Lecture 10

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1 Continued from the last lecture...

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0.5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 200 \\ 20 \end{pmatrix}$$

with initial conditions

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 200 \\ 80 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned} (-2-\lambda)^2 - \frac{1}{2} &= \lambda^2 + 4\lambda + 4 - \frac{1}{2} \\ &= \lambda^2 + 4\lambda + \frac{7}{2} \end{aligned}$$

where the quadratic formula gives us

$$\lambda = \frac{-4 \pm \sqrt{2}}{2}$$

Now

$$A - \lambda_1 I = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Where the eigenvector is

$$\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$

and similarly

$$\vec{v_2} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

The solution to the homogeneous system is

$$\vec{\phi}(t) = c_1 e^{-\frac{4+\sqrt{2}}{2}t} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} + c_2 e^{-\frac{4-\sqrt{2}}{2}t} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

Adding the equilibrium solution, we have

$$\vec{x}(t) = c_1 e^{-\frac{4+\sqrt{2}}{2}t} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} + c_2 e^{-\frac{4-\sqrt{2}}{2}t} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 120 \\ 40 \end{pmatrix}$$

Plugging the initial conditions gives us

$$\vec{x}(0) = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 120 \\ 40 \end{pmatrix} = \begin{pmatrix} 200 \\ 80 \end{pmatrix}$$

which simplifies to

$$\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 80 \\ 40 \end{pmatrix}$$

Taking the inverse of the matrix gives us

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{40}{\sqrt{2}} - 20 \\ \frac{40}{\sqrt{2}} + 20 \end{pmatrix}$$

c_1, c_2 can be substituted to yield the general solution.

2 Behaviour of System

If A has 2 real eigenvalues, we can characterise the solutions by the eigenvalues and eigenvectors of A. How do the phase potraits behave?

Example 2.1.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -13 & 6 \\ 2 & -2 \end{pmatrix} \vec{x}$$

The eigenvalues are given by

$$\begin{aligned}(-13-\lambda)(-2-\lambda)-12 &= \lambda^2+15\lambda+26-12\\ &= \lambda^2+15\lambda+14\\ &= (\lambda+1)(\lambda+14)\end{aligned}$$

$$A-\lambda_1 I = \begin{pmatrix}1 & 6\\ 2 & 12\end{pmatrix}$$

whose eigenvector is

$$\vec{v_1} = \begin{pmatrix}-6\\ 1\end{pmatrix}$$

And the second eigenvector is

$$\vec{v_2} = \begin{pmatrix}1\\ 2\end{pmatrix}$$

The general solution is then

$$\vec{\phi}(t) = c_1 e^{-14t}\begin{pmatrix}-6\\ 1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\ 2\end{pmatrix}$$

As t goes to ∞, the solution tends to 0. As t goes to $-\infty$¡ the solution behaves like its first term. If one were to draw a phase potrait (remind me to add one after the annotated slides are out), there would be an equilibrium at the point $(0,0)$. However, non equilibrium solutions can never reach the point; they only tend to it. If this were a nonhomogeneous system, the equilibrium would be shifted by $\vec{x_{eq}}$.

Example 2.2.

$$\frac{d\vec{x}}{dt} = \begin{pmatrix}0 & 1\\ -6 & 5\end{pmatrix}\vec{x}$$

The general solution is

$$\vec{\phi}(t) = c_1 e^{2t}\begin{pmatrix}1\\ 2\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}1\\ 3\end{pmatrix}$$

The phase potrait diverges away from the equilibrium. This gives us an unstable equilibrium.