

# Lecture 2

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## 1 Definitions and Terminology

**Definition 1.1.** The order of a differential equation is the order of the highest derivative, ordinary or partial, in the equation.

**Example 1.1.**

$$ay''' + by'' = f(t)$$

This is a 3<sup>rd</sup> order ODE.

**Example 1.2.**

$$u_t = \frac{d}{dx}(u_x + y_y + u_z)$$

This is a 1<sup>st</sup> order PDE.

**Definition 1.2.**  $u^{(n)}$  refers to the  $n^{\text{th}}$  derivative of  $u$ .

The most general equation of an  $n^{\text{th}}$  order derivative ODE is:

$$F[t, u, u', u'', \dots, u^{(n)}] = 0$$

**Definition 1.3.** We say an  $n^{\text{th}}$  order ODE is linear if it can be expressed as

$$a_0(t)u(t) + a_1(t)u'(t) + \dots + a_n(t)u^{(n)}(t) = g(t)$$

If  $g(t) = 0$  for all  $t$ , the equation is homogeneous, otherwise it is nonhomogeneous.

**Example 1.3.** Are the following differential equations linear? If so, are they homogeneous or not?

- $t \sin(t^3)u + 56u''' + t^4 = 0$

- $t^3u''' + et^2u'' + 4u = 0$

- $tu + 5\sqrt{u'} + 10 \sin(u) = 0$

The first equation is linear nonhomogeneous, due to the  $t^4$  term. The second equation is linear homogeneous, and the third equation is non linear, due to the second and third terms.

How could we extend these definitions to PDEs?

Consider

$$a_0(x, y)u_x + a_1(x, y)u_y + a_2(x, y)u_{xx} + a_3(x, y)u_{xy} + a_4(x, y)u_{yx} + a_5(x, y)u_{yy} + a_6(x, y)u = g(x, y)$$

This is a linear equation. If  $g(x, y) = 0 \forall x, y$ , it is homogeneous.

**Definition 1.4.** An ODE is *autonomous* if it does not explicitly depend on the independent variable.

**Example 1.4.**      •  $\frac{dy}{dt} = \sin(y) + y^3 \ln(y) + e^y$

- $\frac{dy}{dt} = y' \cos(y) + \frac{y^2}{1+e^y}$

Does there exist a linear autonomous nonhomogeneous ODE?

Consider

$$y + y' = 1$$

For it to be nonhomogeneous, a  $g(t)$  must exist. It must be independent of  $t$ , so it can only be a constant (since  $g'(t)$  must be 0).

**Definition 1.5.** A first order ODE is separable if it can be expressed as

$$\frac{du}{dt} = f(u)g(t)$$

**Example 1.5.** Which of the following equations are separable?

- $\frac{dx}{dt} = t^2 + \frac{\arccos(t)}{\ln(t)}$
- $u'' = ut^2e^t$
- $\frac{dv}{dt} = tv + t^2$
- $\frac{du}{dt} = \frac{1}{2}(\sin(u+t) - \sin(u-t)) + \sin(t)$

Yes, no (second order), no, yes (trig identity).

## 2 Systems of Differential Equations

This occurs when two or more dependent variables interact with one another.

**Example 2.1.** The Lotka-Volterra equations for a predator-prey model. The assumptions are that the prey will only die when eaten, and predators either naturally die or relocate.

$u_1(t)$  : Population count of prey

$u_2(t)$  : Population count of predator

Considering the birth and death of prey,

$$\frac{du_1}{dt} = \alpha u_1 - \beta u_1 u_2$$

Considering the birth and death of predators,

$$\frac{du_2}{dt} = \gamma u_1 u_2 - \delta u_2$$

Intuition tells us that this should be a system of periodic functions, with  $u_2$  lagging behind  $u_1$ .

### 3 Initial Value Problems

In general, is one initial value enough?

No, it depends on the order. E.g. a second order ODE requires 2 initial values.

**Example 3.1.** Consider

$$\frac{d^2u}{dt^2} = 0$$

Integrating both sides with respect to  $t$ , we have

$$\frac{du}{dt} = C_0$$

and integrating again gives us

$$u = C_0t + C_1$$

Therefore, two initial values are required to solve for the two constants.

**Definition 3.1.** An initial value problem for an  $n^{\text{th}}$  order ODE consists of the ODE itself and  $n$  initial conditions.

$$u(t_0) = a_0, u'(t_0) = a_1, \dots, u^{(n-1)}(t_0) = a_{n-1}$$