Lecture 24

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1 Integral Tansforms

Example 1.1. Suppose you are playing minecraft. Points A and B are separated by 10000 blocks, with a base speed on 4.317 blocks per second. 38.61 minutes of real life is needed. Is there an alternative to save time? Consider the nether, where moving 1 block there is equivalent to moving 8 blocks on the surface.

The same can be done with ODEs. We can transform to a new space, where it is easier to solve, then transform back.

2 Laplace Transform

Definition 2.1. Let f be a function defined on $[0, \infty)$. The Laplace transform is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

The domain of F are the values of s is where the integral is defined and convergent. We also denote F as $\mathcal{L}\{f\}$.

It is technically a change of basis.

Example 2.1.

$$\mathcal{L}\{1\}, t \geq 0$$

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\Big|_0^\infty \\ &= \frac{1}{s}\end{aligned}$$

only where $s > 0$.

Example 2.2.

$$\mathcal{L}\{e^{at}\}, t \geq 0$$

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-(s-a)t}dt \\ &= \frac{1}{s-a}\end{aligned}$$

as above. Again, this only holds where $s - a > 0$.

Example 2.3.

$$\mathcal{L}\{e^{(a+bi)t}\}, t \geq 0$$

$$\begin{aligned}\mathcal{L}\{e^{(a+bi)t}\} &= \int_0^\infty e^{-(s-a-bi)t}dt \\ &= \frac{1}{s-a-bi}\end{aligned}$$

Again, this only holds where $s - a > 0$. This is because the complex exponential can be split into the product of the real and imaginary exponential, where the former requires $s-a > 0$ to converge to 0 and the latter is bounded and oscillates.

Theorem 2.1. *Linearity of Laplace Transform.*
Suppose $\mathcal{L}\{f_1\}$ is defined for $s \geq s_1$ and $\mathcal{L}\{f_2\}$ is defined for $s \geq s_2$. Then

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$$

defined for $s \geq max(s_1, s_2)$

Example 2.4.

$$\mathcal{L}\{\sin t\}, t \geq 0$$

$$\begin{aligned}
\mathcal{L}\{\sin t\} &= \mathcal{L}\left\{\frac{e^{it} - e^{-it}}{2i}\right\} \\
&= \frac{1}{2i}\mathcal{L}\left\{e^{it}\right\} - \frac{1}{2i}\mathcal{L}\left\{e^{-it}\right\} \\
&= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) \\
&= \frac{1}{s^2+1}
\end{aligned}$$

For the Laplace transform to exist on (a, ∞), the integrand has to be integrable and the improper integral has to converge.

Definition 2.2. A function f is *piecewise continuous* if its domain can be partitioned into intervals where on each interval

- f is continuous on the open subinterval
- f has a finite limit at the endpoints of each interval when approached from inside the interval

Heuristically, f can grow at most exponentially fast.

Definition 2.3. A function $f(t)$ is of *exponential order* if there exists real constants $M \geq 0$, $K > 0$, and a such that

$$|f(t)| \leq Ke^{at} \forall t \geq M$$

Theorem 2.2. *The Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > a$ if*

- *f is piecewise continuous on the interval $t \in [0, A]$ for any positive A*
- *f is of exponential order*

We can use the comparison test to prove this.