Homework 4

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1. Let V be a real or complex inner product space, and $v, w \in V$.

   (a) Show $||v+w||^2 + ||v-w||^2 = 2||v||^2 + 2||w||^2$.

   Solution:

$$\begin{aligned}
||v+w||^2 + ||v-w||^2 &= \langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\
&= \langle v, v+w\rangle + \langle w, v+w\rangle + \langle v, v-w\rangle - \langle w, v-w\rangle \\
&= \overline{\langle v+w, v\rangle} + \overline{\langle v+w, w\rangle} + \overline{\langle v-w, v\rangle} - \overline{\langle v-w, w\rangle} \\
&= \overline{\langle v, v\rangle} + \overline{\langle w, v\rangle} + \overline{\langle v, w\rangle} + \overline{\langle w, w\rangle} + \overline{\langle v, v\rangle} - \overline{\langle w, v\rangle} - \overline{\langle v, w\rangle} + \overline{\langle w, w\rangle} \\
&= ||v||^2 + \langle v, w\rangle + \langle w, v\rangle + ||w||^2 + ||v||^2 - \langle v, w\rangle - \langle w, v\rangle + ||w||^2 \\
&= 2||v||^2 + 2||w||^2
\end{aligned}$$

   (b) Show $||v-w|| \geq ||v|| - ||w||$

   Solution:

$$\begin{aligned}
||v-w|| &= \sqrt{\langle v-w, v-w\rangle} \\
&= \sqrt{\langle v, v-w\rangle - \langle w, v-w\rangle} \\
&= \sqrt{\overline{\langle v-w, v\rangle} - \overline{\langle v-w, w\rangle}} \\
&= \sqrt{\overline{\langle v, v\rangle} - \overline{\langle w, v\rangle} - \overline{\langle v, w\rangle} + \overline{\langle w, w\rangle}} \\
&= \sqrt{||v||^2 - \langle v, w\rangle - \overline{\langle v, w\rangle} + ||w||^2} \\
&= \sqrt{||v||^2 - 2\Re\langle v, w\rangle + ||w||^2} \\
&\geq \sqrt{||v||^2 - 2|\langle v, w\rangle| + ||w||^2} \\
&\geq \sqrt{||v||^2 - 2\|v\|\|w\| + ||w||^2} \\
&\geq ||v|| - ||w||
\end{aligned}$$

   (c) Show that $v = w$ if and only if $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in V$.

   Solution: It is obvious that $v = w$ implies

$$\langle v, x\rangle = \langle w, x\rangle \forall x \in V$$

To prove the "if" part, we need an orthonormal basis of V, which can be formed from any basis using Gram-Schmidt then normalising. Denoting the basis as $v_1, \ldots, v_n$, note that if we let

$$v = \sum_{i=1}^{n} a_i v_i$$

and

$$w = \sum_{i=1}^{n} b_i v_i$$

We get

$$\langle v, v_j \rangle = \left\langle \sum_{i=1}^{n} a_i v_i, v_j \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, v_j \rangle = a_j$$

Similarly,

$$\langle w, v_j \rangle = b_j$$

Putting x to be each of $v_1, \ldots, v_n$, we get $a_j = b_j$ for j from 1 to n, which means $v = w$.

2. Let $V = P_2(\mathbb{R})$ be the vector space of polynomials of degree ≤ 2, with inner product

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

Use Gram-Schmidt to find an orthonormal basis for the subspace spanned by the polynomials $p_1(x) = x$, $p_2(x) = x^2$.

Solution: Both polynomials are obviously linearly independent, so the subspace has a dimension of 2. Using the Gram-Schmidt process, the first vector is p_1 normalised to 1. Then

$$\begin{aligned} ||kp_1|| &= 1 \\ \langle kp_1, kp_1 \rangle &= 1 \\ k^2 \langle p_1, p_1 \rangle &= 1 \\ p_1(0)p_1(0) + p_1(1)p_1(1) + p_1(2)p_1(2) &= \frac{1}{k^2} \\ 5 &= \frac{1}{k^2} \\ k &= \frac{1}{\sqrt{5}} \end{aligned}$$

So the first basis vector u_1 is $\frac{x}{\sqrt{5}}$. Using the Gram-Schmidt process, the second (yet to be normalised) basis vector is given by

$$\begin{aligned} p_2 - \text{proj}_{u_1}(p_2) &= x^2 - \frac{\langle p_2, u_1 \rangle}{||u_1||^2} u_1 \\ &= x^2 - (p_2(0)u_1(0) + p_2(1)u_1(1) + p_2(2)u_1(2))u_1 \\ &= x^2 - \frac{9}{5}x \end{aligned}$$

