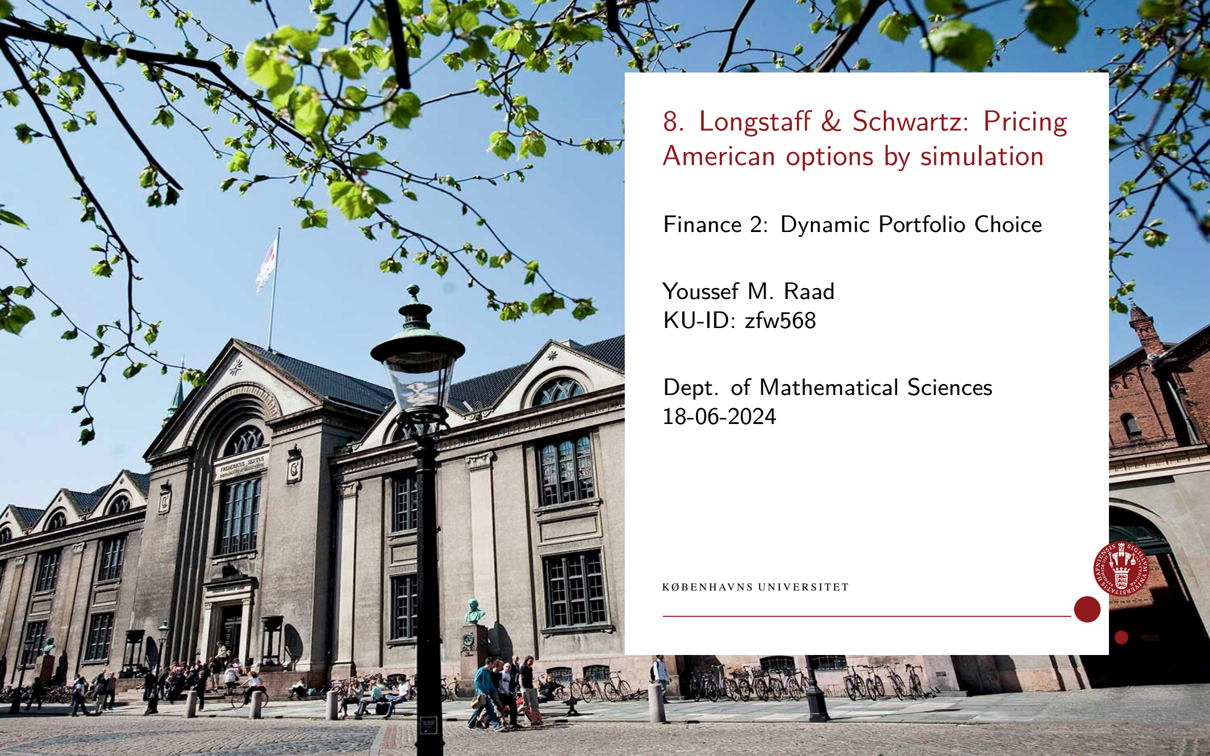

8. Longstaff & Schwartz: Pricing American options by simulation

Finance 2: Dynamic Portfolio Choice

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Overview

Model Setup and American Options (Longstaff and Schwartz 2001)

The holder of an American option has the right but not the obligation to exercise the option at any time until expiry-T. The price of an American option is given by the Snell Envelope of the payoff function $g(S_t)$:

$$\pi_t^{\text{AMR}} = \begin{cases} g(S_T) & t = T \\ \max\left\{ g(S_t), \frac{1}{1+r_t}\mathbb{E}_t^{\mathbb{Q}}\left[\pi_{t+1}^{\text{AMR}}\right]\right\} & 0 \leq t \leq T \end{cases}.$$

Problem: How do we find (Extra Slide 1 and 2 for how not to) $\mathbb{E}_t^{\mathbb{Q}}\left[\pi_{t+1}^{\text{AMR}}\right]$ and thus value the American option at time-0?

