Lecture 25

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November 14, 2022

1 Laplace Transform

Recall

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Given the Laplace Transform of $f(t)$, how does it change when $f(t)$ itself is changed?

Example 1.1.

$$f(t) \to e^{ct} f(t)$$

Then

$$\begin{aligned}
\mathcal{L}\{e^{ct} f(t)\}(s) &= \int_0^\infty e^{-st} e^{ct} f(t) dt \\
&= \int_0^\infty e^{-(s-c)t} f(t) dt \\
&= F(s-c)
\end{aligned}$$

Which is defined when $s > a + c$.

Example 1.2.

$$f(t) \to f'(t)$$

Then

$$\begin{aligned}
\mathcal{L}\{f'(t)\}(s) &= \int_0^\infty e^{-st} f'(t) dt \\
&= e^{-st} f(t)\Big|_0^\infty + s \int_0^\infty e^{-st} f(t) dt \\
&= sF(s) - f(0)
\end{aligned}$$

assuming f and f' are of exponential order.

Doing this twice,

$$\mathcal{L}\{f''(t)\}(s) = s^2F(s) - sf(0) - f'(0)$$

assuming f, f' and f'' are of exponential order, and are defined on $[0, \infty)$. Furthermore, using induction, one can show

$$\mathcal{L}\{f^{(n)}(s)\} = s^nF(s) - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0)$$

Example 1.3.

$$f(t) \to t^n f(t)$$

Then

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st} f(t)dt \\
\frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s) &= \int_0^\infty (-1)^n t^n e^{-st} f(t)dt \\
(-1)^n\mathcal{L}\{t^n f(t)\}(s) &= F^{(n)}(s) \\
\mathcal{L}\{t^n f(t)\}(s) &= (-1)^n F^{(n)}(s)
\end{aligned}$$

Putting $f(t) = 1$, we get

$$\begin{aligned}
\mathcal{L}\{t^n\} &= (-1)^n \frac{d^n}{ds^n}\frac{1}{s} \\
&= \frac{n!}{s^{n+1}}
\end{aligned}$$

2 Applications of the Laplace Transform

Example 2.1.

$$y'' + 2y' + 5y = e^{-t}, y(0) = 1, y'(0) = -3$$

Applying the Laplace Transform on both sides,

$$\begin{aligned}
s^2Y(s) - s + 3 + 2sY(s) - 2 + 5Y(s) &= \frac{1}{s+1} \\
Y(s) &= \frac{s^2}{(s+1)(s^2+2s+5)}
\end{aligned}$$