# Problem Set 4

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1. Consider a region with a uniform electrostatic field of intensity $E$. If the electric scalar potential at the point $A$ is zero, the potential at the point $B$ equals

**Solution:**
$$V_B = -Ed\cos\alpha$$

2. A point charge $Q$ is situated in free space. The line integral of the electric field intensity vector $\vec{E}$ due to this charge along the contour $C$, composed of two circular parts of radii $a$ and $2a$, respectively, and two radial parts of length $a$, amounts to

**Solution:** $\vec{E}$ is conservative, so the integral is 0.

3. What happens to electric potentials and voltages in an electrostataic system after a new reference point is adopted for the potential?

**Solution:** Potentials change by the same value and voltages remain unchanged.

4. The electrostatic potential $V$ in a aregion is a function of the rectangular coordinate $x$ only. Consider the electric field intensities at points $A, B, C, D$, and $E$. The largest field intensity is at point

**Solution:** It is where the slope is maximum, which is $C$.

5. Consider an electrostatic field in a region of space and the following two statements. Which of the statements is true?
   (a) If the electric scalar potential at a point in the region is zero, then the electric field vector at that point must be zero as well.
   (b) If the electric field vector at a point in zero, then the potential at the same point must be zero.

**Solution:** None of the statements are true. The statements simply discuss if there is any implication between $x = 0$ and $x' = 0$, where obvious counterexamples can be found.

6. An uncharged thin metallic rod is introduced into a uniform electrostatic field, of intensity vector $\vec{E}_0$, in free space, such that it is either perpendicular or parallel to $\vec{E}_0$. The rod affects the original field

> **Solution:** Less in case (a). As the rod is a conductor, the electric field becomes close to zero, depending on conductivity.

7. A uniform electric field, of intensity vector $\vec{E}_0$, is established in the air-filled space between two metallic electrodes. If an uncharged (thick) metallic slab is then inserted in this space, without touching the electrodes, the electric field intensity vector in region 3 in the new electrostatic state is

> **Solution:**
> $$E = \frac{V}{d}$$
> where $d$ is the distance. Substituting $V$ with $\frac{V}{2}$ and $d$ with $\frac{d}{3}$ gives
> $$\vec{E}_3 = \frac{3\vec{E}_0}{2}$$

8. A negatively charged small body is situated inside an uncharged spherical metallic shell. The distribution of induced charges on the outer surface of the shell can be represented as in

> **Solution:** A

9. In order to protect body $B$ from the electrostatic field due to a charged body $A$, an ungrounded closed metallic screen is introduced. The protection is achieved for

> **Solution:** b only.

10. point charge $Q$ is situated in air at a height $h$ above a grounded conducting plane. Relative to the plane, the electric force on this charge is

> **Solution:** Always attractive.

11. The electrostatic potential $V$ in a region is a function of a single rectangular coordinate $x$, $V(x)$ and is shown in the figure below. Sketch the components of the electric field intensity $E$ in this region

**Solution:**

| $x$ | 0-1 | 1-2 | 2-3 | 3-4 | 4-5 | 5-6 | 6-7 | 7-8 | 8-9 | 9-10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $E(x)$ | 2 | 1 | 0 | -1 | -2 | -2 | -1 | 0 | 1 | 2 |

12. For the three charges in the figure below, calculate the electric potential at points defined by
    (a) (0,0,2)

(b) (1,1,1)

**Solution:** $(0,0,2)$:

$$\frac{2\times 10^{-6}k}{1}-\frac{2\times 10^{-6}k}{\sqrt{5}}+\frac{10^{-6}k}{\sqrt{5}}=10^{-6}k\left(2-\frac{1}{\sqrt{5}}\right)$$

$(1,1,1)$:

$$\frac{(1-2+2)10^{-6}k}{\sqrt{2}}=\frac{10^{-6}k}{\sqrt{2}}$$

13. For the semi-circular line charge in the figure below, the electric field at an arbitrary point on the $z$-axis has an $x$ and a $z$ component (confirm). Find the $z$-component of the field from the potential $V(0,0,z)$. Can you find the $x$-component too using $V(0,0,z)$?

