Homework 8

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1. Let $T \in \mathcal{L}(V)$ be an operator on a finite-dimensional complex inner product space. Show that T is unitary if and only if and only if all its singular values are equal to 1.

Solution: If T is unitary, let v be any eigenvectors with eigenvalue λ. Then

$$v = Iv = T^*Tv = \lambda\overline{\lambda}v = |\lambda|^2 v \Rightarrow |\lambda| = 1$$

Since T is unitary, T is normal, and so V has an orthonormal basis of eigenvectors of T, namely $v_1, \ldots, v_n$. As shown above, $v_1, \ldots, v_n$ are also eigenvalues of T^*T with eigenvalues 1. Thus all singular values are $\sqrt{1} = 1$.
If all singular values of T are equal to 1, then let $U = \sqrt{T^*T}$. U is normal, with all eigenvalues equal to 1. Then let $v_1, \ldots, v_n$ be an orthonormal basis of V that are eigenvectors of U. Then

$$U\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i v_i = I\left(\sum_{i=1}^{n} a_i v_i\right)$$

So $U = I$. Then

$$T^*T = U^2 = I^2 = I$$

Also note that T is invertible, as I is invertible. Then T is unitary.

2. Consider the following complex matrix

$$A = \begin{pmatrix} 1 & i & 1 \\ -1 & 0 & -2 \\ 1 & 2i & 0 \end{pmatrix}$$

(a) Find a singular value decomposition of A. That is, give matrices U_1, U_2, D, where U_1, U_2 are unitary and D is diagonal, with positive diagonal entries, such that

$$A = U_2 D U_1^{-1}$$

(b) Using (a), find the polar decomposition of A.

Solution:

$$A^*A = \begin{pmatrix} 1 & -1 & 1 \\ -i & 0 & -2i \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & i & 1 \\ -1 & 0 & -2 \\ 1 & 2i & 0 \end{pmatrix} = \begin{pmatrix} 3 & 3i & 3 \\ -3i & 5 & -i \\ 3 & i & 5 \end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned}
q(z) &= (z-3)(z-5)(z-5) - 3i(-i)3 - 3(-3i)i - (z-3)(-i)i - 3i(-3i)(z-5) - 3\times(z-5)\times 3 \\
&= z^3 - 13z^2 + 55z - 75 - 9 - 9 - z + 3 - 9z + 45 - 9z + 45 \\
&= z^3 - 13z^2 + 36z \\
&= z(z-4)(z-9)
\end{aligned}$$

The eigenvector and eigenvalue pairs are

$$v_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ -i \\ 1 \end{pmatrix}, \lambda_1 = 0, v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \lambda_2 = 4, v_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}, \lambda_3 = 9$$

Hence

$$w_2 = \frac{1}{2}\begin{pmatrix} 1 & i & 1 \\ -1 & 0 & -2 \\ 1 & 2i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

and

$$w_3 = \frac{1}{3}\begin{pmatrix} 1 & i & 1 \\ -1 & 0 & -2 \\ 1 & 2i & 0 \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Then we can extend this to form a basis by defining

$$w_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Now

$$U_1 = \begin{pmatrix} -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{i}{\sqrt{6}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Its inverse is its adjoint, which is its complex conjugate. Hence

$$A = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

where the first matrix is U_2, the second matrix is D, and the third is U_1^{-1}.
We also get the polar decomposition for free.

$$A = U_2 D U_1^{-1} = (U_2 U_1^{-1})(U_1 D U_1^{-1}) = UR$$

where U is unitary and R is positive. Then

$$U = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2i}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{i}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2i}{3} & -\frac{1}{3} \end{pmatrix}$$

and

$$R = \begin{pmatrix} -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{i}{\sqrt{6}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & i & 1 \\ -i & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

so

$$A = \begin{pmatrix} -\frac{1}{3} & \frac{2i}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{i}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2i}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & i & 1 \\ -i & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

3. Let $T \in \mathcal{L}(V, W)$ be a linear operator between finite-dimensional inner product spaces. For any orthonormal basis $v_1, \dots, v_n$ of V, prove that

$$||Tv_1||^2 + \dots ||Tv_n||^2 = s_1^2 + \cdots + s_n^2$$

where $s_1, \dots, s_n$ are the singular values of T.

