Multiperiod Investment in Discrete Time [*]

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1 Introduction

In this note we will look at multiperiod investment. It is assumed that the investor faces a finite time horizon. So the goal of the investor is to maximize his terminal wealth. How should this be done in a model with uncertainty? The first idea could be to maximize expected wealth, but as the following example illustrates this could have disastrous consequences.

Example 1 [1] *We consider two investment possibilities with independently and identical returns each period:*

$$
\begin{aligned}
X_1 &= \begin{cases} 0.9 & \textit{with probability } \frac{1}{2} \\ 1.2 & \textit{with probability } \frac{1}{2} \end{cases} \\
X_2 &= \begin{cases} 0.4 & \textit{with probability } \frac{1}{2} \\ 2.0 & \textit{with probability } \frac{1}{2} \end{cases}
\end{aligned}
$$

If we put all our wealth in the first asset, then the expected return is only 5% compared to the second which yields 20%. Thus to maximize the expected terminal wealth one should only invest in asset number two. However, if we look at a good period followed by a bad period then the return for asset one is $\sqrt{1.2 \cdot 0{:}9} - 1 = 3.923\%$ *per period compared to the return of asset two by* $\sqrt{2 \cdot 0.4} - 1 = -10.557\%$ *per period. It can easily be verified by the Law of Large numbers that this is in fact the long term average return per period. Thus when we only hold asset number two, we will see as* $n \to \infty$,

$$
\begin{aligned}
E\left[S(n)\right] &= \prod_{k=1}^{n} E\left[S(1)\right] = 1.2^n \to \infty \\
S(n) &\to 0 \ a.s.
\end{aligned}
$$

This might at first appear counter-intuitive: Expected terminal wealth explodes but we will almost surely go bankrupt!

The problem in the example is that the investor does not take the higher risk of asset number 2 into account. Therefore, he does not realize that in most

[*] This is Chapters 2 and 3 from the Master Thesis:"Optimal Multiperiod Investment Behavior" by Martin Blædel and Brian Huge, University of Copenhagen.

[1] This example is inspired by Hakansson and Ziemba [10].

scenarios he will loose a lot of money and only with a very small probability he will become a very wealthy man.

Instead we will in the following use utility functions to describe the behavior of the investors. In Section 2 we will define a class of utility functions, and in Section 3 we will solve the multiperiod investment problem for this class of utility functions. In Section 4 we will show some properties of the logarithmic utility function.

2 Risk Aversion

We are interested in utility functions for money and the choice made under uncertainty. We will deduce and use the measure of risk aversion introduced by Pratt [14][2]. It is natural to try to group utility functions with the same properties. In what follows, we will especially be interested in the classes which demonstrate constant absolute or relative risk aversion. These will be generalized to the class of hyperbolic risk aversion, which covers just about any standard choice of cardinal utility function.

It is known that positive affine transformations of utility functions preserve their ordering. Hence we will say that two utility functions $U^1(\cdot)$ and $U^2(\cdot)$ are **equivalent**, and write $U^1 \approx U^2$ if there exist constants $\alpha_1 > 0$ and α_2 such that $U^1(S){=}\alpha_1 \cdot U^2(S) + \alpha_2$.

2.1 Measuring Risk Aversion.

We will now discuss and try to measure risk aversion. Risk aversion is revealed when an investor must choose between a risky asset Z and a non-random amount c. Consider therefore an investor with known wealth S and a utility function for wealth $U(\cdot)$. Assume that Z has third moment, U is three times continuously differentiable with $E\left[U(S+Z)\right]$ well-defined. Furthermore we must be able to differentiate under the expectation twice. We will work with the definition from Pratt [14]:

Definition 1 *Given wealth S the amount c is termed **the cash equivalent** for Z if*

$$E\left[U(S+Z)\right] = U(S+c) \tag{1}$$

The cash equivalent c can be decomposed into

$$c = E\left[Z\right] - \pi^a \tag{2}$$

*and we will term π^a the **risk premium**.*

The interpretation is that given wealth S the cash equivalent is the smallest amount for which the investor is willing to sell the asset if he had it[3] and the

[2]Independently K. J. Arrow and R. Schlaifer have worked with the same concepts. The complete works of Arrow on risk aversion is found in Kenneth J. Arrow(1971):"Essays in the theory of risk bearing", North-Holland, Amsterdam. See also Borglin [4].

[3]This is also termed the "ask" price. Notice that this is to be distinguished from the "bid" price c_b, the largest amount the investor will be willing to pay for obtaining Z, defined by Pratt [14]:

$$U(S) = E\left[U(S+Z-c_b)\right]$$

risk premium is the amount the investor has to be compensated for taking the risky asset. The definition of the risk premium allows us to work with risky assets with mean zero without loss of generality, since a mean different from zero can be incorporated by changing the wealth.

The cash equivalent and the risk premium are functions of wealth and the distribution of Z. We illustrate that by writing $c(S, \mathcal{D}(Z))$ and $\pi^a(S, \mathcal{D}(Z))$. Generally we will not expect to find a simple closed form expression for them. However we can try to approximate. Assume that $E[Z] = 0$ and $Var[Z] = E[Z^2] = \sigma^2$. We will use $O(\cdot)$ meaning "at most" and $o(\cdot)$ meaning "of smaller order than" when we are interested in the magnitude order of terms. Expanding (1) around S, the left hand side to second and the right-hand side to first order, we obtain:

$$E\left[U(S) + U'(S) \cdot Z + \frac{1}{2}U''(S) \cdot Z^2 + o\left(Z^3\right)\right] \tag{3}$$

$$= \quad U(S) + U'(S) \cdot c(S, \mathcal{D}(Z)) + o\left(c(S, \mathcal{D}(Z))^2\right) \tag{4}$$

$$\Leftrightarrow \quad \frac{1}{2}U''(S) \cdot \sigma^2 + E\left[o\left(Z^3\right)\right] = U'(S) \cdot c(S, \mathcal{D}(Z)) + o\left(c(S, \mathcal{D}(Z))^2\right) \tag{5}$$

$$\Leftrightarrow \quad -c(S, \mathcal{D}(Z)) = \pi^a(S, \mathcal{D}(Z)) = \underbrace{-\frac{U''(S)}{U'(S)}}_{\equiv \rho^a(S)} \cdot \frac{1}{2}\sigma^2 + E\left[o\left(Z^3\right)\right] \tag{6}$$

(We indirectly assume that $o\left(c(S, \mathcal{D}(Z))^2\right)$ is of smaller order than $E\left[o\left(Z^3\right)\right]$)

The term $\rho^a(S)$ can be interpreted as twice the fee the decision maker requires per unit of variance or infinitesimal risks, and thus we will term it the *local (absolute) risk aversion* at the point S under utility function U. If the investor (given S) is risk averse, he wants compensation for taking Z and hence we will expect him to demand a *positive* risk premium. The riskier Z, measured by the variance of Z, the higher risk premium is demanded. We shall term a utility function risk averse if $\rho^a(S) > 0$ for all S. For negative $\rho^a(S)$ we will instead speak of *risk love*.

Since we expect utility functions to be strictly increasing[4], i.e. $U'(S) > 0$, we have $\rho^a(S) > 0$ if and only if U is strictly concave, i.e. $U''(S) < 0$. Thus one might set out to define aversion to risk as concavity as done in Borglin [4]. However it is clear that $U''(S)$ is not in itself a meaningful measure of concavity. A positive affine transformation of U does not alter the behavior of the utility function, but does alter $U''(S)$ and the curvature of $U(S)$. The only feature of $U''(S)$ that does have a meaning is its sign. A positive (negative) sign of $U''(S)$ implies (un)willingness to accept small risks with the asset Z. However, our measure is independent of positive affine transformations of the utility function. Thus if $U^1(S) \approx U^2(S)$ we have $\rho_1^a(S) = \rho_2^a(S)$ with self-explanatory notation.

Notice that the measure of risk aversion is found as a local approximation and it is therefore, as such, not a global measure. However, global comparisons can be made: it follows directly that if we have two utility functions, $U^1(S)$ and $U^2(S)$, then $\rho_1^a(S) \geq \rho_2^a(S)$ for all S if and only if $\pi_1^a \geq \pi_2^a$ for all S and Z, and hence $U^1(S)$ is more risk averse than $U^2(S)$ in a global sense.

[4] At least we want the utility function to be strictly increasing in a neighborhood of S. We choose not to exclude utility functions with a (distant) satiation point like the quadratic utility function.

2.2 The CARA Class

The first family of utility functions we will define is that of *constant absolute risk aversion*. This class of utility functions satisfy:

$$\rho^a(S) = -\frac{U''(S)}{U'(S)} = \alpha_1 \tag{7}$$

The differential equation is easily solved, and is seen to be satisfied by functions of the form

$$U(S) \approx \begin{cases} \exp(-\alpha_1 S) & \text{if } \alpha_1 < 0 \\ S & \text{if } \alpha_1 = 0 \\ -\exp(-\alpha_1 S) & \text{if } \alpha_1 > 0 \end{cases}$$

We will term these functions the CARA class[5], and when speaking of utility functions with a specific constant absolute risk aversion we will use the notation $U \in CARA(\alpha_1)$. Since the risk aversion is constant locally it is also constant globally, i.e. a change in assets makes no change in preference among risks (observe that $U(S+k) \approx U(S)$ for $k \in \mathbb{R}$). Therefore it makes sense to speak of "constant risk aversion" without the qualification "local" or "global".

