Lecture 19

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1 System of Linear Equations

From the previous lecture, since we have $X(0) = \mathbb{I}$, we know that each column vector is linearly independent, so X contains all solutions. We also have some matrix exponential properties

- $e^{0t} = \mathbb{I}$
- $e^{A(t+\tau)} = e^{At}e^{A\tau}$
- $Ae^{At} = e^{At}A$
- $\left(e^{At}\right)^{-1} = e^{-At}$
- $e^{(A+B)t} = e^{At}e^{Bt}$ if $AB = BA$

To prove the last identity, we first observe $e^{(A+B)t}$ satisfies

$$X'(t) = (A+B)X(t)$$

However,

$$\begin{aligned}\frac{d}{dt}e^{At}e^{Bt} &= Ae^{At}e^{Bt} + e^{At}Be^{Bt} \\ &= Ae^{At}e^{Bt} + Be^{At}e^{Bt} \\ &= (A+B)e^{At}e^{Bt}\end{aligned}$$

Moreover, obviously $e^{A0}e^{B0} = \mathbb{I}$, so by existence and uniqueness, we can demonstrate they are equal. Note that we have

$$e^{At}B = Be^{At}$$

since e^{At} is a polynomial with A^k terms. One can easily show through mathematical induction that $A^kB = BA^k$ if we have $AB = BA$.
To compute A^k, if it is diagonalisable, then

$$A = VDV^{-1} \Rightarrow A^k = VD^kV^{-1}$$

so

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k \\ &= V \left(\sum_{k=0}^{\infty} \frac{D^k}{k!} t^k \right) V^{-1} \\ &= V \begin{bmatrix} e^{\lambda_1 t} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{\lambda_n t} \end{bmatrix} V^{-1} \end{aligned}$$

2 Second Order differential Equations

Example 2.1.

- Spring-Mass System

$$my''(t) + \gamma y'(t) + ky(t) + F(t)$$

- Linearised Pendulum

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \theta = 0$$

- Airy Equation

$$y''(t) = ty(t)$$

Definition 2.1. A second order ODE

$$y''(t) = f(t, y, y')$$

is linear if it can be written as

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

It is homogeneous if $g(t) = 0$, and it has constant coefficients if $p(t)$ and $q(t)$ are constants.

Definition 2.2. An initial value probem for a second order ODE on an interval $I = (\alpha, \beta)$ is

$$y''(t) = f(t, y, y'), y(t_0) = y_0, y'(t_0) = y_1$$

where $t_0 \in (\alpha, \beta)$ and $y_0, y_1 \in \mathbb{R}$.

If we define $x_1(t) = y(t), x_2(t) = y'(t)$, the ODE can be reexpressed as

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

where

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Therefore, a second order ODE is equivalent to a first order ODE! Using the existance and uniqueness theorem for linear systems, the specific theorem for second order ODEs is

Theorem 2.1. *For a second order ODE, if*

- $q(t), p(t)$ *are continuous on* I
- $g(t)$ *is continuous on* I
- $t_0 \in I$

Then there exists a unique solution in the interval (α, β).

If we consider the homogeneous case ($g(t) = 0$), the same theory applies, so we have the

- Superposition Principle
- Linear Independence
- Wronskian