Lecture 18

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1 Superposition Principle

A: $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ is linearly dependent iff $\exists t_0 \in I$ such that the only solution for

$$\sum_i c_i \vec{x_i} = 0$$

is $c_i = 0$
B: $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}$ is linearly dependent iff $\forall t_0 \in I$, the only solution for

$$\sum_i c_i \vec{x_i} = 0$$

is $c_i = 0$

Theorem 1.1. *Let $\vec{x_1}(t), \vec{x_2}(t), \ldots, \vec{x_n}(t)$ be solutions to*

$$\frac{d\vec{x}}{dt} = P(t)\vec{x}$$

Where $P(t) \in \mathbb{R}^{n \times n}$ is continuous with respect to t on an interval of I. Then

$$W[\vec{x_1}(t), \vec{x_2}(t), \ldots, \vec{x_n}(t)] \neq 0$$

iff $\vec{x_1}(t), \vec{x_2}(t), \ldots, \vec{x_n}(t)$ are linearly independent.

Proof:
$\Leftarrow$:

Proof by contradiction. If the Wronskian is zero, the column vectors $\vec{x_i}(t_0)$ are linearly independent, so $\exists$ nontrivial c_i such that

$$\sum_i c_i \vec{x_i}(t_0) = 0$$

Define

$$\vec{y}(t) = \sum_i c_i \vec{x_i}(t)$$

Since it is a nontrivial linear combination of linearly independent vectors, $\vec{y}(t)$ cannot be the 0 function. Plugging this into the system (as $\vec{y}(t)$ is a solution by superposition), we realise this is a solution for the initial value $\vec{x}(t_0) = 0$. However, the 0 function is also a solution, so this contradicts uniqueness.
$\Rightarrow$:
Proof by Contradiction: if the vectors are not linearly independent, the Wronskian would obviously be 0 (using the same linear operation on the determinant would yield a 0 column).

Definition 1.1. The fundamental matrix is defined as

$$X(t) = \begin{bmatrix} \vec{x_1}(t) & \vec{x_2}(t) & \dots & \vec{x_n}(t) \end{bmatrix}$$

where $\vec{x_i}(t)$ form a basis of the solution space.

Then

$$P(t)X(t) = X'(t)$$

Definition 1.2. Given $\frac{d\vec{x}}{dt} = P(t)\vec{x}$ we call $X(t)$ the special fundamental matrix for $t_0 \in I$ if $X(t)$ is a fundamental matrix and $X(t_0) = \mathbb{I}$.

If $P(t) = A$, then $X'(t) = AX(t)$, so through the magical power of abuse of notation, the solution is obviously

$$X(t) = e^{At}$$

where the definition for the exponention of a matrix is

Definition 1.3.

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

This converges $\forall A, t$.

Then

$$\begin{aligned}
X'(t) &= \frac{d}{dt}\left[e^{At}\right] \\
&= \frac{d}{dt}\left[\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{d}{dt}\frac{A^k}{k!}t^k\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{A^k}{k!}kt^{k-1}\right] \\
&= A\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k \\
&= Ae^{At} \\
&= AX(t)
\end{aligned}$$

A general solution is then given by

$$\vec{x}(t) = X(t)\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Substituting the initial condition,

$$\vec{x}(t_0) = X(t_0)\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{x_0}$$

As $X(t_0) = \mathbb{I}$. Therefore

$$x\vec{(t)} = X(t)\vec{x_0} = e^{At}\vec{x_0}$$