Lecture 3

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1 Questions from last lecture

1.1 What if integral curves intersect?

If we lack uniqueness, the IVP is ill-posed, usually because the model is incorrect.

Example 1.1.

$$\frac{du}{dt} = u^{\frac{2}{3}}, u(0) = 0$$

Both $u(t) = 0$ and $u(t) = \left(\frac{t}{3}\right)^3$ are solutions.

1.2 Initial values are imposed on higher derivatives instead of the dependent variable

$$u'' = -u$$

The solution is in the form of

$$u = C_0 \sin(t) + C_1 \cos(t)$$

If initial conditions are imposed on u, there may not be a solution, e.g.

$$\begin{cases} u(0) = 0 \\ u(\pi) = 1 \end{cases}$$

2 Global and Local solutions

Example 2.1.

$$\frac{du}{dt} = 1 + u^2, u(0) = 0$$

We know that $\tan(t)$ is a solution, but it cannot be global, as it diverges at $\pm\frac{\pi}{2}$.

3 Direction (Slope) Fields

$$\frac{du}{dt} = f(t, u)$$

f is the slope of u. By plotting a $t - u$ plane, and drawing out the slopes f, we get adirection field of the ODE. In fact, if the equation is autonomous, the plot is independent of t, which simplifies it.

3.1 Slope Fields vs Integral Curves

You do not need to solve for the general solution when drawing slope fields. However, they are less "accurate", as they are only a first order Taylor approximation (cooler way of saying "tangent").

4 Equilibrium

Definition 4.1. Consider the first order ODE

$$\frac{du}{dt} = f(u)$$

Equilibrium solutions are those satisfying

$$f(u) = 0$$

Stable equilibria are when solutions "near" it tend towards them.
Unstable equilibria are when solutions "near" it tend away from them.

Example 4.1.

$$\frac{du}{dt} = \cos(u)$$

Then at $u = \frac{\pi}{2}$, solutions slightly below will increase (positive slope) and those above will decrease (negative slope), so it is a stable equilibrium. Conversely, $u = \frac{3\pi}{2}$ is an unstable equilibrium.

There are asymptomatically stable equilibrium points, unstable equilibrium points, and two types of semistable equilibrium points, where solutions tend to increase/decrease (type i and ii) when they are "near" the equilibrium.

Example 4.2. An ODE with a semistable equilibrium point (type i):

$$\frac{du}{dt} = u^2$$

An ODE with a semistable equilibrium point (type ii):

$$\frac{du}{dt} = -u^2$$

An ODE with infinitely many semistable equilibrium points (type i):

$$\frac{du}{dt} = \sin(u) + 1$$

An ODE with infinitely many semistable equilibrium points (type ii):

$$\frac{du}{dt} = \sin(u) - 1$$