Course Code: PEU 228
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Non-Orthogonal Curvilinear Coordinates

0 Introduction

This project is our attempt to set the stage for the usage of non-orthogonal curvilinear coordinates in any application. We aim to reach general formulas for the grad, div, and curl in $\mathbb{R}^3$ in terms of the entries of the metric matrix (discussed in the next section).

1 The Metric Matrix

To start, we will introduce the coordinate transformations as $q_1 = q_1(x, y, z)$, $q_2 = q_2(x, y, z)$, $q_3 = q_3(x, y, z)$. These coordinates are necessarily independent so that they may span the entire $\mathbb{R}^3$. Any infinitesimal displacement can be written as (using the multivariable chain rule):

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3 \tag{1}$$

We will define the vectors in the direction of change of the coordinate q_i as:

$$\vec{e}_i = \frac{\partial \vec{r}}{\partial q_i} \tag{2}$$

In the later sections this convetion might seem confusing as the scale factors are swallowed up in the $\vec{e}_i$. By that we mean $\vec{e}_i = h_i \hat{e}_i$.

The infinitesimal displacement then becomes:

$$d\vec{r} = \vec{e}_i dq_i \qquad \text{(Using summation convention)} \tag{3}$$

We will now define the metric matrix g_{ij} in index notation as:

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j \tag{4}$$

By this definition, the diagonals of the metric would be the square of the usual scale factors. The metric is symmetric due to the commutativity of the dot product.
We can now write the infinitesimal distance as (using summation convention):

$$dr = |d\vec{r}| = \sqrt{d\vec{r} \cdot d\vec{r}} = \sqrt{\vec{e}_i dq_i \cdot \vec{e}_j dq_j} \tag{5}$$

$$= \sqrt{\vec{e}_i \cdot \vec{e}_j dq_i dq_j} = \sqrt{g_{ij} dq_i dq_j} \tag{6}$$

The determinant of the metric would be:

$$|g| = g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23} \tag{7}$$

Throughout this formulation, we will assume that $|g|$ is non-zero $\implies$ g_{ij} is invertible. This assumption is logical as it just implies that there is a reverse transformation from the q coordinates to the cartesian coordinates.
Another useful quantity that we will need later on is the inverse metric $(g_{ij})^{-1}$ which we will denote with superscripted indices g^{ij}:

$$g^{ij} = \frac{1}{|g|}\mathrm{Adj}(g_{ij}) \tag{8}$$

$$g^{ij} = \frac{1}{|g|}\begin{bmatrix} \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} & \begin{vmatrix} g_{32} & g_{33} \\ g_{12} & g_{13} \end{vmatrix} & \begin{vmatrix} g_{12} & g_{13} \\ g_{22} & g_{23} \end{vmatrix} \\ \\ \begin{vmatrix} g_{31} & g_{33} \\ g_{21} & g_{23} \end{vmatrix} & \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} & \begin{vmatrix} g_{21} & g_{23} \\ g_{11} & g_{13} \end{vmatrix} \\ \\ \begin{vmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{vmatrix} & \begin{vmatrix} g_{31} & g_{32} \\ g_{11} & g_{12} \end{vmatrix} & \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \end{bmatrix} \tag{9}$$

$$g^{ij} = \frac{1}{|g|}\begin{bmatrix} g_{22}g_{33} - g_{23}^2 & g_{23}g_{13} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ \\ g_{13}g_{23} - g_{12}g_{33} & g_{11}g_{33} - g_{13}^2 & g_{12}g_{13} - g_{23}g_{11} \\ \\ g_{12}g_{23} - g_{13}g_{22} & g_{12}g_{13} - g_{23}g_{11} & g_{11}g_{22} - g_{12}^2 \end{bmatrix} \tag{10}$$

2 Gradient

We will define the gradient from the following relation:

$$df = \vec{\nabla} f \cdot d\vec{r} \tag{11}$$

We choose this equation as df is a scalar, meaning it is invariant under any transformation of coordinates.
We can write df using the multivariable chain rule as:

$$df = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \frac{\partial f}{\partial q_3}dq_3 \tag{12}$$

We will write the gradient with undetermined coefficients a_i:

$$\vec{\nabla} f = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 \tag{13}$$

