Training Report

Bounds on Higher Derivative Couplings In The Effective Field Theory of Inflation

Prepared by:

Temmam Youssef

2nd year Master : Astronomie, Astrophysique & Ingénierie Spatiale (AAIS) - track Astrophysics
Université Paris Science et Lettres (PSL) - Observatoire de Paris

Supervised by:

Dr. Apruzzi Fabio & Pr. Reffert Susanne

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1 Résumé

This report presents the work I conducted in my internship project with Dr.Apruzzi Fabio and Pr.Reffert Susanne at the Institute for Theoretical Physics-Albert Einstein Center for Fundamental Physics. I will report the work done from the begining of the internship on the 4th of April and the 22nd of June.

We will start by presenting the scientific context of the topic, and the host laboratory. After, we will review the classic model of inflation in cosmology. We will also review the formalism of particle scattering in quantum field theory and we will establish the effective Lagrangian for inflation taking into consideration the interaction inflation-gravity that will enable us to compute higher derivative terms. Later, we will apply Bootstrap techniques, namely unitary analyticity, and crossing symmetry of the scattering matrix, to bound the coupling coefficient of those higher derivative terms. Finally we will end by exposing the results of this study and the promising perspectives that it has opened.

2 Introduction

2.1 Scientific context and project goal

"How did start the universe?" a question that have been raised since the dawn of science. From Babylonian era to nowadays, many those who tried to answer it including theologians, philosophers, scientists... But in the fifth century BC, the universe was thought, for the first time as a time evolving structure with a proper beginning by the philosopher Pythagoras. He also introduced the word "*Kosmos*" from "κόσμος" to refer to the universe and the laws that governs its order.

Many centuries later, after the formulation of general relativity, Albert Einstein attempted to study the universe by applying his new theory in 1915. In his model, the universe was a static positive curvature hyper-sphere with no beginning. Einstein's model of cosmology had been adopted until the time when the experimental data began to call into question the validity of static hypothesis. For instance, after Eddington measurements in 1924, the Belgian astronomer "George Lemaitre" conjectured the thesis that the universe is not static and expanded from a single point to get as big as it is. This thesis took the name of *the big bang theory*. Thanks to the evolution of observational techniques, many evidences came to confirm this theory, amongst them we recall Hubble observations on galaxies, CMB radiation... However, the big bang theory describes only the evolutionary model of the universe on a global scale but not the initial "Bang" itself [1]. So what did happen exactly during the that bang? And how did the universe expand during those first moments after his birth?

It was in this exactly moment when The theory of *Inflation* came into play, precisely to answer this very specific question on how the universe expanded from a single point to the size of galaxies at superluminal rate in the first moments of the universe. We know from observations (namely the CMB) that the bang was far homogeneous on large scales, but we don't know the mechanism behind.

This theory was formulated until late 70s, by a theoretical physic group composed of Alexei Starobinsky, Alan Guth and Andrei Linde in order to explain the mechanism behind the big band and also solve many other problems in cosmology: isotropy of the CMB, flatness of the universe, absence of magnetic mono-poles... Particularly, this theory relies on a particle physics model that induces a field called *inflaton* and undergoes the rules of quantum field theory (QFT).

The inflaton field has been studied in different ways, but the most convenient way to study it is through the approach of quantum scalar fields. This field can be coupled to other fields like gravity. In cosmology, this coupling will manifest by the presence of gravity terms in the inflation Lagrangian. In QFT, the coupling of two fields or more can be extended to higher order terms in order to increase the precision of the considered model. However, those higher derivative terms can be added to the s-matrix up to certain coefficients that can take only some values that we have to define. For this purpose, we will constrain the s-matrix using the *Bootstrap techniques* [2] that consist of unitary, analyticity, and stability by crossing symmetry. Those constrains would enable us to establish *dispersion relation* between the low and high energy regimes defined by an energy scale M. However, a pole appears in the low energy part of the dispersion relation formula due to the graviton interaction. To remediate this issue, we will follow the approach described in [3] and will extend it to the case of 4-dimensional space-time by introducing an *Infrared cutoff*. This is the first time in literature where we would bound the higher derivative terms of the gravity interaction in 4-dimensional space-time using Bootstrap techniques.

The plan of this report is the following: we will start with a review of inflation in section 3 where we will define the inflaton field. Then we will derive in section 4 the s-matrix in QFT. In section 5, we will examine EFT, reconsider inflation in this theory and establish the s-matrix including higher derivative terms. In section 6 we will constrain the s-matrix and derive dispersion relation. Finally, in section 7 we will work out the algorithm to get the bounds on the coupling coefficients of the higher derivative terms.

2.2 Hosting Laboratory

Albert Einstein Center For Fundamental physics (AEC) is a laboratory affiliated to the University of Bern. It is a collaboration mainly focused on experimental and theoretical particle physics and its applications. It includes the Institute for Theoretical Physics (ITP) and of the Laboratory for High-Energy Physics (LHEP) of the University of Bern. With its over 100 members the AEC operating in a range fields of physics, from astrophysics and astronomy to string theory and Lattice Gauge theory. As for the ITP [1], there are 7 groups specializing in different topics of theoretical physics:

Thermal Field Theory and Particle Cosmology,

Effective Field Theories - Chiral Perturbation Theory and Non-relativistic QFT,

Particle Physics at Colliders,

Non-perturbative Methods of QFTs,

Fields, Strings and Dualities,

General Relativity and String Theory,

Supersymmetric Field Theories, Supergravity and Superstring Unified Theories.

[1] https://www.itp.unibe.ch/research/research_fields/index_eng.html

3 Review of Inflation

The theory of inflation has essentially emerged to deal with the cosmological shortcomings of flatness and horizon and the accelerating phase of the universe. Those problems motivated scientists such as Guth to look for inflation, cosmological perturbations became part of the story well after inflation was formulated.

The fact that inflation could source primordial perturbations was indeed realized only shortly after the formulation of inflation. At that time, CMB perturbations were not yet observed, but the fact that we observed galaxies today, and the fact that matter grows as $\delta \propto a$ in a matter dominated universe predicted that some perturbations had to exist on the CMB. The way inflation produces these perturbations is both compelling. This theory is that stimulating because it shows that quantum effects, that are usually relegated to the hardly experienciable world of the small distances, can be exponentiated in the peculiar inflationary space-time to become actually the source of all the cosmological perturbations, and ultimately of the galaxies and of all the structures that are present in our universe. With inflation, quantum effects are at the basis of the formation of the largest structures in the universe.

3.1 Inflaton field

The main component behind the theory of inflation is the QFT that happens when we introduce a quantum field in an accelerating space-time. It makes predictions that we are actually testing right now in the universe. The quantization of this field, induces the *inflaton* particle that is represented in the QFT, by a real scalar field $\varphi(x)$[2]. Since it's a scalar field, the dynamics of this field are ruled by the Klein-Gordon equation. We set $\hbar = c = 1$ for simplification.

In Minkowski space-time [3], the Lagrangian takes the form:

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi), \tag{3.1}$$

where $V(\varphi)$ is a potential. We note that different potentials describe different models of inflation theory. We do not know yet the exact form of the inflationary potential. But in our case, we will work with the simplest possible potential in QFT:

$$V(\varphi) = \frac{1}{2}m^2\varphi^2, \tag{3.2}$$

with m is the mass of the field. It follows that the action takes the form:

$$S = \int d^4x \left(-\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right). \tag{3.3}$$

[2] $x = x^\mu = (t, \mathbf{x})$ is a position in the space-time

[3] note that we use the convention $g = \mathrm{diag}(-1, 1, 1, 1)$.

The minimisation of this action yields to the equation of motion, also called *Klein-Gordon* equation:

$$\partial^\mu \partial_\mu \varphi - \frac{\partial V}{\partial \varphi} = 0. \tag{3.4}$$

We want to promote this equation to describe the expanding universe in his homogeneous and isotropic description. So we shall use the FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right), \tag{3.5}$$

where $k \in \{-1, 0, 1\}$ denotes the three possible spacial geometries. In this metric the action becomes:

$$S = \int \frac{a(t)^3 r \sin(\theta)}{\sqrt{1 - kr^2}} dt\, dr\, d\theta\, d\phi \left(\frac{1}{2} \partial_0 \varphi^2 - \frac{1}{2} \nabla \varphi^2 - V(\varphi) \right), \tag{3.6}$$

with:

$$\nabla = \left| \begin{array}{c} \frac{\sqrt{1-kr^2}}{a(t)} \frac{\partial}{\partial r} \\ \frac{1}{a(t) r} \frac{\partial}{\partial \theta} \\ \frac{1}{a(t) r \sin(\theta)} \frac{\partial}{\partial \phi} \end{array} \right., \tag{3.7}$$

is the gradient in the FRW manifold.

