# Probelm Set 5

niceguy

March 8, 2023

1. An infinitely large dielectric slab of thickness $d = 2a$ is uniformly polarized through-out its volume such that the polarization vector, $\vec{P}$, is perpendicular to the faces (boundary surfaces) of the slab. The surrounding medium is air. The electric field intensity vector (due to bound charges of the slab) at a point inside the slab is

   **Solution:** Nonzero and is directed oppositely to $\vec{P}$.

2. Consider a polarized dielectric body with no free charge, in free space. The outward flux of the electric field intensity vector, $\vec{E}$, through a closed surface $S$ that completely encloses the body is

   **Solution:** By Gauss' Law, 0.

3. Consider a boundary surface between two dielectric media, with relative permittivities $\varepsilon_{r1} = 4$ and $\varepsilon_{r2} = 2$ respectively. Assuming that there is no surface charge on the boundary, which of the cases represent possible electric field intensity vectors on the two sides of the boundary?

   **Solution:** Case (a) only.

4. The figure shows lines of an electrostatic field near a dielectric-dielectric boundary that is free of charge ($\rho x = 0$). Which of the following is a possible combination of the two media?

   **Solution:** Medium 1 is water ($\varepsilon_{r1} = 81$) and Medium 2 is air ($\varepsilon_{r1} = 1$).

5. Consider a rectangular dielectric parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$. The polarization vector in the dielectric is given by:
$$\vec{P} = P_0 \left( \frac{x}{a}\hat{a}_x + \frac{y}{b}\hat{a}_y + \frac{z}{c}\hat{a}_z \right)$$
   where $P_0$ is a constant.

   (a) Find the densities of volume and surface bound (polarization) charge in the parallelepiped.

   (b) Show that the total bound charge in the parallelepiped is zero.

**Solution:**

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \frac{\varepsilon_r}{\chi_e} \vec{P}$$

Hence

$$\begin{aligned} \rho_v &= -\vec{\nabla} \cdot \vec{P} \\ &= -P_0 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \end{aligned}$$

And

$$\begin{aligned} \rho_{p,s} &= \hat{a}_n \cdot \vec{P} \\ &= \begin{cases} P_0 & x = a \text{ or } y = b \text{ or } z = c \text{ given } x, y, z \neq 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Total charge is then

$$-P_0 \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) abc + P_0(ab + ac + bc) = P_0(-bc - ac - ab + ab + ac + bc) = 0$$

6. A very (infinitely) long homogeneous dielectric cylinder, of radius $a$ and relative dielectric permittivity $\varepsilon r$, is uniformly charged with free charge density $\rho$ throughout its volume. The cylinder is surrounded by air.
   (a) Calculate the voltage between the axis and the surface of the cylinder.
   (b) Find the bound charge distribution in the cylinder.

**Solution:** Note the cylindrical symmetry. Using Gauss' Law,

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho \\ \frac{1}{r} \frac{\partial r D_r}{\partial r} &= \rho \\ r D_r &= \frac{\rho r^2}{2} + C \\ D_r &= \frac{\rho r}{2} + \frac{C}{r} \end{aligned}$$

For $D_r$ to exist at $r = 0$, we need $C = 0$, so substituting,

$$E_r = \frac{\rho r}{2 \varepsilon_0 \varepsilon_r}$$

$$\begin{aligned} V &= -\int \vec{E} \cdot d\vec{l} \\ &= -\int_0^a E_r dr \\ &= -\frac{\rho a^2}{4 \varepsilon_0 \varepsilon_r} \end{aligned}$$

The bound charges are then

$$\begin{aligned}\rho_{p,v} &= -\vec{\nabla}\cdot\vec{P}\\ &= -\chi_e\varepsilon_0\vec{\nabla}\cdot\vec{E}\\ &= -(\varepsilon_r-1)\varepsilon_0\times\frac{1}{r}\frac{\rho r}{\varepsilon_0\varepsilon_r}\\ &= -\frac{\rho(\varepsilon_r-1)}{\varepsilon_r}\end{aligned}$$

and

$$\begin{aligned}\rho_{p,s} &= \vec{a}_n\cdot\vec{P}\\ &= \frac{\rho r(\varepsilon_r-1)}{2\varepsilon_r}\end{aligned}$$

7. The polarization in a dielectric cube of side $L$ centred at the origin is given by $\vec{P} = P_0(\hat{a}_x x + \hat{b}_y y + \hat{c}_z z)$.
   (a) Determine the surface and volume bound-charge densities.
   (b) Show that the total bound charge is zero.