**Solution:** By symmetry, there is no $y$ component.

$$\begin{aligned}V(0,0,z) &= \int \frac{kdQ}{r} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ka\rho_l d\phi}{\sqrt{a^2+z^2}} \\ &= \frac{ka\rho_l\pi}{\sqrt{a^2+z^2}} \\ &= \frac{kQ}{\sqrt{a^2+z^2}}\end{aligned}$$

Then the $z$-component is

$$\begin{aligned}E_z &= -\frac{dV(0,0,z)}{dz} \\ &= \frac{kQ}{2(a^2+z^2)^{3/2}}\times 2z \\ &= \frac{kQz}{(a^2+z^2)^{3/2}}\end{aligned}$$

The $x$-component cannot be found.

14. Two point charges $Q_1 = 7\mu\,\mathrm{C}$, and $Q_2 = -3\mu\,\mathrm{C}$, are located on two non-adjacent vertices of a square contour $a = 15\mathrm{cm}$ on a side. Find the voltage between any of the remaining two vertices of the square and the square center.

**Solution:** By symmetry, both voltages are the same.

$$\frac{(7-3)\times 10^{-6}k}{0.15}-\frac{(7-3)\times 10^{-6}k}{0.15\div\sqrt{2}}=-99.3\mathrm{kV}$$

15. Obtain the expression for the electric field intensity of a point charge at the origin from its potential.

**Solution:** Division by $r$.

16. Determine the work done in carrying a $-2\mu$C charge from $P_1(2,1,-1)$, to $P_2(8,2,-1)$ in the field $\vec{E} = \hat{a}_x y + \hat{a}_y x$
    (a) Along the parabola $x = 2y^2$
    (b) Along the straight line joining $P_1$ and $P_2$

**Solution:**

$$\begin{aligned} E &= -\int \vec{F} \cdot d\vec{l} \\ &= 2 \times 10^{-6} \int (\hat{a}_x y + \hat{a}_y x) \cdot d\vec{l} \end{aligned}$$

For the parabola,

$$E = 2 \times 10^{-6} \int_1^2 (\hat{a}_x y + \hat{a}_y 2y^2) \cdot (4y\hat{a}_x + \hat{a}_y) dy = 2 \times 10^{-6} \int_1^2 (4y^2 + 2y^2) dy = 28 \times 10^{-6}$$

For the line,

$$E = 2 \times 10^{-6} \int_1^2 (\hat{a}_x y + \hat{a}_y (6y-4)) \cdot (6\hat{a}_x + \hat{a}_y) dy = 2 \times 10^{-6} \int_1^2 (6y + 6y - 4) dy = 28 \times 10^{-6}$$

17. Three charges ($+q$, $-2q$, and $+q$) are arranged along the $z$-axis at $z = \frac{d}{2}$, $z = 0$, and $z = -\frac{d}{2}$, respectively.
    (a) Determine $V$ and $\vec{E}$ at a distant point $P(R, \theta, \phi)$.
    (b) Find the equations for equipotential surfaces and streamlines.
    (c) Sketch a family of equipotential llines and streamlines.

**Solution:**

$$V = \frac{kQ}{r} = \frac{kq}{\sqrt{(R\cos\theta - \frac{d}{2})^2 + (R\sin\theta)^2}} - \frac{2kq}{R} + \frac{kq}{\sqrt{(R\cos\theta + \frac{d}{2})^2 + (R\sin\theta)^2}}$$

Note that the denominator can be written as

$$\sqrt{R^2 \pm d\cos\theta R + \frac{d^2}{4}} = R\sqrt{1 \pm \frac{d\cos\theta}{R} + \frac{d^2}{4R^2}}$$