3.2 Uniform inflation field

Particularly, in the case of a flat space ($k = 0$) and a uniform field (only time dependant) φ, the equation of motion reduces to:

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0, \tag{3.8}$$

where H is the *Hubble parameter*. In the analogy with the harmonic oscillator, the second term of this last equation resembles a friction term. It is sometimes referred to as *Hubble friction* or *Hubble drag.*

The action in 3.6 accounts only the first order of interaction between the inflation field and the gravitational field. What about the higher orders? How can we bound them? To this end, we will resort to the effective field theory (EFT). But first let us review some useful concepts in QFT that will enable us to compute the EFT.

4 S-matrix in QFT

In field theories, scattering is the process how particles (fields) interact. They can either interact with themselves or with another particle from the same or another field. This interaction is translated by an additional term in the Lagrangian.

4.1 Interaction Lagrangian

In our model we consider the inflaton field as scalar. In the Minkowskian space-time of d dimensions, the Lagrangian take the form of:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{Int}}, \tag{4.1}$$

$$\mathcal{L}_{\text{Int}} = -\sum_{n\geq 3} \frac{\lambda_n}{n!}\varphi^n, \tag{4.2}$$

$$\mathcal{L}_0 = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2, \tag{4.3}$$

where $\mathcal{L}_0$ is the Lagrangian density of free scalar field that we have seen in 3.1, and $\mathcal{L}_{\text{Int}}$ is the interaction Lagrangian density. Note that we will promote the total Lagrangian density $\mathcal{L}$ to one that includes interaction with gravity in section 5.

The coefficients λ_n are *the coupling coefficients.* Let us pause for a while to analyse them further. If we choose the mass of the field as a unit of dimension, then length L and time T have M^{-1} dimension. ($T = [c^{-1}]L = L = [\hbar c^{-1}]M^{-1} = M^{-1}$). Therefore, any quantity A can be thought of as having units of mass to some power (positive, negative, or zero) that we will call $[A]$. For example:

$$[m] = +1, \tag{4.4}$$

$$[x^\mu] = +1, \tag{4.5}$$

$$[\partial^\mu] = +1, \tag{4.6}$$

$$[d^dx] = +d. \tag{4.7}$$

$$\tag{4.8}$$

Since the action is dimensionless, it yields to $\mathcal{L} = d$, $[\varphi] = \frac{1}{2}(d-2)$ and $[\lambda_n] = d - \frac{1}{2}(d-2)$. renormalization theories in [4], chapter.13, imply that $[\lambda_n] = d - \frac{1}{2}(d-2) \leq 0$. Particularly in case of $d = 4$, n can only take $n \in \{3, 4\}$. Furthermore, φ^3 and φ^4 interactions are the ones contributing to the majority of real scalar scattering.

Once we derived the terms of our Lagrangian, we can move to define the scattering amplitudes.

4.2 Correlation function

An easy way to derive those amplitudes is using the correlation function. This latter is obtained through Feynman path integrals. This method was used for quantum mechanics and was generalized to QFT as described in [**?**], chapter.9.

In quantum mechanics

The evolution of a free particle between (t', q') and (t'', q''), also known as the *correlation function*, is given by [5] section.1:

$$\langle q'', t'' \mid q', t' \rangle_f = \int \mathcal{D}q \exp\left[i \int_{t'}^{t''} dt L_0(\dot{q}(t), q(t)) \right], \tag{4.9}$$

where $\mathcal{D}q$ is the measure over the position space and $L_0(\dot{q}(t), q(t))$ is the classical Lagrangian of a free particle. In the Heisenberg picture [4], when the particle reaches the position $q(t_1)$ [5] at time $t' < t_1 < t''$, the particle amplitude is obtained by:

$$\langle q'', t'' \mid Q(t_1) \mid q', t' \rangle_f = \int \mathcal{D}q\mathcal{D}p\, q(t_1) \exp(iS). \tag{4.10}$$

By tuning the Lagrangian using the functional derivative

$$L_0(\dot{q}(t), q(t)) \to L_0(\dot{q}(t), q(t)) + f(t)q(t), \tag{4.11}$$

we can obtain the same result as following:

$$\begin{aligned} \frac{1}{i}\frac{\delta}{\delta f(t_1)} \langle q'', t'' \mid q', t' \rangle_f &= \int \mathcal{D}q\, q(t_1) \exp\left[i \int_{t'}^{t''} dt (L_0 + fq) \right] \\ &= \langle q'', t'' \mid Q(t_1) \mid q', t' \rangle_f . \end{aligned} \tag{4.12}$$

One must note that f can be also interpreted as an external force applied to the particle. Now if we take $t' \to -\infty$, and $t'' \to +\infty$, we get the evolution of the particle on the whole space-time:

$$\begin{aligned} \langle 0 \mid 0 \rangle_f &= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int dq'' dq' \langle q'' \mid 0 \rangle^* \langle q'', t'' \mid q', t' \rangle_f \langle q' \mid 0 \rangle \\ &= \int \mathcal{D}q \exp\left[i \int_{-\infty}^{+\infty} dt (L_0(\dot{q}(t), q(t)) + f(t)q(t)) \right]. \end{aligned} \tag{4.13}$$

We shall define the *Functional integral* as:

$$Z_0(f) = \int \mathcal{D}q \exp\left[i \int_{-\infty}^{+\infty} dt (L_0(\dot{q}(t), q(t)) + f(t)q(t)) \right], \tag{4.14}$$

thus we can write:

$$\langle 0 \mid Q(t_1) \mid 0 \rangle_f = \frac{1}{i}\frac{\delta}{\delta f(t_1)} Z_0(f) \bigg|_{f=0}. \tag{4.15}$$

[4] operators are time dependent and the state of the system is time independent, as opposed to the more familiar Schrodinger picture where the system is time dependent.

[5] $q(t_1)$ is the eigen value of the position operator Q is the position operator.

In quantum field theory

Similarly, we can generalize those results in QFT. Indeed, the position operator Q upgrades to the field φ and the source f becomes J:

$$Z_0(J) \equiv \langle 0 \mid 0 \rangle_J = \int \mathcal{D}\varphi e^{i \int d^4x [\mathcal{L}_0 + J\varphi]}, \tag{4.16}$$

where:

$$\mathcal{D}\varphi \propto \prod_x d\varphi(x), \tag{4.17}$$

is the functional measure. Now we have established the functional, we can derive the scattering amplitude. For $2 \longrightarrow 2$ scattering (2 incoming particles with respective 4-momentum k_1 and k_2 that scatter into 2 outgoing particles with respective 4-momentum k_3 and k_4), represented in figure 1. Thereby, the correlation function is written as:

$$\begin{aligned}
\langle 0 \mid T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \mid 0 \rangle &= \delta_1\delta_2\delta_3\delta_4 Z(J)\Big|_{J=0} && (4.18)\\
&= \delta_1\delta_2\delta_3\delta_4 \int \mathcal{D}\varphi e^{i \int d^4x [\mathcal{L}_{\text{Int}} + \mathcal{L}_0 + J\varphi]}\Big|_{J=0} && (4.19)\\
&= \delta_1\delta_2\delta_3\delta_4 \left(\exp\left[i \int d^4x \mathcal{L}_{\text{Int}} \left(\frac{\delta}{\delta J(x)} \right) \right] Z_0(J) \right)\Big|_{J=0}, && (4.20)
\end{aligned}$$

because $\int d^4x \mathcal{L}_{\text{Int}} \left(\frac{\delta}{\delta J(x)} \right)$ does not depend on φ and can be taken outside the integral in 4.19. T is *the time ordering operator* that orders fields from the recent to the old, from left to right.

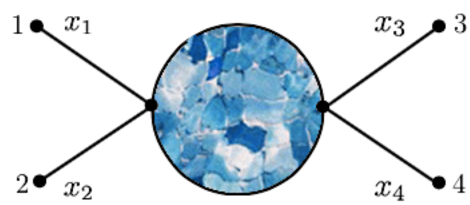


Figure 1: Representative diagram for $2 \longrightarrow 2$ scattering, corresponding to the correlation function $\langle 0 \mid T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \mid 0 \rangle$. The mosaic circle inside represent the internal interactions that we haven't yet specified.

