Lyapunov's Direct Method

A detailed analysis of Lyapunov's direct method for autonomous systems, and applying it to the Lotka-Volterra model

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1.1.1 Theorem

If for a system like 1.3 there exists a Lyapunov function for an isolated critical point $\vec{x}_0$, and $\dot{V}(\vec{x}) \leq 0$ except at $\dot{V}(\vec{x}_0) = 0$, then $\vec{x}_0$ is stable.

1.1.2 Proof

Since we need to show stability, we need to develop a scheme to find a δ (to define a region where we should choose our initial values for our solution) for every $\epsilon > 0$ to fulfill the conditions of stability 1.6. All our discussion will be on an arbitrary ϵ to avoid loss of generality.
First, as $V(\vec{x})$ is a continuous function, we can find a minimum of $V(\vec{x})$ on the set $\mathrm{C}_\epsilon(\vec{x}_0)$, which we call μ. Hence, $\forall \vec{x} \in \mathrm{C}_\epsilon(\vec{x}_0)$, $V(\vec{x}) \geq \mu$, where $\mu > 0$. As $V(\vec{x})$ is continuous and $V(\vec{x}_0) = 0$, there must exist $\delta > 0$ such that $\forall\ \vec{x} \in \mathrm{B}_\delta(\vec{x}_0)$, $V(\vec{x}) < \mu$. We will let the δ we just described be the one we were looking for, we are left to show that for any solution $\vec{\phi}(t)$ to our autonomous system that has $\vec{\phi}(t_0) \in \mathrm{B}_\delta(\vec{x}_0)$, it must never go outside $\mathrm{B}_\epsilon(\vec{x}_0)$. If the solution escapes $\mathrm{B}_\epsilon(\vec{x}_0)$, there must be some time t_1 where it touches the boundary $\mathrm{C}_\epsilon(\vec{x}_0)$. We will now show that the requirement that $\dot{V}(\vec{x}) \leq 0$ prevents that from occurring.

$$V(\vec{\phi}(t)) - V(\vec{\phi}(t_0)) = \int_{V(\vec{\phi}(t_0))}^{V(\vec{\phi}(t))} dV$$

By the gradient theorem, we have:

$$= \int_{\vec{\phi}(t_0)}^{\vec{\phi}(t)} \nabla V \cdot d\vec{x} = \int_{t_0}^{t} \nabla V \cdot \frac{d\vec{x}}{dt} dt$$

$$= \int_{t_0}^{t} \nabla V \cdot \vec{F}(\vec{x}) dt = \int_{t_0}^{t} \dot{V}(\vec{x}) dt \leq 0$$

Hence, we have:

$$V(\vec{\phi}(t)) \leq V(\vec{\phi}(t_0))$$

As we chose $\vec{\phi}(t_0) \in \mathrm{B}_\delta(\vec{x}_0)$ we must have $V(\vec{\phi}(t_0)) < \mu$, so $V(\vec{\phi}(t)) < \mu$. Since μ is the minimum value for all points on $\mathrm{C}_\epsilon(\vec{x}_0)$, $\phi(t)$ can never belong to $\mathrm{C}_\epsilon(\vec{x}_0)$ for any $t \in [t_0, \infty)$.
Hence, $\phi(t)$ will always stay in $\mathrm{B}_\epsilon(\vec{x}_0)$, and Lyapunov stability is satisfied. $\square$

1.2 Lyapunov's Stability Theorem (II)

1.2.1 Theorem

If for a system like 1.3 there exists a Lyapunov function for an isolated critical point $\vec{x}_0$, and $\dot{V}(\vec{x}) < 0$ except at $\dot{V}(\vec{x}_0) = 0$, then $\vec{x}_0$ is asymptotically stable.

