Problem Set 8

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1. A point charge Q is moving in the air with a velocity $\vec{v}$ near a straight wire conductor with a time-invariant current of intensity I. Referring to three cases with different directions of $\vec{v}$ shown in Fig. Q4.1, the magnetic force on Q is zero for

> **Solution:** B. $\vec{F} = q\vec{v} \times \vec{B}$, which vanishes when $\vec{v}$ and $\vec{B}$ are parallel. By the right hand rule, $\vec{B}$ goes into the page, so force vanishes when $\vec{v}$ also goes into/out of the page.

2. A charged particle moves with velocity $\vec{v}$ in a vacuum. An applied magnetic fields of flux density $\vec{B}$ can change

> **Solution:** C, the direction of $\vec{v}$ but not its magnitude. This is because $\vec{F}$ is always perpendicular to $\vec{v}$.

3. A steady current is established in a straight metallic wire conductor in a nonmagnetic medium. The magnetic vector potential due to this current at an arbitrary point in space that is not on the wire axis is

> **Solution:** A. Parallel to the wire. If $\vec{A}$ only has a $\hat{a}_z$ component, then
>
> $$\vec{B} = \vec{\nabla} \times A = \frac{\partial A_z}{\partial y}\hat{a}_x - \frac{\partial A_z}{\partial x}\hat{a}_y$$
>
> which matches with reality, as $\vec{B}$ has no $\hat{a}_z$ component, as it is always perpendicular to current.

4. Consider a square loop with a steady current of intensity I, in free space. Let A_1 denote the magnitude of the magnetic vector potential at the loop center due to the current along one of the square sides. The magnitude of the total magnetic vector potential at the center equals

> **Solution:** Noncritical, will do later/never

5. A DC voltage of 6V applied to the ends of 1 km of a conducting wire of 0.5 mm radius results in a current of $\frac{1}{6}$ A. Find
   (a) the conductivity of the wire
   (b) the electric field intensity in the wire

(c) the power dissipated in the wire

(d) the electron drift velocity, assuming electron mobility in the wire to be 1.4×10^{-3}

Solution:

$$R = \frac{V}{I} = 6 \div \frac{1}{6} = 36, \sigma = \frac{l}{RS} = \frac{1000}{36\pi 0.0005^2} = 3.54 \times 10^7$$

Combining $I = JS$ and $\vec{J} = \sigma\vec{E}$,

$$E = \frac{V}{l} = 6 \times 10^{-3}$$

Power dissipated is $P = VI = 1$.

$$\rho_e u = \sigma E \Rightarrow u = \frac{\sigma}{\rho_e}E = \mu_e E = 1.4 \times 10^{-3} \times 6 \times 10^{-3} = 8.4 \times 10^{-6}$$

6. Refer to the flat conducting quarter-circular washer in Example 5-6 and Fig. 5-8. Find the resistance between the curved sides.

Solution: Now, we ignore fringing effects, so V is a function of r. Then

$$\begin{aligned}
\vec{\nabla}^2 V &= 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) &= 0 \\
\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) &= 0 \\
r\frac{\partial V}{\partial r} &= C_1 \\
\frac{\partial V}{\partial r} &= \frac{C_1}{r} \\
V &= C_1 \ln r + C_2
\end{aligned}$$

Letting $V = 0$ at $r = a$ and $V = V_0$ at $r = b$, we get

$$V = \frac{V_0 \ln \frac{r}{a}}{\ln \frac{b}{a}}$$

and

$$\begin{aligned}
\vec{J} &= \sigma\vec{E} \\
&= -\sigma\vec{\nabla}V \\
&= -\sigma\frac{\partial V}{\partial r}\hat{a}_r \\
&= -\sigma\frac{V_0}{r\ln\frac{b}{a}}\hat{a}_r \\
&= -\frac{\sigma V_0 \ln a}{r \ln \frac{b}{a}}\hat{a}_r
\end{aligned}$$

Current is then

$$
\begin{aligned}
I &= \iint_S \vec{J} \cdot d\vec{S} \\
&= \int_0^h \int_0^{\frac{\pi}{2}} -\frac{\sigma V_0}{r \ln \frac{b}{a}} r d\phi dz \\
&= -\frac{\sigma \pi V_0 h}{2 \ln \frac{b}{a}}
\end{aligned}
$$

Now resistance can be found to be

$$
\begin{aligned}
R &= \frac{V}{I} \\
&= \frac{2 \ln \frac{b}{a}}{\sigma \pi h}
\end{aligned}
$$

7. A positive point charge q of mass m is injected with a velocity $\vec{u}_0 = \hat{a}_y u_0$ into the $y > 0$ region where a uniform magnetic field $\vec{B} = \hat{a}_x B_0$ exists. Obtain the equation of motion of the charge, and describe the path that the charge follows.