Approximating the square root,

$$\left(1 \pm \frac{d\cos\theta}{R} + \frac{d^2}{4R^2}\right)^{\frac{1}{2}} \approx 1 \mp \frac{d\cos\theta}{2R} - \frac{d^2}{8R^2} + \frac{3d^2\cos^2\theta}{8R^2} + \frac{3d^4}{128R^4} \pm \frac{3d^3\cos\theta}{16R^3} \approx 1 \mp \frac{d\cos\theta}{2R} - \frac{d^2}{8R^2} + \frac{3d^2\cos^2\theta}{8R^2}$$

Then

$$V = \frac{kq}{R}\left(1 - \frac{d\cos\theta}{2R} - \frac{d^2}{8R^2} + \frac{3d^2\cos^2\theta}{8R^2} - 2 + 1 + \frac{d\cos\theta}{2R} - \frac{d^2}{8R^2} + \frac{3d^2\cos^2\theta}{8R^2}\right) = \frac{kqd^2}{4R^3}\left(3\cos^2\theta - 1\right)$$

Note that due to symmetry, $\vec{E}$ has no $\hat{a}_\phi$ component.

$$\begin{aligned}
\vec{E} &= -\vec{\nabla} V \\
&= -\frac{\partial V}{\partial R}\hat{a}_R - \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{a}_\theta \\
&= \frac{3kqd^2}{4R^4}(3\cos^2\theta - 1)\hat{a}_R + \frac{kqd^2}{4R^4}(6\cos\theta\sin\theta)\hat{a}_\theta \\
&= \frac{3kqd^2}{4R^4}((3\cos^2\theta - 1)\hat{a}_R + \sin 2\theta\hat{a}_\theta)
\end{aligned}$$

18. Given the finite line charge $\rho_l$ of length $L$, find the potential on the cylindrical surface of radius $b$ as a function of $x$ and plot it.

**Solution:**

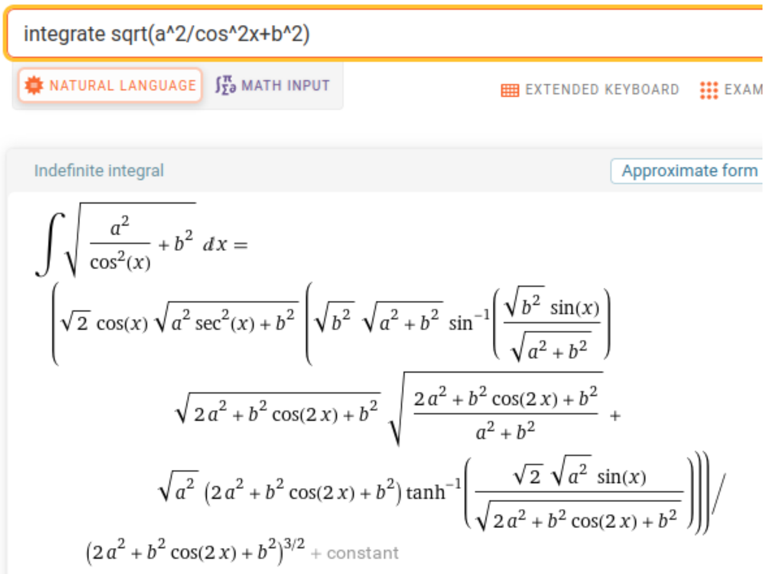
$$\begin{aligned}
V(X) &= \int_0^L \frac{k\rho_l dx}{\sqrt{(x-X)^2 + b^2}} \\
&= k\rho_l \int_{\arctan\left(-\frac{X}{b}\right)}^{\arctan\left(\frac{L-X}{b}\right)} \frac{b\sec^2\theta d\theta}{b\sec\theta} \\
&= k\rho_l \ln|\sec\theta + \tan\theta|\Big|_{\arctan\left(-\frac{X}{b}\right)}^{\arctan\left(\frac{L-X}{b}\right)} \\
&= k\rho_l \ln\frac{\sqrt{(L-X)^2 + b^2} + L - X}{\sqrt{X^2 + b^2} - X}
\end{aligned}$$

19. A charge $Q$ is distributed uniformly over an $L \times L$ square plate. Determine $V$ and $\vec{E}$ at a point on the axis perpendicular to the plate and through its center.

