6. The Martingale Method

Finance 2: Dynamic Portfolio Choice

Youssef M. Raad
KU-ID: zfw568

Dept. of Mathematical Sciences
18-06-2024

KØBENHAVNS UNIVERSITET

Overview

Model Setup (1/2) (Pascucci & Runggaldier, 2012)

- Time is discrete and finite with time points $n = 0, ..., N$. We let:

$$t_0 < t_1 < \ldots < t_N,$$

represent the trading dates: $t_0 = 0$ today and $t_N = N$ the expiry date.

- Finite filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^{N}, \mathbb{P}\right)$, i.e finite state space $\Omega = \{\omega_1, ..., \omega_M\}$ (paths).

- d risky assets, $S^1, \ldots, S^d$. If S_n^i denotes the price at time t_n of the i-th asset:

$$\begin{cases} S_0^i \in \mathbb{R}_+, \\ S_n^i = S_{n-1}^i \left(1 + \mu_n^i\right), \quad n = 1, \ldots, N, \end{cases}$$

where μ_n^i is a real random variable that represents the rate of return of the i-th asset in the n-th period $[t_{n-1}, t_n]$.

- One risk-free asset (bank/money account) B where:

$$\begin{cases} B_0 = 1, \\ B_n = B_{n-1}(1 + r_n^f) \quad , n = 1, \ldots, N, \end{cases}$$

where $r_n^f > -1$ denotes the risk-free rate in the n-th period $[t_{n-1}, t_n]$.

Model Setup (2/2) (Pascucci & Runggaldier, 2012)

- Utility function of class C^1:

$$u : I \to \mathbb{R}, \quad I =]a, \infty[, \quad \text{where } a \leq 0,$$

strictly increasing, strictly concave.

Define (without consumption):

- Value of a trading strategy $(\alpha, \beta) = (\alpha_n^1, \ldots, \alpha_n^d, \beta_n)_{n=1,\ldots,N}$ at time n:

$$V_n^{(\alpha,\beta)} = \alpha_n S_n + \beta_n B_n,$$

where α_n^i (resp., β_n) represents the amount of asset S_i (resp., of bond) kept in the portfolio during the n-th period.

$((\alpha_n, \beta_n)$ composition at $t_{n-1})$.

- The trading strategy (α, β) is self-financing if:

$$V_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1}$$

- Proportions of the total value invested in asset i through recursive relation:

$$\pi_n^i = \frac{\alpha_n^i S_n^i}{V_{n-1}^i} : \quad V_n \underbrace{=}_{\text{prop. 1.9 P\&R}} V_{n-1} \left(1 + r_n + \sum_{i=1}^{d} \pi_n^i (\mu_n^i - r_n) \right)$$

The Dynamic Programming Method (Pascucci & Runggaldier, 2012)

Consider the stochastic process $(V_n)_{n=0,\dots,N}$ (think value of port.), and suppose it holds recursively that (V_n) depends on the choice of a control process and something stochastic i.e:

$$V_k = G_k\left(\underbrace{V_{k-1}, \mu_k}_{\text{stochastic}}; \underbrace{\eta_{k-1}\left(V_{k-1}\right)}_{\text{control}}\right), \quad k = 1, \dots, N, \tag{1}$$

where:

- $\mu_1, \dots, \mu_N$: d-dim independent stochastic variables (risk factors that drive the dynamics of the asset prices),
- $\eta_0, \dots, \eta_N$: controls given as $\eta_k : \mathbb{R} \to \mathbb{R}^\ell$ (think strategy (stock fractions)),
- $G_k : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^\ell \to \mathbb{R}$ for $k = 1, \dots, N$.

Denote by $(V_k^{n,v})_{k=n,\dots,N}$ the process with initial value $V_n^{n,v} = v$ and for $k > n$ recursively by (1).