Solution: Let U be the matrix

$$U = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}$$

Then note that U is a change of basis, hence it is invertible. Also

$$\begin{aligned}\langle Ux, Uy \rangle &= \left\langle U\left(\sum_i a_i e_i\right), U\left(\sum_i b_i e_i\right)\right\rangle \\ &= \left\langle \sum_i a_i v_i, \sum_i b_i v_i \right\rangle \\ &= \sum_i a_i b_i \\ &= \left\langle \sum_i a_i e_i, \sum_i b_i e_i \right\rangle \\ &= \langle x, y \rangle\end{aligned}$$

Thus U is unitary. Note that for any unitary matrix W with columns $w_1, \dots, w_n$, note that

$$1 = \langle e_i, e_i \rangle = \langle We_i, We_i \rangle = \langle w_i, w_i \rangle$$

so all columns have a norm of 1. Now for $v_1, \dots, v_n$ to be orthonormal, the norm of v_i has to be 1. We define $f_1, \dots, f_n$ to be the orthonormal basis consisting of eigenvectors of $\sqrt{T^*T}$. Defining v in terms of f with coefficients a,

$$v_i = \sum_j a_{ji} f_j$$

where a_{ij} is the ijth component of U.

$$\begin{aligned}||v_i|| &= 1 \\ \langle v_i, v_i \rangle &= 1 \\ \left\langle \sum_j a_{ji} f_j, \sum_j a_{ji} f_j \right\rangle &= 1 \\ \sum_j |a_{ji}|^2 &= 1\end{aligned}$$

Similarly, $U^* = \overline{U^t}$ is also unitary, with columns

$$w_i = \sum_j \overline{a_{ij}} f_j$$

Doing the same to w_i, we see

$$\sum_j |a_{ij}|^2 = \sum_j |\overline{a_{ij}}|^2 = 1$$

where the first equality is because a complex number and its conjugate share the same norm. Now

$$\begin{aligned}
\sum_i ||Tv_i||^2 &= \sum_i \langle Tv_i, Tv_i \rangle \\
&= \sum_i \langle v_i, T^*Tv_i \rangle \\
&= \sum_i \left\langle \sum_j a_{ji} f_j, \sum_j a_{ji} s_j^2 f_j \right\rangle \\
&= \sum_i \sum_j |a_{ji}|^2 s_j^2 \\
&= \sum_j s_j^2 \sum_i |a_{ji}|^2 \\
&= \sum_j s_j^2
\end{aligned}$$

4. Let V be a finite-dimensional inner product space. For $T \in \mathcal{L}(V)$ define the 'operator norm'

$$||T|| = \sup\{||Tv|| : v \in V, ||v|| = 1\}$$

(a) For $T \in \mathcal{L}(V)$ is positive, prove that $||T||$ equals the largest eigenvalue of T.

Solution: Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors of T with eigenvalues $\lambda_1, \ldots, \lambda_n$, such that λ_1 is (one of the) largest eigenvalue. Then let $||v|| = 1$ where

$$v = \sum_i a_i v_i$$

and

$$\sum_i |a_i|^2 = 1$$

Now

$$||Tv||^2 = \langle Tv, Tv \rangle = \left\langle \sum_i a_i \lambda_i v_i, \sum_i a_i \lambda_i v_i \right\rangle = \sum_i \lambda_i^2 |a_i|^2 \leq \sum_i \lambda_1^2 |a_i|^2 = \lambda_1^2$$

Taking the square root of both sides,

$$||Tv|| \leq \lambda_1$$

Putting $v = v_1$, it is obvious that $||Tv|| = \lambda_1$, so the maximum value of $||Tv||$ where $||v|| = 1$ is λ_1. Hence $||T|| = \lambda_1$, which is the largest eigenvalue of T.

(b) For $T \in \mathcal{L}(V)$ arbitrary, prove that $||T||$ equals the largest singular value of T.

Solution: Using polar decomposition, $T = UR$. We use the same v as defined in the previous part. Since U is unitary,

$$\delta_{ij} = \langle v_i, v_j \rangle = \langle Uv_i, Uv_j \rangle$$

So $Uv_i, \ldots, Uv_n$ is an orthonormal basis. Define

$$f_i = Uv_i$$

Now R is a diagonal matrix with diagonal entries $s_1, \ldots, s_n$, which are all singular values of T, with the greatest singular value being s_k. Then

$$||Tv||^2 = \langle Tv, Tv \rangle = \left\langle \sum_i a_i s_i f_i, \sum_i a_i s_i f_i \right\rangle = \sum_i |a_i|^2 s_i^2 \leq \sum_i |a_i|^2 s_k^2 = s_k^2$$

So $||Tv|| \leq s_k$. Again, it is obvious that $||Tv_k|| = s_k$, so the maximum value of $||Tv|| with ||v|| = 1$ is s_k, or $||T|| = s_k$, the largest singular value of T.

(c) Prove that

$$||T^*T|| = ||T||^2$$

Solution: Similar to above, let $s_1, \ldots, s_n$ be the singular values of T, with s_k being the largest singular value. By the first part, since T^*T is always positive for any T, we know $||T^*T|| = s_k^2$. From the second part, $||T|| = s_k$. Then obviously

$$||T^*T|| = s_k^2 = ||T||^2$$