Lecture 11

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1 Recap

For $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we have an inner product $\langle , \rangle$ where

- It is linear in the first argument
- $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- $\langle v, v \rangle \geq 0$
- $\langle v, v \rangle = 0$ iff $v = 0$

For the inner product spaces V and W with finite dimensions, we have $T \in \mathcal{L}(V, W)$

2 Adjoints

Theorem 2.1. *There exists a unique $T^* : W \to V$ such that*

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V$$

Recall that for $f \in V^*$, $\exists v^* \in V$ such that

$$f(v) = \langle v, v^* \rangle$$

Proof. Fixing $w \in W$, define

$$\phi(v) = \langle T(v), w \rangle_W \in \mathbb{F}$$

Then $\exists! v^* \in V$ such that
$$\phi(v) = \langle v, v^* \rangle_V$$
This gives $T^* : W \to V, w \mapsto v^*$ and
$$\langle T(v), w \rangle_W = \langle v, v^* \rangle_V = \langle v, T^*(w) \rangle_V$$
To show that T^* is linear, not that
$$\begin{aligned}
\langle v, T^*(kw_1 + w_2) \rangle_V &= \langle T(v), kw_1 + w_2 \rangle_W \\
&= \overline{k} \langle T(v), w_1 \rangle_W + \langle T(v), w_2 \rangle_W \\
&= \overline{k} \langle v, T^*(w_1) \rangle_V + \langle v, T^* w_2 \rangle_V \\
&= \langle v, kT^*(w_1) + T^* w_2 \rangle_V
\end{aligned}$$
Then setting v to be all the unit vectors in V completes the proof. $\square$

Let $V = \mathbb{C}^n, W = \mathbb{C}^m, T \in \mathcal{L}(V, W)$ and A be the matrix of T. Then
$$\begin{aligned}
\langle T(v), w \rangle &= (Av)^t \overline{w} \\
\langle v, T^* w \rangle &= v^t A^t \overline{w} \\
v^t \overline{T^* w} &= v^t \overline{\overline{A^t} w}
\end{aligned}$$

Hence
$$T^* = \overline{A^t}$$
Note that this is only when we are using an orthonormal basis on both sides.

2.1 Infinite Dimensions

Let V be the set of all complex continuous functions on $\mathbb{R}$ where
$$f(x + 2\pi) = f(x) \forall x \in \mathbb{R}$$
Define
$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt$$

Then

$$\begin{aligned}
\langle T(f), g\rangle &= \int_{-\pi}^{\pi} f'(x)\overline{g(x)}dx \\
&= f(x)\overline{g(x)}|_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x)\overline{g'(x)}dx \\
&= \langle f, -g'\rangle
\end{aligned}$$

So $T^* = -T$.

2.2 Properties

- $(T^*)^* = T$
- $(T_1 + T_2)^* = T_1^* + T_2^*$
- $(\lambda T)^* = \overline{\lambda} T^*$
- $I^* = I$
- $(ST)^* = T^* S^*$

Proof. For the first property,

$$\langle w, T(v)\rangle_W = \langle T^*(w), v\rangle_V = \langle w, (T^*)^* v\rangle_W$$

The second and third can be proven similarly, using linearity. The fourth is trivial. Finally, let $T : V \to W, S : W \to X$. Then

$$\langle STv, x\rangle_X = \langle Tv, S^*x\rangle_W = \langle v, T^*S^*x\rangle_V$$

$\square$

Theorem 2.2.

$$\begin{aligned}
range(T^*) &= null(T)^{\perp} \\
range(T) &= null(T^*)^{\perp} \\
rank(T) &= rank(T^*)
\end{aligned}$$

Proof.

$V \in \text{null}(T) \Leftrightarrow \langle T(v), w\rangle = 0 \forall w \in W \Leftrightarrow \langle v, T^*(w)\rangle = 0 \forall w \in W \Leftrightarrow v \in \text{range}(T^*)^\perp$

Assuming finite dimensions, taking the $^\perp$ on both side yields the desired equality. Letting $S = T^*$ and noting $S^* = T$ gives the second equality.

$$\text{dim}V = \text{dim range}(T^*) + \text{dim null}T = \text{dim}V - \text{dim null}(T^*) + \text{dim null}T$$

so the nullspaces of T and T^* have the same dimension, and they are of the same rank. □

Corollary 2.1.

$$\text{null}T = 0 \Leftrightarrow \text{range}(T^*)^\perp = 0 \Leftrightarrow \text{range}(T^*) = V$$

So T is injective iff T^* is surjective, and vice versa.

2.3 Projections

Consider a projection P which gives the direct sum

$$V = X \oplus Y$$

where V is an inner product space. Then

$$P^* = (P^2)^* = P^* P^*$$

so P^* is also a projection.
If P is an orthogonal projection,

$$\langle Pv_1, v_2\rangle = \langle x_1, x_2 + y_2\rangle = \langle x_1, x_2\rangle = \langle v_1, x_2\rangle = \langle v_1, Pv_2\rangle$$

So $P = P^*$. Moreover, the converse also holds.

$$\langle Pv_1, v_2\rangle = \langle v_1, Pv_2\rangle$$
$$\langle x_1, x_2 + y_2\rangle = \langle x_1 + y_1, x_2\rangle$$
$$\langle x_1, y_2\rangle = \langle y_1, x_2\rangle$$

And if any x_1, y_2 pair has a non zero inner product, replacing y_1 with $2y_1$ produces a contradiction. Hence X and Y are orthogonal.

Theorem 2.3. *Suppose V is a finite dimensional inner product space, and $T \in \mathcal{L}(V)$. Then*

$$T = T^* \Rightarrow T \textit{ is diagonalisable}$$

We will prove this next time (hopefully).