Lecture 7

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1 Inner Product Spaces

For field $\mathbb{F}$, define dot product on F^n

$$v = \sum_{i=1}^{n} a_i e_i$$

$$w = \sum_{i=1}^{n} b_i e_i$$

Then

$$v \cdot w = \sum_{i=1}^{n} a_i b_i$$

So

$$\mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}, (v, w) \mapsto v \cdot w$$

Geometric interpretation for $\mathbb{F} = \mathbb{R}$: The length of v is

$$||v|| = \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{v \cdot v}$$

For $n = 2$, the dot product describes angle between vectors.

$$v = r \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, w = s \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$$

where $r, s \geq 0$. Then

$$\begin{aligned} v \cdot w &= rs(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= rs\cos(\alpha - \beta) \\ &= ||v||||w||\cos(\alpha - \beta) \end{aligned}$$

For $n > 2$, we use this to define the angle between non-zero vectors, namely by

$$\cos\theta = \frac{v \cdot w}{||v||||w||}$$

Note that this only defines $\cos\theta$ not θ, and we have to prove that RHS has absolute value ≤ 1.

More generally,

Definition 1.1. Let V be a vector space over $\mathbb{F} = \mathbb{R}$. An *inner product* on V is a positive definite symmetric bilinear form on V. That is, it is a bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, (v, w) \mapsto \langle v, w \rangle$$

with

1. Symmetry: $\langle v, w \rangle = \langle w, v \rangle \forall v, w$
2. Positivity: $\langle v, v \rangle \geq 0 \forall v$
3. Definite: $\langle v, v \rangle = 0$ iff $v = 0$

V with $\langle \cdot, \cdot \rangle$ is called *inner product space*. The associated norm is

$$||v|| = \sqrt{\langle v, v \rangle}$$

Example 1.1. $V = \mathbb{R}^n$, then $\langle v, w \rangle = v \cdot w$

Example 1.2. $V = \mathcal{P}_n(\mathbb{R})$, polynomials of degree $\leq n$. For $a < b \in \mathbb{R}$,

$$\langle p, q \rangle = \int_a^b p(x)q(x)dx$$

Same formula defined on $\mathcal{P}(\mathbb{R})$.

Example 1.3. $V = C([a, b])$, the set of continuous functions on $[a, b]$.

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

Example 1.4. $V = \mathcal{P}_n(\mathbb{R})$. Fix distinct points $x_0, \ldots, x_n$, put

$$\langle p, q \rangle = \sum_{i=0}^{n} p(x_i)q(x_i)$$

Note we need $n + 1$ points for the definite property.

Example 1.5. $V = M_{n\times n}(\mathbb{R})$. Then

$$\langle A, B \rangle = \mathrm{tr}(AB^t)$$

It is obviously a bilinear form. Symmetry holds as $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for square matrices. For positivity and definite, expand it at home ig?

For $\mathbb{F} = \mathbb{C}$, we have to modify things a bit. Recall $z = x + yi$ has $|z| = \sqrt{x^2 + y^2}$. Given

$$v = \sum_{i=1}^{n} a_i e_i, a_i \in \mathbb{C}, v \in \mathbb{C}^n$$

define

$$||v|| = \sqrt{\sum_{i=1}^{n} |a_i|^2}$$

Note that this extends $|| \cdot ||$ from $\mathbb{R}^n$ to $\mathbb{C}^n$. In terms of $\mathbb{C}^n \cong \mathbb{R}^{2n}$, one recovers $|| \cdot ||$ on $\mathbb{R}^{2n}$. But note that we **do not have**

$$||v||^2 = v \cdot v$$

Example 1.6. Vector $v \in \mathbb{C}^2$ with $||v|| = 1, v \cdot v = 0$ is

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Instead, we have $||v||^2 = v \cdot \bar{v}$

$$\langle v, w \rangle = v \cdot \bar{w} = \sum_{i=1}^{n} a_i \bar{b}_i$$

More generally,

Definition 1.2. Let V be a vector space on $\mathbb{F} = \mathbb{C}$. An inner product on V is a map

$$\langle , \rangle : V \times V \to \mathbb{C}$$

with properties

1. $\langle \cdot, \cdot \rangle$ is linear in the *first* argument
2. Conjugate symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$
3. Positivity: $\langle v, v \rangle \in \{0\} \cup \mathbb{R}^+$
4. Definite: $\langle v, v \rangle = 0$ iff $v = 0$

Note that this extends $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n \subset \mathbb{C}^n$. It does not agree with $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

Example 1.7. The first two properties imply

$$\begin{aligned}
\langle v, w_1 + w_2 \rangle &= \overline{\langle w_1 + w_2, v \rangle} \\
&= \overline{\langle w_1, v \rangle} + \overline{\langle w_2, v \rangle} \\
&= \langle v, w_1 \rangle + \langle v, w_2 \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle v, \lambda w \rangle &= \overline{\langle \lambda w, v \rangle} \\
&= \overline{\lambda} \langle v, w \rangle
\end{aligned}$$

Hence it does not agree for the reals and complex sets.

Note that many authors use different notations, where it is linear in the second argument, and conjugate linear in the first.