2.3 The CRRA Class

So far we have been concerned with risks that remained fixed while assets varied, but it will sometimes be useful to view everything as a proportion of assets. Denoting the price *relative* for the asset by $X = 1 + R$, $E[R] = 0$ and $E[R^2] = \sigma^2$ the choice will be between SX and $c(S, \mathcal{D}(X))\, S$, and the decomposition of the cash equivalence (2) should be rewritten using a relative risk premium termed π^r :

$$c(S, \mathcal{D}(X)) = E[X] - \pi^r(S, \mathcal{D}(X)) \tag{8}$$

Inspecting the calculations (3) to (6) we see that $\pi^r(S, \mathcal{D}(X))$ can be expressed as

$$\pi^r(S, \mathcal{D}(X)) = \underbrace{-\frac{U''(S)S}{U'(S)}}_{\equiv \rho^r(S)} \cdot \frac{1}{2}\sigma^2 + E\left[o\left(R^3\right)\right] \tag{9}$$

By analogy with the absolute risk aversion we will term $\rho^r(S)$ the *local relative risk aversion* at the point S under utility function U. Its interpretation is like that of $\rho^a(S)$. This inspires the definition of the class of *constant relative risk aversion* containing utility functions that satisfy

$$\rho^r(S) = -\frac{U''(S)S}{U'(S)} = \alpha_2 \tag{10}$$

We note that this definition is independent of the unit of measurement. The measure can also be interpreted as the elasticity of the first derivative of U

[5]The mention of CARA and CRRA is taken from Blanchard and Fischer [3].

w.r.t. wealth (See Blanchard and Fischer [3] or Borglin [4]. This is why the utility functions of this class is also termed *isoelastic.* The differential equation is easily solved, and is seen to be satisfied by functions of the form

$$U(S) \approx \begin{cases} \frac{1}{1-\alpha_2} S^{1-\alpha_2} & \text{if } \alpha_2 < 1 \\ \log(S) & \text{if } \alpha_2 = 1 \\ \frac{1}{1-\alpha_2} S^{1-\alpha_2} & \text{if } \alpha_2 > 1 \end{cases}$$

We will term these functions the CRRA class, and when speaking of utility functions with a specific constant relative risk aversion we will use the notation $U \in CRRA(\alpha_2)$. As true for the CARA class, it also qualifies for these functions just to speak of "constant relative risk aversion", without the "local" or "global" (observe that $U(k \cdot S) \approx U(S)$, $k \in \mathbb{R}_+$).

2.4 The HARA class

It is convenient also to define the class of *hyperbolic absolute risk aversion* as functions satisfying

$$\rho^a(S) = -\frac{U''(S)}{U'(S)} = \left(\alpha_1^{-1} + \alpha_2^{-1} S\right)^{-1} \tag{11}$$

We will term these functions the HARA class[6], and when speaking of utility functions with a specific hyperbolic absolute risk aversion, we will use the notation $U \in HARA(\alpha_1, \alpha_2)$. We will speak of α_1 as the **absolute risk aversion coefficient** and α_2 as the **relative risk aversion coefficient**. It is seen that the class for *positive* wealth is composed of utility functions of the form listed in Table 1 (Some of the functions are only well defined for S in some subset of $\mathbb{R}$). By allowing for parameters to be equal to infinity we obtain:

	$\alpha_1 = \infty$	$\alpha_1 > 0$	$\alpha_1 < 0$
$\alpha_2 = \infty$	Not defined	$-\exp(-\alpha_1 S)$	$\exp(-\alpha_1 S)$
$\alpha_2 > 1$	$\frac{1}{1-\alpha_2} S^{1-\alpha_2}$	$\frac{\alpha_2}{1-\alpha_2}\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)^{1-\alpha_2}$	$\frac{\alpha_2}{1-\alpha_2}\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)^{1-\alpha_2}$
$\alpha_2 = 1$	$\log(S)$	$\log(S + \frac{1}{\alpha_1})$	$\log(S + \frac{1}{\alpha_1})$
$0 < \alpha_2 < 1$	$\frac{1}{1-\alpha_2} S^{1-\alpha_2}$	$\frac{\alpha_2}{1-\alpha_2}\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)^{1-\alpha_2}$	$\frac{\alpha_2}{1-\alpha_2}\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)^{1-\alpha_2}$
$\alpha_2 < 0$	$\frac{1}{1-\alpha_2} S^{1-\alpha_2}$	$\frac{\alpha_2}{1-\alpha_2}\left(\frac{1}{\alpha_1} + \frac{S}{\alpha_2}\right)^{1-\alpha_2}$	No solution

Table 1: The HARA functions. Furthermore, if either α_1 or α_2 is equal to 0 then the utility function is affine or $U(S) \approx S$.

$$\begin{aligned} CARA(\alpha_1) &= HARA(\alpha_1, \infty) \\ CRRA(\alpha_2) &= HARA(\infty, \alpha_2) \end{aligned}$$

As seen the class covers a wide range of functions. $U(S)$ will be a hyperbola for $\alpha_2 > 1$, a root-function for $\alpha_2 \in]0, 1[$ and a 'parabolic' for $\alpha_2 < 0$. Note that $HARA(\frac{2}{\alpha}, -1)$ is equal to the quadratic utility class, i.e. utility functions which satisfy $U(S) \approx \alpha S - S^2$.

[6] The term HARA originates from Merton [12].

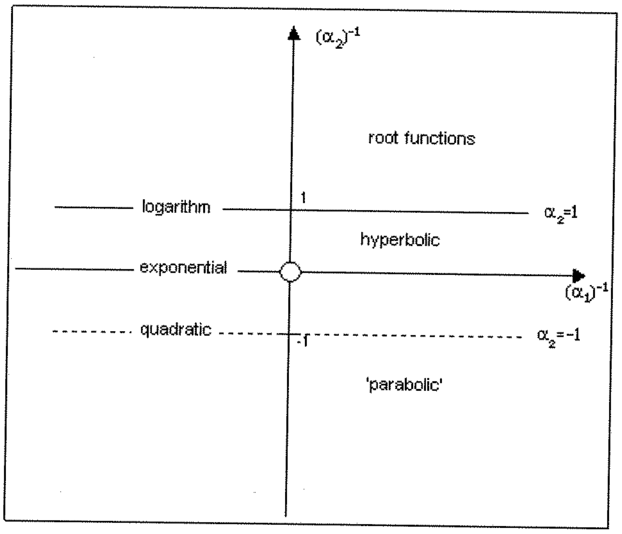

Figure 1: Graphical illustration of the functional forms of the HARA class.

2.5 A Realistic Utility Function

The HARA class is obviously a very large class of utility functions. As already noted it includes the CARA and CRRA class as well as the quadratic utility functions. The functional form can be exponential, logarithmic or power function. But what would we expect a realistic utility function to be like?

First, could we imagine an investor being a risk lover at some level of wealth and risk averse for another level? The investor could be extremely risk averse as a poor man, afraid of starving to death, and then act as a gambler once his basic needs are fulfilled. We choose to think that the common investor is the more risk averse type. If he is interested in gambles such as casinos, "Tips" or "Lotto" he will only use a fraction of his wealth at them. His main concern will be to pay the terms and save for the holidays and pension. Thus we think of the utility function as risk averse.

It is not enough to say that the utility function should be concave. We would expect an investor's risk premium to be decreasing (or at least non-increasing) as his wealth increases when he considers a given investment. A bet of 1000 kr. is not so frightening to a billionaire as it is to a (poor) student. Thus utility functions with increasing absolute risk aversion are an unattractive description of behavior towards risk. This is equivalent to assuming that $\alpha_2 \geq 0$. As can be seen, this also excludes the quadratic utility function[7].

It is commonly said that institutional investors are so well diversified, that they behave almost risk neutral to new investment possibilities. This would be the same as saying that the relative risk aversion was low and that the absolute risk aversion is abating.

However it is very difficult to access the magnitude of risk aversion. It is hard to deduce from empirical findings. Firstly, we can never have access to the

[7]Still the quadratic utility function has played a significant role in financial theory, especially in the Markowitz-Tobin Mean-Variance analysis and in the CAPM model.

information that an investor uses in order to asses an investment, and therefore we cannot work out the utility function used to make an investment simply by looking at all trades at the stock exchange. Secondly, since the invested wealth of private and institutional investors differ significantly in magnitude, we are not even in a position to conclude whether all groups use the same utility function or not. Different behavior could be caused by the fact that the decisions were made very far from each other on the same utility curve.

3 The Multiperiod Problem

In this section we will formulate the investor's problem as a sequential investment problem aiming at maximizing expected utility of terminal wealth. It will be shown that the utility functions in the CRRA class have several appealing properties. Most importantly, they make a myopic policy optimal, i.e. reduce a multistage problem to a sequence of single period problems each solved as if it was the last period. Second, the optimal portfolio choice has relative portfolio weights on each risky asset independent of initial wealth. Third, we will see this class as a limit of a larger class of utility functions, when the investment horizon goes to infinity.

