Arbitrages

Finance 2: Dynamic Portfolio Choice

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Overview

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Introduction and arbitrages

We consider a one-period binomial model with N assets, a finite number of future states S.

Definitions (Lando & Poulsen, 2023, p. 10):

- Price vector: $\pi \in \mathbb{R}^N$.
- Matrix of payoffs for each given asset and each possible future state: $D \in \mathbb{R}^{N \times S}$.
- Portfolio of assets: $\theta \in \mathbb{R}^N$, and where θ_i is the amount invested in asset i.

Define 4 arbitrages (not exhaustive: Platen & Tappe, 2023, p. 5) as a portfolio θ (with time-0 portfolio price $\pi^\top \theta \in \mathbb{R}$, time-$1$ payment stream $D^\top \theta \in \mathbb{R}^S$) that satisfies :

- **Strong arbitrage** ("Printing Money": type 2): $\pi^\top \theta < 0, \quad D^\top \theta \geq 0.$
- **Weak arbitrage** ("Free lottery tickets": type 1): $\pi^\top \theta = 0, \quad D^\top \theta > 0.$
- **Semi-Strong arbitrage** ("All the lottery tickets for free"): $\pi^\top \theta = 0, \quad D^\top \theta >> 0.$
- **Risk-free arbitrage**: $\pi^\top \theta \leq 0, \quad (D^\top \theta)_j = c > 0$

Cross-currency betting arbitrage

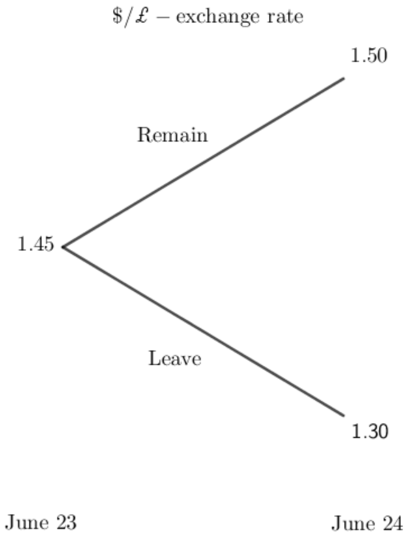
Bookmakers odds on Remain and Leave shortly before the British EU - referendum on June 23, 2016:

EU-Referendum Outcome	Remain	Leave
Decimal Odds	1.25	4.80

Betting $£\frac{1}{1.25}$ on Remain and $£\frac{1}{4.80}$ on Leave guarantees a payoff of $£1$, but at the cost of $£\frac{1}{1.25} + £\frac{1}{4.8} = £1.008\overline{3}$ i.e no free lunch here (cut to bookmaker, denoted as c).

Well, then what? Assume:

- The punter is allowed to choose which currency the bet is made in and is repaid in the same currency (GBP or USD).
- The choice of currency does not affect the betting-odds.
- Only long positions allowed (we are punters).
- That the exchange rate is affected by the outcome of the referendum in a binomial fashion (next page):



Assumption (more in Extra Slide 2 for estimation and what if): *"[...] perfect ex ante information on conditional (i.e., outcome-dependent) exchange rate reacttions after the respective events [...]"* - (Hanke et al., 2019, p. 2)

Example of Cross-Betting Arbitrage

Intuitively, to achieve "something extra":

- Bet Remain in £: Outcome = Remain ⇒ Strengthens £ relative to \$.
- Bet Leave in \$: Outcome = Leave ⇒ Strengthens \$ relative to £.

Looking for arbitrage (in £)		
23.jun	Borrow (£)	1,000
	£ bet on Remain (free)	0,811
	\$ bet on Leave (residual)	0,274
	Check: Total cost in £	1,000
24.jun	£ payoff if Remain	**1,0134**
	£ payoff if Leave	**1,0134**

Looking for arbitrage (in \$)		
23.jun	Borrow (\$)	1,000
	£ bet on Remain (residual)	0,543
	\$ bet on Leave (free)	0,212
	Check: Total cost in \$	1,000
24.jun	\$ payoff if Remain	**1,0187**
	\$ payoff if Leave	**1,0187**

Risk-free arbitrage is possible to achieve in both strategies!

