Lecture 6

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1 Companion Matrix

Let A be a companion matrix. Then A^2 shifts the diagonal of "1"s towards the bottom left by 1. It is easy to see that, then,

$$I, A, A^2, \ldots, A^{n-1}$$

are linearly independent. Since its characteristic equation is of degree n, we know its characteristic equation is its minimal polynomial.

2 Jordan Normal Form

Let $T \in \mathcal{L}(V)$, choose Jordan basis so that A has Jordan normal form. Then the characteristic polynomial is

$$q_T(z) = q_A(z) = \prod_{j=1}^{r} (z - \lambda_j)^{k_j}$$

where k_j is the size of the jth block.

$$p_T(z) = p_A(z) = \prod_{\lambda} (z - \lambda)^{k_\lambda}$$

gives the minimal polynomial, where k_λ is the largest Jordan block for λ.

Example 2.1.

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

Then

$$q_A(z) = (z-2)^5(z+3)$$

and

$$p_A(z) = (z-2)^2(z+3)$$

Comments:
We have $p_T(z) = q_T(z)$ iff there exists unique Jordan blocks for each eigenvalue. In other words, every eigenspace is 1-dimensional.
T is diagonalisable iff all Jordan blocks have a size of 1, iff the minimal polynomial has no repeated roots.
Another version. Suppose $T \in \mathcal{L}(V)$ satisfies $p(T) = 0$ for some non zero polynomial p. Then

- The eigenvalues of T appear among roots of p
- If p has no repeated roots, then T is diagonalisable

Example 2.2.

$$T^2 = T$$

Then its only possible eigenvalues are 0 and 1. Then $p(T) = 0$ for $p(z) = z^2 - z$, and it is diagonalisable.

$$T^2 = I$$

We have $p(z) = z^2 - 1$ which gives a similar result.
In general, if $T^k = I$ then $P(T) = 0$ for $p(z) = z^k - 1$. p has k roots spread uniformly along the unit circle. Hence there are no repeated roots, and T is diagonalisable.

3 Cyclic Subspace Decomposition

($\mathbb{F}$ is any field, not just $\mathbb{C}$), $T \in \mathcal{L}(V)$. A *cyclic subspace* of V is a T-invariant subspace $W \subseteq V$ containing a cyclic vector for $T|_W$, i.e. $v \in W$ where $v, Tv, T^2v, \dots$ span W.

Theorem 3.1. *For all $T \in \mathcal{L}(V)$ there exists a direct sum decomposition*

$$V = V_1 \oplus \cdots \oplus V_r$$

where all V_j are cyclic subspaces. Hence, there exists basis such that matrix of T is A whose diagonals A_j are companion matrices.

Remark:
One can show that every companion matrix A is similar to its transpose

$$A^{-1} = CAC^{-1}$$

Using the theorem, we can see that every matrix is similar to its transpose.
Proof:
Difficulty: in general, T-invariant subspaces don't admint invariant complements.
Let $W \subseteq V$ be a cyclic subspace of largest possible dimension. If $W = V$ we are done. Else construct a T-invariant compound as follows. Let $v \in W$ be a $T|_W$-cyclic vector, so $v_1 = v, v_2 = Tv, \dots, v_k = T^{k-1}v$ is a basis of W. Extend to basis $v_1, \dots, v_m$ of V. Define $f : V \to F$ by

$$f(v_i) = \begin{cases} 0 & i \leq k \\ 1 & i = k \end{cases}$$

Put

$$P : V \to F^k, x \mapsto \begin{pmatrix} f(x) \\ f(Tx) \\ \vdots \\ f(T^{k-1}x) \end{pmatrix}$$

Claim: the kernel of P is a T-invariant complement to W.
Observe $P|_W : W \to F^k$ is an isomorphism.
Continued in next lecture...