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CHAPMAN & HALL/CRC FINANCIAL MATHEMATICS SERIES

Introduction to Stochastic Calculus Applied to Finance

Second Edition (2008)

Damien Lamberton
Bernard Lapeyre

Chapman & Hall/CRC
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Chapter 2

Optimal stopping problem and American options

The purpose of this chapter is to address the pricing and hedging of American options and to establish the link between these questions and the optimal stopping problem. To do so, we will need to define the notion of stopping time, which will enable us to model exercise strategies for American options. We will also define the Snell envelope, which is the fundamental concept used to solve the optimal stopping problem. The application of these concepts to American options will be described in Section 2.5.

2.1 Stopping time

The buyer of an American option can exercise his or her right at any time until maturity. The decision to exercise or not at time n will be made according to the information available at time n. In a discrete-time model built on a finite filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{0\leq n\leq N}, \mathbb{P})$, the exercise date is described by a random variable called a stopping time.

Definition 2.1.1. A random variable ν taking values in $\{0,1,2,\ldots,N\}$ is a stopping time if, for any $n \in \{0,1,\cdots,N\}$,

$$\{\nu = n\} \in \mathscr{F}_n.$$

Remark 2.1.2. As in the previous chapter, we assume that $\mathscr{F} = \mathscr{P}(\Omega)$ and $\mathbb{P}(\{\omega\}) > 0, \forall \omega \in \Omega$. This hypothesis is nonetheless not essential: if it does not hold, the results presented in this chapter remain true almost surely. However, we will not assume $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_N = \mathscr{F}$, except in Section 2.5, dedicated to finance.

Remark 2.1.3. The reader can verify, as an exercise, that ν is a stopping time if and only if, for any $n \in 0, 1, \ldots, N$,

$$\{\nu \leq n\} \in \mathscr{F}_n.$$

We will use an equivalent definition to generalize the concept of stopping time to the continuous-time setting.

Let us introduce now the concept of a *sequence stopped at a stopping time*. Let $(X_n)_{0\leq n\leq N}$ be a sequence adapted to the filtration $(\mathscr{F}_n)_{0,\leq n\leq N}$ and let ν be a stopping time. The sequence stopped at time ν is defined as

$$X_n^\nu(\omega) = X_{\nu(\omega)\wedge n(\omega)},$$

i.e. on the set $\nu = j$ we have

$$X_n^\nu = \begin{cases} X_j \text{ if } j \leq n \\ X_n \text{ if } j > n. \end{cases}$$

Note that $X_N^\nu(\omega) = X_{\nu(\omega)}(\omega) (= X_j \text{ on } \{\nu = j\})$.

Proposition 2.1.4. *Let (X_n) be an adapted sequence and ν be a stopping time. The stopped sequence $(X_n^\nu)_{0\leq n\leq N}$ is adapted. Moreover, if (X_n) is a martingale (resp. a supermartingale), then (X_n^ν) is a martingale (resp. a supermartingale).*

Proof. We see that, for $n \geq 1$, we have

$$X_{\nu\wedge n} = X_0 + \sum_{j=1}^{n} \phi_j (X_j - X_{j-1}),$$

where $\phi_j = \mathbf{1}_{\{j\leq\nu\}}$. Since $\{j \leq \nu\}$ is the complement of the set $\{\nu < j\} = \{\nu \leq j-1\}$, the process $(\phi_n)_{0\leq n\leq N}$ is predictable.

It is clear then that $(X_{\nu\wedge n})_{0\leq n\leq N}$ is adapted to the filtration $(\mathscr{F}_n)_{0\leq n\leq N}$. Furthermore, if (X_n) is a martingale, $(X_{\nu\wedge n})$ is also a martingale with respect to $(\mathscr{F}_n)$, since it is the martingale transform of (X_n). Similarly, we can show that if the sequence (X_n) is a supermartingale (resp. a submartingale), the stopped sequence is still a supermartingale (resp. a submartingale) using the predictability and the non-negativity of $(\phi_j)_{0\leq j\leq N}$. □

2.2 The Snell envelope

In this section, we consider an adapted sequence $(Z_n)_{0\leq n\leq N}$ and define the sequence $(U_n)_{0\leq n\leq N}$ as follows:

$$\begin{cases} U_N = Z_N \\ U_n = \max(Z_n, \mathbb{E}(U_{n+1}|\mathscr{F}_n)), \quad n = 0, \ldots, N-1. \end{cases}$$

The study of this sequence is motivated by our first approach of American options (see Section 1.3.3). We already know, by Proposition 1.3.6, that $(U_n)_{0\leq n\leq N}$ is the smallest supermartingale that dominates the sequence $(Z_n)_{0\leq n\leq N}$. We call it the Snell envelope of the sequence $(Z_n)_{0\leq n\leq N}$.

