Lecture 15

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1 Euler's Method

If we partition $[t_0, T]$ such that the distance between adjacent points are constant, we can let that distance be $h = t_{n+1} - t_n$, and approximations are given by

$$y_{n+1} = y_n + hf(t_n, y_n)$$

1.1 Euler's Method as an Integral Approximation

$$\begin{aligned} y'(t) &= f(y,t) \\ \int_{t_n}^{t_{n+1}} y'(t)dt &= \int_{t_n}^{t_{n+1}} f(y,t)dt \\ y(t_{n+1}) - y(t_n) &= \int_{t_n}^{t_{n+1}} f(y,t)dt \\ y(t_{n+1}) &\approx y(t_n) + hf(y_n, t_n) \end{aligned}$$

where we use the approximation

$$f(y,t) \approx f(y_n, t_n)$$

in the range $[t_n, t_{n+1}]$. More formally, we define

$$y_{n+1} = y_n + hf(y_n, t_n)$$

and use the approximation

$$y(t_{n+1}) \approx y_{n+1}$$

1.2 Euler's Method as an Integral Approximation

Assuing y has a Taylor series, we can approximate y using its first order Taylor polynomial

$$y(t) \approx y(t_n) + y'(t_n)(t - t_n)$$

which is equivelant to Euler's Method.

1.3 Improving Euler's method

If we treat it as a forward difference quotient, this can be improved by taking a better approximation of the derivative, which gives us the **Runge-Kutta method**.
If we treat it as an integral approximation, we can use better integral approximations, which gives us the **Improved Euler Method**.
If we treat it as a Taylor Polynomial, we can improve it by taking more terms, which gives us the **Power Series Solution**.

2 Sources of Errors

2.1 Global Truncation Error

$$E_n = y(t_n) - y_n$$

The error stacks, as we use y_n and not $y(t_n)$ for our next approximation.

2.2 Local Truncation Error

When calculating e_{n+1}, we use $y(t_n)$ instead of y_n to calculate the error. This is the error due to linear approximation.

2.3 Error Bounding

Taylor's Remainder Theorem is

$$y(t) = y(t_n) + y'(t_n)(t - t_n) + \frac{y''(\xi)}{2}(t - t_n)^2$$

where $t \in [t_n, t_n + h]$. To calculate for e_{n+1}, we substitute $t = t_{n+1}$ to get

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + y'(t_n)(t_{n+1} - t_n) + \frac{y''(\xi)}{2}(t_{n+1} - t_n)^2 \\ y(t_{n+1}) &= y_{t_{n+1}} + \frac{y''(\xi)}{2}h^2 \\ |e_{n+1}| &= \frac{y''(\xi)}{2}h^2 \\ &= \frac{M}{2}h^2 \end{aligned}$$

where M is chosen such that

$$|y''(t)| \leq M \forall t \in [t_n, t_{n+1}]$$

$$\begin{aligned} y' &= f(t, y) \\ y'' &= f_t(t, y(t)) + f_y(t, y(t)) \times y'(t) \\ &= f_t(t, y) + f_y(t, y) f(t, y) \end{aligned}$$

What remains is to bound this expression.
Our assumptions are that y is twice continuously differentiable, and f_t, f_y, f are continuous functions.
Even if we don't have access to the solution, a bound may still be obtained.

Example 2.1.

$$y'(t) = \arctan(y) + e^{-t}, y(0) = 1$$

for $t \in [0, 4]$. Then

$$\begin{aligned} f &= \arctan(y) + e^{-t} \\ f_t &= -e^{-t} \\ f_y &= \frac{1}{1 + y^2} \end{aligned}$$

As the arctan function is bounded, ad e^{-t} is obviously bounded in the region, f is bounded, f_t is bounded, and f_y is bounded (it is always smaller than or equal to 1). Therefore $\exists$ an upper bound M.