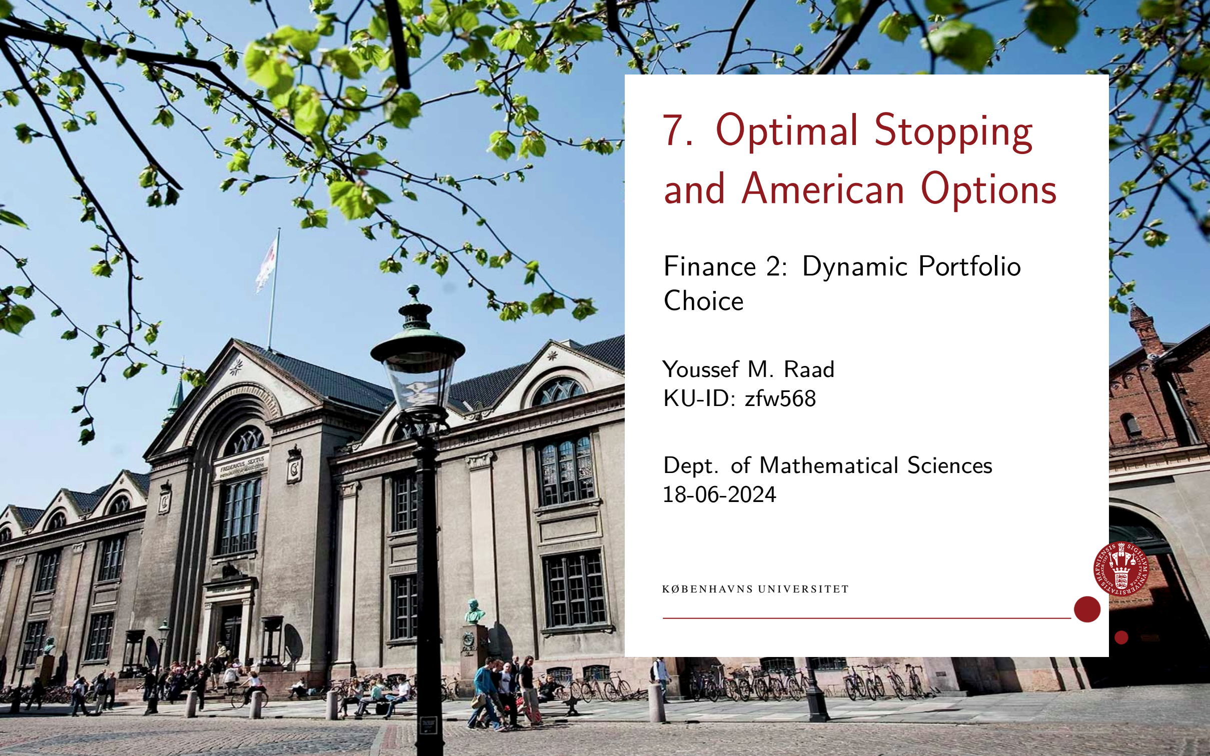

7. Optimal Stopping and American Options

Finance 2: Dynamic Portfolio Choice

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Overview

Model setup and American option

Setup

- Finite filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^N, \mathbb{P})$
- Discrete time $n = 0, \ldots, N$
- Complete and arbitrage free market (existence and uniqueness of EMM, $\mathbb{Q}$)

- We seek to establish the link between American options and the problem of optimal stopping!
- The buyer of an American option has the right, but not the obligation, to exercise the option at any given time until expiry (unlike EU).
- We want (and will) show that:

$$\pi_n^{\text{AMR}} = \max\left\{g(S_n), \mathbb{E}^{\mathbb{Q}}\left[\pi_{n+1}^{\text{AMR}} \mid \mathcal{F}_n\right]\right\},$$

  is the time-n arbitrage-free price of the American option $(r = 0)$ and where $g(S_n)$ is the *intrinsic value* of the option, $\mathbb{E}^{\mathbb{Q}}\left[\pi_{n+1}^{\text{AMR}} \mid \mathcal{F}_n\right]$ the continuation value (Poulsen, 2017, p. 12).
- To hold or not to hold? Clearly a problem of optimal stopping time!

Stopping time

Definition 2.1.1 (Lamberton & Lapeyre (2008))

A random variable τ that takes values in $\{0, 1, ..., N\}$ is a stopping time if for any $n \in \{0, 1, ..., N\}$:

$$\{\tau = n\} \in \mathcal{F}_n.$$

Given an adapted process $(X_n)_{0\leq n\leq N}$ and a stopping time, τ, then the stopped sequence, X_n^τ, is defined as

$$X_n^\tau = X_{\min(\tau,n)} = \begin{cases} X_\tau & \tau \leq n, \\ X_n & \tau > n. \end{cases}$$

So: the process is frozen/flatlined at the stopping time $\rightarrow$ stays at X_τ.

