Functional Programming

Week1 :

call by value vs call by name :

Call-by-value : evaluate the parameters *then* execute the function on these values.

e.g :

```
sumOfSquares(3, 2+2)
sumOfSquares(3, 4) // evaluated 2+2 BEFORE executing sumOfSquares
square(3) + square(4)
3 * 3 + square(4)
9 + square(4)
9 + 4 * 4
9 + 16
25
```

Call-by-value has the advantage that it evaluates every function argument only once.

Call-by-name : Apply the function to unreduced arguments.

e.g :

```
sumOfSquares(3, 2+2)
square(3) + square(2+2) // executing sumeOfSquare before evaluating 2+2
3 * 3 + square(2+2)
9 + square(2+2)
9 + (2+2) * (2+2)
9 + 4 * (2+2)
9 + 4 * 4
```

Call-by-name has the advantage that a function argument is not evaluated if the corresponding parameter is unused in the evaluation of the function body.

e.g :

```
def loop : Int = loop
def f(a:Int , b : Int ) : Int = a // Call-by-value
f(1,loop) // does not terminate
def f(a:Int, b : => Int) : Int = a // Call-by-Name ( => )
f(1,loop) // terminates
```

Both strategies reduce to the same final values as long as
▶ the reduced expression consists of pure functions, and
▶ both evaluations terminate.

Conditionals :

```
def abs(x: Int) = if x ≥ 0 then x else -x
```

`if-then-else` in Scala is an expression (has a value) not a statement .

Blocks , Scopes and first functional program :

Sqrt with Newton's method :

```
def sqrt(x: Double) = {
    def sqrtIter(guess: Double, x: Double): Double =
        if isGoodEnough(guess, x) then guess
        else sqrtIter(improve(guess, x), x)
    def improve(guess: Double, x: Double) =(guess + x / guess) /
    def isGoodEnough(guess: Double, x: Double) =abs(square(guess) - x) < 0.001

    sqrtIter(1.0, x)
}
```

`sqrtIter` , `improve` and `isGoodEnough` should not be outside sqrt because :

- they pollute the global scope.
- they should not be accessible to the user of the sqrt code given they are specific to sqrt .

the {} could be removed for more clarity.

Tail recursion :

```
def gcd(a: Int, b: Int): Int =if b == 0 then a else gcd(b, a % b)
```

```
gcd(14, 21)
→ if 21 == 0 then 14 else gcd(21, 14 % 21)
→ if false then 14 else gcd(21, 14 % 21)
→ gcd(21, 14 % 21)
→ gcd(21, 14)
→ if 14 == 0 then 21 else gcd(14, 21 % 14)
→» gcd(14, 7)
```

`gcd(14,21) = gcd(21,14) = gcd(14,7) = gcd(.., ...) = ... = 7` : flat structure, `gcd` only calls `gcd` => **tail recursion.**

```
def factorial(n: Int): Int = if n == 0 then 1 else n * factorial(n - 1)
```

```
factorial(4)
→ if 4 == 0 then 1 else 4 * factorial(4 - 1) 3-> ↠ 4 * factorial(3)
↠ 4 * (3 * factorial(2))
↠ 4 * (3 * (2 * factorial(1)))
↠ 4 * (3 * (2 * (1 * factorial(0)))
↠ 4 * (3 * (2 * (1 * 1)))
```

`factorial` calls factorial *and multiplies* , nested structure `⇒` **not tail recursion**

To tell the compiler than a function is tail recursive:

```
import scala.annotation.tailrec
@tailrec
def gcd(a: Int, b: Int): Int = ...
```

Week 2 :

Higher order functions :

Def : Functions that take other functions as parameters and return functions.

example :

```
def sum( f : Int ⇒ Int )( a : Int , b : Int ) : Int
```

- takes as argument : the function `f : Int ⇒ Int` .
- returns : a function that has 2 arguments `a:Int,b:Int` and returns `Int`

Function types : `A⇒B` is the type of a function that takes an argument of type A and returns a result of type B. e.g : `Int⇒Int` maps integers to integers.

Anonymous Functions :

```
(x: Int, y: Int) ⇒ x + y
```

Use case :

```
def sumCubes(a: Int, b: Int) = sum(x ⇒ x * x * x, a, b) // type of function is inferred by the compiler
```

Currying :

Suppose we have this header for `sum` : `def sum(f : Int=>Int , a : Int , b : Int)`

```
def sumInts(a: Int, b: Int) = sum(x => x, a, b)
def sumCubes(a: Int, b: Int) = sum(x => x * x * x, a, b)
def sumFactorials(a: Int, b: Int) = sum(fact, a, b)
```

Notice that a and b get passed unchanged from `sumInts` to `sum` .

Q : Can we do it shorter ?

```
def sum(f: Int => Int): (Int, Int) => Int =
    def sumF(a: Int, b: Int): Int =
        if a > b then 0
        else f(a) + sumF(a + 1, b)
sumF

def sumInts = sum(x => x)
def sumCubes = sum(x => x * x * x)
def sumFactorials = sum(fact)
```

`sum` is now a function that takes 1 argument (`f`) and returns a function that takes 2 arguments `a` and `b` .

We can still do shorter :

```
def sum(f: Int => Int)(a: Int, b: Int): Int =
    if a > b then 0 else f(a) + sum(f)(a + 1, b)
```

Q : What is the type of `sum` ?

A : `(Int=>Int) => ((Int,Int)=>Int)` equivalent to

`(Int=>Int) => (Int,Int)=>Int`

Note that function types **associate to the right** so it is equivalent to :

`Int => Int => Int` is equivalent to `Int => (Int => Int)`

Classes :

we can define classes in Scala :

```
class Rational(x: Int, y: Int):
    def numer = x // numer and denom here are functions (recalculated each call)
    def denom = y

    // adding operations
    def addRational(r: Rational, s: Rational):
        Rational = Rational(r.numer * s.denom + s.numer * r.denom,r.denom *
```

```scala
    // overriding toString
    override def toString = s"$numer/$denom" // s means formatted string
    // what is after a $ is evaluated

val x = Rational(1, 2) // x: Rational = Rational@2abe0e27
x.numer // 1
x.denom // 2
val y = Rational(5, 7)
val z = Rational(3, 2)
x.add(y).add(z)
```

Classes are very similar to Java.

Suppose that `numer` and `denom` need more calculations , like the following :

```scala
class Rational(x: Int, y: Int):
    private def gcd(a: Int, b: Int): Int =
        if b == 0 then a else gcd(b, a % b)
    def numer = x / gcd(x, y) // these are recalcaled at each call
    def denom = y / gcd(x, y) // BAD IDEA

    // BETTER IDEA : IMMUTABLE VARIABLES :
    val numer = x / gcd(x, y)
    val denom = y / gcd(x, y)
```

It is possible to use `this.(..)` like is Java.

Substitution and Extensions :

1. Substitution :

Suppose that we have a class definition :

`class C(x1, ..., xm){ ... def f(y1, ..., yn) = b ... }`

Q : How is the following expression evaluated ? `C(v1, ..., vm).f(w1, ..., wn)`

A : Three substitutions happen written :

`[w1/y1, ..., wn/yn][v1/x1, ..., vm/xm][C(v1, ..., vm)/this] b`

- Substitution of `y1, ..., yn` of f by the arguments `w1, ..., wn` .
- Substitution of the `x1, ..., xm` of the class C by the class arguments `v1, ..., vm` .
- Substitution of the self reference `this` by the value of the object `C(v1, ..., vn)` .

2. Extension methods (1) :

Having to define all methods that belong to a class inside the class itself can lead to very large classes, and is not very modular.

Methods that do not need to access the internals of a class can alternatively be defined as extension methods

For instance, we can add min and abs methods to class Rational like this:

```
extension (r: Rational):
    def min(s: Rational): Boolean = if s.less(r) then s else r
    def abs: Rational = Rational(r.numer.abs, r.denom)
```

- Extension methods **CANNOT** access internals (`private`) or `this.(..)` .

3. Operators :

▶ We write `x + y` , if x and y are integers, but

▶ We write `r.add(s)` if r and s are rational numbers.

In Scala, we can eliminate this difference with **operators**.

```
extension (x: Rational):
    def + (y: Rational): Rational = x.add(y)
```

Precedence rules :

```
(all letters)
|
^
&
< >
= !
:
+ -
* / %
(all other special characters)
```

(all other special characters) having the **highest priority**.

Week3 :

Abstract classes :

```
abstract class IntSet:
    def incl(x: Int): IntSet  // ABSTRACT MEMBERS : no implementation provided
    def contains(x: Int): Boolean
// incl returns the union of {x} and this set
```

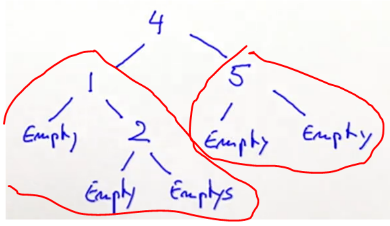
Like in Java, abstract classes cannot be instantiated.

We will try to implement a set as a binary tree.