1.2.2 Proof

By 2.1 we already know that $\vec{x}_0$ is stable. We are left to show that 1.7 is satisfied. We need to show that for every solution that starts inside some $\mathrm{B}_\eta(\vec{x}_0)$ for $\eta > 0$, $||\vec{\phi}(t) - \vec{x}_0|| \to 0$ as $t \to \infty$. We choose $\eta = \delta$ for the ϵ in the proof of 2.1. In summary, the proof we are going to do just says that since $V(\vec{\phi}(t))$ is strictly decreasing as long as we are not at the critical point, and $V(\vec{\phi}(t))$ is always positive except at the critical point, therefore the solution will have to always approach the critical point. Now we will write this proof a little more rigorously.
We will prove by contradiction that for any solution with $\vec{\phi}(t_0) \in \mathrm{B}_\eta(\vec{x}_0)$, the solution must approach the critical point. We will first assume that $\vec{\phi}(t)$ never touches the critical point and show that this assumption causes a contradiction. If $\vec{\phi}(t)$ never touches the critical point, there must exist $0 < \alpha < \delta$ such that the trajectory of $\vec{\phi}(t)$ never crosses into $\mathrm{B}_\alpha(\vec{x}_0)$, so it always stays in $\bar{\mathrm{A}}_{\alpha,\epsilon}(\vec{x}_0)$. As $\dot{V}(\vec{x})$ is continuous and $\dot{V}(\vec{x}) < 0$, we can find a maximum for $\dot{V}(\vec{x})$ in $\bar{\mathrm{A}}_{\alpha,\epsilon}(\vec{x}_0)$ which we will denote as μ ($\mu < 0$), we then have $\forall\ \vec{x} \in \bar{\mathrm{A}}_{\alpha,\epsilon}(\vec{x}_0)$, $\dot{V}(\vec{x}) \leq \gamma$. Using the previous result and the process we did in the previous proof we find:

$$\int_{t_0}^{t} \dot{V}(\vec{\phi}(t))dt \leq \int_{t_0}^{t} \gamma$$

$$V(\vec{\phi}(t)) - V(\vec{\phi}(t_0)) \leq \gamma(t - t_0)$$

$$V(\vec{\phi}(t)) \leq V(\vec{\phi}(t_0)) + \gamma(t - t_0)$$

Hence, we have that as $t \to \infty$, $V(\vec{\phi}(t)) \to -\infty$ (because $\gamma < 0$), which can't happen as $V(\vec{x})$ is positive definite ($V(\vec{x}) > 0$).
Therefore, our initial assumption that $\vec{\phi}(t)$ never approaches the critical point is wrong by contradiction, showing that our solution's trajectory must terminate at the critical point due to $\dot{V}(\vec{x})$ being negative definite. Hence, we have shown asymptotic stability. $\square$

1.3 Lyapunov's Instability Theorem

1.3.1 Theorem

If for a system like 1.3 there exists a continuous function with continuous partial derivatives (slightly varied from 1.8) $V(\vec{x})$ and $\vec{x}_0$ is an isolated critical point. If $\forall\ \epsilon > 0$, $\exists\ \vec{a} \in \mathrm{B}_\epsilon(\vec{x}_0)$ where $V(\vec{a}) > 0$ and $V(\vec{x}_0) = 0$. If also $\dot{V}(\vec{x}) > 0$ except at the origin where $\dot{V}(\vec{x}_0) = 0$, then $\vec{x}_0$ is unstable.

1.3.2 Proof

Since the condition of Lyapunov stability 1.6 requires that for a given $\epsilon > 0$, there must exist a $\delta > 0$ such that for every solution $\vec{\phi}(t)$ with initial conditions satisfying

$$||\vec{\phi}(t_0) - \vec{x}_0|| < \delta$$

must also satisfy

$$||\vec{\phi}(t) - \vec{x}_0|| < \epsilon$$

I will first give a rundown of the proof then we will show it rigorously. To show instability, we will show that for a given ϵ no matter how small you choose δ, there will always be a point in the neighborhood of $\mathrm{B}_\delta(\vec{x}_0)$ such that if you have your initial condition lying there, it must leave $\mathrm{B}_\epsilon(\vec{x}_0)$. That point will be the point that we have enforced its existence in the theorem at which $V(\vec{x}) > 0$. Since $V(\vec{x})$ is always increasing due to the condition that $\dot{V}(\vec{x}) > 0$, if $V(\vec{\phi}(t_0))$ starts out larger than 0, then it can never go to 0, and we know that $V(\vec{x}_0) = 0$. Hence, if the solution's $\dot{V}(\phi(\vec{t}))$ always stays larger than 0 as it never touches the critical point (and the critical point is isolated and we are only concerned with the region which only contains this critical point), then $V(\phi(\vec{t}))$ will keep increasing till it leaves our region of interest. Now we will show this rigorously.

