Homework 1

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1. Calculate the determinant of the following complex matrix.

$$\begin{pmatrix} 0 & i & 2 & -1 \\ i & 5 & i & i \\ 0 & 3 & 1+i & 2 \\ 0 & -2i & 1 & 4-i \end{pmatrix}$$

Solution: Expanding along the first column, the determinant is equal to

$$-i \det \begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$$

where the determinant of the 3 by 3 matrix is

$$i(1+i)(4-i) + 2 \times 2(-2i) - 1 \times 3 \times 1 - i \times 2 \times 1 - 2 \times 3(4-i) + 1(1+i)(-2i) = -i - 28$$

The desired determinant is then

$$-i(-i-28) = -1 + 28i$$

2. For $n = 1, 2, \ldots$ consider the $n \times n$ matrix

$$A_n = \begin{pmatrix} 2\cos\theta & 1 & 0 & \ldots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \ldots & 0 & 0 \\ 0 & 1 & 2\cos\theta & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2\cos\theta & 1 \\ 0 & 0 & 0 & \ldots & 1 & 2\cos\theta \end{pmatrix}$$

(a) Show that $\det(A_{n+2}) - 2\cos\det(A_{n+1}) + \det(A_{n+1}) = 0$

Solution: Expanding along the first row,

$$\begin{aligned} \det(A_{n+2}) &= 2\cos\theta\det(A_{n+1}) - \det\begin{pmatrix} 1 & P \\ Q & A_n \end{pmatrix} \\ &= 2\cos\theta\det(A_{n+1}) - \det(A_n) \end{aligned}$$

$$\det(A_{n+2}) - 2\cos\theta\det(A_{n+1}) + \det(A_n) = 0$$

Where P is the $1 \times n$ row matrix where the first entry is 1 and the rest are 0, and Q is the $n \times 1$ column matrix whose entries are all 0. The secondequality comes from expanding along the first column.

(b) Use (a) and induction to show

$$\det(A_n) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

Solution: For $n = 1$,

$$A_1 = \begin{pmatrix} 2\cos\theta \end{pmatrix}$$

where the determinant is obviously $2\cos\theta$. Then

$$\frac{\sin(2\theta)}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta} = 2\cos\theta$$

so the identity holds for $n = 1$.
For $n = 2$,

$$A_2 = \begin{pmatrix} 2\cos\theta & 1 \\ 1 & 2\cos\theta \end{pmatrix}$$

where the determinant is obviously $4\cos^2\theta - 1$. Then

$$\frac{\sin(3\theta)}{\sin\theta} = \frac{\sin(2\theta)\cos\theta + \sin\theta\cos(2\theta)}{\sin\theta} = \frac{2\sin\theta\cos^2\theta + \sin\theta(2\cos^2\theta - 1)}{\sin\theta}$$

which simplifies to

$$4\cos^2\theta - 1$$

Let this identity hold for $n = k$ and $n = k + 1$. Then

$$\begin{aligned}
\det(A_{k+2}) &= 2\cos\theta\det(A_{k+1}) - \det(A_k) \\
&= \frac{2\cos\theta\sin((k+2)\theta) - \sin((k+1)\theta)}{\sin\theta} \\
&= \frac{\sin((k+3)\theta) + \sin((k+1)\theta) - \sin((k+1)\theta)}{\sin\theta} \\
&= \frac{\sin((k+3)\theta)}{\sin\theta}
\end{aligned}$$

which shows that the identity holds for $n = k + 3$. By mathematical induction, it holds for all $n = 1, 2, \ldots$.

3. Let $T \in \mathcal{L}(V)$ be a linear transformation, and $T^* \in \mathcal{L}(V^*)$ the dual transformation. Show that

$$\det(T^*) = \det(T)$$

Solution: The expression of the determinant involves the constants A_{ij}. For T defined by

$$T(\hat{e}_i) = \vec{v}_i$$

the determinant of T is the determinant of its matrix

$$\det(v_1, v_2, \ldots, v_n)$$

where the constants are

$$\vec{v}_j = \sum_i A_{ij}\hat{e}_i$$

Similarly for the dual, we have

$$T^*(\phi_i) = \phi_i \circ T$$

Note that for an arbitrary $\hat{e}_j$,

$$\begin{aligned} T^*(\phi_i) &= \phi_i \circ T(\hat{e}_j) \\ &= \phi_i(\vec{v}_j) \\ &= \phi_i \left(\sum_k A_{kj} \hat{e}_k \right) \\ &= \sum_k A_{kj} \phi_i \hat{e}_k \\ &= A_{ij} \end{aligned}$$

