Homework 6

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1. Let V be a complex inner product space, $\dim V < \infty$, and $T \in \mathcal{L}(V)$.
   (a) Prove that T may be uniquely written as

$$T = T_1 + iT_2$$

where T_1, T_2 are Hermitian.

Solution: First we prove existence. Note that

$$T = \frac{T + T^*}{2} + i\frac{T - T^*}{2i}$$

Then letting $T_1 = \frac{T+T^*}{2}$ and $T_2 = \frac{T-T^*}{2i}$, note that

$$\langle T_1x, y\rangle = \left\langle \frac{T + T^*}{2}x, y\right\rangle = \frac{1}{2}\langle Tx, y\rangle + \frac{1}{2}\langle T^*x, y\rangle = \frac{1}{2}\langle x, T^*y\rangle + \frac{1}{2}\langle x, Ty\rangle = \left\langle x, \frac{T + T^*}{2}y\right\rangle$$

Thus T_1 is Hermitian/self-adjoint. Similarly,

$$\langle T_2x, y\rangle = \langle\frac{T - T^*}{2i}x, y\rangle = \frac{1}{2i}\langle Tx, y\rangle - \frac{1}{2i}\langle T^*x, y\rangle = \frac{1}{2i}\langle x, T^*y\rangle - \frac{1}{2i}\langle x, Ty\rangle = \left\langle x, \frac{T - T^*}{2i}y\right\rangle$$

Hence T_2 is also Hermitian.
Then we prove uniqueness. Assume $T = T_1 + iT_2$. Then

$$\langle Tx, y\rangle = \langle T_1x, y\rangle + \langle iT_2x, y\rangle = \langle x, T_1y\rangle + \langle x, -iT_2y\rangle = \langle x, (T_1 - iT_2)y\rangle = \langle x, T^*y\rangle$$

Then considering the last inequality, we have

$$\langle x, (T_1 - iT_2 - T^*)y\rangle = 0$$

Which implies $T_1 - iT_2 - T^* = 0$, or else letting

$$x = (T_1 - iT_2 - T^*)y$$

for some y that gives a nonzero x would yield a contradiction. Then

$$T^* = T_1 - iT_2$$

implying

$$T - T^* = 2iT_2$$

which uniquely determines T_2. Similarly,

$$T + T^* = 2T_1$$

uniquely determinds T_1. Hence the representation is unique.

(b) Prove that T is normal if and only if T_1, T_2 commute.

Solution:

$$\begin{aligned} T_1T_2 - T_2T_1 &= \left(\frac{T+T^*}{2}\right)\left(\frac{T-T^*}{2i}\right) - \left(\frac{T-T^*}{2i}\right)\left(\frac{T+T^*}{2}\right) \\ &= \frac{(T+T^*)(T-T^*)}{4i} - \frac{(T-T^*)(T+T^*)}{4i} \\ &= \frac{T^2 + T^*T - TT^* - (T^*)^2 - T^2 + T^*T - TT^* + (T^*)^2}{4i} \\ &= \frac{T^*T - TT^*}{2i} \end{aligned}$$

Then if T is normal, $T_1T_2 - T_2T_1 = 0$, hence T_1, T_2 communtes. If T_1, T_2 commutes, this implies $T^*T - TT^* = 0$, hence T is normal.

(c) Prove that T is unitary if and only if, furthermore, $T_1^2 + T_2^2 = I$.

Solution: We assume T is normal, i.e. T_1, T_2 commute. Then we have shown that

$$\langle Tx, y\rangle = \langle x, (T_1 - iT_2)y\rangle$$

Letting $y = Tz$ for arbitrary $z \in V$,

$$\langle Tx, Tz\rangle = \langle x, (T_1 - iT_2)(T_1 + iT_2)z\rangle = \langle x, (T_1^2 + T_2^2)z\rangle$$

If $T_1^2 + T_2^2 = I$, then

$$\langle Tx, Tz\rangle = \langle x, z\rangle$$

so T is unitary. If T is unitary, then

$$\langle x, (T_1^2 + T_2^2 - I)z\rangle = 0$$

in general, meaning

$$T_1^2 + T_2^2 - I = 0 \Rightarrow T_1^2 + T_2^2 - I$$

2. Let V be a complex inner product space, $\dim V < \infty$, and $T \in \mathcal{L}(V)$ an involution: That is, $T^2 = I$. Prove that the following are equivalent:
   (a) T is self-adjoint
   (b) T is unitary
   (c) T is normal