4.3 Scattering Amplitude

Now we have assembled all the needed pieces to work out the scattering amplitude. In our scattering model, we consider that the incoming particle comes from the far past, and the outgoing particles goes after scattering to the far future. In other words, the initial state in the far past is expressed as:

$$\begin{aligned} \mid i\rangle &= \lim_{t\to-\infty} \mid \mathbf{k}_1\mathbf{k}_2\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\mid 0\rangle \\ &= a_1^\dagger(-\infty)a_2^\dagger(-\infty)\mid 0\rangle, \end{aligned} \tag{4.21}$$

and the final state as:

$$\begin{aligned} \mid f\rangle &= \lim_{t\to+\infty} \mid \mathbf{k}_3\mathbf{k}_4\rangle = \lim_{t\to+\infty} a_3^\dagger(t)a_4^\dagger(t)\mid 0\rangle \\ &= a_3^\dagger(+\infty)a_4^\dagger(+\infty)\mid 0\rangle, \end{aligned} \tag{4.22}$$

where $a^\dagger$ is the creation operator for the field φ. Therefore, the scattering amplitude is:

$$\langle f \mid i\rangle = \langle 0 \mid a_3(+\infty)a_4(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\mid 0\rangle. \tag{4.23}$$

In order to link this formula to the correlation function, we must express a and $a^\dagger$ in terms of φ. We note that:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt\partial_0 a_1^\dagger(t), \tag{4.24}$$

by using equation A.4 from the appendix, we get:

$$\begin{aligned} a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= -i\int_{-\infty}^{+\infty} dt\partial_0 \int d^3x e^{ik_1x}\overleftrightarrow{\partial}_0\varphi(x) \\ &= -i\int d^3k\delta^3(\mathbf{k}-\mathbf{k}_1)\int d^4x e^{ikx}\left(\partial_0^2+\omega^2\right)\varphi(x), \end{aligned} \tag{4.25}$$

and using the 4-momentum amplitude ($k^2 = -\omega^2 + \mathbf{k}^2 = -m^2$), we get:

$$\begin{aligned} a_1^\dagger(+\infty) - a_1^\dagger(-\infty) &= -i\int d^3k\delta^3(\mathbf{k}-\mathbf{k}_1)\int d^4x e^{ikx}\left(\partial_0^2+\mathbf{k}^2+m^2\right)\varphi(x) \\ &= -i\int d^3k\delta^3(\mathbf{k}-\mathbf{k}_1)\int d^4x e^{ikx}\left(\partial_0^2-\vec{\nabla}^2+m^2\right)\varphi(x) \\ &= -i\int d^3k\delta^3(\mathbf{k}-\mathbf{k}_1)\int d^4x e^{ikx}\left(-\partial^2+m^2\right)\varphi(x). \end{aligned} \tag{4.26}$$

Similarly, we get for a:

$$a_3(+\infty) - a_3(-\infty) = i\int d^3k\delta^3(\mathbf{k}-\mathbf{k}_3)\int d^4x e^{-ikx}\left(-\partial^2+m^2\right)\varphi(x). \tag{4.27}$$

By inserting equations 4.26 and 4.27 into the scattering amplitude in 4.23, and rearranging using the time ordering operator to move the annihilation operators to the right, and the creation operators to the left, the scattering amplitude reduces to:

$$\begin{aligned}\langle f \mid i\rangle = i^4 \int & d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(k_1x_1+k_2x_2-k_3x_3-k_4x_4)} \\ & \times \left(-\partial_1^2+m^2\right)\left(-\partial_2^2+m^2\right)\left(-\partial_3^2+m^2\right)\left(-\partial_4^2+m^2\right) \\ & \times \langle 0 \mid \mathrm{T}\varphi\left(x_1\right)\varphi\left(x_2\right)\varphi\left(x_3\right)\varphi\left(x_4\right)\mid 0\rangle ,\end{aligned} \tag{4.28}$$

which is called *Lehmann-Symanzik-Zimmermann reduction* (LSZ) formula.

4.4 S-matrix

Now we shall define the s-matrix. But first, we should discuss its structure. On one hand, if the particles do not interact, the incoming particles are the same as the outgoing, then the s-matrix is the identity. On the other hand, if particles interact, then the s-matrix involves an additional matrix $\mathcal{T}$ encoding the interactions as:

$$\mathcal{S} = \mathbb{I} + i\mathcal{T}. \tag{4.29}$$

Since only the $\mathcal{T}$ that transcribes the scattering process, we can relate it to the scattering amplitude by:

$$\langle f \mid i\rangle = (2\pi)^4\delta^4(k_{\text{in}} - k_{\text{out}})i\mathcal{T}, \tag{4.30}$$

with:

$$\begin{aligned} k_{\text{in}} &= k_1 + k_2, \\ k_{\text{out}} &= k_4 + k_3. \end{aligned} \tag{4.31}$$

The Dirac's delta reads the conservation of the total momentum of the interacting particles (ie $k_{\text{in}} = k_{\text{out}}$).

To compute the scattering matrix, one can proceed through calculations using LSZ formula in 4.28. However, the calculations are long and tedious, thus we resolve to *Feynman Diagrams*. We will talk about those diagrams in details in the next section after we compute the effective Lagrangian for inflation.

5 Effective Field Theory for Inflation

The Standard Model (SM) of particle physics is an extremely successful theory making accurate predictions over a wide range of energies—from phenomena at the eV scale as in atomic physics to high-energy processes in colliders at scales of hundreds of GeV. The spectrum of the SM contains particles with very different masses. In order to make accurate predictions of processes at very low energies, we do not need to take into account the entire SM, considering the effects of the heavy electroweak particles as a perturbative correction. This approach is an example of an EFT.

5.1 Principles

According to [6] the construction of an EFT is based on the following three core principles:

Degrees of freedom In a first step, we need to determine the degrees of freedom that are relevant to describe the physical system we are interested in. These degrees of freedom are the building blocks of the Lagrangian. We keep the fields that are describing light particles, while dropping the fields that represent heavy particles.

Symmetries In a second step, it is crucial to know the symmetries of the problem at hand, which constrain possible interactions. In case that the underlying theory is not known, one will write down all possible interactions that are compatible with the assumed symmetries.

Expansion parameters An EFT turns a certain type of approximation into a systematic expansion of a specific parameter. In order to do so, we need to the small parameter that, for instance this can be a light mass. In the end, we need to determine a dimensionless parameter that describes what is small and we perform a perturbative expansion in powers of this small dimensionless quantity.

5.2 Effective Lagrangian for inflation

In this section, we will establish the effective Lagrangian for both inflation field and gravitational field interacting with each other. But first let us discuss the potential of inflation $V(\varphi)$. Since we are working in 4 dimensional space-time, and from the discussion in section 3.1, the only possible expression for the potential is:

$$V(\varphi) = \frac{\lambda_3}{3!}\varphi^3 + \frac{\lambda_4}{4!}\varphi^4, \tag{5.1}$$

With λ_3 and λ_4 two coupling coefficients. This potential represents inflaton-inflaton interaction. For the inflaton-graviton interaction, the lowest order of the gravitational field involves only the metric $g^{\mu\nu}$. Therefore the possible interactions in the Lagrangian up to the 4th derivative of the field are:

$$\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right)^2, \left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right)\left(\Box\varphi\right)^2, \left(\Box\varphi\right)^4, \tag{5.2}$$

Where $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$.

Consequently the action takes the expression:

$$\begin{aligned} S = \int d^4x\mathcal{L} = \int d^4x\sqrt{g}\Big(&-\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda_3}{3!}\varphi^3 - \frac{\lambda_4}{4!}\varphi^4 \\ &+ \frac{g_2}{2}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right)^2 + \frac{g_3}{3}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right)\left(\Box\varphi\right)^2 + 4g_4\left(\Box\varphi\right)^4 + \dots\Big), \end{aligned} \tag{5.3}$$

with $g = -\operatorname{Det}(g_{\mu\nu})$ and $M_P = 1/\sqrt{8\pi G}$. In this Lagrangian, we consider also the gravitational field as a quantum field. This approach to the gravitational field is known as *quantum gravity* [7]. Eventually, the particle arisen from quantization is called *graviton*. In our model, both inflaton and graviton particles are masseless.

Each term in the Lagrangian can be interpreted as follows: the first term in corresponds to the gravitational field, second term to the interaction inflaton-graviton, and the third and the fourth to inflaton-inflaton interaction.

5.3 Feynman diagrams

We have seen in section 4.4 that, in addition to the LSZ procedure, there is a much simpler one to compute the s-matrix, using Feynman diagrams. To compute those diagrams, we follow the ensuing steps:

1. Initially, we draw an external line for each particle in the initial state $\mid i\rangle$ and each particle in the final state $\mid f\rangle$. We will choose solid straight lines with an arrow indicating the 4-momentum for the scalar field, and curled lines for the gravitation field represented by the metric as in the figure. Since we are computing $2 \longrightarrow 2$ inflaton scattering, the external lines will always be inflatons.

2. Than we join the external lines together within vertices. Depending on the Lagrangian, there can be different types of vertices. Precisely, the types of vertices are defined according to the interaction term in the Lagrangian. Moreover, the total momenta is conserved in a vertex. To rephrase it, the sum of momenta of the incoming particles (incoming lines) to the vertex is equal to the sum of momenta of outgoing particles (outgoing lines) of the vertex.

Consequently, the only possible vertices from 5.3 are:

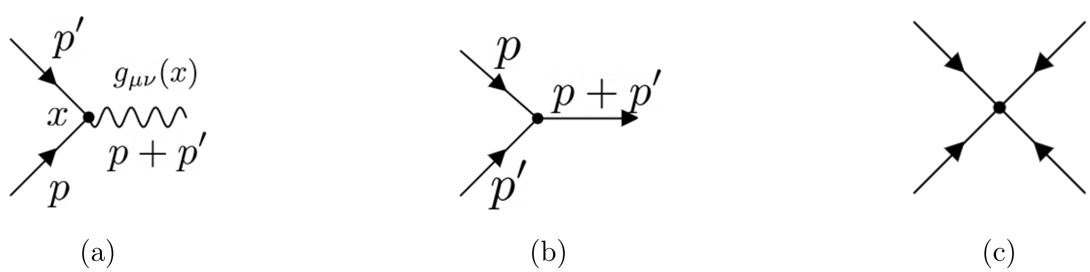


Figure 2: Possible vertices for the Lagrangian 5.3.