We would then have df as:

$$df = \vec{\nabla} f \cdot d\vec{r} = (a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3) \cdot (dq_1\vec{e}_1 + dq_2\vec{e}_2 + dq_3\vec{e}_3) \tag{14}$$

$$= (a_i\vec{e}_i) \cdot (dq_j\vec{e}_j) \tag{15}$$

$$= a_i dq_j \vec{e}_i \cdot \vec{e}_j \tag{16}$$

$$= a_i dq_j g_{ij} \tag{17}$$

$$= a_i g_{ij} dq_j \tag{18}$$

As dq_i are independent, we will equate their coefficients individually. Giving us:

$$g_{ij}a_j = \frac{\partial f}{\partial q_i} \tag{19}$$

To get the coefficients, we multiply by the inverse of g_{ij}:

$$g^{ki}g_{ij}a_j = g^{ki}\frac{\partial f}{\partial q_i} \tag{20}$$

$$\delta_{kj}a_j = g^{ki}\frac{\partial f}{\partial q_i} \tag{21}$$

$$a_k = g^{ki}\frac{\partial f}{\partial q_i} \tag{22}$$

$$\vec{a} = \frac{1}{|g|}\begin{bmatrix} g_{22}g_{33} - g_{23}^2 & g_{23}g_{13} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ g_{13}g_{23} - g_{12}g_{33} & g_{11}g_{33} - g_{13}^2 & g_{12}g_{13} - g_{23}g_{11} \\ g_{12}g_{23} - g_{13}g_{22} & g_{12}g_{13} - g_{23}g_{11} & g_{11}g_{22} - g_{12}^2 \end{bmatrix}\begin{bmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial q_2} \\ \frac{\partial f}{\partial q_3} \end{bmatrix} \tag{23}$$

$$\vec{a} = \frac{1}{|g|}\begin{bmatrix} \frac{\partial f}{\partial q_1}(g_{22}g_{33} - g_{23}^2) + \frac{\partial f}{\partial q_2}(g_{23}g_{13} - g_{12}g_{33}) + \frac{\partial f}{\partial q_3}(g_{12}g_{23} - g_{13}g_{22}) \\ \frac{\partial f}{\partial q_1}(g_{13}g_{23} - g_{12}g_{33}) + \frac{\partial f}{\partial q_2}(g_{11}g_{33} - g_{13}^2) + \frac{\partial f}{\partial q_3}(g_{12}g_{13} - g_{23}g_{11}) \\ \frac{\partial f}{\partial q_1}(g_{12}g_{23} - g_{13}g_{22}) + \frac{\partial f}{\partial q_2}(g_{12}g_{13} - g_{23}g_{11}) + \frac{\partial f}{\partial q_3}(g_{11}g_{22} - g_{12}^2) \end{bmatrix} \tag{24}$$

$$\begin{aligned} \vec{\nabla} f = {} & \frac{\frac{\partial f}{\partial q_1}(g_{22}g_{33} - g_{23}^2) + \frac{\partial f}{\partial q_2}(g_{23}g_{13} - g_{12}g_{33}) + \frac{\partial f}{\partial q_3}(g_{12}g_{23} - g_{13}g_{22})}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}}\vec{e}_1 \\ & + \frac{\frac{\partial f}{\partial q_1}(g_{13}g_{23} - g_{12}g_{33}) + \frac{\partial f}{\partial q_2}(g_{11}g_{33} - g_{13}^2) + \frac{\partial f}{\partial q_3}(g_{12}g_{13} - g_{23}g_{11})}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}}\vec{e}_2 \\ & + \frac{\frac{\partial f}{\partial q_1}(g_{12}g_{23} - g_{13}g_{22}) + \frac{\partial f}{\partial q_2}(g_{12}g_{13} - g_{23}g_{11}) + \frac{\partial f}{\partial q_3}(g_{11}g_{22} - g_{12}^2)}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}}\vec{e}_3 \end{aligned} \tag{25}$$

For orthogonal coordinate systems, we have g_{ij}:

$$g_{ij} = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{bmatrix} \tag{26}$$

When substituting with this you get back the regular gradient in orthogonal coordinates:

$$\vec{\nabla} f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3 \tag{27}$$

$$\vec{\nabla} f = \frac{1}{h_1^2}\frac{\partial f}{\partial q_1}\vec{e}_1 + \frac{1}{h_2^2}\frac{\partial f}{\partial q_2}\vec{e}_2 + \frac{1}{h_3^2}\frac{\partial f}{\partial q_3}\vec{e}_3 \tag{28}$$

