Lecture 19

niceguy

March 1, 2023

1 Laplace and Poisson Equations

Example 1.1 (Electric Field inside a capacitor)**.** Ignoring fringing fields, $\vec{E} = E_x\hat{a}_x$. Hence from Laplace's Equation,

$$\begin{aligned}\vec{\nabla}^2 V &= 0\\ \frac{d^2V}{dx^2} &= 0\\ V &= ax + b\end{aligned}$$

Plugging $V(0)$ and $V(L)$ gives the constants a and b. For $V(0) = V_0, V(d) = 0$, we get

$$V = -\frac{V_0}{d}x + V_0$$

Example 1.2 (Coaxial Cable)**.** Find the solution for the variation of the electric scalar potential, $V(r)$, within a coaxial cable which lies along the z-axis. The outer conductor ($r = b$) is grounded, while the inner conductor ($r = a$) is held at a potential of V_0. For a 1 m long section of this coaxial cable, determine the capacitance and stored electrostatic energy.
We use Laplace's equation

$$\vec{\nabla}^2 V = 0$$

Ignoring fringing fields, considering symmetry, and using

$$\vec{\nabla}^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

We get that

$$V = A \ln r + B$$

Where the boundary values $V(r=a) = V_0, V(r=b) = 0$ give

$$V(r) = \frac{V_0}{\ln \frac{b}{a}} \ln \frac{b}{r}$$

Then the energy stored is

$$\begin{aligned}
W &= \frac{1}{2} \iiint \varepsilon_0 \varepsilon_r |\vec{E}|^2 dv \\
&= \frac{1}{2} \int_a^b \int_0^{2\pi} \int_0^L \varepsilon_0 \varepsilon_r \left(\frac{V_0}{\ln \frac{b}{a}} \right)^2 \frac{1}{r} dz d\phi dr \\
&= \frac{1}{2} \times 2\pi L \varepsilon_0 \varepsilon_r \times \frac{V_0^2}{\ln^2 \frac{b}{a}} \ln \frac{b}{a} \\
&= \frac{\pi \varepsilon_0 \varepsilon_r V_0^2 L}{\ln \frac{b}{a}}
\end{aligned}$$

Thus

$$C = \frac{2W}{V_0^2} = \frac{2\pi \varepsilon_0 \varepsilon_r L}{\ln \frac{b}{a}}$$