By definition, U_n is greater than Z_n (with equality for $n = N$) and in the case of a strict inequality, $U_n = \mathbb{E}(U_{n+1}|\mathscr{F}_n)$. This suggests that, by stopping adequately the sequence (U_n), it is possible to obtain a martingale, as the following proposition shows.

Proposition 2.2.1. *The random variable defined by*

$$\nu_0 = \inf\{n \geq 0 \mid U_n = Z_n\} \tag{2.1}$$

is a stopping time and the stopped sequence $(U_{n\wedge\nu_0})_{0\leq n\leq N}$ is a martingale.

Proof. Since $U_N = Z_N$, ν_0 is a well-defined element of $\{0, 1, \ldots, N\}$ and we have

$$\{\nu_0 = 0\} = \{U_0 = Z_0\} \in \mathscr{F}_0,$$

and, for $k \geq 1$,

$$\{\nu_0 = k\} = \{U_0 > Z_0\} \cap \cdots \cap \{U_{k-1} > Z_{k-1}\} \cap \{U_k = Z_k\} \in \mathscr{F}_k.$$

To demonstrate that $(U_n^{\nu_0})$ is a martingale, we write as in the proof of Proposition 2.1.4:

$$U_n^{\nu_0} = U_{n\wedge\nu_0} = U_0 + \sum_{j=1}^{n} \phi_j \Delta U_j,$$

where $\phi_j = \mathbf{1}_{\{\nu_0 \geq j\}}$. So that, for $n \in \{0, 1, \ldots, N-1\}$,

$$\begin{aligned} U_{n+1}^{\nu_0} - U_n^{\nu_0} &= \phi_{n+1}(U_{n+1} - U_n) \\ &= \mathbf{1}_{\{n+1\leq\nu_0\}}(U_{n+1} - U_n). \end{aligned}$$

By definition, $U_n = \max(Z_n, \mathbb{E}(U_{n+1}|\mathscr{F}_n))$ and on the set $\{n+1 \leq \nu_0\}, U_n > Z_n$. Consequently

$$U_n = \mathbb{E}(U_{n+1}|\mathscr{F}_n)$$

and we deduce

$$U_{n+1}^{\nu_0} - U_n^{\nu_0} = \mathbf{1}_{\{n+1\leq\nu_0\}}(U_{n+1} - \mathbb{E}(U_{n+1}|F_n))$$

and taking the conditional expectation on both sides of the equality

$$\mathbb{E}((U_{n+1}^{\nu_0} - U_n^{\nu_0})|\mathscr{F}_n) = \mathbf{1}_{\{n+1\leq\nu_0\}}\mathbb{E}((U_{n+1} - \mathbb{E}(U_{n+1}|\mathscr{F}_n))|\mathscr{F}_n)$$

because $\{n+1 \leq \nu_0\} \in \mathscr{F}_n$ (since the complement of $\{n+1 \leq \nu_0\}$ is $\{\nu_0 \leq n\}$). Hence

$$\mathbb{E}((U_{n+1}^{\nu_0} - U_n^{\nu_0})|\mathscr{F}_n) = 0$$

which proves that U^{ν_0} is a martingale. $\square$

In the remainder, we shall denote by $\mathscr{T}_{n,N}$ the set of stopping times taking values in $\{n, n+1, \dots, N\}$. Notice that $\mathscr{T}_{n,N}$ is a finite set since Ω is assumed to be finite. The martingale property of the sequence U^{ν_0} gives the following result, which relates the concept of Snell envelope to the optimal stopping problem.

Corollary 2.2.2. *The stopping time ν_0 satisfies*

$$U_0 = \mathbb{E}(Z_{\nu_0}|\mathscr{F}_0) = \sup_{\nu \in \mathscr{T}_{0,N}} \mathbb{E}(Z_\nu|\mathscr{F}_0).$$

If we think of Z_n as the total winnings of a gambler after n games, we see that stopping at time ν_0 maximises the expected gain given $\mathscr{F}_0$.

