Lecture 5

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1 Triple Integrals

$$\iiint f(x,y,z)dV$$

Given a continuous $w = f(x,y,z)$ over a region Q with a volume V, one can break$V$ into n subintervals (ΔV_i's). $\forall$ sample point $P_i(x_i^*, y_i^*, z_i^*) \in \Delta V_i$, the upper and lower limits can be defined using the maximum M_i and minimum m_i of f in ΔV_i.

$$\text{Lower Sum: } \sum_{i=1}^{n} m_i \Delta V_i$$

$$\text{Upper Sum: } \sum_{i=1}^{n} M_i \Delta V_i$$

If f is continuous in V,

$$\lim_{||P||\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta V_i = \iiint_Q f(x,y,z)dV$$

In rectangular coordinates, we have

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$$

so

$$\iiint_Q f(x,y,z)dV = \iiint_Q f(x,y,z)dxdydz$$

Example 1.1. Suppose $f(x, y, z)$ is a continuous function defined on the box region Q, given by

$$Q = \{(x, y, z)|a \le x \le b, c \le y \le d, r \le z \le s\}$$

The limits can then be written as

$$\iiint_Q f(x, y, z)dV = \int_r^s \int_c^d \int_a^b dxdydz$$

Example 1.2. Same as above, but

$$Q = \{(x, y, z)|(x, y) \in \mathbb{R} \text{ and } g_1(x, y) \le z \le g_2(x, y)\}$$

The limits can then be written as

$$\iiint_Q f(x, y, z)dV = \int_c^d \int_a^b \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dzdxdy$$

Example 1.3. Evaluate $\iiint_Q 6xydV$ where Q is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 4$.

$$\begin{aligned}
\iiint_Q 6xydV &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} 6xydzdydx \\
&= \int_0^2 \int_0^{4-2x} 24xy - 12x^2y - 6xy^2dydx \\
&= \int_0^2 48x^3 - 192x^2 + 192x - 24x^4 + 96x^3 - 96x^2 + 16x^4 - 96x^3 + 192x^2 - 128xdx \\
&= \int_0^2 -8x^4 + 48x^3 - 96x^2 + 64xdx \\
&= -8 \times \frac{2^5}{5} + 12 \times 16 - 32 \times 8 + 32 \times 4 \\
&= \frac{64}{5}
\end{aligned}$$

Example 1.4. Evaluate the same integral, but integrate with respect to x

first.

$$\iiint_Q 6xy dV = \int_0^4 \int_0^{4-z} \int_0^{2-\frac{y}{2}-\frac{z}{2}} 6xy dx dy dz$$
$$= \frac{64}{5}$$

Example 1.5. Using a triple integral, find the volume of the solid bounded by the surface $z = 4 - y^2$ and planes given by $x + y = 4$, $x = 0$ and $y = 0$.

$$\begin{aligned}
\iiint_Q dV &= \int_{-2}^{2} \int_0^{4-y^2} \int_0^{4-z} dx dz dy \\
&= \int_{-2}^{2} \int_0^{4-y^2} 4 - z dz dy \\
&= \int_{-2}^{2} 16 - 4y^2 - \frac{y^4}{2} + 4y^2 - 8 dy \\
&= 64 - \frac{32}{5} - 32 \\
&= \frac{128}{5}
\end{aligned}$$

It is less convenient to integrate with respect to z first, or else the region will have to be spit in 2.

Example 1.6. Change the order of integration in the following triple iterated integral such that the integrations are performed in the order x, y, z with appropriate limits.

$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x,y,z) dz dy dx = \int_0^1 \int_0^{z-1} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z) dx dy dz$$

2 Applications of Triple Integrals

2.1 Mass

Total mass of a volume is given by

$$m = \iiint_Q \rho(x,y,z) dV$$

where ρ is the density.

2.2 Centre of Mass

$$\overline{x} = \frac{\iiint_Q x\rho(x,y,z)dV}{m}$$
$$\overline{y} = \frac{\iiint_Q y\rho(x,y,z)dV}{m}$$
$$\overline{z} = \frac{\iiint_Q z\rho(x,y,z)dV}{m}$$

2.3 Centroid

$$x_c = \frac{\iiint_Q xdV}{V}$$
$$y_c = \frac{\iiint_Q ydV}{V}$$
$$z_c = \frac{\iiint_Q zdV}{V}$$

2.4 Moment of inertia

$$I = \iiint_Q \rho(x,y,z)[r(x,y,z)]^2 dV$$

Example 2.1. Find the centre of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and

$x = 1$.

$$
\begin{aligned}
m &= \iiint_Q \rho dV \\
&= \rho \int_{-1}^{1} \int_{y^2}^{1} \int_0^x dz dx dy \\
&= \rho \int_{-1}^{1} \int_{y^2}^{1} x dx dy \\
&= \rho \int_{-1}^{1} \frac{1}{2} - \frac{y^4}{2} dy \\
&= \rho \left(1 - \frac{1}{5}\right) \\
&= \frac{4}{5}\rho
\end{aligned}
$$

$$
\begin{aligned}
\overline{x} &= \frac{1}{m} \int_{-1}^{1} \int_{y^2}^{1} \int_0^x x dz dx dy \\
&= \frac{1}{m} \int_{-1}^{1} \int_{y^2}^{1} x^2 dx dy \\
&= \frac{1}{m} \int_{-1}^{1} \frac{1}{3} - \frac{y^6}{3} dy \\
&= \frac{1}{m} \left(\frac{2}{3} - \frac{2}{21}\right) \\
&= \frac{5}{7\rho}
\end{aligned}
$$

Due to symmetry, $\overline{y} = 0$.

$$\begin{aligned}
\overline{z} &= \frac{1}{m}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} z dz dx dy \\
&= \frac{1}{m}\int_{-1}^{1}\int_{y^2}^{1} \frac{x^2}{2} dx dy \\
&= \frac{1}{m}\int_{-1}^{1} \frac{1}{6} - \frac{y^6}{6} dx \\
&= \frac{1}{m}\left(\frac{1}{3} - \frac{1}{21}\right) \\
&= \frac{5}{14\rho}
\end{aligned}$$

Example 2.2. Find the moment of inertia of a cylinder about its axis, given the density ρ is a constant.

$$\begin{aligned}
I_z &= 4\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{h} \rho(x^2+y^2) dz dy dx \\
&= 4\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}} h\rho(x^2+y^2) dy dx \\
&= 4\int_{0}^{2\pi}\int_{0}^{a} h\rho r^3 dr d\theta \\
&= \int_{0}^{2\pi} h\rho a^4 d\theta \\
&= 2\pi h\rho a^4 \\
&= 2ma^2
\end{aligned}$$