Lecture 26

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November 15, 2022

1 Applications of the Laplac Transform

Example 1.1.

$$y'' + 2y' + 5y = e^{-t}$$

Then

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

from last lecture. What remains is to "undo" the transform to give us $y(t)$.

Limitations of the Laplace Transform: we need the initial values, and we need to transform to be defined in the first place.

Example 1.2. Consider the Laplace Transform of both functions

$$f(t) = 0, g(t) = \begin{cases} 0 & t \neq 5 \\ 1 & t = 5 \end{cases}$$

However, the Laplace Transform of both are equal to 0. It is not injective!

For the cases above, the "nicer" functions can be picked, as it is differentiable, so this is not a big issue.

Theorem 1.1. *If $f(t)$ and $g(t)$ are piecewise continuous and of exponential order on $[0, \infty)$ and $f = G$ where $F = \mathcal{L}\{f\}$ and $G = \mathcal{L}\{G\}$ then $f(t) = g(t)$ at all points where both f and g are continuous.*

Then if f and g differ on an interval of positive length, there is a subinterval where both f and g are continuous, which leads to a contradiction.

Definition 1.1. If $f(t)$ is a piecewise continuous and of exponential order and $\mathcal{L}\{f(t)\} = F(s)$, then we call f the inverse Laplace Transform of F, and denote it by

$$f = \mathcal{L}^{-1}\{F\}$$

Where any 2 inverses differ at most at finite points.

Theorem 1.2. *If $f_1 = \mathcal{L}\{F_1\}$ and $f_2 = \mathcal{L}\{F_2\}$ are piecewise continuous and of exponential order, then for any constants c_1 and c_2,*

$$\mathcal{L}^{-1}\{c_1F_1 + c_2F_2\} = c_1\mathcal{L}^{-1}\{F_1\} + c_2\mathcal{L}^{-1}\{F_2\} = c_1f_1 + c_2f_2$$

There is a formula for inverse Laplace Transforms, namely

$$\mathcal{L}^{-1}\{F\} = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds$$

In practice a lookup table and partial fractions are easier.

Example 1.3. In the previous example,

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

Decomposing into partial fractions,

$$\begin{aligned}\frac{s^2}{(s+1)(s^2+2s+5)} &= \frac{s^2}{(s+1)[(s+1)^2+4]}\\ &= \frac{A}{s+1} + \frac{B(s+1)}{(s+1)^2+4} + \frac{2C}{(s+1)^2+4}\end{aligned}$$

This gives $A = \frac{1}{4}, B = \frac{3}{4}, C = -1$, so

$$Y(s) = \frac{1}{4(s+1)} + \frac{3(s+1)}{4[(s+1)^2+4]} - \frac{2}{(s+1)^2+4}$$

From the lookup table,

$$y(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{-t}\cos(2t) - e^{-t}\sin(2t)$$