A set can either be :

1. A tree for the **empty set**.

2. A tree consisting of **one** integer with two subtrees.



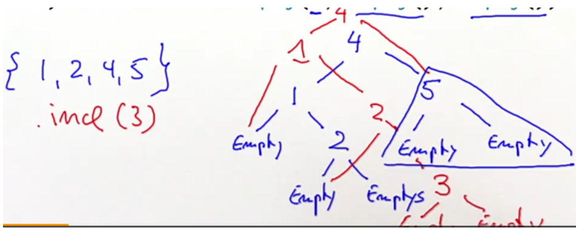

```
class Empty() extends IntSet:
    def contains(x: Int): Boolean = false
    def incl(x: Int): IntSet = NonEmpty(x, Empty(), Empty())

class NonEmpty(elem: Int, left: IntSet, right: IntSet) extends IntSet:
        def contains(x: Int): Boolean =
            if x < elem then left.contains(x)
            else if x > elem then right.contains(x) else true

        def incl(x: Int): IntSet =
        if x < elem then NonEmpty(elem, left.incl(x), right)
        else if x > elem then NonEmpty(elem, left, right.incl(x))
    else this
end NonEmpty
```

1. **Persistence :**

   Note that we the sets are immutable, we always return a new tree while **reusing some subtrees.**(blue one in the following e.g)



A data structure that is creating by maintaining the old one is called **persistent**.

2. **Dynamic binding :**

```
class Empty() extends IntSet:
    ...
    def union(other : Inset): Intset = other
class NonEmpty((elem: Int, left: IntSet, right: IntSet)) extends Inset :
    ...
    def union(other : Inset ): Intset =
lelf.union(right).union(that).incl(elem)

// Why does this terminate ? Union is called with strictly smaller sets each
time so union( "an empty set " ) will be called at some point.
```

Note that a call to `union` doesn't execute the same function if s is `Empty` or `NonEmpty`. That is called Dynamic **binding**. (Polymorphism)

Object definition :

In the `IntSet` example, one could argue that there is really **only a single** empty `IntSet`.

This defines a **singleton** object named Empty. No other Empty instance can be (or needs to be) created.

```
 object Empty extends IntSet:
    def contains(x: Int): Boolean = false
    def incl(x: Int): IntSet = NonEmpty(x, Empty, Empty) end Empty
```

Companion object :

If a class and object with the same name are given in the same sourcefile, we call them companions. Example:

```

class IntSet ...
object IntSet:
    def singleton(x: Int) = NonEmpty(x, Empty, Empty)
```

Similar to Java, static nested class.

How is naming a class and an object the same name possible ? Scala has **two** namespaces , one for objects and the other for classes.

Programs :

Like Java , scale source files can have a main methods

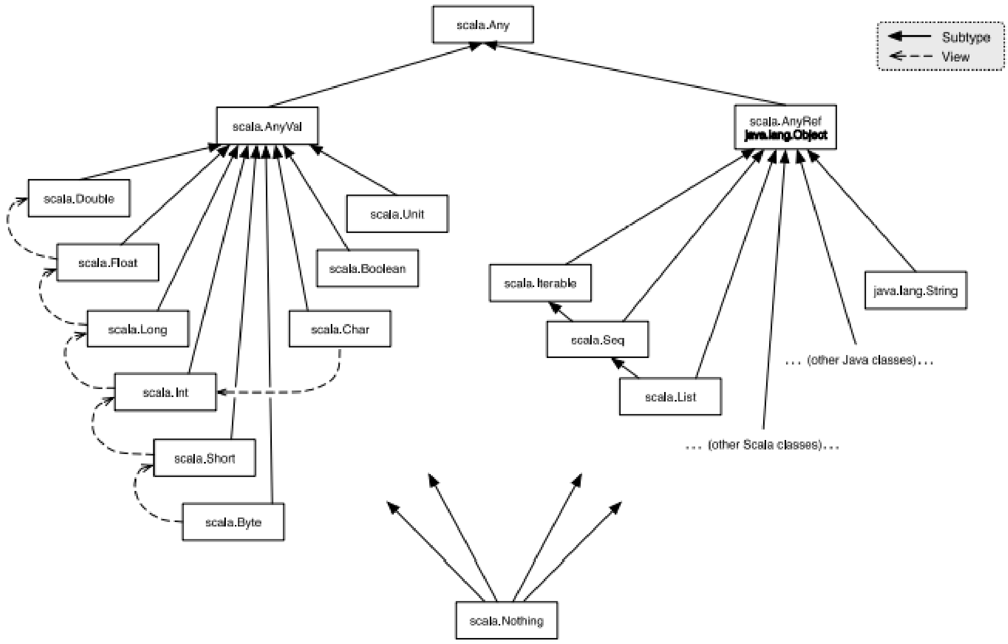
```
object Hello:
    def main(args: Array[String]): Unit = println("hello world!")
// written Shorter as the following :
@main def birthday(name: String, age: Int) =
println(s"Happy birthday, $name! $age years old already!")
```

to run : `scala Hello`

Traits :

Java `Interface` but stronger :they can have parameters and can contain fields and concrete methods.

```
trait Planar:
    def height: Int
    def width: Int
    def surface = height * width
class Square extends Shape, Planar, Movable ...
```



`- - ->` : can be converted (viewed) as

Cons List :

A fundamental data structure in many functional languages is the **immutable linked list.**

It is constructed from two building blocks:

1. `Nil` :the empty list
2. `Cons` :a cell containing an element and the remainder of the list

```
trait IntList ...
class Cons(val head: Int, val tail: IntList) extends IntList ...
// NOTE THE USE OF val
// this defines at the same time a parameter and a field of a class
class Nil() extends IntList ...
```

A `IntList` is either a `Nil()` or a `Cons(x,xs)` .

A complete definition with Generics :

```
trait List[T]:
    def isEmpty: Boolean
    def head: T
    def tail: List[T]
class Cons[T](val head: T, val tail: List[T]) extends List[T]:
    def isEmpty = false
class Nil[T] extends List[T]:
    def isEmpty = true
    def head = throw new NoSuchElementException("Nil.head")
    def tail = throw new NoSuchElementException("Nil.tail")
```

Generic functions :

```scala
def singleton[T](elem: T) = Cons[T](elem, Nil[T])
// We can then write:
singleton[Int](1)
singleton[Boolean](true)
```

Pure Object Orientation :

Def : A pure object-oriented language is one in which every value is an object.

Q : Is Scala a pure OO language ?

At first primitive types and functions seem like exceptions.

Primitive types are in fact not implemented as Class. (E.g `scala.Int` : 32bits , `scala.Boolean` as Java Boolean) , for reasons of efficiency.

But they are not *conceptually* treated differently. In fact one can implement them using classes :

```scala
package idealized.scala
abstract class Boolean extends AnyVal:
    def ifThenElse[T](t: => T, e: => T): T
    def && (x: => Boolean): Boolean = ifThenElse(x, false)
    def || (x: => Boolean): Boolean = ifThenElse(true, x)
    def unary_!: Boolean = ifThenElse(false, true)
    def == (x: Boolean): Boolean = ifThenElse(x, x.unary_!)
    def != (x: Boolean): Boolean = ifThenElse(x.unary_!, x)
...
end Boolean
object true extends Boolean:
def ifThenElse[T](t: => T, e: => T) = t
object false extends Boolean:
def ifThenElse[T](t: => T, e: => T) = e
```

Here is how to implement `Scala.Int` .

Functions : In scala functions are `objects` with `apply` methods

The function type `A => B` is just an abbreviation for the class `scala.Function1[A, B]` , which is defined as follows.

```scala
package scala
trait Function1[A, B]:
    def apply(x: A): B
```

Example :

```scala
f = (x:Int) => x * x
// is expended to
f = new Function1[Int, Int]:
    def apply(x: Int) = x * x
```

This anonymous class can itself expend to

```
{ class $anonfun() extends Function1[Int, Int]:
def apply(x: Int) = x * x
$anonfun()
}
f(7) // expends to
f.apply(7)
```

Week 4 :

Decomposition :

`Expr` Suppose you want to write a small interpreter for arithmetic

/ \ expressions.

`Number` `Sum` `Sum(Number(1) , Number(2)).eval` = 3

```
trait Expr :
    def eval(e: Expr): Int =
        if e.isNumber then e.numValue
        else if e.isSum then eval(e.leftOp) + eval(e.rightOp)
        else throw Error("Unknown expression " + e)
```

1. `isNumber` , `isSum` gets quickly gets tedious when adding other datatypes .
2. There's no static guarantee you use the right accessor functions.
   You might hit an Error case if you are not careful.

Non solution :Type Tests and Type casts

use `def isInstanceOf[T]: Boolean` to type *test* and `def asInstanceOf[T]: T` to cast.

Their use in Scala is discouraged, because there are better alternatives

Solution 1 : Object Oriented Decomposition

```
trait Expr:
def eval: Int
class Number(n: Int) extends Expr:
    def eval: Int = n
class Sum(e1: Expr, e2: Expr) extends Expr:
    def eval: Int = e1.eval + e2.eval
```

▶ OO decomposition mixes data with operations on the data.

▶Good for encapsulation and data abstraction.

▶Dependencies between classes => Increases **Complexity**.

▶ It makes it easy to add new kinds of data but **hard** to add new kinds of operations

OO decomposition only works well if operations are on a single object.
NOT for expressions of type `a * b + a * c -> a * (b + c)`

Solution : Pattern Matching :

```
def eval(e: Expr): Int = e match
    case Number(n) => n // pattern => expression
    case Sum(e1, e2) => eval(e1) + eval(e2)
```

A `MatchError` exception is thrown if no pattern matches the value of the selector.