**Solution:** The bound charges are

$$\begin{aligned}\rho_{p,v} &= -\vec{\nabla}\cdot\vec{P}\\ &= -3P_0\end{aligned}$$

and

$$\begin{aligned}\rho_{p,s} &= \hat{a}_n\cdot\vec{P}\\ &= \begin{cases} LP_0 & x = L \text{ or } y = L \text{ or } z = L \text{ given } x, y, z \neq 0\\ 0 & \text{else}\end{cases}\end{aligned}$$

Total bound charge is then

$$-3P_0L^3 + LP_O\times 3L^2 = 0$$

8. Determine the boundary conditions for the tangential and the normal components of $\vec{P}$ at an interface between two perfect dielectric media with dielectric constants $\varepsilon_{r1}$ and $\varepsilon_{r2}$.

**Solution:**

$$\vec{E} = \frac{1}{(\varepsilon_r-1)\varepsilon_0}\vec{P}$$

$$\vec{D} = \varepsilon_0\varepsilon_r\vec{E} = \frac{\varepsilon_r}{\varepsilon_r-1}\vec{P}$$

Hence for the normal component,

$$\frac{\varepsilon_{r1}}{\varepsilon_{r1}-1}\vec{P_{n1}} = \frac{\varepsilon_{r2}}{\varepsilon_{r2}-1}\vec{P_{n2}}$$

For the tangential component,

$$\frac{1}{(\varepsilon_{r1}-1)\varepsilon_0}\vec{P_{t1}} = \frac{1}{(\varepsilon_{r2}-1)\varepsilon_0}\vec{P_{t2}}$$

9. What are the boundary conditions that must be satisfied by the electric potential at an interface between two perfect dielectrics with dielectric constants $\varepsilon_{r1}$ and $\varepsilon_{r2}$?

**Solution:** The normal component of $\vec{D}$ must be constant, hence

$$\varepsilon_{r1}(\vec{\nabla}V_1)\cdot\hat{a}_n = \varepsilon_{r2}(\vec{\nabla}V_2)\cdot\hat{a}_n$$

Voltage at the boundary has to agree, so

$$V_1 = V_2$$

at the boundary.

10. Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\vec{P} = P_0\hat{a}_z$ exists.

**Solution:**

$$\rho_{p,s} = \hat{a}_n\cdot\vec{P} = -\hat{a}_R\cdot P_0\hat{a}_z = -P_0\cos\theta$$

Letting the radius be $a$, the field is given by

$$\begin{aligned}
\vec{E} &= \int_0^{2\pi}\int_0^{\pi}\frac{dQ}{4\pi\varepsilon_0 a^2}(-\hat{a}_R)\\
&= \int_0^{2\pi}\int_0^{\pi}\frac{-P_0\cos\theta a^2\sin\theta d\theta d\phi}{4\pi\varepsilon_0 a^2}(-a)(\sin\theta\cos\phi\hat{a}_x+\sin\theta\sin\phi\hat{a}_y+\cos\theta\hat{a}_z)\\
&= \int_0^{2\pi}\int_0^{\pi}\frac{P_0 a\sin\theta\cos\theta d\theta d\phi}{4\pi\varepsilon_0}(\sin\theta\cos\phi\hat{a}_x+\sin\theta\sin\phi\hat{a}_y+\cos\theta\hat{a}_z)\\
&= \int_0^{2\pi}\int_0^{\pi}\frac{P_0 a\sin\theta\cos^2\theta d\theta d\phi}{4\pi\varepsilon_0}\hat{a}_z\\
&= \int_0^{2\pi}\frac{P_0 a d\phi}{6\pi\varepsilon_0}\hat{a}_z\\
&= \frac{P_0 a}{3\varepsilon_0}\hat{a}_z
\end{aligned}$$

Where the $\hat{a}_x$ and $\hat{a}_y$ components vanish as the integrals of $\sin\phi$ and $\cos\phi$ from 0 to $2\pi$ are 0.

11. Assume that the $z = 0$ plane separates two lossless dielectric regions with $\varepsilon_{r1} = 2$ and $\varepsilon_{r2} = 3$. Let the electric field $\vec{E}_1 = 2y\hat{a}_x - 3x\vec{a}_y + (5+z)\hat{a}_z$ in region 1. Find the electric field $\vec{E}_2$ and the electric flux density $\vec{D}_2$ in region 2.