Finally, for functions $u_0, \dots, u_N$, $u_n : \mathbb{R} \times \mathbb{R}^\ell \to \mathbb{R}$ (think utility), define:

$$U^{n,v}\left(\eta_n, \dots, \eta_N\right) = \mathbb{E}\left[\sum_{k=n}^{N} u_k\left(\underbrace{V_k^{n,v}}_{\in\mathbb{R}}, \underbrace{\eta_k\left(V_k^{n,v}\right)}_{\in\mathbb{R}^\ell}\right)\right]. \tag{$\star$}$$

The Bellman equation

Objective: Optimal expected utility of $(\star)$ over the controls and optimal controls (when they exist) for which the supremum is attained, i.e:

$$\sup_{\eta_0,\dots,\eta_N} U^{0,v}(\eta_0,\dots,\eta_N) = \sup_{\eta_0,\dots,\eta_N} \mathbb{E}\left[\sum_{k=0}^{N} u_k\left(V_k^{0,v}, \eta_k\left(V_k^{0,v}\right)\right)\right],$$

"*If the control is optimal over an entire sequence of periods, then it has to be optimal over each single period*". Theorem 2.32 (Pascucci & Runggaldier, 2012, p. 90) DP solves optimization problem $(\star)$:

Theorem 2.32

For each $n = 0, \dots, N$ we have:

$$\sup_{\eta_n,\dots,\eta_N} U^{n,v}(\eta_n,\dots,\eta_N) = W_n(v),$$

where W_n is defined recursively by ([the Bellman equation]):

$$\begin{cases} W_N(v) &= \sup_{\xi\in\mathbb{R}^\ell} u_N(v,\xi), \qquad \text{and, for } n = N,\dots,1 \\ W_{n-1}(v) &= \sup_{\xi\in\mathbb{R}^\ell} \left(u_{n-1}(v,\xi) + \mathbb{E}\left[W_n\left(G_n(v,\mu_n;\xi)\right)\right]\right). \end{cases}$$

([ξ : Imagine we are at time $n-1 \to$ We then have to choose 2 quantities: consum. and inv.])

So: DP is a deterministic algorithm in which at every step we determine (by backwards recursion), the optimal value and the optimal controls by standard maximization procedure!

Example 2.4.2 - Terminal Utility in the Binomial Model: DP Method (1/3)

Objective: (See Extra Slides 2 for numerical example) Maximizing expected utility from terminal wealth where $u(x) = \log(x)$ in a standard binomial model with N periods and riskless interest rate r.

Firstly, dynamics of portolio value:

$$V_n = G_n(V_{n-1}, \mu_n; \pi_n) = \begin{cases} \text{(Up):} & V_{n-1}(1 + r + \pi_n(u - 1 - r)), \quad \text{if } \mu_n = u - 1, \\ \text{(Down):} & V_{n-1}(1 + r + \pi_n(d - 1 - r)), \quad \text{if } \mu_n = d - 1, \end{cases} \tag{2}$$

where the controls, π_n, denote the fraction of wealth held in the risky asset (*not a percentage*).

If $V_{n-1} > 0$, then $V_n > 0$ if $d < 1 + r < u$ - absence of arbitrage holds, or equivalently:

$$\pi_n \in D =]a, b[, \qquad \text{where } a = -\frac{1+r}{u-1-r}, \quad b = \frac{1+r}{1+r-d}.$$

From theorem 2.32 and *Example 2.33 - Maximization of expected utility of terminal wealth* (Pascucci & Runggaldier, 2012, pp. 90-91) $\sup_{\pi_{n+1},\dots,\pi_N} \mathbb{E}\left[u(V_N^{v,n})\right] = W_n(v)$ where:

$$\begin{cases} W_N(v) = u(v) \qquad \text{and, for } n = N, \dots, 1 \\ W_{n-1}(v) = \sup_{\bar{\pi}_n \in \mathbb{R}^d} \mathbb{E}\left[W_n\left(G_n(v, \mu_n; \bar{\pi}_n)\right)\right]. \end{cases}$$

(W: Think indirect utility: represents the maximal expected utility that can be achieved with given initial wealth)

Example 2.4.2 - Terminal Utility in the Binomial Model: DP Method (2/3)