To be able to do pattern matching `Number` and `Sum` should be **case classes**.

```
trait Expr
case class Number(n: Int) extends Expr
case class Sum(e1: Expr, e2: Expr) extends Expr
```

Lists and more pattern matching :

▶ the empty list `Nil`, and
▶ the construction operation `::` (pronounced cons):
`x :: xs` gives a new list with the first element x, followed by the elements of `xs`

```
val fruit: List[String] = List("apples", "oranges", "pears")
fruit = "apples" :: ("oranges" :: ("pears" :: Nil)) // similar
```

Right associativity convention : Operators ending in ":" associate to the right.
`A :: B :: C` is interpreted as `A :: (B :: C).`

All operations on lists can be expressed in terms of the following three:

- `head` the first element of the list
- `tail` the list composed of all the elements except the first.
- `isEmpty` true if the list is empty, false otherwise .

example of using lists and pattern matching :

```
def isort(xs: List[Int]): List[Int] = xs match
    case List() => List()
    case y :: ys => insert(y, isort(ys))

def insert(x: Int, xs: List[Int]): List[Int] = xs match
    case List() => List(x)
    case y :: ys => if x > y then y :: insert(x,ys) else x :: xs
```

Enum :

```
Here's our case class hierarchy for expressions again:
trait Expr
object Expr:
    case class Var(s: String) extends Expr
    case class Number(n: Int) extends Expr
    case class Sum(e1: Expr, e2: Expr) extends Expr
```

This is so common is scala that there's a shorthand for that.

```
enum Expr:
    case Var(s: String)
    case Number(n: Int)
    case Sum(e1: Expr, e2: Expr)
    case Prod(e1: Expr, e2: Expr)
```

There's more to Enumeration , they can take parameters and can define methods.

```
enum Direction(val dx: Int, val dy: Int):
    case Right extends Direction( 1, 0)
    case Up extends Direction( 0, 1)
    case Left extends Direction(-1, 0)
    case Down extends Direction( 0, -1)
def leftTurn = Direction.values((ordinal + 1) % 4)
end Direction

val r = Direction.Right
val u = x.leftTurn // u = Up
val v = (u.dx, u.dy) // v = (1, 0)
```

Type Bounds :

`assertPos` should return the set itself if all elements are positive and throw `exception` otherwise.

```
def assertAllPos(s: IntSet): IntSet
```

The above definition doesn't show that `assertAllPos` of an `Empty` returns an `Empty` and `assertAllPos` of `NonEmpty` returns an `NonEmpty` . There's better :

```
def assertAllPos[S <: IntSet](r: S): S = ...
```

Here, `<: IntSet` is an upper bound of the type parameter S.

- `S <: T` means: S is a **subtype** of T, and
- `S >: T` means: S is a **supertype** of T, or T is a subtype of S.

We can *mix* them : `[S >: NonEmpty <: IntSet]`

> **The Liskov Substitution Principle**
>
> If A <: B, then everything one can to do with a value of type B
> one should also be able to do with a value of type A.

Variance :

Given: `NonEmpty <: IntSet` is `List[NonEmpty] <: List[IntSet]` ? yes.

when `A <: B => C[A] <: C[B]` we call `C[T]` **covariant**.

Does this work for all types ? let's consider `Arrays` (**mutable**)

```
val a: Array[NonEmpty] = Array(NonEmpty(1, Empty(), Empty()))
val b: Array[IntSet] = a // TYPE ERROR HEERE
b(0) = Empty() // CAN DO WITH ARRAY[INTSET] BUT NOT WITH ARRAY[NONEMPTY]
val s: NonEmpty = a(0)
```

Type Error Line 2 : `Array[NonEmpty]` not a subtype of `Array[Inset]` .

Why ? Because otherwise it would contradict *Liskov Principle* **not** all you can do with `Array[Inset]` , you can do with `Array[NonEmpty]` . (Line 3,4)

Say `C[T]` is a parameterized type and A, B are types such that `A <: B` .

- `C[A] <: C[B]` C is **covariant** (e.g `List`)
- `C[A] >: C[B]` C is **contravariant**
- neither `C[A]` nor `C[B]` is a subtype of the other C is **nonvariant** (e.g `Array`)

```
class C[+A] { ... } C is covariant
class C[-A] { ... } C is contravariant
class C[A] { ... } C is nonvariant
```

> **Function types :**
>
> `A1 => B1 <: A2 => B2` if `A2 <: A1 and B1 <: B2` .
>
> **Covariant** in return type. **Contravariant** in input type

```
trait Function1[-T, +U]:
    def apply(x: T): U
```

> **Variance rules** :

- covariant type parameters can only appear in method results.
- contravariant type parameters can only appear in method parameters.
- invariant type parameters can appear anywhere.

Week 5 :

List methods :

`xs.length` , `xs(n) <=> xs.apply(n)` , `xs.reverse` , `xs.updated(n, x)` , `xs.indexOf(x)` (returns -1 if `x` not in `xs`) , `xs.contains(x)` .

Creating new Lists	All of the following are LINEAR
`xs.last`	The list's last element, exception if `xs` is empty.
`xs.take(n)`	A list consisting of the first n elements of `xs` , or `xs` itself if it is shorter than n.
`xs.drop(n)`	The rest of the collection after taking n elements.
`xs ++ ys`	Concatenation. `def ++ (ys: List[T]): List[T] = xs match case Nil => ys case x :: xs1 => x :: (xs1 ++ ys)`
`xs(n)`	(or, written out, `xs.apply(n)`). The element of `xs` at index n.`
`splitAt(n)`	pair of : (`xs[1 to n]` , `xs[n to xs.length]`)

Simple merge sort implementation :

```
def msort[T](xs: List[T])( (T, T) => Boolean ): List[T] =
    val n = xs.length / 2
    if n == 0 then xs
    else
        def merge[T](xs: List[T], ys: List[T]) = (xs,ys) match
            case (Nil,ys) => ys
            case (xs,Nil) => xs
            case (x :: xs1 , y :: ys1 ) =>
                if lt(x,y) then x :: merge(xs1,ys)
                else y :: merge(xs,ys1)

        val (fst, snd) = xs.splitAt(n)
        merge(msort(fst), msort(snd))
```

`lt : (T, T) => Boolean` is a comparison function.

Pairs :

```scala
val label = pair._1
val value = pair._2

val (label, value) = pair
```

Filtering and mapping :

methods	all LINEAR
`xs.filter(p)`	Elements of `xs` verifying `p.`
`xs.filterNot`	Same as `xs.filter(x => !p(x))` ;
`xs.partition(p)`	Same as `(xs.filter(p), xs.filterNot(p))` , but computed in a single traversal of the list `xs`
`xs.takeWhile(p)`	The longest prefix of list `xs` consisting of elements that all satisfy the predicate `p` .
`xs.dropWhile(p)`	The remainder of the list `xs` after any leading elements satisfying `p` have been removed.
`xs.span(p)`	Same as `(xs.takeWhile(p), xs.dropWhile(p))` but computed in a single traversal of the list `xs` .
`xs.map`	create a new list with `f` applied to all elements of `xs` .

Reductions :

Combining elements of lists with a given operator.

```scala
def sum(xs: List[Int]) = (0 :: xs).reduceLeft((x, y) => x + y)

([0,xs1,xs2, ... ,xsn])
=> [ 0+xs1 , xs2 ,  ...  , xsn ]
=> [ 0+xs1+xs2 ,  ...  , xsn ]
=> 0+xs1+xs2+ ... +xsn
```

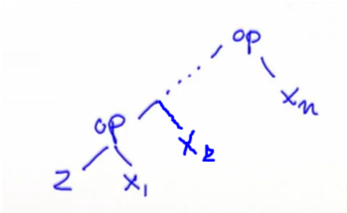
In more generality: `List(x1, ..., xn).reduceLeft(op) = x1.op(x2). ... .op(xn)`

In the same way there's `reduceRight` :

`List(x1, ..., x{n-1}, xn).reduceRight(op) = x1.op(x2.op( ... (x{n-1}.op(xn)) )`

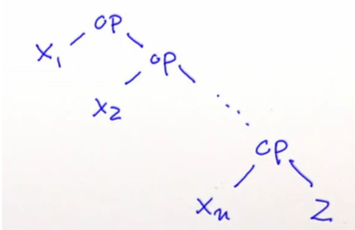
FoldLeft :

```
def reduceLeft(op: (T, T) => T): T = this match
    case Nil => throw IllegalOperationException("Nil.reduceLeft")
    case x :: xs => xs.foldLeft(x)(op)
// takes an accumulator
def foldLeft[U](z: U)(op: (U, T) => U): U = this match
    case Nil => z
    case x :: xs => xs.foldLeft(op(z, x))(op)
```



FoldRight:

```
def reduceRight(op: (T, T) => T): T = this match
    case Nil => throw UnsupportedOperationException("Nil.reduceRight")
    case x :: Nil => x
    case x :: xs => op(x, xs.reduceRight(op))
// takes an accumulator
def foldRight[U](z: U)(op: (T, U) => U): U = this match
    case Nil => z
    case x :: xs => op(x, xs.foldRight(z)(op))
```



`FoldLeft` and `FoldRight` are equivalent when `op` is associative and commutative.

`FoldLeft` is `tailrec` so more efficient .