LSM algorithm provides pathwise approximation of optimal stopping rule by exploiting cross-sectional information in the simulated paths to estimate the conditional expectation function:

- Choose an initial stock price S_0 and simulate N paths with K steps (start at step t_{K-1}).
- Let X be the vector of stock prices for paths which the option is ITM (increased efficiency).
- Let Y be the discounted realised value of continuation to t_K in these paths, i.e. discounted cash flows received at next time point if option lives.
- Perform least squares regression on Y using M basis functions on X.
- For each path calculate the regression value and use as estimate for the conditional expectation.
- Exercise immediately if $g(S_{t_{(K-1)}}) \geq \frac{1}{1+r_{t_{K-1}}}\mathbb{E}_{t_{K-1}}^{\mathbb{Q}}\left[\pi_{t_{(K-1)}+1}^{\text{AMR}}\right]$ (intrinsic value $\geq$ continuation value), otherwise let the option live for another time step.
- American option price estimated by averaging discounted payoffs from earlier decisions.

Simple Numerical Example by Simulation (1/4)

Consider an American put option i.e $g(S_t) = (K - S_t)^+$, where $K = 1.10$, $r = 0.06$, $S_0 = 1$, and $\delta \equiv 0$.

Discretization of time $\Rightarrow$ $\underbrace{\text{Bermudan option}}_{\text{predetermined } t\text{'s}}$ $\Rightarrow$ Approximately an American option with K large enough.

For simplicity, consider eight sample paths generated/simulated under the risk-neutral measure for $t \in \{0, 1, 2, 3\}$ (Extra Slides 10 and 11 for `Python` figures):

Stock price paths

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

Cash-flow matrix at time 3

Path	$t=1$	$t=2$	$t=3$
1	—	—	.00
2	—	—	.00
3	—	—	.07
4	—	—	.18
5	—	—	.00
6	—	—	.20
7	—	—	.09
8	—	—	.00

Regression at time 2

Path	Y	X
1	.00 × .94176	1.08
2	—	—
3	.07 × .94176	1.07
4	.18 × .94176	.97
5	—	—
6	.20 × .94176	.77
7	.09 × .94176	.84
8	—	—

$t = 3$: Essentially European put.

$t = 2$: Exercise if $g(S_{t_{(K-1)}}) \geq \frac{1}{1+r_{t_{K-1}}} \mathbb{E}^{\mathbb{Q}}_{t_{K-1}} \left[\pi^{\text{AMR}}_{t_{(K-1)}+1} \right]$.

So: X: Stock prices at time-2 for ITM paths and Y: Discounted cash flows of ITM paths received at time-3 if the put is not exercised at time-2 (next page).

Simple Numerical Example by Simulation (2/4)

Estimate expected cashflow from continuation: regress Y on a constant, X and X^2 (Extra Slide 2). For $t = 2 \to$ Conditional expectation function:

$$\mathbb{E}[Y \mid X] = -1.070 + 2.983X - 1.813X^2.$$

Stock price paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

Optimal early exercise decision at time 2

Path	Exercise	Continuation
1	.02	.0369
2	—	—
3	.03	.0461
4	.13	.1176
5	—	—
6	.33	.1520
7	.26	.1565
8	—	—

Cash-flow matrix at time 2

Path	$t = 1$	$t = 2$	$t = 3$
1	—	.00	.00
2	—	.00	.00
3	—	.00	.07
4	—	.13	.00
5	—	.00	.00
6	—	.33	.00
7	—	.26	.00
8	—	.00	.00

- $g(S_{t_{(3-1)}}) =$ Strike $- S_{t_{(3-1)}}$, i.e path 1 has $t = 2$ price $S_{t_{(3-1)}} = 1.08$:

$$\text{Exercise: } 1.1 - 1.08 = 0.02.$$

- $\frac{1}{1+r_{t_{3-1}}}\mathbb{E}^{\mathbb{Q}}_{t_{K-1}}\left[\pi^{\text{AMR}}_{t_{(3-1)}+1}\right] = X$ into $\mathbb{E}[Y \mid X]$, i.e path 1 has $t = 2$ price $S_{t_{(3-1)}} = 1.08$:

$$\text{Continuation: } -1.070 + 2.983 \cdot 1.08 - 1.813 \cdot (1.08)^2 = 0.0369.$$

- Cash-flow: If $g(S_{t_{(3-1)}}) \geq \frac{1}{1+r_{t_{3-1}}}\mathbb{E}^{\mathbb{Q}}_{t_{K-1}}\left[\pi^{\text{AMR}}_{t_{(3-1)}+1}\right] \Rightarrow$ gain $g(S_{t_{(3-1)}})$ else .00, i.e path 1: 00. If exercised $t = 3$ becomes 00.