For initial wealth $v > 0$ and log-utility:

$$\sup_{\pi_{n+1},\dots,\pi_N} \mathbb{E}\left(\log(V_N^{n,v})\right) = W_n(v),$$

where:

$$\begin{cases} W_N(v) & = \log(v) \qquad \text{and, for } n = N-1, \\ W_{N-1}(v) & = \max_{\bar{\pi}_N \in D} \mathbb{E}\left[\log(G_N(v,\mu_N;\bar{\pi}_N))\right] = \underbrace{\log(v) + \max_{\bar{\pi}_N \in D} f(\bar{\pi}_N)}_{\text{product property of } \log}, \end{cases}$$

where f from (2) (remember, W_{N-1} is the expectation of $V_n = \log(G_n)$):

$$f(\pi) = \underbrace{p}_{\text{up}} \log\left(1 + r + \pi(u-1-r)\right) + \underbrace{(1-p)}_{\text{down}} \log\left(1 + r + \pi(d-1-r)\right).$$

We find the global maximizer, $\bar{\pi}$, of f:

$$f'(\pi) = p\frac{u-1-r}{1+r+\pi(u-1-r)} + (1-p)\frac{d-1-r}{1+r+\pi(d-1-r)} = 0$$

$$\Rightarrow$$

$$\bar{\pi} = \frac{(1+r)(pu+(1-p)d-1-r)}{(u-1-r)(1+r-d)} \in D, \qquad \text{for } p \in]0,1[\text{ and } d < 1+r < u$$

$$\Rightarrow \max_{\bar{\pi}_N \in D} f(\bar{\pi}_N) = f(\bar{\pi}), \qquad \text{as } \lim_{\pi \to a^+} f(\pi) = \lim_{\pi \to b^-} f(\pi) = -\infty.$$

Example 2.4.2 - Terminal Utility in the Binomial Model: DP Method (3/3)

Lastly, substitute $\bar{\pi}$ into $f(\pi)$:

$$\max_{\bar{\pi}_N \in D} f(\pi) = f(\bar{\pi}) = p \log\left(\frac{p(u-d)}{1+r-d}\right) + (1-p)\log\left(\frac{(1-p)(u-d)}{u-1-r}\right) + \log(1+r).$$

Next step $(N-2)$: The indirect utility function yields (remember, backwards recusion):

$$W_{N-2}(v) = \underbrace{\max_{\bar{\pi}_{N-1}} E\left[\log(G_{N-1}(v, \mu_{N-1}; \bar{\pi}_{N-1}))\right] + \max_{\bar{\pi}_{N-1} \in D} f(\pi)}_{\text{same problem but 2 steps/ticks}}$$
$$= \log(v) + 2f(\bar{\pi}),$$

i.e. recursively at generic step n:

$$W_{N-n}(v) = \underbrace{\log(v) + nf(\bar{\pi})}_{\text{same problem but } n \text{ steps/ticks}},$$

and specifically the optimal value of expected utility, starting from an initial $v > 0$ is:

$$W_0(v) = \log(v) + Nf(\bar{\pi}),$$

with a "constant" optimal strategy, $\pi_n^{\max}(v) = \bar{\pi}$.

Implication and Discussion

Constant optimal strategy:

$$\bar{\pi} = \frac{(1+r)(pu+(1-p)d-1-r)}{(u-1-r)(1+r-d)}$$

Result: In any period the optimal strategy involves holding the same **ratio** in the risky asset, independent of stock price, wealth and time horizon ("myopia") and thus the solution for $n \leq N-2$ is the same. Same ratio in wealth in every period does **not** imply, expressed in units of the assets kept in the portfolio, **remains constant**.

(same result as MG-method, requires DT!).

By proposition 2 (Blædel & Huge, 1999, p. 12): Can extend the analysis to any utility function with constant RRA.

Question: Should young investors (long time horizon) hold a higher proportion of stocks?

- Yes! Stock returns outperform bonds in the long run (out−performance probability).
- Yes! Future income/salaries is like a big bond investment $\Rightarrow$ more should be invested in risky asset.
- No! Human capital is maybe not that risk-free - unemployment (for months), lower salary after unemployment (stocks relatively uncorrelated between periods), i.e words of Rolf: "really nasty".
- No! We have rigorous and robost results in dynamic opimization that says our ratio should be constant.

Conclusion: Inconclusive: it is not obvious whatsoever if you should have more in risky assets.

References

Blædel, N., & Huge, H. (1999). Multiperiod investment in discrete time.

Pascucci, A., & Runggaldier, W. J. (2012). *Financial mathematics: Theory and problems for multi-period models*. Springer Science & Business Media.

