Problem Set 11

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1. Consider a circular copper loop carrying a steady current, in air, and the following changes to the loop, one at the time: (a) change the wire material from copper to aluminum, (b) double the loop radius, (c) extend the loop so it becomes an ellipse with the major to minor axis ratio of four but the same circumference, (d) add a ferromagnetic core so that the loop encircles it, (e) double the current of the loop, and (f) reverse the direction of the loop current. Which of these changes would result in a change of the loop inductance?

Solution: C, changes (b)-(d) only. We know

$$L = \frac{N\Phi}{I}$$

This is independent of material, so (a) is false. $\vec{B}$ and hence Φ changes with (b) or (c), with all else constant, so both are true. For (c), μ changes, so $\vec{B}$ and hence Φ changes, so it is true. (e) would double Φ and I, where the changes cancel out. Finally, (f) does not change inductance, which is always positive.

2. If a piece of a ferromagnetic material of relative permeability μ_r is placed as a core of a wire loop, as indicated in Fig. Q7.3, the inductance of the loop, L, is related that, L_0, of the same loop with no core as follows

Solution: B, $L_0 < L < \mu_r L_0$. Note that L is proportional to Φ, which is proportional to $\vec{B}$, which is proportional to μ. Depending on the region, i.e. air or the core, $\mu = \mu_0$ or $\mu = \mu_r\mu_0$. We can then take a weighted average and apply it to all $\vec{B}$ to get the same mathematical result for L. Since $\mu = \mu_0$ for L_0, the constant by which it is scaled to get L is between 1 and μ_r.

3. A coil with N turns of wire is wound uniformly and densely about a thin toroidal core made from a linear ferromagnetic material of relative permeability μ_r. Consider the magnetic flux density, $\vec{B}$, inside the core and inductance, L, of the coil. If the diameter of the wire in the coil is halved and N is doubled, while the current I in the coil is kept the same, we have that

Solution: D, $\vec{B}$ doubles and L quadruples. The diameter has no effect, but doubling N would double $\vec{B}$, hence Φ. Then the numerator of L quadruples, with the denominator kept constant.

4. Two conducting wire contours, C_1 and C_2, in air carry slowly time-varying currents of intensities $i_1(t)$ and $i_2(t)$, respectively, as shown in Fig. Q7.4. The mutual inductance L_{21} between the contours will not change if

Solution: D, both contour remain the same. This is because

$$L_{12} = \frac{N_2}{I_1} \int \vec{B}_1 \cdot d\vec{S}_2$$

The fraction is unchanged. If only C_2 changes, the domain of the integral changes, so L_{12} need not stay constant. The same happens if only C_1 changes. Although it is possible, there is no guarantee mutual inductance is kept constant when both change.

5. The mutual inductance L_{12} of the two contours in Fig. Q7.4 will change if

Solution: E, none of the above cases. If I is scaled by a nonzero constant, this is reflected only in $\vec{B}_1$ and I_1 in the equation, where the constant is cancelled out. Hence no scaling of current would change mutual inductance.

6. If in Fig. Q.7.4 the orientation of the contour C_1 is reversed and that of C_2 remains the same, which of the mutual inductances L_{12} and L_{21} of the contours will change?

Solution: C, both inductances. Consider L_{21}, where the direction of $d\vec{S}_1$ is reversed. This causes L_{21} to become its negative. Noting that $L_{12} = L_{21}$, we know both inductances change. Alternatively, flipping the contour would flip the direction of current, flipping the sign of $\vec{B}_1$ and hence L_{12}.

7. Out of the four mutual positions of two circular wire loops shown in Fig. Q7.5, the magnitude of the mutual inductance between the loops is largest in

Solution: B. N and I are constant, so only Φ matters. Note the direction of $\vec{B}$. Case (c) is out as it is perpendicular to the normal of the surface. Case (b) is out, as its $\vec{B}$ is not parallel to the normal, and is lesser in magnitude than in case (a), all else being equal. Finally, case (d) is also out, as it has a smaller $\vec{B}$. (It is inversely proportional to radius squared, which is minimised in case (a).)

8. Fig. Q7.6 shows two coils wound on a cardboard one. The mutual inductance L_{12} of the coil is

Solution: A, positive. Without loss of generality, assume current goes clockwise, i.e. the positive voltage is on the right. Then $\vec{B}$ goes counterclockwise on the cardboard coil, cause by both coils. In both cases, $\vec{B}$ goes in the same direction as the normal of the surface, according to the right hand rule. Hence mutual inductance is positive.