FoldLeft/FoldRight are very useful :

```
def concat[T](xs: List[T], ys: List[T]): List[T] =
    xs.foldRight( ys )(_::_)
```

```
def reverse[a](xs: List[T]): List[T] =
    xs.foldLeft( List[T]() ) ( ( x , zs ) => x :: zs )

// List[T]()  is neccesary for type inference
```

```
def mapFun[T, U](xs: List[T], f: T => U): List[U] =
    xs.foldLeft( List[T]()  )( (x,z) => f(x)::z )
def lengthFun[T](xs: List[T]): Int =
    xs.foldRight(0)((y, n) => n + 1)
```

Structural Induction :

We would like to verify that concatenation is associative, and that it admits the empty list Nil as neutral element to the left and to the right:

```
(xs ++ ys) ++ zs = xs ++ (ys ++ zs)
xs ++ Nil = xs
Nil ++ xs = xs
```

We will use *Structural Induction* : To prove a property `P(xs)` for all lists `xs`,

- show that `P(Nil)` holds (base case),
- for a list `xs` and some element x, show the induction step: if `P(xs)` holds, then `P(x :: xs)` also holds

recall the implementation of `++` :

```
extension [T](xs: List[T])
def ++ (ys: List[T]) = xs match
    case Nil => ys
    case x :: xs1 => x :: (xs1 ++ ys)
```

two facts :

1. `Nil ++ ys` = `ys`
2. `(x :: xs1) ++ ys` = `x :: (xs1 ++ ys)`

Proof : Induction on `xs`

- **base case** : Nil

  (Nil ++ ys) ++ zs = ys ++ zs by 1st clause

  Nil ++ (ys ++ zs) = ys ++ zs also by 1st clause
- **Inductive step :** suppose (xs ++ ys) ++ zs = xs ++ (ys ++ zs) holds , prove it on x :: xs

  on LHS :

  ((x :: xs) ++ ys) ++ zs

  = (x :: (xs ++ ys)) ++ zs // by 2nd clause of ++

  = x :: ((xs ++ ys) ++ zs) // by 2nd clause of ++

  = x :: (xs ++ (ys ++ zs)) // by induction hypothesis

  on RHS :

  (x :: xs) ++ (ys ++ zs)

Week 6 :

Collections :

Collection Hierarchy :

```
                     Iterable
                    /    |     \
                   /     |      \
     ...........Seq     Set      Map
     .         /   \
     .        /     \
   Array    List    Vector
```

Arrays and String :

- Arrays and Strings support the same operations as `Seq` (`filter` , `map` ...)
- They cannot be subclasses of `Seq` because they come from Java.

Sequences :

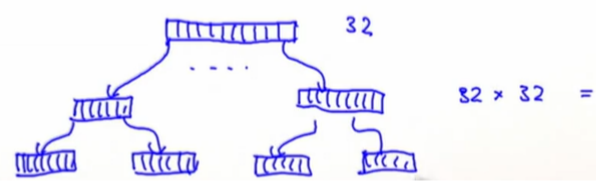
Range :

```
val r: Range = 1 until 5 // 5 EXCLUDED
val s: Range = 1 to 5 // 5 INCLUDED
1 to 10 by 3 // 3 IS THE STEP
6 to 1 by -2
```

Vector :

`List` has *linear* time access : Access to the middle element is slower than first...

On the contrary, `Vector` has similar access time for all its elements.



It is implement (as shown above) in the following way : (n=number of elements)

- if n <= 32 then it is an array of 32 elements
- if n <= 32*32 then it is an array of 32 elements each containing an array of 32 elements .
- if n <= 32 *32 *32 : array of array of array

Sequence	

Sequence operations :	
`xs.exists(p)`	true if there is an element x of xs such that p(x) holds, false otherwise.
`xs.forall(p)`	true if p(x) holds for all elements x of `xs`, false otherwise.
`xs.zip(ys)`	A sequence of pairs drawn from corresponding elements of sequences xs and ys.
`xs.unzip`	Splits a sequence of pairs `xs` into two sequences consisting of the first, respectively second halves of all pairs.
`xs.flatMap(f)`	Applies collection-valued function f to all elements of `xs` and concatenates the results. `xs.flatMap(f) = xs.map(f).flatten`
`xs.sum`, `xs.product`, `xs.max`, `xs.min`	the sum,product,min and max

Examples showing use of `seq` operations for conciseness :

```
def scalarProduct(xs: Vector[Double], ys: Vector[Double]): Double =
    xs.zip(ys).map(_ * _).sum

def isPrime(n: Int): Boolean =
(2 to n - 1).forall(d => n % d == 0)
```

Combinatorial Search and For-Expressions :

Generate all pairs (i , j) 1 <= j < i <= n :

```
(1 until n).flatMap(i =>
    (1 until i).map(j => (i, j)))
```

`For` helps for more conciseness and clarity :

```
for
i <- 1 until n
j <- 1 until i
if isPrime(i + j)
yield (i, j)

// THIS RETURNS A LIST OF THE (i,j)
```

> **IMPORTANT**
>
> The for-expression maybe seem similar to loops in imperative languages, except that it **builds a list** of the results of all iterations.

The type of the sequence returned by the for loop **is the type of the sequence on which the first for loop is applied.**

For and If :

```
for p ← persons if p.age > 20 yield p.name
// EQUIVALENT TO
persons
.filter(p ⇒ p.age > 20)
.map(p ⇒ p.name)
```

> **IMPORTANT 2 :** for
>
> `i <- 1 until n`
>
> `j <- 1 until n`
>
> is equivalent to **nested** loops.

```
def scalarProduct(xs: List[Double], ys: List[Double]) : Double =
(for (x, y) ← xs.zip(ys) yield x * y).sum

NOT EQUIVALENT TO

(for x ← xs; y ← ys yield x * y).sum // HERE ALL PAIRS (X,Y) are considered
```

Maps :

Class `Map[Key, Value]` extends the collection type `Iterable[(Key, Value)]`.

```
val romanNumerals = Map("I" → 1, "V" → 5, "X" → 10)
val capitalOfCountry = Map("US" → "Washington", "Switzerland" → "Bern")
```

The syntax `key -> value` is just an alternative way to write the pair `(key, value)`.

`toList` on a `Map` produces a `List` of pairs `(key,value)`.

Query on map :

```
capitalOfCountry("Andorra")
// java.util.NoSuchElementException: key not found: Andorra

capitalOfCountry.get("US") // Some("Washington")
capitalOfCountry.get("Andorra") // None
```

- The `Option` type :

```
trait Option[+A]
    case class Some[+A](value: A) extends Option[A]
    object None extends Option[Nothing]
```

we can decompose the `Option` type with pattern matching :

```
def showCapital(country: String) =
capitalOfCountry.get(country) match
    case Some(capital) => capital
    case None => "missing data"


showCapital("US") // "Washington"
showCapital("Andorra") // "missing data"
```

- Updates :
  - `m + (k -> v)` : The map that takes key `k` ‘ to value `v` and is otherwise equal to `m` .
  - `m ++ kvs` The map `m` updated via `+` with all key/value pairs in `kvs`.

    ```
    val m1 = Map("red" -> 1, "blue" -> 2) // > m1 = Map(red -> 1, blue -> 2)
    val m2 = m1 + ("blue" -> 3) // > m2 = Map(red -> 1, blue -> 3)
    val m3 = m1 ++  ("blue" -> 0 ,  "yellow" -> 4) // m2 = Map(red -> 1, blue -
    > 0 , yellow -> 4 )
    ```

  - `m - k` : The map `m` without key `k`.
- Default values : `capitalOfCountry.withDefaultValue( v )` producuces a new map that has `v` as defaut value.

Example use of maps on `Polynoms` (`+` operation) :

```
class Polynom(nonZeroTerms: Map[Int, Double]):

    def terms = nonZeroTerms.withDefaultValue(0.0)
    // ===== Solution 1 : More efficient
    def + (other: Polynom) =
        Polynom(other.terms.foldLeft(terms)(addTerm))
    def addTerm(terms: Map[Int, Double], term: (Int, Double)) =
        val (exp, coeff) = term
        terms + (exp, coeff + terms(exp))
    // ===== Solution 2 : More concise
    def + (other: Polynom) =
        Polynom(terms ++ other.terms.map((exp, coeff)
                                    => (exp, terms(exp) + coeff)))
```

Week 7 :

Queries with For :

We can use `for` to query information instead of using a combination of `filter` , `map` and `mapReduce` .

`authors` is a list of names (`string`).

Query the name of all the authors that have more than a book:

```
for
    b1 <- books
    b2 <- books
    if b1 != b2
    a1 <- b1.authors
    a2 <- b2.authors
    if a1 == a2
yield a1

// if authors a1="Daniel" wrote b1 and b2 this will return
// List("Daniel","Daniel") because it considers (b1,b2) and (b2,b1)

// SOLUTION 1 : use the function distinct
val repeated =
for
    b1 <- books
    b2 <- books
    if b1.title < b2.title
    a1 <- b1.authors
    a2 <- b2.authors
    if a1 == a2
yield a1
repeated.distinct
```

Better alternative : for returns a sequence of the same type as the sequence we iterate over. so Iterate over set => result will be a set.

```
val bookSet = books.toSet
    for
    b1 <- bookSet
    b2 <- bookSet
    if b1 != b2
    a1 <- b1.authors
    a2 <- b2.authors
    if a1 == a2
    yield a1
```

How are FOR expressions translated ?

3 rules :

1. `for x <- e1 yield e2` is translated to `e1.map(x => e2)`
2. `for x <- e1 if f; s yield e2` is translated to `for x <- e1.withFilter(x => f); s yield e2`
   - `s` is a (possibly empty) sequence of generators (`for`)and filters(`if`).
   - `withFilter` is a variation of `filter` than doesn't produce a new list but instead just executes the next `map/flatMap` on elements that pass the filter.
3. is translated to

Example :

```
for b ← books; a ← b.authors if a.startsWith("Bird")
yield b.title
⇒
books.flatMap( b ⇒ for a ← b.authors if a.startsWith("Bird") yield b.title)

⇒ books.flatMap( b ⇒
              for a ← b.authors.withFilter(a⇒a.startsWith("Bird"))
              yield b.title)

⇒ books.flatMap(b⇒b.authors.withFilter(a⇒a.startsWith("Bird")).
                 map(a⇒b.title))
```

Generalizing for expressions :

Interestingly, the translation of for is not limited to lists or sequences, or even collections .

