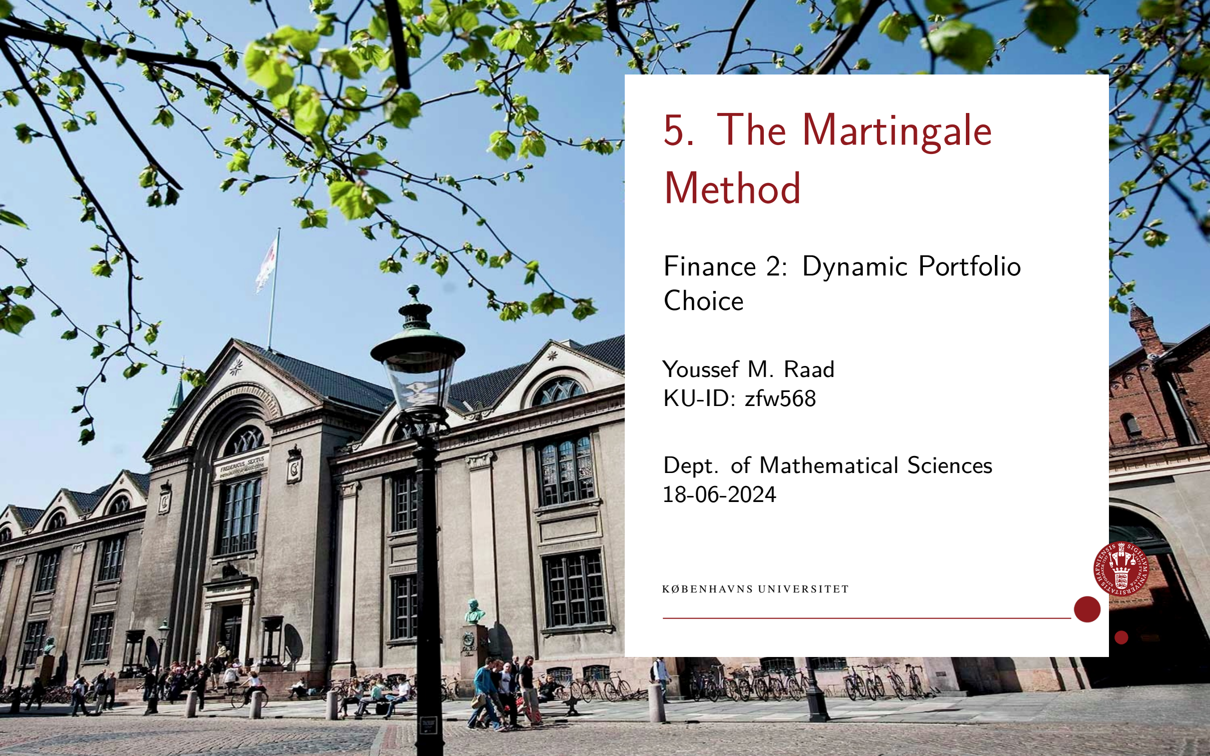

5. The Martingale Method

Finance 2: Dynamic Portfolio Choice

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18-06-2024

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Model setup (1/2) (Pascucci & Runggaldier, 2012)

- Time is discrete and finite with time points $n = 0, ..., N$. We let:

$$t_0 < t_1 < \ldots < t_N,$$

represent the trading dates: $t_0 = 0$ today and $t_N = N$ the expiry date.

- Finite filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{N}, \mathbb{P}\right)$, i.e finite state space $\Omega = \{\omega_1, ..., \omega_M\}$ (paths).

- d risky assets, $S^1, \ldots, S^d$. If S_n^i denotes the price at time t_n of the i-th asset:

$$\begin{cases} S_0^i \in \mathbb{R}_+, \\ S_n^i = S_{n-1}^i \left(1 + \mu_n^i\right), \quad n = 1, \ldots, N, \end{cases}$$

where μ_n^i is a real random variable that represents the rate of return of the i-th asset in the n-th period $[t_{n-1}, t_n]$.

