Lecture 6

niceguy

September 20, 2022

1 Review

A general first order IVP $u' = f(t, u)$ defined on an open rectangle with continuous f and f_u has a unique solution on some $S_h(t_0)$ where t_0 is the location of the initial value.

2 Picard's Iterations

Define a sequence of functions

$$\begin{cases} u_0(x) = y_0 \\ u_{n+1}(x) = y_0 + \int_{y_0}^{x} f(t, u_n(t))dt \end{cases}$$

It can be shown that given the assumptions above, $\exists u : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} u_n = u$$

This u will then be the solution.

3 Applications of Existence and Uniqueness

$$\begin{cases} u' = \frac{3t^2+4t+2}{2(u-1)} \\ u(0) = -1 \end{cases}$$

We have that $(t_0, u_0) = (0, -1)$. Noting that

$$f_u = -\frac{3t^2 + 4t + 2}{2(u-1)^2}$$

Both f and f_u are continuous everywhere at $u \neq 1$, so we can choose

$$(\alpha, \beta) \times (\gamma, \delta) = (-\infty, \infty) \times (-\infty 1)$$

We can then conclude a unique solution exists on some interval $(t_0 - h, t_0 + h)$. To solve this explicitly,

$$\begin{aligned} 2(u-1)\frac{du}{dt} &= 3t^2 + 4t + 2 \\ (u-1)^2 &= t^3 + 2t^2 + 2t + C \\ u &= 1 \pm \sqrt{t^3 + 2t^2 + 2t + C} \end{aligned}$$

Putting the initial value in,

$$u(0) = 1 \pm \sqrt{C} = -1 \to C = 4$$

and that the negative sign is taken. Hence

$$u(t) = 1 - \sqrt{t^3 + 2t^2 + 2t + 4}$$

The domain is where the square root is non-negative, i.e. $(-2, \infty)$. To show this, note that the polynomial inside the square root has a positive derivative ($3t^2 + 4t + 2$ has no real roots and is positive at $t = 0$, so it must always be positive), so it intersects the x axis at most one point. Trial and error tells us the square root is 0 at $t = -2$, hence the domain.
Now consider

$$\begin{cases} u' = \frac{u \sin t}{t^2+1} + \cos t \\ u(0) = 1 \end{cases}$$

We have $g(t) = \cos t$ and $p(t) = -\frac{\sin t}{t^2+1}$. Both are continuous $\forall t \in \mathbb{R}$, so we can choose

$$(\alpha, \beta) = (-\infty, \infty)$$

Now consider a similar problem as above.

$$\begin{cases} u' = \frac{3t^2+4t+2}{2(u-1)} \\ u(0) = 1 \end{cases}$$

The initial value is given at where u' and f_u are not defined! Therefore, the theorem tells us nothing (there are no rectangles containing t_0 where f_u is continuous). If we use the general solution, there will be 2 solutions!

$$u(t) = 1 \pm \sqrt{t^3 + 2t^2 + 2t}$$

4 Nonlinear Cases

A global solution is not guaranteed. Why? Consider

$$\begin{cases} u' = 1 + u^2 \\ u(0) = 0 \end{cases}$$

We have already found the solution

$$u(t) = \tan(t)$$

But this has a domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. And it is not immediately obvious that the solution is only defined on a proper subset of $\mathbb{R}$.

5 Continuity of f_u

The continuity of f alone is enough to guarantee existance. Uniqueness is guaranteed by the continuity of f_u.
Consider

$$\begin{cases} u' = u^{\frac{2}{3}} \\ u(0) = 0 \end{cases}$$

$$f_u = \frac{2}{3} u^{-\frac{1}{3}}$$

Which is not continuous at $u = 0$. f is obviously continuous. However, both

$$u(t) = 0$$

and

$$u(t) = \left(\frac{t}{3}\right)^3$$

are solutions, meaning we do have existence, but not uniqueness.