Lecture 2

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September 19, 2022

1 Definitions and Terminology

Definition 1.1. The order of a differential equation is the order of the highest derivative, ordinary or partial, in the equation.

Example 1.1.

$$ay''' + by'' = f(t)$$

This is a 3$^{\text{rd}}$ order ODE.

Example 1.2.

$$u_t = \frac{d}{dx}(u_x + y_y + u_z)$$

This is a 1$^{\text{st}}$ order PDE.

Definition 1.2. $u^{(n)}$ refers to the n$^{\text{th}}$ derivative of u.

The most general equation of an n$^{\text{th}}$ order derivative ODE is:

$$F[t, u, u', u'', \dots, u^{(n)}] = 0$$

Definition 1.3. We say an n$^{\text{th}}$ order ODE is linear if it can be expressed as

$$a_0(t)u(t) + a_1(t)u'(t) + \cdots + a_n(t)u^{(n)}(t) = g(t)$$

If $g(t) = 0$ for all t, the equation is homogeneous, otherwise it is nonhomogeneous.

Example 1.3. Are the following differntial equations linear? If so, are they homogeneous or not?

- $t\sin(t^3)u + 56u''' + t^4 = 0$

- $t^3u''' + et^2u'' + 4u = 0$

- $tu + 5\sqrt{u'} + 10\sin(u) = 0$

The first equation is linear nonhomogeneous, due to the t^4 term. The second equation is linear homogeneous, and the third equation is non linear, due to the second and third terms.

How could we extend these definitions to PDEs?
Consider

$$a_0(x,y)u_x + a_1(x,y)u_y + a_2(x,y)u_{xx} + a_3(x,y)u_{xy} + a_4(x,y)u_{yx} + a_5(x,y)u_{yy} + a_6(x,y)u = g(x,y)$$

This is a linear equation. If $g(x,y) = 0 \forall x, y$, it is homogeneous.

Definition 1.4. An ODE is *autonomous* if it does not explicitly depend on the independent variable.

Example 1.4.

- $\frac{dy}{dt} = \sin(y) + y^3\ln(y) + e^y$

- $\frac{dy}{dt} = y'\cos(y) + \frac{y^2}{1+e^y}$

Does there exist a linear autonomous nonhomogeneous ODE?
Consider

$$y + y' = 1$$

For it to be nonhomogeneous, a $g(t)$ must exist. It must be independent of t, so it can only be a constant (since $g'(t)$ must be 0).

Definition 1.5. A first order ODE is separable if it can be expressed as

$$\frac{du}{dt} = f(u)g(t)$$

Example 1.5. Which of the following equations are separable?

- $\frac{dx}{dt} = t^2 + \frac{\arccos(t)}{\ln(t)}$

- $u'' = ut^2e^t$

- $\frac{dv}{dt} = tv + t^2$

- $\frac{du}{dt} = \frac{1}{2}\left(\sin(u+t) - \sin(u-t)\right) + \sin(t)$

Yes, no (second order), no, yes (trig identity).

2 Systems of Differential Equations

This occurs when two or more dependent variables interact with one another.

Example 2.1. The Lotka-Volterra equations for a predator-prey model. The assumptions are that the prey will only die when eaten, and predators either naturally die or relocate.

$$u_1(t) : \text{Population count of prey}$$
$$u_2(t) : \text{Population count of predator}$$

Considering the birth and death of prey,

$$\frac{du_1}{dt} = \alpha u_1 - \beta u_1 u_2$$

Considering the birth and death of predators,

$$\frac{du_2}{dt} = \gamma u_1 u_2 - \delta u_2$$

Intuition tells us that this should be a system of periodic functions, with u_2 lagging behind u_1.

3 Initial Value Problems

In general, is one initial value enough?
No, it depends on the order. E.g. a second order ODE requires 2 initial values.

Example 3.1. Consider

$$\frac{d^2u}{dt^2} = 0$$

Integrating both sides with respect to t, we have

$$\frac{du}{dt} = C_0$$

and integrating again gives us

$$u = C_0 t + C_1$$

Therefore, two initial values are required to solve for the two constants.

Definition 3.1. An initial value problem for an n^{th} order ODE consists of the ODE itself and n initial conditions.

$$u(t_0) = a_0, u'(t_0) = a_1, \ldots, u^{(n-1)}(t_0) = a_{n-1}$$