Homework 2

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1. If a letter is chosen at random from the English alphabet, find the probability that the letter
   (a) is a vowel exclusive of y;
   (b) is listed somewhere ahead of the letter j;
   (c) is listed somewhere after the letter g

Solution: $\frac{5}{26}, \frac{9}{26}, \frac{19}{26}$

2. Prove that

$$P(A' \cap B') = 1 + P(A \cup B) - P(A) - P(B)$$

Solution:

$$\begin{aligned} P(A' \cap B') &= P((A \cup B)') \\ &= 1 - P(A \cup B) \\ &= 1 + P(A \cap B) - P(A) - P(B) \end{aligned}$$

where the first equality comes from De Morgan.

3. Pollution of the reivers in the United States has been a problem for many years. Consider the following events:

$A:$ the river is polluted

$B:$ a sample of water tested detects pollution

$C:$ fishing is permitted

Assume $P(A) = 0.3, P(B|A) = 0.75, P(B|A') = 0.20, P(C|A \cap B) = 0.20, P(C|A' \cap B) = 0.15, P(C|A \cap B') = 0.80$, and $P(C|A' \cap B') = 0.90$.

   (a) Find $P(A \cap B \cap C)$
   (b) Find $P(B' \cap C)$
   (c) Find $P(C)$
   (d) Find the probability that the river is polluted, given that fishing is permitted and the sample tested did not detect pollution.

Solution:

$$\begin{aligned} P(A \cap B \cap C) &= \frac{P(A \cap B \cap C)}{P(A \cap B)} \times P(A \cap B) \\ &= P(C|A \cap B)P(B|A)P(A) \\ &= 0.20 \times 0.75 \times 0.3 \\ &= 0.045 \end{aligned}$$

Given $P(A) = 0.3$ and $P(B|A) = 0.75$, we can deduce $P(A \cap B) = 0.225$. Similarly, from $P(B|A')$ we obtain $P(A' \cap B) = 0.14$. Summing we have $P(B) = 0.365$. This gives $P(A \cap B') = 0.075$ and $P(A' \cap B') = 0.56$. From $P(C|A \cap B)$ we get $P(A \cap B \cap C) = 0.045$. Similarly, $P(A' \cap B \cap C) = 0.021, P(A \cap B' \cap C) = 0.06, P(A' \cap B' \cap C) = 0.504$. Summing we have $P(C) = 0.63$ and $P(B' \cap C) = 0.564$. Then finally

$$\begin{aligned} P(A|C \cap B') &= \frac{P(A \cap B' \cap C)}{P(B' \cap C)} \\ &= \frac{0.06}{0.564} \\ &= \frac{5}{47} \\ &= 0.106 \end{aligned}$$

4. Suppose the diagram of an electrical system is as given in Figure 2.10. What is the probability that the system works? Assume the components fail independently.

Solution:

$$0.95 \times (1 - 0.2 \times 0.3) \times 0.9 = 0.8037$$

5. A paint-store chain produces and sells latex and semigloss paint. Based on long-range sales, the probability that a customer will purchase latex paint is 0.75. Of those that purchase latex paint, 60% also purchase rollers. But only 30% of semigloss paint buyers purchase rollers. A randomly selected buyer purchases a roller and a can of paint. What is the probability that the paint is latex?

Solution: Let A be the purchase of latex paint, B be the purchase of rollers, and C the the purchase of semigloss paint. Then

$$\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{P(B|A)P(A)}{P(B)} \\
&= \frac{0.6 \times 0.75}{P(B|A)P(A) + P(B|C)P(C)} \\
&= \frac{0.45}{0.45 + 0.3 \times 0.25} \\
&= \frac{0.45}{0.525} \\
&= 0.857
\end{aligned}$$

6. Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each sample point assign a value w of W.

Solution:

$$S = \{HHH(3), HHT(1), HTH(1), HTT(-1), THH(1), THT(-1), TTH(-1), TTT(-3)\}$$

7. The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability that over a period of one year, a family runs their vacuum cleaner

(a) less than 120 hours

(b) between 50 and 100 hours

Solution: For less than 120 hours, $x \in [0, 1.2]$, so the probability is

$$0.5 + (1 + 0.8) \times 0.1 = 0.68$$

For between 50 and 100 hours, the probability is

$$0.5 \times 0.75 = 0.375$$

8. Suppose it is known from large amounts of historical data that X, the number of cars that arrive at a specific intersection during a 20-second time period, is characterized by the following discrete probability function:

$$f(x) = e^{-6}\frac{6^x}{x!} \forall x \in \mathbb{N}$$

(a) Find the probability that in a specific 20-second time period, more than 8 cars arrive at the intersection.

(b) Find the probability that only 2 cars arrive.

Solution: For more than 8 cars to arrive, the probability is

$$1 - \sum_{i=0}^{8} f(i) = 0.153$$

For two cars to arrive, the probability is

$$f(2) = 0.0446$$

9. Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature (°F) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f(x, y) = \begin{cases} 4xy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(a) $P(0 \leq X \leq \frac{1}{2} \cap \frac{1}{4} \leq Y \leq \frac{1}{2})$

(b) $P(X < Y)$

Solution: The first probability is given by

$$\int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^{\frac{1}{2}} 4xydydx = 0.046875$$

For the second probability, note that $P(X = Y) = 0$, so by symmetry

$$P(X < Y) = 0.5$$

10. The joint probability density function of the random variables X, Y, and Z is

$$f(x, y, z) = \begin{cases} \frac{4xyz^2}{9} & 0 < x, y < 1, 0 < z < 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(a) the joint marginal density function of Y and Z

(b) the marginal density of Y

(c) $P(\frac{1}{4} < X < \frac{1}{2}, Y > \frac{1}{3}, 1 < Z < 2)$

(d) $P(0 < X < \frac{1}{2} | Y = \frac{1}{4}, Z = 2)$

Solution: The joint marginal density is

$$g(y,z) = \int_0^1 \frac{4xyz^2}{9} dx = \frac{2yz^2}{9}$$

The marginal density is

$$h(y) = \int_0^3 \int_0^1 \frac{4xyz^2}{9} dxdz = 2y$$

The first desired probability is

$$\int_1^2 \int_{\frac{1}{3}}^1 \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{4xyz^2}{9} dxdydz = 0.0432$$

The second desired probability is

$$\frac{\int_0^{\frac{1}{2}} \frac{4x}{9} dx}{g\left(\frac{1}{4}, 2\right)} = \frac{1}{18} \div \frac{2}{9} = \frac{1}{4}$$