Lecture 34

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1 1-D Heat Equation

From the last lecture, we assumed $\lambda > 0$. This does not work in general. However, no solution is found if $\lambda \leq 0$, so it can be discarded. Substituting into the equation for X yields

$$X(\sqrt{\lambda}x) = 0$$

and

$$\lambda = \frac{n^2\pi^2}{L^2}$$

$$\begin{aligned}
T'(t) + \alpha^2\lambda T(t) &= 0 \\
T'(t) &= -\alpha^2\lambda T(t) \\
T(t) &= Ce^{-\alpha^2\lambda t} \\
&= Ce^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}
\end{aligned}$$

The solution is then their product

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}$$

Using the initial condition

$$u(x,0) = f(x)$$

we have

$$f(x) = c_n \sin\left(\frac{n\pi x}{L}\right)$$

However, this "fixes" our initial condition, which should not be possible.

Theorem 1.1. *If $u_1(x,t)$ and $u_2(x,t)$ solve*

$$\begin{cases} u_t(x,t) = \alpha^2 u_{xx}(x,t) \\ u(0,t) = 0 \qquad\qquad\qquad u(L,t) = 0 \end{cases}$$

then $c_1 u_1(x,t) + c_2 u_2(x,t)$ also solves the above equation for any coefficients $c_1, c_2 \in \mathbb{R}$.

Then

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

f is odd, but it does not matter, as the solution is only defined on $x \in [0, L]$. The coefficients c_n can easily be found, given an intial condition, which yields the solution.

2 Final Review

Consider the autonomous ODE

$$y'(t) = y^2(y-5)(y+10)$$

The equilibrium solutions are 0, 5, -10. Given the initial value $y(0) = 1$,

$$\lim_{t\to\infty} y(t) = 0$$

and

$$\lim_{t\to-\infty} y(t) = 5$$