Lecture 28

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1 Window Functions

The first n periods of a periodic function is then

$$f_{nT}(t) = \sum_{k=0}^{n-1} f_T(t - kT)u_{kT}(t)$$

Now $f(t)$ can be represented as

$$f(t) = \sum_{k=0}^{\infty} f_T(t - kT)u_{kT}(t)$$

Theorem 1.1. *If f is periodic with period T and is piecewise continuous on $[0, T]$, and $F_T(s) = \mathcal{L}\{f_T\}$ is the Laplace Transform of the window function. Then*

$$\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1 - e^{-sT}} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

Example 1.1. The sawtooth waveform is given by

$$f(t) = \begin{cases} t, & t \in [0, 1) \\ 0, & t \in [1, 2) \end{cases}$$

and $f(t)$ has period 2. Find the Laplace Transform of $f(t)$.

$$\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \frac{\int_0^1 e^{-st}tdt}{1-e^{-sT}} \\
&= \frac{-\frac{1}{s}e^{st}t\Big|_{t=0}^{t=1} + \int_0^1 \frac{1}{s}e^{-st}dt}{1-e^{-2s}} \\
&= \frac{-\frac{1}{s}e^{s} - \frac{1}{s^2}e^{-st}\Big|_{t=0}^{t=1}}{1-e^{-2s}} \\
&= \frac{-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}}{1-e^{-2s}}
\end{aligned}$$

2 Forcing Functions

Visualising an ODE as a physical description, where

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

we call $g(t)$ the forcing function. Now we attempt to solve the ODE where the forcing function is not differentiable. Existence and Uniqueness does not guarantee a solution, but a solution may still exist.

Example 2.1.

$$y'' + 4y = g(t), y(0) = 0, y'(0) = 0$$

where

$$g(t) = \begin{cases} 0, & t \in [0,5) \\ \frac{1}{5}(t-5), t \in [5,10) \\ 1, & t \in [10,\infty) \end{cases}$$

We decompose the intial value problem into the 3 intervals defined for g. In the first interval, obviously $y(t) = 0$. For the second interval,

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{t}{20} - \frac{1}{4}$$

For the third interval,

$$y(t) = c_1\cos(2) + c_2\sin(2t) + \frac{1}{4}$$

Solving for the solution using the Laplace Transform, we first note that

$$g(t) = \frac{1}{5}(t-5)u_{5,10}(t) + u_{10}(t)$$
$$= \frac{1}{5}(t-5)[u_5(t) - u_{10}(t)] + u_{10}(t)$$
$$= \frac{1}{5}[(t-5)u_5(t) - (t-10u_{10}(t)]$$

Then

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \frac{1}{5}[e^{-5s}\mathcal{L}\{t\} - e^{-10s}\mathcal{L}\{t\}] \\
&= \frac{e^{-5s} - e^{-10s}}{5s^2} \\
\mathcal{L}\{y''(t) + 4y\} &= \mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} \\
&= s^2Y(s) - sy(0) - y'(0) + 4Y(s) \\
&= (s^2+4)Y(s) \\
Y(s) &= \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)} \\
&= \frac{1}{5}\left(\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right)\left(e^{-5s} - e^{-10s}\right) \\
&= \frac{1}{5}\left(u_5(t)h(t-5) + u_{10}(t)h(t-10)\right)
\end{aligned}$$

Where h is the inverse Laplace of the partial fractions, i.e.

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$