For the region $\mathrm{B}_\epsilon(\vec{x}_0)$, since $V(\vec{x})$ is continuous, there must exist a bound for it M such that $\forall \vec{x} \in \mathrm{B}_\epsilon(\vec{x}_0)$, $|V(\vec{x})| \leq M$, $M > 0$ (M cannot be zero as there always exists a positive $V(\vec{x})$ in any neighborhood of $\vec{x}_0$). We will now take an arbitrarily small $\delta > 0$ and show that no matter how small we take it, we will reach the same conclusion. For a solution $\vec{\phi}(t)$, let its initial point $\vec{\phi}(t_0) \in \mathrm{B}_\delta(\vec{x}_0)$, we will choose $\vec{\phi}(t_0)$ such that $V(\vec{\phi}(t_0)) > 0$ which we have enforced on our $V(\vec{x})$ that such a point must exist. Since $\dot{V}(\vec{x}) > 0$, we must have:

$$\int_{t_0}^{t} \dot{V}(\vec{\phi}(t))dt > \int_{t_0}^{t} 0$$

$$V(\vec{\phi}(t)) - V(\vec{\phi}(t_0)) > 0$$

$$V(\vec{\phi}(t)) > V(\vec{\phi}(t_0))$$

Since $V(\vec{\phi}(t_0)) > 0$, and $V(\vec{x}_0) = 0$ and $V(\vec{x})$ is continuous, there must exist a region $\mathrm{B}_\lambda(\vec{x}_0)$ where for $\forall \vec{x} \in \mathrm{B}_\lambda(\vec{x}_0)$, $|V(\vec{x})| < V(\vec{\phi}(t_0))$. Hence, $V(\vec{\phi}(t))$ can never enter or exist in that region for $t \in [t_0, \infty)$. Now, we will add the assumption that $\vec{\phi}(t)$ stays in $\mathrm{B}_\epsilon(\vec{x}_0)$. If that assumption is correct, $V(\vec{\phi}(t)) \in \mathrm{A}_{\lambda,\epsilon}(\vec{x}_0)$. As $\dot{V}(\vec{x}) > 0$ except at $\dot{V}(\vec{0}) = 0$, and $\dot{V}(\vec{x})$ is continuous, we then must have a minimum bound for $\dot{V}(\vec{x})$ in $\mathrm{A}_{\lambda,\epsilon}(\vec{x}_0)$. Hence, we have that $\forall \vec{x} \in \mathrm{A}_{\lambda,\epsilon}(\vec{x}_0)$, $\dot{V}(\vec{x}) \geq \mu$, $\mu > 0$. We therefore have:

$$\int_{t_0}^{t} \dot{V}(\vec{\phi}(t))dt > \int_{t_0}^{t} \mu$$

$$V(\vec{\phi}(t)) - V(\vec{\phi}(t_0)) > \mu(t - t_0)$$

$$V(\vec{\phi}(t)) > V(\vec{\phi}(t_0)) + \mu(t - t_0)$$

At a sufficiently large t, we must have $V(\vec{\phi}(t)) > M$, contradicting our initial bound $|V(\vec{x})| \leq M$ in $\mathrm{B}_\epsilon(\vec{x}_0)$. Hence, we have reached a contradiction. As the only assumption we made is that $\vec{\phi}(t)$ stays inside $\mathrm{B}_\epsilon(\vec{x}_0)$, that assumption must be incorrect. We have shown that no matter the choice of $\delta > 0$, a solution will always diverge out of $\mathrm{B}_\epsilon(\vec{x}_0)$. Hence, $\vec{x}_0$ is unstable. $\square$

