Modeling Pulsars as Rotating Neutron Stars

A thesis to be submitted by

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ID. No. 201900300

for the award of the degree of

Bachelor of Science
in
Physics/Astrophysics

Under the supervision of

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Zewail City University of Science and Technology
Zewail City, 6 October, Egypt
June 2023

Acknowledgement

The value of this undergraduate journey would not be achieved without the faithful efforts of professors like **Dr. Hisham Anwar** and **Dr. Ali Nassar**, who dedicated a great deal of their time to flame the love and passion towards science inside their students. Thank you from the bottom of the heart.

The completion of this stressful academic stage would not be possible without the companion of colleagues that were a true academic help for me like **Tarek Mohamed** and **Salma Mousa**. I am grateful for your unconditioned help.

Last but not least, I want to thank my parents, especially my father **Dr. Waleed Gebreel**, for believing in me and for their continuous love and support.

Contents

1 Abstract

When discovered by *Hewish* in 1968, pulsars were one of the most interesting mysteries to astronomers. Those periodic x- ray pulsations that has a period that ranges from 0.033 to 3.7*secs* were not understood and their sources were unknown. Among many systems that can produce pulses of radiation, it turns out the rotating neutron stars are the most plausible models that can account for this specific range of periods that is characteristic to these pulsars. In this thesis, the model of *Ostriker and Gunn* of pulsars as a rotating neutron stars will be adopted. It could predict the age of the *Crab Pulsar* within a 25% accuracy and derived its surface magnetic field within a factor of 2 of its observed value, but some basic features and characteristics of neutron stars will be expanded in detail first in the sake of a sufficient understanding of the model.

2 Introduction

During their observation of the flickering coming from quasars and passing through solar wind in 7967, *Bell and Hewish* noticed some scruff that appears repeatedly on the roll of their strip chart recorder as shown in fig 1. At first, they thought it might be a signal from aliens, but when they direct their telescope to other orientations in the sky, so they excluded the source of these signals to be from aliens and attributed this x-ray periodic pulses to some unknown sources which they called **Pulsars**.(HEWISH et al., 1968)

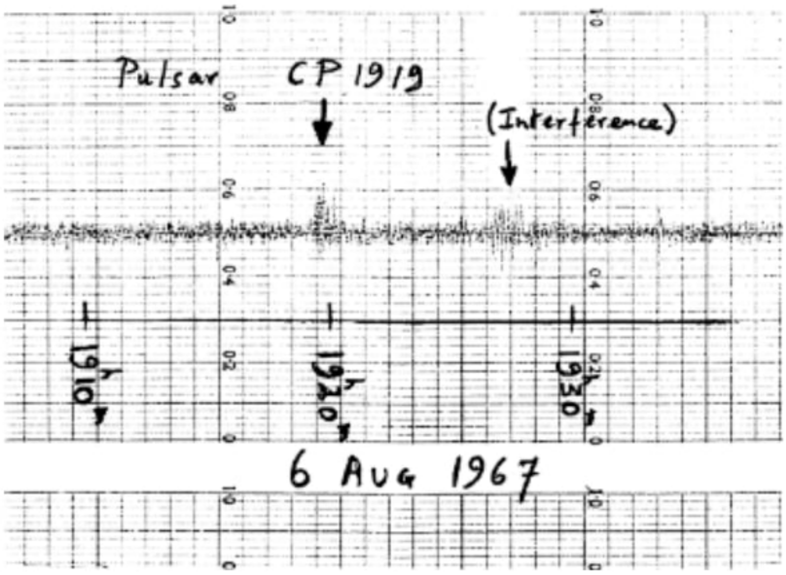

Figure 1: First Detection of pulsars(Lyne & Graham-Smith, 2012)

The main two characteristics of the pulsar radiation are their short periods $\mathbf{P}$, which falls in the range $0.033 \leq \mathbf{P} \leq 3.7$ sec, and the increase of them over time $0.0 \leq \frac{d\mathbf{P}}{dt} \leq 4.3 \times 10^{-13} sec$ with an approximate life time($\dot{\mathbf{P}}/\mathbf{P}$) of 10^7 yrs.(Ostriker & Gunn, 1969)
In order to account for the source of pulsars, there are only three different mechanism in astronomy that can produce periodic pulses.
The first one is the *pulsating stars*. Seeking equilibrium, stars' outer layers expand and contract in a regular way that's called **Stellar Pulsation**. It's known from the period - density relation that the period of star's pulsation is inversely proportional to the square root of the density as follows,

