Lecture 10

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1 Orthogonal Projections

Recall $P \in \mathcal{L}(V)$ is a projection iff

$$P^2 = P$$

Then $I - P$ is also a projection, and all vectors v can be decomposed by

$$v = Pv + (I - P)v = v_1 + v_2$$

or

$$V = V_1 \oplus V_2$$

where

$$V_2 = \text{ran}(I - P) = \text{null}(P)$$

Conversely, for any direct sum decomposition with two subspaces $V = V_1 \oplus V_2$, we get a P.

If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, we call P an orthogonal projection iff there is an orthogonal decomposition. Conversely, if $V = W \oplus W^\perp$, we get a corresponding $P = P_W$.

From the last lecture, given an orthogonal projection P, then

$$||Pv|| \leq ||v||$$

Proposition 1.1. *Let V be an inner product space, $P \in \mathcal{L}(V)$ a projection. Then P is an orthogonal projection iff $||Pv|| \leq ||v|| \forall v \in V$.*

Proof. We proved the "only if" part in last lecture. For "if", we want to show

$$V = \operatorname{ran}(P) \oplus \operatorname{null}(P)$$

is an orthogonal decomposition, so every $v \in \operatorname{null}(P)$ is orthogonal to $\operatorname{ran}(P)$. Since both have the same dimensions, it suffices to show

$$\operatorname{null}(P)^{\perp} \subseteq \operatorname{ran}(P)$$

Supposed $v \in \operatorname{null}(P)^{\perp}$. Write

$$Pv = v - (I - P)v$$

This is an orthogonal decomposition, since $v \in \operatorname{null}(P)^{\perp}$, $(I - P)v \in \operatorname{ran}(I - P) = \operatorname{null}(P)$. Therefore

$$||Pv||^2 = ||v||^2 + ||(I - P)v||^2 \geq ||v||^2$$

Then

$$||v|| \geq ||Pv|| \geq ||v|| \Rightarrow ||Pv|| = ||v||$$

i.e. $||(I - P)v|| = 0$. So $Pv = v$, and $v \in \operatorname{ran}(P)$. □

Proposition 1.2. *Let V be an inner product space, $W \subseteq V$, where W is finite dimensiona. Let $v_1, \ldots, v_n$ be an orthonormal basis of W. Then*

$$P_w(v) = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$$

Proof. Use $V = W \oplus W^{\perp}$. The formula is true for all $v \in W^{\perp}$ and $v \in W$. □

Theorem 1.1. *Let $V = W \oplus W^{\perp}$, $v \in V, w \in W$. Then*

$$P_w(v) - v|| \leq ||w - v||$$

with equality iff $w = P_w(v)$.

Proof.

$$w - v = (P_w(w) - v) + (w - P_w(v))$$

where the first term is an element of $W^{\perp}$ and the second term is an element of W. Then the Pythagorean theorem shows that

$$||w - v||^2 = ||P_w(v) - v||^2 + ||v - P_w(v)||^2 \geq ||P_w(v) - v||^2$$

with equality iff the dropped term is 0. □

Example 1.1. Let V be continuous real functions on $[-\pi, \pi)$ with

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$

Last time, we showed that

$$f_n(x) = \frac{1}{\sqrt{\pi}} \sin(nx), n \in \mathbb{N}$$

are all orthonormal. Then which linear combination

$$\sum_{i=1}^{n} a_i f_i(x)$$

is the best approximation to $f(x) = x$? I.e. such that

$$||f - \sum_{i=1}^{n} a_i f_i||$$

is minimized.
By 1.1, the best approximation is the orthogonal projection of f on

$$W = \text{span}\{f_1, \ldots, f_n\}$$

so

$$P_w(f) = \sum_{i=1}^{n} \langle f, f_i \rangle f_i$$

So the oordinate a_i become

$$\begin{aligned} a_i &= \langle f, f_i \rangle \\ &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin(nx)dx \\ &= -\frac{1}{n\sqrt{\pi}}(x\cos(nx))|_{-\pi}^{\pi} + \frac{1}{n\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(nx)dx \\ &= \frac{2\sqrt{\pi}}{n}(-1)^{n+1} \end{aligned}$$

So

$$P_w(f) \approx 2 \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} \sin(ix)$$

2 Adjoint Operators

Consider $\mathbb{F}^n$, where $\mathbb{F}$ is real or complex, and

$$\mathcal{L}(\mathbb{F}^m, \mathbb{F}^n) = M_{n\times m}(\mathbb{F})$$

For $A \in M_{m\times n}(\mathbb{F})$ define the conjugate transpose

$$A^* = \overline{A}^t$$

If $\langle , \rangle$ is the standard inner product on $\mathbb{F}^n, \mathbb{F}^m$, we have

$$\langle Av, w\rangle = \langle v, A^* w\rangle$$

For all $v \in \mathbb{F}^m, w \in \mathbb{F}^n$.

Proof.

$$\begin{aligned}
\langle Av, w\rangle &= (Av)^t \overline{w} \\
&= v^t A^t \overline{w} \\
&= v^t \overline{\overline{A^t} w} \\
&= v^t \overline{A^* w} \\
&= \langle v, A^* w\rangle
\end{aligned}$$

□

Theorem 2.1. *Let V, W be inner product spaces with finite dimensions. $forall T \in \mathcal{L}(V, W)$ there is a unique $T^* \in \mathcal{L}(W, V)$ such that*

$$\langle Tv, w\rangle = \langle v, T * w\rangle$$