Proof. Since U^{ν_0} is a martingale, we have

$$U_0 = U_0^{\nu_0} = \mathbb{E}(U_N^{\nu_0}|\mathscr{F}_0) = \mathbb{E}(U_{\nu_0}|\mathscr{F}_0) = \mathbb{E}(Z_{\nu_0}|\mathscr{F}_0).$$

On the other hand, if $\nu \in \mathscr{T}_{0,N}$, the stopped sequence U^ν is a supermartingale, so that

$$\begin{aligned} U_0 &\geq \mathbb{E}(U_N^\nu|\mathscr{F}_0) = \mathbb{E}(U_\nu|\mathscr{F}_0) \\ &\geq \mathbb{E}(Z_\nu|\mathscr{F}_0), \end{aligned}$$

which yields the result. $\square$

Remark 2.2.3. An immediate generalization of Corollary 2.2.2 gives

$$\begin{aligned} U_n &= \sup_{\nu \in \mathscr{T}_{n,N}} \mathbb{E}(Z_\nu|\mathscr{F}_n) \\ &= \mathbb{E}(Z_{\nu_n}|\mathscr{F}_n), \end{aligned}$$

where $\nu_n = \inf\{j \geq n | U_j = Z_j\}$.

Definition 2.2.4. A stopping time ν^* is called optimal for the sequence $(Z_n)_{0 \leq n \leq N}$ if

$$\mathbb{E}(Z_{\nu^*}|\mathscr{F}_0) = \sup_{\nu \in \mathscr{T}_{0,N}} \mathbb{E}(Z_\nu|\mathscr{F}_0).$$

We can see that ν_0 is optimal. The following result gives a characterization of optimal stopping times that shows that ν_0 is the smallest optimal stopping time.

Theorem 2.2.5. *A stopping time ν is optimal if and only if*

$$\begin{cases} Z_\nu = U_\nu \\ \textit{and } (U_{\nu \wedge n})_{0 \leq n \leq N} \textit{ is a martingale.} \end{cases} \tag{2.2}$$

Proof. If the stopped sequence U^ν is a martingale, $U_0 = \mathbb{E}(U_\nu|\mathscr{F}_0)$ and consequently, if (2.2) holds, $U_0 = \mathbb{E}(Z_\nu|\mathscr{F}_0)$. Optimality of ν is then ensured by Corollary 2.2.2.

Conversely, if ν is optimal, we have

$$U_0 = \mathbb{E}(Z_\nu|\mathscr{F}_0) \leq \mathbb{E}(U_\nu|\mathscr{F}_0).$$

But, since U^ν is a supermatingale,

$$\mathbb{E}(U_\nu|\mathscr{F}_0) \leq U_0.$$

Therefore

$$\mathbb{E}(U_\nu|\mathscr{F}_0) = \mathbb{E}(Z_\nu|\mathscr{F}_0)$$

and since $U_\nu \geq Z_\nu, U_\nu = Z_\nu$.

Since $\mathbb{E}(U_\nu|\mathscr{F}_0) = U_0$ and from the inequalities

$$U_0 \geq \mathbb{E}(U_{\nu\wedge n}|\mathscr{F}_0) \geq \mathbb{E}(U_\nu|\mathscr{F}_0)$$

(based on the supermartingale property of (U_n^ν)) we get

$$\mathbb{E}(U_{\nu\wedge n}|\mathscr{F}_0) = \mathbb{E}(U_\nu|\mathscr{F}_0) = \mathbb{E}(\mathbb{E}(U_\nu|\mathscr{F}_n)|\mathscr{F}_0).$$

But we have $U_{\nu\wedge n} \geq \mathbb{E}(U_\nu|\mathscr{F}_n)$, therefore $U_{\nu\wedge n} = \mathbb{E}(U_\nu|\mathscr{F}_n)$, which proves that (U_n^ν) is a martingale. $\square$

2.3 Decomposition of supermartingales

The following decomposition (commonly called the 'Doob decomposition') is used in viable complete market models to associate any supermartingale with a trading strategy for which consumption is allowed (see Exercise 8 for that matter).