Solution: noncritical, will do later/never

8. The magnetic flux density $\vec{B}$ for an infinitely long cylindrical conductor has been found in Example 6-1. Determine the vector magnetic potential $\vec{A}$ both inside and outside the conductor from the relation $\vec{B} = \vec{\nabla} \times \vec{A}$.

Solution: noncritical, will do later/never

9. In a certain region, the magnetic vector potential is given as the following function in a spherical coordinate system: $\vec{A} = 2R^2 \hat{a}_\phi$.
   (a) Find the magnetic flux density vector in this region.
   (b) Obtain the magnetic flux through a circular contour 1 m in adius that lies in the plane $z = 0$ and is centered at hte coordinate origin.
   (c) Check the results by evaluating the circulation of $\vec{A}$ along the contour.

Solution:

$$
\begin{aligned}
\vec{B} &= \vec{\nabla} \times \vec{A} \\
&= \frac{1}{R \sin\theta} \frac{\partial(\sin\theta A_\phi)}{\partial\theta} \hat{a}_R - \frac{1}{R} \frac{\partial(R A_\phi)}{\partial R} \hat{a}_\theta \\
&= \frac{2 \cos\theta R^2}{R \sin\theta} \hat{a}_R - \frac{6R^2}{R} \hat{a}_\theta \\
&= 2 \cot\theta R \hat{a}_R - 6R \hat{a}_\theta
\end{aligned}
$$

Note that the given surface is in the $-\hat{a}_\theta$ direction. Then

$$\begin{aligned}\iint_S \vec{B}\cdot d\vec{S} &= \int_0^{2\pi}\int_0^1 6R\times RdRd\phi \\ &= 2\pi\times 2 \\ &= 4\pi\end{aligned}$$

We use the counterclockwise direction.

$$\begin{aligned}\oint_C \vec{A}\cdot d\vec{l} &= \int_0^{2\pi} 2R^2\hat{a}_\phi\cdot\hat{a}\phi d\phi \\ &= \int_0^{2\pi} 2d\phi \\ &= 4\pi \\ &= \iint_S \vec{B}\cdot d\vec{S}\end{aligned}$$

10. Find the resistance of two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$). The space in between is filled with a material of conductivity σ.

Solution: Now V is a function of R with only an $\hat{a}_R$ component. Hence

$$\begin{aligned}\vec{\nabla}^2 V &= 0 \\ \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) &= 0 \\ \frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) &= 0 \\ R^2\frac{\partial V}{\partial R} &= C_1 \\ \frac{\partial V}{\partial R} &= \frac{C_1}{R^2} \\ V &= -\frac{C_1}{R}+C_2\end{aligned}$$

Let $V = 0$ at R_1 and $V = V_0$ at R_2. Substituting,

$$V = \frac{V_0R_2}{R_2-R_1}\left(1-\frac{R_1}{R}\right)$$

Electric field intensity is

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\vec{V} \\ &= -\frac{\partial V}{\partial R}\hat{a}_R \\ &= -\frac{V_0R_1R_2}{(R_2-R_1)R^2}\hat{a}_R\end{aligned}$$

Then current is

$$\begin{aligned}
I &= \iint_S \vec{J} \cdot d\vec{S} \\
&= \iint_S \sigma \vec{E} \cdot d\vec{S} \\
&= \int_0^{2\pi} \int_0^{\pi} -\sigma \frac{V_0 R_1 R_2}{(R_2 - R_1) R^2} R^2 \sin\theta d\theta d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} -\sigma \frac{V_0 R_1 R_2}{R_2 - R_1} \sin\theta d\theta d\phi \\
&= -\frac{4\sigma\pi V_0 R_1 R_2}{R_2 - R_1}
\end{aligned}$$

Resistance is then

$$\begin{aligned}
R &= \frac{V}{I} \\
&= \frac{R_2 - R_1}{4\sigma\pi R_1 R_2}
\end{aligned}$$

11. Find the resistance between the surfaces R_1 and R_2 of a truncated conical block defined by $R_1 \leq R_2$ and $0 \leq \theta \leq \theta_0$. The two spherical surfaces ($R = R_1$ and $R = R_2$) are perfect electric conductors (PECs), while the rest of the block has conductivity σ. You can neglect edge effects.

