Homework 7

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1. Let $A \in M_{n\times n}(\mathbb{C})$ be an upper triangular matrix. Show that A is normal if and only if A is diagonal.

Solution: If A is diagonal, then obviously the unit vectors e_i are eigenvectors with eigenvalue A_{ii}. Hence there is an orthonormal basis composed of eigenvectors, implying A is normal.
Then assume A is normal. Now $A^* = \overline{A^t}$. So

$$AA^*_{ik} = \sum_j A_{ij}A^*_{jk} = \sum_j A_{ij}\overline{A_{kj}}$$

and

$$A^*A_{ik} = \sum_j A^*_{ij}A_{jk} = \sum_j \overline{A_{ji}}A_{jk}$$

Since A is normal, $AA^* = A^*A$. By induction, one can prove that the non diagonal elements of each row are all zero. For the first row, put $i = k = 1$, then

$$\sum_j A_{1j}\overline{A_{1j}} = \sum_j \overline{A_{j1}}A_{j1}$$
$$\sum_j |A_{1j}|^2 = |A_{11}|^2$$
$$\sum_{j>1} |A_{1j}|^2 = 0$$

Since A is upper triangular, the latter terms of the sum on the right hand side vanish. So $A_{1j} = 0$ for $j > 1$. The non diagonal elements of the first row vanish. Now assume this holds for $i = k = 1, \ldots, m$. For $i = k = m + 1$,

$$\sum_j A_{m+1,j}\overline{A_{m+1,j}} = \sum_j \overline{A_{j,m+1}}A_{j,m+1}$$
$$\sum_j |A_{m+1,j}|^2 = |A_{m+1,m+1}|^2$$
$$\sum_{j>m+1} |A_{m+1,j}|^2 = 0$$

Since A is upper triangular, the terms for $j > m+1$ on the sum on the right hand side vanish. The terms for $j < m + 1$ are also zero, as they are non diagonal elements of previous rows, which are assumed to be zero for $i = k = 1, \ldots, m$. So $A_{m+1,j} = 0$ for $j > m + 1$. It is also zero for $j < m+1$ as A is upper triangular. Hence the non diagonal elements of the $m+1$th row vanish. By induction, only the diagonal elements of A can be nonzero, meaning A is diagonal.

2. Let V be a finite dimensional complex inner product space, and $P \in \mathcal{L}(V)$ be a projection. Show that

P is normal if and only if it is an orthogonal projection.

Solution: If P is normal, then

$$P = \sum_{\lambda} \lambda P_{\lambda}$$

and

$$P^2 = \sum_{\lambda} \lambda^2 P_{\lambda}$$

Let v be an eigenvector with eigenvalue λ. Then for P to be a projection, $P^2 = P$, so

$$P^2 v = \lambda^2 v = Pv = \lambda v$$

So $\lambda = 0$ or $\lambda = 1$. This holds for all eigenvalues. Then we can arrange the eigenvectors as $v_1, \ldots, v_k, v_{k+1}, v_n$ where $v_1, \ldots, v_k$ all have eigenvalues of 1, and the rest have eigenvalues of 0. Then P is an orthogonal projection on the subspace $\operatorname{span}\{v_1, \ldots, v_k\} \subseteq V$.
Now if P is an orthogonal projection, let $v_1, \ldots, v_k$ be an orthonormal basis for $\operatorname{ran}(P)$. Then $Pv_i = v_i \forall 1 \leq i \leq k$. Extend this to an orthonormal basis for V, namely $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$. As P is a projection, $Pv_i = 0 \forall i > k$. Then all of v_i are eigenvectors with eigenvalues 0 or 1. Since V has an orthonormal basis of eigenvectors, P is normal.

3. Let V be a finite dimensional complex inner product space, and $T \in \mathcal{L}(V)$.
   (a) Show that if v is an eigenvector of T^*T, with nonzero eigenvalue, then Tv is an eigenvector of TT^*, with the same eigenvalue.
   (b) Prove that TT^* and T^*T have the same eigenvalues, with the same multiplicities.