- One risk-free asset (bank/money account) B where:

$$\begin{cases} B_0 = 1, \\ B_n = B_{n-1}(1 + r_n^f), \quad n = 1, \ldots, N, \end{cases}$$

where $r_n^f > -1$ denotes the risk-free rate in the n-th period $[t_{n-1}, t_n]$.

Model setup (2/2) (Pascucci & Runggaldier, 2012)

- Utility function of class C^1:

$$u : I \to \mathbb{R}, \quad I =]a, \infty[, \quad \text{where } a \leq 0,$$

strictly increasing, strictly concave. Denote by $\mathcal{I} := (u')^{-1}$ continuous, strictly decreasing.

Define (without consumption):

- Value of a trading strategy $(\alpha, \beta) = (\alpha_n^1, \dots, \alpha_n^d, \beta_n)_{n=1,\dots,N}$ at time n:

$$V_n^{(\alpha,\beta)} = \alpha_n S_n + \beta_n B_n,$$

where α_n^i (resp., β_n) represents the amount of asset S_i (resp., of bond) kept in the portfolio during the n-th period $[t_{n-1}, t_n]$.

$((\alpha_n, \beta_n)$ composition at $t_{n-1})$.

- The trading strategy (α, β) is self-financing if:

$$V_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1}.$$

- (α, β) is predictable i.e (α_n, B_n) is $\mathcal{F}_{n-1}$-measurable for any $n = 1, ..., N$.

The Martingale Method: Terminal Wealth (Pascucci & Runggaldier, 2012, p. 72)

Consider an arbitrage-free and complete model with strictly increasing and strictly concave utility function.

For fixed $V_0 = v \in \mathbb{R}_+$: Agent seeks to find a strategy (α, β) that maximizes expected utility of terminal wealth i.e:

$$\max_{\alpha} \mathbb{E}^{\mathbb{P}} \left[u \left(V_N^{(\alpha)} \right) \right], \tag{$\star$}$$

where α is a predictable process. This is done in three steps (Pascucci & Runggaldier, 2012, p. 72):

The Martingale Method

(P1) Determine reachable set of terminal values by a self-financing and predictable strategy given initial wealth:

$$\mathcal{V}_v = \left\{ V \mid V = V_N^{(\alpha)},\ \alpha \text{ predictable},\ V_0^{(\alpha)} = v \right\}.$$

(P2) Determine the optimal terminal reachable value, $\bar{V}_N$, namely the value that realizes the maximum in $(\star)$

(P3) Determine the self-financing strategy, $\bar{\alpha}$, such that $V_N^{(\bar{\alpha})} = \bar{V}_N$

Solving the Three Steps

(P1): Market free of arbitrage and complete $\iff$ Exists a unique EMM $\mathbb{Q} \to$ By replication theorem 1.21 (Pascucci & Runggaldier, 2012, p. 12) terminal values reachable by SF strat, v initial:

$$\mathcal{V}_v = \left\{ V \mid \mathbb{E}^{\mathbb{Q}} \left[\frac{V}{B_N} \right] = v \right\}.$$

(P2): Theorem 2.18 (Pascucci & Runggaldier, 2012, p. 74):

Theorem 2.18

Assume that the market is complete, free of arbitrage and $u'(I) = \mathbb{R}_+$ ([bijective]). The optimal terminal value in the problem of maximizing expected utility of terminal wealth, starting from an initial capital $v \in \mathbb{R}_+$ will then be given by:

$$\bar{V}_N = \mathcal{I}\left(\lambda \tilde{L}\right),$$

where $\tilde{L} = B_N^{-1} L$ with $L = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ ([the likelihood/Radon−Nikodym derivative]), $\mathbb{Q}$ being the martingale measure. $\lambda \in \mathbb{R}$ is determined by the equation (["budget equation"]):

$$\mathbb{E}^{\mathbb{P}} \left[\mathcal{I}\left(\lambda \tilde{L}\right) \tilde{L} \right] = v.$$

([Note that $\mathcal{I} = (u')^{-1}$ and $I =]a, +\infty[$, where $a \leq 0$]).

(P3): Hedging problem where the replicated pay-off is the optimal terminal value.

