Rolf Poulsen

American π: Piece of Cake?

Textbooks tell you that pricing an American option in the context of the binomial model is a lot easier than it sounds. Is it really simple and obvious? Yes...and no.

An American option can be exercised by its holder at any time he wishes, not just at the expiration date T. Textbooks tell you that pricing it in the context of the binomial model is a lot easier than it sounds: at any point in the stock lattice check if the option is worth more dead than alive and act accordingly. Or, in a half-abstract formula:

$$\pi_t^{AMR} = max\left(g\left(S_t\right), \tfrac{1}{1+r}E_t^Q\left(\pi_{t+1}^{AMR}\right)\right) \qquad (*)$$

where g is the payoff function [$g(x) = (K - x)^+$ for a put option] and we call $g(S_t)$ the option's intrinsic value. Implementing this is half a line of extra code. And the argument sounds convincing. But is it really that simple and obvious? Yes, it is that simple. No, it is not that obvious. To see the latter: rigorous derivative pricing courses spend time and effort proving that for a model to be arbitrage-free, there must exist a (martingale) probability measure, Q, such that for any traded contingent claim, its price satisfies the local characterization

$$\pi_t = \frac{1}{1+r_t}E_t^Q\left(\pi_{t+1} + \delta_{t+1}\right),$$

where δ denotes dividend payments. Unless the whole thing were trivial, what we claim to be the American option price in (*) violates this local characterization. Where the rabbit goes into the hat is that the American option is not a *contingent claim*, which is something whose cash flows (and prices), although random, are determined exogenously in our model; i.e. by nature or the market, not specifically by our counterparty in the trade.[1] So, even in the simple setting of a binomial model, there is work to do in proving that (*) is indeed the only arbitrage-free price of the American option.[2] To do this let us first assume that the interest rate is zero and define Z as the so-called Snell envelope of the intrinsic value, $Z_t = max(g(S_t), E_t^Q(Z_{t+1}))$.

It is clear that Z is a supermartingale. Hence it can be Doob decomposed, $Z_t = M_t - A_t$, where M is a martingale and A is a non-decreasing, predictable, 0-at-0 process. The stopping time $\tau^* = min\,\{t | Z_t = g(S_t)\}$ has the very particular property that the stopped process defined via $Z_t^{\tau^*} = Z_{min(t,\tau^*)}$ is a martingale. This is surprising, because in general stopped supermartingales are only supermartingales. By considering the following two cases, we conclude that (*) gives the only arbitrage-free price:

1. $\pi_t^{AMR} < Z_t$. We buy the American option, finance it (with money to spare) by selling a self-financing trading strategy that replicates Z^{τ^*} (for this to be possible, the martingale property is crucial), and exercise according to τ^*. This is an arbitrage strategy.
2. $\pi_t^{AMR} > Z_t$. We sell the American option. This can (with money to spare) finance buying a self-financing trading strategy that replicates the martingale part of Z's Doob decomposition. Because Z dominates the option's payoff and A is positive (its predictability isn't used), we can always cover our liabilities by liquidating the trading strategy. Again, an arbitrage. (This differs from case 1 because we no longer control the exercise of the American option.)

Applying the optional sampling theorem (twice) gives us an abstract representation of the American option price via an optimal stopping problem,

$$\pi_t^{AMR} = sup_{\tau\in T_{t,T}}E_t^Q\left(\tfrac{g(S_\tau)}{R_{t,\tau}}\right) \qquad (**)$$

where $T_{t,T}$ denotes the set of stopping times with values in $[t;T]$ and $R_{t,u} = \prod_{v=t}^{u-1}(1 + r_v)$ is the natural discount factor. An advantage of (**) – whose validity is a theorem, a result; not a definition or an assumption – is pattern recognition; essentially the same formulation works in continuous-time models. Contrarily, the continuous-time analogue of (*) isn't obvious but turns out to be a free-boundary-value problem.

To exercise or not? Since Merton (1973) we have known that if the interest rate is positive and the underlying pays no dividends during the life of the option, then it is never optimal to exercise an American call option early and its value is therefore the same as that of a European call option. There is a dual result for the put option: zero interest rate and positive dividends make us hold it until expiry. However, these results are dependent on a frictionless market; no borrowing or short-selling constraints, no transaction costs. Without a liquid market for the American option, or if replication is costly, the more efficient way for the holder to cash in on his option may be to exercise it. Ahn and Wilmott (2003) and Jensen and Pedersen (2016) deliver and build on this insight – with disjoint bibliographies.

