Lecture 13

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1 Surface Integrals of Vector Fields

Definition 1.1. An *orientable* surface is one that is **two-sided**.

Example 1.1. A plane is orientable. A Möbius strip is non-orientable.

Given the surface

$$S : \vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

We construct a normal vector to the surface by

$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

The unit normal is then

$$\vec{n} = \frac{\vec{N}}{||\vec{N}||} = \frac{\vec{r_u} \times \vec{r_v}}{||\vec{r_u} \times \vec{r_v}||}$$

Example 1.2. Imagine a fluid with density $\rho(x,y,z)$ and velocity field $\vec{V}(x,y,z)$. Then the mass flow rate passing through that surface is

$$\iint_S \rho\vec{V} \cdot \vec{n} dS$$

We define flux as $\vec{F} = \rho\vec{V}$, which gives us the mass flow rate as

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_S \vec{F} \cdot d\vec{S}$$

This can be simplified as

$$\vec{n}dS = \frac{\vec{r_u} \times \vec{r_v}}{||\vec{r_u} \times \vec{r_v}||} \times ||\vec{r_u} \times \vec{r_v} dudv = \vec{r_u} \times \vec{r_v} dudv$$

so

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r_u} \times \vec{r_v}) dudv$$

Example 1.3. Calculate the flux of the vector field $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ for the closed cylindrical surface S, given by $x^2 + y^2 = a^2, -h \leq z \leq h$.
The lateral component can be parameterised by θ, z, which gives

$$\begin{aligned} I &= \int_{-h}^{h} \int_{0}^{2\pi} (a\cos\theta\hat{i} + a\sin\theta\hat{j} + z\hat{k}) \cdot (a\cos\theta\hat{i} + a\sin\theta\hat{j}) d\theta dz \\ &= \int_{-h}^{h} \int_{0}^{2\pi} a^2 d\theta dz \\ &= 4\pi a^2 h \end{aligned}$$

The top and bottom surfaces are more trivial, as the $\hat{k}$ component is constant. Hence

$$\begin{aligned} I &= \int_{0}^{2\pi} \int_{0}^{a} (a\cos\theta\hat{i} + a\sin\theta\hat{j} + h\hat{k}) \cdot \hat{k} r dr d\theta \\ &= \int_{0}^{2\pi} \int_{0}^{a} h r dr d\theta \\ &= \pi a^2 h \end{aligned}$$

This is for the top surface. For the bottom surface, both h and $\hat{k}$ become negative, which cancels out, giving a total flux of $6\pi a^2 h$.

Example 1.4. Find the flux of $\vec{F} = \frac{2x\hat{i} + 2y\hat{j}}{x^2+y^2} + \hat{k}$ through the surface S defined parametrically as

$$\vec{r} = u\cos\theta\hat{i} + u\sin\theta\hat{j} + u\hat{k}, 0 \leq u \leq 1, 0 \leq \theta \leq 2\pi$$

taking the downwards face as positive. Then we have

$$\begin{aligned} \vec{r_u} \times \vec{r_\theta} &= (\cos\theta\hat{i} + \sin\theta\hat{j} + \hat{k}) \times (-u\sin\theta\hat{i} + u\cos\theta\hat{j}) \\ &= -u\cos\theta\hat{i} - u\sin\theta\hat{j} + u\hat{k} \end{aligned}$$

Using the downwards direction, we have

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^1 \left(\frac{2\cos\theta}{u}\hat{i} + \frac{2\sin\theta}{u}\hat{j} + \hat{k}\right)\cdot(u\cos\theta\hat{i} + u\sin\theta\hat{j} - u\hat{k})dud\theta \\
&= \int_0^{2\pi}\int_0^1 2\cos^2\theta + 2\sin^2\theta - udud\theta \\
&= \int_0^{2\pi} 2\cos^2\theta + 2\sin^2\theta - \frac{1}{2}d\theta \\
&= 3\pi
\end{aligned}$$

So the net outflow is 3π.

2 Divergence and Curl

Definition 2.1.

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Gradient operations are then

Definition 2.2. The *gradient* of a scalar function is

$$\vec{\nabla} f \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Definition 2.3. The *divergence* of a vector function $\vec{f} = P\hat{i} + Q\hat{j} + R\hat{k}$ is

$$\vec{\nabla}\cdot\vec{f} \equiv \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Physically, the divergence of a vector function is the measure of how much a function "sinks" into or "flows" from a point.

Definition 2.4. The *curl* of a vector function $\vec{f} = P\hat{i} + Q\hat{j} + R\hat{k}$ is

$$\vec{\nabla}\times\vec{f} \equiv \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

Fun fact: the divergence of a curl or the curl of a gradient is always zero! (Proof left to reader as exercise)

Definition 2.5. The Laplace operator is defined as

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

3 Stokes' Theorem

Stokes' Theorem is a 3D version of Green's Theorem.

Theorem 3.1. *Let S be an orientable, piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C having positive orientation. If $\vec{F}$ is a vector field with continuous first partial derivatives over S then*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot \vec{n} dS$$

If $\vec{F} = \vec{V}$ (ie velocity), we have

$$\oint_C \vec{V} \cdot \vec{T} ds = \iint_S \vec{w} \cdot \vec{n} dS$$

where $\vec{T}$ is the unit tangent vector and $\vec{w}$ is as defined.

Example 3.1. Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ such that $z \geq 0$, and let c be the trace of S on the xy-plane. Verify the Stokes' theorem for the vector field $\vec{F} = 3z\hat{i} + 4x\hat{j} + 2y\hat{k}$.

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (12\cos t\hat{j} + 6\sin t\hat{k}) \cdot (-3\sin t\hat{i} + 3\cos t\hat{j}) dt \\ &= \int_0^{2\pi} 36\cos^2 t dt\end{aligned}$$

$$36\pi$$

And

$$\vec{\nabla} \times \vec{F} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

so

$$
\begin{aligned}
I &= \iint (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \cdot ((\hat{\imath} - 2x\hat{k}) \times (\hat{\jmath} - 2y\hat{k})) dxdy \\
&= \iint (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) \cdot (2x\hat{\imath} + 2y\hat{\jmath} + \hat{k}) dxdy \\
&= \iint 4x + 6y + 4dxdy \\
&= \int_0^{2\pi} \int_0^3 4r^2 \cos\theta + 6r^2 \sin\theta + 4rdrd\theta \\
&= \int_0^{2\pi} 36\cos\theta + 36\sin\theta + 18d\theta \\
&= 36\pi
\end{aligned}
$$