Proposition 2.1.4 ("Stopped martingales are martingales") (Lamberton & Lapeyre (2008))

Let (X_n) be an adapted sequence and τ a stopping time. The stopped sequence $(X_n^\tau)_{0\leq n\leq N}$ is adapted. Moreover, if (X_n) is a martingale (resp. a supermartingale) then $(X_n^\tau)_{0\leq n\leq N}$ is a martingale (resp. a supermartingale).

The Snell Envelope and Proposition 2.2.1 (Lamberton & Lapeyre, 2008, p. 39)

The fundamental concept for solving the optimal stopping problem → **Snell Envelope**: Let Z_n be an adapted process. Now define the Snell Envelope of Z_n as

$$U_n = \begin{cases} Z_N & n = N, \\ \max\left\{Z_n, \mathbb{E}\left[U_{n+1} \mid \mathcal{F}_n\right]\right\} & 0 \leq n \leq N. \end{cases}$$

Clearly, U_n is a supermartingale (decreasing in expectation) as:

$$U_n \overset{n\leq N}{=} \max\left\{Z_n, \mathbb{E}\left[U_{n+1} \mid \mathcal{F}_n\right]\right\} \geq \mathbb{E}\left[U_{n+1}|\mathcal{F}_n\right],$$

and the smallest supermartingale dominating Z_n (more of a fun fact, really). Surprisingly:

Proposition 2.2.1 (Lamberton & Lapeyre (2008))

The random variable defined by:

$$\tau^* = \inf\left\{n \geq 0 \mid U_n = Z_n\right\},$$

is a stopping time and the stopped sequence $(U_n^{\tau^*})_{0\leq n\leq N}$ is a martingale.

Proof of Proposition 2.2.1 (1/2)

Stopping time: Since $U_N = Z_N$, τ^* is a well-defined element of $\{0, 1 \ldots, N\}$ and:

$$\begin{aligned}
&\text{For } k = 0: && \{\tau^* = 0\} = \{U_0 = Z_0\} \in \mathcal{F}_0, \\
&\text{For } k \geq 1: && \underbrace{\{\tau^* = k\} = \{U_0 > Z_0\} \cap \ldots \cap \{U_{k-1} > Z_{k-1}\}}_{U \text{ dominates } Z} \cap \underbrace{\{U_k = Z_k\}}_{\text{first hitting time}} \in \mathcal{F}_k
\end{aligned}$$

Martingale: Writing the stopped process via the process $\{\phi_j\}_{0 \leq j \leq N}$ with $\phi_j = 1_{\{\tau^* \geq j\}}$:

$$U_n^{\tau^*} = U_0 + \sum_{j=1}^{n} \phi_j (U_j - U_{j-1})$$

$$\overset{\text{telescoping sum}}{\Longrightarrow}$$

$$\begin{aligned}
U_{n+1}^{\tau^*} - U_n^{\tau^*} &= \phi_{n+1} \left(U_{n+1} - U_n \right) \\
&= \phi_{n+1} \left(U_{n+1} - \underbrace{\mathbb{E}_n^{\mathbb{Q}} \left[U_{n+1} \right]}_{\bigstar} \right).
\end{aligned}$$

$\bigstar$: If $\{\tau^* \geq n+1\} \Rightarrow U_n > Z_n$, so $U_n = \mathbb{E}_n^{\mathbb{Q}}[U_{n+1}]$ by definition of U_n (intuition: haven't hit equality by $n+1 \Rightarrow$ haven't hit equality at n and thus $U_n > Z_n$ by definition of τ^*).
$\bigstar$: If $\{\tau^* \leq n\} \Rightarrow \phi_{n+1} = 0$ and everything vanishes (intuition: we reached equality at latest n, so by definition the process $\{\phi_j\}_{0 \leq j \leq N}$ is 0 at $j = n+1$).

