Lecture 10

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1 Line Integrals with Vector Fields

Example 1.1. Given a force field $\vec{F} = y\hat{i} - x\hat{j}$, find the amount of work done in moving a particle from $(1, 0)$ to $(0, -1)$ along the straight line segment joining these points, and $\frac{3}{4}$ of the circle of unit radius centred at the origin and traveled counterclockwise.
Along the straight line segment, we have

$$\vec{r}(t) = (1-t)\hat{i} - t\hat{j}$$

Then

$$\begin{aligned} I &= \int_0^1 \vec{F} \cdot \vec{r'}(t) dt \\ &= \int_0^1 (-t\hat{i} - (1-t)\hat{j}) \cdot (-\hat{i} - \hat{j}) dt \\ &= \int_0^1 t + 1 - t dt \\ &= 1 \end{aligned}$$

Along the circle, we have

$$\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j}$$

Then

$$\begin{aligned}
I &= \int_0^{\frac{3\pi}{2}} (\sin t\hat{i} - \cos t\hat{j}) \cdot (-\sin t\hat{i} + \cos t\hat{j})dt \\
&= \int_0^{\frac{3\pi}{2}} -\sin^2 t - \cos^2 t dt \\
&= \int_0^{\frac{3\pi}{2}} -1dt \\
&= -\frac{3\pi}{2}
\end{aligned}$$

Example 1.2. Let $\vec{F}(x,y) = y\hat{i} + x\hat{j}$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along the straight line $y = x$ and the curve $y = x^2$.
Along the straight line,

$$\vec{r}(t) = t\hat{i} + t\hat{j}$$

Then

$$\begin{aligned}
I &= \int_0^1 (t\hat{i} + t\hat{j}) \cdot (\hat{i} + \hat{j})dt \\
&= \int_0^1 2tdt \\
&= 1
\end{aligned}$$

Along the curve,

$$\vec{r}(t) = t\hat{i} + t^2\hat{j}$$

Then

$$\begin{aligned}
I &= \int_0^1 (t^2\hat{i} + t\hat{j}) \cdot (\hat{i} + 2t\hat{j})dt \\
&= \int_0^1 t^2 + 2t^2 dt \\
&= 1
\end{aligned}$$

2 Fundamental Theorem for Line Integrals

Definition 2.1. A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar f. In this situation, the scalar function is called a potential function of $\vec{F}$.

$$\vec{F} = \vec{\nabla} f$$

Suppose

$$\vec{f}(x, y, z) = \vec{\nabla} f(x, y, z)$$

Let C be a smooth curve given by

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

where

$$\int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int_C \vec{\nabla} f(x, y, z) \cdot d\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r'}(t) dt$$

Then

$$\begin{aligned}
\vec{\nabla} f(\vec{r}(t)) \cdot \vec{r'}(t) &= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\
&= \frac{df}{dt}
\end{aligned}$$

So

$$\int_a^b \vec{\nabla} f(\vec{r'}(t)) dt = \int_a^b \frac{d}{dt}(f(\vec{r'}(t))) dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

The integral is path independent!

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$