

Problem Set 10

niceguy

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1. A coaxial cable carries a time-invariant current I . A thin layer of a ferromagnetic material is placed near the outer conductor, and the rest of the space between the conductor is air-filled. With respect to the notation in the Fig. Q5.1 showing the cable cross section, the magnetic field exist only in

Solution: D, regions 1, 2, 3, 4. This is because

$$\oint_C \vec{B} \cdot d\vec{l} = \mu I$$

For all of these regions, enclosed I is nonzero, so there is a magnetic field. For region 5, enclosed current cancels out.

2. The conduction current density vector at a point of a homogenous magnetic material of relative permeability μ_r is \vec{J} . The curl of the magnetic flux density vector, $\vec{\nabla} \times \vec{B}$, at that point equals

Solution: D, $\mu_r \mu_0 \vec{J}$. This is because

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}$$

3. Consider a boundary surface between two magnetic media, with relative permeabilities $\mu_{r1} = 600$ and $\mu_{r2} = 300$, respectively. Assuming that no conduction current exist on the boundary ($\vec{J}_s = 0$), which of the cases shown in Fig. Q5.2 represent possible magnetic field intensity vectors on the two sides of the boundary?

Solution: B. Tangential component of \vec{H} is preserved, and normal component of \vec{B} is preserved. Since $\vec{B} = \mu \vec{H}$, and μ is greater in region 1, \vec{H}_2 has to have a greater normal component to balance it out.

4. Repeat the previous question but assuming that the vectors in Fig. Q5.2 are magnetic flux density vectors, \vec{B}_1 and \vec{B}_2 , in place of \vec{H}_1 and \vec{H}_2 , respectively, and determine which of these cases shown represent possible vectors B on the two sides of the interface (consider the answers provided in the previous question).

Solution: D. Similarly, normal component of \vec{B} is preserved. \vec{H} scales with μ . Since the tangential component of \vec{H} is preserved, but μ is greater in region 1, the tangential component of \vec{B} is also greater in region 1.

5. Fig. Q5.3 shows magnetic field lines near a boundary between two linear magnetic media that is free of conduction currents ($\vec{J}_s = 0$). Which of the two media has higher permeability?

Solution: A, medium 1. Consider the last question, where μ is greater in region 1. Now, \vec{B} becomes steeper upon crossing to region 2, which matches the case.

6. In the structure in Fig.Q5.6, a coil of wire is wound uniformly and densely about a thin toroidal magnetic core. The current of the coil is I , and the magnetic field intensity and flux density in the core are \vec{H} and \vec{B} , respectively. If I is doubled,

Solution: F, need more information. For a hard core, both would double, but for a soft core, it would follow the hysteresis curve.

7. Assume that the core in Fig.Q5.6 is made of iron. If the current I of the coil is increased gradually and smoothly from zero to a very large value, the value of the magnetic flux density, \vec{B} , in the core

Solution: C, rises rapidly first and more slowly later. This is according to the hysteresis curve.

8. Let the core in Fig.Q5.6 be cut from iron and the current I of the coil be established at a very large (positive) value. If I is then reduced to zero, the values of the magnetic field intensity (\vec{H}) and flux density (\vec{B}) in the core encounter the following changes:

Solution: D, \vec{H} drops to zero and \vec{B} drops to a nonzero value. This is because the curl of \vec{H} is equal to \vec{J} , which goes to zero, thus so does \vec{H} . However, \vec{B} drops, but not to zero, according to the hysteresis curve.

9. In analysis of magnetic circuits, a set of approximations is introduced to simplify the computation. However, consider the following possible assumptions: (a) The magnetic flux is concentrated exclusively inside the branches of the ferromagnetic core and air gaps; (b) Magnetic materials of the core can be considered to be linear; (c) Magnetic materials of the branches are never in the state of saturation; (d) There is no magnetic field in air gaps; (e) Lengths of air gaps can be considered to be zero; (f) The fringing magnetic flux near the edges of air gaps can be neglected; (g) The magnetic field intensity (\vec{H}) is the same in all branches of the circuit; (h) The magnetic field is uniform throughout the volume of each branch of the circuit; (i) Magnetic fluxes are the same in all branches of the circuit. Which of the assumptions constitute the set of approximations used in the analysis?