Proposition 2.3.1. *Every supermartingale $(U_n)_{0 \leq n \leq N}$ has the unique following decomposition:*

$$U_n = M_n - A_n,$$

where (M_n) is a martingale and (A_n) is a non-decreasing, predictable process, null at 0.

Proof. It is clearly seen that the only solution for $n = 0$ is $M_0 = U_0$ and $A_0 = 0$. Then we must have

$$U_{n+1} - U_n = M_{n+1} - M_n - (A_{n+1} - A_n),$$

so that, conditioning both sides with respect to $\mathscr{F}_n$ and using the properties of M and A,

$$-(A_{n+1} - A_n) = \mathbb{E}(U_{n+1}|\mathscr{F}_n) - U_n$$

and

$$M_{n+1} - M_n = U_{n+1} - \mathbb{E}(U_{n+1}|\mathscr{F}_n).$$

(M_n) and (A_n) are entirely determined using the previous equations and we see that (M_n) is a martingale and that (A_n) is predictable and non-decreasing (because (U_n) is a supermartingale).

□

Suppose now that (U_n) is the Snell envelope of an adapted sequence (Z_n). We can then give a characterization of the largest optimal stopping time for (Z_n) using the non-decreasing process (A_n) of the Doob decomposition of (U_n):

Proposition 2.3.2. *The largest optimal stopping time for (Z_n) is given by*

$$\nu_{\max} = \begin{cases} N & if A_N = 0 \\ \inf\{n, A_{n+1} \neq 0\} & if A_N \neq 0. \end{cases}$$

Proof. It is straightforward to see that $\nu_{\max}$ is a stopping time using the fact that $(A_n)_{0 \le n \le N}$ is predictable. From $U_n = M_n - A_n$ and because $A_j = 0$, for $j \le \nu_{\max}$, we deduce that $U^{\nu_{\max}} = M^{\nu_{\max}}$ and conclude that $U^{\nu_{\max}}$ is a martingale. To show the opimality of $\nu_{\max}$, it is sufficient to prove

$$U_{\nu_{\max}} = Z_{\nu_{\max}}.$$

We note that

$$\begin{aligned} U_{\nu_{\max}} &= \sum_{j=0}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} U_j + \mathbf{1}_{\{\nu_{\max}=N\}} U_N \\ &= \sum_{j=0}^{N-1} \mathbf{1}_{\{\nu_{\max}=j\}} \max(Z_j, \mathbb{E}(U_{j+1}|\mathscr{F}_j)) + \mathbf{1}_{\{\nu_{\max}=N\}} Z_N, \end{aligned}$$

We have $\mathbb{E}(U_{j+1}|\mathscr{F}_j) = M_j - A_{j+1}$ and, on the set $\{\nu_{\max} = j\}, A_j = 0$ and $A_{j+1} > 0$, so $U_j = M_j$ and $\mathbb{E}(U_{j+1}|\mathscr{F}_j) = M_j - A_{j+1} < U_j$. It follows that $U_j = \max(Z_j, \mathbb{E}(U_{j+1}|\mathscr{F}_j)) = Z_j$, so that finally

$$U_{\nu_{\max}} = Z_{\nu_{\max}}.$$

It remains to show that it is the greatest optimal stopping time. If ν is a stopping time such that $\nu \ge \nu_{\max}$ and $\mathbb{P}(\nu > \nu_{\max}) > 0$, then

$$\mathbb{E}(U_\nu) = \mathbb{E}(M_\nu) - \mathbb{E}(A_\nu) = \mathbb{E}(U_0) - \mathbb{E}(A_\nu) < \mathbb{E}(U_0)$$

and U^ν cannot be a martingale, which establishes the claim. □

2.4 Snell envelope and Markov chains

The aim of this section is to compute Snell envelopes in a Markovian setting. A sequence $(X_n)_{n \ge 0}$ of random variables taking their values in a finite

set E is called a Markov chain if, for any integer $n \ge 1$ and any elements $x_0, x_1, \ldots, x_{n-1}, x, y$ of E, we have

$$\mathbb{P}(X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = \mathbb{P}(X_{n+1} = y \mid X_n = x).$$

