Lecture 8

niceguy

December 5, 2022

1 Examples on Change of Variables

Example 1.1. Change the variables of a double integral from rectangular to polar coordinates.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Differentiating,

$$x_r = \cos\theta$$

$$x_\theta = -r\sin\theta$$

$$y_r = \sin\theta$$

$$y_\theta = r\cos\theta$$

The Jacobian is

$$x_r y_\theta - x_\theta r_y = r\cos^2\theta + r\sin^2\theta = r$$

Hence $dxdy = rdrd\theta$.

Example 1.2. Evaluate the integral $\iint_R (x^2 + 2xy)dA$ where R is the region bounded by the lines $y = 2x + 3$, $y = 2x + 1$, $y = 5$, $y = 2 - x$.
Let $u = y - 2x$ and $v = x + y$. Then

$$x = \frac{v - u}{3}$$

$$y = \frac{u + 2v}{3}$$

$$x_u = -\frac{1}{3}$$
$$x_v = \frac{1}{3}$$
$$y_u = \frac{1}{3}$$
$$y_v = \frac{2}{3}$$
$$J = |x_u y_v - x_v y_u| = \frac{1}{3}$$

The integral is given by

$$\begin{aligned}
I &= \int_2^5 \int_1^3 \frac{v^2 - 2vu + u^2 + 4v^2 - 2vu - 2u^2}{9} \frac{1}{3} dudv \\
&= \frac{1}{27} \int_2^5 \int_1^3 5v^2 - 4vu - u^2 dudv \\
&= \frac{1}{27} \int_2^5 10v^2 - 16v - \frac{26}{3} dv \\
&= \frac{1}{27}(390 - 168 - 26) \\
&= \frac{196}{27}
\end{aligned}$$

Example 1.3. Evaluate $\int_R xydxdy$ where R is the first quadrant region bounded by the curves: $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $x^2 - y^2 = 1$,$x^2 - y^2 = 4$.
Define

$$u = x^2 + y^2$$
$$v = x^2 - y^2$$

Then

$$x = \sqrt{\frac{u+v}{2}}$$
$$y = \sqrt{\frac{u-v}{2}}$$
$$x_u = x_v = \frac{1}{2\sqrt{2(u+v)}}$$

$$y_u = \frac{1}{2\sqrt{2(u-v)}}$$

$$y_v = -\frac{1}{2\sqrt{2(u-v)}}$$

The Jacobian is then

$$J = |x_u y_v - x_v y_u| = \frac{1}{4\sqrt{(u+v)(u-v)}}$$

And the integral is given by

$$\begin{aligned} I &= \int_1^4 \int_4^9 \frac{\sqrt{(u+v)(u-v)}}{2} \times \frac{1}{4\sqrt{(u+v)(u-v)}} dudv \\ &= \int_1^4 \int_4^9 \frac{1}{8} dudv \\ &= \frac{15}{8} \end{aligned}$$

2 More on Jacobians

2.1 In 3 Dimensions

The Jacobian with 3 variables is similar to that of 2 variables, where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

2.2 Inverses of Jacobians

Let R be a region on the xy plane, S be the equivalent on the uv plane and T be the equivalent on the pq plane. Considering

$$dxdy = J_{R \to S} dudv$$

and similar equations between R, S, and T, it is obvious that

$$J_{R \to S} = J_{R \to T} \times J_{T \to S}$$

In other words, Jacobians behave like fractions

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(p,q)} \times \frac{\partial(p,q)}{\partial(u,v)}$$

Similarly, it follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}$$