$$\mathbf{P} = \sqrt{\frac{3\pi}{2\gamma G\rho}} \tag{1}$$

where $\mathbf{P}$ is the period, γ is the adiabatic index, G is the gravitational constant and ρ is the star's density. So as the density of the object increases its period decreases. Since we seek short periods, we should choose very dense objects. When checking the periods of white dwarfs, we find that their periods range from 100 to 1000 seconds (Carroll & Ostlie, 2021), and since the density of the more dense objects, neutron stars, is about 10^8 times greater, then from the period - density relation, their periods will be 10^4 times smaller than white dwarfs. Therefore the periods of neutron star will range from 0.01 to 0.1 second, which are also out of the observed range of pulsars (0.033-3.7 sec).
The second possible sources of regular pulsations in astronomy are the *binary systems*. Turning out the argument, If we plug in the average observed period of pulsars ($\mathbf{P} = 0.79$ sec) into Kepler's third law:

$$a = (\frac{GMT^2}{4\pi^2})^{\frac{1}{3}} \tag{2}$$

with a system of two white dwarf stars of mass $1M_{\odot}$, the mass of the B sirius white dwarf, we get separation distance $a = 1.6 \times 10^6 m$, which is less than the observed separation value of the sirius B star by about 4 factors. Let us take a farther step and consider the periods of a binary of neutron stars. It turns out that they will be comparable to the desired range of periods, however according to Einstein theory of general relativity, this binary system will radiate gravitational waves over time and thus the two stars will come closer, so their separation distance a will decrease and hence their periods will decrease over time according to Kepler's third law, the consequence that contradicts with the observation, since the observed periods of pulsars are increasing over time. Consequently this model of binary system can not be the source of pulsars.(Carroll & Ostlie, 2021)
The third possible source of pulsars is the *rotating neutron stars*. The next section will be dedicated to explain the features of neutron stars and at the last of the section I will prove that the rotating neutron stars are good candidate for being pulsars, then its following section to the famous model of *Ostriker and Gunn* of pulsars as rotating neutron stars.

3 Basic Characteristics of Neutron Stars

After the postulation of the possible existence of neutrons by *James Chadwick* (Chadwick, 1932) that was confirmed thereafter, *Walter Baade and Fritz Zwicky* hypothesized the existence of neutron stars as supernova remainants.(Baade & Zwicky, 1934). In order to understand the adopted model of pulsars as rotating neutron stars, basic feature and characteristics oof neutron stars will be expanded in this section first in detail.

3.1 Mass -Radius Relation

From the study of statistical mechanics, the pressure of a non-relativistic degenerate matter is given by,

$$P = \frac{\pi^3 \hbar^2}{15m} (\frac{3N}{\pi V})^{\frac{5}{3}} \tag{3}$$

Where m is the mass of the degenerate fermions (i. e. electrons, neutrons), N is the number of degenerate particles and V is the volume containing these particles.
The energy of a self gravitating star of radius R and mass M can be derived as follows,

$$\begin{aligned} E &= \int_0^R -G\frac{m(r)}{r}\, dm \\ &= \int_0^R -G\frac{(\frac{4}{3}\rho\pi r^3)(4\pi r^2)}{r}\, dr \\ &= -\frac{3}{5}\frac{GM^2}{R} = \frac{-3}{5}GM^2(\frac{4\pi}{3})^{\frac{1}{3}}V^{-\frac{1}{3}} \end{aligned} \tag{4}$$

Where $m(r)$ is the mass of the star up to radius r and ρ is the density of the star and it is a constant quantity.
Differentiating equation 4 with respect to volume, we get the pressure due to gravity of a star P_g,

$$P_g = -\frac{1}{5}GM^2(\frac{4\pi}{3})V^{-\frac{4}{3}} \tag{5}$$

For hydrostatic equilibrium to be achieved, the degenerate pressure of the star must equal its pressure due to gravity. Therefore, setting $P = P_g$ for a neutron star gives the following relation between mass and radius of the neutron star, which is called the **Mass - Radius Relation**

$$\boxed{R_n = \frac{(18\pi)^{\frac{2}{3}}}{10}\frac{\hbar^2}{GM_n^{\frac{1}{3}}}(\frac{1}{m_H})^{\frac{8}{3}}} \tag{6}$$

Where the subscript n indicates a neutron star quantity and we take the mass of the neutron to be approximately equal to the mass of the hydrogen atom $m_n \approx m_H$.