The chain is said to be homogeneous if the value $P(x,y) = \mathbb{P}(X_{n+1} = y \mid X_n = x)$ does not depend on n. The matrix $P = (P(x,y))_{(x,y) \in E \times E}$, indexed by $E \times E$, is then called the transition matrix of the chain. The matrix P has non-negative entries and satisfies $\sum_{y \in E} P(x,y) = 1$ for all $x \in E$; it is said to be a stochastic matrix. On a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{0 \le n \le N}, \mathbb{P})$, we can define the notion of a Markov chain with respect to the filtration:

Definition 2.4.1. A sequence $(X_n)_{0 \le n \le N}$ of random variables taking values in E is a homogeneous Markov chain with respect to the filtration $(\mathscr{F}_n)_{0 \le n \le N}$, with transition matrix P, if (X_n) is adapted and if for any real-valued function f on E, we have

$$\mathbb{E}(f(X_{n+1})|\mathscr{F}_n) = Pf(X_n),$$

where Pf represents the function that maps $x \in E$ to

$$Pf(x) = \sum_{y \in E} P(x,y) f(y).$$

Note that, if one interprets real-valued functions on E as matrices with a single column indexed by E, then Pf is indeed the product of the two matrices P and f. It can also be easily seeen that a Markov chain, as defined at the beginning of the section, is a Markov chain with respect to its natural filtration, defined by $\mathscr{F}_n = \sigma(X_0, \ldots, X_n)$.

The following proposition is an immediate consequence of the latter definition and the definition of a Snell envelope. It is the basis for the effective computation of American option prices in discrete models (see Exercise 7).

Proposition 2.4.2. *Let (Z_n) be an adapted sequence defined by $Z_n = \psi(n, X_n)$, where (X_n) is a homogeneous Markov chain with transition matrix P, taking values in E, and ψ is a function from $\mathbb{N} \times E$ to $\mathbb{R}$. Then, the Snell envelope (U_n) of the sequence (Z_n) is given by $U_n = u(n, X_n)$, where the function u is defined by*

$$u(N,x) = \psi(N,x) \quad \forall x \in E$$

and, for $n \le N-1$,

$$u(n,\cdot) = \max(\psi(n,\cdot), Pu(n+1,\cdot)).$$

2.5 Application to American options

>From now on, we will work in a viable complete market. The modeling will be based on the filtered space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{0 \le n \le N}, \mathbb{P})$ and, as in Sections 1.3.1

and 1.3.3, we will denote by $\mathbb{P}^*$ the unique probability under which the discounted asset prices are martingales.

2.5.1 Hedging American options

In Section 1.3.3, we defined the value process (U_n) of an American option described by the sequence (Z_n) by the system

$$\begin{cases} U_N = Z_N \\ U_n = \max(Z_n, S_n^0 \mathbb{E}^*(U_{n+1}/S_{n+1}^0 | \mathscr{F}_n)) \quad \forall n \leq N-1. \end{cases}$$

Thus, the sequence $(\tilde{U}_n)$ defined by $\tilde{U}_n = U_n/S_n^0$ (discounted price of the option) is the Snell envelope, under $\mathbb{P}^*$, of the sequence $(\tilde{Z}_n)$. We deduce from the above Section 2.2 that

$$\tilde{U}_n = \sup_{\nu \in \mathscr{T}_{n,N}} \mathbb{E}^*(\tilde{Z}_\nu | \mathscr{F}_n)$$

and consequently

$$U_n = S_n^0 \sup_{\nu \in \mathscr{T}_{n,N}} \mathbb{E}^* \left(\frac{Z_\nu}{S_\nu^0} | \mathscr{F}_n \right).$$

>From Section 2.3, we can write

$$\tilde{U}_n = \tilde{M}_n - \tilde{A}_n,$$

where $\tilde{M}_n$ is a $\mathbb{P}^*$-martingale and $(\tilde{A}_n)$ is an increasing predictable process, null at 0. Since the market is complete, there is a self-financing strategy ϕ such that

$$V_N(\phi) = S_N^0 \tilde{M}_N,$$

i.e. $\tilde{V}_N(\phi) = \tilde{M}_N$. Since the sequence $(\tilde{V}_n(\phi))$ is a $\mathbb{P}^*$-martingale, we have

$$\begin{aligned} \tilde{V}_n(\phi) &= \mathbb{E}^*(\tilde{V}_N(\phi)|\mathscr{F}_n) \\ &= \mathbb{E}^*(\tilde{M}_N|\mathscr{F}_n) \\ &= \tilde{M}_n, \end{aligned}$$

and consequently

$$\tilde{U}_n = \tilde{V}_n(\phi) - \tilde{A}_n.$$

Therefore

$$U_n = V_n(\phi) - A_n,$$

where $A_n = S_n^0 \tilde{A}_n$. From the previous equality, it is obvious that the writer of the option can hedge himself perfectly: once he receives the premium $U_0 = V_0(\phi)$, he can generate a wealth equal to $V_n(\phi)$ at time n which is bigger than U_n and *a fortiori* Z_n.