Normalising,

$$\begin{aligned}
\langle k(x^2 - \frac{9}{5}x), k(x^2 - \frac{9}{5}x)\rangle &= 1 \\
k^2\langle x^2 - \frac{9}{5}x, x^2 - \frac{9}{5}x\rangle &= 1 \\
0 + (1 - \frac{9}{5})^2 + (4 - \frac{18}{5})^2 &= \frac{1}{k^2} \\
k &= \frac{\sqrt{5}}{2}
\end{aligned}$$

Thus the second basis vector is

$$u_2 = \frac{\sqrt{5}}{2}x^2 - \frac{9}{2\sqrt{5}}x$$

3. Let $T \in \mathcal{L}(V)$ be a linear operator on a finite-dimensional complex vector space V.
   (a) Let $R \geq 0$ a constant such that
   $$||Tv|| \leq R||v||$$
   for all $v \in V$. Show that all eigenvalues λ of T satisfy $|\lambda| \leq R$.
   (b) Suppose there exists a constant $r > 0$ such that
   $$||Tv|| \geq r||v||$$
   for all non-zero $v \in V$. Show that T is invertible.

Solution: Let λ be an eigenvalue, and v be any corresponding nonzero eigenvector. Then

$$\begin{aligned}
||Tv|| &\leq R||v|| \\
||\lambda v|| &\leq R||v|| \\
\sqrt{\langle \lambda v, \lambda v\rangle} &\leq R||v|| \\
\sqrt{\lambda\bar{\lambda}\langle v, v\rangle} &\leq R||v|| \\
|\lambda|.||v|| &\leq R||v|| \\
|\lambda| &\leq R
\end{aligned}$$

Note that we made use of the fact that v being nonzero implies it has a nonzero norm. Since λ is set arbitrarily, all eigenvalues satisfy this property.
For the second part, observe if T is not invertible, then there exists a non zero v such that $Tv = 0$. Then $||Tv|| = 0$, but $r||v|| > 0$, since r is positive by definition and $||v|| > 0 \forall v \neq 0$. This leads to a contradiction. Hence T is invertible.

4. A symmetric bilinear form $\langle \cdot, \cdot \rangle$ on vector space V is called nondegenerate if the only vector $v \in V$ satisfying $\langle v, w\rangle = 0$ for all $w \in V$ is the zero vector $v = 0$.
   (a) Show that the bilinear form on $\mathbb{R}^n$, given on vectors $v = \sum_i a_i e_i, w = \sum_j b_j e_j$ by
   $$\langle v, w\rangle = a_1b_1 + \cdots + a_rb_r - a_{r+1}b_{r+1} - \cdots - a_nb_n$$
   is nondegenerate.

Solution: Let v be a vector such that $\langle v, w\rangle = 0 \forall w \in V$. Consider $w = e_k$, where $k \leq r$. Then $\langle v, w\rangle = a_k = 0$. Setting this for $k = 1$ to $k = r$ gives

$$a_1 = a_2 = \cdots = a_r = 0$$

Then similarly, letting $k > r$, we have $\langle v, w\rangle = -a_k = 0$. Setting this for $k = r+1$ to $k = n$ gives

$$a_{r+1} = a_{r+2} = \cdots = a_n = 0$$

Hence $v = 0$, and the symmetric bilinear form is nondegenerate.

(b) Given a nondegenerate symmetric bilinear form on a real vector space V, prove that there exists a basis $v_1, \ldots, v_n$ and some r with $0 \leq r \leq n$ such that

$$\langle v_j, v_k\rangle = \begin{cases} 0 & j \neq k \\ 1 & j = k \leq r \\ -1 & j = k > r \end{cases}$$

Solution: Similar to inner product spaces, we define v and w to be orthogonal when

$$\langle v, w\rangle = 0$$

We can also define projections

$$\text{proj}_u(v) = \frac{\langle v, u\rangle}{\langle u, u\rangle} u$$

when $\langle u, u\rangle \neq 0$. Note that for an arbitrary nondegenerate subspace of V,

$$\begin{aligned} \langle v + w, v + w\rangle &= \langle v, v\rangle + \langle w, w\rangle + 2\langle v, w\rangle \\ 2\langle v, w\rangle &= \langle v, v\rangle + \langle w, w\rangle - \langle v + w, v + w\rangle \end{aligned}$$

