Lecture 12

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October 14, 2022

1 Surface Integrals

If a surface is parametrised as

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

a small section of the surface area dA is given by

$$dA \approx ||(\vec{r}(u+\delta u, v) - \vec{r}(u,v)) \times (\vec{r}(u, v+\delta v) - \vec{r}(u,v))|| \approx \left|\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|\right| \Delta u \Delta v$$

Thus the surface area is given by

$$S = \iint_D ||\vec{r_u} \times \vec{r_v}|| du dv$$

Example 1.1. Find the surface area of a sphere of radius a.
We parametrise the sphere as

$$\begin{aligned}
x &= a \sin\phi \cos\theta \\
y &= a \sin\phi \sin\theta \\
z &= a \cos\phi
\end{aligned}$$

The cross product is

$$\begin{aligned}
\vec{r_\phi} \times \vec{r_\theta} &= (a\cos\phi\cos\theta\hat{i} + a\cos\phi\sin\theta\hat{j} - a\sin\phi\hat{k}) \times (-a\sin\phi\sin\theta\hat{i} + a\sin\phi\cos\theta\hat{j}) \\
&= a^2\sin^2\phi\cos\theta\hat{i} + a^2\sin^2\phi\sin\theta\hat{j} + a^2\sin\phi\cos\phi\hat{k}
\end{aligned}$$

And its magnitude is

$$a^2 \sin \phi$$

Thus the surface area is

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi} a^2 \sin \phi d\phi d\theta \\ &= 2a^2 \int_0^{2\pi} d\theta \\ &= 4\pi a^2 \end{aligned}$$

In the special case where $z = f(x, y)$, we have

$$\begin{aligned} \vec{r_x} &= \hat{\imath} + f_x \hat{k} \\ \vec{r_y} &= \hat{\jmath} + f_y \hat{k} \end{aligned}$$

And the magnitude of their cross product is

$$\begin{aligned} ||\vec{r_x} \times \vec{r_y}|| &= ||(\hat{\imath} + f_x \hat{k}) \times (\hat{\jmath} + f_y \hat{k})|| \\ &= || - f_x \hat{\imath} - f_y \hat{\jmath} + \hat{k}|| \\ &= \sqrt{f_x^2 + f_y^2 + 1} \end{aligned}$$

So

$$S = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

which corresponds with our previously derived equations.

2 Surface Integrals of Scalar Functions

Using what we derived above, the surface integral of a scalar function $f(x, y, z)$ is

$$\iint_S f dS = \iint_S f(x(u, v), y(u, v), z(u, v)) ||\vec{r_u} \times \vec{r_v}|| du dv$$

Example 2.1. Evaluate $\int_S \sqrt{x^2+y^2+1}dS$ where S is the surface given parametrically by $\vec{r}(u,v) = (u\cos v, u\sin v, v)$ where $u \in [0,1], v \in [0,2\pi]$.

$$\begin{aligned}
\iint_S \sqrt{x^2+y^2+1}dS &= \int_0^{2\pi}\int_0^1 \sqrt{u^2\cos^2 v + u^2\sin^2 v + 1}||(\cos v\hat{i} + \sin v\hat{j}) \\
&\quad \times (-u\sin v\hat{i} + u\cos v\hat{j} + \hat{k})||dudv \\
&= \int_0^{2\pi}\int_0^1 \sqrt{u^2+1}||\sin v\hat{i} - \cos v\hat{j} + u\hat{k}||dudv \\
&= \int_0^{2\pi}\int_0^1 \sqrt{u^2+1}\sqrt{u^2+1}dudv \\
&= \int_0^{2\pi} \frac{4}{3}dv \\
&= \frac{8\pi}{3}
\end{aligned}$$

If S is a piecewise smooth surface

$$S = \bigcup_i S_i$$

we can add up the integrals

$$I = \sum_i \iint_{S_i} f dS$$