Proof of Theorem 2.18 (1/2)

Standard constrained optimization problem in $\mathbb{R}^M$, where M is the cardinality of Ω. The problem can now be reformulated to a problem of maximizing, with notation $\mathbb{P}_i = \mathbb{P}(\{\omega_i\})$ and $\mathbb{Q}_i = \mathbb{Q}(\{\omega_i\})$, $i = 1, \ldots, M$:

$$f(V) := \sum_{i=1}^{M} u(V_i)\mathbb{P}_i = \mathbb{E}^{\mathbb{P}}\left[u(V)\right],$$

s.t $V \in \mathcal{V} \cap I^M$ (or less abstractly: set of terminal values V_i's reachable from initial wealth v).
By **(P1)** characterization: $\mathcal{V}_v = \left\{V \mid \mathbb{E}^{\mathbb{Q}}\left[B_N^{-1}V\right] = v\right\} \iff \left\{V \mid \mathbb{E}^{\mathbb{Q}}\left[B_N^{-1}V\right] - v = 0\right\}$:

$$g(V) := \sum_{i=1}^{M} B_N^{-1} V_i \mathbb{Q}_i - v = \mathbb{E}^{\mathbb{Q}}\left[B_N^{-1}V\right] - v = 0 \quad \text{and} \quad V_i > a.$$

(Assumed) Arbitrage free $\iff$ Exists a solution $\bar{V}_N \in \mathcal{V}_v$ to $(\star)$ by theorem 2.12 (Pascucci & Runggaldier, 2012, p. 67). Set up Lagrangian over the constraint $\mathcal{V}_v = \{g = 0\}$:

$$\mathcal{L}(V, \lambda) = f(V) - \lambda g(V) = \left(\sum_{i=1}^{M} u(V_i)\mathbb{P}_i\right) - \lambda\left(\sum_{i=1}^{M} B_N^{-1} V_i \mathbb{Q}_i - v\right),$$

resulting in the following system of F.O.C's:

$$\partial_{V_i}\mathcal{L}(V, \lambda) = u'(V_i)\mathbb{P}_i - B_N^{-1}\lambda\mathbb{Q}_i = 0, \quad i = 1, \ldots, M, \tag{1}$$

$$\partial_{\lambda}\mathcal{L}(V, \lambda) = \sum_{i=1}^{M} B_N^{-1} V_i \mathbb{Q}_i - v = 0. \tag{2}$$

Proof of Theorem 2.18 (2/2)

(Assumed) $u'(I) = \mathbb{R}_+$ (bijective) $\Leftrightarrow$ (1) has an unique solution $(\bar{V}_N)_i$. Isolating u' and taking the inverse:

$$(\bar{V}_N)_i = (u')^{-1}\left(B_N^{-1}\lambda\frac{\mathbb{Q}_i}{\mathbb{P}_i}\right) \equiv \mathcal{I}\left(B_N^{-1}\lambda\frac{\mathbb{Q}_i}{\mathbb{P}_i}\right) = \mathcal{I}\left(\lambda\tilde{L}\right), \quad i = 1, \dots, M. \tag{3}$$

Substituting (3) into (2) to determine λ:

$$h(\lambda) := \sum_{i=1}^{M} B_N^{-1}\mathcal{I}\left(\lambda\tilde{L}\right)\mathbb{Q}_i \overset{(*)}{=} \mathbb{E}^{\mathbb{P}}\left[\mathcal{I}(\lambda\tilde{L})\tilde{L}\right] = v, \tag{4}$$

where $(*)$ is from multiplication by $\frac{\mathbb{P}_i}{\mathbb{P}_i}$.

h is continuous and strictly decreasing remark 17 (assumptions of the utility function u) (Pascucci & Runggaldier, 2012, p. 73) $\iff$ For each $v \in \mathbb{R}_+$, there exists a unique solution to (4).