3.1 Formulation of the problem and notation

We will now look at a model where the investor has determined a certain point in the future (his horizon or the terminal point) at which he plans to consume all his wealth[8]. The horizon is termed n. The objective is, given the initial wealth, to construct a self-financing portfolio such that the expected terminal utility is maximized. For each period k, $k = 0, ..., n$, we use the following notation:

Letters in bold indicate vectors. We take full advantage of the matrix-notation and interpret a vector plus a scalar as adding the scalar to each element in the vector. Let $I_m = \{\mathbf{e}_i\}_{i=1}^m$ denote the identity-matrix with dimension m, $\mathbb{R}_+^m = [0, \infty[^m$ and $\mathbf{1}_m = (1, ..., 1)^t \in \mathbb{R}^m$.

We will write $u \succ v$ and say that **u is bounded above v** when there exist $\varepsilon > 0 : u \geq v + \varepsilon$.

For functions $u, v : \mathbb{N} \to \mathbb{R}$ we write $u(n) \sim v(n)$ and say that **u and v are asymptotically equal** when $lim_{n\to\infty} \frac{u(n)}{v(n)} = 1$

Let $S(k)$ be wealth at the end of the period k.

$\mathbf{X}(k) = (X_1(k), ..., X_m(k))^t \in \mathbb{R}_+^m$ is the price relatives for m non-dividend paying risky assets.

$\mathcal{F}(k) = \sigma(\mathbf{X}(1), \ldots, \mathbf{X}(k))$ as the information set in period k. We will use the notation $E_k(\cdot) = E(\cdot | \mathcal{F}(k))$ for the conditional expectation.

When forming a portfolio of risky assets we denote the *absolute* portfolio weights by $\mathbf{h}(k) = (h_1(k), ..., h_m(k))^t$ (the amount of money invested in each asset) and *relative* portfolio weights by $\mathbf{b}(k) = (b_1(k), ..., b_m(k))^t$. We denote the optimal weights by adding a '•'. Any set of portfolio weights, $\{\mathbf{h}(k)\}_{k=1}^n$ or $\{\mathbf{b}(k)\}_{k=1}^n$, will be termed a **strategy**. A strategy that satisfies $\mathbf{h}(k) \in \mathcal{F}(k-1)$ (or $\mathbf{b}(k) \in \mathcal{F}(k-1)$) for all k is called **non-anticipating**.

[8]For simplicity we disregard intermediate consumption.

We assume that there exists a possibility to make a riskless investment at each period with return $r(k)$ (unity plus the rate of interest). Assume $r(k) \geq 1$ and that $r(k)$ is deterministic. Then we can formulate:

Definition 2 *A Multiperiod Problem (MPP) is a maximization problem of the form*

$$\max_{\mathbf{h}(1),\ldots,\mathbf{h}(n)} E_0\left[U\left(S\left(n\right)\right)\right] \tag{12}$$

$$\begin{aligned} s.t. \quad & S(k) = \mathbf{h}^t(k)\,\mathbf{X}(k) + \left(S(k-1) - \mathbf{h}^t(k)\,\mathbf{1}\right) r(k) \qquad (13) \\ & \textit{for every } k = 1, \ldots, n \\ & S(0) \quad \textit{given} \end{aligned}$$

We will term any non-anticipating strategy satisfying (13) **self-financing**[9]. We interpret $h_i(k) < 0$ as short-selling asset i. Alongside the risky investment it is possible to make a riskless investment. This can be thought of as using a money market account. If $\mathbf{h}^t(k)\,\mathbf{1} < S(k-1)$ then we wish to lend, and for $\mathbf{h}^t(k)\,\mathbf{1} > S(k-1)$ we wish to borrow. We will denote the amount put in the money market account, with signs, $h_0(k) = S(k-1) - \mathbf{h}^t(k)\,\mathbf{1}$.

We could have chosen to include the riskless asset as one of the m assets, so (13) would have been

$$S(k) = \mathbf{h}^t(k)\,\mathbf{X}(k)$$

Riskless borrowing would then be included by allowing short sale in the riskless asset and the constraint $\mathbf{h}^t(k)\,\mathbf{1} = S(k-1)$. However, by using the original formulation it is easier to isolate the effects of having a riskless asset using equation (13) and since this is a standard formulation when using dynamic programming, the results are easier to compare with the literature.

We are interested in the properties of the optimal solution to the MPP when it exists and is unique. First, we want to know when the optimal relative portfolio weights are independent of initial wealth allowing for a simplifying normalization. Thus we can solve the relative problem and scale by the initial wealth to find the optimal solution. Second, is it possible to reduce the multi-stage problem to a sequence of single period problems each solved as if it were the last period? This would clearly simplify the art of portfolio management. Third, we will examine the first period decision as the horizon tends to infinity.

All this will be conducted under the assumption that there are no constraints on the use of assets. When we investigate the problem in continuous time we will include different constraints on the investment possibilities, to see how much the results are changed.

3.2 Solving the Multiperiod Problem

We will now turn to solving the problem. It is clear that for a multiperiod problem it is rarely optimal, if at all possible, to specify a sequence of single-period decisions once and for all in period 0. Nor could it generally be optimal to simply make the first period decision and neglect the investment possibilities

[9]There is an obvious problem in defining self-financing in a relative setup. We cannot control what happens with fortune earned from other sources like dividends. That is why we have precluded this.

for the rest of the periods. Rather, any sequence of portfolio decisions must be contingent upon the outcome of the previous periods and at the same time take into account the information of the future probability distributions.

To solve the problem we will apply the technique of dynamical programming introduced by Bellman. His "principle of optimality" states that

> "an optimal strategy has the property that whatever the initial state and the initial decision, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision" [10]

Noting that the problem facing the investor at period $n-1$ is a standard single period investment problem given the wealth $S(n-1)$, and we can solve the problem conditioning on the past recursively backwards. Hence the strategy is to ~~the~~ solve a sequence of problems of the following structure for $k = 1, ..., n$:

problem [illegible] $k-1$

$$\max_{h(k)} E_{k-1}\left[V_k\left(S\left(k\right)\right)\right] \tag{14}$$

$$s.t. \quad S\left(k\right) = \mathbf{h}^t\left(k\right)\mathbf{X}\left(k\right) + \left(S\left(k-1\right) - \mathbf{h}^t\left(k\right)\mathbf{1}\right)r\left(k\right) \tag{15}$$

$$S\left(k-1\right) \text{ given}$$

where

$$V_{k-1}(S(k-1)) = \max_{\mathbf{h}(k)} E_{k-1}\left[V_k(S(k))\right] \quad k = 1, \ldots, n \tag{16}$$

$$V_n(S(n)) = U(S(n)) \tag{17}$$

We denote $V_k(\cdot)$ the **indirect utility function**. We refer to $V_0(\cdot)$ as the first period decision function. The representation of the stochastic variable $\max_{\mathbf{h}(k+1)} E_k\left[V_{k+1}\left(S\left(k+1\right)\right)\right]$ by $V_k(S(k))$ is the essence of dynamical programming. $E_k\left[V_{k+1}\left(S\left(k+1\right)\right)\right]$ is well defined but we need some assumptions to assure existence of a maximum. We will return to this. The rather technical issue of measurability of the solution to the maximization problem will be neglected (in a finite space this is trivial, but it can be rather troublesome when the state space is non-countable).

We are now ready for our main result of this section: The HARA class of utility functions has the property that the indirect utility functions are also in the HARA class. We will solve for the indirect utility functions, and this will reveal the structure of the optimal solution and enable us to draw conclusions on myopia and proportional portfolios.

This is stated in three propositions - one for each type of HARA function. The strategy is the same for the three types, each done by induction. We assume that there exists a unique solution (criteria will be discussed after the theorem), and then we try to rewrite the utility function, so that the part of maximization is isolated in a term independent of wealth, A. To find the optimal solution, one only has to solve the maximization problem assigned to A. The solution to this is termed $\beta^\bullet$ and is the *pseudo* relative optimal portfolio weight for the

[10] We refer to Bertsekas [2] for a thorough presentation of (discrete-time) dynamical programming. The common model is with intermediate consumption, and a quick introduction is given in Duffie [6] chapter 2 or Groth [7]. The "principle of optimality" is from the original Bellman article (we took it from Hakansson [8]).

investor's problem. The optimal (absolute) solution is then given as an *affine* transformation of $\beta^\bullet$ by the formula:

$$\mathbf{h}^\bullet(k) = \left(\alpha_2^{-1} S(k-1) + \alpha_1^{-1} \prod_{j=k}^{n} r^{-1}(j) \right) \beta^\bullet(k) \tag{18}$$

The approach taken in the proof differs from the approach in Bertsekas [2] and Mossin [13]. This is done, because we feel that this approach is more illuminative, we avoid further assumptions on the process of price relatives $\mathbf{X}$ and finally we obtain the exact form of the constants A. In the case of the power function, the constant A_{pow} will be useful when we turn to convergence of the first period decision function later in this section.

Rewriting (13) yields the identity:

$$\begin{aligned} S(k) &= \mathbf{h}^t(k)\mathbf{X}(k) + \left(S(k-1) - \mathbf{h}^t(k)\mathbf{1}\right) r(k) \\ &= r(k) S(k-1) + \mathbf{h}^t(k)\left[\mathbf{X}(k) - r(k)\right] \end{aligned} \tag{19}$$

Assume that the interest rates are deterministic and that the initial wealth, $S(0)$, is given. Consider an arbitrary process $\mathbf{X}(k)$ and a strategy $\{\mathbf{h}(k)\}_{k=1}^{n}$ for which $E[U(S(n))] < \infty$.

