Lecture 6

Tensors

- A collection of properties that do not depend on a basis
- Example:

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

  which can be written more compactly as

$$J_1 = \sigma_{ij} E_j$$

  where the rank is equal to the number of subscripts. The principal components are the diagonal elements σ_{ii}.

Anisotropic case

E is along the x axis, so

$$J_1 = \sigma_{11} E_1$$
$$J_2 = \sigma_{21} E_2$$
$$J_3 = \sigma_{31} E_3$$

where J_1 is the principal component, and the rest are transverse components.

Einstein Convention If the same dummy variable appears more than once, a summation is implied

Transformation of Axes

From Cartesian Coordinates, the transformation (rotation) matrix is given by $A = [a_{ij}]$ where a_{ij} denotes the angle between the new axis x'_i and the old axis x_j.

Example Our original equation is

$$J_p = \sigma_{pq} E_q$$

Combining this with transformation of axes

$$J'_i = a_{ip} J_p$$

we have

$$J'_i = a_{ip} \sigma_{pq} E_q$$

Tensor Property Transformation Law

$$\sigma'_{ij} = a_{ip} a_{jq} \sigma'_{pq}$$

$$T'_{ij} = a_{ip} a_{jq} T'_{pq}$$

Note: m rank tensor related to n rank tensor by $(m+n)$ rank tensor.

Third Rank Tensors

- Piezoelectricity: stress produces electric dipole moment
- Isotropic case: $D = d\sigma$, where d is the piezoelectric modulus
- Anisotripic case: $D_i = d_{ijk}\sigma_{jk}$
- Normal stresses: s_{ii}
- Transverse (shear) stresses: $s_{ij}, i \neq j$
- No turning moment: $\sigma_{ij} = \sigma_{ji}$

Symmetrical Tensors (rank 2 tensors)

$$T_{ij} = T_{ji} = [S_{ij}]$$

where

$$[S_{ij}] = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix}$$

which has only 6 independent terms.
In fact, piezoelectric tensors are symmetric for the latter 2 terms, ie

$$d_{ijk} = d_{ikj}$$

which gives us 18 independent d_{ijk} terms