Lecture 22

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1 Finding Particular Solutions

$g(t)$	Ansatz
αe^{kt}	Ae^{kt}
$a\sin kt$ or $a\cos kt$	$A\sin(kt) + B\cos(kt)$

2 Superposition Principle

Theorem 2.1. *Supposed $y_1(t)$ is a solution to*

$$ay''(t) + by'(t) + cy(t) = g_1(t)$$

and $y_2(t)$ is a solution to

$$ay''(t) + by'(t) + cy(t) = g_2(t)$$

Then $y(t) = y_1(t) + y_2(t)$ is a solution to

$$ay''(t) + by'(t) + cy(t) = g(t)$$

The proof is trivial. We then can split the nohomogeneous term.

Example 2.1.

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t$$

We split this into equations

$$y'' - 3y' - 4y = 3e^{2t} \tag{1}$$

and

$$y'' - 3y' - 4y = 2\sin t \tag{2}$$

Let the particular solution for equation (n) be $y_n(t)$, then the particular solution for the equation is

$$y(t) = y_1(t) + y_2(t)$$

Example 2.2.

$$y'' - 3y' - 4y = 2e^{-t}$$

If we try the particular solution $y_p(t) = Ae^{-t}$, we get

$$\begin{aligned} Ae^{-t} + 3Ae^{-t} - 4Ae^{-t} &= 2e^{-t} \\ 0 &= 2e^{-t} \end{aligned}$$

which is bad. One can try $y(t) = Ate^{-t}$ by consulting the solution manual. Therefore, a different ansatz is needed if it happens to be the general solution.

3 Variation of Parameters

The method of undetermined coefficients is not a general approach, as ansatz are needed for every case, yet not every case is covered, and ansätze do not work if they happen to be the general solution.

Now consider

$$\frac{d\vec{x}}{dt} = P(x)\vec{x}(t) + \vec{g}(t)$$

and assume we have linearly independent vectors $\vec{x_1}(t), \vec{x_2}(t)$ that are solutions to the homogeneous equation. Then our guess of a particular funtion is

$$\vec{x}(t) = u_1(t)\vec{x_1}(t) + u_2(t)\vec{x_2}(t) = X(t)\vec{u}(t)$$

This is similar to the general solution for the homogeneous equation, but only with functions instead of constants as coefficients. Plugging into the ODE,

$$\begin{aligned} \frac{d\vec{x}}{t} &= P(t)\vec{x}(t) + \vec{g}(t) \\ X'(t)\vec{u}(t) + X(t)\vec{u}'(t) &= P(t)X(t)\vec{u}(t) + \vec{g}(t) \\ P(t)X(t)\vec{u}(t) + X(t)\vec{u}'(t) &= P(t)X(t)\vec{u}(t) + \vec{g}(t) \\ X(t)\vec{u}'(t) &= \vec{g}(t) \\ \vec{u}'(t) &= X^{-1}(t)\vec{g}(t) \end{aligned}$$

Note that on the third line, we substitute the first term back into the homogeneous equation, and on the last line, $X(t)$ is invertible as its determiant is nonzero $\forall t$. Integrating, we then have

$$\vec{u}(t) = \vec{c} + \int X^{-1}(t)\vec{g}(t)dt$$

and then

$$\vec{x}(t) = X(t)\vec{c} + X(t) \int X^{-1}(t)\vec{g}(t)dt$$

The first term is just the general solution, so the particular solution can be reduced to the second term.

Example 3.1.

$$\frac{d\vec{x}}{t} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 10\cos t \\ 2e^{-t} \end{bmatrix}$$

The general solution is spanned by

$$\vec{x}_1(t) = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$\vec{x}_2(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The inverse matrix is

$$X^{-1}(t) = \begin{bmatrix} e^{t} & -e^{t} \\ -e^{3t} & 2e^{3t} \end{bmatrix}$$

So

$$\begin{aligned} \vec{x}_p(t) &= X(t) \int X^{-1}(t)\vec{g}(t)dt \\ &= X(t) \int \begin{bmatrix} e^{t} & -e^{t} \\ -e^{3t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} 10\cos t \\ 2e^{-t} \end{bmatrix} dt \\ &= \begin{bmatrix} 7\cos t + 9\sin t + 2(1-2t)e^{-t} \\ 2\cos t + 4\sin t + 2(1-t)e^{-t} \end{bmatrix} \end{aligned}$$

Note that some steps were skipped.

This can be applied to second order ODEs, as they are but special cases of linear first order ODEs. Then for

$$ay''(t) + by'(t) + cy(t) = g(t)$$

and solutions $y_1(t), y_2(t)$,

$$X(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$