The vertex represented in figure 2a corresponds to the interaction inflaton-graviton term $\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$ which includes 3 fields: 1 graviton field $g^{\mu\nu}$, and 2 inflaton fields $\partial_\mu\varphi$, and $\partial_\nu\varphi$.

The vertex represented in figure 2b corresponds to the interaction inflaton-inflaton term $\lambda_3\varphi^3$, which includes 3 inflaton fields φ.

Finally, the vertex represented in figure 2c corresponds to the interaction inflaton-inflaton term $\lambda_4\varphi^4$ which includes 4 inflaton fields φ.

As a result, the possible diagrams to our $2 \longrightarrow 2$ scattering are represented as:

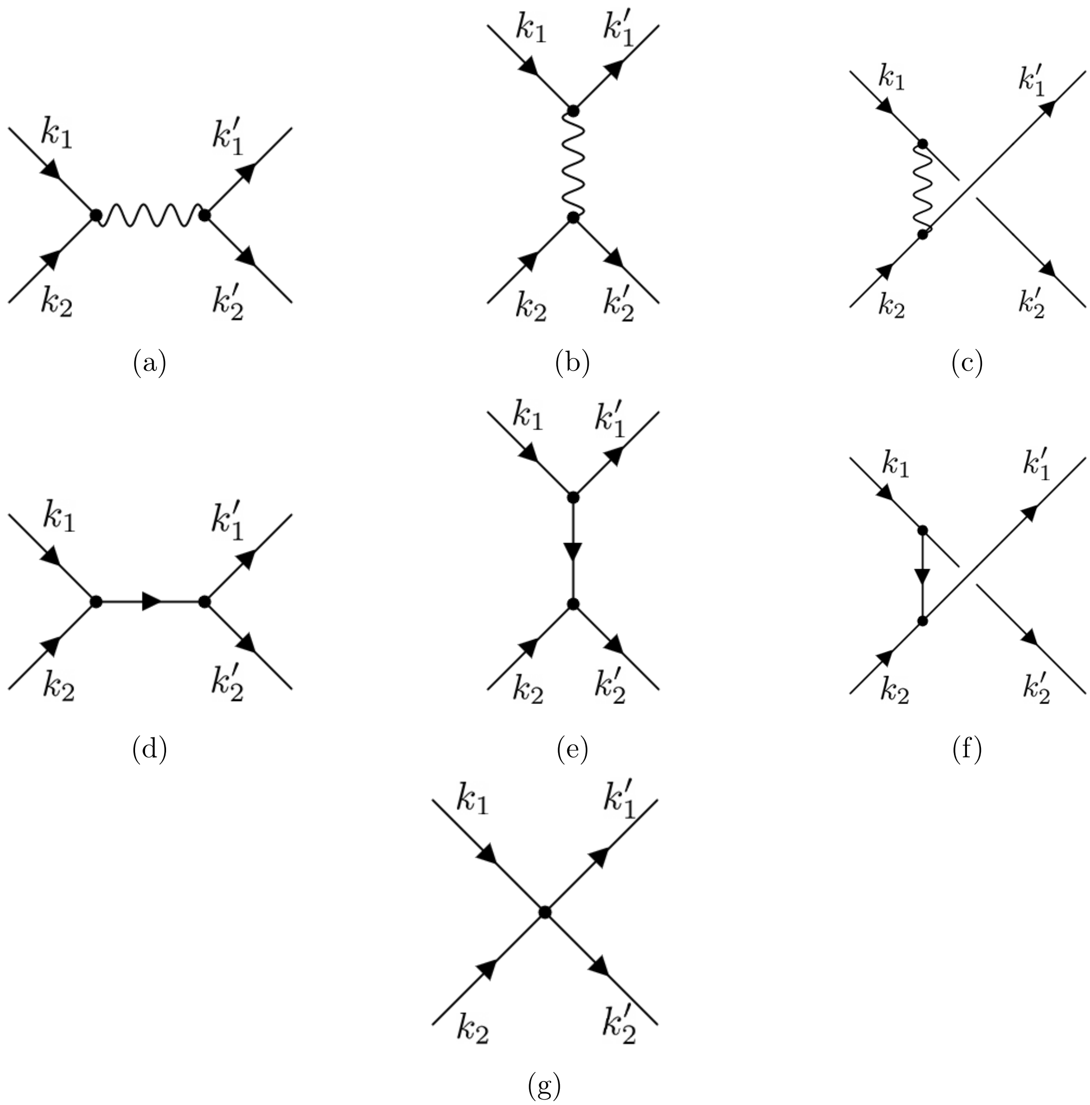


Figure 3: Possible diagrams for the Lagrangian 5.3.

Those diagrams can be separated into 3 types depending on the scattering:

Type 1 diagrams (figures [3a,3c]) represents to the scattering of inflatons via a graviton, and it corresponds to the second term of the Lagrangian 5.3.

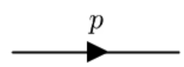
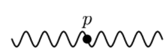
Type 2 diagram (figures [3d,3f]) correspond to the scattering of inflatons via another inflaton, and it corresponds to the third term of the Lagrangian. This term only has 3 diagrams because of the possible symmetries in the scattering amplitudes.

Type 3 diagram (figure 3g) corresponds to last interaction in the Lagrangian.

All those diagrams are call *tree level diagrams* because they represent the lowest order in the expansion of the exponential function in the scattering amplitude 4.18.The higher order contributing diagrams contain loops and we ignore them because they are much unlikely to happen (95% less to happen compared to the likelihood of 1st order diagram scattering), and also more complicated to compute.

5.4 Feynman rules

Now we shall establish the Feynman rules to compute the scattering amplitudes. For the graviton rules, we use those given in [7] section 4.

1. For each scalar propagator, we have:  $= -\frac{i}{p^2 - i\epsilon}$.
2. For each graviton propagator, we have:  $= -\frac{iP^{\alpha\beta\gamma\delta}}{p^2 - i\epsilon}$

   $$= -\frac{i}{p^2 - i\epsilon}\frac{1}{2}\left[\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right].$$
3. For each scalar vertex of φ^3 interaction (figure 2b), we have: $= i\lambda_3$.
4. For each scalar vertex of φ^4 interaction (figure 2c), we have: $= i\lambda_4$.
5. For each graviton vertex in (figure 2a), we have: $= -\frac{i}{M_P}\big[\left(p_\mu p'_\nu + p'_\mu p_\nu\right)$

   $$-\eta_{\mu\nu}\left(p \cdot p' - m^2\right)\big].$$

5.5 S-matrix

At this time, We will establish in this section s-matrix $i\mathcal{T}$. We start by computing the s-matrix for each type of diagrams $i\mathcal{T}_k$ separately using the rules from the previous subsection, then we will $i\mathcal{T}$ as:

$$i\mathcal{T} = \sum_{k=1}^{3} i\mathcal{T}_k. \tag{5.4}$$

We start by Type 3 diagram, because it is implemented directly as:

$$i\mathcal{T}_3 = i\lambda_4. \tag{5.5}$$

Next we compute Type 2 diagrams. By summing the s-matrix of the 3 possible diagrams, we find:

$$i\mathcal{T}_2 = i\lambda_3^2 \left[\frac{1}{(k_1+k_2)^2 - i\epsilon} + \frac{1}{(k_1-k_1')^2 - i\epsilon} + \frac{1}{(k_1-k_2')^2 - i\epsilon}\right], \tag{5.6}$$

where each term corresponds respectively to the diagrams [3d,3f] in the layout order.

