Homework 3

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1. A coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

Solution: Then $P(H) = 0.75, P(T) = 0.25$. The expected number is

$$0 \times 0.75^2 + 1 \times 2 \times 0.75 \times 0.25 + 2 \times 0.25^2 = 0.5$$

2. The density function of coded measurements of the pitch diameter of threads of a fitting is

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of X.

Solution:

$$\begin{aligned} E(X) &= \int_0^1 \frac{4x dx}{\pi(1+x^2)} \\ &= \frac{4}{\pi} \int_0^1 \frac{x}{1+x^2} \\ &= \frac{2 \ln 2}{\pi} \end{aligned}$$

3. Assume that two random variables (X, Y) are uniformly distributed on a circle with radius a. Then the joint probability density function is

$$f(x, y) = \begin{cases} \frac{1}{\pi a^2} & x^2 + y^2 \leq a^2 \\ 0 & \text{otherwise} \end{cases}$$

Find μ_X, the expected value of X.

Solution: As they are distributed on the circle, $x^2 + y^2 \leq a^2$ has to be true. Then $f(x, y)$ only takes on one value on the circle, so the expected value is 0 by symmetry.

4. A continuous random variable X has the density function

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of $g(X) = e^{2X/3}$

Solution:

$$\begin{aligned} E(g(X)) &= \int_0^\infty e^{2x/3} e^{-x} dx \\ &= \int_0^\infty e^{-x/3} dx \\ &= -3[e^{-x/3}]_0^\infty \\ &= 3 \end{aligned}$$

5. Suppose that X and Y have the following joint probability function:

y/x	2	4
1	0.10	0.15
3	0.20	0.30
5	0.10	0.15

(a) Find the expected value of $g(X, Y) = XY^2$.

Solution:

$$0.10(2 \times 1^2 + 2 \times 5^2) + 0.15(4 \times 1^2 + 4 \times 5^2) + 0.20 \times 2 \times 3^2 + 0.30 \times 4 \times 3^2 = 35.2$$

(b) Find μ_X and μ_Y.

Solution:

$$\mu_X = 2 \times (0.10 + 0.20 + 0.10) + 4 \times (0.15 + 0.30 + 0.15) = 3.2$$

$$\mu_Y = 1 \times (0.10 + 0.15) + 3 \times (0.20 + 0.30) + 5 \times (0.10 + 0.15) = 3$$

6. The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport is a random variable $Y = 3X - 2$, where X has the density function

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the mean and variance of the random variable Y.

Solution: The mean is

$$\begin{aligned}\mu &= \int_0^\infty \frac{1}{4}(3x-2)e^{-x/4}dx \\ &= -(3x-2)e^{-x/4}\Big|_0^\infty + 3\int_0^\infty e^{-x/4}dx \\ &= 10\end{aligned}$$

The variance is

$$\begin{aligned}\sigma^2 &= E((Y-\mu)^2) \\ &= \int_0^\infty (3x-12)^2 \times \frac{1}{4}e^{-x/4}dx \\ &= \frac{9}{4}\int_0^\infty (x^2-8x+16)e^{-x/4} \\ &= 18\int_0^\infty xe^{-x/4}dx - 18\int_0^\infty xe^{-x/4} + 144 \\ &= 144\end{aligned}$$

7. For a laboratory assignment, if the equipment is working, the density function of the observed outcome X is

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the variance and standard deviation of X

Solution: The mean is

$$\begin{aligned}\mu &= \int_0^1 2x(1-x)dx \\ &= \int_0^1 2x - 2x^2 dx \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3}\end{aligned}$$

Then the variance is

$$\begin{aligned}\sigma^2 &= E(X^2) - \mu^2 \\ &= \int_0^1 2x^2(1-x)dx - \frac{1}{9} \\ &= \int_0^1 2x^2 - 2x^3 dx - \frac{1}{9} \\ &= \frac{2}{3} - \frac{1}{2} - \frac{1}{9} \\ &= \frac{1}{18}\end{aligned}$$

The standard deviation is then

$$\sigma = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$$

8. Random variables X and Y follow a joint distribution

$$f(x, y) = \begin{cases} 2 & 0 < x \leq y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the correlation coefficient between X and Y.