Proposition 1 (Logarithmic utility.) *Assume that the terminal utility function is logarithmic*

$$U(S(n)) = \log\left(\alpha_1^{-1} + S(n)\right)$$

Then the indirect utility function for $k = 1, ..., n$ will be of the form

– [illegible] k-1

$$V_{k-1}(S(k-1)) = \log\left(\alpha_1^{-1} \prod_{j=k}^{n} r^{-1}(j) + S(k-1) \right) + A_{\log}(k-1) \tag{20}$$

uafh. af S(k-1)

where

$$\begin{aligned} A_{\log}(k-1) &\equiv \max_{\beta(k)} E_{k-1}\left[\log\left(\beta^t(k)\left[\mathbf{X}(k) - r(k)\right] + r(k)\right) \right. \\ &\qquad\qquad \left. + A_{\log}(k)\right] \\ A_{\log}(n) &= 0 \end{aligned} \tag{21}$$

Furthermore the optimal solution of the multi period problem (20) in period k can be written as

$$\mathbf{h}^\bullet(k) = \left(S(k-1) + \alpha_1^{-1} \prod_{j=k}^{n} r^{-1}(j) \right) \beta^\bullet(k)$$

where $\beta^\bullet(k)$ is the optimal solution to (21)

$S(n-1)\,\beta^t(n)\,(X(n)-r(n))$
$+ \frac{\alpha_1^{-1}}{r(n)}\,\beta^t(n)\,(X(n)-r(n))$
$= \underbrace{\left(S(n-1)+\frac{\alpha_1^{-1}}{r(n)}\right)\beta^t(n)}_{h^t(n)}\,(X(n)-r(n))$

Proof.
step 1) OK for k = n

$$
\begin{aligned}
& V_{n-1}(S(n-1)) \\
\equiv\ & \max_{\mathbf{h}(n)} E_{n-1}\left[\log\left(\alpha_1^{-1}+S(n)\right)\right] \\
=\ & \max_{\mathbf{h}(n)} E_{n-1}\left[\log\left(\alpha_1^{-1}+\mathbf{h}^t(n)\left[\mathbf{X}(n)-r(n)\right]+r(n)\,S(n-1)\right)\right] \\
=\ & \max_{\beta(n)} E_{n-1}\left[\log\left(\left(\frac{\alpha_1^{-1}}{r(n)}+S(n-1)\right)\left(\begin{array}{c}\beta^t(n)\left[\mathbf{X}(n)-r(n)\right] \\ +r(n)\end{array}\right)\right)\right] \\
=\ & \log\left(\frac{\alpha_1^{-1}}{r(n)}+S(n-1)\right)+A_{\log}(n-1)
\end{aligned}
$$

where

$$
A_{\log}(n-1) \equiv \max_{\beta(n)} E_{n-1}\left[\log\left(\beta^t(n)\left[\mathbf{X}(n)-r(n)\right]+r(n)\right)\right]
$$

The optimal portfolio choice in period n is then given by

$$
\mathbf{h}^{\bullet}(n)=\left(S(n-1)+\frac{\alpha_1^{-1}}{r(n)}\right)\beta^{\bullet}(n)
$$

where $\beta^{\bullet}(n)$ is the optimal solution to

$$
\max_{\beta(n)} E_{n-1}\left[\log\left(\beta^t(n)\left[\mathbf{X}(n)-r(n)\right]+r(n)\right)\right]
$$

step k) under OK for k, is OK for k−1

Define $A_{\log}(k)$ by (21) and assume

$$
V_k(S(k))=\log\left(\alpha_1^{-1}\prod_{j=k+1}^{n} r^{-1}(j)+S(k)\right)+A_{\log}(k).
$$

$$
\begin{aligned}
& V_{k-1}(S(k-1)) \\
\equiv\ & \max_{\mathbf{h}(k)} E_{k-1}\left[V_k(S(k))\right] \\
=\ & \max_{\mathbf{h}(k)} E_{k-1}\left[\log\left(\alpha_1^{-1}\prod_{j=k+1}^{n} r^{-1}(j)+S(k)\right)+A_{\log}(k)\right] \\
=\ & \max_{\mathbf{h}(k)} E_{k-1}\left[\log\left(\begin{array}{c}\mathbf{h}^t(k)\left[\mathbf{X}(k)-r(k)\right] \\ +r(k)\,S(k-1)+\alpha_1^{-1}\prod_{j=k+1}^{n} r^{-1}(j)\end{array}\right)+A_{\log}(k)\right] \\
=\ & \max_{\beta(k)} E_{k-1}\left[\log\left(\alpha_1^{-1}\prod_{j=k}^{n} r^{-1}(j)+S(k-1)\right)\right. \\
& \qquad \left. +\left(\beta^t(k)\left[\mathbf{X}(k)-r(k)\right]+r(k)\right)+A_{\log}(k)\right] \\
=\ & \log\left(\alpha_1^{-1}\prod_{j=k}^{n} r^{-1}(j)+S(k-1)\right)+A_{\log}(k-1)
\end{aligned}
$$

The optimal portfolio choice is then given by

$$\mathbf{h}^{\bullet}(k)=\left(\alpha_1^{-1}\prod_{j=k}^{n}r^{-1}(j)+S(k-1)\right)\beta^{\bullet}(k)$$

where $\beta^{\bullet}(k)$ is the optimal solution to (21).

■

Proposition 2 (Power utility) *Assume that the terminal utility function is a power function of the form*

$$U(S(n))=\frac{\alpha_2}{1-\alpha_2}\left(\alpha_1^{-1}+\alpha_2^{-1}S(n)\right)^{1-\alpha_2}$$

Then the indirect utility function for $k=1,...,n$ will be of the form

$$V_{k-1}(S(k-1))=\frac{\alpha_2 A_{pow}(k-1)}{1-\alpha_2}\left(\frac{1}{\alpha_1\prod_{j=k}^{n}r(j)}+\frac{S(k-1)}{\alpha_2}\right)^{1-\alpha_2} \tag{22}$$

where

$$\begin{aligned}
&A_{pow}(n)=1\\
&A_{pow}(k-1)\\
\equiv\;&\begin{cases}\max\limits_{\beta(k)}E_{k-1}\left[\left(\beta^t(k)\frac{\mathbf{X}(k)-r(k)}{\alpha_2}+r(k)\right)^{1-\alpha_2}A_{pow}(k)\right] & \text{for } \alpha_2<1\\ \min\limits_{\beta(k)}E_{k-1}\left[\left(\beta^t(k)\frac{\mathbf{X}(k)-r(k)}{\alpha_2}+r(k)\right)^{1-\alpha_2}A_{pow}(k)\right] & \text{for } \alpha_2>1\end{cases}
\end{aligned} \tag{23}$$

Furthermore the optimal solution of the multiperiod problem (22) *in period k can be written as*

$$\mathbf{h}^{\bullet}(k)=\left(\alpha_1^{-1}\prod_{j=k}^{n}r^{-1}(j)+\alpha_2^{-1}S(k-1)\right)\beta^{\bullet}(k)$$

where $\beta^{\bullet}(k)$ is the optimal solution to (23).

Proof.
step 1)

$$\begin{aligned}
&V_{n-1}(S(n-1))\\
\equiv\;&\max_{\mathbf{h}(n)}E_{n-1}\left[\frac{\alpha_2}{1-\alpha_2}\left(\alpha_1^{-1}+\alpha_2^{-1}S(n)\right)^{1-\alpha_2}\right]\\
=\;&\max_{\mathbf{h}(n)}E_{n-1}\left[\frac{\alpha_2}{1-\alpha_2}\left(\alpha_1^{-1}+\alpha_2^{-1}\left(\begin{array}{c}\mathbf{h}^t(n)\left[\mathbf{X}(n)-r(n)\right]\\+r(n)S(n-1)\end{array}\right)\right)^{1-\alpha_2}\right]\\
=\;&\max_{\beta(n)}E_{n-1}\left[\frac{\alpha_2}{1-\alpha_2}\left(\left(\frac{\alpha_1^{-1}}{r(n)}+\alpha_2^{-1}S(n-1)\right)\cdot\right.\right.\\
&\left.\left.\left(\beta^t(n)\frac{\mathbf{X}(n)-r(n)}{\alpha_2}+r(n)\right)\right)^{1-\alpha_2}\right]\\
=\;&\frac{\alpha_2}{1-\alpha_2}\left(\frac{\alpha_1^{-1}}{r(n)}+\alpha_2^{-1}S(n-1)\right)^{1-\alpha_2}A_{pow}(n-1)
\end{aligned}$$

where $A_{pow}(n-1)$ is defined in (23).