Therefore,

$$T^*(\phi_i) = \sum_j A_{ij} \phi_j$$

Then the formula for the $\det(T*)$ is the same as that of $\det(T)$, so

$$\det(T) = \det(T^*)$$

4. (a) Suppose $A \in M_{n \times n}(F)$ has 'block upper triangular diagonal form'

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}$$

where $A' \in M_{k \times k}(F)$ and $A'' \in M_{l \times l}(F)$, while $*$ stands for 'anything'. Prove that

$$\det(A) = \det(A') \det(A'')$$

Solution: Let S be the set of permutations. Let $\mathcal{S} \subset S$ be the subset where $\forall \sigma \in \mathcal{S}, i \in (1, 2, \ldots, k)$,

$$\sigma(k) \in (1, 2, \ldots, k)$$

Then define $\mathcal{S}' = S - \mathcal{S}$ (or $S \backslash \mathcal{S}$). The left hand side then becomes

$$\det(A) = \sum_{\sigma \in \mathcal{S}} \text{sign}(\sigma) \prod_i A_{\sigma(i),i} + \sum_{\sigma \in \mathcal{S}'} \text{sign}(\sigma) \prod_i A_{\sigma(i),i}$$

Note that for all permutations in $\mathcal{S}'$, there exists an $i \in (1, 2, \ldots, k)$ where

$$\sigma(i) \in (k+1, k+2, \ldots, k+l)$$

i.e. $A_{\sigma(i),i} = 0$. Thus the second term goes to zero.
Note that permutations τ for $(1, 2, \ldots, k)$ and τ' for $(1, 2, \ldots, l)$ can be combined to exactly form all permutations in $\mathcal{S}$. Moreover, if τ and τ' are combined to form σ, then

$$\text{sign}(\tau) \times \text{sign}(\tau') = \text{sign}(\sigma)$$

Then the right hand side of the equation becomes

$$\begin{aligned} \det(A') \det(A'') &= \sum_\tau \text{sign}(\tau) \prod_i A'_{\tau(i),i} + \sum_{\tau'} \text{sign}(\tau') \prod_i A''_{\tau'(i),i} \\ &= \sum_{\sigma \in \mathcal{S}} \text{sign}(\sigma) \prod_i A_{\sigma(i),i} \\ &= \det(A) \end{aligned}$$

(b) Let V be a finite-dimensional vector space, and $T \in \mathcal{L}(V)$ a linear transformation. Suppose $W \subseteq V$ is a T-invariant subspace. Let

$$S = T|_W \in \mathcal{L}(W)$$

be the restriction, and

$$U \in \mathcal{L}(V/W)$$

the induced transformation on the quotient space (i.e., U takes $v + W$ to $Tv + W$). Prove that

$$\det(T) = \det(S)\det(U)$$

Solution: We wish to show that T is in the form of A in the part above, where A' is the matrix for S and A'' is the matrix for U. The desired result follows in this case. We use the basis $(e_1, e_2, \ldots, e_m, \ldots, e_n)$, where $(e_1, e_2, \ldots, e_m)$ is the basis for W. Then we define X as the space spanned by $(e_{m+1}, e_{m+2}, \ldots, e_n)$. Note that since W is T-invariant, this corresponds to the 0 matrix in A, as $T(w)\forall w \in W$ does not map to any entry in X, or $T(e_i)$ does not map to any vector with a nonzero e_j component where $1 \leq i \leq n$ and $n + 1 \leq j \leq m$. The restriction S then obviously has the matrix A', as $S(e_i) = T(e_i)$ where i is defined as above. Finally, let $v = w + x$ where $w \in W$ and $x \in X$. Define $T' \in \mathcal{L}(V)$ such that if

$$T(v) = \sum_{i=1}^{n} a_i e_i$$

then

$$T'(v) = \sum_{i=m+1}^{n} a_i e_i$$

Then

$$U(v + W) = U(x + W) = Tx + W = T'x + W$$

since the components of W can be absorbed into W. Then defining the basis vectors as

$$e'_i = e_i + W$$

for $m + 1 \leq i \leq n$, then

$$U(e'_i) = T'e_i + W$$

So U and T' share the same matrix. Since T' is the linear transformation T whose domain and range are restricted to X, the matrix for T' is A''.