Lecture 23

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1 Variation of Parameters

Since we define

$$\begin{cases} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \end{cases}$$

A second order ODE can be rewritten as a linear first order ODE

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g(t)}{a} \end{bmatrix}$$

If $y^*(t)$ is a solution to the second order ODE, then

$$x_1(t) = y^*(t), x_2(t) = (y^*)'(t)$$

The particular solution as developed in the previous lecture was

$$\vec{x}_p(t) = X(t) \int X^{-1}(t)\vec{g}(t)dt$$

where

$$X(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$X^{-1}(t) = \frac{1}{y_1y_2' - y_2y_1'} \begin{bmatrix} y_2' & -y_2 \\ -y_2'y_1 & \end{bmatrix}$$

$$\vec{g}(t) = \begin{bmatrix} 0 \\ \frac{g(t)}{a} \end{bmatrix}$$

Assuming $a = 1$, the particular solution is

$$-\begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \int W^{-1} g y_2 dt + \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} \int W^{-1} g y_1 dt$$

Reverting to the second orer linear ODE, we take the first coordinate

$$y_p = -y_1 \int W^{-1} g y_2 dt + y_2 \int W^{-1} g y_1 dt$$

Example 1.1.

$$y'' + 4y = \frac{3}{\sin t}$$

Obviously,

$$y_1(t) = \cos(2t)$$

and

$$y_2(t) = \sin(2t)$$

The Wronskian is then

$$\begin{aligned} W &= \det \begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \\ &= 2 \end{aligned}$$

Plugging things in,

$$y_p = -\cos(2t) \int \frac{3\sin(2t)}{2\sin t} dt + \sin(2t) \int \frac{3\cos(2t)}{2\sin t} dt$$