Lecture 4

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1 Applications of Double Integrals

Example 1.1. A rectangular plate of mass m, length L and width W rotate about the y axis lying on the left edge of the plate. Find the moment of intertia of the plate about that line:

1. given that the plate has uniform density

2. given that the density at a point on the plate is directly proportional to the square of the distance from the rightmost side.

$$\begin{aligned}
I_y &= \frac{m}{LW}\iint_R \rho r^2 dydx \\
&= \frac{m}{LW}\int_0^L\int_0^W x^2 dydx \\
&= \frac{m}{LW} int_0^L W x^2 dx \\
&= \frac{m}{LW}\frac{WL^3}{3} \\
&= \frac{mL^2}{3}
\end{aligned}$$

and

$$
\begin{aligned}
I_y &= \iint_R \rho r^2 dydx \\
&= k \int_0^L \int_0^W (1-x)^2 x^2 dydx \\
&= kW \int_0^L x^4 - 2x^3 + x^2 dydx \\
&= kW \left(\frac{L^5}{5} - \frac{L^4}{2} + \frac{L^3}{3} \right) \\
&= \frac{kWL^3(6L^2 - 15L + 10)}{30}
\end{aligned}
$$

Example 1.2. Same plate, with constant density. Calculate its rotation about the centre.

$$
\begin{aligned}
I_0 &= \iint_R \rho r^2 dydx \\
&= \int_{-\frac{W}{2}}^{\frac{W}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 + y^2 dydx \\
&= \int_{-\frac{W}{2}}^{\frac{W}{2}} Lx^2 + \frac{L^3}{12} dx \\
&= \frac{LW^3}{12} + \frac{WL^3}{12} \\
&= \frac{WL(L^2 + W^2)}{12}
\end{aligned}
$$

2 Surface Area

The area can be considered as the sum of small parallelograms projected from the xy plane. The new vectors $\vec{x'}$ and $\vec{y'}$ can be expressed as

$$
\vec{x'} = \Delta x \hat{i} + f_x \Delta x \hat{k}
$$

$$
\vec{y'} = \Delta y \hat{j} + f_y \Delta y \hat{k}
$$

And the area is the magnitude of their cross products, which is the magnitude of

$$-f_x \Delta x \Delta y \hat{i} - f_y \Delta x \Delta y \hat{j} + \Delta x \Delta y \hat{k}$$

which is

$$\sqrt{f_x^2 + f_y^2 + 1} \Delta x \Delta y$$

Hence the surface area is given by

$$S = \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA$$

Example 2.1. Find the surface area of a sphere $x^2 + y^2 + z^2 = a^2$.
Using symmetry, we consider the first octant only.

$$\begin{aligned}
V &= 8 \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA \\
&= 8 \iint_R \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2} dA \\
&= 8 \iint_R \sqrt{\frac{a^2}{a^2 - x^2 - y^2}} dA \\
&= 8a \iint_R \frac{dA}{\sqrt{a^2 - x^2 - y^2}} \\
&= 8a \int_0^{\frac{\pi}{2}} \int_0^a \frac{r dr d\theta}{\sqrt{a^2 - r^2}} \\
&= -8a \int_0^{\frac{\pi}{2}} \sqrt{a^2 - r^2}\Big|_0^a d\theta \\
&= -8a \int_0^{\frac{\pi}{2}} -a d\theta \\
&= 4a^2 \pi
\end{aligned}$$

Where polar coordinates were used to simplify dA.

Example 2.2. Let R be the triangular region $(0,0,0),(0,1,0),(1,1,0)$. Find the surface area of $z = 3x + y^2$ that lies over R.

$$
\begin{aligned}
A &= \int_0^1 \int_0^y \sqrt{1+9+4y^2}dxdy \\
&= \int_0^1 y\sqrt{10+4y^2}dy \\
&= \frac{1}{12}(10+4y^2)^{\frac{3}{2}}\Big|_0^1 \\
&= \frac{1}{12}\left(14^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) \\
&\approx 1.7
\end{aligned}
$$