When is a Free Lunch Possible?

Deterministic Conditional Returns: Arbitrage Opportunities (Hanke et al., 2019)

Let the bookmakers cut be given as: $c = \frac{1}{\text{odds}_L} + \frac{1}{\text{odds}_R} - 1$. A betting arbitrage can be constructed if and only if one (or both) of the two following conditions hold:

(A) $1 - c \cdot \text{odds}_L > d$

(B) $1 - c \cdot \text{odds}_R > \frac{1}{u}$

where u and d denotes the factor multiplied onto the initial value after an up (u) or down (d) move, respectively, in the one-period binomial model.

In our example we have (source: trust me) $c = 0.008\overline{3} \Rightarrow$:

(A) $1 - c \cdot \text{odds}_L = 1 - 0.008\overline{3} \cdot 4.80 = 0.96 > 0.90 = \frac{1.30}{1.45}$

(B) $1 - c \cdot \text{odds}_R = 1 - 0.008\overline{3} \cdot 1.25 = 0.99 > 0.97 = \frac{1}{\frac{1.50}{1.45}}$

Both conditions hold and two (risk-free) arbitrages are present!

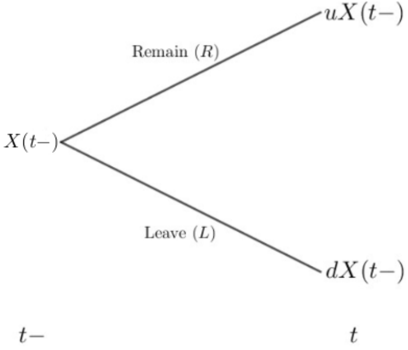
Proof of the bounds for arbitrage (1/3)

Idea: Track units (fysik i gymnasiet: "Regn i enheder, så kommer det ud i enheder") and book keep!

Assume the choice of currency does not affect the betting-odds:

$$\text{odds}_R^{\$} = \text{odds}_R^{£} =: \text{odds}_R \quad \text{and} \quad \text{odds}_L^{\$} = \text{odds}_L^{£} =: \text{odds}_L.$$

Let the $(\$/£)$-exchange rate be denoted as seen below ($t-$ is before the result, t just after):



Proof of the Bounds for Arbitrage (2/3)

(A)

- Borrow £ 1
- Bet £ $\frac{1}{\text{odds}_R}$ on Remain
- Bet the rest exchanged in dollars on Leave (remember: $X = \$/£\text{-ER}$):

$$\$\underbrace{\left(1 - \frac{1}{\text{odds}_R}\right)}_{\text{Rest in } £} X(t-) = \$\underbrace{\left(\frac{1}{\text{odds}_L} - c\right)}_{\text{As } c = \frac{1}{\text{odds}_L} + \frac{1}{\text{odds}_R} - 1} X(t-)$$

- Payoff if Remain:

$$£\underbrace{\frac{1}{\text{odds}_R}}_{\text{Bet in } £} \cdot \underbrace{\text{odds}_R}_{\text{Winning odds}} = £1.$$

- Payoff if Leave (using ex-post $\$/£$-ER $\Rightarrow$ Reciprocal yields $£/\$$-ER):

$$£\underbrace{\left(\frac{1}{\text{odds}_L} - c\right) X(t-)}_{\text{Bet in } \$} \cdot \underbrace{\text{odds}_L}_{\text{Winning odds}} \cdot \underbrace{\frac{1}{X(t-)d}}_{\text{Convert to } £} = £(1 - c \cdot \text{odds}_L) \cdot \frac{1}{d}.$$

- We thus achieve weak arbitrage if: $(1 - c \cdot \text{odds}_L) \cdot \frac{1}{d} > 1 \Longleftrightarrow 1 - c \cdot \text{odds}_L > d$, i.e (A).

