

# Lecture 4

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December 5, 2022

## 1 Applications of Double Integrals

**Example 1.1.** A rectangular plate of mass  $m$ , length  $L$  and width  $W$  rotate about the  $y$  axis lying on the left edge of the plate. Find the moment of inertia of the plate about that line:

1. given that the plate has uniform density
2. given that the density at a point on the plate is directly proportional to the square of the distance from the rightmost side.

$$\begin{aligned} I_y &= \iint_R \rho r^2 dy dx \\ &= \frac{m}{LW} \int_0^L \int_0^W x^2 dy dx \\ &= \frac{m}{LW} \int_0^L W x^2 dx \\ &= \frac{m}{LW} \frac{WL^3}{3} \\ &= \frac{mL^2}{3} \end{aligned}$$

and

$$\begin{aligned}
I_y &= \iint_R \rho r^2 dy dx \\
&= k \int_0^L \int_0^W (1-x)^2 x^2 dy dx \\
&= kW \int_0^L x^4 - 2x^3 + x^2 dy dx \\
&= kW \left( \frac{L^5}{5} - \frac{L^4}{2} + \frac{L^3}{3} \right) \\
&= \frac{kWL^3(6L^2 - 15L + 10)}{30}
\end{aligned}$$

**Example 1.2.** Same plate, with constant density. Calculate its rotation about the centre.

$$\begin{aligned}
I_0 &= \iint_R \rho r^2 dy dx \\
&= \int_{-\frac{W}{2}}^{\frac{W}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 + y^2 dy dx \\
&= \int_{-\frac{W}{2}}^{\frac{W}{2}} Lx^2 + \frac{L^3}{12} dx \\
&= \frac{LW^3}{12} + \frac{WL^3}{12} \\
&= \frac{WL(L^2 + W^2)}{12}
\end{aligned}$$

## 2 Surface Area

The area can be considered as the sum of small parallelograms projected from the  $xy$  plane. The new vectors  $\vec{x}'$  and  $\vec{y}'$  can be expressed as

$$\begin{aligned}
\vec{x}' &= \Delta x \hat{i} + f_x \Delta x \hat{k} \\
\vec{y}' &= \Delta y \hat{j} + f_y \Delta y \hat{k}
\end{aligned}$$

And the area is the magnitude of their cross products, which is the magnitude of

$$-f_x \Delta x \Delta y \hat{i} - f_y \Delta x \Delta y \hat{j} + \Delta x \Delta y \hat{k}$$

which is

$$\sqrt{f_x^2 + f_y^2 + 1} \Delta x \Delta y$$

Hence the surface area is given by

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA$$

**Example 2.1.** Find the surface area of a sphere  $x^2 + y^2 + z^2 = a^2$ . Using symmetry, we consider the first octant only.

$$\begin{aligned} V &= 8 \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA \\ &= 8 \iint_R \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2} dA \\ &= 8 \iint_R \sqrt{\frac{a^2}{a^2 - x^2 - y^2}} dA \\ &= 8a \iint_R \frac{dA}{\sqrt{a^2 - x^2 - y^2}} \\ &= 8a \int_0^{\frac{\pi}{2}} \int_0^a \frac{r dr d\theta}{\sqrt{a^2 - r^2}} \\ &= -8a \int_0^{\frac{\pi}{2}} \sqrt{a^2 - r^2} \Big|_0^a d\theta \\ &= -8a \int_0^{\frac{\pi}{2}} -a d\theta \\ &= 4a^2 \pi \end{aligned}$$

Where polar coordinates were used to simplify  $dA$ .

**Example 2.2.** Let  $R$  be the triangular region  $(0, 0, 0), (0, 1, 0), (1, 1, 0)$ . Find the surface area of  $z = 3x + y^2$  that lies over  $R$ .

$$\begin{aligned} A &= \int_0^1 \int_0^y \sqrt{1 + 9 + 4y^2} dx dy \\ &= \int_0^1 y \sqrt{10 + 4y^2} dy \\ &= \frac{1}{12} (10 + 4y^2)^{\frac{3}{2}} \Big|_0^1 \\ &= \frac{1}{12} \left( 14^{\frac{3}{2}} - 10^{\frac{3}{2}} \right) \\ &\approx 1.7 \end{aligned}$$