Solution: First we assume T is self-adjoint. Then

$$\langle Tx, Ty\rangle = \langle x, T^2y\rangle = \langle x, y\rangle$$

For a finite dimensional V, this is sufficient to show that T is unitary.
Then assuming T is unitary,

$$\langle x, y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle$$

Hence

$$\langle x, (T^*T - I)y\rangle = 0 \forall x, y \in V \Rightarrow T^*T = I$$

Then T and T^* are inverses of each other, so $T^*T = TT^*$, and T is normal.
If T is normal, then V has an orthonormal basis consisting of eigenvectors of T. Since T is an involution, any eigenvalue of T must satisfy

$$v = Iv = T^2v = \lambda^2 v$$

so $\lambda = \pm 1$. We choose the orthonormal basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ where the eigenvalues for $v_1, \ldots, v_k$ are 1 and the eigenvalues for $v_{k+1}, \ldots, v_n$ are -1. Then letting $x = \sum_i a_i v_i$ and $y = \sum_i b_i v_i$, we have

$$\begin{aligned}\langle Tx, y\rangle &= \langle \sum_{i\leq k} a_i v_i - \sum_{i>k} a_i v_i, \sum_i b_i v_i\rangle \\ &= \sum_{i\leq k} a_i b_i - \sum_{i>k} a_i b_i\end{aligned}$$

And

$$\begin{aligned}\langle x, Ty\rangle &= \langle \sum_{i} a_i v_i, \sum_{i\leq k} b_i v_i - \sum_{i>k} b_i v_i\rangle \\ &= \sum_{i\leq k} a_i b_i - \sum_{i>k} a_i b_i \\ &= \langle Tx, y\rangle\end{aligned}$$

Since this holds for arbitrary $x, y \in V$, T is self-adjoint.

3. Let V be a complex inner product space, $\dim V < \infty$, and $T \in \mathcal{L}(V)$ is a normal operator.
   (a) Show that $V = \text{null}T \oplus \text{ran}T$
   (b) Show that for any $S \in \mathcal{L}(V)$ (not necessarily normal), if $ST = TS$ then $ST^* = T^*S$.

Solution: Let $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be an orthonormal eigenbasis of V, where $v_1, \ldots, v_k$ are the only basis vectors with eigenvalue 0. Then $\{v_1, \ldots, v_k\}$ span null(T) and $\{v_{k+1}, \ldots, v_n\}$ span ran(T). Then any vector $v \in V$ can be written as

$$v = \sum_{i=1}^n a_i v_i = \left(\sum_{i=1}^k a_i v_i\right) + \left(\sum_{i=k+1}^n a_i v_i\right)$$

where the first term is an element in null(T) and the second term is an element in ran(T), so the union of both sets is V. Since both sets are spanned by distinct linearly independent basis vectors, their intersection is $\{0\}$. This justifies the use of $\oplus$.
Then if $ST = TS$, let v be an eigenvector of T with eigenvalue λ,

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda Sv$$

Then Sv is also an eigenvector with eigenvalue λ.
In lecture, we proved that if v is an eigenvector with eigenvalue λ in T, then it is also an eigenvector with eigenvalue $\overline{\lambda}$ in T^*. For an arbitrary eigenvector v_i with eigenvalue λ_i, we know Sv_i is also an eigenvector with eigenvalue λ_i, so

$$ST^*v_i = S(\overline{\lambda_i} v_i) = \overline{\lambda_i} Sv_i$$

and

$$T^* S v_i = \overline{\lambda_i} S v_i = S T^* v_i$$

This holds for all eigenvectors of T, hence this holds for all basis vectors for V, thus

$$ST^* = T^* S$$

4. Let V be a complex vector space, $\dim V < \infty$. For any $T \in \mathcal{L}(V)$ such that -1 is not an eigenvalue of T, one defines the Cayley transform

$$C(T) = (I + T)^{-1}(I - T)$$

   (a) Show that if T does not have -1 as an eigenvalue, then $C(T)$ does not have -1 as an eigenvalue, and

$$C(C(T)) = T$$

Solution: Proof by contradiction. Let -1 be an eigenvalue of $C(T)$, so $\exists v \neq 0 \in V$ such that

$$\begin{aligned} C(T)v &= -v \\ (I+T)^{-1}(I-T)v &= -v \\ (I+T)^{-1}(v - Tv) &= -v \\ v - Tv &= (I+T) - v \\ v - Tv &= -v - Tv \\ v &= 0 \end{aligned}$$

Which contradicts $v \neq 0$. Hence -1 is not an eigenvalue of $C(T)$.
Now let $v \in V$, and define $w = C(C(T))v$. Then

$$\begin{aligned} C(C(T))v &= w \\ \left(I + (I+T)^{-1}(I-T)\right)^{-1}\left(I - (I+T)^{-1}(I-T)\right)v &= w \\ \left(I - (I+T)^{-1}(I-T)\right)v &= \left(I + (I+T)^{-1}(I-T)\right)w \\ v - (I+T)^{-1}(v - Tv) &= w + (I+T)^{-1}(w - Tw) \\ v + Tv - v + Tv &= w + Tw + w - Tw \\ 2Tv &= 2w \\ Tv &= w \end{aligned}$$

This implies

$$(C(C(T)) - T)v = w - w = 0$$

Since this holds for arbitrary v, we know

$$C(C(T)) - T = 0 \Rightarrow C(C(T)) = T$$

   (b) Suppose V has an inner product, so that adjoints are defined. Show that if T is skew-adjoint, then $C(T)$ is unitary, and if T is unitary, then $C(T)$ is skew-adjoint.

