Lecture 20

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1 Complexification

We can complexify real vectors, polynomials, vector spaces, linear maps,...

Example 1.1 (Matrices)**.** For $A \in M_{m\times n}(\mathbb{R})$, let $A_{\mathbb{C}} \in M_{m\times n}(\mathbb{C})$ be the same matrix. Then for $m = n$,

- $\operatorname{tr}(A_{\mathbb{C}}) = \operatorname{tr}(A)$
- $\det(A_{\mathbb{C}}) = \det(A)$
- The characteristic polynomial of $A_{\mathbb{C}}$ is the complexification of that of A (see Example 1.2)

Example 1.2 (Polynomials)**.** For the real polynomial $p(t) = \sum_{i=0}^{n} a_i t^i, a_i \in \mathbb{R}$, define

$$p_{\mathbb{C}}(t) = \sum_{i=0}^{n} a_i t^i$$

with the domain $\mathbb{C}$.

Proposition 1.1. *The minimal polynomial of $A_{\mathbb{C}}$ is the complexification of the minimal polynomial of A.*

Proof. The minimal polynomial of A is the unique shortest monic polynomial

$$p(t) = \sum_{i=0}^{n} a_i t^i$$

such that $p(A) = 0$. This implies

$$p_{\mathbb{C}}(A_{\mathbb{C}}) = p(A) = 0$$

So $p_{\mathbb{C}}$ is divisible by the minimal polynomial. Now suppose

$$r(z) = \sum_{i=0}^{k} c_i z_i, c_i \in \mathbb{C}, k \leq m$$

is the minimal polynomial of $A_{\mathbb{C}}$. Since all entries of $A_{\mathbb{C}}$ are real, then the real part of $r(A) = 0$ becomes

$$r_{\mathbb{R}}(A) = \sum_{i=0}^{k} \Re(c_i) A^i = 0$$

Since $r_{\mathbb{R}}(A) = 0$, it cannot be shorter than the minimal polynomial of A, so $k \geq m$. Combining this with $k \leq m$, we see $k = m$. □

Proposition 1.2. *For $A \in M_{n\times n}(\mathbb{R})$, in the Jordan normal form for $A_{\mathbb{C}}$, the number of Jordan blocks of size k and type λ eqauls the number of Jordan blocks of size k and type $\overline{\lambda}$.*

Then the eigenvalues appear in complex conjugate pairs, with the same geometric and algebraic multiplicity. Equivalently,

$$\dim(\text{null}(\lambda I - A_{\mathbb{C}})^k) = \dim(\text{null}(\overline{\lambda} I - A_{\mathbb{C}})^k) \forall \lambda \in \mathbb{C}, k \in \mathbb{N}$$

Proof. By Jordan normal form theorem, we have an invertible $C \in M_{n\times n}(\mathbb{C})$ with $CA_{\mathbb{C}}C^{-1} = \mathcal{J}$. Taking the complex conjugate,

$$\overline{C} A_{\mathbb{C}} \overline{C^{-1}} = \overline{\mathcal{J}}$$

Since $A_{\mathbb{C}}$ only has 1 Jordan normal form, up to rearrangement of the boxes, $\mathcal{J}$ and $\overline{\mathcal{J}}$ share the same λs that are conjugated. □

By complex conjugation,

$$(\lambda I - A_{\mathbb{C}})v = 0 \Leftrightarrow (\overline{\lambda} I - A_{\mathbb{C}})\overline{v} = 0$$

Example 1.3.
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$
The characteristic polynomial is
$$q(t) = (t-1)^2 + 1 = (t-(1+i))(t-(1-i))$$
Then the eigenvalues of $A_{\mathbb{C}}$ are $1+i$ and $1-i$, with eigenvectors $\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ i \end{pmatrix}$, where the pairs of eigenvalues and eigenvectors are complex conjugates.

2 Complexification of Vector Spaces

Let V be a real vector space. We put
$$V_{\mathbb{C}} = V \times V$$
with elements denotes $(u, w) = u + iw$. Addition is defined as
$$(u_1 + iw_1) + (u_2 + iw_2) = (u_1 + u_2) + i(w_1 + w_2)$$
and multiplication
$$(a + ib)(u + iw) = (au - bw) + i(aw + bu)$$

Lemma 2.1. *$V_{\mathbb{C}}$ with the addition and scalar multiplication as defined above is a complex vector space.*

Proof. Trivial. □

Example 2.1. $(\mathbb{R}^n)_{\mathbb{C}} = \mathbb{C}^n, (M_{m\times n}(\mathbb{R}))_{\mathbb{C}} = M_{m\times n}(\mathbb{C})$

We can consider V as a subset of $V_{\mathbb{C}}$ with vectors $u + i0$. Similarly we define an imaginary part, such that for all complex vector spaces V, it is the direct sum of hts real and complex subspaces. We can then use this decomposition to define complex conjugates of vectors in $V_{\mathbb{C}}$.

