Lecture 16

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March 14, 2023

1 Normal Operators

Recall for a finite dimensional complex space V with an inner product,

$$TT^* = T^*T \Leftrightarrow V \text{ has an orthonormal basis of eigenvectors of } T$$

Let P_λ be orthogonal projections to eigenspaces $E(\lambda, T)$.

Theorem 1.1 (Spectral Resolution). *For a normal T, all of the following hold.*

$$P_\lambda P_\mu = \begin{cases} 0 & \lambda \neq \mu \\ P_\lambda & \lambda = \mu \end{cases}$$

$$\sum_\lambda P_\lambda = I$$

$$T = \sum_\lambda \lambda P_\lambda$$

Proof. Trivial. □

If $W \subseteq \mathbb{C}^n$ is a subspace with orthonormal basis $v_1, \ldots, v_k \in W$, then the orthogonal projection to W is

$$P = \sum_{i=1}^k v_i v_i^*$$

This is because

$$Pv_j = \sum_i v_i v_i^* v_j = \sum_i v_i \langle v_j, v_i \rangle = \begin{cases} v_j & v_j \in W \\ 0 & v_j \notin W \end{cases}$$

Example 1.1. Find the spectral resolution of $A \in M_{2\times 2}(\mathbb{C})$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

The eigenvalues are $a \pm ib$ with unit eigenvectors $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. The orthogonal projections are then

$$P_{\lambda_1} = v_1 v_1^* = \frac{1}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$P_{\lambda_2} = v_2 v_2^* = \frac{1}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Thus the spectral resolution is

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (a+ib)\frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + (a-ib)\frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

The spectral resolution is useful in finding us the adjoint, as

$$T^* = \sum_\lambda \overline{\lambda} P_\lambda$$

and that

$$T^k = \sum_\lambda = \lambda^k P_\lambda$$

More generally,

$$q(T) = \sum_\lambda q(\lambda) P_\lambda$$

Recall $\mathrm{Spec}(T)$ is the set of eigenvalues of T.

Definition 1.1 (Functional Calculus)**.** For any $f : \mathrm{Spec}(T) \to \mathbb{C}$, define, for normal T,

$$f(T) = \sum_\lambda f(\lambda) P_\lambda$$

If $v \in V$ is an eigenvalue of T, eigenvalue λ, then it is also an eigenvalue for $f(T)$ with eigenvalue $f(\lambda)$ and $f(T)$ is the unique operator with this property. In particular $f(T)$ is normal.

Example 1.2. The properties of $f(T)$ are as follows.

- $f(\lambda) = \lambda \Rightarrow f(T) = T$
- $f(\lambda) = 1 \Rightarrow f(T) = 1$
- $f(\lambda) = q(\lambda) \Rightarrow f(T) = q(T)$
- $f(\lambda) = \overline{\lambda} \Rightarrow f(T) = T^*$
- $(f+g)(T) = f(T) + g(T)$
- $(af)(T) = af(T)$
- $(f \cdot g)(T) = f(T)g(T)$
- $f(T)^* = \overline{f}(T^*)$
- $\text{Spec}(f(T)) = f(\text{Spec}(T))$

Now we can define all kinds of functions of normal operators.

Example 1.3. $|A|$ from the first example becomes

$$|a+ib|\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} + |a-bi|\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$$

$$\exp\begin{pmatrix}a & b\\ -b & a\end{pmatrix} = e^{a+ib}\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} + e^{a-ib}\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} = e^a\begin{pmatrix}\cos b & \sin b\\ -\sin b & \cos b\end{pmatrix}$$

For a finite dimensional vector space $V = \mathbb{R}$ or $V = \mathbb{C}$, $T \in \mathcal{L}(V)$,

$$\exp(T) = e^T$$

is always defined (it converges, but one has to show it). More generally, $f(T)$ is defined $\forall f(z) = \sum_{n=0}^{\infty} a_n z^n$ with an ∞ radius of convergence.

2 Positive Operators

For a finite dimensional complex space V,

Proposition 2.1. *For self adjoint $T \in \mathcal{L}(V)$, the following are equivalent*

1. $\langle Tv, v\rangle > 0 \forall v \in V$
2. *The eigenvalues of T are all* ≥ 0

Proof. The first obviously implies the second; put v to be any eigenvector. Given the second,

$$\langle Tv, v\rangle = \langle T\left(\sum_i a_i e_i\right), \sum_i a_i e_i\rangle = \langle \sum_i a_i \lambda_i e_i, \sum_i a_i e_i\rangle = \sum_i \lambda_i a_i^2 \geq 0$$

where e_i is an orthonormal basis of eigenvectors. $\square$

Definition 2.1. A self-adjoint operator satisfying these properties is called positive. Note that 0 (the operator) is positive.

Example 2.1. I is positive. $\forall T \in \mathcal{L}(V)$, the operators TT^*, T^*T are positive. This is because

$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = ||Tv|| \geq 0$$

and similarly for TT^*. Then for any normal T and any $f : \mathrm{Spec}(\mathbb{C}) \to \mathbb{R}^+$, then $f(T)$ is positive. In particular, $|T|$ is positive.

Proposition 2.2. *If T_1, T_2 are positive, $a_1, a_2 \geq 0$, then $a_1T_1 + a_2T_2$ is positive.*

Proof. Trivial. $\square$

Example 2.2. If $T \in \mathcal{L}(V)$ is any self adjoint operator, then $I + \epsilon T$ is positive for sufficiently small ϵ. This is when $1 - \epsilon\lambda \geq 0 \forall$ eigenvalues λ

Given $T \in \mathcal{L}(V)$, an operator $S \in \mathcal{L}(V)$ is its "suare root" if $S^2 = T$.

Proposition 2.3. *If $T \in \mathcal{L}(V)$ is positive, then it has a unique positive square root.*

Proof. Note that the positive square root function is well defined on $\mathrm{Spec}(T) \subseteq \mathbb{R}^+ \cup \{0\}$. Then $\sqrt{T}$ is a square root. To show uniqueness, let $Tv = \mu^2 v$ and $Sv = kv + u$, where u and v are orthogonal. Since S is positive, $k \geq 0$. Then

$$\mu^2 v = S^2 v = k^2 v + ku + Su \Rightarrow Su = -ku$$

which means $k = 0$ is the only possible value of k. Then S and T share the same eigenvectors. This uniquely determines the eigenvalues of S as the square roots of that of T. □

This can be applied to polar decomposition. Let $T \in \mathcal{L}(V)$ be any operator.

Theorem 2.1. *Suppose $T \in \mathcal{L}(V)$ is invertible. Then there exists a unique unitary operator $U \in \mathcal{L}(V)$ and positive $R \in \mathcal{L}(V)$ such that $T = UR$.*

Proof.

$$T^*T = R^*U^*UR = R^*R = R^2$$

We can take $R = \sqrt{T^*T}$. Then we have a unique $U = TR^{-1}$. Note that

$$U^*U = (R^{-1})^*T^*TR^{-1} = R^{-1}R^2R^{-1} = I$$

□

As a consequence, if T is not normal, then U, R do nto commute. We also have $T = \sqrt{TT^*}U_1$, where U_1 is unitary.