The optimal portfolio choice in period n is then given by

$$\mathbf{h}^{\bullet}(n) = \left(\frac{\alpha_1^{-1}}{r(n)} + \alpha_2^{-1} S(n-1)\right) \beta^{\bullet}(n)$$

where $\beta^{\bullet}(n)$ is the optimal solution to (23).

step k)

Define $A_{pow}(k)$ by (23) and assume

$$V_k(S(k)) = \frac{\alpha_2}{1-\alpha_2}\left(\alpha_1^{-1}\prod_{j=k+1}^{n} r^{-1}(j) + \alpha_2^{-1} S(k)\right)^{1-\alpha_2} A_{pow}(k)$$

$$\begin{aligned}
& V_{k-1}(S(k-1)) \\
\equiv\ & \max_{\mathbf{h}(k)} E_{k-1}\left[V_k(S(k))\right] \\
=\ & \max_{\mathbf{h}(k)} E_{k-1}\left[\frac{\alpha_2 A_{pow}(k)}{1-\alpha_2}\left(\alpha_1^{-1}\prod_{j=k+1}^{n} r^{-1}(j) + \alpha_2^{-1} S(k)\right)^{1-\alpha_2}\right] \\
=\ & \max_{\mathbf{h}(k)} E_{k-1}\left[\frac{\alpha_2 A_{pow}(k)}{1-\alpha_2}\left(\alpha_1^{-1}\prod_{j=k+1}^{n} r^{-1}(j)\right.\right. \\
& \left.\left. +\alpha_2^{-1}\left(\mathbf{h}^t(k)\left[\mathbf{X}(k) - r(k)\right] + r(k) S(k-1)\right)\right)^{1-\alpha_2}\right] \\
=\ & \max_{\beta(k)} E_{k-1}\left[\frac{\alpha_2 A_{pow}(k)}{1-\alpha_2}\left(\left(\alpha_1^{-1}\prod_{j=k}^{n} r^{-1}(j) + \alpha_2^{-1} S(k-1)\right)\cdot\right.\right. \\
& \left.\left.\left(\beta^t(k)\frac{\mathbf{X}(k) - r(k)}{\alpha_2} + r(k)\right)\right)^{1-\alpha_2}\right] \\
=\ & \frac{\alpha_2}{1-\alpha_2}\left(\alpha_1^{-1}\prod_{j=k}^{n} r^{-1}(j) + \alpha_2^{-1} S(k-1)\right)^{1-\alpha_2} \cdot A_{pow}(k-1)
\end{aligned}$$

The optimal portfolio choice in period k is then given by

$$\mathbf{h}^{\bullet}(k) = \left(\alpha_1^{-1}\prod_{j=k}^{n} r^{-1}(j) + \alpha_2^{-1} S(k-1)\right) \beta^{\bullet}(k)$$

where $\beta^{\bullet}(k)$ is the optimal solution to (23).

■

Proposition 3 (Exponential utility) *Assume that the terminal utility function is of the exponential form*

$$U(S(n)) = -\exp(-\alpha_1 S(n))$$

Then the indirect utility function for $k = 1, ..., n$ will be of the form

$$V_{k-1}(S(k-1)) = -\exp\left(-\alpha_1 \prod_{j=k}^{n} r(j) S(k-1)\right) A_{\text{exp}}(k-1) \tag{24}$$

where

$$\begin{aligned} A_{\text{exp}}(k-1) &\equiv \min_{\beta(k)} E_{k-1}\left[\exp\left(-\beta^t(k)\left[\frac{\mathbf{X}(k)}{r(k)} - 1\right]\right) A_{\text{exp}}(k)\right] \\ A_{\text{exp}}(n) &= 1 \end{aligned} \tag{25}$$

Furthermore the optimal solution of the multi period problem (24) *in period k can be written as*

$$\mathbf{h}^{\bullet}(k) = \left(\alpha_1 \prod_{j=k}^{n} r(j)\right)^{-1} \beta^{\bullet}(k)$$

where $\beta^{\bullet}(k)$ is the optimal solution to (25).

Proof.
step 1)

$$\begin{aligned} & V_{n-1}(S(n-1)) \\ \equiv\; & \max_{\mathbf{h}(n)} E_{n-1}\left[-\exp\left(-\alpha_1 S(n)\right)\right] \\ =\; & \max_{\mathbf{h}(n)} E_{n-1}\left[-\exp\left(-\alpha_1\left(\begin{array}{c} \mathbf{h}^t(n)\left[\mathbf{X}(n) - r(n)\right] \\ +r(n) S(n-1) \end{array}\right)\right)\right] \\ =\; & \max_{\beta(n)} E_{n-1}\left[-\exp\left(-\alpha_1 r(n) S(n-1) - \beta^t(n)\left[\frac{\mathbf{X}(n)}{r(n)} - 1\right]\right)\right] \\ =\; & -\exp\left(-\alpha_1 r(n) S(n-1)\right) \underbrace{\min_{\beta(n)} E_{n-1}\left[\exp\left(-\beta^t(n)\left[\frac{\mathbf{X}(n)}{r(n)} - 1\right]\right)\right]}_{=A_{\text{exp}}(n-1)} \end{aligned}$$

The optimal portfolio choice in period n is then given by

$$\mathbf{h}^{\bullet}(n) = \left(\alpha_1 r(n)\right)^{-1} \beta^{\bullet}(n)$$

where $\beta^{\bullet}(n)$ is the optimal solution to

$$\min_{\beta(n)} E_{n-1}\left[\exp\left(-\beta^t(n)\left[\frac{\mathbf{X}(n)}{r(n)} - 1\right]\right)\right]$$

step k)
Define $A_{\text{exp}}(k)$ by (25) and assume

$$V_k(S(k)) = -\exp\left(-\alpha_1 \prod_{j=k+1}^{n} r(j) S(k)\right) A_{\text{exp}}(k)$$

$$
\begin{aligned}
& V_{k-1}(S(k-1)) \\
\equiv\ & \max_{\mathbf{h}(k)} E_{k-1}\left[V_k(S(k))\right] \\
=\ & \max_{\mathbf{h}(k)} E_{k-1}\left[-\exp\left(-\alpha_1 \prod_{j=k+1}^{n} r(j)\, S(k)\right) A_{\exp}(k)\right] \\
=\ & \max_{\mathbf{h}(k)} E_{k-1}\left[-\exp\left(-\alpha_1 \prod_{j=k+1}^{n} r(j)\left(\mathbf{h}^t(k)\left[\mathbf{X}(k)-r(k)\right]\right.\right.\right. \\
& \qquad \left.\left.\left. +r(k)\, S(k-1)\right)\right)\cdot A_{\exp}(k)\right] \\
=\ & \max_{\beta(k)} E_{k-1}\left[-\exp\left(-\alpha_1 \prod_{j=k}^{n} r(j)\, S(k-1) - \beta^t(k)\left[\frac{\mathbf{X}(k)}{r(k)}-1\right]\right)\right. \\
& \qquad \left.\cdot A_{\exp}(k)\right] \\
=\ & -\exp\left(-\alpha_1 \prod_{j=k}^{n} r(j)\, S(k-1)\right) A_{\exp}(k-1)
\end{aligned}
$$

The optimal portfolio choice in period k is then given by

$$
\mathbf{h}^{\bullet}(k) = \left(\alpha_1 \prod_{j=k}^{n} r(j)\right)^{-1} \beta^{\bullet}(k)
$$

where $\beta^{\bullet}(k)$ is the optimal solution to (25).

■

Note for all proofs that the sign of the function plays no role, so these three main type of functions determine the form of the indirect utility function for all HARA utility functions. We have proved that it is necessary for a HARA utility function to have indirect utility functions of the same class with the absolute risk aversion parameter being discounted by the riskless rate of interest. Since our results hold for any positive affine transformation, this is, however, also sufficient because the constants A are all positive, where A is short for one of the constants $A_{\log}$, $A_{\exp}$ or A_{pow}. To sum up, we can formulate:

Theorem 1 *Consider an arbitrary process* $\mathbf{X}(k)$. *If* $E[U(S(n))] < \infty$ *and interest rates are deterministic then we have*

$$
U \in HARA(\alpha_1, \alpha_2) \Leftrightarrow k = 1, \ldots, n : V_{k-1} \in HARA\left(\alpha_1 \prod_{j=k}^{n} r(j), \alpha_2\right)
$$

and in that case the optimal absolute portfolio weights will be of the form

$$
\mathbf{h}^{\bullet}(k) = \left(\alpha_1^{-1} \prod_{j=k}^{n} r^{-1}(j) + \alpha_2^{-1} S(k-1)\right) \beta^{\bullet}(k)
$$

where $\beta^{\bullet}(k)$ *can be found by maximizing the proper constant defined in* (21), (23) *or* (25).

Observe that the absolute risk aversion coefficient α_1 is not used to find the optimal portfolio-mix β, and therefore it plays no role in determining A. Second we note that if $r(k) \succ 1$ then the discounting will diminish its importance in the optimal absolute portfolio weight. This will be pursued later when we look at convergence for the first period decision function.

To interpret A we rewrite the three indirect utility functions for period k:

$$V_k^{\log}(S(k)) = \underbrace{\log\left(\tfrac{1}{\alpha_1} + \prod_{j=k+1}^{n} r(j)\,S(k)\right)}_{=U\left(\prod_{j=k+1}^{n} r(j)S(k)\right)} + A_{\log}(k) - \sum_{j=k+1}^{n} \log(r(j)) \quad (26)$$

$$V_k^{pow}(S(k)) = \underbrace{\tfrac{\alpha_2}{1-\alpha_2}\left(\tfrac{1}{\alpha_1} + \frac{\prod_{j=k+1}^{n} r(j)S(k)}{\alpha_2}\right)^{1-\alpha_2}}_{=U\left(\prod_{j=k+1}^{n} r(j)S(k)\right)} A_{pow}(k)\left(\prod_{j=k+1}^{n} \tfrac{1}{r(j)}\right)^{1-\alpha_2} \quad (27)$$

$$V_k^{\exp}(S(k)) = \underbrace{-\exp\left(-\alpha_1 \prod_{j=k+1}^{n} r(j)\,S(k)\right)}_{=U\left(\prod_{j=k+1}^{n} r(j)S(k)\right)} A_{\exp}(k) \quad (28)$$

Thus the indirect utility functions can be decomposed into one part that expresses the utility obtained if the entire wealth were invested in the riskless asset until the terminal date and a part that contains A. We interpret this as A being an adjustment to the expected utility of the sure return. The formulation (26) – (28) suggests that A for some utility functions could be expressed as measuring the maximum expected utility of *excess return* from risky investments $\beta^t(k)[\mathbf{X}(k) - r(k)]$.