Solution: The means are

$$\begin{aligned} \mu_x &= \int_0^1 \int_x^1 2x dy dx \\ &= \int_0^1 2x(1-x) dx \\ &= \int_0^1 2x - 2x^2 dx \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} \mu_y &= \int_0^1 \int_x^1 2y dy dx \\ &= \int_0^1 1 - x^2 dx \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

Then the covariance is

$$\begin{aligned} \sigma_{XY} &= E(XY) - \mu_X \mu_Y \\ &= \int_0^1 \int_x^1 2xy dy dx - \frac{2}{9} \\ &= \int_0^1 x(1-x^2) dx - \frac{2}{9} \\ &= \int_0^1 x - x^3 dx - \frac{2}{9} \\ &= \frac{1}{2} - \frac{1}{4} - \frac{2}{9} \\ &= \frac{1}{36} \end{aligned}$$

Then the standard deviations are given by

$$\begin{aligned}
\sigma_x^2 &= E(X^2) - \mu_x^2 \\
&= \int_0^1 \int_x^1 2x^2 dy dx - \frac{1}{9} \\
&= \int_0^1 2x^2(1-x) dx - \frac{1}{9} \\
&= \int_0^1 2x^2 - 2x^3 dx - \frac{1}{9} \\
&= \frac{2}{3} - \frac{1}{2} - \frac{1}{9} \\
&= \frac{1}{18} \\
\sigma_x &= \frac{1}{3\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
\sigma_y^2 &= E(Y^2) - \mu_y^2 \\
&= \int_0^1 \int_x^1 2y^2 dy dx - \frac{4}{9} \\
&= \frac{2}{3} \int_0^1 1 - x^3 dx - \frac{4}{9} \\
&= \frac{2}{3}(1 - \frac{1}{4}) - \frac{4}{9} \\
&= \frac{1}{18} \\
\sigma_y &= \frac{1}{3\sqrt{2}}
\end{aligned}$$

Combining, the correlation coefficient is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1}{2}$$

9. Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find $E(X)$ and $E(X^2)$ and then, using these values, evaluate $E[(2X+1)^2]$.

Solution: We have

$$E(X) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$$

and

$$E(X^2) = 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = \frac{93}{2}$$

Combining,

$$E[(2X+1)^2] = E(4X^2+4X+1) = 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = 209$$

10. Let X represent the number that occurs when a red die is tossed and Y the number that occurs when a green die is tossed. Find
    (a) $E(X+Y)$
    (b) $E(X-Y)$
    (c) $E(XY)$

Solution:

$$E(X+Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$$
$$E(X-Y) = E(X) - E(Y) = 0$$
$$E(XY) = E(X)E(Y) = 3.5 \times 3.5 = 12.25$$

11. Let X represent the number that occurs when a green die is tossed and Y the number that occurs when a red die is tossed. Find the variance of the random variable
    (a) $2X - Y$
    (b) $X + 3Y - 5$

Solution: The mean of $2X - Y$ is obviously $2 \times 3.5 - 3.5 = 3.5$. Then the variance is

$$\begin{aligned} E[(2X-Y-3.5)^2] &= E(4X^2 + Y^2 + 12.25 - 4XY - 14X + 7Y) \\ &= 4 \times \frac{91}{6} + \frac{91}{6} + 12.25 - 4 \times 3.5^2 - 14 \times 3.5 + 7 \times 3.5 \\ &= \frac{175}{12} \end{aligned}$$

The mean of $X + 3Y - 5$ is obviously $3.5 + 3 \times 3.5 - 5 = 9$. Then the variance is

$$\begin{aligned} E[(X+3Y-5-9)^2] &= E[(X+3Y-14)^2] \\ &= E(X^2 + 9Y^2 + 196 + 6XY - 28X - 84Y) \\ &= \frac{91}{6} + 9 \times \frac{91}{6} + 196 + 6 \times 3.5^2 - 28 \times 3.5 - 84 \times 3.5 \\ &= \frac{175}{6} \end{aligned}$$