Lecture 12

niceguy

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1 Adjoints

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and $T \in \mathcal{L}(V)$. Using induction, one can prove that

$$\langle T^n(v), w\rangle = \langle v, (T^*)^n(w)\rangle \forall n \in \mathbb{N}$$

Extending this to the minimal polynomial,

$$\begin{aligned} 0 &= \langle p(T)v, w\rangle \\ &= \langle v, \overline{p}(T^*)w\rangle \end{aligned}$$

However, this is true for arbitrary $v, w \in V$. Hence $\overline{p}$ is the minimal polynomial for T^*. (Obviously $\overline{p}(T^*) = 0$. If there is a polynomial with a smaller degree, one obtains a p' of that lower degree such that $p'(T) = 0$, which is a contradiction, as p is the minimal polynomial).
Recall with a orthonormal basis, if A is the matrix of T, then the matrix of T^* is $\overline{A}^t$. Assuming this, consider the characteristic polynomial

$$q_T(z) = \det(zI - T) = \det(zI - A)$$

$$q_{T^*}(z) = \det(zI - T^*) = \det(zI - \overline{A}^t) = \det\left((zI - \overline{A})^t\right) = \det(zI - \overline{A}) = \overline{\det(\overline{z}I - A)}$$

Then

$$q_{T^*}(z) = \overline{q_T(\overline{z})}$$

Now, let J be the Jordan Normal Form of T. Then

$$\begin{aligned} A &= CJC^{-1} \\ \overline{A} &= \overline{C}\overline{J}\overline{C^{-1}} \\ \overline{A}^t &= \overline{C^{-1}}^t\overline{J}^t\overline{C}^t \\ &= \overline{C^{-1}}^t\overline{S}\overline{J}\overline{S^{-1}}\overline{C}^t \end{aligned}$$

Undoing the change of basis, the Jordan Normal Form of T^* is $\overline{J}$.
Suppose $T^* = T$. Then

1. if $W \subseteq V$ is T invariant, then $W^\perp$ is alto T invariant
2. The eigenvalues of T are real
3. T has at least one eigenvector

Theorem 1.1. *T is diagnoalisable if $T = T^*$.*

Proof. By 3 there is an eigenvector v_1 of T. Then $W = fv_1 \forall f \in \mathbb{F}$ is a T invariant subspace. So $W^\perp$ is also T invariant, and

$$T|_{W^\perp} \in \mathcal{L}(W^\perp)$$

Then

$$(T|_{W^\perp})^* = (T|_{W^\perp})$$

By induction, we get $v_1, v_2, \ldots, v_n$ where n is the dimension of V. All v_i are eigenvectors, so T is diagonalisable. $\square$