Example: Terminal Utility in the Binomial Model: MG Method (1/3)

Standard binomial model setup (Extra Slide 1 for example with numbers):

- N: Amount of periods.
- log-utility:

$$u(x) = \ln(x) \iff u'(x) = \frac{1}{x} = y \iff x = (u')^{-1}(y) = \frac{1}{y} \equiv \mathcal{I}(y).$$

- $\omega \in \Omega$: Paths through the tree.
- ν_n : Number of up moves in n periods (or, equivalently, the number of 1 among the first n elements of the N-tuple ω, i.e $\omega = (0, 1, 0, 0, 1, 0, \dots)$).
- S_0 and $S_n = u^{\nu_n} d^{n-\nu_n} S_0$ where u with probability p, d with probability $1 - p$.

The martingale measure in the binomial model is defined as (Pascucci & Runggaldier, 2012, p. 75):

$$\mathbb{Q} = q^{\#\text{up-moves}}(1-q)^{\#\text{down-moves}} = q^{\nu_N}(1-q)^{N-\nu_N}, \quad q = \frac{1+r-d}{u-d}.$$

μ_n^i (rate of return of asset i) is $\mathbb{Q}$-independent theorem 1.26 (as q constant) (Pascucci & Runggaldier, 2012, p. 14), thus the Radon-Nikodym derivative is:

$$L = \frac{d\mathbb{Q}}{d\mathbb{P}} = \left(\frac{q}{p}\right)^{\nu_N} \left(\frac{1-q}{1-p}\right)^{N-\nu_N}.$$

Example: Terminal Utility in the Binomial Model: MG Method (2/3)

Using theorem 2.18 (Pascucci & Runggaldier, 2012, p. 74) and $\mathcal{I}(y) = \frac{1}{y}$:

$$v = \mathbb{E}^{\mathbb{P}}\left[\mathcal{I}\left(\lambda\tilde{L}\right)\tilde{L}\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\lambda\tilde{L}}\tilde{L}\right] = \frac{1}{\lambda} \Leftrightarrow \lambda = \frac{1}{v}.$$

The optimal terminal wealth by theorem 2.18 (Pascucci & Runggaldier, 2012, p. 74) and $\lambda = 1/v$:

$$\bar{V}_N = \mathcal{I}\left(\lambda\tilde{L}\right) = \frac{1}{\lambda\tilde{L}} = \frac{1}{\frac{1}{v}\tilde{L}} = \frac{v}{\tilde{L}} = v\underbrace{(1+r)^N}_{\text{discount to find } \tilde{L}}\left(\frac{p}{q}\right)^{\nu_N}\left(\frac{1-p}{1-q}\right)^{N-\nu_N}.$$

and the optimal value of expected utility is:

$$\begin{aligned}\mathbb{E}\left[\log \bar{V}_N\right] &= \log v + N\log(1+r) + \mathbb{E}[\nu_N]\log\frac{p}{q} + (N - \mathbb{E}[\nu_N])\log\frac{1-p}{1-q} \\ &= \log v + N\log(1+r) + \underbrace{Np}_{=\mathbb{E}[\nu_N]}\log\frac{p}{q} + \underbrace{N(1-p)}_{=N-\mathbb{E}[\nu_N]}\log\frac{1-p}{1-q}.\end{aligned}$$

Example: Terminal Utility in the Binomial Model: MG Method (3/3)

Lastly, determine a self-financing portfolio (α, β) which replicates $\bar{V}_N$. Last period $\rightarrow$ Portfolio replication condition:

$$\alpha_N S_N + \beta_N B_N = \bar{V}_N. \tag{5}$$

Consider time $N-1$: Either up-move or a down-move to period N. Assuming that $S_{N-1} = S_0 u^k d^{N-1-k}$ (i.e $\nu_{N-1} = k$ up-ticks for $k < N$), rewrite (5) as a system of equations in α_N, β_N:

$$\begin{cases} \text{(Up):} & \alpha_N u S_{N-1} + \beta_N B_N = v\,(1+r)^N \left(\dfrac{p}{q}\right)^{k+1} \left(\dfrac{1-p}{1-q}\right)^{N-(k+1)}, \\ \text{(Down):} & \alpha_N d S_{N-1} + \beta_N B_N = v\,(1+r)^N \left(\dfrac{p}{q}\right)^{k} \left(\dfrac{1-p}{1-q}\right)^{N-k}. \end{cases}$$

