Lecture 7

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1 Divergence and Curl of a Vector Field

Definition 1.1. The divergence is defined as

$$\vec{\nabla} \cdot \vec{D} = \lim_{\Delta S \to 0} \frac{\oiint_S \vec{D} \cdot d\vec{S}}{\Delta S}$$

This is equivalent to our definition of $\vec{\nabla}$ in Cartesian coordinates by considering the flux out of one face of an infitesimal cube and applying symmetry.

Definition 1.2. The curl is defined as

$$\vec{\nabla} \times \vec{E} = \lim_{\Delta S \to 0} \frac{\left(\hat{a}_n \oint_C \vec{E} \cdot d\vec{l}\right)_{\max}}{\Delta S}$$

where $\hat{a}_n$ is chosen to maximise the expression.
In "proving" Gauss' Law, we assumed

1. Field had specific components
2. Gaussian surface was chosen so that $\vec{D} \cdot d\vec{S} = 0$ or $\vec{D} \cdot d\vec{S} = DdS$

This helps us determine the different regions where Gauss' Law is applied separately, and what type of Gaussian surface is needed. The surface has to satisfy

1. S has to be closed (cube, cylinder, sphere)
2. S has to be oriented such that $\vec{D} \cdot d\vec{S} = 0$ and/or $\vec{D} \cdot d\vec{S} = DdS$ over different points of the surface

3. Over points of S where $\vec{D} \cdot d\vec{S} = DdS$, we need $|\vec{D}|$ to be a constant

Example 1.1. Consider the charge density given by ρ_v at $R \leq a$ (and 0 everywhere else), which is a constant. Find the electric flux density everywhere.
In the region $R \leq a$, we get

$$\oiint_S \vec{D} \cdot d\vec{S} = D_R A = D_R(4\pi R^2)$$

while

$$Q = \iiint \rho_v dV = \rho_v \left(\frac{4}{3}\pi R^3\right)$$

Letting both sides be equal gives

$$\vec{D} = \frac{\rho_v R}{3}\hat{a}_R$$

For everywhere else, i.e. $R > a$, we obtain the same expression on the left hand side, and

$$Q = \rho_v \left(\frac{4}{3}\pi a^3\right)$$

So

$$\vec{D} = \frac{\rho a^3}{3R^2}\hat{a}_R = \frac{Q}{4\pi R^2}\hat{a}_R$$