What is the optimal date to exercise the option? The date of exercise is to be chosen among all the stopping times. For the buyer of the option, there is no point in exercising at time n when $U_n > Z_n$, because he would trade an

asset worth U_n (the option) for an amount (Z_n) (by exercising the option). Thus an optimal date τ of exercise is such that $U_\tau = Z_\tau$. On the other hand, there is no point in exercising after the time

$$\nu_{\max} = \inf\{j, A_{j+1} \neq 0\}$$

(which is equal to $\inf\{j, \tilde{A}_{j+1} \neq 0\}$) because, at that time, selling the option provides the holder with a wealth $U_{\nu_{\max}} = V_{\nu_{\max}}(\phi)$ and, following the strategy ϕ from that time, he creates a portfolio whose value is strictly bigger than the option's at times $\nu_{\max}+1, \nu_{\max}+2, \dots, N$. Therefore we set, as a second condition, $\tau \leq \nu_{\max}$, which allows us to say that U^τ is a martingale. As a result, optimal dates of exercise are optimal stopping times for the sequence $(\tilde{Z}_n)$, under probability $\mathbb{P}^*$. To make this point clear, let us consider the writer's point of view. If he hedges himself using the strategy ϕ as defined above, and if the buyer exercises at a non-optimal time τ, then $U_\tau > Z_\tau$ or $A_\tau > 0$. In both cases, the writer makes a profit $V_\tau(\phi) - Z_\tau = U_\tau + A_\tau - Z_\tau$, which is positive.

2.5.2 American options and European options

Proposition 2.5.1. *Let C_n be the value at time n of an American option described by an adapted sequence $(Z_n)_{0 \leq n \leq N}$ and let c_n be the value at time n of the European option defined by the $\mathscr{F}_N$-measurable random variable $h = Z_N$. Then, we have $C_n \geq c_n$.*

Moreover, if $c_n \geq Z_n$ for any n, then

$$c_n = C_n \quad \forall n \in \{0, 1, \dots, N\}.$$

The inequality $C_n \geq c_n$ makes sense since the American option entitles the holder to more rights than its European counterpart.

Proof. Since the discounted value $(\tilde{C}_n)$ is a supermartingale under $\mathbb{P}^*$, we have

$$\tilde{C}_n \geq \mathbb{E}^*(\tilde{C}_N|\mathscr{F}_n) = \mathbb{E}^*(\tilde{c}_N|\mathscr{F}_n) = \tilde{c}_n.$$

Hence $C_n \geq c_n$.

If $c_n \geq Z_n$ for any n, then the sequence $(\tilde{c}_n)$, which is a martingale under $\mathbb{P}^*$, appears to be a supermartingale (under $\mathbb{P}^*$) and an upper bound for the sequence $(\tilde{Z}_n)$ and consequently

$$\tilde{C}_n \leq \tilde{c}_n \quad \forall n \in \{0, 1, \dots, N\}.$$

□

Remark 2.5.2. One checks readily that if the relationships of Proposition 2.5.1 did not hold, there would be some arbitrage opportunities by trading the options.

To illustrate the last proposition, let us consider the case of a market with a single risky asset, with price S_n at time n and a constant riskless interest rate, equal to $r \geq 0$ on each period, so that $S_n^0 = (1+r)^n$. Then, with the notations of Proposition 2.5.1, if we take $Z_n = (S_n - K)_+$, c_n is the price at time n of a European call with maturity N and strike price K on one unit of the risky asset and C_n is the price of the corresponding American call. We have

$$\begin{aligned}\tilde{c}_n &= (1+r)^{-N}\mathbb{E}^*((S_N - K)_+ | \mathscr{F}_n)\\ &\geq \mathbb{E}^*(\tilde{S}_n - K(1+r)^{-N} | \mathscr{F}_n)\\ &= \tilde{S}_n - K(1+r)^{-N},\end{aligned}$$

using the martingale property of $(\tilde{S}_n)$. Hence, $c_n \geq S_n - K(1+r)^{-(N-n)} \geq S_n - K$, for $r \geq 0$. As $c_n \geq 0$, we also have $c_n \geq (S_n - K)_+$ and, by Proposition 2.5.1, $C_n = c_n$. There is equality between the price of the European call and the price of the corresponding American call.