Simple Numerical Example by Simulation (3/4)

$t = 1$: Repeat for ITM paths: X as path stock prices at time $t = 1$ and Y as discounted payoff for $t = 2$.
For $t = 1 \to$ Conditional expectation function:

$$\mathbb{E}[Y \mid X] = 2.038 - 3.335X + 1.356X^2.$$

Stock price paths

Path	$t=0$	$t=1$	$t=2$	$t=3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

Regression at time 1

Path	Y	X
1	.00 × .94176	1.09
2	—	—
3	—	—
4	.13 × .94176	.93
5	—	—
6	.33 × .94176	.76
7	.26 × .94176	.92
8	.00 × .94176	.88

Optimal early exercise decision at time 1

Path	Exercise	Continuation
1	.01	.0139
2	—	—
3	—	—
4	.17	.1092
5	—	—
6	.34	.2866
7	.18	.1175
8	.22	.1533

Note: For Y at $t = 1$: Use actual realized cash flows along each path; not the conditional expected value of Y regression estimated at $t = 2$ (i.e for $t = 1$ use ITM paths from realized cash flow

in $t = 2$ table) (to avoid upward bias (Longstaff and Schwartz 2001, p. 118)).

Substituting the values of X into this regression gives the estimated conditional expectation function (continuation column).

Note: Future cash flows occur only once, i.e. at time 2 or at time 3. Cash flows received at time 2 (resp. time 3) are discounted back one period (resp. two periods) to time 1 (here only $t = 2$).

Simple Numerical Example by Simulation

This yields cash flow matrix and optimal stopping rule scheme/strategy (1 yields a stop, 0 continue to ride):

Stopping rule

Path	$t=1$	$t=2$	$t=3$
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

Option cash flow matrix

Path	$t=1$	$t=2$	$t=3$
1	.00	.00	.00
2	.00	.00	.00
3	.00	.00	.07
4	.17	.00	.00
5	.00	.00	.00
6	.34	.00	.00
7	.18	.00	.00
8	.22	.00	.00

Discounting by $\eta := \mathrm{e}^{-0.06} = 0.94176$ back to time 0 and averaging over all paths yields the value of the American option at $t = 0$ (note stopping first column: 1 discount, last coulmn: 3 discounts):

$$\pi_0^{\mathrm{AMR}} = \frac{1}{8} \times \left(0 + 0 + \eta^3 \times 0.07 + \eta \times 0.17 + 0 + \eta \times 0.34 + \eta \times 0.18 + \eta \times 0.22\right) = 0.1144,$$

$$\pi_0^{\mathrm{EU}} = \frac{1}{8} \times \left(\underbrace{0 + 0 + \eta^3 \times 0.07 + \eta^3 \times 0.18 + 0 + \eta^3 \times 0.20 + \eta^3 \times 0.09 + 0}_{t=3 \text{ cash flows discounted}}\right) = 0.0564.$$

Note: $\pi_0^{\mathrm{AMR}} = 0.1144 > 0.1 = (S_0 - K)$, i.e. at time 0 the option is worth more alive than dead!

Convergence Results - Upper limit (Longstaff and Schwartz 2001, p. 124) (1/2)

Proposition 1 (Longstaff and Schwartz 2001)

Let $V(X)$ define the true value of the American option. For any finite choice of basis functions, M, and time points, K, $V(X)$ has the following property, almost surely

$$V(X) \overset{\text{a.s.}}{\geq} \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} LSM(\omega_i; M, K),$$

where $LSM(\omega_i; M, K)$ denotes the discounted cash flow resulting from following the LSM rule.

Intuition:

- Upper limit: $V(X)$ based on stopping scheme that maximizes the discounted expected payoff; all other stopping rules (including the one implied by LSM) result in values less than or equal to that implied by the optimal stopping rule.
- Rule of Thumb: Increase the number of basis functions, M, until the value implied by LSM algorithm no longer increases.

Difficult with general convergence rule (discretization points, basis functions, paths among plenty named in (Longstaff and Schwartz 2001, p. 124)).

Convergence Results - Nearing True Value (Longstaff and Schwartz 2001, p. 125) (2/2)

Proposition 2 (Longstaff and Schwartz 2001)

Consider a single state variable X with support on $(0, \infty)$, which follows a Markov process, and an American option dependent of X **which can only be exercised at times $\mathbf{t_1}$ and $\mathbf{t_2}$**. Then $\forall \epsilon > 0$, $\exists M < \infty$ such that

$$\lim_{N \to \infty} P\left[\left| V(X) - \frac{1}{N} \sum_{i=1}^{N} LSM(\omega_i; M, K) \right| > \epsilon \right] = 0.$$

Intuition:

- Choosing a large enough number of basis functions M (need not be ∞ as $N \to \infty$!), then LSM algorithm results in a value for the American option within ε of the true value!