Solution: E, Assumptions (a), (f), and (h).

10. A thin toroidal core, made of a ferromagnetic material of permeability μ , has an air gap, as shown in Fig.Q5.7. There is a time-invariant current through the winding. The magnitude of the magnetic field intensity vector in the ferromagnetic with respect to the clockwise reference direction is H . The magnitude of the magnetic field intensity vector in the gap (H_0) with respect to the same reference direction is

Solution: E, $H_0 = \frac{\mu H}{\mu_0}$. In this case, B is constant, so this answer is obtained by rearranging

$$\mu_0 H_0 = \mu H$$

11. In a magnetic circuit with several nodes and branches, nonlinear magnetization curves of the core materials, and more than one magnetomotive force (winding with current), it is possible to have a branch with a zero magnetic flux.

Solution: A, True. The forces can act in opposite directions, leading to a zero magnetic flux.

12. A thin ferrromagnetic disk of radius a and thickness d ($d \ll a$) in air has a nonuniform magnetization, given by $M = M_0 \left(\frac{r}{a}\right)^2 \hat{a}_z$ (Fig. Q5.37), where M_0 is a constant. Calculate
 (a) the distribution of magnetization currents of the disk, and
 (b) the magnetic flux density vector along the z -axis.

Solution: The volume current magnetization vector is given by

$$\begin{aligned} \vec{J}_m &= \vec{\nabla} \times \vec{M} \\ &= \vec{\nabla} \times M_0 \left(\frac{r}{a}\right)^2 \hat{a}_z \\ &= -\frac{\partial}{\partial r} M_0 \left(\frac{r}{a}\right)^2 \hat{a}_\phi \\ &= -\frac{2M_0 r}{a^2} \hat{a}_\phi \end{aligned}$$

and the surface current is

$$\begin{aligned} \vec{J}_{m,s} &= \vec{M} \times \hat{a}_n \\ &= \vec{M} \times \hat{a}_r \\ &= M_0 \hat{a}_\phi \end{aligned}$$

which holds for the circumference, with normal in the \hat{a}_r direction. The only other surfaces both point in the \hat{a}_z direction, both of which lead to a cross product of 0.

Now consider the $d\vec{B}$ contributions of each ring with inner radius r and thickness dr . The integral is

$$\begin{aligned} \oint_c \frac{rd\phi \hat{a}_\phi \times (-r\hat{a}_r + z\hat{a}_z)}{(r^2 + z^2)^{\frac{3}{2}}} &= \int_0^{2\pi} \frac{(r^2 \hat{a}_z + rz\hat{a}_r)d\phi}{(r^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{2\pi r^2}{(r^2 + z^2)^{\frac{3}{2}}} \hat{a}_z \end{aligned}$$

where the \hat{a}_r component is ignored, as it has a constant coefficient, and integrating the unit vector gives 0. Then we can write

$$d\vec{B} = \frac{\mu dI}{4\pi} \frac{2\pi r^2}{(r^2 + z^2)^{\frac{3}{2}}} = \frac{\mu r^2}{2(r^2 + z^2)^{\frac{3}{2}}} \left(-\frac{2dM_0 r dr}{a^2} \right) = -\frac{d\mu M_0 r^3 dr}{a^2 (r^2 + z^2)^{\frac{3}{2}}}$$

