Lecture 24

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1 Wave in an Open Channel

A piston is pushing a body of fluid horizontally with velocity δv. Let c_0 be wavespeed, y be depth, δy be height of wave, and b be width. Consider a control volume that moves with wavefront. As mass is conserved,

$$
\begin{aligned}
\dot{m}_1 &= \dot{m}_2 \\
\rho c_0 y b &= \rho(c_0 - \delta V)(y + \delta y)b \\
c_0 \delta y &= \delta V(y + \delta y) \\
\delta V &= c_0 \frac{\delta y}{y + \delta y}
\end{aligned}
$$

Considering the momentum equation,

$$
\begin{aligned}
\Delta F &= \dot{m}\Delta v \\
\frac{1}{2}\rho g(y + \delta y)^2 b - \frac{1}{2}\rho g y^2 b &= \rho c_0 y b \delta v \\
g(1 + \frac{\delta y}{2y})\delta y &= c_0 \delta v
\end{aligned}
$$

Combining both, we have

$$c_0^2 = gy(1 + \frac{\delta y}{2y})(1 + \frac{\delta y}{y})$$

Assuming $\delta y << y$, we have

$$c_0 = \sqrt{gy}$$

Definition 1.1. The Froude number is

$$\mathrm{Fr} = \frac{V}{\sqrt{gy}}$$

When it is less than one, flow is subcritical. When it is equal to one, flow is critical. When it is greater than one, flow is supercritical.

where c_0 is the wave speed and y is the water depth.

2 Compressible Flow

Similarly, for sound waves, we have

$$\begin{aligned}
\dot{m}_{\text{in}} &= \dot{m}_{\text{out}} \\
\rho c A &= (\rho + d\rho)(c - dv)A \\
\rho c &= \rho c - \rho dv + c d\rho \\
\rho dv &\approx c d\rho
\end{aligned}$$

Using the momentum equation,

$$\begin{aligned}
(p + dp)A - pA &= \rho c A dv \\
dp &= \rho c dv
\end{aligned}$$

Combining,

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

at constant s.
Under isentropic conditions,

$$\frac{p}{\rho^\gamma} = C$$

thus

$$c^2 = C\gamma\rho^{\gamma-1} = \gamma\frac{p}{\rho} = \gamma RT$$

for ideal gas, where the simplified equation

$$c = \sqrt{\gamma RT}$$

can be used.
More general, the bulk modulus

$$E_v = \frac{dp}{\frac{d\rho}{\rho}}$$

can be used, giving

$$c = \sqrt{\frac{E_v}{\rho}}$$

Definition 2.1. The Mach number is defined as

$$M = \frac{v}{c}$$

where c is the speed of sound in the same fluid. Flow is compressible only if the free stream Mach number $M_\infty > 0.3$.

Similarly, we define subsonic, sonic and supersonic flow. We have transonic flow when $M \in [0.8, 1]$ and hypersonic flow when $M \in [5, \infty)$.

Example 2.1. Consider steady isentropic flow of 1D compressible flow.

$$\begin{aligned} \dot{m} &= C \\ \rho v A &= 0 \\ vAd\rho + \rho v dA + \rho A dv &= 0 \\ \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v} &= 0 \end{aligned}$$

Example 2.2. Steady Isentropic Flow of Compressible Flows
We first assume work done from friction is 0, and work done from shaft work is 0 under steady state. Work done by pressure is then

$$\begin{aligned} W &= W_1 + W_2 \\ &= p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t \\ \dot{W} &= p_1 A_1 v_1 - p_2 A_2 v_2 \end{aligned}$$