9. Repeat the previous question but for two coils shown in Fig. Q7.7.

Solution: B, negative. Without loss of generality, we let current flow downwards. Then in this case, $\vec{B}$ points upwards, yet the normal of the surface points downwards for both coils. Hence mutual inductance is negative.

10. Two linear inductors of inductances L and $2L$, respectively, have the same magnetic flux, Φ. The magnetic energy stored in the inductor with twice as large inductance is

> **Solution:** C, half of that stored in the other inductor. For convenience, since this holds for all cases, this holds for $N = 1$. Then current in the inductor with the smaller inductance is twice as large. Now consider
>
> $$L = \frac{2W_m}{I^2}$$
>
> Since current is squared, W_m must be doubled for the inductor with the smaller inductance to maintain the ratio of inductance.

11. Two linear inductors contain the same amounts of magnetic energy. If the magnetic field intensity ($\vec{H}$) at every point in the first inductor becomes twice larger, while the magnetic flux density ($\vec{B}$) at every point in the second inductor is halved, the energy stored in the first inductor in the new steady state is

> **Solution:** D, 16 times. Note that W_m is the integral of the dot product of both, and that there is a linear relationship between $\vec{B}$ and $\vec{H}$. Then the energy in the first inductor quadruples, as both are doubled. Similarly, the energy in the second inductor becomes 4 times smaller.

12. In two equally sized pieces of different ferromagnetic materials, a uniform magnetic field is first established, at the same intensity (H_m), and then reduced to zero ($H = 0$), during which process the operating point describes the respective paths shown in Fig.Q7.10. The net magnetic energy spent in the magnetization-demagnetization of the piece in case(a) is

> **Solution:** C, a half. W_m is proportional to BH, and B is half as great at the end in case (a), so net energy used is also half.

13. Refer to Example 6-16. Determine the inductance per unit length of the air coaxial tranmission line assuming that its outer conductor is not very thin but is of a thickness d.

> **Solution:** In the outer conductor,
>
> $$I_{\text{enc}} = I \times \frac{\pi(b+d)^2 - \pi r^2}{\pi(b+d)^2 - \pi b^2} = I \times \frac{(b+d)^2 - r^2}{(b+d)^2 - b^2}$$
>
> Hence
>
> $$\vec{B}_3 = \frac{\mu_0 I}{2\pi r} \times \frac{(b+d)^2 - r^2}{(b+d)^2 - b^2}\hat{a}_\phi$$
>
> Then the magnetic energy per length stored in the outer conductor is
>
> $$W_m = \frac{1}{2}\int_0^{2\pi}\int_b^{b+d} \frac{|\vec{B}_3|^2 r dr d\phi dz}{\mu_0} = \frac{\mu_0 I^2}{4\pi}\left(\frac{(b+d)^4}{((b+d)^2 - b^2)^2}\ln\left(1 + \frac{d}{b}\right) + \frac{b^2 - 3(b+d)^2}{4((b+d)^2 - b^2)}\right)$$
>
> Adding this to inductance found in the example, we have
>
> $$L = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a} + \frac{(b+d)^4}{((b+d)^2 - b^2)^2}\ln\left(1 + \frac{d}{b}\right) + \frac{b^2 - 3(b+d)^2}{4((b+d)^2 - b^2)}\right)$$

14. calculate the mutual inductance per unit length between two parallel two-wire transmission lines $A-A'$ and $B-B'$ separated by a distance D. Assume the wire radius to be much smaller than D and the wire spacing d.

Solution: Note: this is a weird integral I don't quite get even after literally copying the answer key. To find inductance, we use

$$L = \frac{N\Phi}{I}$$

Where

$$\Phi = \int \vec{B} \cdot d\vec{S}$$

The surface we pick is the surface in the $\hat{a}_\phi$ direction, from the circle centred at A touching B to another concentric circle touching B', i.e. from $r = D$ to $r = \sqrt{D^2 + d^2}$. This is so that the surface "covers" the entirety of B and B' in the point of view of A. The same process is also done for A', but the result is the same by symmetry. Moving on to the maths, note that for a long wire,

$$\vec{B} = \frac{\mu I}{2\pi r}\hat{a}_\phi$$

The surface as defined has area

$$dS = Ldr = dr$$

where $L = 1$ is taken to be the unit length (in/out of page). Now

$$\Phi = \int_D^{\sqrt{D^2+d^2}} \frac{\mu I}{2\pi r}\hat{a}_\phi \cdot dr\hat{a}_\phi = \frac{\mu I}{2\pi} \ln \frac{\sqrt{D^2+d^2}}{D}$$