Proof of Proposition 2.2.1

For $n \in \{0, 1, \ldots, N-1\}$, we have $\{\tau^* \geq n+1\} = (\{\tau^* \leq n\})^{\complement} \in \mathcal{F}_n$, then taking conditional $\mathbb{Q}$-expectation using linearity of the operator:

$$\begin{aligned}
\mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}^{\tau^*} - U_n^{\tau^*}\right] &= \mathbb{E}_n^{\mathbb{Q}}\left[\underbrace{1_{\{\tau^* \geq n+1\}}}_{\mathcal{F}_n-\text{measureable}}\left(U_{n+1} - \mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}\right]\right)\right] \\
&= \phi_{n+1}\mathbb{E}_n^{\mathbb{Q}}\left[\left(U_{n+1} - \mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}\right]\right)\right] \\
&= \phi_{n+1}\left(\mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}\right] - \mathbb{E}_n^{\mathbb{Q}}\left[\mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}\right]\right]\right) \\
&\overset{\dagger}{=} 0,
\end{aligned}$$

where $\dagger$ follows from The Law of Iterated Expectation. Consequently, the stopped process $\left\{U_n^{\tau^*}\right\}$ satisfies:

$$U_n^{\tau^*} = \mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}^{\tau^*}\right],$$

and is therefore a martingale (as the above equals 0).

Optimal Stopping

Corollary 2.2.2 - Optimal stopping time (Lamberton & Lapeyre (2008))

Let $\mathcal{J}_{0,N}$ denote the set of stopping times taking values in $\{0, ..., N\}$.

The stopping time $\tau^* = \inf\{n \mid U_n = Z_n\}$ satisfies ([solves the optimization problem]):

$$U_0 = \mathbb{E}_0^{\mathbb{Q}}[Z_{\tau^*}] = \sup_{\tau \in \mathcal{J}_{0,N}} \mathbb{E}_0^{\mathbb{Q}}[Z_\tau].$$

Proof: For any $\tau \in \mathcal{J}_{0,N}$, $\{U_n^\tau\}$ is a supermartingale (proposition 2.1.4) (decreasing in expectation), so:

$$U_0 = U_0^\tau \geq \mathbb{E}_0^{\mathbb{Q}}[U_N^\tau] \equiv \mathbb{E}_0^{\mathbb{Q}}[U_{\tau \wedge N}] = \mathbb{E}_0^{\mathbb{Q}}[U_\tau] \overset{\dagger}{\geq} \mathbb{E}_0^{\mathbb{Q}}[Z_\tau],$$

where † follows from $U_\tau \geq Z_\tau$. For the stopping time τ^*, $\{U_n^{\tau^*}\}$ is a martingale (proposition 2.2.1), so:

$$U_0 = U_0^{\tau^*} = \mathbb{E}_0^{\mathbb{Q}}\left[U_N^{\tau^*}\right] \equiv \mathbb{E}_0^{\mathbb{Q}}[U_{\tau^* \wedge N}] = \mathbb{E}_0^{\mathbb{Q}}[U_{\tau^*}] \overset{\dagger\dagger}{=} \mathbb{E}_0^{\mathbb{Q}}[Z_{\tau^*}],$$

where †† follows from $U_{\tau^*} = Z_{\tau^*}$. Hence τ^* solves the maximization problem.

This corollary implies that τ^* is the optimal stopping time for the process $\{Z_n\}$ given the information $\mathcal{F}_0$. So, if we can find a Snell envelope that is also a martingale, then it is the optimal stopping time!

Back to American Options: Assembling All the Avengers

We have established a link between the Snell Envelope and the optimal stopping problem with τ^* as the optimal stopping time for the process $\{Z_n\}$ given the information $\mathcal{F}_n$ for some $n \in [0, N]$.

Now: Apply the theory to American Options by letting $U_n =$ Snell Envelope of the stochastic process $\{g(S_n)\}$, (intrinsic value). Now, interpreting Z_n as the option's payoff ($Z_n \sim g(S_n)$), corollary 2.2.2 can be generalized to, for some $n \in [0, N]$:

$$U_n = \sup_{\tau \in \mathcal{J}_{n,N}} \mathbb{E}_n^{\mathbb{Q}} [g(S_\tau)] = \mathbb{E}_n^{\mathbb{Q}} [g(S_{\tau^*})].$$

The Snell Envelope can now be written as:

$$U_n = \max\left\{g(S_n), \mathbb{E}_n^{\mathbb{Q}}\left[U_{n+1}\right]\right\} \left(= \max\left(\underbrace{g(S_n)}_{\text{intrinsic value}}, \underbrace{\mathbb{E}_n^{\mathbb{Q}}\left[\pi_{n+1}^{\text{AMR}}\right]}_{\text{continuation value}} \right) = \pi_n^{\text{AMR}} \right),$$

which is obviously a supermartingale! We now wish to show that $U_n = \pi_n^{\text{AMR}}$ is in fact the the (only) arbitrage-free time-n price of the American Option!