3.2 Rapid Rotation

Conservation of angular momentum imposes the rotation of the neutron star to be very rapid. Assuming that the progenitor of the neutron star is an iron - rich core white dwarf, an assumption that's consistent with observations, then the radius of this progenitor white dwarf can be derived by the same procedure adopted in section 3.1 and it turns out to be,

$$R_w = \frac{(18\pi)^{\frac{2}{3}}}{10} \frac{\hbar^2}{G m_e M_w^{\frac{1}{3}}} [(\frac{Z}{A}) \frac{1}{m_H}]^{\frac{5}{3}} \tag{7}$$

Where the index w represents white dwarfs, the index e represents electrons, since white dwarfs are electron degenerate stars, and where,

$$m_e \approx \frac{m_H A}{Z} \tag{8}$$

For A represents the mass number and Z represents the atomic number of the element that is approximately constitute the core of the progenitor white dwarf, which is taken to be Fe. Therefore, from equations (7) and (6), the ration between the progenitor white dwarf to the newly formed neutron star will be,

$$\frac{R_w}{R_n} \approx \frac{m_n}{m_e} (\frac{Z}{A})^{\frac{5}{3}} \tag{9}$$

Since the ratio of $\frac{Z}{A} = \frac{26}{56}$ for iron and the mass of neutron and electron are both very well known, hence the ratio of radii can be calculated to be,

$$\frac{R_w}{R_n} \approx 512 \tag{10}$$

Now, applying *conservation of angular momentum* to this system will give,

$$I_i \Omega_i = I_f \Omega_f \tag{11}$$

Where the index i represents the initial state, which is the progenitor white dwarf WD, and the index f represents the final state, which is the newly formed neutron star NS.

Treating both the WD and the NS as spheres, will give a moment of inertia of the form,

$$I = CMR^2 \tag{12}$$

Where C is a constant that determines the mass distribution in the sphere. Therefore substituting with eq (12) into eq (11) will yield,

$$C_i M_i R_i^2 \Omega_i = C_f M_f R_f^2 \Omega_f \tag{13}$$

Taking both the mass distribution and the total mass of both WD and NS to be the same ($C_i = C_f$, $M_i = M_f$), then eq (13) will yield,

$$\Omega_f = \Omega_i (\frac{R_i}{R_f})^2 \tag{14}$$

Substituting with eq (10) into eq (14) will give,

$$\boxed{\Omega_f = 2.62 \times 10^5 \Omega_i} \tag{15}$$

This result shows that neutron stars rotates with an angular velocity that's huge by a factor of $2.6 * 10^5$ relative to their progenitor, so their pulsation periods are expected to be very small.
Since the period $\mathbf{P}$ is the inverse of the angular momentum, then taking the inverse of eq (15) gives,

$$\mathbf{P}_n = 3.8 \times 10^{-6} \mathbf{P}_w \tag{16}$$

For the sake of estimation, substituting in eq (16) by the period of the white dwarf 40 Eridani B, which is 1350 s, will give a value of the period of its subsequent neutron star which will be abut 5 ms. <u>Therefore</u>, *rapidly rotating neutron stars represents good candidates for pulsars.* This model will be established from scratch in the section 3.

3.3 Strong Magnetic Field

Neutron stars are very compact dense objects. Their density reaches 6.65×10^{17} kg/m^3. Having this extreme density, neutron stars are almost seas of closely backed neutrons, consequently their core is highly conducting and hence the magnetic field lines freeze in, never being lost, in them. Therefore, *magnetic flux is conserved in neutron stars.* Applying conservation of magnetic flux on an iron - rich progenitor white dwarf that turns to a neutron star, ignoring the geometry of magnetic field lines, taking only the components that are perpendicular to the surface area of the star, in order to get only a rough estimation of the strength of the newly formed neutron star magnetic field will yield,

$$4\pi R_w^2 B_w = 4\pi R_n^2 B_n \tag{17}$$

Hence the magnetic field of the neutron star will be,

$$B_n = \left(\frac{R_w}{R_n}\right)^2 B_w \tag{18}$$

For getting an upper bound, the largest observed white dwarf magnetic field, which is about 5×10^4 T, can be used in addition to substituting with equation 10 for the ratio of white dwarf to neutron star radius we get,

