Lecture 20

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1 Eigenvalues for second order ODEs

We have
$$\vec{x}' = A\vec{x}, \vec{x} = \begin{bmatrix} y \\ y' \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$$
for
$$ay'' + by' + cy = 0$$
The characteristic equation is then
$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$$
The null space has a dimension of 1, as there is a nontrivial element, and $A \neq \vec{0}$. Considering the first row, we have the eigenvector
$$\vec{v} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

Example 1.1.
$$y'' + 5y' + 6y = 0$$
The eigenvalues are -3, -2, with eigenvectors $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Then
$$\vec{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
So
$$y(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

Example 1.2.

$$y'' + y' + y = 0$$

Then

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

The characteristic equation gives us

$$\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{3}i}{2}$$

And

$$\vec{v} = \begin{bmatrix} 1 \\ \frac{-1+\sqrt{3}i}{2} \end{bmatrix}$$

Giving us

$$\vec{x}(t) = c_1 e^{\mu t}[\cos(\nu t)\vec{a} - \sin(\nu t)\vec{b}] + c_2 e^{\mu t}[\sin(\nu t)\vec{a} + \cos(\nu t)\vec{b}]$$

Plugging in the values,

$$y(t) = c_1 e^{-0.5t} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-0.5t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

From the above example, we know that the first component of the eigenvector never has an imaginary part, so in general,

$$y(t) = c_1 e^{\mu t} \cos(\nu t) + c_2 e^{\mu t} \sin(\nu t)$$

Example 1.3.

$$y'' - \frac{y' + 1}{4} y = 0$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix}$$

Where we have $\lambda = \frac{1}{2}$ and $\vec{v} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$. Then generalised eigenvector is

$$(A - \lambda I)\vec{w} = \vec{v}$$

$$\begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \vec{w} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Which gives us

$$y(t) = c_1 e^{0.5t} + c_2 t e^{0.5t}$$

We know that in general,

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} + \begin{bmatrix} \frac{b}{2a} & 0 \\ 0 & \frac{b}{2a} \end{bmatrix} \\ &= \begin{bmatrix} \frac{b}{2a} & 1 \\ -\frac{c}{a} & -\frac{b}{2a} \end{bmatrix} \end{aligned}$$

So we know $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by substitution (or, the second column is equal to $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$).