This property does not hold for the put, nor in the case of calls on currencies or dividend paying stocks.

Notes: For further discussions on the Snell envelope and optimal stopping, one may consult Neveu (1972), Chapter VI, and Dacunha-Castelle and Duflo (1986a), Chapter 5, Section 1. For the theory of optimal stopping in continuous time, see El Karoui (1981), Shiryayev (1978) and Peskir and Shiryaev (2006).

2.6 Exercises

Exercise 4 Let ν be a stopping time with respect to a filtration $(\mathscr{F}_n)_{0 \leq n \leq N}$. We denote by $\mathscr{F}_\nu$ the set of events A such that $A \cap \{\nu = n\} \in \mathscr{F}_n$ for any $n \in \{0, \ldots, N\}$.

1. Show that $\mathscr{F}_\nu$ is a sub-σ-algebra of $\mathscr{F}_N$. $\mathscr{F}_\nu$ is often called 'σ-algebra of events determined prior to the stopping time ν'.
2. Show that the random variable ν is $\mathscr{F}_\nu$-measurable.
3. Let X be a real-valued random variable. Prove the equality
$$\mathbb{E}(X|\mathscr{F}_\nu) = \sum_{j=0}^{N} \mathbf{1}_{\{\nu=j\}}\mathbb{E}(X|\mathscr{F}_j).$$
4. Let τ be a stopping time such that $\tau \geq \nu$. Show that $\mathscr{F}_\nu \subset \mathscr{F}_\tau$.
5. Under the same hypothesis, show that if (M_n) is a martingale, we have
$$M_\nu = \mathbb{E}(M_\tau | \mathscr{F}_\nu).$$
(Hint: first consider the case $\tau = N$.)

Exercise 5 Let (U_n) be the Snell envelope of an adapted sequence (Z_n). Without assuming that $\mathscr{F}_0$ is trivial, show that

$$\mathbb{E}(U_0) = \sup_{\nu \in \mathscr{T}_{0,N}} \mathbb{E}(Z_\nu),$$

and more generally

$$\mathbb{E}(U_n) = \sup_{\nu \in \mathscr{T}_{n,N}} \mathbb{E}(Z_\nu).$$

Exercise 6 Show that ν is optimal according to Definition 2.2.4 if and only if

$$\mathbb{E}(Z_\nu) = \sup_{\tau \in \mathscr{T}_{0,N}} \mathbb{E}(Z_\tau).$$

Exercise 7 The purpose of this exercise is to study the American put in the model of Cox-Ross-Rubinstein. Notations are those of Chapter 1.

1. Show that the price $\mathscr{P}_n$, at time n, of an American put on a share with maturity N and strike price K can be writen as
$$\mathscr{P}_n = P_{am}(n, S_n),$$
where $P_{am}(n,x)$ is defined by $P_{am}(N,x) = (K-x)_+$ and, for $n \leq N-1$,
$$P_{am}(n,x) = \max\left((K-x)_+, \frac{f(n+1,x)}{1+r}\right),$$
with
$$f(n+1,x) = pP_{am}(n+1,xd) + (1-p)P_{am}(n+1,xu)$$
and $p = (u-1-r)/(u-d)$.
2. Show that the function $P_{am}(0,.)$ can be expressed as
$$P_{am}(0,x) = \sup_{\nu, \in \mathscr{T}_{0,N}} \mathbb{E}^*((1+r)^{\tau}(K - xV_\nu)_+),$$
where the sequence of random variables $(V_n)_{0 \leq n \leq N}$ is defined by $V_0 = 1$ and for $n \geq 1$, $V_n = \prod_{i=1}^n U_i$, where the U_i's are some random variables. Give their joint distribution under $\mathbb{P}^*$.
3. From the last formula, show that the function $x \mapsto P_{am}(0,x)$ is convex and non-increasing.
4. We assume $d < 1$. Show that there is a real number $x^* \in [0, K]$ such that, for $x \leq x^*$, $P_{am}(0,x) = (K-x)_+$ and, for $x^* < x < K/d^N$,
$$P_{am}(0,x) > (K-x)_+.$$

5. An agent holds the American put at time 0. For which values of the spot S_0 would he rather exercise his option immediately?