$$B_n = 1.3 \times 10^{10} T$$

<u>Hence</u> *neutron stars are formed with extremely strong magnetic fields.*

4 Ostriker and Gunn Model of Pulsars as Rotating Neutron Stars

4.1 Theory of The Model

It is well established that any *non-symmetric* object that rotates in vacuum has a time - varying **gravitational, magnetic and electric fields**. This time variation in these different fields force the object to radiate energy and consequently lose some of its angular momentum through time evolution. Assuming that neutron stars are not symmetric objects, they have to be radiating **magnetic dipole radiation** and **gravitational quadruple radiation**. The energy losses due to these two kinds of radiation is given by,

$$\frac{dE_{md}}{dt} = -\frac{2}{3}\frac{m_\perp^2 \Omega^4}{c^3} \tag{19}$$

$$\frac{dE_{gq}}{dt} = -\frac{1}{45}\frac{G D_\perp^2 \Omega^4}{c^5} \tag{20}$$

Where $m_\perp$ and $D_\perp$ are the components of the magnetic dipole moment and the mass quadruple moment perpendicular to the axis of rotation.
Assuming that the pulsars are *slowly rotating neutron stars* with nearly *spherical equipotential surfaces*, their moment of inertia to be very close to those of non-rotating neutron stars and that the axis of rotation does not align with the magnetic moment axis, then we can say that the total energy loss emerge in the form of loss in the rotational kinetic energy of the pulsar as follows,

$$\frac{dE}{dt} = \frac{dT}{dt} = \frac{d}{dt}(\frac{1}{2}I\Omega^2) = \frac{dE_{md}}{dt} + \frac{dE_{gq}}{dt} = -\frac{2}{3}\frac{m_\perp^2 \Omega^4}{c^3} - \frac{1}{45}\frac{G D_\perp^2 \Omega^4}{c^5} \tag{21}$$

Defining the following set of variables to simplify the differential equation,

$$\tau_m \equiv \frac{3c^3 I}{2m_\perp^2 \Omega_0^2} \tag{22a}$$

$$\tau_g \equiv \frac{45c^3 I}{G D_\perp^2 \Omega_0^4} \tag{22b}$$

$$x \equiv \frac{\Omega_0}{\Omega} \tag{22c}$$

$$\eta \equiv \frac{\tau_g}{\tau_m} \tag{22d}$$

Where τ_m and τ_g have dimensions of time and represent physically the characteristic magnetic and gravitational time and Ω_0

Where τ_m and τ_g have dimensions of time and represent physically the characteristic magnetic and gravitational time and Ω_0 is taken to be the angular velocity of the neutron star at the present time.

Substituting with eqs 22 int eq 21 gives,

$$\frac{1}{2}I\Omega_0^2\frac{d}{dt}(x^{-2}) = -(\frac{I\Omega_0^2}{\tau_m x^4} + \frac{I\Omega_0^2}{\tau_g x^6})$$
$$x^4\frac{d}{dt}(x^{-2}) = -\frac{2}{\tau_m}(1 + \frac{1}{\eta x^2})$$

$$\boxed{\frac{d}{dt}(x^2) = \frac{2}{\tau_m}(1 + \frac{1}{\eta x^2})} \tag{23}$$

Integrating eq 23 over time, we get,

$$\frac{d(x^2)}{1 + \frac{1}{\eta x^2}} = \frac{2}{\tau_m}dt$$
$$\int \frac{\eta x^2}{\eta x^2 + 1}\, d(x^2) = \int \frac{2}{\tau_m}\, dt$$

$$\boxed{t = \frac{\tau_m}{2}\left[x^2 - \frac{1}{\eta}\ln\left(\eta x^2 + 1\right)\right]} \tag{24}$$

Taking t_0 to be the taken from the formation of the neutron star until the present moment (Physically the **Age of the neutron star**)at which ($\Omega_0 = \Omega$) and thus ($x = 1$), then t_0 will be given by,

$$\boxed{t_0 = \frac{\tau_m}{2}\left(1 - \frac{1}{\eta}\ln\left(1 + \eta\right)\right)} \tag{25}$$

Defining the characteristic scale time (τ_0) of the star to be,

$$\tau_0 = -\left(\frac{1}{\Omega}\frac{d\Omega}{dt}\right)_0^{-1} = +\left(\frac{1}{\mathbf{P}}\frac{d\mathbf{P}}{dt}\right)_0^{-1} \tag{26}$$

