Lecture 18

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1 Singular Value Decomposition

Recall the singular values of $T \in \mathcal{L}(V, W)$ are the eigenvalues of $\sqrt{T^*T} \in \mathcal{L}(V)$, with V, W being finite dimensional inner product spaces.

$$V = \text{null}(T) \oplus \text{ran}(T^*)$$

$$W = \text{null}(T^*) \oplus \text{ran}(T)$$

T restricts to an isomorphism on $\text{ran}(T^*) \to \text{ran}(T)$, and

$$\text{null}(T^*T) = \text{null}(T)$$

the same holds for T^* (use $S = T^*$ and consider how the above holds for S). In the homework problems, we showed that T gives an isomorphism from $E(\lambda, T^*T) \to E(\lambda, TT^*)$. We see that

$$v \in E(\lambda, T^*T) \Rightarrow Tv \in (E\lambda, TT^*)$$

because

$$TT^*(Tv) = T(T^*Tv) = T(\lambda v) = \lambda Tv$$

We can use this to get the normal form for T. Pick an orthonormal basis $v_1, \ldots, v_n$ of $\text{ran}(T^*)$ consisting of eigenvectors of T^*T, with eigenvalues $s_1^2, \ldots, s_n^2$, the strictly positive singular values.

Lemma 1.1. *The vectors $w_i = \frac{1}{s_i}Tv_i$ form an orthonormal basis of ran(T), consisting of eigenvectors of TT^* and eigenvalues s_i^2.*

Proof.

$$\begin{aligned}
\langle w_i, w_j \rangle &= \frac{1}{s_i s_j} \langle T v_i, T v_j \rangle \\
&= \frac{1}{s_i s_j} \langle v_i, T^* T v_j \rangle \\
&= \frac{s_j}{s_i} \langle v_i, v_j \rangle \\
&= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}
\end{aligned}$$

We have already checked that T takes $E(s^2, T^*T)$ to $E(s^2, TT^*)$. □

Rearranging yields

$$T v_i = s_i w_i$$

so

$$T(v) = \sum_{i=1}^{k} s_i \langle v, v_i \rangle w_i$$

Theorem 1.1 (Singular Value Decomposition)**.** *Let V, W be finite dimensional inner product spaces. Then any $T \in \mathcal{L}(V, W)$ can be written as*

$$T(v) = \sum_{i=1}^{k} s_i \langle v, v_i \rangle w_i$$

where $v_1, \dots, v_k$ and $w_1, \dots, w_k$ are orthonormal sets of vectors, and $s_i > 0$. In the expression, s_i are the singular values of T, v_i are eignevectors of T^*T for s_i^2 and w_i are eignevectors of TT^* for s_i^2.

Proof. Given T, we have shown how to find such a decomposition. We take s_i, the singular values, v_i, the eigenvalues of T^*T, and put $w_i = \frac{1}{s_i} T v_i$. Conversely, given the expression in the theorem,

$$\begin{aligned}
\langle T^* w_i, v_j \rangle &= \langle w_i, T v_j \rangle \\
&= s_j \langle w_i, w_j \rangle \\
&= \begin{cases} 0 & i \neq j \\ s_i & i = j \end{cases}
\end{aligned}$$

To show that T^*w is a multiples of v_i, in fact $T^*w_i = s_i v_i$. Then this gives

$$T^*T(v_i) = T^*(s_i w_i) = s_i^2 v_i$$

$$TT^*(w_i) = T(s_i v_i) = s_i^2 w_i$$

$\square$

Note that we do not need to compute $\sqrt{T^*T}$, and the decomposition is not really unique, as it is dependent on a choice of unit eigenvectors. In the case of normal operators T, we can take v_i to be the unit eigenvectors of T, eigenvalue λ. Singular values are $s_i = |\lambda_i|$ and

$$w_i = \frac{1}{s_i} T v_i = \frac{\lambda_i}{|\lambda_i|} v_i$$

Indeed, singular value decomposition is essentially spectral resolution.

2 Singluar Value Decomposition For Matrices

Suppose $A \in M_{n\times n}(\mathbb{F})$ is invertible, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then all singular values are strictly positive. Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors of A^*A for the eigenvalues $s_1^2, \ldots, s_n^2$. Let $w_i = \frac{1}{s_i} A v_i$. Let U_1 be the unitary matrix having $v_1, \ldots, v_n$ as columns, and similarly U_2 be the unitary matrix having $w_1, \ldots, w_n$ as columns. Let D be a matrix such that $D_{ij} = \delta_{ij} s_i$. Then $Av_i = s_i w_i$ means

$$AU_1 = U_2 D$$

or

$$A = U_2 D U_1^{-1}$$

Theorem 2.1 (Singular Value Decomposition for Invertible Matrix)**.** *Every invertible $A \in M_{n\times n}(\mathbb{F})$ can be written as*

$$A = U_2 D U_1^{-1}$$

where U_1, U_2 are unitary, and D is diagonal with strictly positive entries.

Note that this is not unique.

Example 2.1. Find the singular value decomposition of

$$A = \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix}$$

We have

$$A^*A = \begin{pmatrix} 4 & 3 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} 25 & -15 \\ -15 & 25 \end{pmatrix}$$

with the eigenvalues $s_1^2 = 40, s_2^2 = 10$. The eigenvectors are

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

So

$$w_1 = \frac{1}{2\sqrt{10}} \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$w_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 4 & 0 \\ 3 & -5 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So

$$A = U_2 D U_1^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \sqrt{10} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

If you have the SVD, you also get polar decomposition for free.

$$A = U_2 D U_1^{-1} = (U_2 U_1^{-1})(U_1 D U_1^{-1}) = UR$$

where the first matrix is unitary and the second is positive if you look hard enough.

More generally, consider $A \in M_{m\times n}(\mathbb{F})$. THink of it as $A : \mathbb{F}^n \to \mathbb{F}^m$. As before pick an orthonormal basis $v_1, \dots, v_k$ of $\operatorname{ran}(A^*) = \operatorname{null}(A)^\perp$, and let $w_i = \frac{1}{s_i} A v_i$ where s_i denote the singular values. Extend $v_1, \dots, v_k$ to an orthonormal basis $v_1, \dots, v_n$, and extend $w_1, \dots, w_k$ similarly to $w_1, \dots, w_m$. Let $U_1 \in M_{n\times n}(\mathbb{F})$ have $v_1, \dots, v_n$ as columns, and $U_2 \in M_{m\times m}(\mathbb{F})$ have $w_1, \dots, w_m$ as columns. Let $D \in M_{m\times n}(\mathbb{F})$ be the matrix with

$$D_{ij} = \begin{cases} s_i & i = j \le k \\ 0 & \text{else} \end{cases}$$

Then

Theorem 2.2 (Singular Value Decomposition for Non Square Matrices). *Every $A \in M_{m\times n}(\mathbb{F})$ can be written as*

$$A = U_2 D U_1^{-1}$$

where $U_1 \in M_{n\times n}(\mathbb{F})$, $U_2 \in M_{m\times m}(\mathbb{F})$ are unitary, and $D \in M_{m\times n}(\mathbb{F})$ has only nonzero entries at $D_{ii} = s_i$, the strictly positive singular values.

Example 2.2.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$A^* = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

with singular values $s_1 = 1, s_2 = \sqrt{2}$ and eigenvectors $\begin{pmatrix} 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix}$. Now

$$w_1 = \frac{1}{s_1} A v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, w_2 = \frac{1}{s_2} A v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We complete the orthonormal basis by

$$w_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Then

$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$