The Type 1 diagrams is a little bit more complicated, because it involves the tensor $P^{\alpha\beta\gamma\delta}$. By applying the rules of the previous subsection, we get:

$$\begin{aligned}
i\mathcal{T}_1 = \frac{i}{M_P^2}\Bigg[&\frac{(k_{1\mu}k_{2\nu} + k_{2\mu}k_{1\nu} - \eta_{\mu\nu}k_1\cdot k_2)\, P^{\mu\nu\alpha\beta}\left(k'_{2\alpha}k'_{1\beta} + k'_{1\alpha}k'_{2\beta} - \eta_{\alpha\beta}k'_1\cdot k'_2\right)}{(k_1+k_2)^2 + i\epsilon} \\
&+ \frac{\left(k_{1\mu}k'_{1\nu} + k'_{1\mu}k_{1\nu} - \eta_{\mu\nu}k_1\cdot k'_1\right) P^{\mu\nu\alpha\beta}\left(k_{2\alpha}k'_{2\beta} + k'_{2\alpha}k_{2\beta} - \eta_{\alpha\beta}k_2\cdot k'_2\right)}{(k_1-k'_1)^2 + i\epsilon} \\
&+ \frac{\left(k_{1\mu}k'_{2\nu} + k'_{2\mu}k_{1\nu} - \eta_{\mu\nu}k_1\cdot k'_2\right) P^{\mu\nu\alpha\beta}\left(k_{2\alpha}k'_{1\beta} + k'_{1\alpha}k_{2\beta} - \eta_{\alpha\beta}k_1\cdot k'_2\right)}{(k_1-k'_2)^2 + i\epsilon}\Bigg].
\end{aligned} \tag{5.7}$$

Using the symmetries of the graviton propagator tensor:

$$P^{\mu\nu\alpha\beta} = P^{\nu\mu\alpha\beta} = P^{\alpha\beta\mu\nu}, \tag{5.8}$$

and after some simplifications, the equation 5.7 reduces to:

$$i\mathcal{T}_1 = -\frac{i}{M_P^2}\left[\frac{(k_1-k'_1)^2(k_1-k'_2)^2}{(k_1+k_2)^2} + \frac{(k_1+k_2)^2(k_1-k'_2)^2}{(k_1-k'_1)^2} + \frac{(k_1-k'_1)^2(k_1+k_2)^2}{(k_1-k'_2)^2}\right]. \tag{5.9}$$

Before we write the expression of $i\mathcal{T}$, let us define a set of variables called *Mandelstam variables* that would lightens considerably the computations:

$$\begin{aligned}
s &= -(k_1+k_2)^2 = -(k'_1+k'_2)^2, \\
t &= -(k_1-k'_1)^2 = -(k_2-k'_2)^2, \\
u- &= (k_1-k'_2)^2 = -(k_2-k'_1)^2.
\end{aligned} \tag{5.10}$$

Note also that since the inflaton field is massless, we have:

$$s + t + u = 0. \tag{5.11}$$

By injecting equations 5.5, 5.6, 5.9 into 5.4, using the Mandelstam variables with 5.11, and setting $\epsilon \to 0$ the s-matrix becomes:

$$i\mathcal{T} = i\left(\frac{1}{{M_P}^2}\left[\frac{st}{u} + \frac{su}{t} + \frac{tu}{s}\right] - \lambda_3^2\left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right] - \lambda_4\right), \tag{5.12}$$

which involves the scattering of the first derivative order, and the type 2 and 3 interactions of 5.3.

6 Bootstrap techniques for bounds on the coupling coefficients

We remind the goal of this project is to constrain the higher derivative coupling terms in the effective Lagrangian for inflation. In this paragraph, we will provide a brief summary about the procedure.

Since this implementation has never been done before, we will use the formalism described in [2]. Particularly, we will follow the steps of [8] that are only valid for a space-time with a dimension superior than 5. But we will use [3] to present an approach to extend the previous procedure to a 4-dimensional space-time. The low energy term contains a pole that we will overcome by introducing an impact parameter of scattering.

6.1 Higher derivative terms

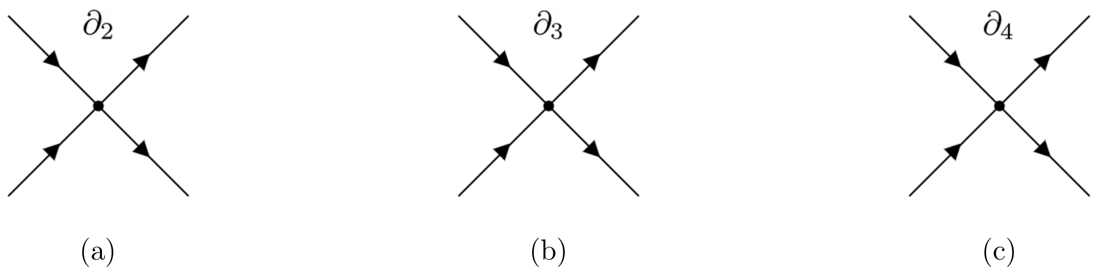


Figure 4: Diagrams corresponding to first 4 higher derivative terms.

Having established the s-matrix in 5.12, we can now apply finally the Bootstrap techniques. But first we need to implement the higher order terms. There are two ways do so, either using the corresponding diagrams then deriving the s-matrix, or compute them directly in the s-matrix expression. We will proceed according to the second method. We suppose that we are in the case of weak coupling. Using the crossing symmetry constrain, each higher derivative term is expressed in terms of powers of the polynomials $(s^2 + u^2 + t^2)$ and (stu) with respect to the spin $J \in 2\mathbb{N}$. Precisely, for a certain spin J, we have:

$$\mathcal{T}_{a,b}(s,u) = \sum_{k=0}^{\frac{J}{2}} g_{J+2k}(s^2 + u^2 + t^2)^{\frac{J}{2}-k}(stu)^k, \tag{6.1}$$

where $(g_{(J+2k)})$ are the coupling coefficients we aim to bound. At the 4th higher derivative term, $\mathcal{T}_{a,b}(s,u)$ takes the from:

$$\mathcal{T}_{a,b}(s,u) = g_2(s^2 + t^2 + u^2) + g_3(stu) + g_4(s^2 + t^2 + u^2)^2 + \dots, \tag{6.2}$$

which are represented by the diagrams of figure 4.

Hence the new s-matrix including those new terms:

$$\mathcal{T} = \left(\frac{1}{{M_P}^2} \left[\frac{st}{u} + \frac{su}{t} + \frac{tu}{s} \right] - \lambda_3^2 \left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right] + \lambda_4 \right) \\ + g_2(s^2+t^2+u^2) + g_3(stu) + g_4(s^2+t^2+u^2)^2 + \dots \quad (6.3)$$

In order to constrain those coefficients we will establish the s-matrix in both low and high energy regimes, and we will equate those terms using the dispersion relation.

6.2 Constrains on the s-matrix

First we start by the *Partial-Wave decomposition* as portrayed in [2] of the s-matrix $\mathcal{T}$ in respect to the scattering angle θ in appendix B:

$$\mathcal{T}(s,u) = \sum_{J \text{ even}} n_J c_J(s) \mathcal{P}_J \left(1 + \frac{2u}{s}\right), \quad (6.4)$$

with $\mathcal{P}_J$ are *Legendre* polynomials, c_J the spherical *Bessel* function, and the normalizing coefficients:

$$n_J = 2^4 \pi (J+1)_0 (2J+1), \quad (6.5)$$

We define the spectral density $\rho_J(s) = \mathrm{Im}(c_J(s))$, therefore the imaginary part of equation 6.4:

$$\mathrm{Im}\left(\mathcal{T}(s,u)\right) = \sum_{J \text{ even}} n_J \rho_J(s) \mathcal{P}_J \left(1 + \frac{2u}{s}\right). \quad (6.6)$$

Let us return now to the density $\rho_J(s)$. We constrain the scattering matrix $\mathcal{S}$ in equation 4.29 to be unitary (ie. $\mathcal{S}\mathcal{S}^\dagger = \mathbb{I}$), we can show that $0 \leqslant \rho_J(s) \leqslant 2$, which implies that $\mathrm{Im}(\mathcal{T}(s,u))$ is also positive since n_J and $\mathcal{P}_J$ are positive.

To establish this relation, we need to make more constrains on the s-matrix [6]:

1. $\mathcal{T}$ has to be stable by crossing symmetry,
2. $\mathcal{T}$ has to be analytic for s in the half plan $\mathrm{Im}(s) > 0$.
3. $\mathcal{T}$ has to be real analytic, $\mathcal{T}(s^*, u^*) = \mathcal{T}^*(s,u)$. Therefore we extend the analyticity domain to whole $\mathbb{C}^*$,
4. $\mathcal{T}$ has to be bounded at ∞ as the following:

$$\lim_{|s| \to \infty} \left(\frac{\mathcal{T}(s,u)}{s^2} \right) = 0. \quad (6.7)$$

[6] Constraining $\mathcal{S}$ is the same as $\mathcal{T}$ because of equation 4.29

6.3 Dispersion relation

The dispersion relation is derived by applying the previous constrains together on the s-matrix. By proceeding accordingly and using complex integrals, the limit in 6.7 turns into:

$$\oint_{\infty} \frac{ds'}{2\pi i\,(s'-s)} \frac{\mathcal{T}\,(s',u)}{s'\,(s'+u)} = 0. \tag{6.8}$$

Now we set an energy scale M. Indeed, this scale will be defined as the border supposed to separate approximately the low energy regime ($s \lesssim M^2$) and from the high energy one ($s \gtrsim M^2$). This scale can be defined where the threshold where $\mathrm{Im}\,(\mathcal{T}(s,u)) > 0$.

