Lecture 11

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1 Behaviour of System: One Positive One Negative Eigenvalue

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

Solving for the eigenvalues and eigenvectors, the general solution is

$$\vec{\phi}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

As $t \to \infty$, the solution would behave like $c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and as $t \to -\infty$, it would behave like $c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. In both cases, the solution diverges. Considering the first term only, the vector lies on a straight line from the origin to top right. Considering the second term only, we get a straight line from bottom right to the origin. The general solution is therefor a combination of both vectors. For $c_1, c_2 > 0$, it goes from bottom right (negative values) to $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $(t = 0)$ to top right (positive values). The other cases are drawn similarly.

Consider

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix} \vec{x}$$

The general solution is

$$\vec{\phi}(x) = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Since the first term is a constant, we get a parallel series of lines from top left to bottom right which point inwards to the origin.

2 Complex Eigenvalues

If the characteristic polynomial

$$\lambda^2 + b\lambda + c$$

gives us complex roots, ie

$$b^2 < 4c$$

$$\begin{aligned}
A\vec{v_1} &= \lambda_1 \vec{v_1} \\
\overline{A\vec{v_1}} &= \overline{\lambda_1 \vec{v_1}} \\
A\vec{v_2} &= \lambda_2 \vec{v_2}
\end{aligned}$$

So λ_1 and λ_2 are conjugates of each other. Letting $\lambda_1 = \mu + i\nu$ and $\vec{v_1} = \vec{a} + i\vec{b}$, we have

$$\begin{aligned}
\vec{\phi_1}(t) &= e^{\mu t} e^{i\nu t}(\vec{a} + i\vec{b}) \\
&= e^{\mu t}(\cos(\nu t) + i\sin(\nu t))(\vec{a} + i\vec{b})
\end{aligned}$$

Which gives us

$$\vec{\phi_1}(t) = \vec{u}(t) + i\vec{w}(t)$$

where

$$\vec{u}(t) = e^{\mu t}(\vec{a}\cos(\nu t) - \vec{b}\sin(\nu t))$$

and

$$\vec{w}(t) = e^{\mu t}(\vec{a}\sin(\nu t) + \vec{b}\cos(\nu t))$$

In fact, $\vec{u}$ and $\vec{w}$ are linearly independent solutions to the ODE! Readers can verify this an an exercise. (Hint: they are linearly independent as the Wronskian is given by the determinant of $\vec{v_1}\vec{v_2}$, which are conjugates).

Consider the following system

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 8 & -2 \end{pmatrix} \vec{x}$$

The eigenvalues and eigenvectors are

$$\begin{aligned} \lambda_1 &= 6i \\ \vec{v_1} &= \begin{pmatrix} 5 \\ 2-6i \end{pmatrix} \\ \lambda_2 &= -6i \\ \vec{v_2} &= \begin{pmatrix} 5 \\ 2+6i \end{pmatrix} \end{aligned}$$

Using our previous notation,

$$\vec{u}(t) = \left(\cos(6t) \begin{pmatrix} 5 \\ 2 \end{pmatrix} - \sin(6t) \begin{pmatrix} 0 \\ -6 \end{pmatrix} \right)$$

and

$$\vec{w}(t) \left(\sin(6t) \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \cos(6t) \begin{pmatrix} 0 \\ -6 \end{pmatrix} \right)$$