Modeling Kepler's Laws of Planetary Motion

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1 Introduction

Kepler's laws of planetary motion are observational laws that *Johannes Kepler* had concluded from a huge amount of accurate data collected by *Tycho Brahe*. These laws have become the cornerstone of the modern astronomy science and were proven theoretically by *sir Isaac Newton* later.[1]
Kepler's First Law states that each planet moves around the sun in an elliptical orbit with the sun at one focus of this ellipse such that the sum of the distance from the planet to the sun (Pf_1) and the distance from the planet to the other focus of the ellipse (Pf_2) is always constant[2], as follows:

$$Pf_1 + Pf_2 = constant \tag{1}$$

textbfKepler's Second Law states that each planet orbiting the sun in an elliptical orbit must sweep equal areas (dA) in equal times (dt) during its rotation around the sun, which means that each planet's **Areal Velocity** ($\frac{dA}{dt}$) must remain constant during its rotation around the sun[2], as follows:

$$\frac{dA}{dt} = constant \tag{2}$$

Kepler's Third Law states that the square of each planet's **Period** (T^2), the time it takes the planet to complete a whole cycle around the sun, is directly proportional to the cube of the orbit's **semi-major axis** (a^3)[2], as the following equation shows:

$$T^2 = \frac{4\pi^2 a^3}{GM} \tag{3}$$

Where G is the gravitational constant and M is the mass of the sun.
The three law of Kepler are shown in fig1.
The aim of this project is to model these three laws using a python program, as shown in the coming sections.

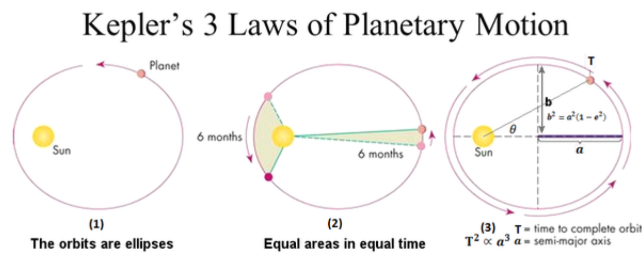

Figure 1

2 Methodology

In this **Python Project**,*first*: considering the modeling of the first and third law, the user is asked to enter the period (T) and the eccentricity (C) of each planet, and consequently the program is expected to display the orbit of the planet around the sun as a plot in the x-y plane with the sun at the origin.*Second*: considering the modeling of the second law, the user is asked to enter the previous data, in addition to any two equal time intervals that belongs to the planet's period, and consequently the program is expected to display the orbit of the planet around the sun in the x-y plane with the sun at the origin, with the two areas swept by the planet in these two equal intervals of time, in addition to displaying the value of each area, which verifies Kepler's second law.[3]

2.1 Kepler's First and Third Law Modeling

First, the program divides the period (T) of the planets evenly into a huge number of equal time sub-intervals. *then*, substituting with the period (T) and setting the initial time (τ) to be zero into the following equation, will give the values of each **Mean Anomaly** (M) corresponding to each value of the partitioned times(t).

$$M = \frac{2\pi}{T}(t - \tau) \tag{4}$$

Second, the program takes the values of the mean anomaly and employ them in the following **Kepler's Equation** to obtain the values of the eccentric anomaly corresponding to each value of the mean anomaly.

$$M = E - e \sin E \tag{5}$$

To do so, the program implement a numerical algorithm based on the **Fixed Point Theory** to solve Kepler's equation and get values of the eccentric anomaly. *Third*, the program takes the values of the eccentric anomaly and implement them into the following equation to obtain values of the **True Anomaly**.

$$\theta = 2\arctan[(\frac{1+e}{1-e})^{0.5} tan(\frac{E}{2})] \tag{6}$$

Then, substituting with the values of the true anomaly (θ) and eccentricity (e) and the semi-major axis (a) will give the corresponding value of the vector(r) from the sun to the point on the orbit, that the planet should exist at.

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \tag{7}$$

Finally, analyzing the position vector of the planet (r) into two components in the x and y direction by the following equations and plotting these x and y coordinates will produce the orbit of the planet around the sun with the sun at the origin.[3]

$$x = r\cos\theta, y = r\sin\theta \tag{8}$$

2.2 Kepler's Second Law Modeling

In Kepler's second law modeling, the part of the modeling corresponding to plotting the orbit of the planet is as shown in the section 2.1. Concerning the new part of modeling areas swept by planets: *first*, the program takes the initial and final times of two separate equal intervals($t1, t2, t3, t4$, and proceeds to obtain their mean, eccentric and true anomaly by the same procedure explained in section 2.1. *Second*, the program produces four position vectors(the two of each interval are in the same color) that corresponds to the position of the planets at these chosen times.[3]
Finally, to calculate the area swept in each time interval, the following equation calculates the area of a sector of an ellipse(s) swept from a focus and from the positive x-axis:

$$s = \frac{1}{2}abM \tag{9}$$

So, to verify Kepler's second law, the area swept in the first interval ($s_2 - s_1$) should equal the area swept in the second interval ($s_3 - s_4$).
To obtain the parameters of equation 9, the magnitude of the position vector from the sun to the perihelion is calculated by the following equation substituting the true anomaly by zero.

$$r(\theta) = \frac{1-e^2}{1+e\cos\theta} \tag{10}$$

then the semi-major axis (a) can be obtained by the following equation:

$$a = \frac{r_p}{1-e} \tag{11}$$

then the semi-minor axis (b) can be obtained by the following equation:

$$b = \sqrt{a^2(1 - e^2)} \tag{12}$$

3 Simulation of Results

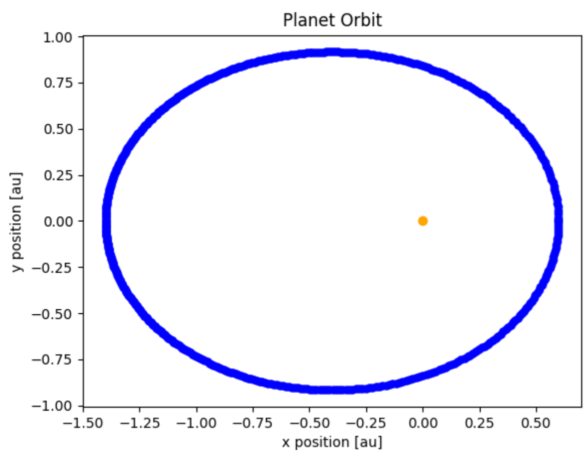

Figure 2: Simulation of first and third laws of Kepler

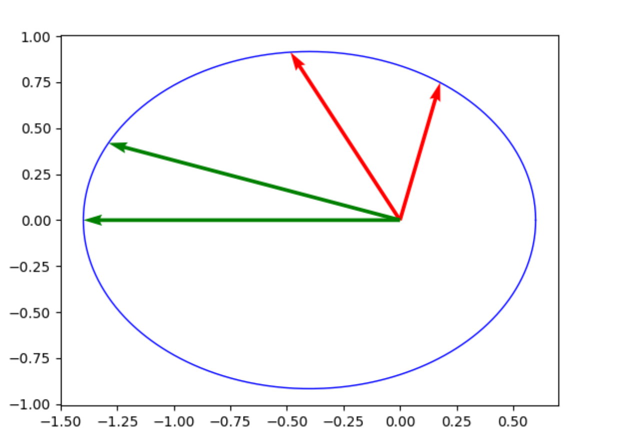

Figure 3: Simulation of second law of Kepler

4 Reference

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