Lecture 17

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1 Moments and Moment-Generating Functions

Definition 1.1. The rth moment about the origin of the random variable X is

$$\mu_r = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx$$

Then obviously the mean is the first moment. The second moment is

$$\mu_2 = E[X^2] = \sigma^2 + \mu^2$$

Definition 1.2. The moment-generating function of the random variable X is

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Note the rth derivative of $M_X(t)$. For the discrete case,

$$\begin{aligned}
\left.\frac{d^r M_X(t)}{dT^r}\right|_{t=0} &= \left.\frac{d^r}{dt^r}\sum_x e^{tx} f(x)\right|_{t=0} \\
&= \left.\sum_x f(x) x^r e^{tx}\right|_{t=0} \\
&= \sum_x x^r f(x) \\
&= \mu_r
\end{aligned}$$

Similar for the continuous case.

Example 1.1. X is normal with mean μ and variance σ^2. Then

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2-2(\mu+t\sigma^2)x+\mu^2}{2\sigma^2}} dx \end{aligned}$$

Completing the square,

$$x^2 - 2(\mu + t\sigma^2)x + \mu^2 = (x - (\mu + t\sigma^2))^2 - 2\mu t\sigma^2 - t^2\sigma^4$$

Thus

$$M_X(t) = e^{\frac{2\mu t + t^2\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}} dx$$

This is just a normal PDF with mean $\mu + t\sigma^2$ and variance σ^2. Hence it integrates to 1, and

$$M_x(t) = e^{\frac{2\mu t + t^2\sigma^2}{2}}$$

2 Linear Combinations of Random Variables

Consider the random variable X with distribution $f(x)$. The distribution of $Y = aX$ is then

$$\begin{aligned} h(y) &= P(Y = y) \\ &= P(aX = y) \\ &= f\left(\frac{y}{a}\right) \end{aligned}$$

In the continuous case,

$$\begin{aligned} H(y) &= P(Y \leq y) \\ &= P(X \leq \frac{y}{a}) \\ &= \int_{-\infty}^{\frac{y}{a}} f(t)dt \end{aligned}$$

Setting $s = at$,

$$\int_{-\infty}^{\frac{y}{a}} f(t)dt = \int_{-\infty}^{y} \frac{1}{a} f\left(\frac{s}{a}\right) ds$$

Hence

$$h(y) = \frac{1}{|a|} f\left(\frac{y}{a}\right)$$

Supposed X has moment-generating function $M_X(t)$. Then

$$\begin{aligned}
M_Y(t) &= \int_{-\infty}^{\infty} e^{ty} h(y) dy \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{ty} f\left(\frac{y}{a}\right) dy \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} e^{taz} f(z) a dz \\
&= \int_{-\infty}^{\infty} e^{taz} f(z) dz \\
&= M_X(at)
\end{aligned}$$

Now supposed X and Y are independent with distributions $f(x)$ and $g(y)$. The distribution of their sum is

$$h(z) = \int_{-\infty}^{\infty} f(w) g(z-w) dw$$

Which is a convolution.

Example 2.1. The changes of rolling an 8 from 2 dice is

$$h(8) = \sum_{k=-\infty}^{\infty} f(k)g(8-k) = f(2)g(6)+f(3)g(5)+f(4)g(4)+f(5)g(3)+f(6)g(2)$$

Example 2.2. Suppose X and Y have moment-generating functions $M_X(t)$ and $M_y(t)$. The moment-generating function of $Z = X + Y$ is

$$\begin{aligned}
M_Z(t) &= \sum_{z=-\infty}^{\infty} e^{tz} h(z) \\
&= \sum_{z=-\infty}^{\infty} e^{tz} \sum_{w=-\infty}^{\infty} f(w) g(z-w) \\
&= \sum_{w=-\infty}^{\infty} f(w) \sum_{z=-\infty}^{\infty} e^{tz} g(z-w)
\end{aligned}$$

Letting $k = z - w$,

$$\begin{aligned} M_Z(t) &= \sum_{w=-\infty}^{\infty} f(w) \sum_{k=-\infty}^{\infty} e^{t(k+w)g(k)} \\ &= \sum_{w=-\infty}^{\infty} e^{tw} f(w) \sum_{k=-\infty}^{\infty} e^{tk} g(k) \\ &= M_X(t) M_Y(t) \end{aligned}$$

So the moment generating function for $X + Y$ is $M_X(t)M_Y(t)$.