Then inductance becomes

$$L = \frac{\mu}{\pi} \ln \frac{\sqrt{D^2+d^2}}{D}$$

15. Determine the mutual inductance between a very long, straight wire and a conducting circular loop.

Solution: Consider the flux of the long straight wire,

$$B = \frac{\mu_0 I}{2\pi r}$$

Then the flux is given by

$$\begin{aligned}\Phi &= \int B dS \\ &= \int_{-b}^{b} \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{\mu I}{2\pi(x+d)} dy dx \\ &= \int_{-b}^{b} \frac{\mu I \sqrt{b^2-x^2}}{\pi(x+d)} dx\end{aligned}$$

The integral is analytical, but quite involved, according to WolframAlpha. According to the solution manual, which gives an integral that is most definitely wrong, we get

$$\Phi = \mu I(d - \sqrt{d^2 - b^2})$$

Then inductance is

$$L = \mu(d - \sqrt{d^2 - b^2})$$

16. Find the mutual inductance between two coplanar rectangular loops with parallel sides. Assume that $h_1 >> h_2 (h_2 > w > d)$.

Solution: We ignore the shorter sides because they are shorter. Since $h_1 >> h_2$, we can ignore fringing effects, and assume

$$B = \frac{\mu_0 I}{2\pi r}$$

Then the flux according to the contributions of both long wires to the smaller rectangular loop is

$$\begin{aligned}
\Phi &= \int \vec{B} \cdot d\vec{S} \\
&= \int_0^{w_2} \frac{\mu_0 I}{2\pi} \left(\frac{1}{d+x} - \frac{1}{w_1 + d + x} \right) h_2 dx \\
&= \frac{\mu_0 I h_2}{2\pi} \left(\ln \frac{d + w_2}{d} - \ln \frac{w_1 + d + w_2}{w_1 + d} \right)
\end{aligned}$$

And

$$L = \frac{\mu_0 h_2}{2\pi} \left(\ln \frac{d + w_2}{d} - \ln \frac{w_1 + d + w_2}{w_1 + d} \right)$$

17. Calculate the force per unit length on each of three equidistant, infinitely long parallel wires 0.15m apart, each carrying a current of 25A in the same direction. Specify the direction of the force.

Solution: Between each pair of wires,

$$B = \frac{\mu I}{2\pi r}$$

The contributions to any wire by the other two wires have an angle of $\frac{\pi}{3}$, so net $\vec{B}$ bisects both contributions, forming an angle of $\frac{\pi}{6}$ with each. Then total contribution becomes

$$B = 2\cos\frac{\pi}{6}\frac{\mu I}{2\pi r} = \frac{\sqrt{3}\mu I}{2\pi r}$$

Since

$$F = I\vec{l} \times \vec{B}$$

it points the wires towards the centre of the triangle. Its magnitude is

$$F = IB = \frac{\sqrt{3}\mu I^2}{2\pi r} = 1.44 \times 10^{-3}$$

per length.

18. Determine the force per unit length between two parallel, long, thin conducting strips of equal width w. The strips are at a distance d apart and carry currents I_1 and I_2 in opposite directions.

Solution: From a previous problem, the magnetic flux density at height y is given by

$$B = -\frac{\mu_0 I_1}{2\pi w}\left[\arctan\frac{y}{d} + \arctan\frac{w-y}{d}\right]$$

Then the force can be integrated as

$$\begin{aligned}
F &= \int_0^w B \times \frac{I_2}{w} dy \\
&= \frac{\mu_0 I_1 I_2}{2\pi w^2}\int_0^w \left[\arctan\frac{y}{d} + \arctan\frac{w-y}{d}\right] dy \\
&= \frac{\mu_0 I_1 I_2}{2\pi w^2}\left[2w\arctan\frac{w}{d} - d\ln\left(1+\frac{w^2}{d^2}\right)\right]
\end{aligned}$$

Where $\vec{F}$ points in the $\hat{a}_x$ direction. Note that the integral uses the identity

$$\int \arctan x = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

which can be derived using integration by parts.

19. Find the force on the circular loop that is exerted by the magnetic field due to an upward current I_1 in the long straight wire. The circular loop carries a current I_2 in the counterclockwise direction.