Solution: If v is an eigenvector of T^*T with eigenvalue λ, then

$$TT^*(Tv) = T(T^*T)v = T(\lambda v) = \lambda Tv$$

Hence Tv is an eigenvector of TT^* with eigenvalue λ.
Then consider $E(\lambda, T^*T)$ where $\lambda \neq 0$. Let $v_1, \ldots, v_m$ be an orthonormal basis for it. Consider $Tv_1, \ldots, Tv_m$. Obviously this set of vectors are all in $E(\lambda, TT^*)$. In fact, they are linearly independent. Or else

$$Tw = 0$$

for some nonzero w which is a linear combination of $v_1, \ldots, v_m$. Then $T^*Tw = 0$. However, as $v_1, \ldots, v_m$ are eigenvectors, this also means $T^*Tw = \lambda w \neq 0$ with both λ and w being nonzero. This yields a contradiction.
Now we show that $Tv_1, \ldots, Tv_m$ span $E(\lambda, TT^*)$. By contradiction, let

$$Tv_1, \ldots, Tv_m, v_{m+1}$$

be a linearly independent list in $E(\lambda, TT^*)$. Substituting $S = T^*$, we can similarly show that

$$STv_1, \ldots, STv_m, Sv_{m+1}$$

is a linearly independent list of length $m + 1$ in $E(\lambda, T^*T)$. Then $v_1, \ldots, v_m$ with a shorter length of m cannot be a basis of $E(\lambda, T^*T)$, which is a contradiction. Hence we know TT^* and T^*T have the same nonzero eigenvalues with the same multiplicities.
Note that both TT^* and T^*T are self adjoint, so they are normal. This means there is an orthonormal

basis of V consisting of eigenvectors of TT^*, same for T^*T. This also means V has a direct sum decomposition of eigenspaces of TT^* for all of its eigenvalues, similar for T^*T. Let k be the sum of dimensions of all nonzero eigenspaces of TT^* or T^*T (they are the same). Then the kernel of both TT^* and T^*T are equal to $n-k$. If $n=k$, then they both do not have 0 as an eigenvalue. Or else, they both have 0 as an eigenvalue, with kernels of the same dimension. In both cases, TT^* and T^*T have the same eigenvalues (including 0) with the same multiplicities.

4. Let $T:\mathbb{C}^3 \to \mathbb{C}^3$ be the operator give on standard basis vectors as

$$Te_1 = e_3, Te_2 = ie_1, Te_3 = -3ie_2$$

(a) Compute $\sqrt{T^*T}$ and $\sqrt{TT^*}$.

(b) Compute $U_1 = T(\sqrt{T^*T})^{-1}$, and verify that it is unitary.

(c) Compute $U_2 = (\sqrt{TT^*})^{-1}T$, and verify that it is unitary.

Solution:

$$\langle T(a_1e_1 + a_2e_2 + a_3e_3), e_1\rangle = \langle a_2ie_1 - 3a_3ie_2 + a_1e_3, e_1\rangle = a_2i = \langle a_1e_1 + a_2e_2 + a_3e_3, T^*e_1\rangle$$

thus $T^*e_1 = ie_2$.

$$\langle T(a_1e_1 + a_2e_2 + a_3e_3), e_2\rangle = \langle a_2ie_1 - 3a_3ie_2 + a_1e_3, e_2\rangle = -3a_3i = \langle a_1e_1 + a_2e_2 + a_3e_3, T^*e_2\rangle$$

thus $T^*e_2 = -3ie_3$.

$$\langle T(a_1e_1 + a_2e_2 + a_3e_3), e_3\rangle = \langle a_2ie_1 - 3a_3ie_2 + a_1e_3, e_3\rangle = a_1 = \langle a_1e_1 + a_2e_2 + a_3e_3, T^*e_3\rangle$$

thus $T^*e_3 = e_1$.
Then

$$T * Te_1 = e_1, T^*Te_2 = -e_2, T^*Te_3 = -9e_3$$

and

$$TT^*e_1 = -e_1, TT^*e_2 = -9e_2, TT^*e_3 = e_3$$

The square roots are then

$$\sqrt{T^*T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 3i \end{pmatrix}$$

and

$$\sqrt{TT^*} = \begin{pmatrix} i & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} U_1 &= T(\sqrt{T^*T})^{-1} \\ &= \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & -3i \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -\frac{i}{3} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

