Lecture 5

niceguy

September 19, 2022

1 Review

Solving a first order ODE

1. Put it into standarm form

2. Integrate $p(t)$

3. The integrating factor is the exponential of the integral of $p(t)$

4. Plug in the formula

Why does the +C term not matter?
When taking the exponential, it results in a nonzero constant coefficient, which is cancelled out on both sides of the equation.

Example 1.1.

$$y' + 2ty = t$$

This is in the right form, so the integrating factor could be immediately found as e^{t^2}.

$$\begin{aligned} ye^{t^2} &= \int te^{t^2} dt \\ ye^{t^2} &= \frac{1}{2}e^{t^2} + C \\ y &= Ce^{-t^2} + \frac{1}{2}, C \in \mathbb{R} \end{aligned}$$

2 Mathematical Modelling: Rocket Ship

Assumptions: air resistance is negligible.
We start with the equations

$$F = ma$$

and

$$w(x) = -\frac{mgR^2}{(R+x)^2}$$

Where $w(x)$ is the gravitational force at height x, and R is Earth's radius.
Equating both sides,

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}$$

There are too many variables!
Using chain rule, we can rewrite

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

So

$$mv\frac{dv}{dx} = -\frac{mgR^2}{(R+x)^2}$$

This is separable, so

$$
\begin{aligned}
v\frac{dv}{dx} &= -\frac{gR^2}{(R+x)^2} \\
\int v dv &= -gR^2 \int \frac{dx}{(R+x)^2} \\
\frac{1}{2}v^2 &= \frac{gR^2}{R+x} + C
\end{aligned}
$$

The initial condition of $v(0) = v_0$ gives us the constant $C = \frac{1}{2}v_0^2 - gR$. The solution is then by

$$
v = \pm\sqrt{\frac{2gR^2}{R+x} + v_0^2 - 2gR}
$$

2.1 Maximum Altitude

The maximum altitude is when $v = 0$, or

$$
\begin{aligned}
\frac{2gR^2}{R+x} + v_0^2 - 2gR = 0 \\
2gR^2 + (R+x)(v_0^2 - 2gR) = 0 \\
R + x &= -\frac{2gR^2}{v_0^2 - 2gR} \\
R + x &= \frac{2gR^2}{2gR - v_0^2} \\
x &= \frac{Rv_0^2}{2gR - v_0^2}
\end{aligned}
$$

2.2 Required initial speed to reach a maximal height of ξ.

$$\frac{2gR^2}{R+\xi} + v_0^2 - 2gR = 0$$

$$\begin{aligned} v_0 &= \pm\sqrt{2gR - \frac{2gR^2}{R+\xi}} \\ &= \pm\sqrt{\frac{2gR\xi}{R+\xi}} \end{aligned}$$

2.3 Exit Velocity

Say we are trying to determine the speed needed to never return to EngSci. We then want to reach a maximal height of "∞", or more formally, as $\xi \to \infty$. We get

$$v_0 = \sqrt{2gR}$$

3 Existance and Uniqueness

Theorem 3.1. *Consider the first order IVP*

$$\begin{cases} u' + p(t)u = g(t) \\ u(t_0) = u_0 \end{cases}$$

and an open interval $I = (\alpha, \beta)$. If $t_0 \in I$, $g(t)$ and $p(t)$ are both continuous on I, then this IVP has a unique solution defined on I.

Proof Sketch:
Let $F(t)$ be the antiderivative of $p(t)$. Since p is continuous, $F(t)$ is continuous and differentiable over I. Letting $\mu(t) = e^{F(t)}$, μ is also defined and continous over I, so the following equation holds $\forall t \in I$.

$$\frac{d}{dt}(\mu(t)u(t)) = \mu(t)g(t)$$

Since μ and g are continous, their product is continuous and hence integrable. So we can integrate both sides to get $\mu(t)u(t) = G(t) + C$ where $C \in \mathbb{R}$ and $G(t)$ is the antiderivative of the right hand side. Since $\mu \neq 0$ (it is an exponential, we can conclude that

$$u(t) = \frac{G(t) + C}{\mu(t)}$$

Theorem 3.2. *Consider the first-order IVP:*

$$\begin{cases} u' = f(t,u) \\ u(t_0) = u_0 \end{cases}$$

and the open rectangle $(\alpha, \beta) \times (\gamma, \delta)$. If $(t_0, u_0) \in (\alpha, \beta) \times (\gamma, \delta)$, f is continuous on the rectangle, and f_u is continuous on the rectangle, then this IVP has a unique solution defined on some subinterval $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$.