Using the Reynolds Transport Theorem,

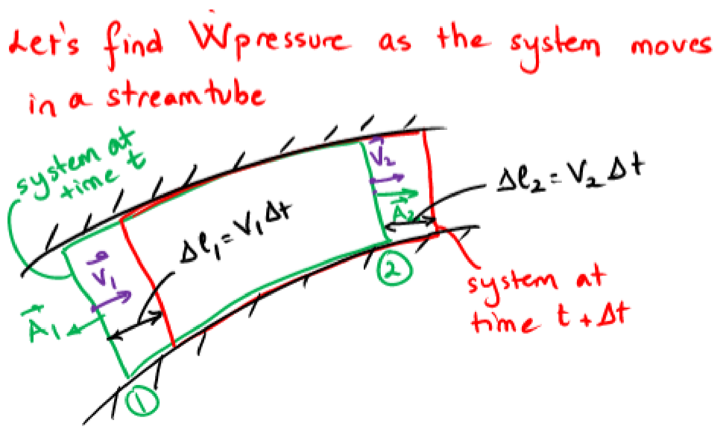

Figure 1: Work Done by Pressure

$$
\begin{aligned}
\frac{dE_{\text{sys}}}{dt} &= \frac{dE_{\text{cv}}}{dt} + \dot{E}_{\text{out}} - \dot{E}_{\text{in}} \\
&= \dot{m}(e_2 + \frac{v_2^2}{2} + gz_2) - \dot{m}(e_1 + \frac{v_1^2}{2} + gz_2) \\
p_1 A_1 v_1 - p_2 A_2 v_2 &= \dot{m}(e_2 + \frac{v_2^2}{2} + gz_2) - \dot{m}(e_1 + \frac{v_1^2}{2} + gz_2) \\
\frac{p_2}{\rho_2} + e_2 + \frac{v_2^2}{2} + gz_2 &= \frac{p_1}{\rho_1} + e_1 + \frac{v_1^2}{2} + gz_1
\end{aligned}
$$

where e_i is the internal energy at i. Substituting enthalpy,

$$h_2 + \frac{v_2^2}{2} + gz_2 = h_1 + \frac{v_1^2}{2} + gz_1$$

For high speed flows, the effect of height is negligible, hence

$$h + \frac{v^2}{2} = C \tag{1}$$

Definition 2.2. We define *stagnation enthalpy* as they enthalpy a fluid element achieves when it is brought to rest adiabetically.

$$h_0 = h(v = 0)$$

Rewriting enthalpy in terms of specific heat and temperature,

$$c_p(T - T_0) + \frac{v^2}{2} = 0$$
$$T_0 = T + \frac{v^2}{2c_p}$$

Where N_0 denotes property N at stagnation.

Definition 2.3. The *dynamic temperature* is defined as

$$\Delta T = \frac{v^2}{2c_p}$$

Example 2.3. The dynamic temperature of air at $100\mathrm{m\,s^{-1}}$ is

$$\Delta T = \frac{100^2}{2 \times 1.005 \times 10^3} = 5\mathrm{K}$$

Therefore, air temperature rises by 5K when it stagnates.

Note that for low speed flow, temperature rise is negligible at stagnation. Noting that $c_p = \frac{\gamma R}{\gamma - 1}$,

$$\begin{aligned}\frac{T_0}{T} &= 1 + \frac{(\gamma - 1)v^2}{2\gamma RT} \\ &= 1 + \frac{(\gamma - 1)v^2}{2c^2} \\ &= 1 + \frac{\gamma - 1}{2}M^2\end{aligned}$$

Substituting isentropic relations

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

we have

$$\begin{aligned}\frac{p_0}{p} &= \left(\frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} \\ \frac{\rho_0}{\rho} &= \left(\frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma-1}}\end{aligned}$$

Example 2.4. Given $M_\infty = 2.2$ and $T_\infty = -30°\text{C}$, find stagnation temperature.

$$\begin{aligned}\frac{T_0}{T} &= 1 + \frac{\gamma - 1}{2}M^2 \\ \frac{243}{T} &= 1 + \frac{1.4 - 1}{2} \times 2.2^2 \\ T &= 478\text{K} \\ &= 205°\text{C}\end{aligned}$$

Example 2.5. We consider flow incompressible when change in density is less than 5%.

$$\begin{aligned}\frac{\rho_0}{\rho} &= \left(\frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma - 1}} \\ 1.05 &= \left(\frac{\gamma - 1}{2}M^2\right)^{\frac{1}{1.4 - 1}} \\ M &= 0.31\end{aligned}$$

Thus we consider flow to be incompressible when $M \leq 3$.

Differentiating Equation 1,

$$\frac{dp}{\rho} + v dv = 0$$

Combining with the continuity equation derived in 2.1,

$$\frac{dA}{A} = -\frac{dv}{v}(1 - M^2)$$

Therefore, at subsonic flow, $\frac{dA}{dV} < 0$, and at supersonic flow, $\frac{dA}{dV} > 0$, where flow accelerates as area increases.