Now we get \vec{B} by integrating this

$$\begin{aligned}
\vec{B} &= \int d\vec{B} \\
&= -\frac{d\mu M_0}{a^2} \int_0^a \frac{r^3 dr}{(r^2 + z^2)^{\frac{3}{2}}} \hat{a}_z \\
&= -\frac{d\mu M_0}{a^2} \int_0^{\arctan \frac{a}{z}} \frac{z^3 \tan^3 \theta \times z \sec^2 \theta d\theta}{z^3 \sec^3 \theta} \hat{a}_z \\
&= -\frac{d\mu M_0}{a^2} \int_0^{\arctan \frac{a}{z}} \frac{z \sin^3 \theta}{\cos^2 \theta} d\theta \hat{a}_z \\
&= -\frac{d\mu M_0}{a^2} \int_0^{\arctan \frac{a}{z}} \frac{z(\sin \theta(1 - \cos^2 \theta))}{\cos^2 \theta} d\theta \hat{a}_z \\
&= -\frac{d\mu M_0}{a^2} \int_0^{\arctan \frac{a}{z}} z(\sec \theta \tan \theta - \sin \theta) d\theta \hat{a}_z \\
&= -\frac{d\mu M_0}{a^2} z \left(\frac{\sqrt{a^2 + z^2}}{z} + \frac{z}{\sqrt{a^2 + z^2}} - 1 \right) \hat{a}_z \\
&= -\frac{d\mu M_0}{a^2} \left(\sqrt{a^2 + z^2} + \frac{z^2}{\sqrt{a^2 + z^2}} - z \right) \hat{a}_z
\end{aligned}$$

Now we consider the contribution from the outer strip. The integral remains the same, substituting $r = a$. Then

$$\begin{aligned}
\vec{B} &= \frac{\mu M_0 d}{4\pi} \frac{2\pi a^2}{(a^2 + z^2)^{\frac{3}{2}}} \hat{a}_z \\
&= \frac{\mu M_0 d}{a^2} \frac{a^4}{(a^2 + z^2)^{\frac{3}{2}}} \hat{a}_z
\end{aligned}$$

So total \vec{B} is

$$\vec{B} = -\frac{d\mu M_0}{a^2} \left(\sqrt{a^2 + z^2} + \frac{z^2}{\sqrt{a^2 + z^2}} - z + \frac{a^4}{(a^2 + z^2)^{\frac{3}{2}}} \right) \hat{a}_z$$

13. Consider a plane boundary ($y = 0$) between air (region 1, $\mu_{r1} = 1$) and iron (region 2, $\mu_{r2} = 5000$).

(a) Assuming $\vec{B}_1 = 0.5\hat{a}_x - 10\hat{a}_y$, find \vec{B}_2 and the angle that \vec{B}_2 makes with the interface.

Solution: Normal components of \vec{B} are conserved, so $B_{2,y} = -10$. Tangential components of \vec{H} are conserved, so $B_{2,x} = B_{1,x} \times \frac{\mu_{r2}}{\mu_{r1}} = 2500$. Hence

$$\vec{B}_2 = 2500\hat{a}_x - 10\hat{a}_y$$

and the angle is

$$\arctan \frac{10}{2500} \approx \frac{1}{250}$$

(b) Assuming $\vec{B}_2 = 10\hat{a}_x + 0.5\hat{a}_y$, find \vec{B}_1 and the angle that \vec{B}_1 makes with the interface.

Solution: The same properties are used. $B_{1,y} = 0.5$ and $B_{1,x} = B_{2,x} \times \frac{\mu_{r1}}{\mu_{r2}} = 0.002$. Then

$$\vec{B}_1 = 0.002\hat{a}_x + 0.5\hat{a}_y$$

and the angle is

$$\arctan \frac{0.5}{0.002} = \arctan 250 = 1.57$$

14. A very long conductor in free space carrying a current I is parallel to, and at a distance d from, an infinite plane interface with a medium.

- (a) Discuss the behaviour of the normal and tangential components of \vec{B} and \vec{H} at the interface
 (a) if the medium is infinitely conducting
 (b) if the medium is infinitely permeable

Solution: If the medium is infinitely conducting, $\vec{B}_2 = \vec{H}_2 = 0$ as there is no current.

- (c) Find and compare the magnetic field intensities \vec{H} at an arbitrary point in the free space for the two cases in part (a).
 (d) Determine the surface current densities at the interface, if any, for the two cases.
15. A uniformly magnetized square ferromagnetic plate of side length a and thickness d ($d \ll a$) is situated in air. With reference to the coordinate system in Fig. 5.36, the magnetization vector in the plate is given by $\vec{M} = M_0\hat{a}_z$, where M_0 is a constant. Determine the magnetic flux density vector at an arbitrary point on the z axis.