Using this regime separation and the *Residue theorem* from complex analysis, we get:

$$\begin{aligned}\frac{\mathcal{T}_{\text{low}}\,(s,u)}{s(s+u)} + \mathrm{Res}_{s'=0}\left[\left(\frac{1}{s'-s}+\frac{1}{s'+s+u}\right)\frac{\mathcal{T}_{\text{low}}\,(s',u)}{s'\,(s'+u)}\right] = \\ \int_{M^2}^{\infty}\frac{ds'}{\pi}\left(\frac{1}{s'-s}+\frac{1}{s'+s+u}\right)\mathrm{Im}\left[\frac{\mathcal{T}_{\text{high}}\,(s',u)}{s'\,(s'+u)}\right].\end{aligned} \tag{6.9}$$

This equation is called the *dispersion relation.* For the low energy part we use the expression 6.3 that reduces thanks to 5.11:

$$\begin{aligned}\frac{\mathcal{T}_{\text{low}}\,(s,u)}{s(s+u)} + \mathrm{Res}_{s'=0}\left[\left(\frac{1}{s'-s}+\frac{1}{s'+s+u}\right)\frac{\mathcal{T}_{\text{low}}\,(s',u)}{s'\,(s'+u)}\right] = \\ \frac{1}{{M_P}^2(-u)} + 2g_2 - g_3 u + 4g_4\left(2u^2+s(s+u)\right) + \ldots\end{aligned} \tag{6.10}$$

Therefore we can use this density to define a measure:

$$\int_{M^2}^{\infty}\frac{dm^2}{m^2}\rho_J\left(m^2\right), \tag{6.11}$$

that will use to average the high energy part of equation 6.9 as:

$$\begin{aligned}\int_{T^2}^{\infty}\frac{ds'}{\pi}\left(\frac{1}{s'-s}+\frac{1}{s'+s+u}\right)\mathrm{Im}\left[\frac{\mathcal{T}_{\text{high}}\,(s',u)}{s'\,(s'+u)}\right] = \\ \sum_{J\text{ even}} n_J\left\langle\frac{(2m^2+u)\,\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{(m^2+u)\,(m^2-s)\,(m^2+s+u)}\right\rangle,\end{aligned} \tag{6.12}$$

where:

$$\langle(\cdots)\rangle = \frac{1}{\pi}\int_{M^2}^{\infty}\frac{dm^2}{m^2}\rho_J\left(m^2\right)(\cdots) \tag{6.13}$$

Consequently, equation 6.9 becomes:

$$\frac{1}{{M_P}^2(-u)} + 2g_2 - g_3 u + 4g_4\left(2u^2 + s(s+u)\right) + \ldots = \sum_{J \text{ even}} n_J \left\langle \frac{\left(2m^2+u\right)\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{\left(m^2+u\right)\left(m^2-s\right)\left(m^2+s+u\right)} \right\rangle. \quad (6.14)$$

This last equation can be seen as an expansion over $s(s+u)$ when $s \to 0$. By moving the left term to the right, we can rearrange the previous equation as:

$$\sum_{n=1}^{\infty} [s(s+u)]^{n-1} \mathcal{C}_{2n,u} = 0 \quad (6.15)$$

where we replaced $J = 2n$ and:

$$\mathcal{C}_{k,u} = \text{Res}_{s'=0}\left[\frac{2s'+u}{s'\left(s'+u\right)} \frac{\mathcal{T}_{\text{low}}\ (s',u)}{\left[s'\left(s'+u\right)\right]^{k/2}}\right] - \left\langle \frac{2m^2+u}{m^2+u} \frac{\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{\left[m^2\left(m^2+u\right)\right]^{k/2}} \right\rangle. \quad (6.16)$$

We can interpret the first term in 6.16 as the low energy component and the second term as the high energy one.By sending $s \to 0$, we get:

$$\mathcal{C}_{2n,u} = 0 \qquad \forall n \in \mathbb{N}. \quad (6.17)$$

7 Setting the bounds

In order to obtain constrains on the coefficients g_2, g_3, and g_4, one can resolve to the *forward limit procedure* as described in [2]. This method consists to send $u \to 0$ and it yields after some few manipulations of $\mathcal{C}_{2n,u}$ coefficients in equation 6.16 to the constrains we are looking for. But in our case, if we send $u \to 0$, the term $\frac{1}{{M_P}^2(-u)}$, corresponding to the graviton coupling, hidden in $\mathcal{C}_{2,u}$ would diverge. Therefore we have to work around this problem in a different manner.

7.1 The impact parameter

Now we introduce an impact parameter b that would suppress the pole of the graviton. We remind that the meaning of the Mandelstam variable u is the transferred 4-momentum amplitude $u = -\left(k_1 - k_2'\right)^2 = -k^2$. In fact, we suppose that incoming particles scatter with an impact parameter b as shown in figure 5.

This impact parameter can be linked to momenta. Indeed, the 4-vector impact $\vec{k} = k_1 - k_1'$ is within the plane of k_1 and k_1'. Therefore it belongs to $\mathbb{R}^2$. This also means that the impact parameter is also in $\mathbb{R}^2$, and can be considered as the Fourier

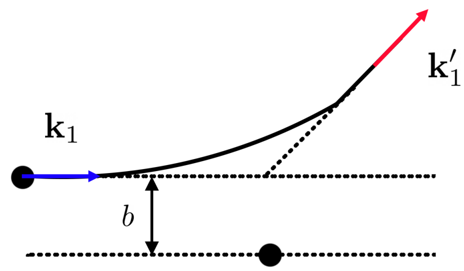


Figure 5: Scattering of inflatons with the impact parameter $\vec{b}$. In this figure, we represented the amplitude of the impact parameter $b = |\vec{b}|$.

transform of $\vec{k}$. For a spherically symmetric momentum-space wave-function $f(k)$, we define the corresponding function $\hat{f}(b)$ on the impact parameter space. Therefore we have:

$$\int_0^\infty db \hat{f}(b) = \int_0^\infty d^2k f(k). \tag{7.1}$$

The integration against the graviton term reads as:

$$\int_0^\infty d^2k \frac{f(k)}{{M_P}^2(-u(k))} = \int_0^\infty d^2k \frac{f(k)}{M_P^2 k^2}, \tag{7.2}$$

and this integral diverge for higher values k, and this is the *graviton pole divergence*. In order to subdue this latter , we must localise this impact parameter of scattering to a certain scale, for instance $b \sim \frac{1}{M}$, that reads in the momentum space as $k \sim M$. Hence the integration bounds become:

$$\int_0^\infty d^2k \frac{f(k)}{M_P^2 k^2} \longrightarrow \int_0^M d^2k \frac{f(k)}{M_P^2 k^2}. \tag{7.3}$$

Let us discuss further the role of the function f. By integrating this function in the momentum space against the high energy part of the $\mathcal{C}_{k,u}$ coefficient on one hand, and against the low energy part on the other hand, and equating those two integrals, the positivity of the first integral yields to the one of the second that contains the coefficients we seek to bound. Consequently the bounding process boils down to the finding of this function f that would generate a positive quantity when we integrate against the high energy part of $\mathcal{C}_{k,u}$.

One way to find f is by using *series expansion*. Indeed:

$$f(k) = \sum_{n \geqslant n_0} a_n k^n. \tag{7.4}$$

For each $n \geqslant n_0$, the integral against the gravitational pole:

$$\int_0^M d^2k \frac{a_n k^n}{M_P^2 k^2}, \tag{7.5}$$

converges in 0, if and only if $n > 1$. Hence the first order $n_0 > 1$.

Having dealt with the graviton pole, there is another divergence emerging at low energies. This divergence shall be called *infrared divergence*. To visualise it, we return to the impact parameter space. However we must set a cutoff for k, because the function $e^{i\vec{b}\cdot\vec{k}} \frac{f(k)}{k \operatorname{vol} S}$ is not integrable in 0 and this is called the *Infra-red divergence*. We set this cutoff as $k_{\min}$, and we define $\hat{f}(b)$ as:

$$\hat{f}(b) = \int_{k_{\min}}^M d\vec{k} e^{i\vec{b}\cdot\vec{k}} \frac{f(k)}{k \operatorname{vol} S} = \int_{k_{\min}}^M d^2k f(k). \tag{7.6}$$

Now the integration of $f(k)$ multiplied to the gravitational pole in the momentum space can be transformed to an integral in the impact parameter space as:

$$\int_{k_{\min}}^M d^2k f(k) \frac{1}{{M_P}^2(-u(k))} = \int_{k_{\min}}^M d^2k f(k) \frac{1}{{M_P}^2 k^2} = \int_{\frac{1}{M}}^{b_{\max}} d^2b \hat{f}(b), \tag{7.7}$$

where $b_{\max}$ can be seen as the maximum impact parameter introduced to avoid the divergence of the integral in the right term. This divergence is called the *infrared divergence* of the gravitational potential in Regge limit[7] [9]. In this regard, $b_{\max}$ is the Fourier conjugate of the cutoff we introduced $k_{\min}$.