Solving for α_N and tons of tedious algebra later (Pascucci & Runggaldier, 2012, pp. 95-96) finds the optimal ratio invested in the risky asset:

$$\pi_n = \frac{\alpha_n S_{n-1}}{V_{N-1}} = \frac{1+r}{u-d}\,\frac{p-q}{q\,(1-q)}, \quad n = 1, \dots N.$$

Implication and Discussion

$$\pi_n = \frac{\alpha_n S_{n-1}}{V_{N-1}} = \frac{1+r}{u-d} \frac{p-q}{q(1-q)}, \quad n = 1, \ldots N.$$

Result: In any period the optimal strategy involves holding the same **ratio** in the risky asset, independent of stock price, wealth and time horizon ("myopia") and thus the solution for $n \leq N-2$ is the same. Same ratio in wealth in every period does **not** imply, expressed in units of the assets kept in the portfolio, **remains constant**.

By proposition 2 (Blædel & Huge, 1999, p. 12): Can extend the analysis to any utility function with constant RRA.

Question: Should young investors (long time horizon) hold a higher proportion of stocks?

- Yes! Stock returns outperform bonds in the long run (out−performance probability).
- Yes! Future income/salaries is like a big bond investment $\Rightarrow$ more should be invested in risky asset.
- No! Human capital is maybe not that risk-free - unemployment (for months), lower salary after unemployment (stocks relatively uncorrelated between periods), i.e words of Rolf: "really nasty".
- No! We have rigorous and robost results in dynamic opimization that says our ratio should be constant.

Conclusion: Inconclusive: it is not obvious whatsoever if you should have more in risky assets.

References

Blædel, N., & Huge, H. (1999). Multiperiod investment in discrete time.

Pascucci, A., & Runggaldier, W. J. (2012). *Financial mathematics: Theory and problems for multi-period models*. Springer Science & Business Media.

Extra Slides 1: Excel-file: GitHub.

1	Model parameters																	
2							Stock price lattice						Optimal wealth					
3	S(0)	100				#up-moves												\tilde(L)
4	alpha	0.05	u	1.419068		4					405.52						1319.896	0.378818
5	sigma	0.3	d	0.778801		3				285.7651	222.5541					1035.493	909.4839	0.549762
6	p	0.5	q	0.407954		2			201.3753	156.8312	122.1403				812.3719	713.5141	626.6864	0.797847
7	Delta t	1	R	1.04		1		141.9068	110.5171	86.0708	67.032			637.3272	559.7707	491.652	431.8228	1.157882
8	N	4				0	100	77.88008	60.65307	47.23666	36.78794		500	439.1549	385.714	338.7764	297.5506	1.680387
9	r	0.04				time	0	1	2	3	4		0	1	2	3	4	
10	v	500											0					
11																		
12	lamda	0.002					Optimal stock position; # stocks						Optimal stock position; weight					
13																		
14						4												
15						3				2.243108						0.61903		
16						2			2.497243	2.81632					0.61903	0.61903		
17						1		2.78017	3.135398	3.536014				0.61903	0.61903	0.61903		
18						0	3.095152	3.490626	3.93663	4.439622			0.61903	0.61903	0.61903	0.61903		
19						time	0	1	2	3	4							

- Ratio (Optimal stock position; weight) is kept constant
- Optimal stock position (# of stocks) is consistently changing to keep wealth ratio constant.
- Likelihood is decreasing in optimal wealth.
- **High Wealth, Low** L: When wealth is high (1319.896065), the Radon-Nikodym derivative L is low (0.378817706). This suggests that in high wealth scenarios, the adjustment from the real-world measure to the risk-neutral measure is smaller, and vice versa.

Extra Slides 2: Incomplete model with terminal wealth (1/2)

Incomplete model: i.e the trinomial-model. Martingale method does not work well in incomplete models because we could cant replicate every claim.