This a purely observable quantity. Getting back t eq 21 to express it in terms of Ω,

$$\frac{d}{dt}(\frac{1}{2}I\Omega^2) = -I\Omega_0^2\left[\frac{1}{\tau_m x^4} + \frac{1}{\tau_g x^6}\right]$$
$$\frac{1}{\Omega}\frac{d\Omega}{dt} = -\left[\frac{1}{\tau_m x^2} + \frac{1}{\tau_g x^6}\right]$$

Setting ($x = 1$) for ($\Omega_0 = \Omega$), then the characteristic time can be written as,

$$\tau_0 = -\left(\frac{1}{\Omega}\frac{d\Omega}{dt}\right)_0^{-1} = \left[\frac{1}{\tau_m} + \frac{1}{\tau_g}\right]^{-1} = \tau_m\left(1 + \frac{1}{\eta}\right)^{-1} \tag{27}$$

From this equation, $\left(\tau_m = \tau_0\left(1 + \frac{1}{\eta}\right)\right)$, therefor substituting with it into eq 25 will give,

$$\boxed{age = t_0 = \frac{\tau_0}{2}\left(1 + \frac{1}{\eta}\right)\left(1 - \frac{1}{\eta}\ln\left(1 + \eta\right)\right)} \tag{28}$$

Since in the absence of gravitational radiation $\frac{1}{\eta}$ approaches zero, then,

$$age = \frac{\tau_0}{2} \qquad \text{For } \frac{1}{\eta} = 0$$

For the Crab pulsar ($\tau = 2340$ yr). Therefore, the predicted age of the Crab pulsar by this model is 1170 yr. Since the observed age of the Crab pulsar is 916 yr, the predicted age of the model in the absence of gravitational radiation has an error of only 25 %. In order to eliminate this error, we should include the effect of gravitational radiation. Substituting ($\eta \approx 5$) will give the actual age of the Crab pulsar. This value of η implies an ellipticity of the neutron star of 3×10^{-4}.(Ostriker & Gunn, 1969)

4.2 Validation of The Model

Now, the most important result of this model is that periods of pulsars depend on the age of the pulsar at the first place much more its dependence on the strength of the the magnetic and gravitational fields strength. This is very evident in eq 28.

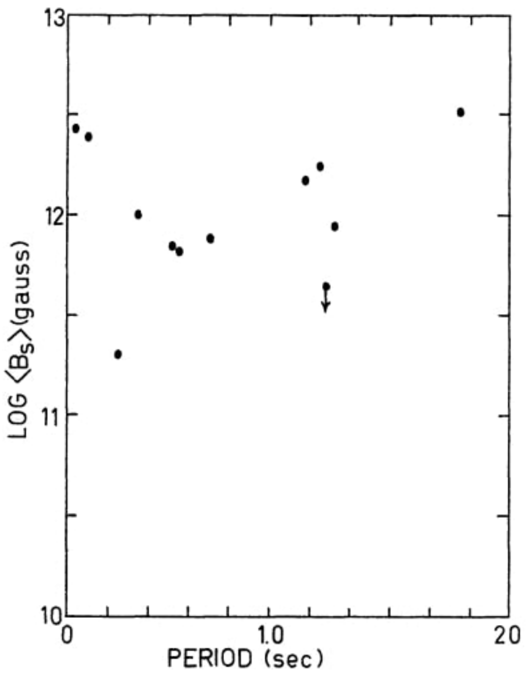

Figure 2: Mean surface magnetic field distribution of pulsars of different periods (Ostriker & Gunn, 1969)

In order to validate this result, the used argument will be turned around such that the characteristic time τ_0 of the pulsar will be used to calculate the mean magnetic surface field $< B >$ via the following equation,

$$< B >= \left(\frac{3c^3 I}{2\tau_m \Omega_0^2 a^6}\right)^{\frac{1}{2}} = 2.4 \times 10^{16} \tau_m^{-\frac{1}{2}} \Omega_0^{-1} \tag{29}$$

Which will become in the absence of gravitational radiation,

$$\boxed{< B >= 2.2 \times 10^{19} \left(\mathbf{P}\frac{d\mathbf{P}}{dt}\right)^{\frac{1}{2}}}$$

Substituting in this equation with short and long periods of different pulsars, our model expect that they must have the same distribution of surface magnetic fields. Applying this criteria using the data of eleven different pulsars (Maran & Cameron 1969) and the result came as already expected by the theory of the model as shown from the distribution of the surface magnetic fields of short and long period pulsars as shown in the figure 2.(Ostriker & Gunn, 1969)

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