Lecture 19

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1 Schur's Theorem and other Decompositions

Theorem 1.1 (Schur's Theorem)**.** *Let V be a finite dimensional complex inner product space, $T \in \mathcal{L}(V)$. Then there exists an orthonormal basis such that the matrix of T is upper triangular.*

Proof. Pick any basis $w_1, \ldots, w_k$ of V such that the matrix of T becomes upper triangular, i.e. $Tw_i \in \text{span}\{w_1, \ldots, w_i\}$ (such as in the Jordan Normal Form). Let

$$W_i = \text{span}\{w_1, \ldots, w_i\}$$

Then W_i is T-invariant, and

$$0 = W_0 \subseteq W_1 \subseteq W_2 \cdots \subseteq W_n = V$$

where $\dim W_i = i$. So, $W_i \cap W_{i-1}^{\perp}$ is 1-dimensional. Comparing dimensions, the intersection has to be at least 1-dimensional. Now if it is more than 1-dimensional, we can find linearly independent v_1, v_2 in the intersection. Either one of them have no w_i component, or they have a nonzero lienar combination that has no w_i component. Then this nonzero vector cannot be in $W_{i-1}^{\perp}$, which is a contradiction. Now let $v_i \in W_i \cap W_{i-1}^{\perp}$ be a unit vector. Then $v_1, \ldots, v_n$ is the desired orthonormal basis.
We could also use the Gram-Schmidt process on $w_1, \ldots, w_n$. □

Note that we have proven that this upper triangular matrix is normal iff it is diagonal.

Theorem 1.2 (Schur's Theorem for Matrices)**.** *Any $A \in M_{n\times n}(\mathbb{C})$ ca be written $A = UBU^{-1}$ where U is unitary and B is upper triangular, with diagonal entries $B_{ii} \geq 0$. If A is invertible, then U, B are unique.*

Proof. Let $w_1, \dots, w_n$ be columns of A. Then $A = UB$ gives

$$w_i = \sum_j B_{ji} v_j$$

where v_j are the columns of U. We want B to be upper triangular, so $B_{ji} = 0 \forall j > i$. Therefore, $w_i \in \operatorname{span}\{v_1, \dots, v_i\}$. So, we want an orthonormal basis $v_1, \dots, v_n$ that satisfies this with the ith coefficient being non-negative. If A is invertible, $w_1, \dots, w_n$ is a basis, so there is a unique solution given by Gram-Schmidt. In general, construct $v_1, \dots, v_n$ by induction. Suppose that $v_1, \dots, v_k$ is an orthonormal set that satisfies. We want a v_{k+1} such that

$$w_{k+1} \in \operatorname{span}\{v_1, \dots, v_{k+1}\}$$

with the $k+1$th coefficient being non-negative. Now, if

$$w_{k+1} \in \operatorname{span}\{v_1, \dots, v_k\}$$

then any unit vector orthogonal to $\operatorname{span}\{v_1, \dots, v_n\}$ suffices. If not, apply Gram-Schmidt on w_{k+1} to produce v_{k+1}. □

This also holds on $\mathbb{R}$. When A is invertible, we can write, one step further, that $B = DN$ where the diagonals of D are all positive, and N is the upper triangular matrix with all diagonal entries 1. In this form, it is called the Iwasawa decomposition.

Theorem 1.3 (Cholesky Decomposition)**.** *Every positive matrix $A \in M_{n\times n}(\mathbb{C})$ can be written as*

$$A = B^* B$$

where B is upper triangular, with diagonal entries $B_{ii} \geq 0$. If A is invertible, this is unique.

Proof. Consider $\sqrt{A}$. Write $\sqrt{A} = UB$ where U is unitary and B is upper triangular with $B_{ii} \geq 0$. Then

$$A = (\sqrt{A})^2 = \sqrt{A}^* \sqrt{A} = B^* U^* U B = B^* B$$

□

Example 1.1.

$$A = \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix}$$

Now A has eigenvectors $w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ with eigenvalues $\lambda_1 = -5, \lambda_2 = 4$. Gram-Schmidt gives

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Giving

$$B = U^{-1}AU = \begin{pmatrix} -5 & 3 \\ 0 & 4 \end{pmatrix}$$

For the Iwasawa decomposition, we write the columns

$$w_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

where Gram-Schmidt gives us

$$v_1 = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}, v_2 = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Then

$$U = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}, B = U^{-1}A = \begin{pmatrix} -5 & 3 \\ 0 & 5 \end{pmatrix}$$

Example 1.2. For

$$P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Find B such that $P = B^*B$. Now here we know

$$B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, B^* = \begin{pmatrix} \overline{a} & 0 \\ \overline{b} & \overline{c} \end{pmatrix}$$

Then

$$B^*B = \begin{pmatrix} |a|^2 & b\overline{a} \\ \overline{b}a & |c|^2 + |b|^2 \end{pmatrix}$$

Which gives

$$a = \sqrt{2}, b = -\frac{1}{\sqrt{2}}, c = \frac{1}{\sqrt{2}}$$

and so

$$B = \begin{pmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$