In addition,

$$P(a \leq X \leq b) = F(b) - F(a)$$

Example 1.4. Triangle PDF. X is a continuous random variable whose probability density function is

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Then

$$\begin{aligned} P(0.5 \le X \le 1) &= \int_{0.5}^{1} x dx \\ &= \frac{x^2}{2}\Big|_{0.5}^{1} \\ &= \frac{3}{8} \end{aligned}$$

The cumulative density function can be derived.

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ -1 + 2x - \frac{x^2}{2} 1 \le x \le 2 & \\ 1 & x \ge 2 \end{cases}$$