Lecture 17

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October 18, 2022

1 Adaptive Step Sizes

1.1 Recap

We want our local truncation error e_{n+1} to be less than a limit $\epsilon > 0$. We can estimate the local truncation error with

$$e_{n+1} = |y_{n+1} - z_{n+1}|$$

where z is a better approximation method. However, in reality, a worse approximation method is used instead.

1.2 New Step Size

$$e_{n+1} \approx |y_{n+1} - z_{n+1}| \approx \frac{|y''(\xi)|}{2} h^2$$

We want to scale h to h' such that

$$\frac{|y''(\xi)|}{2} h'^2 = \epsilon \Rightarrow h' = \sqrt{\epsilon \times \frac{2}{|y''(\xi)|}}$$

Therefore

$$\frac{2}{|y''(\xi)|} \approx \frac{h^2}{|e_{n+1}|}$$

$$h' \approx h \sqrt{\frac{\epsilon}{e_{n+1}}}$$

Therefore,

1. User passes in ϵ, h
2. Approximate local truncation error if we were to take a step-size of h
3. Calculate using step 2 a new step-size h' that ensures our local truncation error is roughly ϵ
4. Take an Euler step with step-size h'

2 Linear Systems in General

We consider the n$^{\text{th}}$ dimension case without the constraint of constant coefficients.

Theorem 2.1.

$$x'(t) = P(t)\vec{x} + \vec{g}(t), \vec{x}(t_0) = \vec{x_0}$$

Assume $P(t)$ and $\vec{g}(t)$ are continuous on an open interval $I = (\alpha, \beta)$. If $t_0 \in I$, $\exists$ a unique solution in (α, β).

For the remainder of this lecture, we shall assume homogeneousity, i.e. $\vec{g}(t) = 0$.

2.1 Superposition Principle

$$\begin{aligned} \frac{d}{dt}\left[\sum_{i=1}^{n} c_i \vec{x_i}(t)\right] &= \sum_{i=1}^{n} c_i \vec{x_i}'(t) \\ &= \sum_{i=1}^{n} c_i P(t) \vec{x_i}(t) \\ &= P(t)\left(\sum_{i=1}^{n} c_i \vec{x_i}(t)\right) \end{aligned}$$

The superposition principle still holds. Thus

Theorem 2.2. *Superposition Principle*

$$\vec{x}' = P(t)\vec{x}$$

Assume $\vec{x_1}(t), \vec{x_2}(t), \ldots, \vec{x_n}(t)$ *are solutions. Then*

$$\sum_{i=1}^{n} c_i \vec{x_i}(t)$$

is also a solution $\forall c_i \in \mathbb{F}$*.*

Definition 2.1. Functions $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ are linearly independent on an interval I if the only constants $c_1, c_2, \ldots, c_n$ such that

$$\sum_{i=1}^{n} c_i \vec{x_i}(t) = \vec{0}$$

$\forall t \in I$ are

$$c_1 = c_2 = \cdots = c_n = 0$$

Suppose we have $\vec{x_1}(t), \vec{x_2}(t), \ldots, \vec{x_n}(t)$ as solutions to the ODE. Now we need a set of constants that would make $\vec{x}(t_0) = \vec{x_0}$. This means we want to solve for

$$\begin{bmatrix} \vec{x_1}(t_0) & \vec{x_2}(t_0) & \ldots & \vec{x_n}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ d_n \end{bmatrix} = \vec{x_0}$$

We are guaranteed a unique solution of $\vec{x_1}(t_0), \vec{x_2}(t_0), \ldots, \vec{x_n}(t_0)$ are linearly independent. Therefore, we have linear independence if the individual vectors at any time t are linearly independent. This is a stronger statement than the above definition, which allows for nonzero coefficients at certain proper subsets of I.