Lecture 4

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1 Review

1.1 Solving Separable Equations

Consider

$$\frac{du}{dt} = f(u)g(t)$$

This is solved by

$$\frac{1}{f(u)}\frac{du}{dt} = g(t)$$
$$\int \frac{1}{f(u)}du = \int g(t)dt + C, C \in \mathbb{R}$$

Note: it might not be possible to find an explicit expression for u.

Example 1.1.

$$\frac{du}{dt} = u^2, u(1) = -1$$
$$\frac{1}{u^2}\frac{du}{dt} = 1$$
$$\int \frac{1}{u^2}du = \int 1dt$$
$$-\frac{1}{u} = t + C$$
$$u = -\frac{1}{t+C}, C \in \mathbb{R}$$

where $u(1) = -1 \Rightarrow C = 0, u = -\frac{1}{t}$. This is a local solution, as the interval $(-\infty, 0)$ is not included. Consider

$$f(t) = \begin{cases} -\frac{1}{t} & t \in (0, \infty) \\ -\frac{1}{2+t} & t \in (-\infty, 0] \end{cases}$$

There is no unique solution for that interval.

2 First Order Linear Equations

$$a_0(t)u(t) + a_1(t)u'(t) = h(t)$$

Assuming $a_1(t) \neq 0$, it can be rearranged as

$$u'(t) + p(t)u(t) = q(t)$$

where $p(t)$ and $q(t)$ are the quotients.
Using the product rule, we have

$$\frac{d}{dt}\mu(t)u(t) = \mu'(t)u(t) + \mu(t)u'(t)$$

Factoring $\mu(t)$ out, we have

$$\mu(t)\left(\frac{\mu'(t)}{\mu(t)}u(t) + u'(t)\right)$$

If $\frac{\mu'(t)}{\mu(t)} = p(t)$ and $\mu(t) \neq 0$, this can be solved conveniently.
Assuming this is true,

$$\frac{d}{dt}(\mu(t)u(t)) = \mu(t)g(t)$$

Integrating both sides, we can solve for u by

$$u(t) = \frac{\int \mu(t)g(t)dt + C}{\mu(t)}$$

What remains is to solve for $\mu(t)$.

$$\begin{aligned}
\frac{1}{\mu}\frac{d\mu}{dt} &= p(t) \\
\int \frac{1}{\mu} d\mu &= \int p(t)dt \\
\ln|\mu| &= \int p(t)dt \\
\mu &= \pm e^{\int p(t)dt}
\end{aligned}$$

Note that the $\pm$ term and C terms are constants that are on both sides. Therefore, they can be ignored. In addition, it is an exponential, so it can never be 0.
Hence, defining $\mu(t) = e^{\int p(t)dt}$, the solution is given by

$$u(t) = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)q(s)ds + C\right)$$

$\mu(t)$ is called the **integrating factor**.

Example 2.1.

$$(t^2 - 1)u' + 2tu = t, u(0) = 1$$

Putting this into standard form,

$$u' + \frac{2t}{t^2 - 1}u = \frac{t}{t^2 - 1}$$

The integrating factor is given by

$$\mu(t) = t^2 - 1$$

Then

$$(t^2 - 1)u(t) = \frac{t^2}{2} + C$$

And rearranging gives us

$$u(t) = \frac{t^2}{2(t^2 - 1)} + \frac{C}{t^2 - 1}$$

Substituting the initial conditions,

$$u(t) = \frac{t^2}{2(t^2 - 1)} - \frac{1}{t^2 - 1}$$