Lecture 17

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1 Polar Decomposition

Let V be a complex inner product space with finite dimensions. As proven in the previous lecture,

Theorem 1.1. *For every invertible $T \in \mathcal{L}(V)$, there are unique unitary $U \in \mathcal{L}(V)$ and positive $R \in \mathcal{L}(V)$ such that $T = UR$.*

Proof. We use $R = \sqrt{T^*T}$ and $U = TR^{-1}$. $\square$

Example 1.1. Find polar decomposition of

$$A = \begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix}$$

Then

$$A^*A = \begin{pmatrix} -2i+1 & 2-i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix}$$

To take the square root, find the eigenvectors and eigenvalues. We get

$$\sqrt{A^*A} = \sqrt{2} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$U = TR^{-1} = \begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix} \frac{1}{-\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

The polar decomposition is

$$\begin{pmatrix} 2i+1 & i \\ 2+i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \sqrt{2} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem 1.2 (Polar Decomposition)**.** *For any $T \in \mathcal{L}(V)$, there exists a unitary $U \in \mathcal{L}(V)$ and a positive $R \in \mathcal{L}(V)$ such that $T = UR$ where R is unique.*

Lemma 1.1. *For $T \in \mathcal{L}(V, W)$, with V, W being inner product spaces, then*

$$V = null(T) \oplus ran(T^*)$$

$$W = null(T^*) \oplus ran(T)$$

Proof. Show that the spaces are orthogonal. $\square$

So for $V = W$, we get two decompositions of V.

Lemma 1.2. *For $T \in \mathcal{L}(V, W)$,*

$$null(T) = null(T^*T), ran(T^*) = ran(T^*T)$$

Proof. If $Tv = 0$ then $T^*Tv = 0$, so $\text{null}(T) \subseteq \text{null}(T^*T)$. Suppose $T^*Tv = 0$. Then

$$\langle T^*Tv, v\rangle = 0 \Rightarrow \langle Tv, Tv\rangle = 0 \Rightarrow Tv = 0$$

so $\text{null}(T^*T) \subseteq \text{null}(T)$. Then by lemma 1.1,

$$\text{ran}(T^*) = \text{null}(T)^\perp = \text{null}(T^*T)^\perp = \text{ran}((T^*T)^*) = \text{ran}(T^*T)$$

$\square$

Now we can prove the polar decomposition theorem 1.2.

Proof. If such a decomposition exists, we must have

$$T^*T = R^*U^*UR = R^*R = R^2 \Rightarrow R = \sqrt{T^*T}$$

Since R is normal, we have

$$\text{null}(R) = \text{nulll}(R^2) = \text{null}(T^*T) = \text{null}(T)$$

We will define $U \in \mathcal{L}(V)$ as a sum of two isomorphisms

$$U_1 : \text{ran}(T^*) \to \text{ran}(T), U_2 : \text{null}(T) \to \text{null}(T^*)$$

For U_2, we can take any isometric isomorphism, and for U_1, consider the restriction

$$T_1 = T|_{\mathrm{ran}(T^*)} : \mathrm{ran}(T^*) \to \mathrm{ran}(T)$$

This is an isomorphism. Put

$$R_1 = \sqrt{T_1^* T_1} : \mathrm{ran}(T^*) \to \mathrm{ran}(T^*)$$

This is the restriction of $R = \sqrt{T^*T}$ to $\mathrm{ran}(T^*)$. Define U_1 by

$$T = U_1 R_1 \Rightarrow U_1 = T_1 R_1^{-1}$$

This is an isometry:

$$U_1^* U_1 = (R_1^{-1})^* T_1^* T_1 R_1^{-1} = R_1^{-1} R_1^2 R_1^{-1} = I_{\mathrm{ran}(T^*)}$$

Finally, define $U \in \mathcal{L}(V)$ by

$$U(v) = U_1(v_1) + U_2(v+2)$$

for $v = v_1 + v_2 \in \mathrm{ran}(T^*) \oplus \mathrm{null}(T) = V$. By the Pythagorean theorem, since $U_1(v_1), U_2(v_2)$ are orthogonal,

$$||U(v)||^2 = ||U_1(v_1)||^2 + ||U_2(v_2)||^2 = ||v_1||^2 + ||v_2||^2 = ||v||^2$$

So U is unitary. Furthermore,

$$URv = URv_1 = U_1 r_1 v_1 = T_1 v_1 = Tv_1 = Tv$$

□

2 Singular Value Decomposition of Operators

For general $T \in \mathcal{L}(V)$, where we do not have an orthonormal basis of eigenvectors, we can get nice results by looking at T^*T.

Definition 2.1. THe eigenvalues of $\sqrt{T^*T}$ are called the singular values of T.

Example 2.1. Find the singular values of $T : \mathbb{C}^4 \to \mathbb{C}^4$ defined by

$$Te_1 = e_2, Te_2 = -2e_3, Te_3 = e_4, Te_4 = 0$$

we find the adjoint to be

$$T^*e_1 = 0, T^*e_2 = e_1, T^*e_3 = -2e_2, T^*e_4 = e_3$$

Then

$$T^*Te_1 = e_1, T^*Te_2 = 4e_2, T^*Te_3 = e_3, T^*Te_4 = 0$$

so the singular values are $1, 2, 1, 0$.