Lecture 7

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1 Taylor Series and Approximations for Two-Variable Functions

A first approximation for a function at somewhere near x_0 can simply be

$$f(x_0 + \Delta x) \approx f(x_0)$$

To make this a better approximation, we could consider the *tangent* at x_0, which gives us the slightly more "accurate" approximation

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x$$

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We can then have a quadratic approximation

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2$$

and so on. In general, the n^{th} degree taylor polynomial is given by

$$f(x_0 + \Delta x) = \sum_{i=1}^{n} \frac{1}{i!} f^{(n)}(x_0)\Delta x^n$$

Let us consider the two dimensional case. Let P be the known point, and Q be the point we wish to approximate. Then the parametric equations of line PQ is

$$x(t) = x_0 + \Delta x t$$
$$y(t) = y_0 + \Delta y t$$

where $t \in [0, 1]$.
We then define

$$F(t) = f(x_0 + \Delta x, y_0 + \Delta y)$$

We now have a single variable function $F(t)$. Note that $F(0) = f(x_0, y_0)$ and $F(1) = f(x_0 + \Delta x, y_0 + \Delta y)$. We want to estimate $F(1)$.

$$F'(t) = \frac{d}{dt}F(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

The second derivative is then

$$F''(t) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right) = \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt}\Delta x + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}\Delta x + \frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}\Delta y + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\Delta y$$

If Clairaut's Theorem holds,

$$F''(t) = \frac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\Delta y^2$$

Applying Taylor's approximation on $F(t)$, we have

$$F(1) = \sum_{i=1}^{n} \frac{1}{i!}F^{(i)}(0)$$

where

$$F^{(n)}(0) = \sum_{i=1}^{n} \binom{n}{i} \frac{\partial^n f}{\partial x^i \partial y^{n-i}} \Delta x^i \Delta y^{n-i}$$

which expands to an ugly sum left to the reader as an exercise. The reader can also combine both equations into one (solving for $F(1)$) or touch grass. Note that the first order approximation gives us the tangent plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1.1. Find the 2^{nd} degree polynomial approximation to the function $f(x,y) = \sqrt{x^2+y^3}$ near $(1,2)$.

$$f(1,2) = 3$$

$$f_x = \frac{x}{\sqrt{x^2+y^3}} \to f_x(1,2) = \frac{1}{3}$$

$$f_y = \frac{3y^2}{2\sqrt{x^2+y^3}} \to f_y(1,2) = 2$$

$$f_{xx} = \frac{y^3}{(x^2+y^3)^{\frac{3}{2}}} \to f_{xx}(1,2) = \frac{8}{27}$$

$$f_{xy} = -\frac{3xy^2}{2(x^2+y^3)^{\frac{3}{2}}} \to f_{xy}(1,2) = -\frac{2}{9}$$

$$f_{yy} = \frac{12x^2y+3y^4}{4(x^2+y^3)^{\frac{3}{2}}} \to f_{yy}(1,2) = \frac{2}{3}$$

Substituting into the formula, we have

$$f(x,y) \approx 3 + \frac{1}{3}(x-1) + 2(y-2) + \frac{4}{27}(x-1)^2 - \frac{2}{9}(x-1)(y-2) + \frac{1}{3}(y-2)^2$$

Example 1.2. Find the third order Taylor Expansion of $f(x,y) = e^{x-2y}$ about $(0,0)$.
The formula gives us

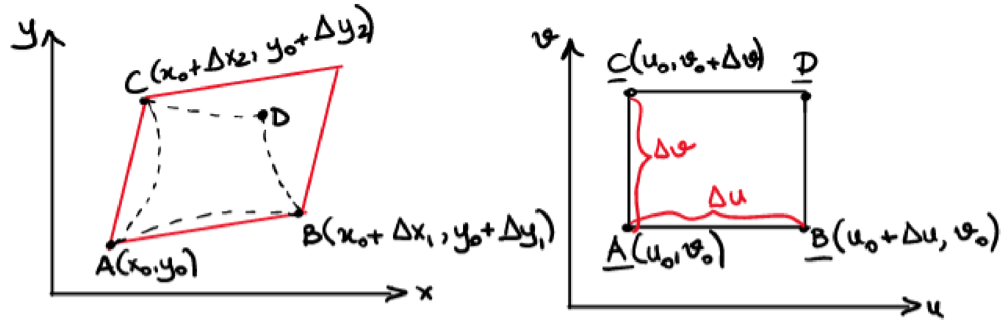
$$f(x,y) = 1 + x - 2y + \frac{1}{2}\left(x^2 - 4xy + 4y^2\right) + \frac{1}{6}\left(x^3 - 6x^2y + 12xy^2 - 8y^3\right)$$

2 Change of Variables

In u substitution, we let u be a function of x to simplify integrations.

Example 2.1.

$$\int_1^3 2x\sqrt{x^2+1}\,dx = \int_2^{10} \sqrt{u}\,du$$



Loosely speaking, we need a $\frac{dx}{dt}$ term to "scale" the integral. Consider 2 different partitions of the same region, one in squares (x and y) and the other in parallelograms (p and q). If we simply convert between $dxdy$ and $dpdq$, we will be off by a scale determined by the ratio between the areas of $||\Delta x \times \Delta y||$ and $||\Delta p \times \Delta q||$ where $\times$ denotes the cross product.

From Fig 2, we let

$$x = g(u, v)$$

and

$$y = h(u, v)$$

We have

$$\Delta x_1 = x_0 + x_1 - x_0 = g(u_0 + \Delta u, v_0) - g(u_0, v_0) = g_u(u_0, v_0)\Delta u$$

Similarly,

$$\begin{aligned}
\Delta x_2 &= g_v(u_0, v_0)\Delta v \\
\Delta y_1 &= h_u(u_0, v_0)\Delta u \\
\Delta y_2 &= h_v(u_0, v_0)\Delta v
\end{aligned}$$

The area in the xy plane is then

$$\begin{aligned}
||AB \times AC|| &= ||(\Delta x_1 \hat{i} + \Delta y_1 \hat{j}) \times (\Delta x_2 \hat{i} + \Delta y_2 \hat{j})|| \\
&= |\Delta x_1 \Delta y_2 - \Delta x_2 \Delta y_1| \\
&= |g_u \Delta u h_v \Delta v - g_v \Delta v h_u \Delta u| \\
&= \left| \det \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix} \Delta u \Delta v \right|
\end{aligned}$$

Definition 2.1. We define the **Jacobian** as

$$J = \left|\det \begin{bmatrix} g_u & g_v \\ h_u & h_v \end{bmatrix}\right| = \left|\det \begin{bmatrix} x_u & x_v \\ y_u & y_u \end{bmatrix}\right| = \frac{\partial(x, y)}{\partial(u, v)}$$

We can then use the Jacobian to change the bases of integrals.