2 Applying the Method

2.1 Finding a suitable Lyapunov function

There is no definite way to choose a Lyapunov function. It is mostly found through trial and error or by looking for a conserved quantity. When a conserved quantity isn't found, a Lyapunov function to show stability is likely to be found in one of these forms:

$$V(x,y) = ax^2 + by^2 \tag{1}$$

$$V(x,y) = ax^4 + by^2 \tag{2}$$

$$V(x,y) = ax^2 + by^4 \tag{3}$$

$$V(x,y) = ax^4 + by^4 \tag{4}$$

The issue will be to determine the value of a and b such that the function becomes a Lyapunov function. We will seek to find values for a and b that make the value of $\dot{V} \leq 0$ to satisfy the conditions of stability.

The most systematic way of finding a Lyapunov function take the following steps though it is not guaranteed to yield a Lyapunov function

1. Check if the origin is an isolated critical point
2. Select one of the 4 equations above based on the values of x and y in your differential equation
3. Calculate $\dot{V}$ while leaving a and b as constants
4. Set $\dot{V} \leq 0$ and find a pair of a and b that satisfy the inequality
5. If such an a and b are found then the origin is a stable critical point

Example: We will show how to apply the method on the following autonomous coupled differential equations

$$\frac{dx}{dt} = -2y^3; \tag{5}$$

$$\frac{dy}{dt} = x - 3y^3 \tag{6}$$

1. The origin is an isolated critical point
2. Assume $V(x,y) = ax^2 + by^4$
3. $\dot{V} = \frac{\partial V}{\partial x}(-2y^3) + \frac{\partial V}{\partial y}(x - 3y^3)$
4. $\dot{V} = 4(b-a)xy^3 - 12by^6$
5. We easily see that if a=b=1 then $\dot{V} = -12by^6$ so $\dot{V} \leq 0$
6. Origin is a stable critical point

3 Lotka-Volterra model

The predator-prey model is a nonlinear ODE that solves for the prey and predators of a population. Its formulas are given by

$$x'(t) = x(a - by) \tag{7}$$

$$y'(t) = y(dx - c) \tag{8}$$

To determine the stability of such a system we seek to find a Lyapunov function. Had this been a mechanical system we could've used energy. We can however find another conserved quantity by getting rid of the time derivative in equations 1 and 2 since a conserved quantity will be independent of time.

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y(dx - c)}{x(a - by)}$$

$$\frac{dy}{dx} = \frac{y(dx - c)}{x(a - by)} = \left(\frac{y}{a - by}\right)\left(\frac{x}{dx - c}\right)$$

Using separation of variables

$$\frac{a - by}{y} dy = \frac{dx - c}{x} dx$$

$$\int \frac{a - by}{y} dy = \int \frac{dx - c}{x} dx + \phi$$

$$a \ln(y) + c \ln(x) - by - dx = \phi$$

Where ϕ is a constant of integration. We now have an equation in terms of x and y that is constant for any time making it an excellent candidate Lyapunov function.
Before proceeding we would like to study the function ϕ. It is a sum of 2 functions $f(x) + g(y)$. We can find the minimum of ϕ by finding the minimum of $f(x)$ and $g(y)$

$$f(x) = c \ln(x) - dx$$

$$f'(x) = \frac{c}{x} - d = 0$$

$$x_{\min} = \frac{c}{d}$$

We can see by symmetry that

$$y_{\min} = \frac{a}{b}$$

Since $f(x)$ and $g(y)$ have a minimum ϕ will also have a minimum at these values that we can calculate by plugging in $x_{\min}$ and $y_{\min}$.

$$\phi_{\min} = a\left(\ln\left(\frac{a}{b}\right) - 1\right) + c\left(\ln\left(\frac{c}{d}\right) - 1\right)$$

