Lecture 8

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February 2, 2023

1 Orthogonal bases and Gram-Schmidt

Consider V, a complex or real inner product space with

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

Recall if $v_1, \ldots, v_n \in V$ are pairwise orthogonal, then they are linearly independent.

Definition 1.1. A basis on an inner product space V is orthonormal if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Example 1.1. The standard basis of $\mathbb{F}^n$ is an orthogonal basis.

Note that any orthogonal basis can be made orthonormal by

$$v_i' = \frac{v_i}{||v_i||}$$

Supposed $v_1, \ldots, v_n$ is an orthonormal basis of V. Then if

$$v = \sum_{i=1}^{n} a_i v_i$$

Its coefficients are

$$a_i = \langle v, v_i \rangle$$

Thus

$$v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$$

Then if $w \in V$

$$\langle v, w \rangle = \sum_{i=1}^{n} \langle v, v_i \rangle \langle v_j, w \rangle$$

A special case is given by

$$||v||^2 = \sum_{i=1}^{n} |\langle v, v_i \rangle|^2$$

So for $v = \sum a_i v_i$ and $w = \sum b_i v_i$

$$\langle v, w \rangle = \sum_{i} a_j \bar{b}_j$$

and

$$||v||^2 = \sum_{i} |a_i|^2$$

Suppose $y_1, \ldots, y_n \in V$ is any basis of V. Then

$$y_k = \sum_{i=1}^{n} \langle y_k, v_i \rangle v_i$$

So the change of basis matrix is given by

$$A_{ij} = \langle y_j, v_i \rangle$$

Consider the standard inner prodct between the jth and kth column.

$$\sum_{i=1}^{n} \langle y_j, v_i \rangle \overline{\langle y_k, v_i \rangle} = \sum_{i=1}^{n} \langle y_j, v_i \rangle \langle v_i, y_k \rangle = \langle y_j, y_k \rangle$$

Where the last equality is shown earlier.
So we see the basis $y_1, \ldots, y_n$ is orthonormal iff the columns of the transformation matrix are orthonormal.

2 Gram-Schmidt

Suppose $y_1, \dots, y_n$ is any basis of an inner product space. Then this can be turned to an orthogonal basis $u_1, \dots, u_n$ (and hence an orthonormal basis) uniquely using the Gram-Schmidt procedure such that

$$\text{span}\{u_1, \dots, u_k\} = \text{span}\{y_1, \dots, y_k\} \forall k \in n$$

and u_k is a linear combination of $y_1, \dots, y_k$ with coefficient of $y_k = 1$. (The second implies the first).
We prove this by induction.
FOr $k = 1$, $u_1 = y_1$ suffices. For the induction step, we put

$$u_{k+1} = y_{k+1} - \sum_{i=1}^{k} \text{proj}_{u_i}(y_{k+1})$$

Now for $j \leq k$,

$$\begin{aligned}
\langle u_{k+1}, u_j \rangle &= \langle y_{k+1}, u_j \rangle - \langle \text{proj}_{u_j}(y_{k+1}), u_j \rangle \\
&= \langle y_{k+1}, u_j \rangle - \left\langle \frac{\langle y_{k+1}, u_j \rangle}{||u_j||^2} u_j, u_j \right\rangle \\
&= \langle y_{k+1}, u_j \rangle - \frac{\langle y_{k+1}, u_j \rangle}{||u_j||^2} \langle u_j, u_j \rangle \\
&= 0
\end{aligned}$$

Uniqueness: suppose $y_{k+1} + a_i u_i + \cdots + a_k u_k$ is orthogonal to u_j, $j \leq k$. Then

$$0 = \langle y_{k+1} + a_1 u_1 + \cdots + a_k u_k, u_j \rangle = \langle y_{k+1}, u_j \rangle + a_j ||u_j||^2$$

So

$$a_j = -\frac{\langle y_{k+1}, u_j \rangle}{||u_j||^2}$$

This produces an orthogonal basis $u_1, \dots, u_n$, and an orthonormal basis can easily be obtained. We can also normalise each step in the process.

Remark: The fact that $\text{span}\{u_1, \dots, u_n\} = \text{span}\{y_1, \dots, y_n\}$ for all k means the change of basis matrix is upper triangular. The diagonals are 1 by construction. Therefore, it has a determinant of 1.

Example 2.1. Let $V = \mathcal{P}(\mathbb{R})$ be the real polynomials with

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx$$

Then $\mathcal{P}(\mathbb{R})$ has the standard basis $p_i(x) = x^i$ (starts from 0). The Gram-Schmidt process gives us the orthogonal basis $q_0, q_1, \ldots$.

$$q_0(x) = 1$$

Then

$$q_1 = p_1 - \mathrm{proj}_{q_0}(p_1) = x - \frac{1}{2}$$

$$q_2 = p_2 - \mathrm{proj}_{q_0}(p_2) - \mathrm{proj}_{q_1}(p_2) = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

Remarks: the more standard convention is to normalise it such that the constant is ± 1. This gives the shifted Legendre polynomials.

$$P_0(x) = 1, P_1(x) = 2x - 1, P_2(x) = 6x^2 - 6x + 1, P_3(x) = 20x^3 - 30x^2 + 12x - 1, \ldots$$