Lecture 14

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1 Midterm Information

The midterm is on Tuesday March 7, 11:00-13:00. Everything until and including unitary operators is covered.

2 Unitary Operators

Definition 2.1. Let V be an inner product space. Then $T \in \mathcal{L}(V)$ is unitary iff

1. T is invertible
2. T is an isometry, i.e.

$$\langle Tv, Tw \rangle = \langle v, w \rangle \forall v, w \in V \Leftrightarrow TT^* = I = T^*T$$

If V is finite dimensional, the second condition implies the first.

Proposition 2.1. *Let $T \in \mathcal{L}(V)$ be unitary, and V be finite dimensional. Then*

- *If $W \subseteq V$ is T-invariant, then $W^{\perp}$ is T-invariant*
- *All eigenvalues of T have an absolute value of 1*
- *Eigenvectors for distinct eigenvalues are orthogonal*

Proof. Since T is invertible, $TW \subseteq W$ means $TW = W$. Then $W = T^{-1}W$, so it is T^{-1} invariant. Let $v \in W^{\perp}$. Then

$$\langle Tv, w \rangle = \langle v, T^* w \rangle = \langle v, T^{-1} w \rangle = 0$$

Then $v \in W^{\perp}$, thus $TW^{\perp} \subseteq W^{\perp}$.
Then, let $v \in V$ be an eigenvector with eigenvalue λ.

$$\langle v, v \rangle = \langle Tv, Tv \rangle = \langle \lambda v, \lambda v \rangle = |\lambda|^2 \langle v, v \rangle$$

Thus the eigenvalue has an absolute value of 1. Similarly, for distinct eigenvalues and eigenvectors, if $\langle v, w \rangle \neq 0$, then

$$\langle v, w \rangle = \langle Tv, Tw \rangle = \langle \lambda v, \mu w \rangle = \lambda \overline{\mu} \langle v, w \rangle \neq \lambda \overline{\lambda} \langle v, w \rangle = \langle v, w \rangle$$

So the eigenvectors are orthogonal by contradiction. □

Theorem 2.1. *Suppose $T \in \mathcal{L}(V)$ is unitary, where V is a complex inner product space with finite dimensions. Then there exists an orthogonal basis of V consisting of eigenvectors of T.*

Proof. By induction, we construct for all $k \leq n$ an orthonormal set $\{v_1, \ldots, v_k\}$ such that $Tv_i = \lambda_i v_i$. Induction starts at $k = 0$. Given $\{v_1, \ldots, v_k\}$, its span is T-invariant, hence $\text{span}\{v_1, \ldots, v_k\}^{\perp}$ is also T-invariant. Taking v_{k+1} to be a unit length eigenvector for the restriction of T to $\text{span}\{v_1, \ldots, v_k\}^{\perp}$ completes the proof. □

Remark:
For $T \in \mathcal{L}(V)$, where V is complex and finite dimensional, the spectrum $\text{Spec}(T)$ is the set of eigenvalues. Then

- If T is self adjoint, $\text{spec}(T) \subseteq \mathbb{R}$
- If T is skew-adjoint, $\text{spec}(T) \subseteq i\mathbb{R}$
- If T is unitary, $\text{spec}(T) \subseteq S^1 \subseteq \mathbb{C}$

3 Normal Operators

Definition 3.1. A operator $T \in \mathcal{L}(V)$ is normal iff

$$TT^* = T^*T$$

Example 3.1. Self adjoint and skew adjoint operators are normal. Unitary maps are also normal.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is normal iff $|b| = |c|, \overline{a}b + \overline{c}d = a\overline{c} + b\overline{d}$. If $\mathbb{F} = \mathbb{R}$, this implies $b = c$, or $b = -c$ and $a = d$.

Properties of normal operators:

1. If T is normal, $\lambda \in \mathbb{F}$, then λT is normal
2. If T is normal, T^{-1} is normal
3. If T is normal, T^k is normal $\forall k \in \mathbb{Z}$
4. If T is normal, $p(T)$ is normal for any polynomial p
5. If T is normal, $p(T)^* = \overline{p}(T^*)$

Proof. For the second property, taking the inverse of both sides of TT^* and T^*T gives

$$(T^*)^{-1} = (T^{-1})^*$$

□