We now have a Lyapunov function with a minimum at x_{min} and y_{min}. Looking at all that we know about ϕ till now we see that:

$$\phi(x_{\min}, y_{\min}) = \phi_{\min}$$

$$\phi \geq \phi_{\min}$$

We now look at

$$\dot{\phi} = \frac{\partial \phi}{\partial x} x'(t) + \frac{\partial \phi}{\partial y} y'(t)$$

$$= \left(\frac{c}{x} - d\right)(x(a - by)) + \left(\frac{a}{y} - b\right)(y(dx - c)) = 0$$

Looking at our new results we see that $\dot{\phi} = 0$ and $\phi \geq \phi_{\min}$. To reach the conditions for Lyapunov stability we demand that $\phi_{\min} = 0$

$$a\left(\ln\left(\frac{a}{b}\right) - 1\right) + c\left(\ln\left(\frac{c}{d}\right) - 1\right) = 0 \tag{9}$$

We consider equation 3 a sufficient requirement for the stability of the system. This means that any predator-prey system with a, b, c, d that satisfy equation 3 is stable.

A Definitions and Conventions

A.1 Norm

The norm that we will be using is the regular Euclidean norm on the vector space $\mathbb{R}^n$. That is for $\vec{x} \in \mathbb{R}^n$:

$$||\vec{x}|| = \sqrt{\sum_{k=0}^{n} x_i^2} \tag{10}$$

A.2 Sets we use

A.2.1 $\mathbf{B}_\alpha(\vec{x}_0)$

Let $\mathrm{B}_\alpha(\vec{x}_0)$ denote the open set of points enclosed inside the hypersphere around $\vec{x}_0$ with radius α without the boundary.

$$\mathrm{B}_\alpha(\vec{x}_0) := \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{x}_0|| < \alpha\} \tag{11}$$

A.2.2 $\bar{\mathbf{B}}_\alpha(\vec{x}_0)$

Let $\bar{\mathrm{B}}_\alpha(\vec{x}_0)$ denote the closed set of points enclosed inside the hypersphere around $\vec{x}_0$ with radius α including the boundary.

$$\bar{\mathrm{B}}_\alpha(\vec{x}_0) := \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{x}_0|| \leq \alpha\} \tag{12}$$

A.2.3 $\mathbf{A}_{\alpha,\beta}(\vec{x}_0)$

Let $\mathrm{A}_{\alpha,\beta}(\vec{x}_0)$ denote the open set of points enclosed inside the hyper-spherical shells around $\vec{x}_0$ with inner radius α and outer radius β without the boundary.

$$\mathrm{A}_{\alpha,\beta}(\vec{x}_0) := \{\vec{x} \in \mathbb{R}^n : \alpha < ||\vec{x} - \vec{x}_0|| < \beta\} \tag{13}$$

A.2.4 $\bar{\mathbf{A}}_{\alpha,\beta}(\vec{x}_0)$

Let $\bar{\mathrm{A}}_{\alpha,\beta}(\vec{x}_0)$ denote the closed set of points enclosed inside the hyper-spherical shells around $\vec{x}_0$ with inner radius α and outer radius β including the boundary.

$$\bar{\mathrm{A}}_{\alpha,\beta}(\vec{x}_0) := \{\vec{x} \in \mathbb{R}^n : \alpha \leq ||\vec{x} - \vec{x}_0|| \leq \beta\} \tag{14}$$

A.2.5 $\mathbf{C}_\alpha(\vec{x}_0)$

Let $\mathrm{C}_\alpha(\vec{x}_0)$ denote the set of points on the hyperspherical shell around $\vec{x}_0$ with radius α.

$$\mathrm{C}_\alpha(\vec{x}_0) := \{\vec{x} \in \mathbb{R}^n : ||\vec{x} - \vec{x}_0|| = \alpha\} \tag{15}$$

A.3 First Order Autonomous System of Differential Equations

For the autonomous (rate of changes are time-invariant) system of differential equations:

$$\dot{\vec{x}} = \vec{F}(\vec{x}(t)) \tag{16}$$

Where $\vec{x} \in \mathbb{R}^n$ is a vector of functions $x_i(t) : \mathbb{R} \to \mathbb{R}$. $\vec{F}(\vec{x}(t)) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector of functions that only depend on the position in the $\vec{x}$-space and describe the rates of change of the xs.

