Lecture 27

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December 13, 2022

1 Fun Functions

Definition 1.1. The Heaviside step function is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

This can be shifted to form

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

It can be used to turn things "on" and "off". If we want a function that turns on from c to d, ie

$$u_{cd}(t) = \begin{cases} 1 & t \in [c, d) \\ 0 & \text{else} \end{cases}$$

it is trivial that

$$u_{cd}(t) = u_c(t) - u_d(t)$$

A triangular pulse can be drawn as e.g.

$$h(t) = (t-1)u_{12}(t) + (3-t)u_{23}(t)$$

where u is defined as above. Expanding yields

$$h(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$$

The Laplace of the step function is

$$\begin{aligned}\mathcal{L}\{u_c(t)\}(s) &= \int_0^\infty e^{-st}u_c(t)dt \\ &= \int_c^\infty e^{-st}dt \\ &= \frac{1}{s}e^{-sc}, s > 0\end{aligned}$$

Similarly,

$$\mathcal{L}\{u_{cd}(t)\}(s) = \frac{1}{s}\left(e^{-sc} - e^{-sd}\right)$$

Note: if we shift a function defined only on $[0,\infty)$ by c to the right, we assume the function has a value of 0 for $t \in [0,c)$. In other words, the function f shifted by c becomes

$$g(t) = u_c(t)f(t-c)$$

We sometimes do this even if f is defined for negative numbers.

Theorem 1.1. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a nonnegative constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), s > a$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

Proof:

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\ &= \int_c^\infty e^{-st}f(t-c)dt \\ &= \int_0^\infty e^{-s(\tau+c)}f(\tau)d\tau \\ &= e^{-sc}\int_0^\infty e^{-s\tau}f(\tau)d\tau \\ &= e^{-sc}F(s)\end{aligned}$$

Example 1.1. Find the Laplace Transform of the triangular pulse.

$$\begin{aligned}\mathcal{L}\{h(t)\} &= \mathcal{L}\{(t-1)u_1(t)\} - 2\mathcal{L}\{(t-2)u_2(t)\} + \mathcal{L}\{(t-3)u_3(t)\} \\ &= e^{-s}\mathcal{L}\{t\} - 2e^{-2s}\mathcal{L}\{t\} + e^{-3s}\mathcal{L}\{t\} \\ &= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2}\end{aligned}$$

Definition 1.2. A function f is periodic with period $T > 0$ if

$$f(t+T) = f(t) \forall t$$

To observe the 1st period, we define the window function to be

Definition 1.3. The window function $f_T(t)$ is

$$f_T(t) = f(t)[1 - u_T(t)]$$