Lecture 1

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1 Integrals Involving a Parameter

Example 1.1.

$$\int_0^1 Cx^3 dx$$

C: constant, x: variable

Example 1.2.

$$\int_0^1 Cx_3 dx = C\frac{x^4}{4}\Big|_0^1 = \frac{1}{4}C$$

Result contains C

Example 1.3.

$$\int_a^b f(x,y)dx = g(y)$$

Definition 1.1. A variable which is kept constant during an integration is called a parameter.

Example 1.4.

$$\int_0^1 x^3 y dx = y\int_0^1 x^3 dx = \frac{y}{4}$$

Where y is the parameter.

1.1 Integrated Integrals (Integral of an Integral)

$z = f(x, y)$ where $x \in [a, b], y \in [c, d]$
Assume $f(x, y) \geq 0$
The area of a vertical slice at a given x is

$$\int_c^d f(x, y)dy = A(x)$$

The volume of said slice is

$$\Delta V(x) = A(x)\Delta x = \left(\int_c^d f(x, y)dy\right)\Delta x$$

Consider a partition of $[a, b]$, and we have

$$V \approx \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} A(_i)\Delta x_i$$

As $\Delta x_i \to 0$, the Riemann sum gives us the integral

$$V = \int_a^b A(x)dx = \int_a^b \int_c^d f(x, y)dydx$$

The same can be done in the reverse order, i.e. with $A'(y)$ and $V = \int_c^d A(y)dy$

Theorem 1.1. *Fubini's Theorem.*

$$\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

Proof: trust me bro

Example 1.5. Find the volume under the surface $z = x^2y, x \in [1, 3], y \in [0, 1]$

Forming the integral by first integrating with respect to y

$$\begin{aligned}
V &= \int_1^3 \int_0^1 x^2 y dy dx \\
&= \int_1^3 \frac{x^2y^2}{2}\Big|_0^1 dx \\
&= \int_1^3 \frac{x^2}{2} dx \\
&= \frac{x^3}{6}\Big|_1^3 \\
&= \frac{13}{3}
\end{aligned}$$

Forming the integral by first integrating with respect to x

$$\begin{aligned}
V &= \int_0^1 \int_1^3 x^2 y dx dy \\
&= \int_0^1 \frac{x^3y}{3}\Big|_1^3 dy \\
&= \int_0^1 \frac{26y}{3} dy \\
&= \frac{26y^2}{6}\Big|_0^1 \\
&= \frac{13}{3}
\end{aligned}$$

Example 1.6. Evaluate the double integral of $f(x, y) = x - 3y^2$ over region $R = \{(x, y)|0 \leq x \leq 2, 1 \leq y \leq 2\}$

$$\int_0^2 \int_1^2 (x - 3y^2) dy dx = \int_0^2 (xy - y^3)\Big|_1^2 dx = \int_0^2 x - 7 dx = \left(\frac{x^2}{2} - 7x\right)\Big|_0^2 = -12$$

In the special case where $f(x, y) = g(x)h(y)$, then

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \int_a^b g(x)h(y) dx dy = \int_c^d h(y) \int_a^g (x) dx dy = \int_a^b g(x) dx \int_c^d h(y) dy$$

Example 1.7.

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin x \cos y dx dy = \int_0^{\frac{\pi}{2}} \sin x dx \int_0^{\frac{\pi}{2}} \cos y dy = -\cos x\Big|_0^{\frac{\pi}{2}} + \sin y\Big|_0^{\frac{\pi}{2}} = 2$$

1.2 Double Integrals over General Regions

Type 1 Region: $R = \{(x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$
If $f(x,y) \geq 0$ on a type 1 region,

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

and similarly, we have

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Type 2 Region: $R = \{(x,y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$
Similarly,

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Example 1.8. Find the volume of the solid that lies under the surface $z = x^2 + y^2$ and above the region R in the xy plane. The region is bounded by the straight line $y = 2x$ and the parabola $y = x^2$.

Integrating with respect to y first,

$$\begin{aligned}
V &= \int_0^2 \int_{x^2}^{2x} x^2 + y^2 dydx \\
&= \int_0^2 x^2 y + \frac{y^3}{3}\Big|_{x^2}^{2x} dx \\
&= \int_0^2 2x^3 - x^4 + \frac{8x^3}{3} - \frac{x^6}{3} dx \\
&= \frac{x^4}{2} - \frac{x^5}{5} + \frac{2x^4}{3} - \frac{x^7}{21}\Big|_0^2 \\
&= 8 - \frac{32}{5} + \frac{32}{3} - \frac{128}{21} \\
&= \frac{216}{35}
\end{aligned}$$

Integrating with respect to x first,

$$\begin{aligned} V &= \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} x^2 + y^2 dxdy \\ &= \int_0^4 \frac{x^3}{3} + xy^2 \Big|_{\frac{y}{2}}^{\sqrt{y}} dx \\ &= \int_0^4 \frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{13}{24} y^3 dy \\ &= \frac{216}{35} \end{aligned}$$

It is sometimes easier to integration with respect to one variable over the other.

Example 1.9. Integrate the surface given by $z = e^{x^2}$ over the region between $y = x$ and $y = 0$ for $x \in [0, 1]$.

If we first integrate with respect to x, this results in an integral which has no elementary antiderivative (though it can still be evaluated).

$$V = \int_0^1 \int_y^1 e^{x^2} dxdy$$

If we first integrate with respect to y,

$$
\begin{aligned}
V &= \int_0^1 \int_0^x e^{x^2} dy dx \\
&= \int_0^1 y e^{x^2} \Big|_0^x dx \\
&= \int_0^1 x e^{x^2} dx \\
&= \frac{1}{2} e^{x^2} \Big|_0^1 \\
&= \frac{e-1}{2}
\end{aligned}
$$