Similar results can be obtained for higher-dimensional problems...

In practice convergence works, but in theory it is a little bit iffy and fluffy.

References

Lind, Peter Pommergård and Rolf Poulsen (2024). "Delta Least Squares Monte Carlo Pricing of American Options". In: *Available at SSRN 4788612.*

Longstaff, Francis A and Eduardo S Schwartz (2001). "Valuing American options by simulation: a simple least-squares approach". In: *The review of financial studies* 14.1, pp. 113–147.

Poulsen, Rolf (2017). "American π: Piece of Cake?" In: *Willmot Magazine*, pp. 12–13.

Extra Slide 1: How Not to Price an American Put Option

Suppose we have simulated some stock price paths $S_{t,i}$.

Simulating prices of the American put as:

$$\text{Put}_{t,i}^{\text{AMR}} = \max\left\{\left(K - S_{t,i}\right)^{+}, \mathrm{e}^{-r}\text{Put}_{t+1,i}^{\text{AMR}}\right\}.$$

- $\text{Put}_{t+1,i}^{\text{AMR}}$ is not a conditional expectation, but a simulated value!
- Cannot "cheat" by looking into the future as we do it every period - we are not oracles.
- These types of errors can be quite difficult to spot.

Fix: Longstaff & Schwartz!

- Trick: Regress across paths.

Extra Slide 2: Example: How to Get it Wrong (Poulsen 2017, p. 13)

```
UpMoves<-cumsum((runif(n) < q))
S[2:(n+1)]<-S0*u^UpMoves*d^((1:n)-UpMoves)
SnellZ<-gS<-pmax(strike-S,0)
for (i in n:1) SnellZ[i]<-
max(gS[i],SnellZ[i+1]/(1+r))
tau.index<-which.min(time.vector[SnellZ==gS])
DiscPayoff<-(1/(1+r))^((tau.index-1))*gS[tau.
index]
```

- ($\mathbb{Q}$)-Conditional expectation forgotten.
- `Snellz` becomes the pathwise remaining maximum of the intrinsic value.
- Effectively exercising our option with perfect knowledge of the future.

These types of errors can be rather tricky to spot - while the resulting price is quite a bit off, it is not *obviously* nonsensical;

- A one-year at-the-money American put option with $r = 0.03$ and $\sigma = 0.2$ should cost 6.74% of the underlying's price.
- The code above (with the same parameters) yields a price of 13.8%.

Extra Slide 3 (1/2): How Do We Regress? (Lind and Poulsen 2024)

$$\text{Put}_{t,i}^{\text{AMR}} = \max\left\{ \left(K - S_{t,i}\right)^+, e^{-r}\mathbb{E}_t^{\mathbb{Q}}\left[\text{Put}_{t+1,i}^{\text{AMR}}\right]\right\}.$$

Longstaff & Schwartz estimate conditional expectations via regression.

- Let X be the vector of stock (path) prices for which the option is in-the-money.
- Let Y be the discounted realised value of continuation to t_K in these paths, i.e. discounted cash flows received at the next time point if option lives.

We regress as follows for e.g. $t = 2$

$$0.00 \times 0.94176 = Y_1 = a + b \cdot 1.08 + c \cdot 1.08^2 + \text{noise}_1,$$

$$0.07 \times 0.94176 = Y_3 = a + b \cdot 1.07 + c \cdot 1.07^2 + \text{noise}_3,$$

$$\vdots$$

$$Y = Z\beta, \quad \hat{\beta}^{\text{OLS}} = \left(Z^\top Z\right)^{-1} Z^\top Y, \quad Z = \begin{bmatrix} 1 & X & X^2 \\ 1 & X & X^2 \\ \vdots & \vdots & \vdots \\ 1 & X & X^2 \end{bmatrix} \in \mathbb{R}^{8\times 3}.$$

Extra Slide 3 (2/2): How Do We Regress? (Lind and Poulsen 2024)

A At the time of decision we only know the intrinsic value and some estimated continuation value (expectation) - if we knew the whole path it might have been better to exercise at some other time point than what we actually decide to do (e.g. see path 1 at $t = 2$).