3.3 Existence of a Solution

The theorem shows the exact form of the indirect utility functions and the constant A gives a guide on how to find the optimal portfolio mix. However we are not sure that a solution does exist. Existence would require some sort of no-arbitrage assumption for the unrestricted problem to have a solution. To understand which, we look at an example.

For all assets with positive price relatives we can choose the initial price to be unity and thus use the theory of absolute prices. We define an **arbitrage** as a portfolio that satisfies:

$$P\left(\frac{S(k)}{S(k-1)} \geq r(k)\right) = 1 \quad (29)$$

$$P\left(\frac{S(k)}{S(k-1)} > r(k)\right) > 0 \quad (30)$$

We rewrite this by using (13):

$$\begin{aligned}\frac{S(k)}{S(k-1)} &= \frac{r(k)\,S(k-1)+\mathbf{h}^t(k)\,[\mathbf{X}(k)-r(k)]}{S(k-1)} \\ &= r(k)+\mathbf{b}^t(k)\,[\mathbf{X}(k)-r(k)] \qquad (31)\end{aligned}$$

Combining (29) – (31) we will denote the model **free of arbitrage**[11] if for any self-financing portfolio with relative weights $\mathbf{b}(k)$ with at least one element different from zero we have

$$P\left(\mathbf{b}^t(k)\,[\mathbf{X}(k)-r(k)]<0\right)>0 \qquad (32)$$

Hakansson [8] proves that this condition secures existence of a solution if a solvency constraint is added to the problem. However, the proof is rather technical, and will be left out.

Example 2 *Consider a model with $n=1$ and two assets. One asset is riskless and pays r. The return of the other asset is defined by*

$$X=\begin{cases} X_1 & \text{with probability } p \\ X_2 & \text{with probability } 1-p \end{cases}$$

where $X_1 \neq X_2$. We consider a utility function of the power function family $U(S)=\frac{\alpha_2}{1-\alpha_2}\left(\alpha_1^{-1}+\alpha_2^{-1}S\right)^{1-\alpha_2}$, $\alpha_2<1$ so by theorem 1 we have the indirect utility function

$$V_0(S)=\frac{\alpha_2 A}{1-\alpha_2}\left(\frac{\alpha_1^{-1}}{r}+\alpha_2^{-1}S\right)^{1-\alpha_2}$$

The optimal portfolio mix is found by maximizing A

$$A=\max_{\beta} E\left[\left(r+\beta\frac{X-r}{\alpha_2}\right)^{1-\alpha_2}\right] \qquad (33)$$

By differentiating $E\left[\left(r+\beta\frac{X-r}{\alpha_2}\right)^{1-\alpha_2}\right]$ w.r.t. β we obtain the first order condition

$$\begin{aligned} E\left[(1-\alpha_2)\left(r+\beta\frac{X-r}{\alpha_2}\right)^{-\alpha_2}(X-r)\right] &= 0 \\ \Leftrightarrow\ (p-1)\left(r+\beta\frac{X_2-r}{\alpha_2}\right)^{-\alpha_2}(X_2-r) &= p\left(r+\beta\frac{X_1-r}{\alpha}\right)^{-\alpha_2}(X_1-r) \\ \Leftrightarrow\ \left(\frac{r+\beta\frac{X_1-r}{\alpha_2}}{r+\beta\frac{X_2-r}{\alpha_2}}\right)^{-\alpha_2} &= \underbrace{-\frac{1-p}{p}\cdot\frac{X_2-r}{X_1-r}}_{\equiv\xi} \end{aligned}$$

[11] Hakansson [8] argues that it may be viewed as a condition that prices of the various assets in the market must satisfy in equilibrium, and therefore he uses the term "no-easy-money" condition. However the definition used in Hakansson [9] or Hakansson and Ziemba [10] is not exactly the same.

Thus we have

$$\begin{aligned}\beta^\bullet &= \alpha_2 \frac{r\left(\xi^{-\alpha_2^{-1}}-1\right)}{(X_1-r)-\xi^{-\alpha_2^{-1}}(X_2-r)} \\ &= \alpha_2 \frac{r\frac{\xi^{-\alpha_2^{-1}}-1}{X_1-r}}{1+\frac{p}{1-p}\xi^{1-\alpha_2^{-1}}}\end{aligned}$$

This would be valid for arbitrary α_2 if $\xi > 0$. The no arbitrage condition simply states that one of the following two cases must be met

$$X_1 > r > X_2 \tag{34}$$

$$X_2 > r > X_1 \tag{35}$$

so $\xi > 0$ if and only if the model is free of arbitrage. Second, assume (34) holds then we see that $\xi < 1$ if and only if $EX > r$. This is a necessary condition for any risk averse investor to be interested in the risky asset ($\beta > 0$).

3.4 Portfolios Independent of Initial Wealth

We are interested in identifying the utility functions that allow portfolio choice to be made independently of the initial wealth, such that a solution can be found once and for all, and then just be scaled. This is why such portfolios are also termed **proportional**. For the single period problem, this is obviously obtained by the CRRA-class.

$$\log\left(rS+\mathbf{h}^t[\mathbf{X}-r]\right) = \log(r+\mathbf{b}^t[\mathbf{X}-r])+\log(S) \tag{36}$$

$$\left(rS+\mathbf{h}^t[\mathbf{X}-r]\right)^{1-\alpha_2} = S^{1-\alpha_2}(r+\mathbf{b}^t[\mathbf{X}-r])^{1-\alpha_2} \tag{37}$$

We now look at the multiperiod problem and follow Bertsekas [2]:

Definition 3 *Consider the optimal absolute portfolio weight $\mathbf{h}^\bullet(k,S(k-1))$, $k=1,\dots,n$. We call it **partially separated** if there exists some real function g so*

$$\mathbf{h}^\bullet(k,S(k-1)) = \beta^\bullet(k)g\left(S\left(k-1\right)\right) \quad \textit{where } \beta^\bullet(k)\in\mathbb{R}^m$$

*We say that $\mathbf{h}^\bullet(k,S(k-1))$ is **completely separated** if there exist some constant α so $g(S(k-1))=\alpha S(k-1)$.*

We term the relative portfolio weights $k=1,\dots,n$ by $\mathbf{b}(k)$ defined by

$$b_i(k) = \frac{h_i(k)}{S(k-1)} \quad \text{for } i=1,\dots,m$$

Lemma 1 *The optimal* relative *portfolio weight, $\mathbf{b}^\bullet(k)$, is independent of wealth, $S(k-1)$, if and only if the optimal* absolute *portfolio weight $\mathbf{h}^\bullet(k,S(k-1))$ is **completely separated***
The quotient

$$\frac{h_i^\bullet}{h_j^\bullet} \quad \textit{for } i,j=1,\dots,m \tag{38}$$

*is independent of wealth, $S(k-1)$, if and only if, $\mathbf{h}^\bullet(k,S(k-1))$, is **partially separated.***

Proof.
The proof is trivial. Simply observe that for $i = 1, ..., m$:

$$b_i^\bullet(k) \equiv \frac{h_i^\bullet(k)}{S(k-1)} = \alpha\beta_i^\bullet(k) \Leftrightarrow h_i^\bullet(k) = \beta_i^\bullet(k) \cdot \alpha S(k-1)$$

and the quotient (38) is independent of wealth if and only if the wealth enters as $h_i^\bullet(k) = \beta_i^\bullet(k) \cdot g(S(k-1))$.

■

We are interested in the initial wealth, $S(0)$. As a corollary of theorem 1 we directly note:

Corollary 1 [12]*Consider a MPP with a utility function $U \in HARA(\alpha_1, \alpha_2)$. The optimal relative portfolio weights for a utility function in the CRRA class are independent of initial wealth.*
For any *utility function in the HARA class the quotient (38) is independent of initial wealth.*

Proof.
From theorem 1 we have $\mathbf{h}^\bullet(k) = \left(\alpha_1^{-1}\prod_{j=k}^{n} r^{-1}(j) + \alpha_2^{-1}S(k-1)\right)\beta^\bullet(k)$. Thus the optimal portfolio weights are partially separated for any HARA function and completely separated for the CRRA class, $\alpha_1 = \infty$. The conclusion then follows by lemma 1 used in period 0.

■

We see that the CRRA class is the only class which allows for two equivalent formulations, either with relative or absolute portfolio weights. To extend this idea to all HARA functions we must preclude the money market account.

It is important to point out that by "equivalent formulation" we do not mean that we can use the exact same utility function in both formulations, but that they are equivalent. Thus one can be expressed as a positive affine transformation of the other. This transformation will probably depend upon initial wealth as Mossin [13] points out and is illustrated in the one period case in equation (36) and (37).

For any utility function in the HARA class we first decide how much to invest in the riskless asset. The rest is then invested in risky assets. However, since (38) is independent of wealth we will invest the same proportion of the rest in each of the risky assets. This can be seen as forming some sort of market portfolio and then deciding on how much to put in the market portfolio and in the riskless asset.