Extra Slides 1: Utility Functions

Special case of hyperbolic absolute risk aversion: Isoelastic function for utility / power utility function / CRRA utility function is given by:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \gamma \geq 0, \\ \ln(c) & \text{if } \gamma = 1, \end{cases}$$

where c represents consumption and γ is the coefficient of relative risk aversion (CRRA).

Special cases:

- When $\gamma = 0$: This corresponds to risk neutrality, because utility is linear in c.
- When $\gamma = 1$: By virtue of l'Hôpital's rule, the limit of $u(c)$ is $\ln c$ as γ goes to 1:

$$\lim_{\gamma \to 1} \frac{c^{1-\gamma} - 1}{1-\gamma} = \ln(c),$$

  which justifies the convention of using the limiting value $u(c) = \ln c$ when $\gamma = 1$. (Merton portfolio)/Growth optimal portfolio/growth rate maximizing investor $\iff$ that has log-utility.
- When $\gamma \to \infty$: This is the case of infinite risk aversion.

CRRA: This utility function has the feature of constant relative risk aversion. Mathematically this means that:

$$\underbrace{\text{RRA}}_{\gamma} = -\frac{cu''}{u'}$$

Extra Slide 2: Numerical Experiment (1/3)

PDF: A Numerical Experiment: The Gain from Dynamic Trading and R-code: DynamicStatic Model setup: not implausible numerical values:

- $u = 1.29$, $d = 0.85$, $p = 0.5$ this corresponds to a yearly expected return of 7% with a standard deviation of 22%.
- $r = 0.04$.
- $T = 30$ it could be your pension savings,

which yields:

$$\hat{\pi} = 0.6568$$

and

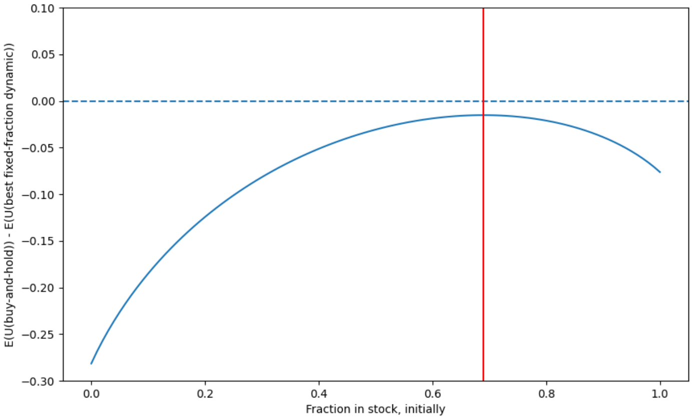
$$W_0(v) = \ln(v) + 30 \times 0.04861$$

The expected utility from a buy-and-hold strategy vs. Optimal dynamic trading is plotted on the next slide.

Extra Slide 3: Numerical Experiment (2/3)

The figure shows $\mathbb{E}(U(\text{buy-and-hold})) - \mathbb{E}(U(\text{best fixed-fraction dynamic}))$ vs. Fraction in stock, initially.

- Full drawn is difference for some fixed fraction, initially.
- Dotted is when both strategies concide.
- Red vertical full drawn line is the optimal $\hat{\pi}_0 \approx 0.69$.
- Below zero everywhere $\rightarrow$ Dynamic trading can improve even the best buy-and-hold strategy.



Extra Slide 4: Numerical Experiment (3/3)

How large a fraction, say f, of his initial wealth would an investor constrained to buy-and-hold strategies be willing to give up in order to trade dynamicall. We can find the certainty equivalent (when are we indifferent) by solving the equation

$$W_0(v \times (1 - f)) = G(v; \hat{\pi}_0)$$

i.e how much of my wealth am I willing to give up to trade dynamically. The solution will be independent on v as the $\ln(v)$-terms cancel out and is:

$$f = 0.0154 \approx 1.5\%,$$

of his initial wealth in order to be allowed to trade dynamically rather than using the best buy-and-hold strategy. On the one hand not a vanishing number, but on the other hand a gain that could easily be eaten up if there are transaction costs when trading dynamically; about 1/20th of a percent per year would seem to do the trick.