By following the optimal strategy τ^*, the American option price is

$$\pi_0^{\text{AMR}} = \mathbb{E}_0^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau^*} g(S_{\tau^*}) \right].$$

The LSM algorithm works backwards since the path of cash flows generated by the option is defined recursively. The cash flow at time t_k might differ from that at time t_{k+1} since it may be optimal to stop at time t_{k+1}, thereby changing all subsequent cash flows along a realized path ω.

Extra Slide 4: All Kinds of Information

- Objective of the LSM algorithm is to provide a pathwise approximation to the optimal stopping rule that maximizes the value of the American option.
- Assume that the American option can only be exercised at the K discrete times $0 < t_1 \leq t_2 \leq \cdots \leq t_k = T$, and consider the optimal stopping policy at each exercise date $\Rightarrow$ For K large enough, LSM algorithm as approximation to the value of continuously exercisable American options.
- Holder chooses at every period (prior to date of expiry) to exercise immediately or to let the option live for another period/day $\Rightarrow$ Value of the option maximized pathwise, and hence unconditionally, if the investor exercises as soon at the immediate exercise value if greater than or equal to the value of continuation.
- Use only in-the-money paths in the estimation since the exercise decision is only relevant when the option is ITM. This limits the region over which the conditional expectation must be estimated, and far fewer basis functions are needed to obtain an accurate approximation to the conditional expectation function $\Rightarrow$ Fewer errors and faster computationally!

Extra Slide 5: All Kinds of Information

- At time t_k, the cash flow from immediate exercise is known, but the cash flow from continuation is not $\Rightarrow$ Problem of optimal exercise reduces to comparing the immediate exercise with the conditional expectation representing the continuation value, and then exercise as soon as the immediate exercise value is positive and greater than or equal to the continuation value.
- Start at time t_{K-1} and once the exercise decision is identified, the option cash flow paths can be approximated. The recursion proceeds by rolling back to time t_{K-2} and repeating the procedure until the exercise decisions at each exercise time along each path have been determined.
- American option valued by starting at time zero, moving forward along each path until the first stopping time occurs, discounting the resulting cash flow from exercise back to time zero, and then taking average over all paths ω.
- Number of basis functions needed to obtain convergence appears to grow much more slowly than exponentially. Experience suggests that the number of basis functions necessary to approximate the conditional expectation function may be very manageable even for high-dimensional problems.

Extra Slide 6: Four Steps as per Rolf

Structurally, the LSM method can be split up into four parts:

1. Simulation of paths.
2. Backward recursion to estimate regression coefficients (i.e. parameters minimizing some loss function least squares) determining the optimal exercise decision at each time point.
3. Forward pass: Simulate paths as in (but independently of) step 1 and use the regression coefficients from step 2 to decide when to exercise the American option hopefully close to optimally.
4. Valuation: Average over the discounted payoffs from the decisions in step 3 to estimate the American option price.

Strictly speaking assuming that our American option is Bermudan, i.e. that it can only be exercised on a discrete, finite set of time points $t_i \in \{t_0 := 0, t_1, ..., t_{N-1}, t_N := T\}$.

Extra Slide 7: Special Case: $\text{Call}^{\text{AMR}} = \text{Call}^{\text{EUR}}$ (Poulsen 2017)

Pricing a AMR Call Option w/ $r > 0$ and $\delta \equiv 0$

Whenever there is a strictly positive interest rate (i.e. $r > 0$) and zero dividends (i.e. $\delta \equiv 0$), then it is optimal to hold the American option until time of expiry T.

Consequently, in this case, the price of an American call option equals that of the European call option.

Proof: Suppose we are standing at some arbitrary time $t < T$ and that $S_t > K$, i.e. that the option is in-the-money, otherwise we would never exercise.

Then:

$$\text{Call}_t^{\text{AMR}} \geq \text{Call}_t^{\text{EUR}} \geq \text{Call}_t^{\text{EUR}} - \underbrace{\text{Put}_t^{\text{EUR}}}_{\geq 0} \underset{\text{P/C-parity}}{=} S_t - \underbrace{e^{-r[T-t]}}_{<1} K > S_t - K = g(S_t),$$

where $g(S_t)$ is the value of exercising the option today.

This means that for $t < T$, the American call option is always worth more alive than dead, i.e. we should not exercise until expiry!

Dual result: $r = 0$, $\delta > 0 \Rightarrow$ Optimal to hold AMR put until expiry.