Simulation. Conventional wisdom was that Monte Carlo simulation could not be used to price American options.[3] The breakthrough came with Longstaff and

Schwartz (2001).[4] A key part of their approach is regression across paths to estimate the continuation value or the early-exercise boundary. Space doesn't permit a detailed description; many such exist in the literature, not least the first seven pages of the original paper.[5] My point here is just that the method has become so part of the fabric of quantitative finance that we forget it's not trivial. The following example – R code[6] based on a true story – shows how to get it wrong:

```
UpMoves<-cumsum((runif(n) < q))
S[2:(n+1)]<-S0*u^UpMoves*d^((1:n)-UpMoves)
SnellZ<-gS<-pmax(strike-S,0)
for (i in n:1) SnellZ[i]<-
max(gS[i],SnellZ[i+1]/(1+r))
tau.index<-which.min(time.vector[SnellZ==gS])
DiscPayoff<-(1/(1+r))^((tau.index-1))*gS[tau.
index]
```

The first lines simulate a stock-price path in the standard binomial model. The last lines find the optimal stopping time and an optimal realization of the term inside the expectation in (**). The error is the highlighted part; the intention is to calculate the Snell envelope, but the conditional expectation on the last term is forgotten. Therefore, `SnellZ` becomes the pathwise remaining maximum of the intrinsic value and we are effectively exercising our option with perfect knowledge of the future. A malicious aspect of this error is that while the resulting price is quite a bit off, it is not *obviously* nonsensical; a one-year, at-the-money American put option in a daily-steps model with $r = 0.03$ and $\sigma = 0.2$ should cost 6.74 percent of the underlying's price; the code above gives a value of 13.8 percent.

Longstaff and Schwartz's approach deals well with model extensions, for instance multi-dimensional stochastic interest rates. Many leading banks use (in-house adapted versions of) it. Another virtue is robustness; small errors in exercise decisions do not matter too much. Intuitively, this is because the critical value is precisely such that the option is worth the same dead as alive.[7] The exact extent of robustness is currently a matter of considerable interest in the deeper echelons of quant departments.[8] Can adjoint differentiation techniques[9] for sensitivity calculations be made to work efficiently; is differentiation through the regression of critical importance?

About the Author

Rolf Poulsen is in the Department of Mathematical Sciences at the University of Copenhagen. His main research interest is quantitative methods for pricing and hedging of derivatives. He will talk about exchange rate markets at length to all who will listen – and some who won't.

ENDNOTES

1. To make European put options fit into this setting, we cheat slightly; we *define* the payoff to be $(K - S_T)^+$, reflecting rational exercise and no physical delivery problems. Another mild imprecision: we really should call π_t^{AMR} the price of the American option given that it has not previously been exercised.
2. The probabilistic results about stopping times I have used are covered excellently in Chapter 2 of Lamberton and Lapeyre (2008). However, from a finance point of view, I think the presentation lacks the "why is this not obvious?" motivation and doesn't quite stick the landing in Section 2.5.
3. Hull (1993) writes on p. 329: "Unlike Monte Carlo simulation they [trees and finite differences] can be used for derivative securities where the holder has early exercise decisions".
4. Longstaff and Schwartz were not the first to propose simulation techniques. Carriere (1996) and Broadie and Glasserman (1997) may feel their thunder was stolen.
5. Honorable mention goes to Stentoft (2004) for extensions of Longstaff and Schwartz's partial convergence results and to Rasmussen (2005) for tricks galore.
6. Code to run is available at: tinyurl.com/z6259fw.
7. The value function is in fact continuously differentiable (once, but not twice) across the exercise boundary; so-called smooth pasting.
8. See Huge and Savine (tinyurl.com/hvz2b26).
9. Sensitivity calculation speed-up via adjoint differentiation is a hot issue in the quant finance industry. Possibly a topic for a future column, but for now I just refer to tinyurl.com/z7gpqj8.

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