**Solution:** Since the tangential components of $\vec{E}$ are conserved,

$$\vec{E}_2^{=} = 2y\hat{a}_x - 3x\hat{a}_y$$

Scaling, the tangential component of $\vec{D}_2$ is

$$\vec{D}_2^{=} = 6\varepsilon_0 y\hat{a}_x - 9\varepsilon_0 x\hat{a}_y$$

The normal components of $\vec{D}$ are conserved, so

$$\vec{D}_2^{\perp} = \vec{D}_1^{\perp} = 10\varepsilon_0\hat{a}_z$$

Scaling gives

$$\vec{E}_2^{\perp} = \frac{10}{3}\hat{a}_z$$

Then the field and flux density are given by

$$\vec{E}_2 = 2y\hat{a}_x - 3x\hat{a}_y + \frac{10}{3}\hat{a}_z$$

and

$$\vec{D}_2 = 6\varepsilon_0 y\hat{a}_x - 9\varepsilon_0 x\hat{a}_y + 10\varepsilon_0\hat{a}_z$$

12. Dielectric lenses can be used to collimate electromagnetic fields. In the diagram below, the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\vec{E}_1$ at point $P(r_o, 45°, z)$ is region 1 is $5\hat{a}_r - 3\hat{a}_\phi$, what must be the dielectric constant of the lens in order that $\vec{E}_3$ in region 3 is parallel to the $x$-axis?

**Solution:** If $\vec{E}_3$ in region 3 is parallel to the $x$-axis, then $\vec{E}_2$ in region 2 must also be parallel to the $x$-axis. Note that at point $P$,

$$\hat{a}_x = \frac{1}{\sqrt{2}}\hat{a}_r - \frac{1}{\sqrt{2}}\hat{a}_\phi$$

We can also see that $\hat{a}_r$ points in the normal direction, and $\hat{a}_\phi$ points in the tangential direction. For $\vec{E}_2$ to be parallel to the $x$-axis, it must be a multiple of $\hat{a}_x$, i.e. its components for $\hat{a}_r$ and $\hat{a}_\phi$ must be equal in magnitude and opposite in sign. Letting the dielectric constant be $\varepsilon_r$, we get

$$\frac{5}{\varepsilon_r} = 3 \Rightarrow \varepsilon_r = \frac{5}{3}$$

13. Refer to Example 3-16 in Cheng (page 119). Assuming the same $r_i$ and $r_o$ and requiring the maximum electric field intensities in the insulting materials not exceed 25% of their dielectric strengths, determine the voltage rating of the coaxial cable
    (a) if $r_p = 1.75r_i$
    (b) if $r_p = 1.35r_i$
    (c) Plot the variations of $E_r$ and $V$ versus $r$ for both parts

**Solution:** Note that $\frac{\rho_l}{2\pi\varepsilon_0}$ is a function of $r_i$, so it takes the same value as the example. Integrating, the voltage is

$$\begin{aligned} V &= -\int_{r_o}^{r_p} E_p dr - \int_{r_p}^{r_i} E_r dr \\ &= \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{\varepsilon_{rp}} \ln \frac{r_O}{r_p} + \frac{1}{\varepsilon_{rr}} \ln \frac{r_p}{r_i}\right) \\ &= 8 \times 10^4 \left(\frac{1}{2.6} \ln \frac{0.832}{r_p} + \frac{1}{3.2} \ln \frac{r_p}{0.4}\right) \end{aligned}$$

For $r_p = 1.75 r_i = 0.7, V = 19.3\text{kV}$. For $r_p = 1.35 r_i = 0.54, V = 20.8\text{kV}$.

14. An infinitely large dielectric slab of thickness $d = 2a$ is polarized so that the polarization vector is $\vec{P} = P_0 \frac{x^2}{a^2} \hat{a}_x$, where $P_0$ is a constant. The medium outside the slab is air. Find the voltage between the boundary surfaces of the slab.

**Solution:**

$$\vec{E} = \frac{P_0}{(\varepsilon_r - 1)\varepsilon_0 a^2} x^2 \hat{a}_x$$

Then the voltage is

$$\begin{aligned} V &= -\int \vec{E} \cdot d\vec{l} \\ &= -\int_{-a}^{a} \frac{P_0}{(\varepsilon_r - 1)\varepsilon_0 a^2} x^2 \\ &= \frac{2P_0 a}{3(\varepsilon_r - 1)\varepsilon_0} \end{aligned}$$