Proposition 2.1. *If $v_1, \ldots, v_n$ is a basis of V, a real vector space, then it is also a basis of $V_{\mathbb{C}}$, which has the same dimension.*

Proof. Any $u + iw$ can be written as

$$u + iw = \sum_j a_j v_j + i \sum_j b_j v_j = \sum_j c_j v_j, c_j = a_j + ib_j$$

So the vectors span $V_{\mathbb{C}}$. Now $u + iw = 0$ implies $u = 0$ and $w = 0$. Since $v_1, \ldots, v_n$ is a basis for V, this means $a_j, b_j = 0 \forall j \Rightarrow c_j = 0 \forall j$. Hence the vectors are linearly dependent. Then they are a basis for $V_{\mathbb{C}}$, also implying the dimensions of V and $V_{\mathbb{C}}$ are equal (equal length of bases). $\square$

3 Complexification of Operators

Let V, W be real vector spaces, and $T \in \mathcal{L}(V, W)$. Then

$$T_{\mathbb{C}}(v_1 + iv_2) = T(v_1) + iT(v_2)$$

and $T_{\mathbb{C}} \in \mathcal{L}(V_{\mathbb{C}}, W_{\mathbb{C}})$.

- $(T_1 + T_2)_{\mathbb{C}} = (T_1)_{\mathbb{C}} + (T_2)_{\mathbb{C}}$
- $(\lambda T)_{\mathbb{C}} = \lambda T_{\mathbb{C}}, \lambda \in \mathbb{R}$
- $(ST)_{\mathbb{C}} = S_{\mathbb{C}} T_{\mathbb{C}}$

In particular, $T_{\mathbb{C}}$ is invertible if and only if T is invertible. In that case,

$$(T_{\mathbb{C}})^{-1} = (T^{-1})_{\mathbb{C}}$$

For $T \in \mathcal{L}(V)$,

- $\det(T_{\mathbb{C}}) = \det(T), \operatorname{tr}(T_{\mathbb{C}}) = \operatorname{tr}(T)$
- The characteristic and minimal polynomials of T are the complexification of the respective polynomials of T
- The eigenvalues of $T_{\mathbb{C}}$ appear in complex conjugate pairs, with the same geometric and algebraic multiplicities
- $\dim \operatorname{null}((\lambda I - T_{\mathbb{C}})^k) = \dim \operatorname{null}((\overline{\lambda} I - T)^k)$

If λ is a real eigenvector of $T_{\mathbb{C}}$, then it is also an eigenvalue of T. In fact,

$$\dim E(\lambda, T_{\mathbb{C}}) = \dim E(\lambda, T)$$

with the same algebraic and geometric multiplicities.

Proof. Suppose λ is a real eigenvalue of $T_{\mathbb{C}}$ with a nonzero eigenvector $v \in V_{\mathbb{C}}$. Then $\Re(v)$ is a real eigenvector, and if v is imaginary, take $\Im(v)$.

We can use the same method to get a basis for $E(\lambda, T)$, which is the same basis for $E(\lambda, T_{\mathbb{C}})$. By induction, let $v_1, \ldots, v_k \in E(\lambda, T)$ be linearly independent. They are also linearly independent in $E(\lambda, T_{\mathbb{C}})$. If they don't span, we pick $v \in E(\lambda, T_{\mathbb{C}})$ that is not in the span of $v_1, \ldots, v_k$. We cannot have both the real and imaginary part of the vector to be in the span, so we obtain at least one "new" vector. □

Proposition 3.1. *If* $\dim V$ *is odd, then* $T \in \mathcal{L}(V)$ *has at least an eigenvalue* $\lambda \in \mathbb{R}$.

Proof. The non-real eigenvalues of $T_{\mathbb{C}}$ appear in pairs, with the same algebraic multiplicities, so the total number of non-real eigenvalues is even. Then there is at least a real eigenvalue, which also has to be an eigenvalue of T. □

Proposition 3.2. *If* $T \in \mathcal{L}(V)$*, where* V *is a real vector space, with a negative determinant, then there is a negative eigenvalue.*

Proof.

$$\begin{aligned}
\det(T) &= \det(T_{\mathbb{C}}) \\
&= \prod \lambda \\
&= \left(\prod_{\lambda \in \mathbb{R}} \lambda\right) \left(\prod_{\Im(\lambda) > 0} \lambda\right) \left(\prod_{\Im(\lambda) < 0}\right) \\
&= \left(\prod_{\lambda \in \mathbb{R}} \lambda\right) \left(\prod_{\Im(\lambda) < 0} |\lambda|^2\right)
\end{aligned}$$

Since the second term is real, there has to be a negative λ in the first term. □

Proposition 3.3. *Let* $T \in \mathcal{L}(V)$*, with* V *being a real vector space. Then there exists a* T*-invariant subspace* $W \subseteq V$ *of dimension 1 or 2.*

Proof. If T has a real eigenvalue, we take W as the subspace spanned by one of its eigenvector(s). If not, $T_{\mathbb{C}}$ has a complex eigenvalue, with eigenvector $v \in V_{\mathbb{C}}$.

$$T_{\mathbb{C}}v = \lambda v, \lambda = a + ib$$

Then

$$\begin{aligned} T(\Re(v)) + T(\Im(v)) &= (a\Re(v) - b\Im(v)) + i(a\Im(v) + b\Re(v)) \\ T(\Re(v)) &= a\Re(v) - b\Im(v) \\ T(\Im(v)) &= a\Im(v) + b\Re(v) \end{aligned}$$

Then take

$$W = \operatorname{span}\{\Re(v), \Im(v)\}$$

□

Proposition 3.4. *Let $T \in \mathcal{L}(V)$, with V being a real vector space. There exists a basis $v_1, \ldots, v_k$ of V such that the companion matrix A of T is block upper triangular with diagonal block of size 1 or 2.*

Proof. The proof is left as an exercise to the reader. □