Solution:

$$\begin{aligned}
\vec{F} &= I\vec{l} \times \vec{B} \\
&= \int_0^{2\pi} I_2 d\phi \hat{a}_\phi \times -\frac{\mu I_1}{2\pi(d + b\cos\theta)}\hat{a}_z \\
&= -\frac{\mu I_1 I_2}{2\pi}\int_0^{2\pi} \frac{d\phi}{d + b\cos\theta}\hat{a}_r \\
&= -\frac{\mu I_1 I_2}{2\pi}\int_0^{2\pi} \frac{(\cos\theta\hat{a}_x + \sin\theta\hat{a}_y)d\phi}{d + b\cos\theta}
\end{aligned}$$

By symmetry, $\vec{F}$ has no net y component, so we can ignore that. The integral becomes

$$\begin{aligned}
\vec{F} &= -\frac{\mu I_1 I_2}{\pi}\int_0^{\pi} \frac{\cos\theta d\theta}{d + b\cos\theta}\hat{a}_x \\
&= \mu I_1 I_2\left(\frac{1}{\sqrt{1-\frac{b^2}{d^2}}} - 1\right)\hat{a}_x
\end{aligned}$$

where the solution to the definite integral comes from the same place Praxis TAs get their marks from.

20. For the question above, assuming that the circular loop is rotated about its horizontal axis by an angle α, find the torque exerted on the circular loop.

Solution:

$$\begin{aligned}
\vec{T} &= \int d\vec{T} \\
&= \int d\vec{M} \times \vec{B} \\
&= I_2 \int d\vec{S} \times \vec{B} \\
&= -I_2 \sin\alpha \int B ds \hat{a}_x \\
&= -\mu I_1 I_2 \sin\alpha (d - \sqrt{d^2 - b^2}) \hat{a}_x
\end{aligned}$$

Where the integral is solved the same as before.

21. Determine the mutual inductance between a very long, straight wire and a conducting equilateral triangle loop, as shown in the figure below

Solution: Mutual inductance is defined as

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$$

The B-field generated by the line current is

$$\vec{B} = -\frac{\mu I}{2\pi x} \hat{a}_z$$

Flux through the triangle is

$$\begin{aligned}
\Phi &= \int \vec{B} \cdot d\vec{S} \\
&= \int_0^{\frac{\sqrt{3}}{2}b} \int_{-\frac{x}{\sqrt{3}}}^{\frac{x}{\sqrt{3}}} \frac{\mu I}{2\pi(x+d)} dy dx \\
&= \frac{\mu I}{\sqrt{3}\pi} \int_0^{\frac{\sqrt{3}}{2}b} \frac{x}{x+d} dx \\
&= \frac{\mu I}{\sqrt{3}\pi} x - d \ln(x+d) \Big|_0^{\frac{\sqrt{3}}{2}b} \\
&= \frac{\mu I}{\sqrt{3}\pi} \left(\frac{\sqrt{3}}{2} b - d \ln\left(1 + \frac{\sqrt{3}b}{2d}\right)\right)
\end{aligned}$$

Inductance is then

$$\frac{\mu}{\sqrt{3}\pi} \left(\frac{\sqrt{3}}{2} b - d \ln\left(1 + \frac{\sqrt{3}b}{2d}\right)\right)$$

22. The cross section of a long thin metal plate and a parallel wire is shown in the figure below. Equal and opposite current I flow in the conductors. Find the force per unit length acting on both conductors.

Solution: Magnetic flux at an arbitrary point on the metal plate is

$$\begin{aligned}
\vec{B} &= -\frac{\mu I}{2\pi r}\hat{a}_\phi \\
&= \frac{\mu I}{2\pi\sqrt{D^2+y^2}}\left(\frac{y\hat{a}_x}{\sqrt{D^2+y^2}} + \frac{D\hat{a}_y}{\sqrt{D^2+y^2}}\right)
\end{aligned}$$

