Coordinate Systems

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1 Conversion

- The rectangular coordinates are called invariant and can be used to reference other system
- Converting from Spherical to Rectangular (define rectangular in terms of spherical)

$$x = r\sin(\theta)\cos(\phi) \tag{1}$$
$$y = r\sin(\theta)\sin(\phi) \tag{2}$$
$$z = r\cos(\theta) \tag{3}$$

- From Rectangular to Spherical

$$r = \sqrt{x^2 + y^2 + z^2} \tag{4}$$
$$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \tag{5}$$
$$\phi = \arctan\left(\frac{y}{x}\right) \tag{6}$$

- If we express a vector as a sum of unit vectors that describe the system, we can convert it by determining the appropriate conversion factors. This is done by taking dot products.
- We get

$$\hat{\mathbf{r}} = \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \tag{7}$$
$$\hat{\phi} = -\sin(\phi)\hat{\mathbf{x}} + \cos(\phi)\hat{\mathbf{y}} \tag{8}$$
$$\hat{\theta} = \cos(\theta)\cos(\phi)\hat{\mathbf{x}} + \cos(\theta)\sin(\phi) - \sin(\theta)\hat{\mathbf{z}} \tag{9}$$
$$\tag{10}$$
$$\hat{\mathbf{x}} = \sin(\theta)\cos(\phi)\hat{\mathbf{r}} + \cos(\theta)\cos(\phi)\hat{\theta} - \sin(\phi)\hat{\phi} \tag{11}$$
$$\hat{\mathbf{y}} = \sin(\theta)\sin(\phi)\hat{\mathbf{r}} + \cos(\theta)\sin(\phi)\hat{\theta} + \cos(\phi)\hat{\phi} \tag{12}$$
$$\hat{\mathbf{z}} = \cos(\theta)\hat{\mathbf{r}} - \sin(\theta)\hat{\theta} \tag{13}$$
$$\tag{14}$$

- Thus we can geometrically deduce the Jacobian matrix which describes the transformation

- An interesting property of the euclidean norm of a vector is that it is similar in rectangular coordinates and spherical coordinates:
$$\|\mathbf{r}\| = \sqrt{r^2 + \phi^2 + z^2} \tag{15}$$

2 Laplace Operator

- The Laplace Operator is a useful one when dealing with waves and is fundemental to the Schrödinger equation
- This section will attempt at converting
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{16}$$
To Cylindrical and Spherical coordinate representations by making heavy use of differentiation rules.
- This process is both tedious and rewarding - only time will tell if this is worth the effort.

2.1 Cylindrical Coordinates

- We begin by establishing cylindrical partial derivatives using the chain rule
$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \tag{17}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \tag{18}$$
$$\tag{19}$$
- Then evaluating partial derivatives for r
$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2} = \cos(\phi) \tag{20}$$
$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2 + y^2} = \sin(\phi) \tag{21}$$
$$\tag{22}$$
- And for ϕ using painful implicit differentiation
$$\frac{\partial}{\partial x}\cos(\phi) = \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2 + y^2}} \tag{23}$$
$$-\sin(\phi)\frac{\partial \phi}{\partial x} = \tag{24}$$
$$\frac{\partial}{\partial y}\sin(\phi) = \frac{\partial}{\partial y}\frac{x}{\sqrt{x^2 + y^2}} \tag{25}$$
$$\cos(\phi)\frac{\partial \phi}{\partial y} = \tag{26}$$
$$\tag{27}$$

- Notice that the z term remains unchanged due to the relationship between polar and cylindrical coordinates. We obtain for free the Laplace operator in polar coordinates by omitting the z term (or conversly, we obtain the cylindrical operator for free by ignoring the z as was essentially done).

2.2 Spherical Coordinates

- More of the same but definitely the most brain racking
- Establishing spherical partials

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \tag{28}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} \tag{29}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} \tag{30}$$