Lecture 3

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October 19, 2022

1 Double Integrals in Polar Coordinates

It is sometimes more convenient to integrate in polar coordinates instead of cartesian coordinates. The relevant equations are

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

Since the area of a sector is given by

$$\Delta A = \frac{1}{2}\Delta\theta r^2$$

the area between two curves is given by

$$\frac{1}{2}\Delta\theta(2r + \Delta r)\Delta r \approx r\Delta\theta\Delta r$$

Alternatively, one can use the Jacobian. Using that, the double integral for $f(x, y) = g(r, \theta)$ can be written as

$$\iint f(x, y)dA = \iint g(r, \theta)rdrd\theta$$

Example 1.1. Evaluate the integral of $3x + 4y^2$ in teh region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\begin{aligned}
I &= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta) r dr d\theta \\
&= \int_0^\pi \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta dr d\theta \\
&= \int_0^\pi r^3\cos\theta + r^4\sin^2\theta \Big|_1^2 d\theta \\
&= \int_0^\pi 7\cos\theta + 15\sin^2\theta d\theta \\
&= 7\sin\theta\Big|_0^\pi + \frac{15}{2}\int_0^\pi 1 - \cos 2\theta d\theta \\
&= \frac{15}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^\pi \\
&= \frac{15}{2}\pi
\end{aligned}$$

Example 1.2. Find the volume of the solid bounded by the $z = 0$ plane and the paraboloid $z = 1 - x^2 - y^2$.

$$\begin{aligned}
I &= \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r - r^3 dr d\theta \\
&= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{4} d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

If we do this using cartesian coordinates,

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 - x^2 - y^2 dydx$$

which is more difficult to solve.

Example 1.3. Find the area enclosed by one petal of the rose given by $r = \cos 3\theta$.

$$\begin{aligned}
I &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{0}^{3\cos\theta} rdrd\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{r^2}{2}\Big|_{0}^{3\cos\theta} d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta d\theta \\
&= \frac{1}{4}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos 6\theta + 1 d\theta \\
&= \frac{1}{4}\left(\frac{\sin 6\theta}{6} + \theta\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \\
&= \frac{\pi}{12}
\end{aligned}$$

Example 1.4. Find the volume trapped between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$

First we find their intersection. Substituting z, we have $r = \frac{1}{\sqrt{2}}$.

$$
\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1-r^2}-r)r dr d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}-r^2 dr d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}(1-r^2)^{\frac{3}{2}}-\frac{r^3}{3}\Big|_0^{\frac{1}{\sqrt{2}}} d\theta \\
&= \int_0^{2\pi} -\frac{1}{3}\times\frac{1}{2\sqrt{2}}+\frac{1}{3}-\frac{1}{6\sqrt{2}} d\theta \\
&= \frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)
\end{aligned}
$$

2 Centre of Mass of a Plate

Recall that moment can be expressed as

$$M = mx$$

where M denotes the moment, m denotes mass, and x denotes distance from axis. Adding the individual moments of all particles using integrals give us

$$M_x = \iint_R y\rho(x,y)dA$$

and

$$M_y = \iint_R x\rho(x,y)dA$$

where $\rho(x,y)$ denotes the density at (x,y).
Denote the centre of mass as $(\bar{x},\bar{y})$. The moments at the centre of mass should be equal to the moments of the plate as a whole, so

$$\bar{x} = \frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA}$$

and

$$\bar{y} = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}$$

It is the centre of mass since when balanced at that point, it will not flip as there is no net moment in either direction.

Example 2.1. Find the centre of mass of the following plate with density function $\rho(x, y) = x + y$. The region is bounded by $x = 0$, $y = 0$ and $y = \sqrt{x}$. Then the mass is given by

$$\begin{aligned} m &= \int_0^1 \int_0^{\sqrt{x}} x + y dy dx \\ &= \int_0^1 xy + \frac{y^2}{2}\Big|_0^{\sqrt{x}} dx \\ &= \int_0^1 x^{\frac{3}{2}} + \frac{x}{2} dx \\ &= \frac{2}{5}x^{\frac{5}{2}} + \frac{x^2}{4}\Big|_0^1 \\ &= \frac{13}{20} \end{aligned}$$

The moment about the y axis is

$$\begin{aligned} M_y &= \int_0^1 \int_0^{\sqrt{x}} x^2 + xy dy dx \\ &= \int_0^1 x^2 y + \frac{xy^2}{2}\Big|_0^{\sqrt{x}} dx \\ &= \int_0^1 x^{\frac{5}{2}} + \frac{x^2}{2} dx \\ &= \frac{2}{7}x^{\frac{7}{2}} + \frac{x^3}{6}\Big|_0^1 \\ &= \frac{19}{42} \end{aligned}$$

Hence $\bar{x} = \frac{190}{273}$

The moment about the x axis is

$$
\begin{aligned}
M_x &= \int_0^1 \int_0^{\sqrt{x}} xy + y^2 dydx \\
&= \int_0^1 \frac{xy^2}{2} + \frac{y^2}{2}\Big|_0^{\sqrt{x}} dx \\
&= \int_0^1 \frac{x^2}{2} + \frac{x}{2} dx \\
&= \frac{x^3}{6} + \frac{x^2}{4}\Big|_0^1 \\
&= \frac{5}{12}
\end{aligned}
$$

Hence $\bar{y} = \frac{25}{39}$

3 Moment of Inertia

From physics, we know

$$v = r\omega$$

and

$$\text{KE} = \frac{1}{2}mv^2$$

Expanding this

$$\text{KE} = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

where I is defined as above.
It is easy to turn this into an integral.

$$I = \iint_R \rho(x,y)[r(x,y)]^2 dydx$$

where r is the distance between (x, y) and the axis. For example,

$$I_x = \iint_R \rho(x,y)y^2 dydx$$

and

$$I_y = \iint_R \rho(x,y)x^2 dydx$$

The moment of inertia about a point is hence $I_x + I_y$ where the x and y axis are translated to intersect at the point, e.g. $I_{x=3}$ and $I_{y=4}$.