Lecture 4

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1 Inverse of a Matrix

Denote columns of A^{-1} by w_i. Then observing

$$v_i = Ae_i$$

we can see

$$w_j = A^{-1}e_j \Leftrightarrow Aw_j = e_j$$

That is, w_j solves the equation $Ax = e_j$. Thus by Cramer's rule,

$$A^{-1}_{ij} = \frac{\det(v_1, \dots, v_{i-1}, e_j, v_{i+1}, \dots, v_n)}{\det(A)}$$

Using cofactor expansion along the jth column

$$A^{-1}_{ij} = \frac{(-1)^{i+j}\det(A^{[ji]})}{\det(A)}$$

Note that i and j are flipped for the cofactor, as it is the jth row that is deleted.

2 More Properties

For $T \in \mathcal{L}(V)$ we defined

$$\det(T) = \det(A)$$

giving the properties

$$\det(TS) = \det(T)\det(S)$$

$$\det(T^{-1}) = \det(T)^{-1}$$
$$\det(T') = \det(T)$$

If $W \subseteq V$ is T-invariant then

$$\det(T) = \det(T|_W)\det(\bar{T})$$

where $\bar{T} \in \mathcal{L}(V/W)$ is $\bar{T}(v+W) = Tv + W$

3 Characteristic Polynomials

Let $T \in \mathcal{L}(V)$

Theorem 3.1. *A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T iff* $\det(\lambda I - T) = 0$

Proof:
The definition of an eigenvalue λ is that $Tv = \lambda v$. Rearranging this, v is in the nullspace of $(\lambda I - T)$, which is equivalent to it being noninvertible, giving it a determinant of 0.
To rephrase this, λ is an eigenvalue iff λ is a root of

$$q(z) = \det(zI - T)$$

Definition 3.1. For $T \in \mathcal{L}(V)$, the polynomial

$$q(z) = \det(zI - T)$$

is called the characteristic polynomial. Similarly, for $A \in M_{n\times n}(\mathbb{F})$ we call

$$q(z) = \det(zI - A)$$

the characteristic polynomial.

Remark:
The definition works for any field $\mathbb{F}$

$$p(z) = \sum_{i=0}^{n} a_i z^i$$

where $a_i \in \mathbb{F}$. Note that the degree of the polynomial is at most n (based on the permutations). For $\mathbb{F} = \mathbb{C}$, we can use the fundamental theorem of algebra to factorise $q(z)$

$$q(z) = \prod_i (z - \lambda_i)$$

where some eigenvalues may repeat. Then this is equivalent to the set of eigenvalues by the theorem above.
Then for any square matrix A, for any upper triangular matrix with $A'_{ii} = \lambda_i$, we see

$$\det(zI - A) = \det(zI - A')$$

Remark:
If A has block upper triangular form

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}$$

then

$$q_A(z) = q_{A'}(z) q_{A''}(z)$$

since

$$zI - A = \begin{pmatrix} zI - A' & * \\ 0 & zI - A'' \end{pmatrix}$$

4 Cyclic Vectors

Suppose $T \in \mathcal{L}(V), v \neq 0 \in V, v_k = T^{k-1}v$. There is a smallest k such that $v_1, \ldots, v_{k+1}$ are linearly dependent, The subspace W spanned by $v_1, \ldots, v_k$ is T-invariant, as v_i maps to v_{i+1} except for $i = k$, where it maps to a linear combination of $v_i, \ldots, v_k$.
We call $v \in V$ a cyclic vector for T if $v_1, \ldots$ spans all of V. In this case, $v_1, \ldots, v_k$ is a basis of V. The matrix of T in this basis is

$$A_{\cdot,i} = \begin{cases} e_{i+1} & i \neq k \\ b & i = k \end{cases}$$

where $A_{\cdot,i}$ is the ith column of A, and

$$b = \sum_{i=1}^{k} -a_i e_i$$

where

$$Tv_k = \sum_{i=1}^{k} -a_i v_i$$