Lecture 1

niceguy

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Definition 0.1. A differential equation is an equation that relates a function with its derivatives.

Example 0.1. ODEs:

- $y'(t) = y(t)$
- $y''(t) = 6y'(t) + 2y(t) + 8$
- $y'''(t) = [y'(t)]^2$
- $u_t(t, x, y, z) = u_{xx} + u_{yy} + u_{zz}$

Differential equations can be used to model physical phenomenons, such as weather forecasting, pricing financial derivatives, population dynamics, etc.

1 Newton's Law of Cooling

Form a *simple* model for the temperature of a cup of coffee sitting in a room. We let $u(t)$ be the temperature of coffee at time t, and T be the (constant) ambient temperature.
First model:

$$\frac{du}{dt} = -(u(t) - T)$$

where the temperature of the cup tends to the ambient temperature at a rate proportional to the difference in temperatures.
Second model:

$$\frac{du}{dt} = -k(u(t) - T), k > 0$$

where k is a transmission coefficient, representing factors such as the thermal conductivity of the cup.

1.1 Question of Interest

1. Existence of a solution

2. Uniqueness

First assume $u \neq T$. Then division gives us

$$\begin{aligned} \frac{1}{u-T}\frac{du}{dt} &= -k \\ \frac{1}{u-T}du &= -kdt \\ \ln|u-T| &= -kt + C \\ u - T &= \pm e^C e^{-kt} \end{aligned}$$

Hence $u(t) = \tilde{C}e^{-kt} + T, \tilde{C} \in \mathbb{R}\backslash\{0\}$

If $u = T$, then there is an equilibrium.
All solutions can be written in the general form of

$$u(t) = \tilde{C}e^{-kt} + T, \tilde{C} \in \mathbb{R}$$

To verify this, we can differentiate $u(t)$ as given by the solution

$$u'(t) = -\tilde{C}ke^{-kt} = -k(u(t) - T)$$

Uniqueness:

Giving just an ODE will not be enough for a unique solution. Initial conditions must be given.

Question: given a differential equation and initial conditions is it easy to see if the solution will be unique or even exist?
Answer: No.