Solution: Ignoring edge effects, V is a function of R. Then

$$\begin{aligned}
\vec{\nabla}^2 V &= 0 \\
\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) &= 0 \\
\frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) &= 0 \\
R^2 \frac{\partial V}{\partial R} &= C_1 \\
\frac{\partial V}{\partial R} &= \frac{C_1}{R^2} \\
V &= -\frac{C_1}{R} + C_2
\end{aligned}$$

Setting $V = 0$ at R_1 and $V = V_0$ at R_2, we get

$$V = \frac{V_0 R_2}{R_2 - R_1} \left(1 - \frac{R_1}{R} \right)$$

Electric field intensity is then

$$\begin{aligned}
\vec{E} &= -\vec{\nabla} V \\
&= -\frac{\partial V}{\partial R} \hat{a}_R \\
&= -\frac{V_0 R_1 R_2}{(R_2 - R_1) R^2} \hat{a}_R
\end{aligned}$$

And current is

$$\begin{aligned}
I &= \iint_S \vec{J} \cdot d\vec{S} \\
&= \iint_S \sigma \vec{E} \cdot d\vec{S} \\
&= \sigma \int_0^{2\pi} \int_0^{\theta_0} -\frac{V_0 R_1 R_2}{(R_2 - R_1) R^2} R^2 \sin\theta d\theta d\phi \\
&= -2\pi(1 - \cos\theta_0)\sigma \frac{V_0 R_1 R_2}{R_2 - R_1} \\
&= -\frac{2\pi(1 - \cos\theta_0)\sigma V_0 R_1 R_2}{R_2 - R_1}
\end{aligned}$$

Then resistance is

$$\begin{aligned}
R &= \frac{V}{|I|} \\
&= \frac{R_2 - R_1}{2\pi(1 - \cos\theta_0)\sigma R_1 R_2}
\end{aligned}$$

12. Find the resistance of two conducting spheres immersed in a lossy dielectric.

Solution: no star no do

13. A d-c voltage V_0 is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ε_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ε_2 and conductivity σ_2 in the region $c < r < b$. Find the current density in each region and the surface charge density on the two conductors and the interface between the dielectrics.

Solution: I am lazy, so I will be brief. From Poisson's equation, we obtain

$$V = C_1 \ln r + C_2$$

Substituting initial conditions,

$$V = \frac{V_0 \ln \frac{r}{a}}{\ln \frac{c}{a}}$$

Differentiating,

$$\vec{E} = \frac{V_0}{r \ln \frac{c}{a}} \hat{a}_r$$

Integrating gives

$$I = \frac{2\pi\sigma_1 L V_0}{\ln \frac{c}{a}}$$

Dividing, the resistance of the first layer ($a < r < c$) is

$$R_1 = \frac{\ln \frac{c}{a}}{2\pi\sigma_1 L}$$

Similarly, the resistance of the second layer is

$$R_2 = \frac{\ln \frac{b}{c}}{2\pi\sigma_2 L}$$

As both layers are in series, total resistance is the sum of the above,

$$R = R_1 + R_2 = \frac{1}{2\pi L}\left(\frac{\ln \frac{c}{a}}{\sigma_1} + \frac{\ln \frac{b}{c}}{\sigma_2}\right)$$

Then current is

$$I = \frac{V_0}{R} = \frac{2\pi L\sigma_1\sigma_2 V_0}{\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}}$$

And current density is

$$J = \frac{I}{S} = \frac{\sigma_1\sigma_2 V_0}{r\left(\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}\right)}$$

The E fields are

$$E_1 = \frac{J}{\sigma_1} = \frac{\sigma_2 V_0}{r\left(\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}\right)}$$

and

$$E_2 = \frac{\sigma_1 V_0}{r\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}}$$

Since there is no $\vec{D}$ outside the conductors, and that $\vec{D}$ is normal to the surface, we can simplify surface charge density as $\pm\vec{D}$. On $r = a$,

$$\rho_s = \varepsilon_1 E_1(r = a) = \frac{\varepsilon_1\sigma_2 V_0}{a\left(\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}\right)}$$

For $r = b$,

$$\rho_s = -\varepsilon_2 E_2(r = b) = -\frac{\varepsilon_2\sigma_1 V_0}{b\left(\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}\right)}$$

Surface charge density is then

$$\rho_s = \varepsilon_2 E_2(r = c) - \varepsilon_1 E_1(r = c) = \frac{(\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2)V_0}{c\left(\sigma_2 \ln \frac{c}{a} + \sigma_1 \ln \frac{b}{c}\right)}$$

14. The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as shown below. Find the resistance between the plates, the surface charge density on the plates, and the amount of charge between the plates.