The presence of `map` , `flatMap` and `withFilter` suffices to be able to use `for` on some type.

Example : random value generator :

we know how to generate a random number with the Java library :

```
val rand = java.util.Random()
rand.nextInt()
```

Let's define a trait `Generator[T]` that generates random values of type `T` :

```
trait Generator[+T]:
    def generate(): T
```

- we will instantiate it to create a random `Int` generator :

```
val integers = new Generator[Int]:
    val rand = java.util.Random()
    def generate() = rand.nextInt()
// this creates a subclass of Generator and instanciates it
```

- `Boolean` generator :

```
val booleans = new Generator[Boolean]:
    def generate() = integers.generate() > 0
```

- `pairs` generator :

```
val pairs = new Generator[(Int, Int)]:
    def generate() = (integers.generate(), integers.generate())
```

```
val booleans = for x ← integers yield x > 0 // we would like to write this
```

so we need to define `map` and `flatMap` for that

```
trait Generator[+T]:
    def generate(): T

extension [T, S](g: Generator[T])

    def map(f: T ⇒ S) = new Generator[S]:
        def generate() = f(g.generate())

    def flatMap(f: T ⇒ Generator[S]) = new Generator[S]:
        def generate() = f(g.generate()).generate()
```

we can know write :

```
val booleans = for x ← integers yield x > 0 // which translates to
val booleans = integers.map ( x ⇒ x>0 ) // which translates to
val booleans = new Generator[Boolean]:
    def generate() = integers.generate() > 0
```

More generators:

```
def single[T](x: T): Generator[T] = new Generator[T]:
    def generate() = x

def range(lo: Int, hi: Int): Generator[Int] =
    for x ← integers yield lo + x.abs % (hi - lo)

def oneOf[T](xs: T*): Generator[T] =
    for idx ← range(0, xs.length) yield xs(idx)

def lists: Generator[List[Int]] =
    for
        isEmpty ← booleans // randomly check if our list should be empty
        list ← if isEmpty then emptyLists else nonEmptyLists
    yield list


def nonEmptyLists =
for
    head ← integers // first element is random
    tail ← lists    // rest of the list is random ( maybe empty)
yield head :: tail
```

Application :

- Come up with some some test inputs to program functions and a postcondition.
- The postcondition is a property of the expected result.
- Verify that the program satisfies the postcondition.

```scala
def test[T](g: Generator[T], numTimes: Int = 100)(test: T => Boolean): Unit =
    for i <- 0 until numTimes do
        val value = g.generate()
        assert(test(value), s"test failed for $value")
    println(s"passed $numTimes tests")
```

```scala
test(pairs(lists,lists))(
    (l1,l2) => l1.size + l2.size == concat(l1 , l2).size
)
// tests if concat preserve the property of the sizes
// if this test fails it means concat has a bug
```

Scala check :

```scala
forAll { (l1: List[Int], l2: List[Int]) =>
    l1.size + l2.size == (l1 ++ l2).size
}
```

Monalds :

Data structures with `map` and `flatMap` seem to be quite common.

These types of data structures are called `Monalds` :

```scala
extension [T, U](m: M[T])
    def flatMap(f: T => M[U]): M[U]

def unit[T](x: T): M[T]
```

Why `unit` ?

let's look at how to implement `map` with the help of `flatMap`

```scala
m.map(f) == m.flatMap(x => unit(f(x)))
         == m.flatMap(f andThen unit)
```

- `List` is a monad with `unit(x) = List(x)`
- `Set` is monad with `unit(x) = Set(x)`
- `Option` is a monad with `unit(x) = Some(x)`
- `Generator` is a monad with `unit(x) = single(x)`

Monad Laws :

To qualify as a monad, a type has to satisfy three laws:

1. **Associativity:** `m.flatMap(f).flatMap(g) == m.flatMap(f(_).flatMap(g))`

   (`==m.flatMap(x=>f(x).flatMap(g))`)

2. **Left unit :** `unit(x).flatMap(f) == f(x)`

Let's check these rules for Option

```
extension [T](xo: Option[+T])
    def flatMap[U](f: T => Option[U]): Option[U] = xo match
    case Some(x) => f(x)
    case None => None
```

- Left Unit :

```
Some(x).flatMap(f) = Some(x) match case Some(x) =>f(x) case  ...
= f(x)
```

- Right Unit :

```
opt.flatMap(Some(x))
=   opt match
    case Some(x) => Some(x)
    case None => None
= opt
```

- Associativity :

```
opt.flatMap(f).flatMap(g)
== (opt match { case Some(x) => f(x) case None => None }) // flatMap(f)
    match { case Some(y) => g(y) case None => None }

== opt match
    case Some(x) =>
        f(x) match { case Some(y) => g(y) case None => None }
    case None =>
        None match { case Some(y) => g(y) case None => None }

== opt match
case Some(x) =>
    f(x) match
    { case Some(y) => g(y) case None => None } // flatMap (g)
 case None => None

== opt match
    case Some(x) => f(x).flatMap(g)
    case None => None
```

```
== opt match
    case Some(x) =>
        f(x) match { case Some(y) => g(y) case None => None }
    case None => None
== opt match
    case Some(x) => f(x).flatMap(g)
    case None => None
== opt.flatMap(x => f(x).flatMap(g))
```

1. Associativity says essentially that one can "inline" nested for expressions :

```
for
    y <- for x <- m; y <- f(x) yield y
    z <- g(y)
    yield z
== for x <- m; y <- f(x); z <- g(y)
    yield z
```

2. Right unit says:

```
for x <- m yield x == m
```

Exceptions :

Exceptions in Scala are defined similarly as in Java.

An exception class is any subclass of `java.lang.Throwable`, which has itself subclasses `java.lang.Exception` and `java.lang.Error`. Values of exception classes can be thrown.

```
try e[throw ex] catch case x: Exc => handler

def validatedInput(): String =
    try getInput()
    catch
        case BadInput(msg) => println(msg); validatedInput()
        case ex: Exception => println("fatal error; aborting");
```

We can treat exceptions like normal types. `scala.util.Try` type enables this.

```
abstract class Try[+T]
    case class Success[+T](x: T) extends Try[T]
    case class Failure(ex: Exception) extends Try[Nothing]

Try(expr) // gives Success(someValue) or Failure(someException)
```

here's an implementation:

```
Here's an implementation of Try.apply:
import scala.util.control.NonFatal
object Try:
def apply[T](expr: => T): Try[T] =
    try Success(expr)
    catch
        case NonFatal(ex) => Failure(ex)
```

`FlatMap` and `map` are defined on exceptions , can use `for`

```scala
for
    x ← computeX
    y ← computeY
yield f(x, y)
// if computeX/computeY returns Success(x)/Success(Y) we get Success(f(x,y))
// else we get Failure(ex)
```

```scala
extension [T](xt: Try[T]):

    def flatMap[U](f: T ⇒ Try[U]): Try[U] = xt match
        case Success(x) ⇒ try f(x) catch case NonFatal(ex) ⇒ Failure(ex)
        case fail: Failure ⇒ fail
    def map[U](f: T ⇒ U): Try[U] = xt match
        case Success(x) ⇒ Try(f(x))
        case fail: Failure ⇒ fail
```

Week 9 :

Structural induction on Trees :

To prove a property $P(t)$ for all trees t of a certain type :

- show that $P(l)$ holds for all leaves l of a tree,
- for each type of internal node t with subtrees $s_1, \ldots, s_n$, show that $P(s_1) \wedge \ldots \wedge P(s_n)$ implies $P(t)$.

Let's show : `s.incl(x).contains(x) = true`

recall `Inset`

```scala
abstract class IntSet:
    def incl(x: Int): IntSet
    def contains(x: Int): Boolean
object Empty extends IntSet:
    def contains(x: Int): Boolean = false
    def incl(x: Int): IntSet = NonEmpty(x, Empty, Empty)

case class NonEmpty(elem: Int, left: IntSet, right: IntSet) extends IntSet:
    def contains(x: Int): Boolean =
        if x < elem then left.contains(x)
        else if x > elem then right.contains(x)
        else true
    def incl(x: Int): IntSet =
        if x < elem then NonEmpty(elem, left.incl(x), right)
        else if x > elem then NonEmpty(elem, left, right.incl(x))
        else this
```

Base case : `Empty` :

```
Empty.incl(x).contains(x)
= NonEmpty(x, Empty, Empty).contains(x) // by definition of Empty.incl
= true                                 // by definition of NonEmpty.contain
```

Inductive step:

for left and right subtrees `l` and `r` , we have :

`l.incl(x).contains(x) = true` and `r.incl(x).contains(x) = true`

- `y == x`

  ```
  NonEmpty(x, l, r).incl(x).contains(x)
  = NonEmpty(x, l, r).contains(x) // by definition of NonEmpty.incl
  = true // by definition of NonEmpty.contain
  ```

- `y < x`

  ```
  NonEmpty(y, l, r).incl(x).contains(x)
  = NonEmpty(y, l, r.incl(x)).contains(x) // by definition of NonEmpty.incl
  = r.incl(x).contains(x) // by definition of NonEmpty.contains
  = true // by the induction hypothesis
  ```

- case `y > x` is analogous

Lazy Lists :

We want to find the n-th prime between `from` and `to` :

```
(4 to 10000).filter(isPrime)(1)
```

This works but is terrible performance wise.

> **Lazy principle :** Avoid computing the elements of a sequence until they are needed for the evaluation result (which might be never).

We define `LazyList` :

- `LazyList.cons` (can be also written `#::` equivalent `to` ::` for normal Lists.
- `LazyList.empty` equivalent to `Nil` for normal Lists.

