May 19, 2012
Rolf Poulsen
Dept' of Math'
Finance 2

A Numerical Experiment: The Gain from Dynamic Trading

Chapter 2 in Pascucci & Runggaldier (P&R in the following) tells us that it is optimal for an investor with constant relative risk aversion to have fixed proportions or fractions (over time) of total wealth invested in the different assets. Such portfolios are *not* buy-and-hold strategies; they do require dynamic trading. We now investigate the magnitude of the gain from dynamic trading experimentally.

We consider the standard binomial model and with a logarithmic utility investor who cares only about terminal (i.e. time-T) consumption.

The indirect utility function, W, represents the maximal expected utility that can be achieved with given initial wealth. The equation on the top of page 98 in P&R (or, alternatively, their equation (2.83)) tells us that in a T-period model

$$W_0(v) = \ln(v) + T \max_{\pi \in \mathbb{R}} \mathbf{E} \left(\ln((1+r) + \pi(\mu - r)) \right),$$

where μ is a random variable with the (common) return distribution. One way to write the optimal stock-investment is as

$$\widehat{\pi} = \frac{(1+r)(pu + (1-p)d - (1+r))}{(u-(1+r))((1+r)-d)}$$

So: The investor constantly changes the number of units of stock he holds in order to keep the fraction of wealth invested in the stock fixed at $\widehat{\pi}$.

With a buy-and-hold strategy the investor buys θ_0 units of the stock at time 0 and holds that fixed over time. The initial fraction of wealth invested in the stock satisfies $v\pi_0 = \theta_0 S(0)$, and we can write the time-T value of the strategy as

$$V(T; v, \pi_0) = v \left(\pi_0 S(T)/S(0) + (1-\pi_0)(1+r)^T \right),$$

This means — using the binomial model structure — that the expected utility from a buy-and-hold strategy is

$$\begin{aligned} G(v;\pi_0) &:= \mathbf{E}(\ln(V(T;v,\pi_0))) \\ &= \ln(v) + \sum_{j=0}^{T} \left(\begin{array}{c} T \\ j \end{array} \right) p^j (1-p)^{T-j} \ln\left(\pi_0 u^j d^{T-j} + (1-\pi_0)(1+r)^T\right). \end{aligned}$$

Some *not implausible* numerical values:

- $u = 1.29$, $d = 0.85$, $p = 0.5$ — this corresponds to a yearly expected return of 7% with a standard deviation of 22%
- $r = 0.04$
- $T = 30$ — it could be your pension savings

In this case

$$\widehat{\pi} = 0.6568$$

and

$$W_0(v) = \ln(v) + 30 \times 0.04861.$$

Figure 1 shows the expected utility that be achieved by optimal dynamic trading compared to what buy-and-hold strategies give. To be precise the figure plots $G(v;\pi_0) - W_0(v)$ (which is independent of v) over a range of π_0's. This curve is visibly below zero everywhere, demonstrating that dynamic trading can improve even the best buy-and-hold strategy. (Of course, buy-hold-strategies are part of what we are optimizing over, so a curve above zero would be a definite sign of us having made at least one error.)

But the numerical magnitude of the difference between G and V is not an economically meaningful measure of the gain from (optimal) dynamic trading, as we can multiply the utility function by any positive number without any change to the "economic content". To give such a meaningful comparison, we can ask: How large a fraction, say f, of his initial wealth would an investor constrained to buy-and-hold strategies be willing to give up in order to trade dynamically? More specifically: Letting $\widehat{\pi}_0$ denote the optimal buy-and-hold strategy (here approximately 0.69, as seen in Figure 1), we calculate a certainty equivalent wealth by solving

$$W_0(v \times (1-f)) = G(v;\widehat{\pi}_0)$$

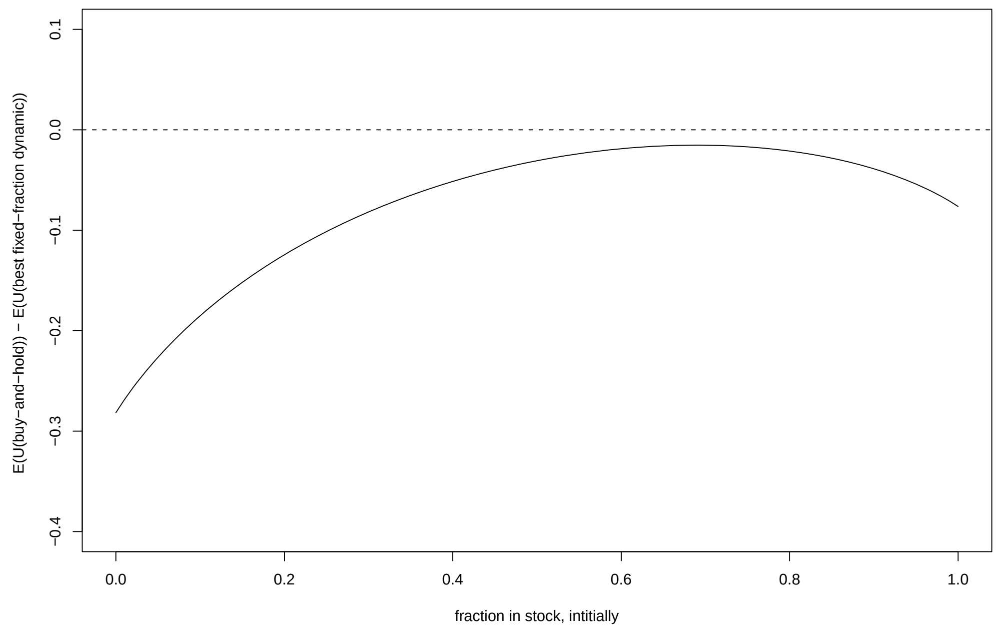

Figure 1: The fully drawn curve is the expected utility from buy-and-hold strategies (with different fractions of initial stock investment along the x-axis; π_0 in the text) *minus* the expected utility from optimal dynamic investment as given by the indirect utility function. (In this log-utility case, this difference is independent of wealth.)

for f which (independently of v) gives

$$f = 0.0154.$$

So: The constrained investor would be willing to give up 1.5% of his initial wealth in order to be allowed to trade dynamically rather than using the best buy-and-hold strategy. On the one hand not a vanishing number, but on the other hand a gain that could easily be "eaten up" if there are transaction costs when trading dynamically; about 1/20th of a percent per year would seem to "do the trick".

There is R-code on the Absalon-page. Here are some extensions for the eager student or lecturer — or the forced TA! — to consider:

- Illustrate that for a log-utility investor with a subsistence level ($U(x) = \ln(x - a)$) the investment in the risk-free asset may (will?) decrease when the risk-free rate goes up. (That would otherwise make the risk-free asset more attractive; so it's a Giffen good-like effect.)
- Re-do the analysis for constant relative risk-aversion power-utility (CRR agents for short). P&R Exercise 2.47 comes in handy. (Solve part i) w/ $r \neq 0$).)
- How risk-averse do CRR agents have to for historically observed average return and standard deviation on stock indices to lead to "sensible" portfolio weights?
- What happens if we take the binomial model to the "smaller and smaller time-steps" limit?
- In the case of 2 or more risky assets: How different are optimal portfolios for CRR agents from mean/variance optimal portfolios? (And why is that question meaningless with just one risky asset?)