Extra Slide 8: Delta LSM: Delta Regularization - An Add-On (Lind and Poulsen 2024)

Extends the classical LSM method by also using the derivative of the discounted cash flow to find the optimal stopping strategy.

The additional complication and computational cost amounts to adding two matrices, which is a relatively small cost since the heavy computation in least-squares lies in the matrix inversion. In return, the Delta LSM method delivers - robustly across market scenarios - better results $\rightarrow$ faster and more safety with a lower variance and without introducing any bias!

The Delta LSM method provides an extension to the backwards recursion step by exploiting the Delta of the continuation value of the American option, i.e. the continuation value's sensitivity w.r.t. the underlying.

As described above, computing the continuation value poses a numerical challenge as it involves the expectation of the option value at a future time conditioned on today's information set - specifically, we must calculate the $\mathbb{Q}$-expected discounted payoff from exercising optimally in the future. We do this by the least squares method, where we use the future payoffs as response variables.

In a perfect world, where we know the true values of the estimated parameters, then we also know the Delta (i.e. the derivative of the continuation value w.r.t. the underlying).

By some algebra (Lind and Poulsen 2024, p. 7-8), we see that the Delta regularization method associated with Delta LSM is an add-on to LSM. However, the usefulness of the regularization approach hinges critically on whether the trick used is computationally feasible (=read fast enough).

Nevertheless, if this is doable, then the Delta LSM method is preferable as it does not add any bias to the estimator but reduces variance, i.e. we get a better estimate of the conditional expectation without having to qualify "better" by some bias-variance-trade-off as is usual for regularization methods.

Extra Slide 9: Forward Pass (Lind and Poulsen 2024)

In the backward induction step, we use discounted future payoffs as the response variable Y to estimate the continuation value. If we use the backward paths themselves to estimate the American option price, then this will create a positive bias, i.e. overvaluation of the American option.

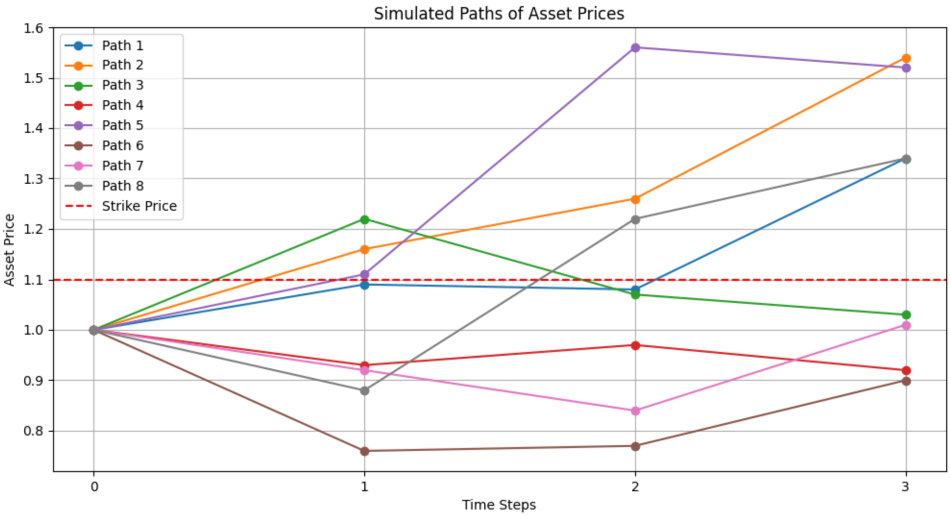
Larger sample for backwards induction $\Rightarrow$ Smaller bias!

Not possible to say what the magnitude of this bias is for a given sample size (e.g. $10,000$ paths).

Simple and computationally cheap fix: Out-of-sample experiment or forward pass:

- Simulate K new independent sample of paths.
- For the k'th path, march forward in time to decide the optimal exercise point, that is the first time $g(S_t) \geq \frac{1}{1+r_t}\mathbb{E}_t^{\mathbb{Q}}[\pi_{t+1}^{\text{AMR}}]$ (and 0 at T if this never happens), which is possible because we have estimated the conditional expectations for the continuation values from the in-sample backward induction.
- No matrix inversion $\Rightarrow$ Computationally cheaper to have a large number of paths in the forward pass than having the same number of paths as in the backward induction.

Extra Slide 10: Asset Prices Using LSM



Extra Slide 11: Cash-Flows Using LSM