Solution: The volume current density is

$$\begin{aligned}\vec{J}_m &= \vec{\nabla} \times \vec{M} \\ &= 0\end{aligned}$$

Since \vec{M} is a constant. The surface current density is

$$\vec{J}_{m,s} = M_0\hat{a}_z \times \hat{a}_n$$

Then on the 4 side surfaces, this has the magnitude of M_0 in the anticlockwise direction. On the top and bottom surfaces, it vanishes. Then by symmetry, \vec{B} on the z -axis only has a \hat{a}_z component. Now we calculate the contribution of each side to \vec{B} . We do this first for the front side,

$$d\vec{l} \times (\vec{R} - \vec{R}') = dx\hat{a}_x \times \left(-x\hat{a}_x + \frac{a}{2}\hat{a}_y + z\hat{a}_z\right) = \left(\frac{a}{2}\hat{a}_z - z\hat{a}_y\right)dx$$

Taking only the \hat{a}_z component and integration,

$$\begin{aligned}\vec{B} &= \frac{\mu_0 M_0 d}{4\pi} \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{adx}{2(x^2 + \frac{a^2}{4} + z^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0 M_0 da^2}{(a^2 + 4z^2)\sqrt{2a^2 + 4z^2}}\end{aligned}$$

Summing them up,

$$\vec{B} = \frac{2\sqrt{2}\mu_0 M_0 da^2}{(a^2 + 4z^2)\sqrt{a^2 + 2z^2}}$$

16. An infinitely-long ferromagnetic cylinder of radius a in air has a nonuniform magnetization. In a cylindrical coordinate system whose z -axis coincides with the cylinder axis, $\vec{M} = M_0 \frac{r}{a} \hat{a}_\phi$ ($0 \leq r \leq a$), where M_0 is a constant. Find the current densities \vec{J}_m in the cylinder and \vec{J}_{ms} on the cylinder, as well as the magnetic flux density \vec{B} inside and outside the cylinder.

Solution:

$$\begin{aligned}
 \vec{J}_m &= \vec{\nabla} \times \vec{M} \\
 &= \vec{\nabla} \times M_0 \frac{r}{a} \hat{a}_\phi \\
 &= \frac{1}{r} \frac{\partial}{\partial r} r M_0 \frac{r}{a} \hat{a}_z \\
 &= \frac{1}{r} \frac{2M_0 r}{a} \hat{a}_z \\
 &= \frac{2M_0}{a} \hat{a}_z
 \end{aligned}$$

For the side surface,

$$\begin{aligned}
 \vec{J}_{ms} &= \vec{M} \times \hat{a}_n \\
 &= M_0 \frac{a}{a} \hat{a}_\phi \times \hat{a}_r \\
 &= -M_0 \hat{a}_z
 \end{aligned}$$

Since \vec{J}_s is constant in the cylinder, pointing upwards, we know \vec{B} is constant given r , and rotates in a counterclockwise direction. Within the cylinder,

$$\begin{aligned}
 \int \vec{B} \cdot d\vec{l} &= \mu I \\
 2\pi r B &= \mu \times \frac{2M_0}{a} \times \pi r^2 \\
 B &= \frac{M_0 \mu r}{a}
 \end{aligned}$$

So

$$\vec{B} = \frac{M_0 \mu r}{a} \hat{a}_\phi$$

Outside the cylinder, we do this similarly, noting the "extra" current on the surface.

$$\begin{aligned}
 \int \vec{B} \cdot d\vec{l} &= \mu I \\
 2\pi r B &= \mu \times \left(\frac{2M_0}{a} \times \pi a^2 - M_0 \times 2\pi a \right) \\
 &= 0 \\
 B &= 0
 \end{aligned}$$

17. Assume that the plane $z = 0$ separates medium 1 ($z > 0$) and medium 2 ($z < 0$), with relative permeabilities $\mu_{r1} = 600$ and $\mu_{r2} = 250$, respectively. The magnetic field intensity vector in medium 1 near the boundary (for $z = 0^+$) is $\vec{H}_1 = (5\hat{a}_x - 3\hat{a}_y + 2\hat{a}_z)$ A/m. Find the magnetic field intensity in medium 2 near the boundary (for $z = 0^-$) if the surface current is either $\vec{J}_s = 0$ or $\vec{J}_s = 3\hat{a}_y$.

