Lecture 33

niceguy

December 2, 2022

1 Fourier Series

For Fourier Transforms, we define convolutions as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

Example 1.1. Express $\mathcal{F}\{f * g\}$ in terms of the Fourier Transform of f and g.

$$\mathcal{F}\{f * g\} = \hat{f}\hat{g}$$

The proof is left to the reader as an exercise.

Properties we won't derive:

- $\mathcal{F}\{f(t-t_0)\} = e^{-it_0\xi}\hat{f}(\xi)$
- $\mathcal{F}\{f(at)\} = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$
- $\mathcal{F}\{t^n f(t)\} = i^n \hat{f}^{(n)}(\xi)$

Example 1.2. Find an expression for the Fourier Transform of the solution to the Airy Equation

$$y''(x) - xy(x) = 0$$

Applying the Fourier Transform on both sides,

$$\begin{aligned}
\mathcal{F}\{y'' - xy\} &= \mathcal{F}\{0\} \\
\mathcal{F}\{y''\} - \mathcal{F}\{xy\} &= 0 \\
-\xi^2 \hat{y}(\xi) - i\hat{y}'(\xi) &= 0 \\
\frac{\hat{y}'(\xi)}{\hat{y}(\xi)} &= i\xi^2 \\
\hat{y}(\xi) &= Ce^{\frac{i\xi^3}{3}}, C \in \mathbb{R}
\end{aligned}$$

2 Inverse Fourier Transform

Recall the Fourier Transform for periodic function

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_{-L}^{L} f(t) e^{-i\xi t} dt$$

In general if f is not necessarily periodic, we need "more" than just countably many coefficients to represent our function. As the $\frac{1}{2\pi}$ term was dropped in the definition, it has to be re-included.

Theorem 2.1. *If the function f and its Fourier Transform $\hat{f}$ are well-behaved then we have that*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi t} d\xi$$

$\forall x \in \mathbb{R}$

Example 2.1. Continuing from 1.2,

$$\begin{aligned}
\hat{y}(\xi) &= Ce^{\frac{i\xi^3}{3}} \\
y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Ce^{\frac{i\xi^3}{3}} e^{i\xi t} d\xi \\
&= \frac{C}{2\pi} \int_{-\infty}^{\infty} e^{i\left(\xi t + \frac{\xi^3}{3}\right)} d\xi
\end{aligned}$$

The Fourier Transform has similar properties to the Laplace Transform, but it has a clear inversion formula. What is its major drawback? It has more restrictions on the solution.

Example 2.2. Use the Fourier Transform to solve

$$y''(x) - y(x) = f(x)$$

The solution is not the general solution.

3 Partial Differential Equations

There are no existence and uniqueness theorems for PDEs. It is hard to tell if equations even have a solution, and it is hard to determine if said solution is unique.

Example 3.1. Consider a heat conducting rod of length L. The initial conditions are

$$\begin{aligned} u_t(x, y) &= \alpha^2 u_{xx}(x, t) \\ u(x, 0) &= f(x) \\ u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned}$$

Using separation of variables, assume

$$u(x, t) = X(x)T(t)$$

Then

$$\begin{aligned} X(x)T'(t) &= \alpha^2 X''(x)T(t) \\ \frac{T'(t)}{\alpha^2 T(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

Both sides have to be constants. Equating them to $-\lambda$,

$$\begin{aligned} X''(x) + \lambda X(x) &= 0 \\ T' + \alpha^2 \lambda T &= 0 \end{aligned}$$

Addressing the initial conditions at both ends of the rod, $X(0) = X(L) = 0$. We can solve for X.

$$X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

And substituting the initial conditions gives

$$B = 0$$

and

$$\lambda = \frac{n^2\pi^2}{L^2}$$

where we assume $\lambda > 0$.