3.5 Myopic Policies

As noted, it is generally suboptimal to make decisions for each period at a time *without* looking ahead. However, there may be utility functions, for which such a procedure is optimal. If we can identify such utility functions, the investor's problem will certainly be simplified. First of all he will not have to make forecasts for the entire investment horizon, but only for the next period. Second

[12]In fact, the point of departure for Mossin [13] is the discussion of 'constant' portfolios, meaning here independent of initial wealth.

the exact length of the investment horizon now becomes unimportant for the investment behavior. Thus he will not behave any differently, if he alters his investment horizon. The next definition by Mossin [13] tries to capture such a policy:

Definition 4 *Consider an investor maximizing terminal wealth. A* ***myopic*** *policy is one where the investor at each point in time optimally can act as if that period was the last.*
A ***partial myopic*** *policy is one, where the investor optimally can act as if the entire resulting wealth would be reinvested in a riskless asset.*

As a corollary of theorem 1 we directly note

Corollary 2 [13] *Consider a MPP with a utility function* $U \in HARA(\alpha_1, \alpha_2)$.
If $\alpha_1 = \infty$ *then a myopic policy is optimal.*
If $\alpha_1 \neq \infty$ *then a partial myopic policy is optimal. (If* $r(k) = 1$ *for all,* $k = 1, ..., n$ *a myopic policy optimal.)*

Proof.
Theorem 1 gives the indirect utility function $V_{k-1} \in HARA\left(\alpha_1 \prod_{j=k}^{n} r(j), \alpha_2\right)$ for $U \in HARA(\alpha_1, \alpha_2)$. Thus the decision in period $k = 1, \ldots, n$ depends only on $S(k-1)$ and the distribution of the risky assets in period k. The future information is put into the constants A. Since we can always choose not to invest or invest all our wealth in the money market account, these will be positive, and can therefore be neglected since we are allowed to make a positive affine transformation.

■

Investment decisions are made according to our risk aversion. So it is clear that maximizing $E[U(S(1))]$ and $E[U(S(n))]$ are not necessarily the same since we have different wealth levels. Now, corollary 2 states that we can scale $S(1)$ by the riskless rate and get the right solution. For CRRA utility functions we do not need the scaling since $U(k \cdot S) \approx U(S)$, hence for CRRA utility functions a myopic policy is optimal.

This is very interesting. If an investor has a CRRA terminal utility function, he can act optimally in every period without any concern to the future, no matter how correlated the asset returns are. Even a postponement of the terminal period will not alter his investment decisions in any period!

In Section 4.2 we will show that the logarithmic utility function optimizes the expected growth rate. So consider the following statement:

> Every agent prefers more to less and the expected logarithm will asymptotically accumulate the highest wealth almost surely, hence the expected logarithm is optimal for every agent with a distant horizon.

[13] This is the main result of Mossin [13]. His proof is based on the two asset case (Justified by the two fund separation property of the HARA class proofed by Cass&Stiglitz in their 1970 article (our reference is Ziemba and Vickson [16])).

Instead we start out with theorem 1 which allow us to establish several positive spin-offs: we can obliterate the assumption of independent stock returns, make the results on myopic policies, proportional portfolio weights and isoelastic convergence appear as corollaries and finally identify the consequences of various constrains directly.

In the light of Corollary 2 this statement is *false*. Corollary 2 states that the decision of an agent with an isoelastic utility function is independent of the horizon. Hence, his decision will not approach a logarithmic decision. As we will see in the next section a larger class of utility functions can be approximated by an isoelastic utility function when the horizon is distant.

3.6 Convergence to an Isoelastic Utility Function

We are interested in the first period decision, i.e. the decision function for today. If a utility function allows for myopia then the first period decision can disregard all future periods. As we have seen, myopia is optimal for the isoelastic utility functions (CRRA class). In this section we will try to uncover the class of functions for which the optimal first period decision can approximately be made myopic. We will search for utility functions where the first period decision function, $V_0(\cdot)$, converge to an isoelastic utility function when the investment-horizon goes to infinity.

First, observe the time effect in the case of partial myopia. As seen from the solutions (20) and (22) α_1 is discounted by all future interest rates. Thus if $\prod_{k=1}^{\infty} r(k) = \infty$ the effect of α_1 is diminishing for the first period decision function as the investment horizon becomes long. Thus the first period decision function of the power function or the logarithmic utility function behaves myopic as the investment horizon tends to infinity. Stated with the equivalent transformation ($A(k)$ is positive) of the first period decision function

$$v_k(S(k), n) \equiv \frac{V_k(S(k), n)}{A(k, n)} \tag{39}$$

where we explicitly have indicated the dependence of the time horizon n, we have

Power function:

$$v_0(S, n) = \frac{\alpha_2}{1-\alpha_2}\left(\underbrace{\frac{1}{\alpha_1}\prod_{j=1}^{n}\frac{1}{r(j)}}_{\to 0}+\frac{S}{\alpha_2}\right)^{1-\alpha_2} \to \frac{\alpha_2}{1-\alpha_2}\left(\frac{S}{\alpha_2}\right)^{1-\alpha_2}$$

Logarithmic:

$$v_0(S, n) = \log\left(\underbrace{\frac{1}{\alpha_1}\prod_{j=1}^{n}\frac{1}{r(j)}}_{\to 0}+S\right) \to \log(S)$$

This observation was first made by Mossin [13].

The following lemma will be of great help in expanding the class of utility functions for which the first period decision function converges to an isoelastic utility function.

Lemma 2 [14]*Let U^1 and U^2 be two continuous utility functions on $]d,\infty[$ with $U^1 \geq U^2$ and assume that the self-financing condition (13) holds. Denote the indirect utility functions V_k^1 and V_k^2 respectively. If the indirect utility functions exist then $V_k^1 \geq V_k^2$ for all $k = 0, \ldots, n-1$:*

Proof.
Denote the optimal solutions $\mathbf{h}^{1\bullet}$ and $\mathbf{h}^{2\bullet}$ for V_k^1 and V_k^2 respectively. Then the inequality follows by

$$
\begin{aligned}
V_{n-1}^2 (S(n-1), n) &= \max_{\mathbf{h}(n)} E_{n-1} \left[U^2 (S(n))\right] \\
&= E_{n-1}\left[U^2 \left(\begin{array}{c} \mathbf{X}^t(n)\, \mathbf{h}^{2\bullet}(n) \\ +\left(S(n-1) - \mathbf{1}^t \mathbf{h}^{2\bullet}(n)\right) r(n) \end{array} \right)\right] \\
&\leq E_{n-1}\left[U^1 \left(\begin{array}{c} \mathbf{X}^t(n)\, \mathbf{h}^{2\bullet}(n) \\ +\left(S(n-1) - \mathbf{1}^t \mathbf{h}^{2\bullet}(n)\right) r(n) \end{array} \right)\right] \\
&\leq E_{n-1}\left[U^1 \left(\begin{array}{c} \mathbf{X}^t(n)\, \mathbf{h}^{1\bullet}(n) \\ +\left(S(n-1) - \mathbf{1}^t \mathbf{h}^{1\bullet}(n)\right) r(n) \end{array} \right)\right] \\
&= \max_{\mathbf{h}(n)} E_{n-1}\left[U^1 (S(n))\right] = V_{n-1}^1 (S(n-1), n)
\end{aligned}
$$

$V_k^1 \geq V_k^2$ now follows by induction by the same tune as above.

■

Now we can formulate:

Corollary 3 *Assume $\prod_{k=1}^{\infty} r(k) = \infty$. Let $U(\cdot)$ be any (arbitrary) terminal utility function with indirect utility functions $V_k(\cdot)$ and let*

$$
v_0(S(0), n) \equiv \frac{V_0(S(0), n)}{A_{pow}(0, n)}
$$

If there exist $\alpha_2 \neq \infty$ and $\alpha_1 > 0$ so we for $U(\cdot)$ (or a positive affine transformation of it) with $S(0) > \alpha_2 \alpha_1^{-1}$ have

$$
\frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1} S - \alpha_1^{-1}\right)^{1-\alpha_2} \leq U(S) \leq \frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1} S + \alpha_1^{-1}\right)^{1-\alpha_2} \tag{40}
$$

then

$$
\lim_{n\to\infty} v_0(S,n) \approx \begin{cases} \frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1} S\right)^{1-\alpha_2} & \alpha_2 \neq 1 \text{ and } \alpha_2 \neq \infty \\ \log(S) & \alpha_2 = 1 \end{cases}
$$

Proof.
By applying lemma 2 on (40) and use theorem 1 on the bounds we get:

$$
\begin{aligned}
& \frac{\alpha_2 A_{pow}(0,n)}{1-\alpha_2} \left(\alpha_2^{-1} S(0) - \alpha_1^{-1} \prod_{j=1}^{n} r^{-1}(j)\right)^{1-\alpha_2} \\
\leq\ & V_0(S(0), n) \\
\leq\ & \frac{\alpha_2 A_{pow}(0,n)}{1-\alpha_2} \left(\alpha_2^{-1} S(0) + \alpha_1^{-1} \prod_{j=1}^{n} r^{-1}(j)\right)^{1-\alpha_2}
\end{aligned}
$$

[14] The lemma is inspired by the lemmas 3 and 4 in Hakansson [9], but we have chosen not to add constrains to the MPP in advance as Hakansson does.

for $S(0) > \alpha_2 \alpha_1^{-1} \prod_{j=1}^{n} r^{-1}(j)$. Dividing by $A_{pow}(0,n)$ and taking the limits yields the desired result:

$$\frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1}S\right)^{1-\alpha_2} \leq \lim_{n\to\infty} v_0(S,n) \leq \frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1}S\right)^{1-\alpha_2}$$

■

The corollary states, that if we for an arbitrary terminal utility function are able to find a power function or an isoelastic logarithmic function ($\alpha_1 = \infty$), that shifted to the right serves as a lower bound to $U(\cdot)$ and left as an upper bound, then the first period decision function converges to that particular function. Thus for such a bounded utility function the first period decision function converges to an isoelastic utility function, which we have proved allows for myopia.