3 Divergence

We define the divergence as the flux over an infinitesimal volume.

$$\text{div } \vec{u} := \lim_{\Delta V \to 0} \frac{\oiint_S \vec{u} \cdot d\vec{A}}{\Delta V} \tag{29}$$

Where $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$
We will work on 2 surfaces of constant q_1 and q_2:
S_1: $q_1 = q_1$

The negative comes from the positive orientation being outward.

$$\vec{u} \cdot d\vec{A} = -(u_i\vec{e}_i) \cdot (\vec{e}_1 \times \vec{e}_2) dq_1 dq_2 \tag{30}$$

$$= -(u_3\vec{e}_3) \cdot (\vec{e}_1 \times \vec{e}_2) dq_1 dq_2 \tag{31}$$

We define $e = (\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3$

$$= -u_3 e dq_1 dq_2 \tag{32}$$

S_2: $q_1 = q_1 + dq_1$

$$\vec{u} \cdot d\vec{A} = \left(u_3 + \frac{\partial u_3}{\partial q_3} dq_3 \right) \left(e + \frac{\partial e}{\partial q_3} dq_3 \right) dq_1 dq_2 \tag{33}$$

$$= u_3 e dq_1 dq_2 + e \frac{\partial u_3}{\partial q_3} dq_1 dq_2 dq_3 + u_3 \frac{\partial e}{\partial q_3} dq_1 dq_2 dq_3 + O(dq^2) \tag{34}$$

$$= u_3 e dq_1 dq_2 + \frac{\partial (u_3 e)}{\partial q_3} dq_1 dq_2 dq_3 \tag{35}$$

$S_1 + S_2$:

$$-u_3 e dq_1 dq_2 + u_3 e dq_1 dq_2 + \frac{\partial (u_3 e)}{\partial q_3} dq_1 dq_2 dq_3 \tag{36}$$

$$= \frac{\partial (u_3 e)}{\partial q_3} dq_1 dq_2 dq_3 \tag{37}$$

By symmetry, the integration over the other 4 surfaces gives:

$$\frac{\partial (u_1 e)}{\partial q_1} dq_1 dq_2 dq_3 + \frac{\partial (u_2 e)}{\partial q_2} dq_1 dq_2 dq_3 + \frac{\partial (u_3 e)}{\partial q_3} dq_1 dq_2 dq_3 \tag{38}$$

$$= \left(\frac{\partial (u_1 e)}{\partial q_1} + \frac{\partial (u_2 e)}{\partial q_2} + \frac{\partial (u_3 e)}{\partial q_3} \right) dq_1 dq_2 dq_3 \tag{39}$$

The divergence can then be expressed as:

$$\text{div } \vec{u} := \lim_{\Delta V \to 0} \frac{\oiint_S \vec{u} \cdot d\vec{A}}{\Delta V} \tag{40}$$

$$= \frac{\left(\frac{\partial (u_1 e)}{\partial q_1} + \frac{\partial (u_2 e)}{\partial q_2} + \frac{\partial (u_3 e)}{\partial q_3} \right) dq_1 dq_2 dq_3}{e dq_1 dq_2 dq_3} \tag{41}$$

$$= \frac{1}{e}\left(\frac{\partial(u_1 e)}{\partial q_1} + \frac{\partial(u_2 e)}{\partial q_2} + \frac{\partial(u_3 e)}{\partial q_3}\right) \tag{42}$$

For orthogonal coordinate systems, we have $e = h_1 h_2 h_3$, which gives:

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(u_1 h_1 h_2 h_3)}{\partial q_1} + \frac{\partial(u_2 h_1 h_2 h_3)}{\partial q_2} + \frac{\partial(u_3 h_1 h_2 h_3)}{\partial q_3}\right) \tag{43}$$

If we instead let $\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$, we have the relation $v_i = u_i h_i$. This gives us:

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(v_1 h_2 h_3)}{\partial q_1} + \frac{\partial(v_2 h_1 h_3)}{\partial q_2} + \frac{\partial(v_3 h_1 h_2)}{\partial q_3}\right) \tag{44}$$

Which is the standard formula we know.

4 Curl

The curl's magnitude in the direction $\hat{n}$ can be defined as the line integral over an infinitesimal area that is perpendicular to $\hat{n}$:

$$\hat{n} \cdot \text{curl } \vec{u} := \lim_{\Delta A \to 0} \frac{\oint \vec{u} \cdot d\vec{r}}{\Delta A} \tag{45}$$

We will start by taking the circulation along q_1 and q_2 to calculate the component of q_3 in the curl. We will divide up the circulation into 4 sections: C_1, C_2, C_3, and C_4.