Solution: Nonessential.

15. An electron is injected with a velocity $\vec{u}_0 = \hat{a}_y u_0$ into a region where both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ exist. Find the velocity of the electron for all time $\vec{u}(t)$.

Solution: Assuming $\vec{E} = \hat{a}_z E_0$ and $\vec{B} = \hat{a}_x B_0$, their contributions to force is

$$\vec{F}_E = q\vec{E} = -eE_0\hat{a}_z$$

and

$$\vec{F}_B = q\vec{u} \times \vec{B} = -eB_0 u_z(t)\hat{a}_y + eB_0 u_y(t)\hat{a}_z$$

Now total force is

$$\vec{F} = -eB_0 u_z(t)\hat{a}_y + e(B_0 u_y(t) - E_0)\hat{a}_z$$

Rewriting force as a derivative of $\vec{u}(t)$,

$$m(u_x'(t)\hat{a}_x + u_y'(t)\hat{a}_y + u_z'(t)\hat{a}_z) = -eB_0 u_z(t)\hat{a}_y + e(B_0 u_y(t) - E_0)\hat{a}_z$$

Then we know $u_x(t)$ is a constant. We get the simultaneous equations

$$\begin{cases} u_y'(t) = -\frac{eB_0}{m} u_z(t) \\ u_z'(t) = \frac{e}{m}(B_0 u_y(t) - E_0) \end{cases}$$

Differentiating the second equation with respect to t and substituting the first,

$$u_z''(t) = -\frac{e^2 B_0^2}{m^2} u_z(t)$$

This gives the solution

$$u_z(t) = A\cos\left(\frac{eB_0}{m}t\right) + B\sin\left(\frac{eB_0}{m}t\right)$$

We know $u_z(0) = 0$ since $\vec{u}(0)$ only has an $\hat{a}_y$ component. Therefore $A = 0$. Substituting back into the first equation,

$$u_y'(t) = -\frac{BeB_0}{m}\sin\left(\frac{eB_0}{m}t\right)$$

Integrating and substituting $u_y(0) = u_0$, we have

$$u_y(t) = B\cos\left(\frac{eB_0}{m}t\right) + u_0 - B$$

Then differentiating $u_z(t)$ gives us

$$u_z'(t) = \frac{BeB_0}{m}\cos\left(\frac{eB_0}{m}t\right)$$

Substituting into the second equation,

$$B = u_0 - \frac{E_0}{B_0}$$

So

$$\vec{u}(t) = \left(\left(u_0 - \frac{E_0}{B_0}\right)\cos\left(\frac{eB_0}{m}t\right) + \frac{E_0}{B_0}\right)\hat{a}_y + \left(u_0 - \frac{E_0}{B_0}\right)\sin\left(\frac{eB_0}{m}t\right)\hat{a}_z$$

Hence if $\frac{E_0}{B_0} = u_0$, then $\vec{u}(t)$ is constant. If $\frac{E_0}{B_0} << u_0$ or the opposite, we have an almost circular motion.

Now if $\vec{E} = -\hat{a}_z E_0$ and $\vec{B} = -\hat{a}_z B_0$, then

$$\begin{aligned} \vec{F} &= q\vec{u} \times \vec{B} + q\vec{E} \\ &= eB_0 u_y(t)\hat{a}_x - eB_0 u_x(t)\hat{a}_y + eE_0\hat{a}_z \\ m(u_x'(t)\hat{a}_x + u_y'(t)\hat{a}_y + u_z'(t)\hat{a}_z) &= eB_0 u_y(t)\hat{a}_x - eB_0 u_x(t)\hat{a}_y + eE_0\hat{a}_z \end{aligned}$$

Comparing like terms, $u_z(t) = \frac{eE_0}{m}t$.

$$\begin{cases} u_x'(t) = \frac{eB_0}{m}u_y(t) \\ u_y'(t) = -\frac{eB_0}{m}u_x(t) \end{cases}$$

Similar to above, we get

$$u_x''(t) = -\frac{e^2B_0^2}{m^2}u_x(t)$$

which yields

$$u_x(t) = A\sin\left(\frac{eB_0}{m}t\right)$$

Substituting into the second equation,

$$u_y(t) = A\cos\left(\frac{eB_0}{m}t\right) + u_0 - A$$

Substituting back into the first equation, $A = u_0$, so

$$\vec{u}(t) = u_0\sin\left(\frac{eB_0}{m}t\right)\hat{a}_x + u_0\cos\left(\frac{eB_0}{m}t\right)\hat{a}_y + \frac{eE_0}{m}t$$

Thus the electron displays helical motion.