Lecture 2

September 19, 2022

1 Formal Definition of Double Integrals

We consider the function $z = f(x, y)$ defined on $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, assume $f(x, y) \geq 0$ over R.

1.1 Formal Definition 1

Similar to the case for integrals,

$$V \approx \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i$$

where ΔA_i is a partition of R.
The Riemann sum is bounded by

$$\sum_{i=1}^{N} m_i \Delta x_i \Delta y_i \leq \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta x_i \Delta y_i \leq \sum_{i=1}^{N} M_i \Delta x_i \Delta y_i$$

The definition then follows similar to the case for integrals.

1.2 Formal Definition 2

We use a double sum

$$V \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$$

and similarly

$$\sum_{j=1}^{m}\sum_{i=1}^{n} m_{ij}\Delta x_i \Delta y_j \leq \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j \leq \sum_{j=1}^{m}\sum_{i=1}^{n} M_{ij}\Delta x_i \Delta y_j$$

Example 1.1. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper corner of each square.

$$\text{Area} = 13 + 10 + 7 + 4 = 34$$

The exact integral is

$$\begin{aligned}
V &= \int_0^2 \int_0^2 16 - x^2 - 2y^2 dydx \\
&= \int_0^2 16y - x^2y - \frac{2y^3}{3}\Big|_0^2 dx \\
&= \int_0^2 32 - 2x^2 - \frac{16}{3}dx \\
&= 32x - \frac{2x^3}{3} - \frac{16x}{3}\Big|_0^2 \\
&= 64 - \frac{16}{3} - \frac{32}{3} \\
&= 48
\end{aligned}$$

1.3 Double Integrals over Non- rectangular Regions

Approximate the region R with its upper and lower bounds (corresponding to upper and lower Riemann sums).

$$V \approx \sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i$$

and

$$\sum_{i=1}^{N} m_i \Delta A_i \leq V \leq \sum_{i=1}^{N} M_i \Delta A_i$$

Alternatively, consider the region itself as an integral. Replace the Riemann sum above with a double sum such as

$$V \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j$$

and take the limit.