Lecture 9

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1 Nonhomogeneous to Homogeneous solutions

Consider a first order lienar system with constant coefficients

$$\vec{x}' = A\vec{x} + \vec{b}$$

where

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Consider an ansatz in the form

$$\vec{x} = \vec{\tilde{x}} + \vec{x_{eq}}$$

where $\vec{x_{eq}}$ is independent of t. Plugging this into the ODE,

$$\vec{\tilde{x}}' = A\vec{\tilde{x}} + A\vec{x_{eq}} + \vec{b}$$

Assuming A is invertible, if we define

$$\vec{x_{eq}} = -A^{-1}\vec{b}$$

we have a homogeneous linear system

$$\vec{\tilde{x}} = A\vec{\tilde{x}}$$

2 Superposition Principle

Let $\phi_1(t)$ and $\phi_2(t)$ be solutions to a first order linear system

$$\vec{x}' = A\vec{x}$$

Then obviously (differentiation and matrix multiplication are linear functions) any linear combination

$$c_1\phi_1(t) + c_2\phi_2(t)$$

is also a solution.

Theorem 2.1. *Suppose $\vec{\phi_1}(t)$ and $\vec{\phi_2}(t)$ are solutions of $\vec{x}' = A\vec{x}$. Then for any coefficients c_1 and c_2,*

$$c_1\vec{\phi_1}(t) + c_2\vec{\phi_2}(t)$$

is also a solution.

Definition 2.1. Suppose we have two solutions $\vec{\phi_1}(t)$ and $\vec{\phi_2}(t)$ for the system $\vec{x}' = A\vec{x}$ defined on I. We say they are linearly independent if $\exists k \in \mathbb{R}$ such that

$$\vec{\phi_1}(t) = k\vec{\phi_2}(t)$$

Otherwise we say they are linearly independent.

Let $\vec{x}(0) = \vec{x_0}$. We can then solve for the constants c_1 and c_2 by

$$\begin{pmatrix} \phi_1^1(t_0) & \phi_2^1(t_0) \\ \phi_2^1(t_0) & \phi_2^2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$$

A unique solution is guaranteed as long as the 2×2 matrix is invertible.

3 Wronskian

Definition 3.1.

$$W[\vec{\phi_1}, \vec{\phi_2}](t) = \det \begin{bmatrix} \phi_1^1(t) & \phi_2^1(t) \\ \phi_1^2(t) & \phi_2^2(t) \end{bmatrix}$$

Theorem 3.1. *Suppose $\vec{\phi_1}(t)$ and $\vec{\phi_2}(t)$ are two solutions to a homogeneous first order linear system*

$$\vec{x}' = A\vec{x}$$

for an interval I. If the Wronskian is non zero $\forall t \in I$, the general solution is given by

$$c_1\vec{\phi_1}(t) + c_2\vec{\phi_2}(t), c_1, c_2 \in \mathbb{R}$$

Theorem 3.2. *Suppose $\vec{\phi_1}(t)$ and $\vec{\phi_2}(t)$ are two solutions to a homogeneous first order linear system*

$$\vec{x}' = A\vec{x}$$

They are linearly independent if and only if the Wronskian is non zero.

Note that it is impossible for the Wronskian to be 0 at only a single point (proof will come later).
From the above, we know all we need is two linearly independent solutions. Consider the ansatz

$$\vec{x} = e^{\lambda t}\vec{v}$$

where $\vec{v}$ is independent of t. Substitution gives us

$$\lambda\vec{v} = A\vec{v}$$

so $\vec{v}$ is an eigenvector, and λ is an eigenvalue.
Now there are several cases. We first consider

$$\lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}$$

Then we have

$$\vec{\phi_1}(t) = e^{\lambda_1 t}\vec{v_1}, \vec{\phi_2}(t) = e^{\lambda_2 t}\vec{v_2}$$

They are independent, as the two eigenvectors must be linearly independent.