**Solution:**

$$\begin{aligned}
V(z) &= \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k\rho_s dxdy}{\sqrt{x^2+y^2+z^2}} \\
&= 8k\rho_s \int_0^{\frac{\pi}{4}}\int_0^{\frac{L}{2\cos\theta}} \frac{rdrd\theta}{\sqrt{r^2+z^2}} \\
&= 8k\rho_s \int_0^{\frac{\pi}{4}} \sqrt{\frac{L^2}{4\cos^2\theta} + z^2} - |z| d\theta
\end{aligned}$$

Where part of the integral is given by this disgusting thing (cancels out for $x = 0$ though).




20. A point charge $2Q$ is placed at the center of an air-filled spherical metallic (perfectly conducting) shell, charged with $Q$ and situated in air. The inner and outer radii of the shell are $a$ and $b$ ($a < b$). What is the total charge on the inner and outer surface of the shell, respectively? Find the potential of the shell.

**Solution:** There is no electric field in the conductor. By Gauss' Law, this means the net enclosed charge $\forall r \in (a, b)$ is 0, meaning a charge of $-2Q$ is concentrated at the inner surface. For the total charge to be $Q$, there is a charge of $3Q$ on the outer surface. Due to Gauss' Law, if there is spherical symmetry, we can treat all the charges to be concentrated at the centre. The potential is then

$$V = \frac{3kQ}{b}$$

21. Prove that the electric potential at an arbitrary point on the $z$-axis produced by the semi-circular line charge of 13 is

$$V = \frac{\rho_l a}{4\varepsilon_0\sqrt{z^2 + a^2}}$$

**Solution:**

$$\begin{aligned} V &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho_l a d\phi}{4\pi\varepsilon_0\sqrt{z^2 + a^2}} \\ &= \frac{\rho_l a}{4\varepsilon_0\sqrt{z^2 + a^2}} \end{aligned}$$

22. Determine the electric potential at the center of a non-uniformly charged spherical surface of radius $a$, with $\rho_s(\theta) = \rho_{s,o}\sin(2\theta)$, with $\rho_{s,o}$ a constant and the angle $0 \le \theta \le \pi$.

**Solution:**

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\pi} \frac{k\rho_{s,o}\sin 2\theta a \sin\theta d\theta d\phi}{a} \\
&= k\rho_{s,o}\int_0^{2\pi}\int_0^{\pi} 2\sin^2\theta d(\sin\theta) d\phi \\
&= k\rho_{s,o}\int_0^{2\pi} 0 d\phi \\
&= 0
\end{aligned}
$$

Or simply, note that for any point on the sphere, its opposite point on the sphere has the opposite charge, hence they all cancel out.

23. Find the distribution of the electric scalar potential inside and outside a sphere $R = \alpha$ with volume charge density: $\rho = \rho_0 \frac{R}{\alpha}$, where $R$ is the radial coordinate of the spherical coordinate system.

**Solution:** Charge inside the sphere at radius $r$ is

$$
\begin{aligned}
Q &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0 \frac{R}{\alpha} R^2 \sin\theta dr d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi} \rho_0 \frac{r^4}{4\alpha} \sin\theta d\theta d\phi \\
&= \int_0^{2\pi} \rho_0 \frac{r^4}{2\alpha} d\phi \\
&= \frac{r^4 \rho_0 \pi}{\alpha}
\end{aligned}
$$

Then using Gauss' Law, $\vec{E}$ inside the sphere is

$$
\begin{aligned}
4\pi r^2 E_r &= \frac{r^4 \rho_0 \pi}{\alpha\varepsilon_0} \\
E_r &= \frac{r^2 \rho_0}{4\alpha\varepsilon_0}
\end{aligned}
$$

