Lecture 6

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1 Triple Integrals in Cylindrical and Spherical Coordinates

Definition 1.1. Cylindrical Coordinates (r, θ, z)
Where r is the (non-negative) distance between the origin and the point projected on the xy plane, θ is the angle between the horizontal and the r vector, and z is the same as that defined in Cartesian coordinates.

To convert from Cartesian coordinates to Cylindrical coordinates,

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{x}{y}\right)$$
$$z = z$$

To convert from Cylindrical coordinates to Cartesian coordinates,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

We usually define the region as

$$Q = \{(x, y, z) | (x, y) \in \mathbb{R}, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where

$$R = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Assuming $f(x,y,z)$ is continuous over Q,

$$\begin{aligned}
\iiint_Q f(x,y,z)dV &= \iint_R \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA \\
&= \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta
\end{aligned}$$

Example 1.1. Evaluate the following triple integral

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2) dz dy dx$$

It is very difficult to evaluate this integral in cartesian coordinates. Therefore we convert it to cylindrical coordinates. From the x and y limits, we can see that the region on the xy plane is a circle centred at the origin with radius 2. Substituting the cylindrical limits,

$$\begin{aligned}
I &= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 dz dr d\theta \\
&= \int_0^{2\pi} \int_0^2 2r^3 - r^4 dr d\theta \\
&= \int_0^{2\pi} 8 - \frac{32}{5} d\theta \\
&= \frac{16}{5}\pi
\end{aligned}$$

1.1 Triple Integrals in Spherical Coordinates

Definition 1.2. Sperical Coordinates.
ρ denotes the (non-negative) distance from the origin to the point. θ denotes the angle between the horizontal and the ρ vector projected on the xy plane. ϕ denotes the angle between the z axis and the ρ vector.

To convert from spherical to cartesian coordinates,

$$
\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\phi
\end{aligned}
$$

To convert from cartesian to spherical coordinates,

$$
\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arctan\left(\frac{y}{x}\right) \\
\phi &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)
\end{aligned}
$$

Approximating a small sperical segment as a cuboid, its base area is $\rho\Delta\theta \times \rho \sin\phi\Delta\phi$, while its height is $\Delta\rho$, so

$$dV = \rho^2 \sin\phi d\rho d\theta d\phi$$

Example 1.2. Find the mass of a half sphere of radius a that has a density $k(2a - \rho)$, where k is a constant and ρ is the distance from the coordinate origin to a point (i.e., the first coordinate of the spherical coordinate system). We let the density be

$$\lambda = k(2a - \rho)$$

Then

$$
\begin{aligned}
m &= \iiint_V \lambda dV \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a k(2a-\rho)\rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{2ak\rho^3 \sin\phi}{3} - \frac{k\rho^4 \sin\phi}{4}\Big|_0^a d\phi d\theta \\
&- \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{5}{12} ka^4 \sin\phi d\phi d\theta \\
&= k \int_0^{2\pi} \frac{15}{12} a^4 \left(-\cos\frac{\pi}{2} + \cos 0\right) d\theta \\
&= \frac{5}{6}\pi k a^4
\end{aligned}
$$