Lecture 10

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February 1, 2023

1 Expectation

For a random variable X with distribution $f(x)$,

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

or

$$E[X] = \sum_x xf(x)$$

where we denote

$$\mu = E[X]$$

2 Variance

Let X be a random variable with $f(x)$. Then

$$\text{var}(X) = E[(X - \mu)^2]$$

which is

$$\int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$$

or

$$\sum_x (x - \mu)^2 f(x)$$

We write

$$\sigma^2 = \text{var}(x)$$

and define

$$\sigma = \sqrt{\sigma^2}$$

to be the standard deviation (which is non-negative).

Example 2.1. Consider the uniform distribution

$$f(x) = \begin{cases} \frac{1}{\alpha} & 0 \leq x \leq \alpha \\ 0 & \text{else} \end{cases}$$

Then the mean is $\frac{\alpha}{2}$. The variance is then

$$\begin{aligned} \int_{-\infty}^{\infty} \left(x - \frac{\alpha}{2}\right)^2 f(x)dx &= \int_0^{\alpha} (x^2 - \alpha x + \alpha^2)\frac{1}{\alpha}dx \\ &= \frac{\alpha^2}{3} - \frac{\alpha^2}{2} + \frac{\alpha^2}{4} \\ &= \frac{\alpha^2}{12} \end{aligned}$$

For $\alpha \to 0$, we get the Dirac Delta function (not a function).

A useful formula is

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f(x)dx \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

3 Covariance

Consider the random variables X and Y, distribution $f(x, y)$ with means μ_x and μ_y.

Definition 3.1. The covariance of X and Y is

$$\sigma_{X,Y} = \text{cov}(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

Observe

$$\mathrm{var}(X) = \sigma_{xx}$$

If X is above its mean when Y is below its mean, we get a negative covariance. Similarly, if X and Y are both above/below their means, we get a positive covariance. We can consider covariance as a correlation.