# Lecture 12

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## 1 Behaviour of System: Complex Eigenvalue, Zero Real Part

Consider

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & -5 \\ 8 & -2 \end{pmatrix}$$

The real part of the eigenvalue is 0. The phase potrait is hence composed of "circles" centred at 0, as $\vec{x}(t)$ is periodic. Substituting (e.g.) $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives us the slope $\begin{pmatrix} -5 \\ -2 \end{pmatrix}$, which means the direction is counterclockwise.

**Example 1.1.** Consider

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \vec{x}$$

The eigenvalues are $-1 \pm 2i$.
The solutions all have a coefficient of $e^{-t}$, so they spiral towards the origin.

**Example 1.2.** Consider

$$\frac{d\vec{x}}{dx} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$$

Where the eigenvalues and eigenvalues are given by $1 \pm 2i$ and $\begin{pmatrix} 1 \\ 1 - i \end{pmatrix}$. We then have

$$\vec{u}(t) = e^t \left( \cos(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)$$

and

$$\vec{w}(t) = e^t \left( \sin(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sin(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)$$

The phase potrait is hence a spiral from the origin. Substituting $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tells us that the spiral is counterclockwise.

# 2 Repeated Eigenvalues, Distinct Eigenvectors

Consider

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x}$$

And the solution is then

$$\vec{\phi}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The behaviour of the phase potrait depends on which coefficient dominates.

**Example 2.1.**

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \vec{x}$$

The eigenvalue is 1, and the eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. However, if we write the system out explicitly,

$$x_1'(t) = x_1(t) + 2x_2(t)$$

and

$$x_2'(t) = x_2(t)$$

One can directly solve for the second equation, which gives us enough information to solve the first equation.

$$x_2(t) = c_2 e^t$$

Substituting,

$$x_1'(t) = x_1(t) + 2c_2 e^t$$

This is a first order linear ODE

$$
\begin{aligned}
e^{-t}x_1(t) &= \int 2c_2 dt \\
e^{-t}x_1(t) &= 2c_2 t + c_1 \\
x_1(t) &= 2c2te^t + c_1 e^t
\end{aligned}
$$

Then simplifying $\vec{\phi_2}(t)$ gives us

$$
\vec{\phi_2}(t) = \vec{\phi_1}(t) + c_2 e^t \left( t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right)
$$

Droppint the $\vec{\phi_1}(t)$ term gives us

$$
\vec{\phi_2}(t) = te^t \vec{v_1} + e^t \vec{w}
$$

We want to generalise this. We try the ansatz $\vec{x}(t) = te^{\lambda t}\vec{v_1} + e^{\lambda t}\vec{w}$. Substituting into

$$
\frac{d\vec{x}}{dt} = A\vec{x}
$$

$$
\begin{aligned}
\text{LHS} &= \frac{d}{dt}\left(te^{\lambda t}\vec{v_1} + e^{\lambda t}\vec{w}\right) \\
&= e^{\lambda t}\vec{v_1} + \lambda te^{\lambda t}\vec{v_1} + \lambda e^{\lambda t}\vec{w} \\
&= e^{\lambda t}\left((\lambda t + 1)\vec{v_1} + \lambda \vec{w}\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} &= A\vec{x} \\
&= A\left(te^{\lambda t}\vec{v_1} + e^{\lambda t}\vec{w}\right) \\
&= e^t\left(\lambda t\vec{v_1} + A\vec{w}\right)
\end{aligned}
$$

Comparing like terms,

$$
A\vec{w} + \lambda t\vec{v_1} = \lambda\vec{w} + (\lambda t + 1)\vec{v_1} \Rightarrow (A - \lambda I)\vec{w} = \vec{v_1}
$$

What remains is to verify linear independence by computing the Wronskian. Factoring out $e^{\lambda t}$, we have

$$\det \begin{bmatrix} \vec{v_1} & t\vec{v_1} + \vec{w} \end{bmatrix}$$

We can remove the constant multiple of $\vec{v_1}$ on the right hand side, yielding

$$\det \begin{bmatrix} \vec{v_1} & \vec{w} \end{bmatrix}$$

This is nonzero because $\vec{v_1}$ and $\vec{w}$ must be linearly independent. If not,

$$\begin{aligned} \vec{v_1} &= k\vec{w} \\ (A - \lambda I)\vec{w} &= k\vec{w} \\ (A - (\lambda + k)I)\vec{w} &= 0 \end{aligned}$$

If $k \neq 0$, there is a second eigenvalue $\lambda + k$, which is a contradiction. If $k = 0$, $\vec{v_1} = 0$, which is also a contradiction.