Lecture 21

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1 More Sampling Distributions

Theorem 1.1 (Central Limit Theorem)**.** *Consider the independently identically distributed sample $X_1, \ldots, X_n$ with μ, σ^2. Defining*

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and

$$Z_n = \frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Then as $n \to \infty$, the distribution of Z_n tends to the normal distribution $n(z; 0, 1)$.

The standard deviation of $\overline{X}_n$ is $\frac{\sigma}{\sqrt{n}}$, so the distribution of $\overline{X}_n$ is $n(\overline{x}; \mu, \frac{\sigma^2}{n})$, and $\overline{x}_n$ is the realisation of $\overline{X}_n$.

2 Sample Variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Recall the χ-squared distribution

$$f(y, \nu) = \begin{cases} \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}} & y > 0 \\ 0 & , y \leq 0 \end{cases}$$

Then

$$\chi^2 = \frac{n-1}{\sigma^2}S^2$$
$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Then χ^2 has χ-squared distribution $\nu = n - 1$. ν is the number of degrees of freedom. For

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$$

the probability density function is χ^2 with $\nu = n$.

3 T distribution

We let

$$T_n = \frac{\overline{X}_n - \mu}{\frac{S}{\sqrt{n}}}, S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X - \overline{X})^2}$$

If $n \geq 30, S \approx \sigma$, then we can use the Central Limit Theorem replacing the standard deviation with S.

Definition 3.1 (T Distribution)**.**

$$H(t) = \frac{\Gamma\left[\frac{\nu+1}{2}\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

4 Comparison of T Distribution and Normal Distribution

The T distribution has "heavy tails", meaning it is more likely for there to be a value far from the mean. The T distribution can be used if there is a normal population, and the Central Limit Theorem ($\sigma = S$) can be used for non normal population given $n \geq 30$.