Lecture 29

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1 Discontinuous Forcing Functions

Example 1.1.

$$y''(t) + \pi^2 y(t) = f(t), y'(0) = 0, y''(0) = 0$$

where $f(t)$ is the square wave.
Note that the window function of $f(t)$ is

$$f_2(t) = \begin{cases} 1, & t \in [0,1) \\ 0, t \in [1,2) \end{cases}$$

Applying the Laplace Transform,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{\mathcal{L}\{f_2(t)\}}{1 - e^{-2s}} \\
&= \frac{\int_0^2 e^{-st} f_2(t) dt}{1 - e^{-2s}} \\
&= \frac{\int_0^1 e^{-st} dt}{1 - e^{-2s}} \\
&= \frac{1}{s(1 + e^{-s})} \\
\mathcal{L}\{y''(t) + \pi^2 y(t)\} &= s^2 Y(s) - sy(0) - y'(0) + \pi^2 Y(s) \\
&= s^2 Y(s) + \pi^2 Y(s) \\
Y(s) &= \frac{1}{s(s^2 + \pi^2)(1 + e^{-s})}
\end{aligned}$$

Define

$$H(s) = \frac{1}{s(s^2 + \pi^2)}$$

Using partial fractions,

$$H(s) = \frac{1}{\pi^2}\left(\frac{1}{s} - \frac{s}{s^2 + \pi^2}\right)$$

Using the lookup table,

$$h(t) := \mathcal{L}^{-1}\{H(s)\} = \frac{1}{\pi^2}(1 - \cos \pi t)$$

We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Therefore

$$\frac{1}{1+e^{-s}} = \sum_{k=0}^{\infty} (-1)^k e^{-ks}$$

Therefore

$$\begin{aligned}
\mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{H(s)}{1+e^{-s}}\right\} \\
&= \mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty} (-1)^k H(s) e^{-Ks}\right\} \\
&= \sum_{k=0}^{\infty} (-1)^k \mathcal{L}^{-1}\left\{e^{-ks} H(s)\right\} \\
&= \sum_{k=0}^{\infty} (-1)^k u_k(t) h(t-K) \\
&= \frac{1}{\pi^2} \sum_{k=0}^{\infty} (-1)^k [1 - \cos(\pi(t-k))] u_k(t)
\end{aligned}$$

Plugging natural numbers,

$$y(n) = \begin{cases} -\frac{n}{\pi^2}, & n \text{ is even} \\ \frac{n+1}{\pi^2}, & n \text{ is odd} \end{cases}$$

This is an oscillating function whose amplitude increases linearly.

2 Impulse Function

Consider

$$\delta_\epsilon(t) = \frac{u_0(t) - u_\epsilon(t)}{\epsilon} = \begin{cases} \frac{1}{\epsilon}, & t \in [0, \epsilon) \\ 0, & t \in (-\infty, 0) \cup [\epsilon, \infty) \end{cases}$$

We want to take the limit as $\epsilon \to 0$. Note that this is not a function.

Example 2.1.

$$y'' + y = \alpha\delta_\epsilon(t), y(0) = 0, y'(0) = 0$$

Using the Laplace Transform,

$$\begin{aligned} \mathcal{L}\{\alpha\delta_\epsilon\} &= \alpha \int_0^\epsilon e^{-st} \times \frac{1}{\epsilon} dt \\ &= \frac{\alpha}{\epsilon} \times \frac{1 - e^{-\epsilon s}}{s} \\ Y(s) &= \frac{\alpha}{\epsilon} \left(\frac{1 - e^{-\epsilon s}}{(s^2+1)s} \right) \end{aligned}$$

Using the lookup table,

$$y_\epsilon = \frac{\alpha}{\epsilon}[u_0(t)(1 - \cos t) - u_\epsilon(t)(1 - \cos(t - \epsilon)] = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{\alpha}{\epsilon}(1 - \cos t), & t \in [0, \epsilon) \\ \frac{\alpha}{\epsilon}(\cos(t - \epsilon) - \cos t), & t \in [\epsilon, \infty) \end{cases}$$

Taking the limit as $\epsilon \to 0$, the second and third cases go to 0.

We then define the Direc Delta function $\delta_0(t)$.

Definition 2.1. The Dirac Delta function is the "function" where

$$\delta_0(t - t_0) = 0 \forall t \neq t_0$$

and $\forall f$ continuous on the interval $[a, b]$ containing t_0,

$$\int_a^b f(t)\delta_0(t - t_0) dt = f(t_0)$$

The intuition behind the second property is that

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_0^\infty f(t)\delta_\epsilon(t-t_0)dt &= \lim_{\epsilon \to 0} \int_0^\epsilon f(t)\delta_\epsilon(t-t_0)dt \\
&= \lim_{\epsilon \to 0} \int_0^\epsilon \frac{f(t)}{\epsilon}dt \\
&= \lim_{\epsilon \to 0} \int_0^\epsilon \frac{f(\epsilon)}{\epsilon}dt \\
&= \lim_{\epsilon \to 0} f(\epsilon) \\
&= f(0)
\end{aligned}$$

Example 2.2. Find the Laplace Transforms of the Dirac Delta function.

$$\begin{aligned}
\mathcal{L}\{\delta_0(t)\} &= \int_0^\infty e^{-st}\delta_0(t)dt \\
&= e^0 \\
&= 1 \\
\mathcal{L}\{\delta_0(t-t_0)\} &= e^{-st_0}
\end{aligned}$$

Example 2.3. Solve the IVP

$$2y''(t) + y'(t) + 2y(t) = \delta_0(t-5), y(0) = 0, y'(0) = 0$$

$$\begin{aligned}
\mathcal{L}\{\delta_0(t-5)\} &= e^{-5s} \\
\mathcal{L}\{2y'' + y' + 2y\} &= 2s^2Y(s) + sY(s) + 2Y(s) \\
Y(s) &= \frac{e^{-5s}}{2s+2+s+2} \\
&= \frac{e^{-5s}}{2} \times \frac{1}{\left(\frac{s+1}{4}\right)^2 + \frac{15}{16}}
\end{aligned}$$