Lecture 11

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1 Expectation Value

Definition 1.1. The expectation value of a function is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

or

$$E[g(X)] = \sum_x g(x)f(x)$$

Definition 1.2. Similarly, for functions with two variables,

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dydx$$

or

$$E[g(X,Y)] = \sum_x \sum_y g(x,y)f(x,y)$$

2 Expectiations of linear combinations of Random Variables

Recall linearity.

Definition 2.1. $p(x)$ is linear if

$$p(ax+y) = ap(x) + p(y)$$

Suppose X and Y are random variables with joint distribution $f(x,y)$ and marginals $g(x)$ and $h(y)$. The expectation of $aX+Y$ is

$$\begin{aligned}E[aX+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax+y)f(x,y)dxdy \\ &= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}yf(x,y)dxdy \\ &= a\int_{-\infty}^{\infty}xg(x)dx + \int_{-\infty}^{\infty}yh(y)dy \\ &= aE[X]+E[Y]\end{aligned}$$

Then expectation is linear.

3 Variance

Suppose X and Y are independent. Then $f(x,y) = g(x)h(y)$. Observe

$$\begin{aligned}E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)dxdy \\ &= \int_{-\infty}^{\infty}xg(x)dx\int_{-\infty}^{\infty}yh(y)dy \\ &= E[X]E[Y]\end{aligned}$$

Since covariance

$$\sigma_{XY} = E[XY] - E[X]E[Y]$$

Independence implies correlation is 0. However, uncorrelated does not imply indepnedence.

Example 3.1. Consider random variables X and Y with joint probability density function

$$f(x,y) = \begin{cases}\frac{1}{\pi} & x^2+y^2 \leq 1 \\ 0 & \text{else}\end{cases}$$

Then

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1-x^2} \end{aligned}$$

And similarly for y by symmetry. However,

$$g(x)h(y) \neq f(x,y)$$

so the are not independent. Intuitively it makes sense, as the value of x limits the range of y for which f is nonzero.

$$\begin{aligned} E[XY] = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\ = &\int_{-1}^{1}\int_{-1}^{1} xyf(x,y)dxdy \\ = &\int_{0}^{1}\int_{0}^{1} xyf(x,y)dxdy + \int_{-1}^{0}\int_{-1}^{0} xyf(x,y)dxdy \\ + &\int_{0}^{1}\int_{-1}^{0} xyf(x,y)dxdy + \int_{-1}^{0}\int_{0}^{1} xyf(x,y)dxdy \\ = &\ 0 \end{aligned}$$

Where the terms cancel out.

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xg(x)dx \\ &= \int_{-1}^{1} xg(x)dx \\ &= \int_{-1}^{0} xg(x)dx + \int_{0}^{1} xg(x)dx \\ &= 0 \end{aligned}$$

And similarly for $E[Y]$ by symmetry. Then

$$\sigma_{XY} = 0$$

even if X and Y are not independent.

Example 3.2.

$$
\begin{aligned}
\sigma^2_{aX+bY+c} &= E[(aX + bY + c - E[aX + bY + c])^2] \\
&= E[(aX + bY + c - (a\mu_X + b\mu_Y + c))^2] \\
&= E[(a(X - \mu_x) + b(Y - \mu_Y))^2] \\
&= E[a^2(X - \mu_x)^2 + b^2(Y - \mu_Y)^2 + 2ab(X - \mu_X)(Y - \mu_Y)] \\
&= a^2\sigma^2_X + b^2\sigma^2_Y - 2ab\sigma^2_{XY}
\end{aligned}
$$