They are only evaluated on demand.

```
def lazyRange(lo: Int, hi: Int): LazyList[Int] =
    if lo ≥ hi then LazyList.empty
    else LazyList.cons(lo, lazyRange(lo + 1, hi))

lazyRange(4,10000).filter(isPrime)(1)
⇒ LazyList.con(4,lazyRange(5,10000)).filter(isPrime)(1) // by def of lazyRange
⇒ lazyRange(5,10000).filter(isPrime)(1)                 // by filter
⇒ LazyList.con(5,lazyRange(6,10000)).filter(isPrime)(1) // by def of lazyRange
⇒ lazyRange(6,10000).filter(isPrime)(0)                 // def of apply
⇒ lazyRange(6,lazyRange(7,10000)).filter(isPrime)(0) // by def of lazyRange
⇒ lazyRange(7,10000).filter(isPrime)(0)                 // def of filter
⇒ LazyList.con(7,lazyRange(8,10000)).filter(isPrime)(0) // def of lazy range
⇒ 7
```

Lazy evaluation :

```
lazy val x = expr // only evaluated the first time it is used , not immediately
```

Ex :

```
def expr =
    val x = {print("x");1}
    lazy val y = {print("y");2}
    def z = {print("z");3}
    z+y+x+z+y+x
// what printed ? xzyz
```

Using `lazy` , we can implement a tail-lazy list:

```
def cons[T](hd: T , tl ⇒ LazyList[T]) = new TaiLazyList[T]:
    def head = hd
    lazy val tail = tl
```

Infinite sequences :

```
def from(n:Int) : LazyList[Int] = n #:: from(n+1)
val nats = from(0)   // all naturals
nats.take(10).toList // :List[Int] = List(1,2, … ,9)
```

A useful application :

```
def sieve(s: LazyList[Int]): LazyList[Int] =
    s.head #:: sieve(s.tail.filter(_ % s.head ≠ 0)) // filter all multiples of
head
val primes = sieve(from(2)) // all primes in a lazyList
```

Week 10 :

Contextual Abstraction :

Let's try to make the sort function more context independent.

```
def msort(xs: List[Int]): List[Int] =    // we want more general
=> def msort[T](xs: List[T]): List[T] = // problem we don't know to compare any
type T
=> def msort[T](xs: List[A])(lessThan: (T, T) => Boolean): List[T] =
                                         // solution : add a comparison function
```

there's already a class that does orderings , it is `scala.math.Ordering[A`

```
def msort[T](xs: List[T])(ord: Ordering[T]): List[T] =
...
... if ord.lt(x, y) then ...
```

Problem: Passing around Ordering arguments is cumbersome.

```
sort(ints)(Ordering.Int)          // most of the time we use the same ordering
sort(strings)(Ordering.String)    // to sort lists of integers , strings ...
```

We can reduce the boilerplate by making `ord` an **implicit** parameter.

```
def sort[T](xs: List[T])(using ord: Ordering[T]): List[T] = ...
```

Then, calls to sort can omit the `ord` parameter:

```
sort(ints)                        // becomes sort[Int](ints)
sort(strings)                     // becomes sort[String](strings)
```

We have seen that the compiler is able to infer types (`Int`) from values (`ints`) => **Type inference**.

```
sort[Int](ints)          // becomes sort[Int](ints)(using Ordering.Int)
sort[String](strings)    // becomes sort[String](strings)(using Ordering.String)
```

The Scala compiler is also able to do the opposite, namely to infer expressions (aka terms) from types. => **Term inference**.

Using clauses :

An implicit parameter is introduced by a `using` parameter clause:

```
def f( x:Int )(using a:Int ) : Int = ...
f(5)(using 3) // we can also omit it and write
f(5)          // it is optional
```

```
//Multiple parameters can be in a using clause:
def f(x: Int)(using a: A, b: B) = ...
f(x)(using a, b)            // notes a AND b are implicit
//Or, there can be several using clauses in a row:
def f(x: Int)(using a: A)(using b: B) = ...
//using clauses can also be freely mixed with regular parameters:
def f(x: Int)(using a: A)(y: Boolean)(using b: B) = ...
f(x)(using a)(y)(using b)
```

Parameters of a using clause can be anonymous:

```
def sort[T](xs: List[T])(using Ordering[T]): List[T] = ...
... merge(sort(fst), sort(snd)) // the ordering will be passed to merge (see
below how)
```

Context bound :

Sometimes one can go further and replace the using clause with a context bound for a type parameter.

```
def printSorted[T](as: List[T])(using Ordering[T]) =
    println(sort(as))
// becomes
def printSorted[T: Ordering](as: List[T]) =
    println(sort(as))
```

It means that "there must be an instance of `Ordering` that is defined on `T`" (it is passed to sort) .

```
def f [T](ps)(using U1[T], . . . ,Un[T]) : // becomes
def f [T : U1 . . . : Un](ps) : R = ...   // T has instances of U1 ... Un defined
on it.
```

Given instances :

For the compiler to find the right value of the implicit (`using`) parameter, there must be a given instance:

```
object Ordering:
//given <name of given istance> : Class[T] with ...
given Int: Ordering[Int] with
    def compare(x: Int, y: Int): Int =
        if x < y then -1 else if x > y then 1 else 0
```

Given instances can be anonymous. Just omit the instance name:

```
given Ordering[Double] with
    def compare(x: Int, y: Int): Int = ...
```

The compiler will synthesize a name for an anonymous instance:

```
given given_Ordering_Double : Ordering[Double] with // example of name generated
    def compare(x: Int, y: Int): Int = ...          // by the compiler
```

```
summon[Ordering[Int]]       // => Ordering.Int                    | in this
summon[Ordering[Double]]    // => Ordering.given_Ordering_Double | case

def summon[T](using x: T) = x // predefined
```

Say, a function takes an implicit parameter of type `T`. The compiler will search a *given instance* that:

- has a type compatible with T . (of type `T` , a subtype of `T` ...)
- is **visible** at the point of the function call, or is defined in a companion object **associated** with `T`.
  - **visible :**
    1. inherited,imported or defined in an enclosing scope.
  - **associated :**
    1. companion objects associated with any of `T`'s inherited types.

       ```
       trait X
       trait Y extends X
       // if a given instance of X is required , compiler will look in Y's
       compiler
       ```

    2. companion objects associated with any type argument in `T` .

       ```
       trait Foo[T]
       trait Bar[T] extends Foo[T]
       // if a given instance of Bar[X] is required, compiler will look in X
       // X here is the type argument
       // Foo[X] will be looked into by the compiler
       ```

    3. if `T` is an inner class, the outer objects in which it is embedded.

If there is a *single* (*most specific*) instance => it is taken as actual arguments Otherwise it's an error.

Importing Given instances :

1. By-name:

```
import scala.math.Ordering.Int
```

2. By-type:

```
import scala.math.Ordering.{given Ordering[Int]}
import scala.math.Ordering.{given Ordering[?]}
```

3. With a wildcard:

```
import scala.math.given
```

Possible errors :

```
//If there is no available given instance matching the queried type, an error
//is reported:
def f(using n: Int) = ()
f
^
//error: no implicit argument of type Int was found for parameter n of method f

// If more than one given instance is eligible, an ambiguity is reported:
trait C:
    val x: Int
    given c1: C with
        val x = 1
    given c2: C with
        val x = 2
    f(using c: C) = ()
f

//error: ambiguous implicit arguments:
//both value c1 and value c2
//match type C of parameter c of method f
```

Priorities :

Several given instances matching the same type don't generate an ambiguity *if one is more specific than the other.*

`given a:A` is more specific than a definition `given b : B` if :

- a is in a closer lexical scope than b, or
- a is defined in a class or object which is a subclass of the class defining b, or
- type A is a generic instance of type B, or
- type A is a subtype of type B.

```
//example 1 :
class A[T](x: T)
    given universal[T](using x: T): A[T](x)
    given specific: A[Int](2)
summon[A[Int]] // specific

// example 2 :
trait A:
    given ac: C
```

```
object O extends B:
val x = summon[C] // bc
// example 3 :
given ac: C
def f() =
    given b: C
    def g(using c: C) = ()
g // b will be chosen
```

Type classes :

Let's consider `Ordering[T]` , it is a recurring pattern and is called **type classes**: generic classes that come with given instances for each type.

```
def sort[A: Ordering](xs: List[A]): List[A] = ...
```

At compile time , the compile resolves the right instance of Ordering to use => It is a form of polymorphism.

Conditional instances :

Q: define an ordering for list.

A: We need ordering for its elements.

```
given listOrdering[A](using ord: Ordering[A]): Ordering[List[A]] with


def sort[A](xs: List[A])(using Ordering[A]): List[A] = ...
val xss: List[List[Int]] = ...
sort(xss)
// Given instances here will be resolved recursively
// ⇒ sort[List[Int]](xss)
// ⇒ sort[List[Int]](xss)(using listOrdering)
// ⇒ sort[List[Int]](xss)(using listOrdering(using Ordering.Int))
```

Type classes and extension methods:

```
trait Ordering[A]:
    def compare(x: A, y: A): Int
    extension (x: A)
        def < (y: A): Boolean = compare(x, y) < 0
        def ≤ (y: A): Boolean = compare(x, y) ≤ 0
        def > (y: A): Boolean = compare(x, y) > 0
        def ≥ (y: A): Boolean = compare(x, y) ≥ 0
```

Q : When are these extensions *visible* ?