Solution: In both cases, the normal components of \vec{B} are constant, so the normal components of \vec{H} scale by μ , i.e. $H_{2,z} = 4.8$. The tangential components depend on \vec{J}_s . In the first case, it vanishes, so the tangential components of \vec{H} are constant. Then

$$\vec{H}_2 = 5\hat{a}_x - 3\hat{a}_y + 4.8\hat{a}_z$$

In the second case,

$$-\hat{a}_z \times (H_2 - H_1) = \vec{J}_s = 3\hat{a}_y$$

Then H_2 and H_1 does not have an \hat{a}_y component. By comparing terms, we get

$$\vec{H}_2 = 2\hat{a}_x - 3\hat{a}_y + 4.8\hat{a}_z$$

18. Consider the magnetic circuit shown in the figure below. A current of 3 A flows through 200 turns of wire on the center leg. Assuming the core has constant cross-sectional area of 10^{-3}m^2 and a relative permeability of 5000, compute the magnetic flux density vector and the magnetic field intensity vector in each leg of the core and in the air gap.

Solution: Reluctance is given by $\frac{l}{\mu S}$. The reluctances of the segments are

$$R_A = R_B = \frac{0.128}{\mu_0}, R_C = R_D = \frac{0.0238}{\mu_0}, R_E = \frac{2}{\mu_0}$$

Then $V = NI = 600$. Total resistance is

$$\frac{1}{2} \times \frac{0.128}{\mu_0} + 2 \times \frac{0.0238}{\mu_0} + \frac{2}{\mu_0} = \frac{2.1116}{\mu_0}$$

Flux through the air gap is just the current, or

$$\Phi = \frac{V}{R} = \frac{600\mu_0}{2.1116} = 0.357\text{mWb}$$

Flux through each leg is half of that, or 0.179mWb.

Magnetic flux density is flux per surface area, or 0.357T through the air gap and 0.179T through the legs.

Magnetic field intensity is scaled by μ , which is 28.4A m^{-1} for the legs and $2.84 \times 10^5\text{A m}^{-1}$ for the air gap.

19. A very large slab of material of thickness d lies perpendicularly to a uniform magnetic field of intensity $\vec{H}_0 = \hat{a}_z H_0$. Find the magnetic field intensity inside the dielectric slab if it has a permeability μ or if it is a permanent magnet having a magnetization vector $M_i = \hat{a}_z M_i$.

Solution: If the slab has permeability μ , then the normal component of \vec{B} is conserved. The tangential component of \vec{H} , which is 0, is also conserved. Then

$$\vec{H} = \frac{\mu_0 H_0}{\mu} \hat{a}_z$$

If it is a permanent magnet,

$$\begin{aligned}\vec{J}_m &= \vec{\nabla} \times \vec{M} \\ &= 0\end{aligned}$$

We find the normal component of \vec{B} .

$$\begin{aligned}\mu_0 H_0 &= \mu_0 (H + M_i) \\ H &= H_0 - M_i\end{aligned}$$

Then

$$\vec{H} = (H_0 - M_i)\hat{a}_z$$

20. A toroidal iron core of relative permeability 3000 has a mean radius $R = 80$ (mm) and a circular cross section with radius $b = 25$ (mm). An air gap $l_g = 3$ (mm) exists, and a current I flows in a 500-turn winding to produce a magnetic flux of 10^{-5} Wb. The geometry of the toroidal core is shown below. Neglecting flux leakage and using mean path length, find the reluctance, magnetic field intensity and magnetic flux density for both the air gap and the iron core, and the required current I .

Solution: Reluctance is given by $\frac{l}{\mu S}$. The reluctances of the air gap and the iron core are 1.22×10^6 and 6.75×10^4 respectively. Then the mmf is found by the electric circuit analogy, where

$$V = \Phi R = 12.8$$

Applying the formula for the mmf, we can find the current $I = \frac{V}{N} = 0.0257\text{A} = 25.7\text{mA}$. The magnetic flux intensity for both the air gap and the iron core is just flux per area, which is 5.09mT. Magnetic field intensity scales by μ , giving 4050A m^{-1} and 1.35A m^{-1} for the air gap and the iron core respectively.