6. Show that the hedging strategy of the American put is determined by a quantity $H_n = \Delta(n, S_{n-1})$ of the risky asset to be held at time n, where Δ can be written as a function of P_{am}.

Exercise 8 Consumption strategies. The self-financing strategies defined in Chapter 1 ruled out any consumption. Consumption strategies can be introduced in the following way: at time n, once the new prices $S_n^0, \ldots, S_n^d$ are quoted, the investor readjusts his positions from ϕ_n to ϕ_{n+1} and selects the wealth γ_{n+1} to be consumed at time $n+1$. With any endowment being excluded and the new positions being decided given prices at time n, we deduce

$$\phi_{n+1}.S_n = \phi_n.S_n - \gamma_{n+1}. \tag{2.3}$$

So, a trading strategy with consumption will be defined as a pair (ϕ, γ), where ϕ is a predictable process taking values in $\mathbb{R}^{d+1}$, representing the numbers of assets held in the portfolio, and $\gamma = (\gamma_n)_{1 \leq n \leq N}$ is a predictable process taking values in $\mathbb{R}^+$, representing the wealth consumed at any time. Equation (2.3) gives the relationship between the processes ϕ and γ and replaces the self-financing condition of Chapter 1.

1. Let ϕ be a predictable process taking values in $\mathbb{R}^{d+1}$ and let γ be a predictable process taking values in $\mathbb{R}^+$. We set $V_n(\phi) = \phi_n.S_n$ and $\tilde{V}_n(\phi) = \phi_n.\tilde{S}_n$. Show the equivalence between the following conditions:

   (a) The pair (ϕ, γ) defines a trading strategy with consumption.

   (b) For any $n \in \{1, \ldots, N\}$,

   $$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j.\Delta S_j - \sum_{j=1}^{n} \gamma_j.$$

   (c) For any $n \in \{1, \ldots, N\}$,

   $$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j.\Delta \tilde{S}_j - \sum_{j=1}^{n} \gamma_j / S_{j-1}^0.$$

2. In the remainder, we assume that the market is viable and complete and we denote by $\mathbb{P}^*$ the unique probability under which the discounted asset prices are martingales. Show that if the pair (ϕ, γ) defines a trading strategy with consumption, then $(\tilde{V}_n(\phi))$ is a supermartingale under $\mathbb{P}^*$.

3. Let (U_n) be an adapted sequence such that $(\tilde{U}_n)$ is a supermartingale under $\mathbb{P}^*$. Using the Doob decomposition, show that there is a trading strategy with consumption (ϕ, γ) such that $V_n(\phi) = U_n$ for any $n \in \{0, \ldots, N\}$.

4. Let (Z_n) be an adapted sequence. We say that a trading strategy with consumption (ϕ, γ) hedges the American option defined by (Z_n) if $V_n(\phi) \geq Z_n$ for any $n \in \{0, 1, \ldots, N\}$. Show that there is at least one trading strategy with consumption that hedges (Z_n), whose value is precisely the value (U_n) of the American option. Also, prove that any trading strategy with consumption (ϕ, γ) hedging (Z_n) satisfies $V_n(\phi) \geq U_n$, for any $n \in \{0, 1, \ldots, N\}$.

5. Let x be a non-negative number representing the investor's endowment and let $\gamma = (\gamma_n)_{1 \leq n \leq N}$ be a predictable strategy taking values in $\mathbb{R}^+$. The consumption process (γ_n) is said to be budget-feasible from endowment x if there is a predictable process ϕ taking values in $\mathbb{R}^{d+1}$, such that the pair (ϕ, γ) defines a trading strategy with consumption satisfying $V_0(\phi) = x$ and $V_n(\phi) \geq 0$, for any $n \in \{0, \ldots, N\}$. Show that (γ_n) is budget-feasible from endowment x if and only if $\mathbb{E}^*(\sum_{j=1}^{N} \gamma_j / S_{j-1}^0) \leq x$.