A : When given instance is visible.

```scala
def merge[T: Ordering](xs: List[T], ys: List[T]): Boolean = (xs, ys) match
    case (Nil, _) => ys
    case (_, Nil) => xs
    case (x :: xs1, y :: ys1) =>
        if x < y then x :: merge(xs1, ys)   else y :: merge(xs, ys1)

//  ▸ There's no need to name and import the Ordering instance to get
//  access to the extension method < on operands of type T.
//  ▸ We have an Ordering[T] instance in scope, that's where the extension
//  method comes from.
```

Type classes provide a way to turn types into values. Unlike class extension, type classes :

- can be defined at any time without changing existing code.
- can be conditional.

Abstract algebra and type classes :

Type classes let one define concepts that are quite abstract, and that can be instantiated with many types. For instance:

```scala
trait SemiGroup[T]:                             //  A set with an associative
    extension (x: T) def combine (y: T): T   //  operation , combine

// we can then define methods that work for any group
def reduce[T: SemiGroup](xs: List[T]): T =
    xs.reduceLeft(_.combine(_))
```

We can have hierarchies in type classes.

```scala
trait Monoid[T] extends SemiGroup[T]:   // a semigroup with unit element
    def unit: T                          // the unit element
```

We want to write `reduce` on this function :

```scala
def reduce[T](xs: List[T])(using m: Monoid[T]): T =
    xs.foldLeft(m.unit)(_.combine(_))
// We can use Context bounds
def reduce[T: Monoid](xs: List[T]): T =                 // there should exist
    xs.reduceLeft(summon[Monoid[T]].unit)(_.combine(_)) // Monoid[t]

// A simpler calling syntax can be obtained if we do some preparation in the
// Monoid trait itself.
trait Monoid[T] extends SemiGroup[T]:
    def unit: T
object Monoid:
    def apply[T](using m: Monoid[T]): Monoid[T] = m
// then
def reduce[T: Monoid](xs: List[T]): T =                 // Monoid[T] is a call
    xs.reduceLeft(Monoid[T].unit)(_.combine(_))         // to apply
```

Important remarks :

- **Context passing :** We often use implicit parameters to pass pieces of context (data) that can change , but rarely do.
- **Tamper proofing :**

```
object ConfManagement:
    type Viewers = Set[Person]
    class Conference(ratings: (Paper, Int)*):
        private val realScore = ratings.toMap
        // if a viewer in viewer is an author we don't return his paper
        def rankings(viewers: Viewers): List[Paper]:
            papers.sortBy(score(_, viewers)).reverse
```

  Problem : One can do `conf.rankings(Set()).takeWhile(conf.score(_, Set()) > 80)` and have all the papers.

  Fix : Make the Viewers type alias `opaque` :

```
opaque type Viewers = Set[Person]
```

  the equality `Viewers = Set[Person]` is known only within the scope where the alias is defined. (in this case, within the `ConfManagement` object).

  Everywhere else `Viewers` is treated as a separate, `abstract` type.

Implicit Function types :

`Viewers ?=> List[Paper]` is called an implicit function type.
It simply means that the argument of type `Viewers` will be implicit.

Week 11 :

Functions and State :

Until now , we have used the substitution **principle of evaluation**.
Expression are evaluated by rewriting some parts of them,

```
sumOfSquares(3, 2+2)
=> sumOfSquares(3, 4 )
=> square(4) + square(4) // substitution

// Generally :
def f(x1, ... , xn) = B; ... f(v1, ... , vn)
→ def f(x1, ... , xn) = B; ... [v1/x1, ... , vn/xn] B

// bigger example
def iterate(n: Int, f: Int => Int, x: Int) =
    if n == 0 then x else iterate(n-1, f, f(x))
def square(x: Int) = x * x
```

```
→ if 1 == 0 then 3 else iterate(1-1, square, square(3))
→ iterate(0, square, square(3))
→ iterate(0, square, 3 * 3)
→ iterate(0, square, 9)
→ if 0 == 0 then 9 else iterate(0-1, square, square(9)) → 9
```

All possible way of rewriting lead to the same result, that is because our programs are *side effect free.*

Stateful Objects : their values may vary , depending on when you call them(values).

```
val account = BankAccount() // account: BankAccount = ...
account.deposit(50) //
account.withdraw(20) // res1: Int = 30
account.withdraw(20) // res2: Int = 10
account.withdraw(15) // java.lang.Error: insufficient funds
```

Identity and Change:

```
val x = BankAccount()
val y = BankAccount()
```

Question: Are x and y the same?

A : It depends how we define "same".

> We define it by the property of **operational equivalence** :
>
> x and y are operationally equivalent if no possible test can distinguish between them.

To test if x and y are the same, we must

- Execute the definitions followed by an arbitrary sequence of operations `S` that involves `x` and `y`, observing the possible outcomes.

```
val x = BankAccount()                    val x = BankAccount()
val y = BankAccount()                    val y = BankAccount()
f(x, y)                                  f(x, x) // should give same result
// S                                     // S'
```

- Then, execute the definitions with another sequence `S` obtained by renaming all occurrences of `y` by `x` in `S`
- If the results are different, then the expressions `x` and y are certainly different.

example :

```
val x = BankAccount()
val y = BankAccount()
x.deposit(30) // val res1: Int = 30
y.withdraw(20) // java.lang.Error: insufficient funds

val x = BankAccount()
val y = BankAccount()
x.deposit(30) // val res1: Int = 30
x.withdraw(20) // val res2: Int = 10
// The final results are different. We conclude that x and y are not the same.
```

Clearly , we cannot use *substitution* here , otherwise we would get:

```
val x = BankAccount()                           val x = BankAccount()
val y = x                       ⇒               val y = BankAccount()
```

which is not correct.

Loops :

Proposition: Variables are enough to model all imperative programs.

It's because we can create loops from functions .

- `while` loops :

```
while i > 0 do { r = r * x; i = i - 1 } // scala build-in while loop
```

```
// if we had to create it from functions
def whileDo(condition: ⇒ Boolean)(command: ⇒ Unit): Unit =
    if condition then
        command
        whileDo(condition)(command)
    else ()                        // function of type Unit that does nothing
```

- `repeatUntil { command } ( condition )` loop :

```
def repeatUntil(command: ⇒ Unit)(condition: ⇒ Boolean) : Unit =
    command
    if !condition then repeatUntil (command)(condition)
    else ()
// Using it
```

- `for` loops :

```
for i ← 1 until 3 do System.out.print(s"$i ") // scala for
// gets translated to

(1 until 3).foreach( i ⇒ System.out.print(s"$i ") )

def foreach(f: T ⇒ Unit): Unit =
    // apply 'f' to each element of the collection

Example:
for i ← 1 until 3; j ← "abc" do println(s"$i $j")
    translates to:
(1 until 3).foreach(i ⇒ "abc".foreach(j ⇒ println(s"$i $j")))
```

Week 12 : Creating an interpreter for functional language

I1 : Arithmetic and if statements :

We will use trees to represent our operations :

```
enum Expr
    case C(c: BigInt) // integer constant
    case BinOp(op: BinOps, e1: Expr, e2: Expr) // binary operation
    case IfNonzero(cond: Expr, trueE: Expr, falseE: Expr)

enum BinOps
    case Plus, Minus, Times, Power, LessEq

// How to write Code in this language ?

val expr1 = BinOp(Times, C(6), C(7)) // 6 * 7
val cond1 = BinOp(LessEq, expr1, C(50)) // expr1 ≤ 50
val expr2 = IfNonzero(cond1, C(10), C(20)) // if (cond1) 10 else 20

// To get their values, apply eval
eval(expr1) // returns  42
```

Our `eval` function will always return `BigInt` :

```
def eval(e: Expr): BigInt = e match
    case C(c) ⇒ c
    case BinOp(op, e1, e2) ⇒
        evalBinOp(op)(eval(e1), eval(e2))
    case IfNonzero(cond, trueE, falseE) ⇒
        if eval(cond) ≠ 0 then eval(trueE) else eval(falseE)
```

I2 : Absolute value and desugaring :

We want to add absolute value to our language:

```
case AbsValue(arg: Expr):
    ...
    case AbsValue(arg: Expr)
```

Notice that absolute value can already be written with elements already present in our language (if , comparison and minus) :

```
IfNonzero(  BinOp(LessEq, x, C(0))   ,  BinOp(Minus, C(0), x)   ,x)
```

so what we are doing is just adding **syntactic sugar** (facilitating syntax for already possible operations).

We can add a `case` in our `eval` function , or we can simply **desugar** `AbsValue` .

```
def desugar(e: Expr): Expr = e match
    case C(c) => e
    case BinOp(op, e1, e2) => BinOp(op, desugar(e1), desugar(e2))
    case IfNonzero(cond, trueE, falseE) =>
      IfNonzero(desugar(cond), desugar(trueE), desugar(falseE))
    case AbsValue(arg) =>
      val x = desugar(arg)
      IfNonzero(BinOp(LessEq, x, C(0))   ,  BinOp(Minus,C(0),x)   ,   x)
// the result will be an expression that uses I1 elements
```

This we will execute this function before using `eval` .

