Homework 5

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1. Let V be the space of real-valued continuous functions with the property $f(x+2\pi) = f(x)$, equipped with the inner product
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$$
Find the linear combination of $1, \sin(x), \cos(x)$ that is closest to $\sin^2(x)$, using the norm defined by this inner product.

Solution: First, note that $1, \sin(x), \cos(x)$ are pairwise orthogonal. For 1 and $\sin(x)$, it suffices to note that $\sin(x)$ is odd. For 1 and $\cos(x)$, note that $\sin(\pi) = \sin(-\pi) = 0$. Finally for $\sin(x)$ and $\cos(x)$, note that their product is $\frac{1}{2}\sin(2x)$, which is odd. Given this, we know that the desired linear combination is

$$\langle \sin^2(x), 1\rangle \times 1 + \langle \sin^2(x), \sin(x)\rangle \sin(x) + \langle \sin^2(x), \cos(x)\rangle \cos(x)$$

The first inner product is given by

$$\int_{-\pi}^{\pi} \sin^2(x)dx = \frac{x}{2} - \frac{\sin(2x)}{4}\Big|_{-\pi}^{\pi} = \pi$$

The second inner product is given by

$$\int_{-\pi}^{\pi} \sin^3 dx = -\int_{-\pi}^{\pi} \sin^2(x)d(\cos(x)) = \int_{-\pi}^{\pi} \cos^2(x) - 1d(\cos(x)) = \frac{\cos^3(x)}{3} - \cos(x)\Big|_{-\pi}^{\pi} = 0$$

The third inner product is given by

$$\int_{-\pi}^{\pi} \sin^2(x)\cos(x)dx = \int_{-\pi}^{\pi} \sin^2(x)d(\sin(x)) = \frac{\sin^3(x)}{3}\Big|_{-\pi}^{\pi} = 0$$

Therefore, the closest linear combination is

$$\pi \times 1 + 0 \times \sin(x) + 0 \times \cos(x) = \pi$$

2. Find a real polynomial p of degree ≤ 2 with the property that
$$\int_0^1 p(x)q(x) = q(2)$$
for all polynomials q of degree ≤ 2.

Solution: Let $q = ax^2 + bx + c, (a,b,c) \in \mathbb{R}^3$. Similarly, $p = dx^2 + ex + f$. Then

$$\int_0^1 p(x)q(x) = q(2)$$

$$\int_0^1 adx^4 + (ae+bd)x^3 + (af+cd+be)x^2 + (bf+ce)x + cf dx = 4a + 2b + c$$

$$\frac{ad}{5} + \frac{ae+bd}{4} + \frac{af+cd+be}{3} + \frac{bf+ce}{2} + cf = 4a + 2b + c$$

$$\left(\frac{d}{5} + \frac{e}{4} + \frac{f}{3}\right) a + \left(\frac{d}{4} + \frac{e}{3} + \frac{f}{2}\right) b + \left(\frac{d}{3} + \frac{e}{2} + f\right) c = 4a + 2b + c$$

Solving the simultaneous equation, $d = 390, e = -372, f = 57$.

3. Let $V = \mathcal{P}_2(\mathbb{R})$ be the space of polynomials of degree at most 2, with the inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$$

Let $T^* \in \mathcal{L}(V)$ be the adjoint of the operator

$$T : V \to V, p \mapsto p' = \frac{dp}{dx}$$

Compute $T * p$ for the polynomial $p(x) = x^2$.

Solution: The basis of V in terms of Legendre polynomials is

$$e_1 = 1, e_2 = x, e_3 = \frac{1}{2}(3x^2 - 1)$$

Then

$$\begin{aligned}
T^*p &= \langle T^*p, e_1 \rangle e_1 + \langle T^*p, e_2 \rangle e_2 + \langle T^*p, e_3 \rangle e_3 \\
&= \langle p, Te_1 \rangle e_1 + \langle p, Te_2 \rangle e_2 + \langle p, Te_3 \rangle e_3 \\
&= \langle x^2, 0 \rangle e_1 + \langle x^2, 1 \rangle e_2 + \langle x^2, 3x \rangle e_3 \\
&= \langle \frac{2}{3}e_3 + \frac{1}{3}e_1, e_1 \rangle e_2 + \langle \frac{2}{3}e_3 + \frac{1}{3}e_1, 3e_2 \rangle e_3 \\
&= \frac{1}{3}e_2 \\
&= \frac{1}{3}x
\end{aligned}$$

4. Let V be a real or complex inner product space, and $W \subset V$ a subspace, with the inner product obtained by restriction of that on V. Let

$$T \in \mathcal{L}(W, V)$$

be the inclusion, taking any vector in W to itself, but regarded as a vector in V.