C_1: $q_1 \to q_1 + \Delta q_1$. $q_2 \to q_2$.

$$\oint_{C_1} \vec{u} \cdot d\vec{r} \approx \vec{u} \cdot \vec{e}_1 \Delta q_1 \tag{46}$$

C_2: $q_1 + \Delta q_1 \to q_1 + \Delta q_1$. $q_2 \to q_2 + \Delta q_2$.

$$\oint_{C_2} \vec{u} \cdot d\vec{r} \approx \frac{\partial(\vec{u} \cdot \vec{e}_2 \Delta q_2)}{\partial q_1}\Delta q_1 + \vec{u} \cdot \vec{e}_2 \Delta q_2 = \frac{\partial(\vec{u} \cdot \vec{e}_2)}{\partial q_1}\Delta q_1 \Delta q_2 + \vec{u} \cdot \vec{e}_2 \Delta q_2 \tag{47}$$

C_3: $q_1 + \Delta q_1 \to q_1$. $q_2 + \Delta q_2 \to q_2 + \Delta q_2$.

$$\oint_{C_3} \vec{u} \cdot d\vec{r} \approx -\vec{u} \cdot \vec{e}_1 \Delta q_1 - \frac{\partial(\vec{u} \cdot \vec{e} \Delta q_1)}{\partial q_2}\Delta q_2 = -\vec{u} \cdot \vec{e}_1 \Delta q_1 - \frac{\partial(\vec{u} \cdot \vec{e})}{\partial q_2}\Delta q_1 \Delta q_2 \tag{48}$$

C_4: $q_1 \to q_1$. $q_2 + \Delta q_2 \to q_2$.

$$\oint_{C_4} \vec{u} \cdot d\vec{r} \approx -\vec{u} \cdot \vec{e}_2 \Delta q_2 \tag{49}$$

Summing all of them gives:

$$\oint_{C} \vec{u} \cdot d\vec{r} \approx \left(\frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1)\right)\Delta q_1 \Delta q_2 \tag{50}$$

Taking the limit in the definition makes the approximation exact.

$$\frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|} \cdot \text{curl } \vec{u} = \lim_{\Delta A \to 0} \frac{\left(\frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1)\right)\Delta q_1 \Delta q_2}{\Delta A} \tag{51}$$

Where the normal is taken to be $\vec{e}_1 \times \vec{e}_2$ as we are doing the circulation on a section where only q_1 and q_2 change.

$$\frac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|} \cdot \text{curl } \vec{u} = \frac{\left(\frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1)\right) dq_1 dq_2}{|\vec{e}_1 \times \vec{e}_2| dq_1 dq_2} \tag{52}$$

$$(\vec{e}_1 \times \vec{e}_2) \cdot \text{curl } \vec{u} = \frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1) \tag{53}$$

Since we are just getting the component in the direction $\vec{e}_3$, let us first write $\vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$. Let us also write curl $\vec{u} = c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$, where the c s are to be figured out.
We then have:

$$(\vec{e}_1 \times \vec{e}_2) \cdot (c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3) = \frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1) \tag{54}$$

$$c_3 e = \frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1) \tag{55}$$

Where $e = (\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3$.

$$c_3 = \frac{1}{e}\left(\frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1)\right) \tag{56}$$

By symmetry, the curl can finally be written as:

$$\vec{\nabla} \times \vec{u} = \frac{1}{e}\left(\frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_3) - \frac{\partial}{\partial q_3}(\vec{u} \cdot \vec{e}_2)\right)\vec{e}_1 + \frac{1}{e}\left(\frac{\partial}{\partial q_3}(\vec{u} \cdot \vec{e}_1) - \frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_3)\right)\vec{e}_2 \tag{57}$$

$$+\frac{1}{e}\left(\frac{\partial}{\partial q_1}(\vec{u} \cdot \vec{e}_2) - \frac{\partial}{\partial q_2}(\vec{u} \cdot \vec{e}_1)\right)\vec{e}_3$$