As the subspace is nondegenerate, there has to be a v, w pair such that $\langle v, w\rangle \neq 0$. Then one of the terms on the right hand side has to be nonzero, meaning there exists v in the subspace where $\langle v, v\rangle \neq 0$. Armed with this, we first obtain $v_1 \in V$ that satisfies this. This means $\text{proj}_{v_1}(w)$ is defined $\forall w \in V$. Then every vector in V can be written as a sum of a scalar multiple of v_1 and a w orthogonal to v. Let $V_1 = V$, and we can define

$$V_2 = \{w \in V_1 | \langle w, v_1\rangle\}$$

where

$$V_1 = \text{span}(v_1) \oplus V_2$$

Now note that V_2 is not degenerate. Or else, let $w \in V_2$ such that $\langle w, v\rangle = 0 \forall v \in V_2$. Then for an arbitrary $u \in V$, u can be written as $av_1 + v_2$ where $v_2 \in V_2, a \in \mathbb{R}$. We get

$$\langle w, u\rangle = \langle w, av_1 + v_2\rangle = \langle w, av_1\rangle + \langle w, v_2\rangle = 0$$

So $V = V_1$ is degenerate, a contradiction. Therefore, V_2 is nondegenerate, and we have a $v_2 \in V_2$ such that

$$\langle v_2, v_2\rangle \neq 0$$

By induction, we can define $v_3, v_4, \ldots, v_n$. Note that for V_i, V_j where $i < j$, V_j is a subset of $V - V_i$, meaning all vectors in V_j, including v_j, are orthogonal to v_i. Thus $v_1, \ldots, v_n$ are pairwise orthogonal. Then rearrange $v_1, \ldots, v_n$ to $w_1, \ldots, w_n$ such that $w_1, \ldots, w_r$ are the unique vectors where $\langle w_i, w_i \rangle = k_i > 0$. We can then normalise using

$$w_i' = \frac{1}{\sqrt{k}} w_i$$

We use the same formula for normalisation if $\langle w_i, w_i \rangle = -k_i < 0$. Then $w_1', \ldots, w_n'$ is the required basis.

(c) Show that a positive symmetric bilinear form $\langle \cdot, \cdot \rangle$ on a real vector space V with $\dim V = n < \infty$ is positive definite if and only if it is nondegenerate.

Solution: If it is an inner product, then let v be a vector such that $\langle v, w \rangle = 0 \forall w \in V$. Then $\langle v, v \rangle = 0$, which implies $v = 0$, as it is definite. Hence, it is nondegenerate. To show the reverse, we consider the basis $v_1, \ldots, v_n$ shown to exist in the previous part. Since it is positive, $\langle v_i, v_i \rangle \geq 0$. This means $r = n$, and gives the stricter condition that $\langle v_i, v_i \rangle > 0$. Then for an arbitrary vector

$$v = \sum_{i=1}^{n} a_i v_i, a_i \in \mathbb{R}$$

we have

$$\langle v, v \rangle = \sum_{i,j} a_i a_j \langle v_i, v_j \rangle = \sum_{i=j} a_i^2 \langle v_i, v_i \rangle = \sum_i a_i^2$$

Which means $\langle v, v \rangle = 0$ if and only if $v = 0$. Note that all the terms with $i \neq j$ vanish, by definition of $v_1, \ldots, v_n$.

5. Let V be a finite-dimensional complex inner product space, and $T \in \mathcal{L}(V)$. Let $v \in V$ with $||v|| = 1$. Prove

$$|\langle Tv, v \rangle| \leq ||Tv||$$

with equality if and only if v is an eigenvector of T.

Solution: By Cauchy-Schwartz,

$$|\langle Tv, v \rangle| \leq ||Tv||.||v|| = ||Tv||$$

If v is an eigenvector, let λ be the eigenvalue, then

$$|\langle Tv, v \rangle| = |\langle \lambda v, v \rangle| = |\lambda \langle v, v \rangle| = |\lambda|$$

From the right hand side,

$$||Tv|| = ||\lambda v|| = |\lambda|.||v|| = |\lambda|$$

Hence equality holds if v is an eigenvector of T. Conversely, given equality, we first assume $Tv \neq 0$, or else v is an eigenvector with eigenvalue 0. Then let $Tv = w + \lambda v$, where λv is its projection on

v. Note that then $\langle w, v\rangle = \langle v, w\rangle = 0$.

$$\begin{aligned}
|\langle Tv, v\rangle| &= ||Tv|| \\
|\langle w + \lambda v, v\rangle| &= \sqrt{\langle w + \lambda v, w + \lambda v\rangle} \\
|\langle w, v\rangle + \lambda\langle v, v\rangle| &= \sqrt{\langle w, w\rangle + \lambda\langle w, v\rangle + \lambda\langle v, w\rangle + |\lambda|^2\langle v, v\rangle} \\
|\lambda| &= \sqrt{||w||^2 + |\lambda|^2}\sqrt{|\lambda|^2} && = \sqrt{||w||^2 + |\lambda|^2} \\
||w|| &= 0 \\
w &= 0
\end{aligned}$$

Then $Tv = \lambda v$, so v is an eigenvector.