Ops... One last Avenger: Doob-decomposition: Every supermartingale has unique decomposition $U_n = M_n - A_n$, where $\{M_n\}$ is Martingale and $\{A_n\}$ is non-decreasing, predictable (i.e. $A_n \in \mathcal{F}_{n-1}$) and 0-at-0 (i.e. $A_0 = 0$).

Proof: The Only Arbitrage-Free Price (1/2)

Assume $\pi_n^{\text{AMR}} < U_n$:

Time n

- Buy American option at price π_n^{AMR}
- Sell self-financing portfolio which replicates $U_n^{\tau^*}$ (possible by martingale property) at price U_n

Leaves us with cash flow:

$$\underbrace{-\pi_n^{\text{AMR}} + U_n > 0}_{\text{by assumption}}, \quad (\text{"money to spare"}).$$

Time τ^*

- Exercise the American option bought at time n and gain $g(S_{\tau^*})$
- Buy back the self-financing portfolio sold at time n at price $U_n^{\tau^*} = g(S_{\tau^*})$

Leaves us with cash flow:

$$g(S_{\tau^*}) - g(S_{\tau^*}) = 0,$$

i.e an arbitrage!

Proof: The Only Arbitrage-Free Price (2/2)

Assume $\pi_n^{\text{AMR}} > U_n (= M_n - A_n)$:

Time n

- Sell American option at price π_n^{AMR}
- Buy self-financing portfolio which replicates M_n (Martingale part of U_n) at price M_n

Leaves us with cash flow:

$$-M_n + \pi_n^{\text{AMR}} = \underbrace{-U_n + \pi_n^{\text{AMR}} > 0}_{\text{by assumption}}, \quad (\text{"money to spare"}).$$

Time $\tau \in \mathcal{J}_{n,N}$

When the holder/counterparty of the American option exercises (see Extra Slide 5 for note):

- Sell self-financing portfolio which replicates M_n at price $M_\tau = U_\tau + A_\tau$

Leaves us with cash flow (seen in $(\star)$) (as U_n dominates options pay off and $A_\tau \geq 0$):

$$M_\tau = U_\tau + A_\tau \underbrace{\geq}_{A_\tau \geq 0} U_\tau \geq g(S_\tau) \quad \Rightarrow \quad \underbrace{U_\tau + A_\tau - g(S_\tau) \geq 0}_{(\star)},$$

i.e an arbitrage!

$\pi_n^{\text{AMR}} = U_n$ is the only arbitrage free time-n price of the American option (although the expression given for π_n^{AMR} could be *abstractly behaving* at the extremes).

References

Lamberton, D., & Lapeyre, B. (2008). *Introduction to stochastic calculus applied to finance (2nd edition)*. Chapman; Hall/CRC.

Poulsen, R. (2017). American π: Piece of cake? *Willmot Magazine*, 12–13.

Extra Slide 1: Observations on Stopping Times

$$\{\omega \ : \ \tau(\omega) = n\} \text{ is } \mathcal{F}_n\text{-measurable.}$$

- The following definition holds as well, and is moreover the right one for continuous time:

$$\{\tau \leq n\} = \underbrace{\{\tau = 0\}}_{\in \mathcal{F}_0} \cup \underbrace{\{\tau = 1\}}_{\in \mathcal{F}_1} \cup \cdots \cup \underbrace{\{\tau = n\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n.$$

- "First time a thing happens" is a stopping time: E.g. $\nu = \min\{n : X_n \in A\}$

$$\{\nu = n\} = \underbrace{\{X_0 \notin A\}}_{\in \mathcal{F}_0} \cup \underbrace{\{X_1 \notin A\}}_{\in \mathcal{F}_1} \cup \cdots \cup \underbrace{\{X_n \in A\}}_{\in \mathcal{F}_n} \in \mathcal{F}_n.$$

- $\{\tau \geq n\} = \Big(\{\tau \leq n-1\}\Big)^c \in \mathcal{F}_{n-1}$, i.e. whether whatever we are looking for happens at time n or later is known at time $n-1$.

Extra Slide 2: Proof of Proposition 2.1.4

Trick: Introduce random variables $\phi_j =_{\{\tau \geq j\}}$ for $j = 0, 1, ..., N$, i.e. ϕ_j is $\mathcal{F}_{j-1}$-measurable.

Write as telescoping sum

$$X_n^\tau = X_{\tau \wedge n} = X_0 + \sum_{j=1}^{n} \phi_j \Big(X_j - X_{j-1} \Big). \qquad (\star)$$

Note: Effectively number of terms in the sum is random, but technically it's not.