For orthogonal coordinate systems, we have $e = h_1h_2h_3$, which gives:

$$\vec{\nabla} \times \vec{u} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_2}(h_3^2u_3) - \frac{\partial}{\partial q_3}(h_2^2u_2)\right)\vec{e}_1 + \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_3}(h_1^2u_1) - \frac{\partial}{\partial q_1}(h_3^2u_3)\right)\vec{e}_2 \tag{58}$$

$$+\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}(h_2^2u_2) - \frac{\partial}{\partial q_2}(h_1^2u_1)\right)\vec{e}_3$$

If we instead let $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$, we have the relation $v_i = u_ih_i$. This gives us:

$$\vec{\nabla} \times \vec{v} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_2}(h_3u_3) - \frac{\partial}{\partial q_3}(h_2u_2)\right)\vec{e}_1 + \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_3}(h_1u_1) - \frac{\partial}{\partial q_1}(h_3u_3)\right)\vec{e}_2 \tag{59}$$

$$+\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}(h_2u_2) - \frac{\partial}{\partial q_2}(h_1u_1)\right)\vec{e}_3$$

$$= \frac{1}{h_2h_3}\left(\frac{\partial}{\partial q_2}(h_3u_3) - \frac{\partial}{\partial q_3}(h_2u_2)\right)\hat{e}_1 + \frac{1}{h_1h_3}\left(\frac{\partial}{\partial q_3}(h_1u_1) - \frac{\partial}{\partial q_1}(h_3u_3)\right)\hat{e}_2 \tag{60}$$

$$+\frac{1}{h_1h_2}\left(\frac{\partial}{\partial q_1}(h_2u_2) - \frac{\partial}{\partial q_2}(h_1u_1)\right)\hat{e}_3$$

Which is the correct formula for the curl in orthogonal systems.

5 Laplacian

The Laplacian of a scalar field is the divergence of its gradient. We can directly compute it from the previously derived formulas.

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) \tag{61}$$

$$= \vec{\nabla} \cdot (\frac{\frac{\partial f}{\partial q_1}(g_{22}g_{33} - g_{23}^2) + \frac{\partial f}{\partial q_2}(g_{23}g_{13} - g_{12}g_{33}) + \frac{\partial f}{\partial q_3}(g_{12}g_{23} - g_{13}g_{22})}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}} \vec{e}_1 \tag{62}$$

$$+ \frac{\frac{\partial f}{\partial q_1}(g_{13}g_{23} - g_{12}g_{33}) + \frac{\partial f}{\partial q_2}(g_{11}g_{33} - g_{13}^2) + \frac{\partial f}{\partial q_3}(g_{12}g_{13} - g_{23}g_{11})}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}} \vec{e}_2$$

$$+ \frac{\frac{\partial f}{\partial q_1}(g_{12}g_{23} - g_{13}g_{22}) + \frac{\partial f}{\partial q_2}(g_{12}g_{13} - g_{23}g_{11}) + \frac{\partial f}{\partial q_3}(g_{11}g_{22} - g_{12}^2)}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}} \vec{e}_3)$$

$$= \frac{1}{e}\frac{\partial}{\partial q_1}\left(e \frac{\frac{\partial f}{\partial q_1}(g_{22}g_{33} - g_{23}^2) + \frac{\partial f}{\partial q_2}(g_{23}g_{13} - g_{12}g_{33}) + \frac{\partial f}{\partial q_3}(g_{12}g_{23} - g_{13}g_{22})}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}} \right) \tag{63}$$

$$+ \frac{1}{e}\frac{\partial}{\partial q_2}\left(e \frac{\frac{\partial f}{\partial q_1}(g_{13}g_{23} - g_{12}g_{33}) + \frac{\partial f}{\partial q_2}(g_{11}g_{33} - g_{13}^2) + \frac{\partial f}{\partial q_3}(g_{12}g_{13} - g_{23}g_{11})}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}} \right)$$

$$+ \frac{1}{e}\frac{\partial}{\partial q_3}\left(e \frac{\frac{\partial f}{\partial q_1}(g_{12}g_{23} - g_{13}g_{22}) + \frac{\partial f}{\partial q_2}(g_{12}g_{13} - g_{23}g_{11}) + \frac{\partial f}{\partial q_3}(g_{11}g_{22} - g_{12}^2)}{g_{11}g_{22}g_{33} - (g_{11}g_{23}^2 + g_{22}g_{13}^2 + g_{33}g_{12}^2) + 2g_{12}g_{13}g_{23}} \right)$$