Solution: In this proof, we make use of the fact that

$$(AB)^* = B^*A^*$$

because

$$\langle ABx, y\rangle = \langle Bx, A^*y\rangle = \langle x, B^*A^*y\rangle$$

If T is skew-adjoint, then

$$\langle (I \pm T)x, y\rangle = \langle x, y\rangle \pm \langle Tx, y\rangle = \langle x, y\rangle \pm \langle x, -Ty\rangle = \langle x, (I \mp T)y\rangle$$

Therefore $(I+T)^* = I - T, (I-T)^* = I + T$. Then

$$\begin{aligned}
(C(T))^*C(T) &= [(I+T)^{-1}(I-T)]^*(I+T)^{-1}(I-T) \\
&= (I-T)^*\left((I+T)^{-1}\right)^*(I+T)^{-1}(I-T) \\
&= (I+T)\left((I+T)^*\right)^{-1}(I+T)^{-1}(I-T) \\
&= (I+T)(I-T)^{-1}(I+T)^{-1}(I-T) \\
&= (I+T)\left((I+T)(I-T)\right)^{-1}(I-T) \\
&= (I+T)\left((I-T)(I+T)\right)^{-1}(I-T) \\
&= (I+T)(I+T)^{-1}(I-T)^{-1}(I-T) \\
&= I
\end{aligned}$$

So

$$\langle C(T)x, C(T)y\rangle = \langle x, (C(T))^*C(T)y\rangle = \langle x, y\rangle$$

hence $C(T)$ is unitary.
If T is unitary, then

$$\langle x, y\rangle = \langle Tx, Ty\rangle = \langle x, T^*Ty\rangle$$

So

$$\langle x, (T^*T - I)y\rangle = 0 \forall x, y \in V \Rightarrow T^*T = I$$

Hence T^* and T are inverses. In addition,

$$\begin{aligned}
\langle (I \pm T)x, y\rangle &= \langle (T \pm T^2)x, Ty\rangle \\
&= \langle Tx, Ty\rangle \pm \langle T^2x, Ty\rangle \\
&= \langle x, y\rangle \pm \langle Tx, y\rangle \\
&= \langle x, y\rangle \pm \langle x, T^*y\rangle \\
&= \langle x, (I \pm T^*)y\rangle
\end{aligned}$$

So $(I+T)^* = I + T^*$ and $(I-T)^* = I-$. Then

$$\begin{aligned}
(C(T))^* + C(T) &= \left((I+T)^{-1}(I-T)\right)^* + (I+T)^{-1}(I-T) \\
&= (I-T)^*\left((I+T)^{-1}\right)^* + (I+T)^{-1} - (I+T)^{-1}T \\
&= (I-T^*)\left((I+T)^*\right)^{-1} + (I+T)^{-1} - (I+T)^{-1}(T^*)^{-1} \\
&= (I+T^*)^{-1} - T^{-1}(I+T^*)^{-1} + (I+T)^{-1} - (T^*+I)^{-1} \\
&= -(T+I)^{-1} + (I+T)^{-1} \\
&= 0
\end{aligned}$$

So $(C(T))^* = -C(T)$, and $C(T)$ is skew-adjoint.

5. Let V be a complex inner product space, $\dim V < \infty$. Let T be a normal operator. Show that the set of numerical values

$$\{\langle Tv, v\rangle | v \in V, ||v|| = 1\}$$

is the convex hull of the spectrum

$$\text{Spec}(T) = \{\lambda \in \mathbb{C} | \lambda \text{ is an eigenvalue of } T\}$$

Solution: Since T is normal, it admits an orthogonal basis of eigenvectors $\{v_1, \ldots, v_n\}$. Let the corresponding eigenvalue for eigenvector v_i be λ_i. Then letting $v = \sum_{i=1}^n a_i v_i$,

$$\langle Tv, v\rangle = \langle \sum_{i=1}^n a_i \lambda_i v_i, \sum_{i=1}^n a_i v_i\rangle = \sum_{i=1}^n a_i^2 \lambda_i$$

Let A be the set defined in the question, and B be the convex hull. Then $\forall a \in A$, using the definitions as above,

$$||v|| = 1 \Rightarrow \sqrt{\sum_{i=1}^n a_i^2} = 1 \Rightarrow \sum_{i=1}^n a_i^2 = 1$$

Letting $t_i = a_i^2$,

$$a = \sum_{i=1}^n a_i^2 \lambda_i = \sum_{i=1}^n t_i \lambda_i \in B$$

Therefore $A \subseteq B$. Conversely, $\forall b \in B$,

$$b = \sum_{i=1}^n t_i \lambda_i = \sum_{i=1}^n a_i^2 \lambda_i = \langle Tv, v\rangle \in A$$

where we define a_i to be a root of $x^2 = t_i$, and $v_i = \sum_{i=1}^n a_i v_i$. Note that this implies $b \in A$ as

$$||v|| = \sqrt{\sum_{i=1}^n a_i^2} = \sqrt{\sum_{i=1}^n t_i} = \sqrt{1} = 1$$

Then $B \subseteq A$. Combining both, we see

$$A = B$$

Or that the set $\{\langle Tv, v\rangle | v \in V, ||v|| = 1\}$ is the convex hull.