(a) What is the adjoint $T^* \in \mathcal{L}(V, W)$?

(b) What is the operator $T^*T \in \mathcal{L}(W)$?

(c) What is the operator $TT^* \in \mathcal{V}$?

Solution: Let e_i denote a set of orthonormal basis vectors of W, then extend it to form an orthonormal basis of V. Then let $e_j \in V, e_k \in W$

$$\begin{aligned} \langle T^* e_j, e_k \rangle &= \langle e_j, T e_k \rangle \\ &= \langle e_j, e_k \rangle \\ &= \delta_{jk} \end{aligned}$$

Then

$$T^* e_j = \sum_k \langle T^* e_j, e_k \rangle e_k = \begin{cases} e_j & e_j \in W \\ 0 & e_j \notin W \end{cases}$$

So T^* is the projection from V to W. More explicitly,

$$T^* \sum_{e_i \in V} a_i e_i = \sum_{e_i \in W} a_i e_i$$

By definition of T,

$$Tw = w \forall w \in W$$

Then T^*T is the identity map of W, and TT^* is the projection from V to W, i.e.

$$TT^* = T^*$$

5. Let V be the vector space of *infinite sequences of finite length.* That is, elements of V are sequences of complex numbers

$$a = (a_1, a_2, a_,3, \dots)$$

with the property that $\{n | a_n \neq 0\}$ is finite. Define an inner product on V by

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}$$

Give an example of an operator $T \in \mathcal{L}(V)$ having the following three properties:

- T admits an adjoint, i.e. there is an operator $T^* \in \mathcal{L}(V)$ with $\langle Ta, b \rangle = \langle a, T^* b \rangle$ for all $a, b \in V$
- T^*T is the identity operator on V
- TT^* is a projection (but not the identity operator)

Solution: Let

$$T : (a_1, a_2, a_3, \dots) \mapsto (0, a_1, a_2, \dots)$$

Then it is linear. Obviously, $T(kv) = kTv$. Also,

$$\begin{aligned} T(v + w) &= T(v_1 + w_1, v_2 + w_2, v_3 + w_3, \dots) \\ &= (0, v_1 + w_1, v_2 + w_2, \dots) \\ &= (0, v_1, v_2, \dots) + (0, w_1, w_2, \dots) \\ &= Tv + Tw \end{aligned}$$

Note that if we define

$$U : (a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots)$$

It is also linear, as $U(kv) = kUv$ is obviously true, and

$$\begin{aligned} U(v + w) &= U(v_1 + w_1, v_2 + w_2, v_3 + w_3, \dots) \\ &= (v_2 + w_2, v_3 + w_3, \dots) \\ &= (v_2, v_3, \dots) + (w_2, w_3, \dots) \\ &= Uv + Uw \end{aligned}$$

Note that U is the adjoint of T, as

$$\begin{aligned} \langle Ta, b\rangle &= \langle T(a_1, a_2, a_3, \dots), (b_1, b_2, b_3, \dots)\rangle \\ &= \langle (0, a_1, a_2, \dots), (b_1, b_2, b_3, \dots)\rangle \\ &= \sum_{i=1}^{\infty} a_i b_{i+1} \end{aligned}$$

And

$$\begin{aligned} \langle a, T^* b\rangle &= \langle (a_1, a_2, a_3, \dots), U(b_1, b_2, b_3, \dots)\rangle \\ &= \langle (a_1, a_2, a_3, \dots), (b_2, b_3, \dots)\rangle \\ &= \sum_{i=1}^{\infty} a_i b_{i+1} \end{aligned}$$

Then

$$T^*T(v) = T^*T(v_1, v_2, v_3, \dots) = T^*(0, v_1, v_2, \dots) = (v_1, v_2, v_3 \dots) = v$$

so T^*T is the identity operator. Now

$$TT^*(v) = TT^*(v_1, v_2, v_3, \dots) = T(v_2, v_3, \dots) = (0, v_2, v_3, \dots)$$

which is not the identity (consider a v where $v_1 \neq 0$). However, it is a projection, as

$$TT^*(TT^*(v)) = TT^*(0, v_2, v_3, \dots) = T(v_2, v_3, \dots) = (0, v_2, v_3, \dots) = TT^*(v)$$

So $(TT^*)^2 = TT^*$.