Legendre Polynomials

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1 What is a Legendre Polynomial

- Legendre polynomials of the first kind are also referred to as zonal harmonics
- They are part of the general solution to Legendre's differential equation
- The nth polynomial is of order n

$$P_n(x) \sim x^n \tag{1}$$

- An interesting property is that Legendre polynomials are orthogonal over the interval $I = [-1; 1]$ and satisfy

$$\int_{-1}^{1} P_m(x) P_n(x) \mathrm{dx} = \frac{2}{2n+1} \delta_{mn} \tag{2}$$

- They also satisfy

$$P_n(1) = 1 \tag{3}$$

2 Generating Legendre Polynomials

- Given the first two Legendre polynomials

$$P_0(x) = 1 \tag{4}$$
$$P_1(x) = x \tag{5}$$

We can generate a sequence of polynomials by generating $n+1$ equations and solving for the coefficients using the inner product of known polynomials

2.1 P_2

- We assume that P_2 has the form $ax^2 + bx + c$

$$\int_{-1}^{1} (1)P_2 \mathrm{dx} = \frac{ax^3}{3} + \frac{bx^2}{2} + cx\Big|_{-1}^{1} \tag{6}$$

$$= \frac{2a}{3} + 2c = 0 \tag{7}$$

$$\int_{-1}^{1} (x)P_2 \mathrm{dx} = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2}\Big|_{-1}^{1} \tag{8}$$

$$= \frac{2b}{3} = 0 \implies b = 0 \tag{9}$$

- Using the property from Equation (3) we also find

$$a + c = 1 \tag{10}$$

Finally we arrive at

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{11}$$

2.2 P_3

- We can similarily find P_3
- Assuming the form $P_3 = ax^3 + bx^2 + cx + d$

$$\int_{1}^{1} (1)P_3 \mathrm{dx} = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\Big|_{-1}^{1} \tag{12}$$

$$= \frac{2b}{3} + 2d = 0 \tag{13}$$

$$\int_{-1}^{1} (x)P_3 \mathrm{dx} = \frac{ax^5}{5} + \frac{bx^4}{4} + \frac{cx^3}{3} + \frac{dx^2}{2}\Big|_{-1}^{1} \tag{14}$$

$$= \frac{2a}{5} + \frac{2c}{3} = 0 \tag{15}$$

$$\int_{-1}^{1} \frac{1}{2}(3x^2 - 1)P_3 \mathrm{dx} = \frac{1}{2}\int_{-1}^{1} 3ax^5 + 3bx^4 + 3cx^3 + 3dx^2 - ax^3 - bx^2 - cx - d \, \mathrm{dx} \tag{16}$$

$$= \frac{3ax^6}{6} + \frac{3bx^5}{5} + \frac{(3c - a)x^4}{4} + \frac{(3d - b)x^3}{3} - \frac{cx^2}{2} - dx\Big|_{-1}^{1} \tag{17}$$

$$= \frac{6b}{5} + \frac{6d - 2b}{3} - 2d = 0 \tag{18}$$

- Again we can find

$$a + b + c + d = 1 \tag{19}$$

- Note that Step (18) and by extension Step (13) yield

$$b = 0 \quad d = 0 \tag{20}$$

- Step (15) and evaluating $P_3(1)$ then yield

$$c = -\frac{3}{2} \quad a = \frac{5}{2} \tag{21}$$

We arrive at

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \tag{22}$$