

# Lecture 16

niceguy

March 14, 2023

## 1 Normal Operators

Recall for a finite dimensional complex space  $V$  with an inner product,

$$TT^* = T^*T \Leftrightarrow V \text{ has an orthonormal basis of eigenvectors of } T$$

Let  $P_\lambda$  be orthogonal projections to eigenspaces  $E(\lambda, T)$ .

**Theorem 1.1** (Spectral Resolution). *For a normal  $T$ , all of the following hold.*

$$P_\lambda P_\mu = \begin{cases} 0 & \lambda \neq \mu \\ P_\lambda & \lambda = \mu \end{cases}$$

$$\sum_{\lambda} P_\lambda = I$$

$$T = \sum_{\lambda} \lambda P_\lambda$$

*Proof.* Trivial. □

If  $W \subseteq \mathbb{C}^n$  is a subspace with orthonormal basis  $v_1, \dots, v_k \in W$ , then the orthogonal projection to  $W$  is

$$P = \sum_{i=1}^k v_i v_i^*$$

This is because

$$Pv_j = \sum_i v_i v_i^* v_j = \sum_i v_i \langle v_j, v_i \rangle = \begin{cases} v_j & v_j \in W \\ 0 & v_j \notin W \end{cases}$$

**Example 1.1.** Find the spectral resolution of  $A \in M_{2 \times 2}(\mathbb{C})$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

The eigenvalues are  $a \pm ib$  with unit eigenvectors  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ . The orthogonal projections are then

$$P_{\lambda_1} = v_1 v_1^* = \frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$P_{\lambda_2} = v_2 v_2^* = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Thus the spectral resolution is

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (a + ib) \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + (a - ib) \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

The spectral resolution is useful in finding us the adjoint, as

$$T^* = \sum_{\lambda} \bar{\lambda} P_{\lambda}$$

and that

$$T^k = \sum_{\lambda} \lambda^k P_{\lambda}$$

More generally,

$$q(T) = \sum_{\lambda} q(\lambda) P_{\lambda}$$

Recall  $\text{Spec}(T)$  is the set of eigenvalues of  $T$ .

**Definition 1.1** (Functional Calculus). For any  $f : \text{Spec}(T) \rightarrow \mathbb{C}$ , define, for normal  $T$ ,

$$f(T) = \sum_{\lambda} f(\lambda) P_{\lambda}$$

If  $v \in V$  is an eigenvector of  $T$ , eigenvalue  $\lambda$ , then it is also an eigenvector for  $f(T)$  with eigenvalue  $f(\lambda)$  and  $f(T)$  is the unique operator with this property. In particular  $f(T)$  is normal.

**Example 1.2.** The properties of  $f(T)$  are as follows.

- $f(\lambda) = \lambda \Rightarrow f(T) = T$
- $f(\lambda) = 1 \Rightarrow f(T) = 1$
- $f(\lambda) = q(\lambda) \Rightarrow f(T) = q(T)$
- $f(\lambda) = \bar{\lambda} \Rightarrow f(T) = T^*$
- $(f + g)(T) = f(T) + g(T)$
- $(af)(T) = af(T)$
- $(f \cdot g)(T) = f(T)g(T)$
- $f(T)^* = \bar{f}(T^*)$
- $\text{Spec}(f(T)) = f(\text{Spec}(T))$

Now we can define all kinds of functions of normal operators.

**Example 1.3.**  $|A|$  from the first example becomes

$$|a + ib|\frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + |a - bi|\frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\exp \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = e^{a+ib}\frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + e^{a-ib}\frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix}$$

For a finite dimensional vector space  $V = \mathbb{R}$  or  $V = \mathbb{C}$ ,  $T \in \mathcal{L}(V)$ ,

$$\exp(T) = e^T$$

is always defined (it converges, but one has to show it). More generally,  $f(T)$  is defined  $\forall f(z) = \sum_{n=0}^{\infty} a_n z^n$  with an  $\infty$  radius of convergence.

## 2 Positive Operators

For a finite dimensional complex space  $V$ ,

**Proposition 2.1.** *For self adjoint  $T \in \mathcal{L}(V)$ , the following are equivalent*

1.  $\langle Tv, v \rangle > 0 \forall v \in V$
2. *The eigenvalues of  $T$  are all  $\geq 0$*

*Proof.* The first obviously implies the second; put  $v$  to be any eigenvector. Given the second,

$$\langle Tv, v \rangle = \langle T \left( \sum_i a_i e_i \right), \sum_i a_i e_i \rangle = \langle \sum_i a_i \lambda_i e_i, \sum_i a_i e_i \rangle = \sum_i \lambda_i a_i^2 \geq 0$$

where  $e_i$  is an orthonormal basis of eigenvectors.  $\square$

**Definition 2.1.** A self-adjoint operator satisfying these properties is called positive. Note that 0 (the operator) is positive.

**Example 2.1.**  $I$  is positive.  $\forall T \in \mathcal{L}(V)$ , the operators  $TT^*, T^*T$  are positive. This is because

$$\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = \|Tv\|^2 \geq 0$$

and similarly for  $TT^*$ . Then for any normal  $T$  and any  $f : \text{Spec}(\mathbb{C}) \rightarrow \mathbb{R}^+$ , then  $f(T)$  is positive. In particular,  $|T|$  is positive.

**Proposition 2.2.** *If  $T_1, T_2$  are positive,  $a_1, a_2 \geq 0$ , then  $a_1T_1 + a_2T_2$  is positive.*

*Proof.* Trivial.  $\square$