I3 : Recursive function definition :

we will add functions to our program :

```
enum Expr
    case C(c: BigInt)
    case BinOp(op: BinOps, e1: Expr, e2: Expr)
    case IfNonzero(cond: Expr, trueE: Expr, falseE: Expr)
    // New expressions N for name and Call

    case N(name: String) // immutable variable
    case Call(function: String, args: List[Expr]) // function is the function's
name

// mapping from list of param to body
case class Function(params: List[String], body: Expr)
type DefEnv = Map[String, Function] // function names to definitions
```

let's create an environment definition , that contains a function definition `square` :

```
val defs : DefEnv = Map[String, Function](
    "square" -> Function(List("n"), // formal parameter "n", body:
```

eval using substitution :

```scala
def eval(e: Expr): BigInt = e match
    case C(c) => c
    case N(n) => error(s"Unknown name '$n'") // should never occur
    case BinOp(op, e1, e2) =>
        evalBinOp(op)(eval(e1), eval(e2))
    case IfNonzero(cond, trueE, falseE) =>
        if eval(cond) != 0 then eval(trueE)
        else eval(falseE)

    // INTERISTING STUFF STARTS HERE
    case Call(fName, args) =>
        defs.get(fName) match // defs is a global map with all functions
        case Some(f) => // f ( body:Expr ,  params:List[String] )

            val evaledArgs = args.map((e: Expr) => C(eval(e)))
            // we need to substitute args in the body of f
            val bodySub = substAll(f.body, f.params, evaledArgs)

            eval(bodySub) // may contain further recursive calls

    // bodySub should no longer have N( ... )

def substAll(e: Expr, names: List[String], replacements: List[Expr]): Expr =
    // replacing multiple arguments
    (names, replacements) match
    case (n :: ns, r :: rs) => substAll(subst(e,n,r), ns, rs)
    case _ => e

def subst(e: Expr, n: String, r: Expr): Expr = e match
    // substiture names n in r

    case C(c) => e                    // nothing to substitute
    case N(s) => if s==n then r else e // substite
    case BinOp(op, e1, e2) => BinOp(op, subst(e1,n,r), subst(e2,n,r))

    case IfNonzero(c, trueE, falseE) =>
        IfNonzero(subst(c,n,r), subst(trueE,n,r), subst(falseE,n,r))

    case Call(f, args) =>
        Call(f, args.map(subst(_,n,r)))
```

eval using environment :

Instead of substituting , we will save results in `env` .

```scala
def evalE(e: Expr, env: Map[String, BigInt]): BigInt = e match
    ... // same as earlier
    case N(n) => env(n) // look up name in the environment
    case Call(fName, args) =>
        defs.get(fName) match        // get function name
            case Some(f) =>
            // evaluate args with current environment
            val evaledArgs = args.map( (e: Expr) => evalE(e,env) )

            // evalue body with env and definitions coming from args

            val newEnv = env ++ f.params.zip(evaledArgs)
            evalE(f.body, newEnv)
```

Higher order functions :

I4 : Naive substittution :

We assume our program will be written in *curried* form that is why all functions take one argument.

```scala
enum Expr
    ... // same as earlier
    case Call(fun: Expr, arg: Expr)          // fun can be expression itself
    case Fun(param: String, body: Expr)      // param => body

def eval(e: Expr): Expr = e match
    case C(c) => e
    case N(n) => eval(defs(n))               // find in global defs, then eval
    ...
    case Fun(_,_) => e                       // functions evaluate to themselves
    case Call(fun, arg) =>
        eval(fun) match
        case Fun(n,body) => eval(subst(body, n, eval(arg)))


def subst(e: Expr, n: String, r: Expr): Expr = e match
    ... // like earlier
    case Call(f, arg) =>
        Call(subst(f,n,r), subst(arg,n,r)) // different here is we substitute in
function
    case Fun(formal,body) =>
        if formal==n then e // do not substitute under (n =>  ... )
                            // because n in body is not n outside
        else Fun(formal, subst(body,n,r)) // otherwise substitute in body
```

there is a problem here : **variable capture**:

We have the *function* `fact` .

FUN : `f  => ( fact => f ( f fact )  )` (takes f and returns fof)

ARG : `fact` (the actual factorial function `(n => (if n then (* n (fact (- n 1))) else 1))`)

We expect the function $(factorial)o(factorial)$ as return value.

```
// CODE EXECUTION
Call( f  ⇒ ( fact ⇒ f ( f fact ) ) , fact )
//  | n |  |        body           | | arg |
//case Fun(n,body) ⇒ eval(subst(body, n, eval(arg)))
eval(  subst ( fact ⇒ f ( f fact ),f,fact)  )

subst ( fact ⇒ f ( f fact ) ) , f , fact)

subst( f ( f fact ) , f , fact )
// this will provoque f ( f fact ) ) to be replaced with
// fact ( fact fact ) where fact is THE FUNCTION not the formal argument
// so the result of substitution is lse 1))
// fact ⇒ fact ( fact fact )
// when we call it with arg = 3
// 3 ⇒ 3 ( 3 3 )
//          ^
//          | function evaluation
java.lang.Exception: Cannot apply non-function 3 in a call
```

Expressing this problem in the general case :

```
case Fun(formal,body) ⇒
    if formal==n then e // do not substitute under (n ⇒  ...)
    else Fun(formal, subst(body,n,r))
```

The last line exhibits **variable capture problem.** If `formal` occurs free in `r`, then it will be captured by `Fun(formal, ...)` even though that outside occurrence of `formal` in `r` has nothing to do with the bound variable in the anonymous function.

I5 : Renaming Bound Variables in Substitution :

```
// old substitution
def subst0(e: Expr, n: String, r: Expr): Expr = e match  ...
    case Fun(formal,body) ⇒
        if formal==n then e else Fun(formal, subst0(body,n,r))

// To avoid problems, we use capture-avoiding substitution:

def subst(e: Expr, n: String, r: Expr): Expr = e match  ...
    case Fun(formal,body) ⇒
        if formal==n then e else
        val fvs = freeVars(r)
        val (formal1, body1) =          // if free in r
            if fvs.contains(formal) then // rename formal
                val formal1 = differentName(formal, fvs)
                (formal1, subst0(body, formal, N(formal1)))
            else (formal, body)
        Fun(formal1, subst(body1,n,r)) // substitute either way
```

What is `freeVars` ?

```scala
def freeVars(e: Expr): Set[String] = e match
    case C(c) => Set()
    case N(s) => Set(s)
    case BinOp(op, e1, e2) => freeVars(e1) ++ freeVars(e2)
    case IfNonzero(cond, trueE, falseE) =>
        freeVars(cond) ++ freeVars(trueE) ++ freeVars(falseE)
    case Call(f, arg) => freeVars(f) ++ freeVars(arg)
    case Fun(formal,body) => freeVars(body) - formal
```

High order with environment :

we will know consider functions as variables that can be returned by `eval`

```scala
enum Value
    case I(i: BigInt)
    case F(f: Value => Value)
type Env = Map[String, Value]

def evalEnv(e: Expr, env: Map[String, Value]): Value = e match
    ... // not very interesting
    case Fun(n,body) => Value.F{(v: Value) =>
          evalEnv(body, env + (n -> v)) }
    case Call(fun, arg) => evalEnv(fun,env) match
          case Value.F(f) => f(evalEnv(arg,env))
          case _ => error(s"Non-function arg $fun in $fun($arg)")
```

I07: Nested recursive definitions using environments:

So far , we could create locally anonymous functions, but without a way to call them recursively.

```scala
def evalEnv(e: Expr, env: Env): Value = e match  ...
    case N(n) => env(n) match // no use of defs, only env
    case Some(v) => v
    case Fun(n,body) => Value.F{(v: Value) => // same as before
        val env1: String => Option[Value] =
        (s:String) => if s==n then Some(v) else env(s)
        evalEnv(body, env1) }
    case Call(fun, arg) => evalEnv(fun,env) match // same
    case Value.F(f) => f(evalEnv(arg,env))
    case Defs(defs, rest) => //
        def env1: Env = // extended environment
        (s:String) =>
        lookup(defs, s) match // list lookup in local defs
    case None => env(s) // fall back to outer scope
    case Some(body) => Some(evalEnv(body, env)) // nonrec
evalEnv(rest, env1)
```

explanation coming soon .

Week 12 :

Church numerals :

Church numerals are alternative ways of expressing natural numbers , using functions :

```
def zero   = (f => x => x)
def one    = (f => x => f x)
def two    = (f => x => f (f x))
def three  = (f => x => f (f (f x)))
```

Number `n` is a function that takes a function `f` and applies it `f` times to `x` .

Number is n expressed by : the function that maps f to it's n-fold composition.

$$f^{\circ n} : f \mapsto \underbrace{f \circ f \ldots \circ f}_{\text{n times}}$$

Addition

```
def addition = (m => n => f => x => m f (n f x))        // n + m in church numerals
```

Notice that `n` and `m` are *church numerals*. let's apply it

```
addition (two) (three) = f => ( x => two f ( three f x ) )
f => x => ((f => (x => (f (f x)))) f) ((f => x => (f (f (f x)))) f x)
//        |          two              |     |         three        |
// let's apply it to F we expect to get x => F F F F F x
x => ((f => (x => (f (f x)))) F) ((f => x => (f (f (f x)))) F x) // <=>
(x => (F (F x))) (F (F (F X)))
(x => F ( F ( F ( F ( F x ))))  // yay
```

How to check if a church numeral is non zero ?

Given a numeral `n` , like one for two: `f => x => f (f x)` How can we apply it to some expressions to get the effect of : `ifNonzero n then eTrue else eFalse`

as `n : f => x => ..` is a function , we will pass to it `f` and `x`

```
(n (arg => _ => eTrue) (_ => eFalse)) d
// suppose n = zero = f => x => x
~>(f => x => x) (arg => _ => eTrue) (_ => eFalse) d
~>( x => x )(_=>eFalse) d
~>(_=>eFalse)d
~>eFalse

// Suppose n is one, f => x => f x. Then:
(f => x => f x) (arg => _ => eTrue) (_ => eFalse) d
~> (arg => _ => eTrue) (_ => eFalse) d
~> eTrue
```