Homework 5

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1. Given a continuous uniform distribution, show that
   (a) $\mu = \frac{A+B}{2}$
   (b) $\sigma^2 = \frac{(B-A)^2}{12}$

Solution: The mean is

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_A^B \frac{x}{B-A}dx \\
&= \frac{B^2-A^2}{2(B-A)} \\
&= \frac{A+B}{2}
\end{aligned}$$

The variance is

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \\
&= \int_A^B \frac{x^2-(A+B)x+\frac{(A+B)^2}{4}}{B-A}dx \\
&= \frac{B^3-A^3}{3(B-A)} - \frac{(A+B)(B^2-A^2)}{2(B-A)} + \frac{(A+B)^2(B-A)}{4(B-A)} \\
&= \frac{A^2+AB+B^2}{3} - \frac{(A+B)^2}{2} + \frac{(A+B)^2}{4} \\
&= \frac{4A^2+4AB+4B^2-6A^2-12AB-6B^2+3A^2+6AB+3B^2}{12} \\
&= \frac{A^2-2AB+B^2}{12} \\
&= \frac{(B-A)^2}{12}
\end{aligned}$$

2. A bus arrives every 10 minutes at a bus stop. It is assumed that the waiting time for a particular individual is a random variable with a continuous uniform distribution.
   (a) What is the probability that the individual waits more than 7 minutes?
   (b) What is the probability that the individual waits between 2 and 7 minutes?

Solution:

$$P[X > 7] = \frac{10 - 7}{10} = 0.3$$
$$P[2 \le X \le 7] = \frac{7 - 2}{10} = 0.5$$

3. Given a standard normal distribution, find the value of k such that
   (a) $P(Z > k) = 0.2946$
   (b) $P(Z < k) = 0.0427$
   (c) $P(-0.93 < Z < k) = 0.7235$

Solution:

$$0.54, -1.72, 1.28$$

4. The loaves of rye bread distributed to local stores by a certain bakery have an average length of 30 centimeters and a standard deviation of 2 centimeters. Assuming that the lengths are normally distributed, what percentage of the loaves are
   (a) longer than 31.7 centimeters?
   (b) between 29.3 and 33.5 centimeters in length?
   (c) shorter than 25.5 centimeters?

Solution:

$$P(X > 31.7) = P(Z > 0.85) = 19.8\%$$
$$P(29.3 < X < 33.5) = P(-0.35 < Z < 1.75) = 59.7\%$$
$$P(X < 25.5) = P(Z < -2.25) = 1.22\%$$

5. f a set of observations is normally distributed, what percent of these differ from the mean by
   (a) more than 1.3σ?
   (b) less than 0.52σ?

Solution:

$$P(|Z| > 1.3) = 19.4\%$$
$$P(|Z| < 0.52) = 39.7\%$$

6. A coin is tossed 400 times. Use the normal curve approximation to find the probability of obtaining
   (a) between 185 and 210 heads inclusive
   (b) exactly 205 heads
   (c) fewer than 176 or more than 227 heads

Solution:

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{X - 200}{10}$$

Then

$$P(184.5 \leq X \leq 210.5) = P(-1.55 \leq X \leq 1.05) = 0.793$$

$$P(204.5 < X < 205.5) = P(0.45 < X < 0.55) = 0.0352$$

$$P(X < 175.5) + P(X > 227.5) = P(Z < -2.45) + P(Z > 2.75) = 0.0101$$

7. A pair of dice is rolled 180 times. What is the probability that a total of 7 occurs
   (a) at least 25 times?
   (b) between 33 and 41 times inclusive?
   (c) exactly 30 times?

Solution: The probability that a total of 7 occurs is $\frac{1}{6}$. Thus

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{X - 30}{5}$$

Then

$$P(X > 24.5) = P(Z > -1.1) = 0.864$$

$$P(32.5 < X < 41.5) = P(0.5 < Z < 2.3) = 0.298$$

$$P(29.5 < X < 30.5) = P(-0.1 < Z < 0.1) = 0.0797$$

8. Use the gamma function with $y = \sqrt{2x}$ to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Solution:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx \\
&= \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx \\
&= \sqrt{2} \int_0^\infty e^{-\frac{y^2}{2}} dy \\
&= \sqrt{2}\sqrt{\frac{\pi}{2}} \\
&= \sqrt{\pi}
\end{aligned}$$

9. Suppose that the service life, in years, of a hearing aid battery is a random variable having a Weibull distribution with $\alpha = \frac{1}{2}$ and $\beta = 2$.
   (a) How long can such a battery be expected to last?
   (b) What is the probability that such a battery will be operating after 2 years?

Solution: The mean is

$$\mu = \alpha^{-\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right) = \sqrt{2}\Gamma(1.5) = \sqrt{2}\int_0^\infty \sqrt{x}e^{-x}dx = \frac{1}{\sqrt{2}}\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}$$

The probability it operates after 2 years is

$$\begin{aligned} P(X>2) &= \int_2^\infty \alpha\beta x^{\beta-1}e^{-\alpha x^\beta}dx \\ &= \int_2^\infty xe^{-\frac{x^2}{2}}dx \\ &= e^{-2} \end{aligned}$$

10. Let X be a random variable with probability

$$f(x) = \begin{cases} \frac{1}{3} & x = 1,2,3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability distribution of the random variable $Y = 2X - 1$.

Solution:

$$g(y) = \begin{cases} \frac{1}{3} & y = 1,3,5 \\ 0 & \text{elsewhere} \end{cases}$$

11. Let X_1 and X_2 be discrete random variables with joint probability distribution

$$f(x_1, x_2) = \begin{cases} \frac{x_1x_2}{18} & x_1 = 1,2; x_2 = 1,2,3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability distribution of the random variable $Y = X_1X_2$.

Solution:

$$g(y) = \begin{cases} \frac{1}{18} & y = 1 \\ \frac{2}{9} & y = 2, y = 4 \\ \frac{1}{6} & y = 3 \\ \frac{1}{3} & y = 6 \end{cases}$$

12. Let X be a random variable with probability distribution

$$f(x) = \begin{cases} \frac{1+x}{2} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability distribution of the random variable $Y = X^2$.

Solution:

$$
\begin{aligned}
G(y) &= P(Y \leq y) \\
&= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} f(t)dt
\end{aligned}
$$

Then

$$
\begin{aligned}
g(y) &= \frac{d}{dy} G(y) \\
&= f(\sqrt{y}) \times \frac{1}{2\sqrt{y}} - f(-\sqrt{y}) \times \frac{-1}{2\sqrt{y}} \\
&= \frac{1+\sqrt{y}}{2} \times \frac{1}{2\sqrt{y}} + \frac{1-\sqrt{y}}{2} \times \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}
\end{aligned}
$$