Lecture 17

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1 Polar Decomposition

Let V be a complex inner product space with finite dimensions. As proven in the previous lecture,

Theorem 1.1. *For every invertible $T \in \mathcal{L}(V)$, there are unique unitary $U \in \mathcal{L}(V)$ and positive $R \in \mathcal{L}(V)$ such that $T = UR$.*

Proof. We use $R = \sqrt{T^*T}$ and $U = TR^{-1}$. $\square$

Example 1.1. Find polar decomposition of

$$A = \begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix}$$

Then

$$A^*A = \begin{pmatrix} -2i+1 & 2-i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$$

To take the square root, find the eigenvectors and eigenvalues. We get

$$\sqrt{A^*A} = \sqrt{2} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$U = TR^{-1} = \begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix} \frac{1}{-\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

The polar decomposition is

$$\begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \sqrt{2} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem 1.2 (Polar Decomposition)**.** *For any $T \in \mathcal{L}(V)$, there exists a unitary $U \in \mathcal{L}(V)$ and a positive $R \in \mathcal{L}(V)$ such that $T = UR$ where R is unique.*

Lemma 1.1. *For $T \in \mathcal{L}(V, W)$, with V, W being inner product spaces, then*

$$V = null(T) \oplus ran(T^*)$$

$$W = null(T^*) \oplus ran(T)$$

Proof. For the first property, note that if T^* maps to a nontrivial vector in the kernel T, then let $T^*w = v \neq 0$ be in the kernel of T. Then

$$0 \neq \langle v, v \rangle = \langle v, T^*w \rangle = \langle Tv, w \rangle = \langle 0, w \rangle = 0$$

which is a contradiction. Then note that T and T^* share the same rank (consider their matrix forms). Then both sets intersect only at 0, and their union has the same dimension as V, so equality holds. Now the second property can be proven by letting $S = T^*$ and applying the first property. □

So for $V = W$, we get two decompositions of V.

Lemma 1.2. *For $T \in \mathcal{L}(V, W)$,*

$$null(T) = null(T^*T), ran(T^*) = ran(T^*T)$$

Proof. If $Tv = 0$ then $T^*Tv = 0$, so $\text{null}(T) \subseteq \text{null}(T^*T)$. Suppose $T^*Tv = 0$. Then

$$\langle T^*Tv, v \rangle = 0 \Rightarrow \langle Tv, Tv \rangle = 0 \Rightarrow Tv = 0$$

so $\text{null}(T^*T) \subseteq \text{null}(T)$. Then by lemma 1.1,

$$\text{ran}(T^*) = \text{null}(T)^\perp = \text{null}(T^*T)^\perp = \text{ran}((T^*T)^*) = \text{ran}(T^*T)$$

□

Now we can prove the polar decomposition theorem 1.2.

Proof. If such a decomposition exists, we must have

$$T^*T = R^*U^*UR = R^*R = R^2 \Rightarrow R = \sqrt{T^*T}$$

Since R is normal, we have

$$\text{null}(R) = \text{nulll}(R^2) = \text{null}(T^*T) = \text{null}(T)$$

We will define $U \in \mathcal{L}(V)$ as a sum of two isomorphisms

$$U_1 : \text{ran}(T^*) \to \text{ran}(T), U_2 : \text{null}(T) \to \text{null}(T^*)$$

For U_2, we can take any isometric isomorphism, and for U_1, consider the restriction

$$T_1 = T|_{\text{ran}(T^*)} : \text{ran}(T^*) \to \text{ran}(T)$$

This is an isomorphism. Put

$$R_1 = \sqrt{T_1^* T_1} : \text{ran}(T^*) \to \text{ran}(T^*)$$

This is the restriction of $R = \sqrt{T^*T}$ to $\text{ran}(T^*)$. Define U_1 by

$$T = U_1 R_1 \Rightarrow U_1 = T_1 R_1^{-1}$$

This is an isometry:

$$U_1^* U_1 = (R_1^{-1})^* T_1^* T_1 R_1^{-1} = R_1^{-1} R_1^2 R_1^{-1} = I_{\text{ran}(T^*)}$$

Finally, define $U \in \mathcal{L}(V)$ by

$$U(v) = U_1(v_1) + U_2(v + 2)$$

for $v = v_1 + v_2 \in \text{ran}(T^*) \oplus \text{null}(T) = V$. By the Pythagorean theorem, since $U_1(v_1), U_2(v_2)$ are orthogonal,

$$||U(v)||^2 = ||U_1(v_1)||^2 + ||U_2(v_2)||^2 = ||v_1||^2 + ||v_2||^2 = ||v||^2$$

So U is unitary. Furthermore,

$$URv = URv_1 = U_1 r_1 v_1 = T_1 v_1 = Tv_1 = Tv$$

$\square$

2 Singular Value Decomposition of Operators

For general $T \in \mathcal{L}(V)$, where we do not have an orthonormal basis of eigenvectors, we can get nice results by looking at T^*T.

Definition 2.1. THe eigenvalues of $\sqrt{T^*T}$ are called the singular values of T.

Example 2.1. Find the singular values of $T : \mathbb{C}^4 \to \mathbb{C}^4$ defined by

$$Te_1 = e_2, Te_2 = -2e_3, Te_3 = e_4, Te_4 = 0$$

we find the adjoint to be

$$T^*e_1 = 0, T^*e_2 = e_1, T^*e_3 = -2e_2, T^*e_4 = e_3$$

Then

$$T^*Te_1 = e_1, T^*Te_2 = 4e_2, T^*Te_3 = e_3, T^*Te_4 = 0$$

so the singular values are $1, 2, 1, 0$.