Lecture 15

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1 Normal Operators

We take V as a finite dimensional complex inner product space in today's lecture.

Proposition 1.1. *$T \in \mathcal{L}(V)$ is normal iff $||Tv|| = ||T^*v|| \forall v \in V$*

Proof. We assume Lemma 1.1. Then

$$\begin{aligned}
T \text{ normal} &\Leftrightarrow TT^* = T^*T \\
&\Leftrightarrow TT^* - T^*T = 0 \\
&\Leftrightarrow \langle v, (TT^* - T^*T)v \rangle = 0 \forall v \in V \\
&\Leftrightarrow \langle T^*v, T^*v \rangle - \langle Tv, Tv \rangle = 0 \forall v \in V \\
&\Leftrightarrow ||T^*v||^2 = ||Tv||^2 \forall v \in V
\end{aligned}$$

Where the backwards implication for the third $\Leftrightarrow$ makes use of the lemma. $\square$

Lemma 1.1. *If $S \in \mathcal{L}(V)$ is self adjoint, then $S = 0$ iff $\langle Sv, v \rangle = 0 \forall v \in V$.*

Proof. The "only if" part is trivial. For the "if" part, note that for S to be self adjoint, it has a basis of eigenvectors. This implies all eigenvalues are 0, so $S = 0$. $\square$

Lemma 1.2. *If $S, T \in \mathcal{L}(V)$ with $ST = TS$, there exists a joint eigenvector $v \neq 0 \in V, Tv = \lambda v, Sv = \mu v$.*

Proof. Pick any eigenvalue λ for T. If v is one of its corresponding eigenvectors, then

$$T(Sv) = STv = \lambda(Sv)$$

So Sv is also an eigenvector. Then the eigenspace for λ is S invariant, and picking any eigenvector w for S restricted to this eigenspace completes the proof. □

Lemma 1.3. *For $T \in \mathcal{L}(V)$, if $W \subseteq V$ is T-invariant then $W^\perp \subseteq V$ is T^*-invariant. In particular, if W is T-invariant and T^*-invariant, the same is true for $W^\perp$.*

Proof. Suppose $T(W) \subseteq W$. Let $v \in W^\perp$. We want to show that $T^*v \in W^\perp$. For all $w \in W$, we have

$$\langle T^*v, w\rangle = \langle v, Tw\rangle = 0$$

Hence $T^*v \in W^\perp$. □

Theorem 1.1 (Spectrial Theorem for Normal Operators)**.** *For $T \in \mathcal{L}(V)$, the following are equivalent:*

- *T is normal*
- *V admits an orthonormal basis consisting of eigenvectors of T*

Proof. Assume T is normal. Let $v_1 \in V$ be a joint unit eigenvector for T, T^* by lemma 1.2. Then $\operatorname{span}\{v_1\}^\perp$ is invariant under T, T^*. Then T is still normal in $\operatorname{span}\{v_1\}^\perp$, and the process is repeated to build a basis.
Suppose $v_1, \ldots, v_n$ is an orthonormal basis with $Tv_i = \lambda_i v_i$. Then

$$\langle v_i, T^*v_j\rangle = \langle Tv_i, v_j\rangle = \langle \lambda_i v_i, v_j\rangle = \langle v_i, \overline{\lambda_i} v_j\rangle$$

For $i \neq j$ we see T^*v_j has no v_i component. Taking $i = j$, we get $T^*v_j = \overline{\lambda}_j v_j$. Now

$$T^*Tv_i = \lambda_i T^*v_i = \lambda_i\overline{\lambda}_i v_i = T(\overline{\lambda_i} v_i) = TT^*v_i$$

This holds for all basis vectors, so $TT^* = T^*T$. □

As a consequence, if T is normal, then the matrix of T has a basis consisting of eigenvectors is diagonal. Moreover, for eigenvalues $\lambda \neq \mu$, their eigenspaces are orthogonal.

2 Spectral Resolution

Suppose $T \in \mathcal{L}(V)$ is normal, let $P_\lambda \in \mathcal{L}(V)$ be the orthogonal projection to the eigenspace $E(\lambda, T)$. Then

$$P_\lambda P_\mu = \begin{cases} 0 & \lambda \neq \mu \\ P_\lambda & \lambda = \mu \end{cases}$$

Now

$$\sum_\lambda P_\lambda = I$$

Hence

$$T = \sum_\lambda \lambda P_\lambda$$

Then

$$T^* = \sum \lambda \bar{\lambda} P_\lambda$$