Solving (P1):

In the case when the market is free of arbitrage and incomplete, the set of martingale measures is infinite. Observe first of all that, on the basis of Theorem 1.21, the set of terminal values that can be obtained starting from an initial wealth v has the following characterization:

$$\mathcal{V}_v = \left\{ V \mid \mathbb{E}^{\mathbb{Q}}[B_N^{-1} V] = v, \text{ for each martingale measure } \mathbb{Q} \right\}.$$

Secondly, the family of martingale measures is the intersection of an affine space of $\mathbb{R}^M$ with the set of strictly positive probability measures:

$$\mathbb{R}_+^M = \{\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_M) \mid \mathbb{Q}_j > 0, \, j = 1, \ldots, M\}$$

In particular, there exist measures $\mathbb{Q}^{(1)}, \ldots, \mathbb{Q}^{(r)} \in \overline{\mathbb{R}_M^+}$ such that every martingale measure $\mathbb{Q}$ can be expressed as a linear combination of the form:

$$\mathbb{Q} = a_1 \mathbb{Q}^{(1)} + \cdots + a_r \mathbb{Q}^{(r)},$$

in which the sum of the weights a_i is one. Consequently, we have:

$$\mathcal{V}_v = \left\{ V \mid \mathbb{E}^{\mathbb{Q}^{(j)}} \left[B_N^{-1} V\right] = v \text{ for } J = 1, \ldots, r \right\}.$$

Extra Slides 2: Incomplete model with terminal wealth (2/2)

Solving (**P2**):
Once the "extremal" measures $\mathbb{Q}^{(1)}, \ldots, \mathbb{Q}^{(r)}$ have been identified, the following result, which generalizes Theorem 2.18, solves the problem P2 of determining the optimal terminal reachable values $\tilde{V}$ when starting from an initial capital $v \in I$.
Theorem 2.21. Under the condition:

$$u'(I) = \mathbb{R}_+,$$

the optimal terminal value is:

$$\tilde{V}_N = \mathcal{I}\left(\sum_{j=1}^{r} \lambda_j \tilde{L}^{(j)}\right),$$

where $\tilde{L}^{(j)} = B_N^{-1} L^{(j)}$ with $L^{(j)} = \frac{d\mathbb{Q}^{(j)}}{d\mathbb{P}}$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ are determined from the system of budget equations:

$$\mathbb{E}^{\mathbb{P}}\left[\mathcal{I}\left(\sum_{k=1}^{r} \lambda_k \tilde{L}^{(k)}\right) \tilde{L}^{(j)}\right] = v, \quad j = 1, \ldots, r.$$

Extra Slides 3: Theorems

Theorem 2.12 (Pascucci & Runggaldie (2012))

For the problem:

$$\max_{\alpha} \mathbb{E}^{\mathbb{P}} \left[u \left(V_N^{(\alpha)} \right) \right],$$

there exists an optimal strategy if and only if the market is free of arbitrage.

Extra Slides 4: Utility Functions

Special case of hyperbolic absolute risk aversion: Isoelastic function for utility / power utility function / CRRA utility function is given by:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \gamma \geq 0, \\ \ln(c) & \text{if } \gamma = 1, \end{cases}$$

where c represents consumption and γ is the coefficient of relative risk aversion (CRRA).

Special cases:

- When $\gamma = 0$: This corresponds to risk neutrality, because utility is linear in c.
- When $\gamma = 1$: By virtue of l'Hôpital's rule, the limit of $u(c)$ is $\ln c$ as γ goes to 1:

  $$\lim_{\gamma \to 1} \frac{c^{1-\gamma} - 1}{1-\gamma} = \ln(c),$$

  which justifies the convention of using the limiting value $u(c) = \ln c$ when $\gamma = 1$. (Merton portfolio)/Growth optimal portfolio/growth rate maximizing investor $\iff$ that has log-utility.
- When $\gamma \to \infty$: This is the case of infinite risk aversion.

CRRA: This utility function has the feature of constant relative risk aversion. Mathematically this means that:

$$\underbrace{\text{RRA}}_{\gamma} = -\frac{cu''}{u'}$$