Now force per unit length is

$$\begin{aligned}
\vec{F} &= I\vec{l} \times \vec{B} \\
&= \int_{-\frac{W}{2}}^{\frac{W}{2}} \frac{I}{W} dy\hat{a}_z \times \frac{\mu I}{2\pi\sqrt{D^2+y^2}}\left(\frac{y\hat{a}_x}{\sqrt{D^2+y^2}} + \frac{D\hat{a}_y}{\sqrt{D^2+y^2}}\right) \\
&= \frac{\mu I^2}{2W\pi}\int_{-\frac{W}{2}}^{\frac{W}{2}} \frac{ydy\hat{a}_y}{D^2+y^2} - \frac{Ddy\hat{a}_x}{D^2+y^2} \\
&= \frac{\mu I^2}{2W\pi} - \arctan\frac{y}{D}\Big|_{-\frac{W}{2}}^{\frac{W}{2}}\hat{a}_x \\
&= \frac{\mu I^2}{2W\pi}(-2)\arctan\frac{W}{2D}\hat{a}_x \\
&= -\frac{\mu I^2}{W\pi}\arctan\frac{W}{2D}\hat{a}_x
\end{aligned}$$

The forces must balance, so force per unit length exerted by the metal plate on the wire is

$$\vec{F} = \frac{\mu I^2}{W\pi}\arctan\frac{W}{2D}\hat{a}_x$$

23. The bar AA' in the figure below serves as a conducting path (such as the blade of a circuit breaker) for the current I in two very long (semi-infinite) parallel lines. The lines have a radius b and are spaced at a distance d apart. Find the direction and the magnitude of the magnetic force on the bar.

Solution: Flux at the bar due to the bottom line is

$$\begin{aligned}
\vec{B} &= \frac{\mu I}{4\pi}\int_0^\infty \frac{-dx\hat{a}_x \times (y\hat{a}_y - x\hat{a}_x)}{(x^2+y^2)^{\frac{3}{2}}} \\
&= -\frac{\mu I}{4\pi}\int_0^\infty \frac{ydx\hat{a}_z}{(x^2+y^2)^{\frac{3}{2}}} \\
&= -\frac{\mu I}{4\pi}\frac{1}{y}\hat{a}_z \\
&= -\frac{\mu I}{4\pi y}\hat{a}_z
\end{aligned}$$

By symmetry, flux due to the top line is

$$\vec{B} = -\frac{\mu I}{4\pi(d-y)}\hat{a}_z$$

Then force acting on the bar is

$$
\begin{aligned}
\vec{F} &= I\vec{l} \times \vec{B} \\
&= \int_0^d I dy \hat{a}_y \times \left(-\frac{\mu I}{4\pi}\right)\left(\frac{1}{y} + \frac{1}{d-y}\right)\hat{a}_z \\
&= -\frac{\mu I^2}{4\pi}\int_b^(d-b)\left(\frac{1}{y} + \frac{1}{d-y}\right) dy\hat{a}_x \\
&= -\frac{\mu I^2}{4\pi}\left(\ln\frac{d-b}{b} - \ln\frac{b}{d-b}\right)\hat{a}_x \\
&= -\frac{\mu I^2}{2\pi}\ln\frac{d-b}{b}\hat{a}_x
\end{aligned}
$$

24. A d-c current $I = 10$A flows in a triangular loop in the xy-plane as in the figure below. Assuming a uniform magnetic flux density $\vec{B} = \hat{a}_y 0.5$T in the region, find the forces and torque on the loop. All dimensions are in cm.

Solution: Segment 1:

$$
\begin{aligned}
\vec{F} &= I\vec{l} \times \vec{B} \\
&= 10 \times 0.2\hat{a}_x \times 0.5\hat{a}_y \\
&= \hat{a}_z
\end{aligned}
$$

Segment 2:

$$
\begin{aligned}
\vec{F} &= I\vec{l} \times \vec{B} \\
&= \left(-\frac{10}{\sqrt{5}}\hat{a}_x + \frac{20}{\sqrt{5}}\hat{a}_y\right) \times 0.5\hat{a}_y \times \frac{\sqrt{5}}{10} \\
&= -0.5\hat{a}_z
\end{aligned}
$$

Segment 3:

$$
\begin{aligned}
\vec{F} &= I\vec{l} \times \vec{B} \\
&= \left(-\frac{10}{\sqrt{5}}\hat{a}_x - \frac{20}{\sqrt{5}}\hat{a}_y\right) \times 0.5\hat{a}_y \times \frac{\sqrt{5}}{10} \\
&= -0.5\hat{a}_z
\end{aligned}
$$

Total force on the loop is then 0. Torque is

$$
\begin{aligned}
\vec{T} &= \vec{m} \times \vec{B} \\
&= NI\vec{A} \times 0.5\hat{a}_y \\
&= 10 \times \frac{1}{2} \times 0.2 \times 0.2\hat{a}_z \times 0.5\hat{a}_y \\
&= -0.1\hat{a}_x
\end{aligned}
$$