Ouside of the sphere,

$$E_r = \frac{\alpha^3 \rho_0}{4\varepsilon_0 r^2}$$

And voltage outside the sphere is

$$V = \frac{\alpha^3 \rho_0}{4\varepsilon_0 r}$$

Voltage inside the sphere is

$$\int_r^{\infty} E dr = \int_r^{\alpha} E dr + \int_{\alpha}^{\infty} E dr = \frac{\rho_0(\alpha^3 - R^3)}{12\varepsilon_0\alpha} + \frac{\rho_0\alpha^2}{4\varepsilon_0} = \frac{\rho_0\alpha^2}{3\varepsilon_0} - \frac{\rho_0 R^3}{12\varepsilon_0\alpha}$$

24. Find the work done by electric forces in moving a charge $Q = 1\text{nC}$ from the coordinate origin to the point (1m, 1m, 1m) in the electrostatic field given by $\vec{E}(x, y, z) = (x\hat{a}_x + y^2\hat{a}_y - \hat{a}_z)$ V/m along the straight line.

**Solution:**

$$\begin{aligned}\text{Work Done} &= -\int 10^{-9}(x\hat{a}_x + y^2\hat{a}_y - \hat{a}_z)\cdot d\vec{l} \\ &= -\int_0^1 10^{-9}(t\hat{a}_x + t^2\hat{a}_y - \hat{a}_z)\cdot(\hat{a}_x + \hat{a}_y + \hat{a}_z)dt \\ &= -\int_0^1 10^{-9}(t^2 + t - 1)dt \\ &= \frac{1}{6}\text{nJ}\end{aligned}$$

25. A finite line charge of length $L$ carrying uniform line charge density $\rho_l$ is coincident with the $x$-axis. In the plane bisecting the line charge ($yz$-plane) determine potential $V$.

**Solution:**

$$\begin{aligned}V &= 2\int_0^L \frac{k\rho_l dx}{\sqrt{x^2+y^2+z^2}} \\ &= 2k\rho_0\left(\ln\left(\sqrt{\frac{L^2}{4}+y^2+z^2}+\frac{L}{2}\right) - \ln\sqrt{y^2+z^2}\right)\end{aligned}$$

26. Charge $Q$ is distributed over the wall of a circular tube of radius $b$ and height $h$. The tube sits on $xy$-plane with its axis coinciding with $z$-axis. Determine $V$ and $\vec{E}$ along the axis of the tube.

**Solution:**

$$\begin{aligned}V(Z) &= \int_0^h \int_0^{2\pi} \frac{k\rho_s b d\phi dz}{\sqrt{(z-Z)^2+b^2}} \\ &= \frac{\rho_s b}{2\varepsilon_0}\int_0^h \frac{dz}{\sqrt{(z-Z)^2 = b^2}} \\ &= \frac{\rho_s b}{2\varepsilon_0}\ln\frac{\sqrt{(Z-h)^2+b^2}-(Z-h)}{\sqrt{Z^2+b^2}-Z}\end{aligned}$$

Differentiating,

$$E = \frac{\rho_s b}{2\varepsilon_0}\left(\frac{1}{\sqrt{(z-h)^2+b^2}} - \frac{1}{\sqrt{z^2+b^2}}\right)$$

27. Consider the system of two metallic spheres connected by a wire. Assume that $a = 5$ cm, $b = 1$ cm, and $d = 1$ m, as well as that the total charge of the two spheres is $Q = 600$pC. Find the potential of the spheres and the electric field intensities $E_a, E_b$ near the surfaces of the spheres.

**Solution:** The surface voltages have to be equal, so

$$
\begin{aligned}
V_a &= V_b \\
\frac{kQ_a}{a} &= \frac{kQ_b}{b} \\
\frac{Q_a}{a} &= \frac{Q_b}{b}
\end{aligned}
$$

Considering the sum of charges,

$$Q_a = 500\text{pC}, Q_b = 100\text{pC}$$

Substituting, the potentials are then 89.9V.
The fields are then $\frac{4.49}{r^2}$ and $\frac{0.899}{r^2}$ respectively.