Example 1.8. In $V = \mathbb{C}^n$, $\langle v, w \rangle = v \cdot \overline{w}$

Example 1.9. For $V = \mathcal{P}_n(\mathbb{C})$,

$$\langle p, q\rangle = \int_a^b p(x)\overline{q(x)}dx$$

Same definition for $V = \mathcal{P}(\mathbb{C}), V = C([a, b])$

Example 1.10. $V = \mathcal{P}_n(\mathbb{C})$, choose distinct $z_0, \dots, z_n \in \mathbb{C}$

$$\langle p, q\rangle = \sum_{i=0}^{n} p(z_j)\overline{q(z_j)}$$

Example 1.11. $V = M_{n\times n}(\mathbb{C})$,

$$\langle A, B\rangle = \mathrm{tr}(A\overline{B}^t)$$

2 Cauchy-Schwarz Inequality

V inner product space ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), $\langle\cdot,\cdot\rangle$ inner product and $||\cdot||$ norm, we have

$$\begin{aligned} ||v + w||^2 &= \langle v + w, v + w\rangle \\ &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= ||v||^2 + ||w||^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} \\ &= ||v||^2 + ||w||^2 + 2\Re\langle v, w\rangle \end{aligned}$$

Definition 2.1. Vectors $v, w \in V$ are orthogonal if $\langle v, w\rangle = 0$.

Pythagorean theorem: If v, w are orthogonal then

$$||v + w||^2 = ||v||^2 + ||w||^2$$

Orthogonality is dependent on choice of inner product!

For general vectos $v, w \in V$, with $w \neq 0$. We want to define $\mathrm{proj}_w(v) = aw, a \in \mathbb{F}$ in such a way that

$$\langle v - \mathrm{proj}_w(v), w\rangle = 0$$

We need to solve

$$\begin{aligned} \langle v - aw, w\rangle &= 0 \\ \langle v, w\rangle - a\langle w, w\rangle &= 0 \\ a &= \frac{\langle v, w\rangle}{||w||^2} \end{aligned}$$

Definition 2.2.

$$\text{proj}_w(v) = \frac{\langle v, w\rangle}{||w||^2} w$$

So, any $v \in V$ decomposes as

$$v = \text{proj}_w(v) + (v - \text{proj}_w(v))$$

a sum of a scalar multiple of w and another vector orthogonal to w. By the Pythagorean theorem,

$$||v||^2 = ||\text{proj}_w(v)||^2 + ||v - \text{proj}_w(v)||^2$$

Theorem 2.1. *(Cauchy-Schwarz inequality). For $v, w \in V$, a complex or real inner product space,*

$$|\langle v, w\rangle| \leq ||v||||w||$$

This is obvious for $w = 0$, so we assume $w \neq 0$. By the Pythagorean theorem,

$$||v||^2 \geq ||proj_w(v)||^2 = \left|\left|\frac{\langle v, w\rangle}{||w||^2} w\right|\right|^2 = \frac{|\langle v, w\rangle|^2}{||w||^2}$$

and rearranging yields the inequality.

Note that we used

$$||\lambda v|| = |\lambda|||v||$$

Example 2.1. $V = \mathbb{C}^n$, and the standard inner product $\langle v, w\rangle = v \cdot \overline{w}$

$$|a_1\bar{b}_1 + \cdots + a_n\bar{b}_n| \leq \left(\sum_{i=1}^{n} |a_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |b_i|^2\right)^{\frac{1}{2}}$$

Example 2.2. $V = \mathcal{P}(\mathbb{C})$

$$\Big|\int_a^b p(x)\overline{q(x)}dx\Big| \leq \left(\int_a^b |p(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_a^b |q(x)|^2 dx\right)^{\frac{1}{2}}$$

Example 2.3. $V = M_{n\times n}(\mathbb{C})$, $\langle A, B\rangle = \text{tr}(A\overline{B}^t)$

$$|\operatorname{tr}(A\overline{B}^t)| \leq \sqrt{\operatorname{tr}(A\overline{A}^t)}\sqrt{\operatorname{tr}(B\overline{B}^t)}$$

Note that equality holds iff v, w are linearly dependent, in which case

$$||v - \text{proj}_w(v)||^2 = 0$$

Theorem 2.2. *(Triangle inequality) For $v, w \in V$ (inner product space),*

$$||v + w|| \leq ||v|| + ||w||$$

For the proof, note that

$$\begin{aligned}
||v + w||^2 &= ||v||^2 + ||w||^2 + 2\Re\langle v, w\rangle \\
&\leq ||v||^2 + ||w||^2 + 2|\langle v, w\rangle| \\
&\leq ||v||^2 + ||w||^2 + 2||v||||w|| \\
&= (||v|| + ||w||)^2
\end{aligned}$$

Definition 2.3. For general complex or real vector spces, we define a norme $|| \cdot ||$ to be a map from V to $\mathbb{R}$ such that

1. $||\lambda v|| = |\lambda|||v||$
2. $||v|| = 0 \Leftrightarrow v = 0$
3. $||v + w|| \leq ||v|| + ||w||$

Note that not every norm can form an inner product.

3 Orthogonal bases

Let V be a real or complex inner product space. Suppose $v_1, \ldots, v_k$ are nonzero, pairwise orthogonal. Then they are linearly independent. This is because

$$\sum_{i=1}^{k} a_i v_i = 0$$

Taking the inner product with v_j on both sides give $a_j = 0$. Then if dim $V = n$, any collection of pairwise orthogonal vectors $v_1, \ldots, v_n$ forms a basis.