All terms on the RHS of $(\star)$ are $\mathcal{F}_n$-measurable, and so X_n^τ is adapted.

Assume X is a (super) martingale, i.e, $X_n \overset{(\geq)}{=} \mathbb{E}_n^{\mathbb{Q}}[X_{n+1}]$.
Consider

$$\mathbb{E}_n^{\mathbb{Q}}[X_{n+1}^\tau] \overset{(\star)}{=} \underset{\textcolor{red}{\bullet}}{X_n^\tau} + \mathbb{E}_n^{\mathbb{Q}}\Big[\phi_{n+1}(X_{n+1} - X_n\Big] \underset{\textcolor{blue}{\bullet}}{=} X_n^\tau + \phi_{n+1}\mathbb{E}_n^{\mathbb{Q}}\Big[X_{n+1} - X_n\Big]$$

$$\underset{\textcolor{green}{\bullet}}{=} X_n^\tau + \phi_{n+1} \underbrace{\Big[\mathbb{E}_n^{\mathbb{Q}}[X_{n+1}] - X_n\Big]}_{=0 \ (\geq 0)} \overset{(\leq)}{=} X_n^\tau,$$

using in $\textcolor{red}{\bullet}$ that terms from zero to n are $\mathcal{F}_n$-measurable, in $\textcolor{blue}{\bullet}$ that ϕ_{n+1} is $\mathcal{F}_n$-measurable, and in $\textcolor{green}{\bullet}$ that X is a (super) martingale.

Extra Slide 3: Observations on American Options

- European option pays $g(S_n)$ to holder at fixed time of expiry n.
- American options pays $g(S_\tau)$ to holder at time $\tau \leq N$, where τ is a stopping time chosen by the holder.
- Pricing function violates the principle of local characterization $\rightarrow$ AMR not a contingent claim, as these are determined solely by the market with no decisions left for the holder.
- π_n^{AMR} is the time-n price of an American option given that the option is still alive at (i.e. not exercised at or before) time n.
- With $r_n \equiv 0$, the American option price, π_n^{AMR}, is the Snell envelope of the intrinsic value of the option, $\{g(S_n)\}$.

Extra Slide 4: Special Case: $\text{Call}^{\text{AMR}} = \text{Call}^{\text{EUR}}$

Pricing a American call Option with $r > 0$ and δ

Assume frictionless market: no borrowing, short-selling constraints and transaction cost. Then, whenever there is a strictly positive interest rate (i.e. $r > 0$) and zero dividends (i.e. $\delta \equiv 0$), then it is optimal to hold the American option until time of expiry N.

Consequently, in this case, the price of an American call option equals that of the European call option.

Proof: Suppose we are standing at some arbitrary time $t > T$ and that $S_n > K$, i.e. that the option is in-the-money, otherwise we would never exercise.

Then:

$$\text{Call}_n^{\text{AMR}} \geq \text{Call}_n^{\text{EUR}} \geq \text{Call}_n^{\text{EUR}} - \underbrace{\text{Put}_n^{\text{EUR}}}_{\geq 0} \underset{\text{P/C-parity}}{=} S_n - \underbrace{^{-r[T-t]}}_{<1} K > S_n - K = g(S_n),$$

where $g(S_n)$ is the value of exercising the option today.

This means that for $n < N$, the American call option is always worth more alive than dead, i.e. we should not exercise until expiry!

Dual result: $r = 0,\ \delta > 0 \Rightarrow$ Optimal to hold AMR put until expiry.

Extra Slide 5: Why Only Hedge Martingale Part When $\pi_n^{\text{AMR}} > U_n$?

Important note!

Whenever $\pi_n^{\text{AMR}} < U_n$, then we buy the American option and finance it buy selling the self-financing portfolio replicating U_n. However, when $\pi_n^{\text{AMR}} > U_n$ then we sell the American option, and so we are longer the holders of the American option. That is, we no longer have any right to say when the option should be exercised. The counterparty might choose to exercise at the optimal date τ^*, but he might choose to exercise at another date.

In this case it is important that we trade only the martingale part of U_n, namely M_n. The reason for this is that for an arbitrary $\tau \in \mathcal{J}_{n,N}$, then U_n is only a supermartingale (and not a martingale). If the counterparty (i.e. the holder of the American option) chooses to exercise later than optimal, then U_n flatlines/freezes.

In turn, this means that our strategy is no longer an arbitrage. If U_n freezes, then we can no longer be sure to cover our liabilities if coutnerparty got lucky, and so we cannot be sure that our strategy is an arbitrage.