Lecture 13

niceguy

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1 Self-Adjoint Operators

Working under finite dimensions,

Definition 1.1. $T : V \to V$ is self adjoint if

$$T = T^*$$

Theorem 1.1 (Spectral Theorem)**.** *If T is self-adjoint, it is diagonalisable. Moreover, it has an orthonormal eigenbasis.*

Proof. Fix an eigenvector v. Consider

$$V = \text{span}\{v\} \oplus (\text{span}\{v\})^\perp$$

This is repeated to form an orthogonal basis, which can be normalised. The existance of an orthogonal basis ensures T is diagonalisable. □

The fact that an eigenvector v always exists, and that V can be written as a direct sum of self-adjoint sets, follows from the following lemmas.

Lemma 1.1. *If T is self-adjoint, and W is T-invariant, then $W^\perp$ is T-invariant.*

Proof. Let $v \in W^\perp, w \in W$. Then $Tw \in W$, and

$$\langle Tv, w\rangle = \langle v, Tw\rangle = 0$$

□

Lemma 1.2. *If T is self-adjoint with eigenvalue λ and eigenvector v, then*

$$\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\lambda}\langle v, v\rangle \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

Lemma 1.3. *If T is self-adjoint, $E_\lambda \perp E_\mu \forall \lambda \neq \mu$.*

Proof. Considering $v \in E_\lambda, w \in E_\mu$, then $\langle Tv, w\rangle = \langle v, Tw\rangle$. This implies $(\lambda - \mu)\langle v, w\rangle = 0$, meaning $\langle v, w\rangle = 0$. $\square$

Lemma 1.4. *If T is self adjoint, it has one eigenvalue.*

Proof. We proved this for complex V. Then let M be a matrix representation of T over $\mathbb{R}$ in an orthonormal basis. Then

$$\langle Mv, w\rangle = \langle v, Mw\rangle \Rightarrow v^t M^t w = v^t M w$$

in general, implying

$$M = M^t$$

Now let $\vec{v} = \vec{x} + i\vec{y}$ where $\vec{v}$ is a complex eigenvector. Then

$$M\vec{x} + iM\vec{y} = \lambda\vec{x} + i\lambda\vec{y}$$

So at least one of $\vec{x}$ and $\vec{y}$ is an eigenvectors (at most one can be zero). $\square$

Example 1.1. Consider the following self adjoint operator.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, d \in \mathbb{R}, c = \overline{b}$$

Then

$$q(z) = z^2 - \operatorname{tr}(A)z + \det(A)$$

And the eigenvalues are given by

$$\lambda = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det(A)}}{2}$$

The determinant is given by

$$(a+d)^2 - 4(ad - |b|^2) = (a-d)^2 + 4|b|^2 \geq 0$$

So the eigenvalue(s) are real.
The eigenvectors are

$$v_1 = \begin{pmatrix} //\lambda_1 - a \end{pmatrix}, v_2 = \begin{pmatrix} \lambda_2 - d \\ c \end{pmatrix}$$

And they are orthogonal

$$\begin{aligned} \langle v_1, v_2 \rangle &= b\overline{\lambda_2 - d} + (\lambda_1 - a)b \\ &= b(\lambda_2 - d + \lambda_1 - a) \\ &= b(\lambda_1 + \lambda_2 - \mathrm{tr}(A)) \\ &= 0 \end{aligned}$$

Where the last equality comes from the fact that the sum of roots is $\mathrm{tr}(A)$.

Example 1.2. Note that the two forms are equivalent

$$ax^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Definition 1.2. $T \in \mathcal{L}(V)$ is unitary if T is invertible and $\langle Tx, Ty \rangle = \langle x, y \rangle$. These are called isometries in general.

Example 1.3. We can have "reflection maps" R, where for a fixed $w \in V$,

$$R_w : v \mapsto v - 2\frac{\langle v, w \rangle}{\langle w, w \rangle} w$$

Which is an example of an isometry.

Lemma 1.5. *If S, T are unitary, then ST is unitary.*

$$\langle STv, STw \rangle = \langle Tv, Tw \rangle = \langle v, w \rangle$$

Lemma 1.6. *If V is finite dimensional, and $T \in \mathcal{L}(V)$ such that*

$$\langle Tv, Tw \rangle = \langle v, w \rangle \forall v, w \in V$$

then T is unitary.

Proof. Note that $||Tv||^2 = ||v||^2$. Hence its kernel is $\{0\}$. For finite dimensions, this implies T is invertible. $\square$

Theorem 1.2. *The following statements are equivalent*

1. *T is unitary*
2. *There exists an orthonormal basis which maps to an orthonormal basis on T*
3. *The above holds for all orthonormal bases*

Proof. Obviously the third statement implies the second. The first implies the third, as

$$\langle Tv_i, Tv_j\rangle = \langle v_i, v_j\rangle$$

where $i \neq j$ shows it is orthogonal, and $i = j$ shows it is orthonormal. Finally, consider the second statement. Let

$$x = \sum x_i v_i, y = \sum y_i v_i$$

Then letting T map from the orthonormal bases v_i to w_i,

$$\begin{aligned}
\langle Tx, Ty\rangle &= \sum_{i,j} x_i \overline{y_i} \langle Tv_i, Tv_j\rangle \\
&= \sum_{i,j} x_i \overline{y_i} \langle w_i, w_j\rangle \\
&= \sum_{i} x_i \overline{y_i} \langle v_i, v_j\rangle \\
&= \langle x, y\rangle
\end{aligned}$$

Thus the second statement implies the first. □

If T is unitary, then

$$\begin{aligned}
\langle v, w\rangle &= \langle Tv, Tw\rangle \\
&= \langle v, T^*Tw\rangle \\
\langle v, w - T^*Tw\rangle &= 0 \\
(I - T^*T)w &= 0 \\
T^{-1} &= T^*
\end{aligned}$$

In fact,

$$\begin{aligned}
1 &= \det(I) \\
&= \det(T^*T) \\
&= \det(T^*)\det(T) \\
&= \overline{\det(T)}\det(T) \\
&= ||\det(T)||^2
\end{aligned}$$

which explains why unitary maps are named so. Note that the inverse doesn't hold; consider

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$