Lecture 30

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1 Maximum Likelihood Estimation

Before $\overline{X}, S^2, \frac{\overline{X}}{n}$.
With data $x_1, \dots, x_n$, the density function

$$f(x_1, \dots, x_n; \theta) = g(x_1; \theta) \times \cdots \times g(x_n; \theta)$$

where the right hand side is the likelihood function. Then

$$\hat{\theta} = \max_{\theta} L(x_1, \dots, x_n; \theta)$$

Taking the log,

$$\hat{\theta} = \max_{\theta} \ln(L)$$

Example 1.1. In the normal case,

$$\begin{aligned}
L(x, \dots, x_n; \mu, \sigma^2) &= n(x; \mu, \sigma^2) \times \cdots \times n(x_n; \mu, \sigma^2) \\
&= \prod_{k=1}^{n} n(x_k; \mu\sigma^2) \\
&= \prod_{k} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x_k - \mu}{\sigma}\right)^2\right) \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2}\sum_{k}\left(\frac{x_k - \mu}{\sigma}\right)^2\right)
\end{aligned}$$

Taking the log, this gives

$$-\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{x_k - \mu}{\sigma}\right)^2$$

Differentiating and setting to zero,

$$\sum_k \frac{x_k - \mu}{\sigma^2} = 0 \Rightarrow \mu = \frac{1}{n} \sum_k x_k = \overline{X}$$

If we differentiate with respect to the variance,

$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_k (x_k - \mu)^2 = 0$$

$$\sigma^2 = \frac{1}{n} \sum_k (x_k - \mu)^2$$

Example 1.2. Recall the gamma distribution

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma[\alpha]} x^{\alpha-1} e^{-\frac{x}{\beta}} & x > 0 \\ 0 & x < 0 \end{cases}$$

Given $x_1, \ldots, x_n$, we can find α, β by

$$\hat{a} = \max_\alpha \prod_k f(x_k; \alpha, \beta)$$

This can be solved by numeric methods.