7.2 The bounding algorithm

We shall detail how we are going to bound the coupling coefficients g_2, g_3 and g_4 using the coefficients $\mathcal{C}_{2n,u}$. We start by bounding g_2, and the other coefficients would be bound similarly. To this end, we consider the first coefficient $\mathcal{C}_{2,u}$. Its low energy part is given by the EFT contribution:

$$\mathcal{C}_{2,u}\big|_{\mathrm{EFT}} = \frac{1}{{M_P}^2(-u)} + 2g_2 - g_3 u + 8g_4 u^2 - 2g_5 u^3 + 24 g_6 u^4 - 4 g_7 u^5 \ldots, \tag{7.8}$$

In order to remove the higher terms, we shall subtract a linear combinations of $\mathcal{C}_{2n,u}$ and their derivatives at $u = 0$ from $\mathcal{C}_{2,u}$. For instance, to remove g_4, we subtract $2u^2\mathcal{C}_{4,0}$ and so on. The general procedure to remove all the other terms is

[7] The Regge theory whose purpose to study the high energy scattering. It relies essentially on the analiticity hypothesis of the s-matrix

through:

$$\mathcal{C}_{2,u}^{\text{improved}} = \mathcal{C}_{2,u} - \sum_{n=2}^{\infty}\left(nu^{2n-2}\mathcal{C}_{2n,0} + u^{2n-1}\mathcal{C}'_{2n,0}\right) = \frac{1}{{M_P}^2(-u)} + 2g_2 - g_3 u$$
$$- \left\langle \frac{\left(2m^2+u\right)\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{m^2\left(m^2+u\right)^2} - \frac{u^2}{m^6}\left(\frac{\left(4m^2+3u\right)\mathcal{P}_J(1)}{\left(m^2+u\right)^2} + \frac{4u\mathcal{P}'_J(1)}{m^4-u^2}\right)\right\rangle. \quad (7.9)$$

Thus we obtain for the low energy component:

$$\mathcal{C}_{2,u}^{\text{improved}}\Big|_{\text{EFT}} = \frac{1}{{M_P}^2(-u)} + 2g_2 - g_3 u, \quad (7.10)$$

which contains only the coefficients g_2 and g_3 that we want to bound, and the high energy component, for $J \in 2\mathbb{N}$, and $m \geqslant M^2$:

$$\mathcal{C}_{2,u}^{\text{improved}}\Big|_{\text{H}}\left[m^2, J\right] = \frac{\left(2m^2+u\right)\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{m^2\left(m^2+u\right)^2} - \frac{u^2}{m^6}\left(\frac{\left(4m^2+3u\right)\mathcal{P}_J(1)}{\left(m^2+u\right)^2} + \frac{4u\mathcal{P}'_J(1)}{m^4-u^2}\right) \quad (7.11)$$

Using the improved coefficient $\mathcal{C}_{2,u}^{\text{improved}}$ in 6.17, we get:

$$\mathcal{C}_{2,u}^{\text{improved}}\Big|_{\text{EFT}} = \left\langle \mathcal{C}_{2,u}^{\text{improved}}\Big|_{\text{H}}\left[m^2, J\right]\right\rangle, \quad (7.12)$$

which is equivalent to:

$$\frac{1}{{M_P}^2(-u)} + 2g_2 - g_3 u =$$
$$\left\langle \frac{\left(2m^2+u\right)\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{m^2\left(m^2+u\right)^2} - \frac{u^2}{m^6}\left(\frac{\left(4m^2+3u\right)\mathcal{P}_J(1)}{\left(m^2+u\right)^2} + \frac{4u\mathcal{P}'_J(1)}{m^4-u^2}\right)\right\rangle. \quad (7.13)$$

Now by integrating over the space momentum k this last equation multiplied by a function f:

$$\int_{k_{\min}}^{M} dk f(k)\, \mathcal{C}_{2,-k^2}^{\text{improved}}\Big|_{\text{H}} = \int_{k_{\min}}^{M} dk f(k)\left[\frac{1}{{M_P}^2 k^2} + 2g_2 + g_3 k^2\right], \quad (7.14)$$

for each $m \geq M$ and $J \in 2\mathbb{N}$. If we find a function where the right integral is positive, than the second integral would be also positive and yield to an inequality for g_2 and g_3 at the same time. We can get stronger bounds if we can add more null terms to 7.9 that would manifest only in the left term of 7.14. Those additional terms would narrow down the domain where $\int_{k_{\min}}^{M} dk f(k)\mathcal{C}^{\text{H}}_{2,-k^2}\left[m^2, J\right]$ would be positive. Such terms are give in [10]. For a given $j \in 2\mathbb{N}^* - \{2\}$ that take the form:

$$\mathcal{X}_{j,u}\left[m^2, J\right] = \frac{2m^2+u}{um^2\left(m^2+u\right)}\frac{m^2\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{\left(um^2\left(m^2+u\right)\right)^{j/2}}$$
$$- \operatorname*{Res}_{u'=0}\frac{\left(2m^2+u'\right)\left(m^2-u'\right)\left(m^2+2u'\right)}{m^2\left(u-u'\right)u'\left(m^2-u\right)\left(m^2+u'\right)\left(m^2+u+u'\right)}\frac{m^2\mathcal{P}_J\left(1+\frac{2u'}{m^2}\right)}{\left(u'm^2\left(m^2+u'\right)\right)^{j/2}}. \quad (7.15)$$

But this would add another set of functions $h_j(k)$ that we will have to determine too.

To rephrase, if we can find a functions f and h_j where:

$$\mathcal{A} : (\forall J \in 2\mathbb{N}, m \geqslant M) \int_{k_{\min}}^{M} dk f(k)\, \mathcal{C}_{2,-k^2}^{\text{improved}} \Big|_{\text{H}} \left[m^2, J\right] \\ + \sum_{j=4,6,\ldots} \int_{k_{\min}}^{M} dk h_j(k) \mathcal{X}_{j,-k^2} \left[m^2, J\right] \geqslant 0, \quad (7.16)$$

implies:

$$\mathcal{B} : \int_{k_{\min}}^{M} dk f(k) \left[\frac{1}{{M_P}^2 k^2} + 2g_2 + g_3 k^2\right] \geqslant 0, \quad (7.17)$$

which bounds g_2 and g_3. We shall refer to this implication $\mathcal{A} \implies \mathcal{B}$ as the *bounding implication*. This latter would constrain the possible values of g_2 and g_3 that can take in so that this integral can be positive.

To find the functions f and h_j, we would use the *Linear programming method*. This method consist on expanding the functions we are looking for using power series. Then we determine the coefficients (a_n) by minimizing f using the positivity of the integral 7.17 for each $m > M$ and $J \in 2\mathbb{N}$. Note that we must do the same procedure for h_j functions too.

Using this procedure, we get for g_2:

$$g_2 \geqslant -\frac{1}{M_P^2 M^2} \times 25 \log\left(0.3 M b_{\max}\right), \quad (7.18)$$

where the impact parameter cutoff $b_{\max}$ is linked to $k_{\min}$ by:

$$k_{\min} = \frac{1}{M {b_{\max}}^3}. \quad (7.19)$$

Up to this point we have bounded the coefficient g_2. In order to bound g_3, we shall repeat the same process, but taking into consideration g_3. For the bounding of g_4, we must consider the coefficient $\mathcal{C}_{4,u}$, and reapply the same techniques we used on them to remove higher coefficients. Then establish the equivalent of equation 7.14, and then find the new function f using Linear programming.

Furthermore, we note also that the bound is related to the cutoff. Therefore, we must choose carefully this cutoff, not very large because this would produce a less accurate bound, and also not very small because of the IR divergence.

7.3 Numerical implementation

Let us return to the simpler version of the bounding implication without the coefficients $\mathcal{X}_{j,u}$:

$$\mathcal{A} : \forall J \in 2\mathbb{N}, m \geqslant M : \int_{k_{\min}}^{M} dk f(k)\, \mathcal{C}^{\text{improved}}_{2,-k^2}\Big|_{\text{H}} \left[m^2, J\right] =$$
$$\int_{k_{\min}}^{M} dk f(k) \left(\frac{(2m^2+u)\,\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{m^2\,(m^2+u)^2} - \frac{u^2}{m^6}\left(\frac{(4m^2+3u)\,\mathcal{P}_J(1)}{(m^2+u)^2} + \frac{4u\mathcal{P}'_J(1)}{m^4-u^2} \right) \right) \geqslant 0$$
$$\implies \mathcal{B} : \int_{k_{\min}}^{M} dk f(k) \left[\frac{1}{{M_P}^2 k^2} + 2g_2 + g_3 k^2 \right] \geqslant 0. \quad (7.20)$$

To implement the linear programming method for f, we have to set for J at certain maximum $J_{\max}$, and an upper bound $m_{\max}$ for m, that we will choose according to the precision we seek. In [8], they set $J_{\max} = 42$. After this value, the higher order contributions were no more relevant.