We will work on the domain $\mathbb{U} \subset \mathbb{R}^n$ which contains an isolated critical point $\vec{x}_0$ and on which $\dot{\vec{x}}$ is continous so that by the existence and uniqueness theorem of differential equations, a unique solution $\vec{\phi}(t)$ exists. This implies that $\vec{F}(\vec{x})$ is also continuous.

A.4 Critical Point

A critical point $\vec{x}_0$ is the position in the $\vec{x}$-space where $\vec{F}(\vec{x}_0) = 0$. If the system has a solution $\vec{\phi}(t)$ for $[t_0, \infty)$, and $\vec{\phi}(t_0) = \vec{x}_0$ it will always stay there for any $t \in [t_0, \infty)$ $(\vec{\phi}(t) = \vec{x}_0)$.

A.5 Isolated Critical Point

An isolated critical point $\vec{x}_{ic}$ is a critical point for which $\exists\ \epsilon > 0$ such that there are no other critical points contained in $\mathrm{B}_\epsilon(\vec{x}_{ic})$. This just means that the critical point doesn't touch other critical points and that we can treat it on its own.

A.6 Lyapunov Stability

We define a critical point $\vec{x}_0$ to be Lyapunov stable if $\forall\ \epsilon > 0$, $\exists\ 0 < \delta < \epsilon$ such that for every $\vec{\phi}(t)$ that is a solution to the system with initial conditions $\vec{\phi}(t_0)$ such that:

$$||\vec{\phi}(t_0) - \vec{x}_0|| < \delta$$

We must also have

$$||\vec{\phi}(t) - \vec{x}_0|| < \epsilon$$

For $t \in [t_0, \infty)$.
Intuitively, this just means that if we want to have the solution to always be bounded in a hypersphere of radius ϵ, there is always an initial point within that hypersphere where we can start from and never leave the hypersphere of radius ϵ.
Furthermore, we define a point to be unstable if it does not fit the preceding criteria. Meaning, for some ϵ it would eventually leave $\mathrm{B}_\epsilon(\vec{x}_0)$ at some $t > t_0$ for any choice of δ.

A.7 Asymptotic Stability

A critical point is asymptotically stable if it is stable and $\exists\ \eta > 0$, such that every solution $\vec{\phi}(t)$ that satisfies:

$$||\vec{\phi}(t_0) - \vec{x}_0|| < \eta$$

also satisfies

$$||\vec{\phi}(t) - \vec{x}_0|| \to 0 \text{ as } t \to \infty$$

A.8 Lyapunov Function

Let $V(\vec{x}) : \mathbb{U} \to \mathbb{R}$, where $\mathbb{U} \subset \mathbb{R}^n$ and $\vec{x}_0 \in \mathbb{U}$ (an isolated critical point of our system 1.3), be a defined continuous scalar function of the position in the $\vec{x}$-space on its domain. Let the first partial derivatives of $V(\vec{x})$ be continuous. Let $V(\vec{x})$ also be a positive definite function, meaning $V(\vec{x}) > 0\ \forall\ \vec{x} \in \mathbb{U}$ except at $\vec{x}_0$ where $V(\vec{x}_0) = 0$. We call such a function a Lyapunov function. Further discussion on finding suitable Lyapunov functions will be presented later.

A.9 $\dot{V}(\vec{x})$

We define $\dot{V}(\vec{x})$ as $\nabla V \cdot \vec{F}(\vec{x})$. We call this a definition and not a direct property as it is motivated by $\frac{dV}{dt} = \nabla V \cdot \dot{\vec{x}}$, but $\vec{x}$ need not be parametrized by time in the definition of $V(\vec{x})$. $\dot{V}(\vec{x})$ is continuous as ∇V and $\vec{F}(\vec{x})$ are continuous as stated above, and the product of two continuous functions at a point is continuous for that point.