# Lecture 31

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## 1 Wrap Up

**Example 1.1.**

$$y'' + 2y' + 5y = g(t)$$

The impulse response is

$$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{2}e^{-t}\sin(2t)$$

by the lookup table.
The homogeneous solution is

$$y_h(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

The particular solution depends on the forced response

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)G(s)\} &= h * g(t) \\ &= \int_0^t h(t-\tau)g(\tau)d\tau\end{aligned}$$

The particular solution is hence

$$y_p(t) = \int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)d\tau$$

Combining, the general solution is

$$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)d\tau$$

Given $y(0) = 1, y'(0) = -3$, we need to find $c_1$ and $c_2$ that satisfy the above. The first substitution gives $c_1 = 1$. Differentiating the integral yields the sum of a zero term and an integral from 0 to $t$, which also vanishes at $t = 0$. Thus

$$y'(0) = -c_1 + 2c_2$$

giving

$$c_2 = -1$$

The general solution is then

$$y(t) = e^{-t}\cos(2t) - e^{-t}\sin(2t) + \int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))g(\tau)d\tau$$

The forced response if $g(t) = t$ is given by the integral

$$\int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))\tau d\tau$$

This is left as an exercise to the reader.

# 2 Fourier Transform

Suppose we have a complex vector space $V$. An inner product on $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is a function that satisfies

- $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$
- $\langle x, x\rangle \geq 0$ with equality only when $x = 0$

We call $(V, \langle\cdot,\cdot\rangle)$ an inner product space. $x$ and $y$ are orthogonal if their inner product is 0.
The vector space we are considering is

$$PC([-L, L]) = f : [-L, L] \to \mathbb{R} : f \text{ is piecewise continuous}$$

And the inner product is defined as

$$\langle f, g\rangle = \int_{-L}^{L} f(t)\overline{g(t)}dt$$

We know that $PC([-L,L])$ is a vector space, and the inner product is actually an inner product.
Proof: trust me bro.

**Definition 2.1.** A Hamel basis is a basis $\mathcal{B}$ such that

- For every finite subset of $\mathcal{B}$, all elements are linearly independent
- every vector can be represented as a finite linear combination of vectors in $\mathcal{B}$

All vector spaces have a Hamel basis if we assume AC.

**Definition 2.2.** A Schauder Basis is a sequence of vectors $\{v_n\} \subseteq V$ such that for any vector $v \in V$, there exists unique coefficients $\{a_n\} \subseteq \mathbb{R}$ such that

$$v = \sum_{n=1}^{\infty} a_n v_n$$

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**Theorem 2.1.** *Suppose $f$ is periodic with period $2L$ and both $f, f'$ belong in $PC([-L,L])$. Then $f$ has a Fourier Series*

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

*where the Fourier coefficients are*

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx$$

*The Fourier series of $f(x)$ converges to $f$ at all points of continuity and to the midpoint of the left and right limit where $f$ is discontinuous.*

**Theorem 2.2.** *The set*

$$\left\{\frac{1}{2}, \sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}, m \in \mathbb{N} - \{0\}\right\}$$

*is an orthogonal family and is a Schauder basis for* $PC([-L, L])$.

Given

$$\vec{v} = \sum_{i=1}^{\infty} a_i \vec{v}_i$$

We can use the inner product to retrieve the coordinates, i.e.

$$a_k = \frac{\langle \vec{v}, \vec{v}_k \rangle}{||v_k||^2}$$

To find the coefficient of $\cos\frac{m\pi x}{L}$, the norm is given by

$$\int_{-L}^{L} \cos^2 \frac{m\pi x}{L} dx = L$$

The coefficient is then

$$\frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} dx$$