Now we return to m. Let us first see the behaviour of the integrand at large values of m. We shall note that the impact parameter is linked to m by:

$$b = \frac{2J}{m}, \quad (7.21)$$

where J is the summing index. In order to maintain b finite, and if m takes a large value, then J does too. Particularly we get:

$$\int_{k_{\min}}^{M} dk f(k) \left\langle \frac{(2m^2+u)\,\mathcal{P}_J\left(1+\frac{2u}{m^2}\right)}{(m^2+u)\,(m^2-s)\,(m^2+s+u)} \right\rangle \sim \frac{2}{m^4} \int_{k_{\min}}^{M} dk f(k) J_0(bp), \quad (7.22)$$

as $m \to \infty$. Therefore, we shall set a certain threshold $m_{\max}$ where we will proceed as:

- if $m < m_{\max}$, then we compute the integral for each m and $J \leqslant J_{\max}$.
- if $m \geqslant m_{\max}$, we use the approximation above for $m \to \infty$ in 7.22.

One must note that we only assumed that b is finite,the integral must be positive and we didn't specify the value of b when $m \to \infty$. In order to be positive as in $\mathcal{A}$, the approximate integral at $m \to \infty$:

$$\int_{k_{\min}}^{M} dk f(k) J_0(bp), \quad (7.23)$$

must be positive for each value of b. Consequently, it is interesting to study the behaviour of the function $b \mapsto \int_{k_{\min}}^{M} dk f(k) J_0(bp)$.

8 Conclusion

To conclude, we have reviewed in the present work the inflation theory. In the standard version of this theory, the inflation Lagrangian was expressed only to the first derivative term coupling inflation to gravity. In order to push the computation to higher orders, we have resorted to EFT.

We have established the necessary background for EFT: firstly, we limited the number of interaction terms for the inflation alone to 2 terms $\lambda_3\varphi^3$ and $\lambda_4\varphi^4$ using renormalization. Secondly, we presented the s-matrix that encodes the information about the scattering process. Thirdly, we presented the classical method to derive this matrix using correlation function and the scattering amplitude.

Then we explained the formalism of Bootstrap techniques and how we will implement it to the scattering matrix to bound the coupling coefficients for the upcoming higher order terms.

Later, we introduced the EFT and its principles, and we expressed the effective Lagrangian for inflation using the terms we derived using re-normalisation. We also reviewed Feynman diagrams and rules to compute the s-matrix. We computed the s-matrix in terms of those variables and added the higher derivative terms we seek, coupled by coefficient that we want to constrain.

The goal of this project was to bound the inflation field to the gravitation field and promote this bounding to higher orders. However, there was some strong constrains (unitary, analyticity and crossing symmetry) that were made in order to bound the coupling coefficients of those higher orders. Some of those constraints like crossing symmetry seems natural because all the s-matrices of $2 \to 2$ scattering processes are stable by crossing symmetry. While other constrains like unitary is less natural and tangible. Concerning the impact parameter, it is always involved in particle scattering, and its introduction is legitimate. Nevertheless if we take a few steps back, we had introduced this cutoff to resolve the main issue of the procedure displayed in [8] coming from the IR divergence due to the graviton pole. However, this issue can be bypassed differently using string theory.

As well as that, we can bound also the inflation field, not only to the gravitation field, but also to the gauge field, by computing a trivalent inflaton-photon-graviton interaction. Such interaction would present the same issue in the low energy regime for the inflaton scattering via a graviton, but we can extend our approach straightforwardly. However, the impact parameter that we introduced for the cutoff would depend on the 3 fields at the same time.

Last but not least, the classical inflation theory supposes that the inflaton is a spin 0 particle. Nevertheless, there has been some implementations of inflation as a fermionic particle in the last few years. Those spinor inflatons are referred as *dark spinors* [11], because of the spinor field's dominant interactions that take place through the Higgs doublet or with gravity. Henceforth it is possible to extend our work to this type of fields too.

A word about this internship

This internship provided me the opportunity to deepen my knowledge about inflation in cosmology. I also learnt so much much about quantum field theory, effective field theory and bootstrap techniques. On the technical aspect, I also enjoyed working on the conceptualization of each model in different theories, and exploring their various aspects and extents. It was also for me to attend to many conferences and seminars that broaden my insight on theoretical physics.

All in all, the subject of this internship was very rich and educating, and we can still produce . I discovered a lot of new feats in term of both knowledge and skills. In general it has gone well and I am grateful to have this opportunity at the ITP.

A Scalar Field

For a given point x in space-time, the canonical quantization of a real scalar field φ gives:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \left[a(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}\right], \tag{A.1}$$

where $a^\dagger$ is the creation operator and a is the annihilation operator, and the 4-momentum $k = (\omega, \mathbf{k})$.

The derivation of in respect to time of φ yields to:

$$\partial_0 \varphi = \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left[-a(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}\right]. \tag{A.2}$$

Hence we get:

$$-i\partial_0\varphi(x) + \omega\varphi(x) = \int \frac{d^3k}{(2\pi)^3} a^\dagger(\mathbf{k})e^{-ikx}. \tag{A.3}$$

Using Fourier inverse transform, we obtain the creation operator:

$$\begin{aligned} a^\dagger(\mathbf{k}) &= \int d^3x e^{ikx} \left[-i\partial_0\varphi(x) + \omega\varphi(x)\right] \\ &= -i \int d^3x e^{ikx} \overleftrightarrow{\partial}_0 \varphi(x), \end{aligned} \tag{A.4}$$

where $f \overleftrightarrow{\partial}_0 g = f\partial_0 g - g\partial_0 f$.

Likewise, the annihilation operator can be expressed as:

$$a(\mathbf{k}) = i \int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \varphi(x), \tag{A.5}$$

B Center of mass frame

The *center of mass frame* is a frame where:

$$\mathbf{k}_1 + \mathbf{k}_2 = 0, \tag{B.1}$$

where $k_1 = (\omega_1, \mathbf{k}_1)$, (respectively $k_1 = (\omega_2, \mathbf{k}_2)$)), is the 4-momentum of the first, (respectively second), incoming inflaton. This frame represents the frame where the scattering process occurs.

We also put $k'_1 = (\omega'_1, \mathbf{k}'_1)$, (respectively $k_1 = (\omega'_2, \mathbf{k}'_2)$)), is the 4-momentum of the first (respectively second) incoming inflaton. Along with the previous equation, and the momenta conservation, one can find equivalently:

$$\mathbf{k}'_1 + \mathbf{k}'_2 = 0. \tag{B.2}$$

Since the field is massless, we have:

$$k_1{}^2 = -\omega_1{}^2 + \mathbf{k}_1{}^2 = 0, \tag{B.3}$$
$$k_2{}^2 = -\omega_2{}^2 + \mathbf{k}_2{}^2 = 0. \tag{B.4}$$
$$k_1'{}^2 = -\omega_1'{}^2 + \mathbf{k}_1'{}^2 = 0, \tag{B.5}$$
$$k_2'{}^2 = -\omega_2'{}^2 + \mathbf{k}_2'{}^2 = 0. \tag{B.6}$$

Equation B.1 yields to $\mathbf{k}_1{}^2 = \mathbf{k}_2{}^2$, whereas equation B.2 yields to $\mathbf{k}_1'{}^2 = \mathbf{k}_2'{}^2$. By combining the first equality with equations B.3 and B.4, and the second with B.5 and B.6 we obtain that:

$$\begin{aligned} \omega_1{}^2 = \omega_2{}^2 = \omega^2, \\ \omega_1'{}^2 = \omega_2'{}^2 = \omega'^2. \end{aligned} \tag{B.7}$$

The conservation of momenta implies again $\omega^2 = \omega'^2$, that yields to:

$$\omega_1{}^2 = \omega_2{}^2 = \omega^2 = \omega_1'{}^2 = \omega_2'{}^2. \tag{B.8}$$

Using the definition of s from 5.10 we get:

$$s = -(k_1 + k_2)^2 = -2k_1 \cdot k_2 = 2\omega^2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2. \tag{B.9}$$

We get also from B.1 $\mathbf{k}_1 \cdot \mathbf{k}_2 = -\mathbf{k}_1{}^2$, and using B.3 and B.8, we get:

$$\mathbf{k}_1 \cdot \mathbf{k}_2 = -\omega_1{}^2 = -\omega^2. \tag{B.10}$$

Injecting this last equation into B.9 we get:

$$s = 4\omega^2. \tag{B.11}$$

Similarly we obtain for u and t:

$$u = -2\omega^2(1 + \mathbf{k}_1 \cdot \mathbf{k}_1'), \tag{B.12}$$
$$t = -2\omega^2(1 - \mathbf{k}_1 \cdot \mathbf{k}_1'), \tag{B.13}$$

For particle 1, we define in this frame the scattering angle θ as the angle by by which the course of the particle has changed. To put it another way, it is the angle between the incoming and the outgoing momenta of the particle:

$$\mathbf{k}_1 \cdot \mathbf{k}_1' = \cos(\theta). \tag{B.14}$$

We can express this angle using Mandesltam variables. From B.11 and B.12, we get:

$$\cos(\theta) = 1 + \frac{2u}{s}. \tag{B.15}$$

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