Proof of the Bounds for Arbitrage (3/3)

(B)

- Borrow \$ 1
- Bet \$ $\frac{1}{\text{odds}_L}$ on Leave
- Bet the rest exchanged in pounds on Remain ($X = \$/£\text{-ER} \Rightarrow$ Reciprocal yields £/\$-ER):

$$£\underbrace{\left(1 - \frac{1}{\text{odds}_L}\right)}_{\text{Rest in }\$}\frac{1}{X(t-)} = £\underbrace{\left(\frac{1}{\text{odds}_R} - c\right)}_{\text{As } c = \frac{1}{\text{odds}_L} + \frac{1}{\text{odds}_R} - 1}\frac{1}{X(t-)}$$

- Payoff if Leave:

$$\$\underbrace{\frac{1}{\text{odds}_L}}_{\text{Bet in }\$} \cdot \underbrace{\text{odds}_L}_{\text{Winning odds}} = \$1.$$

- Payoff if Remain (using ex-post \$/£-ER):

$$\$\underbrace{\frac{\left(\frac{1}{\text{odds}_R} - c\right)}{X(t-)}}_{\text{Bet in }£} \cdot \underbrace{\text{odds}_R}_{\text{Winning odds}} \cdot \underbrace{uX(t-)}_{\text{Convert to }\$} = \$(1 - c \cdot \text{odds}_R) \cdot u$$

- We thus achieve weak arbitrage if: $(1 - c \cdot \text{odds}_R) \cdot u > 1 \iff 1 - c \cdot \text{odds}_R > \frac{1}{u}$, i.e (B).

References

Hanke, M., Poulsen, R., & Weissensteiner, A. (2019). Numeraire dependence in risk-neutral probabilities of event outcomes. *The Journal of Derivatives, 26*(4), 128–143.

Lando, D., & Poulsen, R. (2023). Finance 1 and beyond. *Department of Mathematical Sciences, University of Copenhagen*.

Platen, E., & Tappe, S. (2023). No arbitrage and multiplicative special semimartingales. *Advances in Applied Probability, 55*(3), 1033–1074.

Extra Slide 1: Working Example

Excel document for cross betting arbitrage on GitHub.

Extra Slide 2: Relaxing Assumption and Stochastic Returns

Relaxing the assumption still yields good opportunities:

Relaxing this assumption and acknowledging the risk associated with conditional exchange rate movements turns the arbitrage opportunities into good deals (Cochrane and Saa-Requejo 2002) or approximate arbitrage opportunities (i .e., strategies with a very favorable Sharpe ratio) - (Hanke et al., 2019, p. 9).

Estimate is from options:

[...] we analyze the results of the strategies discussed above for the assumption of stochastic outcome-dependent conditional returns $\tilde{u}$ and $\tilde{d}$ (as seen previously). The empirical data we will use to illustrate this case are estimated from one-month options. - (Hanke et al., 2019, p. 9).

Extra Slide 3: No Arbitrage-Intervals in Incomplete Markets

We consider again a financial market (π, C) that we assume to be arbitrage-free, but possibly (grossly) incomplete. Now introduce a new contract that has $x \in \mathbb{R}^T$ as its payment vector, and whose time-0 price we denote by π_x. In an incomplete model there is (typically) not a unique arbitrage-free price of this new claim, but we will now show that there is considerable and computable structure on the set of arbitrage-free π_x-values.

- If the x-claim can be replicated, then the upper and lower bounds collapse into a single point, which is then the unique arbitrage-free price; technically a closed set.
- If the x-claim cannot be replicated (which is the typical situation where we would apply this whole approach), then at optimum (for both optimization problems) at least one constraint inequality is sharp.
- If the x-claim cannot be replicated, then the x-extended model is arbitrage-free if and only if:

$$\pi_x \in]\pi_{*,L}, \pi_{*,U}[$$

- The good news is that the upper and lower bounds are eminently computable; there are excellent and wide-spread numerical methods for solving linear optimization problems. The bad news is that the arbitrage-free price interval is typically too wide to be of practical use.