Lecture 9

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1 Examples on the Change of Variables in Multiple Integrals

Example 1.1. Evaluate
$int_R e^{(x+y)/(x-y)} dA$ where R is the trapezoidal region with vertices $(1,0),(2,0),(0,-2)$, and $(0,-1)$.
If we let

$$\begin{aligned} u &= x + y \\ v &= x - y \end{aligned}$$

The boundaries will be v from 1 to 2, and u from $-v$ to v. The Jacobian is then

$$\begin{aligned} J &= \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \\ &= \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}^{-1} \\ &= \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= (-2)^{-1} \\ &= -\frac{1}{2} \end{aligned}$$

Then the integral is given by

$$\begin{aligned}
I &= \iint_S e^{\frac{u}{v}} \times \frac{1}{2} dudv \\
&= \int_1^2 \int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}} dudv \\
&= \frac{1}{2} \int_1^2 v(e - e^{-1}) dv \\
&= \frac{1}{2}(e - e^{-1})
\end{aligned}$$

Example 1.2. Evaluate $\iint_R (x^2 - y^2) e^{xy} dxdy$, where the region R is the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 2$ and the lines $y = x$ and $y = x + 2$.
Note that substitution for $x^2 - y^2$ and xy doesn't work (try it!). Instead, we use

$$\begin{aligned}
u &= xy \\
v &= y - x
\end{aligned}$$

The Jacobian is given by

$$\begin{aligned}
J &= \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}^{-1} \\
&= \det \begin{bmatrix} y & x \\ -1 & 1 \end{bmatrix}^{-1} \\
&= \frac{1}{x + y}
\end{aligned}$$

The integral is then

$$\begin{aligned}
I &= \int_0^2 \int_1^4 (x^2 - y^2) e^{xy} \times \frac{1}{x + y} dudv \\
&= \int_0^2 \int_1^4 (x - y) e^{xy} dudv \\
&= \int_0^2 \int_1^4 -ve^{u} dudv \\
&= \int_0^2 v(e - e^4) dv \\
&= 2(e - e^4)
\end{aligned}$$

Example 1.3. Find the volume of the region bounded by the hyperbolic cylinders $xy = 1, xy = 9, xz = 4, xz = 36, yz = 25, yz = 49$. We then let $u = xy, v = xz, w = yz$. The Jacobian is then

$$\begin{aligned} J &= \det \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}^{-1} \\ &= \det \begin{bmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}^{-1} \\ &= -\frac{1}{2xyz} \\ &= -\frac{1}{2\sqrt{uvw}} \end{aligned}$$

The integral is then given by

$$\begin{aligned} I &= \int_{25}^{49} \int_{4}^{36} \int_{1}^{9} \frac{dudvdw}{2\sqrt{uvw}} \\ &= 4(7-5)(6-2)(3-1) \\ &= 64 \end{aligned}$$

2 Line Integrals

We can integrate along lines (not necessarily straight lines) by

$$\int_C f(x, y) ds$$

This can be evaluated by parametrisation

$$\begin{cases} x = x(t) & t \in [a, b] \\ y = y(t) & t \in [a, b] \end{cases}$$

We assume that $f(x, y)$ is continuous over C and that C is smooth, or $\vec{r}'(t)$ is continuous and $\vec{r}(t) \neq \vec{0}$ except at the endpoints. We can then express ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Note that the direction of integration does not matter if f is a scalar function, unlike single integration.

Example 2.1. Find the centre of mass of a semi-circular length of wire $y = \sqrt{a^2 - x^2}, a > 0$. Length density is constant.
By symmetry, $\overline{x} = 0$.
Paramatrisation gives us $x(t) = a\cos t$ and $y(t) = a \sin t$. Then

$$ds = \sqrt{(-a\sin t)^2 + (a\cos t)^2}dt = adt$$

$$\begin{aligned}
m\overline{y} &= \int_C y\rho ds \\
a\pi\overline{y} &= \int_0^{\pi} a\sin t \times adt \\
\overline{y} &= \frac{a}{\pi} \times 2 \\
&= \frac{2a}{\pi}
\end{aligned}$$

Example 2.2. In the special case where C is parallel to the x axis, e.g. from $(a, 0)$ to $(b, 0)$, this can be integrated normally, as $ds = dx$.

2.1 3 Dimensional Case

As expected, ds is now expressed as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

Example 2.3. Find the mass of a spring in the shape of the circular helix defined parametrically by $x2\cos t$, $y = t$, $z = 2\sin t$ for $t \in [0, 6\pi]$ with density of $\rho(x, y, z) = 2y$.

$$ds = \sqrt{(-2\sin t)^2 + 1^2 + (2\cos t)^2}dt = \sqrt{5}dt$$

The mass is then given by

$$\begin{aligned} m &= \int_C \rho ds \\ &= \int_C 2y ds \\ &= \int_0^{6\pi} 2t \times \sqrt{5} dt \\ &= 36\sqrt{5}\pi^2 \end{aligned}$$

2.2 Piecewise smooth curves

Our curve is now

$$C = \bigcup_i C_i$$

where C may not be smooth but C_i is always smooth. Then

$$\int_C f ds = \sum_i \int_{C_i} f ds$$

2.3 Line Integrals of Vector Fields

$$\vec{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

This can be rewritten as

$$\vec{F}(x,y,z) = \vec{F}(\vec{r})$$

Example 2.4. Physical Examples

$$W = \vec{F} \cdot \vec{d}$$

where W stands for work done. Then

$$W = \int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{r}' dt$$

This can also be written as

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_a^b (P\hat{i} + Q\hat{j} + R\hat{k}) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) dt \\ &= \int_C P dx + Q dy + Z dt \end{aligned}$$