Lecture 7

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1 Existence and Uniqueness

Example 1.1.

$$\frac{du}{dt} + \frac{1}{t}u = \frac{1}{t-1}$$

The points of issue are $t = 0, t = 1$, so we want to separate $\mathbb{R}$ into $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ and remove those points.
$u(-1) = 1$:
A unique solution exists. Since g and p are only discountinuous at $t = 0$ and $t = 1$, $(-\infty, 0)$ is the largest interval containing $t_0 = -1$, so by the theorem, it is the domain.
$u(0) = 2$:
This is inconclusive, as the theorem does not tell us anything (p is discontinuous at $t_0 = 0$).
Similarly, a unique solution and corresponding domains exists for $u(2) = 2$ and $u(0.5) = 3$.

2 Cane Toad Example

Assumptions: Cane Toads have no natural predators. Most importantly, they are immortal.
The first model would then be

$$\frac{dp}{dt} = rp$$

Problems with this model

1. Population does not go to infinity in real life, due to limited resources

2. It is expected that population will stabilise eventually at a certain point K

To modify this model, we can change the rate of growth to depend on the current population, i.e.

$$r \to rh(p)$$

We want $h(K) = 0$, and the sign of $h(p)$ to be the opposite of $p - K$. We also want h to be close to 1 when p is small. Our candidate is

$$h(p) = 1 - \frac{p}{K}$$

This gives us

$$\frac{dp}{dt} = r\left(1 - \frac{p}{K}\right)p$$

We want to plot the slope fields in more detail, so we need the points of inflection.

$$\begin{aligned}
\frac{dp}{dt} &= r\left(1 - \frac{p}{K}\right)p \\
\frac{d^2p}{dt^2} &= r\left(p' - \frac{2pp'}{K}\right) \\
&= r^2p\left(1 - \frac{p}{K}\right)\left(1 - \frac{2p}{k}\right)
\end{aligned}$$

So the points of inflection are at $p = 0$, $\frac{p=k}{2}$, $p = k$.
The general solution is

$$
\begin{aligned}
\frac{1}{p}\frac{dp}{dt} &= r\left(\frac{k-p}{k}\right) \\
\frac{dp}{p(k-p)} &= \frac{r}{k}dt \\
-\frac{\ln\left|\frac{k}{p}-1\right|}{k} &= \frac{rt}{k} + C \\
-\ln\left|\frac{k}{p}-1\right| &= rt + C \\
p &= \frac{k}{C'e^{-rt}+1}
\end{aligned}
$$

Putting $p(0) = p_0$, we have $C' = \frac{k}{p_0} - 1$. Substituting into the general solution,

$$p = fracp_0k(k - p_0)e^{-rt} + p_0$$

But in fact, there is a threshold level T where the cane toads will tend towards extinction if population drops below T. Therefore, instead of the rate of change being r, we would like a rate of change of $rg(p)$ where

- $g(p) < 0$ if $p < T$
- $g(p) > 0$ if $p > T$
- $g(p) \approx -1$ if $p \approx 0$

Candidate:

$$g(p) = \frac{p}{T} - 1$$

This gives us our third proposal

$$\frac{dp}{dt} = r\left(\frac{p}{T} - 1\right)p$$

Using a similar approach as above, we can solve for p in terms of its initial condition $p(0) = p_0$

$$p(t) = \frac{p_0 T}{p_0 + (T - p_0)e^{rt}}$$

The problem is, if $p_0 > T$, population explodes to infinity in finite time.

$$\begin{aligned} p_0 + (T - p_0)e^{rt} &= 0 \\ (T - p_0)e^{rt} &= -p_0 \\ e^{rt} &= \frac{p_0}{p_0 - T} \\ rt &= \ln\left|\frac{p_0}{p_0 - T}\right| \\ t &= \ln\left(\frac{p_0}{p_0 - T}\right) \end{aligned}$$

We would therefore like a final model where

$$\frac{dp}{dt} = rq(p)p$$

Where $\exists T < K$ such that

$$\begin{cases} q(p) < 0 & \text{if } p < T \\ q(p) > 0 & \text{if } T < p < K \\ q(p) < 0 & \text{if } p > K \end{cases}$$

A simple solution is

$$\frac{dp}{dt} = r\left(1 - \frac{p}{K}\right)\left(\frac{p}{T} - 1\right)p$$