Lecture 9

niceguy

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1 Joint Distribution

From the density function $f(x, y)$, the marginal distribution of X is

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

or

$$g(x) = \sum_y g(x, y)$$

in the discrete case. Similarly, the marginal distribution of Y can be defined.

2 Independence

X and Y are random variables with joint distribution $f(x, y)$ and marginals $g(x)$ and $h(y)$.

Definition 2.1. X and Y are independent if

$$f(x, y) = g(x)h(y)$$

This implies $g(x)$ and $h(y)$ are probability density functions.

Example 2.1. X and Y are continuous random variables with joint distribution

$$f(x, y) = e^{-x-y} = e^{-x}e^{-y}$$

Then x and y must be independent, as

$$\begin{aligned} g(x) &= \int_0^\infty e^{-x}e^{-y}dy \\ &= e^{-x} \end{aligned}$$

Note that $g(x)$ and $h(y)$ may differ by a constant factor; consider splitting f into $2e^{-x} \times \frac{1}{2}e^{-y}$.

In the discrete case, for A and B to be independent,

$$P(A|B) = P(A)$$

Expaning and rearranging yields

$$P(A \cap B) = P(A)P(B)$$

Letting $A : X = x, B : Y = y$ yields

$$P(X = x \cup Y = y) = P(X = x)P(Y = y)$$

or

$$f(x, y) = g(x)h(y)$$

3 Expectation

Definition 3.1. The expectation value is defined as

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

or

$$E[X] = \sum_x xf(x)$$

for the discrete case.

Example 3.1. For 3 coinflips, and X being the number of heads, then

$$E[X] = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 2 + \frac{1}{8} \times 3 = \frac{3}{2}$$

Let X be a random variable with distribution $f(x)$, and $g(X)$ be some function. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

or

$$E[g(X)] = \sum_{x} g(x)f(x)$$

in the discrete case.

Example 3.2. The power generation for a wind turbine is

$$P = g(X) = aX^3$$

Then

$$E[P] = \int_{-\infty}^{\infty} ax^3 f(x)dx$$

Since integrals are linear, if

$$X = Y_1 + Y_2 + Y_3$$

then

$$E[X] = E[Y_1] + E[Y_2] + E[Y_3]$$

If the right hand side is defined.

Example 3.3. X is a random variable with

$$f(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Then for $g(x) = 3x + 1$,

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{\infty} g(x)f(x)dx \\ &= \int_{0}^{2} (3x+1)\frac{x}{2}dx \\ &= 5 \end{aligned}$$

Let X and Y be random variables with joint distribution $f(x, y)$. Then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

$$E[g(X,Y)] = \sum_{x}\sum_{y} g(x,y)f(x,y)$$