Lecture 17

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1 Capacitors

Example 1.1. Consider the capacitors half filled with a dielectric material (in both senses). It is know that the charge density on the plates is non-uniform. Explain why this is the case, and find an expression for the total capacitance of this structure.
If the bottom half of the capacitor is filled with a dielectric material, then $\vec{D}$ is constant, and is equal to ρ_s. Then

$$\begin{aligned}
\Delta V &= -\int \vec{E} \cdot d\vec{l} \\
&= -E_{\text{air}} \times \frac{d}{2} - E_{\text{diel}} \times \frac{d}{2} \\
&= -\frac{\rho_s}{\varepsilon_0}\frac{d}{2} - \frac{\rho_s}{\varepsilon_0\varepsilon_r}\frac{d}{2} \\
&= -Q \times \frac{d}{2\varepsilon_0 s} - Q \times \frac{d}{2\varepsilon_0\varepsilon_r s} \\
&= -\frac{Q}{C_1} - \frac{Q}{C_2}
\end{aligned}$$

Thus the capacitance of capacitors in series is given by

$$C = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

If they are in parallel,note that tangential $\vec{E}$ is constant. Hence

$$\rho_{\text{air}} = D_{\text{air}} = \varepsilon_0 \frac{V}{d}$$

$$\rho_{\text{diel}} = D_{\text{diel}} = \varepsilon_0 \varepsilon_r \frac{V}{d}$$

And total capacitance is

$$C = C_1 + C_2$$

where

$$C_1 = \frac{\varepsilon_0 s}{2d}$$

and

$$C_2 = \frac{\varepsilon_0 \varepsilon_r s}{2d}$$

2 Electrostatic Potential Energy

Energy can be stored in a charge distribution if there is positive work done in assembling it.

$$W_e = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i$$

Note the factor of 1/2

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Note the factor of $\frac{1}{2}$. Consider arbitrary i and j. Through the sum, we are double counting the effects of Q_i on Q_j and vice versa. However, one charge (WLOG Q_i) is introduced first, so it does not "feel" the effect of Q_j. We divide by 2 to correct for double counting.
If charge distribution is continuous,

$$W = \int_{\Omega} \rho V \, d\Omega$$

Example 2.1. What is the energy stored within a parallel plate capacitor?

$$
\begin{aligned}
W &= \frac{1}{2}\iiint \vec{D}\cdot\vec{E}dV \\
&= \frac{1}{2}\iiint \varepsilon_0\varepsilon_r|\vec{E}|^2 dv \\
&= \frac{1}{2}\frac{\rho_s^2}{\varepsilon_0\varepsilon_r}sd \\
&= \frac{1}{2}\frac{Q^2 d}{\varepsilon_0\varepsilon_r s} \\
&= \frac{1}{2}\frac{Q^2}{C}
\end{aligned}
$$

Thus we also have the relation

$$C = \frac{1}{2}\frac{Q^2}{W} = \frac{2W}{V^2}$$

The other approach is

$$
\begin{aligned}
W &= \frac{1}{2}\iint \rho_s V ds \\
&= \frac{1}{2}\rho_s V s \\
&= \frac{1}{2}QV
\end{aligned}
$$