Then $U_1e_1 = e_3, U_1e_2 = e_1, U_1e_3 = -e_2$. U_1 is obviously invertible, as its determinant is nonzero.

$$\langle a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$$

and

$$\begin{aligned}\langle U_1(a_1e_1 + a_2e_2 + a_3e_3), U_1(b_1e_1, b_2e_2, b_3e_3)\rangle &= \langle a_2e_1 - a_3e_2 + a_1e_3, b_2e_1 - b_3e_2 + b_1e_3\rangle \\ &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= \langle a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3\rangle\end{aligned}$$

So U_1 is unitary.

$$\begin{aligned}U_2 &= (\sqrt{TT^*})^{-1}T \\ &= \begin{pmatrix}-i & 0 & 0\\ 0 & -\frac{i}{3} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & i & 0\\ 0 & 0 & -3i\\ 1 & 0 & 0\end{pmatrix} \\ &= \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 1 & 0 & 0\end{pmatrix}\end{aligned}$$

Then $U_2e_1 = e_3, U_2e_2 = e_1, U_2e_3 = -e_2$. U_2 is obviously invertible, as its determinant is nonzero.

$$\begin{aligned}\langle U_2(a_1e_1 + a_2e_2 + a_3e_3), U_2(b_1e_1 + b_2e_2 + b_3e_3)\rangle &= \langle a_2e_1 - a_3e_2 + a_1e_3, b_2e_1 - b_3e_2 + b_1e_3\rangle \\ &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= \langle a_1e_1 + a_2e_2 + a_3e_3, b_1e_1 + b_2e_2 + b_3e_3\rangle\end{aligned}$$

So U_2 is unitary.

5. Let V be a finite-dimensional complex inner product space. Are the following claims true or false? Justify your answer.
   (a) If $T \in \mathcal{L}(V)$ is a positive operator, and $S \in \mathcal{L}(V)$ with $S^2 = T$, then S must be self-adjoint.
   (b) If $T \in \mathcal{L}(V)$ is self-adjoint, then the operator e^T (defined using functional calculus) is positive.
   (c) If $T \in \mathcal{L}(V)$ is diagonalizable, then T is normal.

Solution: Let T be positive. S does not have to be self adjoint. Note that the zero matrix is a positive operator, as all eigenvalues are 0, which is nonnegative, and it is obviously self adjoint. Then let

$$S = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, T = S^2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Then

$$\left\langle S\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\rangle = 6$$

but

$$\left\langle \begin{pmatrix}1\\2\end{pmatrix}, S\begin{pmatrix}3\\4\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}4\\0\end{pmatrix}\right\rangle = 4 \neq \left\langle S\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\rangle$$

So $S^* \neq S$, and S is not self adjoint.
If T is self adjoint, then e^T is positive. Recall

$$T = \sum_\lambda \lambda P_\lambda$$

with spectrum $\mathrm{Spec}(T) \subseteq \mathbb{R}$. Using functional calculus,

$$f(T) = \sum_{\lambda} f(\lambda) P_\lambda$$

where for $f(T) = e^T$, $f(\lambda) = e^\lambda > 0$ which always holds for positive λ. Now there is an orthonormal basis of eigenvectors of e^T (it shares the same eigenvectors as T which is normal), so e^T is normal. Since all of its eigenvalues are real and positive, it is self adjoint and positive.
Now let T be diagonalizable. It **does not** have to be normal. Consider

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = UDU^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Where U is invertible and D is diagonal as defined above. We see the first and third matrices in the last equality have a product of I, verifying this. Then the only eigenvectors (up to multiplication by a nonzero factor) are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalues 1 and 2 respectively. However, they are not orthonormal, as their inner product can never be zero. Then there is no orthonormal basis of eigenvectors, so T is not normal. This gives a counterexample.