But what does this mean? If the shape of a given terminal utility function is like a power function or a logarithmic function, the decision 10 periods before the terminal day is not myopic, but the decision taken say 100 period before the terminal day can be made myopic.

Now can this result be stated for a larger class of utility functions? If we extended the bounds of (40) by subtracting/adding a constant B, the same procedure can still be used and we would end up with

$$\begin{aligned} & \frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1}S\right)^{1-\alpha_2} - \frac{B}{\lim_{n\to\infty} A_{pow}(0,n)} \\ \leq & \lim_{n\to\infty} v_0(S,n) \\ \leq & \frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1}S\right)^{1-\alpha_2} + \frac{B}{\lim_{n\to\infty} A_{pow}(0,n)} \end{aligned}$$

Thus if we can secure that $\lim_{n\to\infty} A_{pow}(0,n) = \infty$ the answer is affirmative.

Hakansson [9] states the following theorem

Theorem 2 *Let $U(\cdot)$ be any terminal utility function with indirect utility functions $V_k(\cdot)$ and let*

$$v_0(S(0),n) \equiv \frac{V_0(S(0),n)}{A_{pow}(0,n)}$$

Assume that $r(k) \succ 1$ for $k = 1,\dots,n$ and price relatives are independent over time. If there exists $B > 0$ and α_1, α_2 such that the terminal utility function $U(S)$ satisfies:

$$\frac{\alpha_2}{1-\alpha_2}\left(\frac{S}{\alpha_2} - \frac{1}{\alpha_1}\right)^{1-\alpha_2} - B \leq U(S) \leq \frac{\alpha_2}{1-\alpha_2}\left(\frac{S}{\alpha_2} + \frac{1}{\alpha_1}\right)^{1-\alpha_2} + B \quad (41)$$

then

$$\lim_{n\to\infty} v_0(S,n) = \frac{\alpha_2}{1-\alpha_2}\left(\alpha_2^{-1}S\right)^{1-\alpha_2} \text{ for } S > 0$$

Note that we can use different constants for the lower and upper bounds. We further note that (41) can be obtained only by one α_2.

The theorem states that if there for a given terminal utility function exists a power function which shifted any finite distance southeast becomes a lower bound and northwest becomes an upper bound, then the induced first period decision function will converge to that power function.

A natural question is what have we achieved by the corollary 3 and theorem 2? Since we can shift the bounds to the side, the only important aspect of the utility function is its behavior for substantial wealth - not its behavior around zero. If a utility function can be bounded by shapes of a power function or a logarithmic utility function, then it must be rather well behaved. Therefore the class of utility functions captured is perhaps not so interesting, if we can not include any 'generic' new shapes of utility functions. For instance, it is apparent that an exponential utility function will never be bounded by any power function, and thus not treated by theorem 2

It is important to point out, that although we conclude that the risk aversion for the *first period decision function,* $v_0(S,n)$, converges as the horizon, n, goes to infinity, we need not make any assumption about convergence for the risk aversion of the *terminal utility function,* $U(S)$, as wealth, S, goes to infinity. In fact it may oscillate arbitrarily around $\alpha_2 S^{-1}$ indefinitely without violating (41) (Hakansson [9] p213).

3.7 Concluding remarks

This section has stressed that the CRRA class is indeed interesting. Not only does these utility functions allow for a myopic policy and a formulation of the investment problem independent of initial wealth. This class also plays the role as the limit for first period decision functions as the investment horizon goes to infinity for any utility function with the shape of a logarithmic or power function. It appears that it is valid for all utility functions that can be bounded by power functions. On the other hand bounding by a power function does not allow for any 'generically' different utility functions like the exponential class.

One issue has been left out. We have proven that the first period decision functions converge for a class of utility functions, but we have not stated anything about the convergence for the optimal policy. Hakansson [9] states that the optimal policy also converges to the optimal policy of the isoelastic utility function in the sense of theorem 2.

4 Growth Optimality

In the previous chapter the best portfolio was derived by the principle of maximizing expected utility. This gave us a micro economic foundation for our theory and we could adapt the theory exactly to each individual investor. Theoretically, this is the right way to do portfolio selection. In the case of a distant horizon we have seen that we might be able to approximate the investor's utility function by a function in the CRRA class. In that case the investor just has to specify a relative risk aversion coefficient to select the right utility function. However, it is rarely the case that the investors are able to specify their utility functions. Their preferences are not sufficiently elucidated to be captured in a utility function. In the light of this it seems like a good idea to select a more objective goal like growth optimality than to maximize a subjective utility function.

We state two theorems which give the most important properties of the log-optimum portfolio, namely that the expected wealth of a non-anticipating strategy divided by the wealth of the log-optimum strategy is less than one and as time increases this expectation decreases. Surprisingly we do not have to make any assumptions on the stock price process to get these results. If we assume that the process is stationary ergodic the wealth of the log-optimum strategy grows exponentially fast with an asymptotic rate equal to the maximum rate given the infinite past. That is the best possible. We have chosen to ignore the considerations concerning measurability since we feel it is not essential for this note. We refer to Algoet and Cover [1] for a discussion of this topic.

4.1 The Framework

Let $(\Omega, \mathcal{F}, (\mathcal{F}(n))_{n=1}^{\infty}, P)$ be a filtered probability space where $\mathcal{F}(n)$ is defined as in Section 3.1. $\mathbf{X}(k)$ still denotes the price relatives for the m stocks in period k.

Define the **portfolio-space** as

$$B \equiv \left\{ \mathbf{b} \in \mathbb{R}^m \mid \sum_{i=1}^{m} b_i = 1, b_i \geq 0, i = 1, ..., m \right\}.$$

thus we exclude short-sale.

A **portfolio** $\mathbf{b}(k) \in B$ is the proportion of the wealth in period k invested in each of the m stocks. A **strategy** is a sequence of portfolios $\{\mathbf{b}(k)\}_{k=1}^{n}$, and we will call a strategy **non-anticipating** if $\mathbf{b}(k+1) \in \mathcal{F}(k)$ for every $k = 0, 1, ..., n-1$. I.e. the portfolio $\mathbf{b}(k+1)$ must be selected on the basis of the information at the end of period k.

Following the strategy $\{\mathbf{b}(k)\}_{k=1}^{n}$ the achieved **wealth** after n periods is

$$S(n) \equiv s_0 \prod_{k=1}^{n} \left(\mathbf{b}^t(k)\mathbf{X}(k)\right)$$

where $S(0) \equiv s_0$ is the initial wealth.

The rate of growth of $S(n)$ is termed

$$w(n) \equiv \frac{1}{n} \log S(n) = \frac{1}{n} \sum_{k=1}^{n} \log \left(\mathbf{b}^t(k)\mathbf{X}(k)\right)$$

If the strategy of choosing $\{\mathbf{b}(k)\}_{k=1}^{n}$ is self-financing[15], we will term the portfolio **rebalanced**. A rebalanced portfolio $\{\mathbf{b}\}_{k=1}^{n}$ independent of time will be termed **constant**.

We assume that we have perfect capital markets, thus no transaction costs, liquidity constraints or taxes. All stocks are traded and perfectly divisible. The issue of short-sale will briefly be looked at in the concluding remarks.

[15] Using the term self-financing about relative portfolio weights can be a little misleading since we cannot state that a strategy is self-financing just by looking at the portfolio weights.

4.2 Log-optimum

How do we choose a growth optimal strategy? To get an idea we will start by looking at an example.

Example 3 [16] *Let the rate of return be defined by* $r^b(k) \equiv \mathbf{b}^t\mathbf{X}(k) - 1$ *then the wealth after* n *periods of a constant rebalanced portfolio* $\mathbf{b} \in B$ *is defined by*

$$\begin{aligned} S^{\mathbf{b}}(n) &\equiv s_0 \prod_{k=1}^{n} \mathbf{b}^t\mathbf{X}(k) \\ &= s_0 \prod_{k=1}^{n} \left(1 + r^b(k)\right) \\ &= s_0 \exp\left(\sum_{k=1}^{n} \log\left(1 + r^b(k)\right)\right) \\ &= s_0 \exp\left(nw(n)\right) \end{aligned} \tag{42}$$

where $w(n) \equiv \frac{1}{n}\sum_{k=1}^{n} \log\left(1 + r^b(k)\right)$ *is the continuous growth rate over the first* n *periods. Assume that the price relatives,* $\mathbf{X}$*, are stationary ergodic then by the ergodic theorem*

$$\begin{aligned} w(n) &= \frac{1}{n}\sum_{k=1}^{n} \log\left(1 + r(k)\right) \\ &\to E\left[\log\left(1 + r(1)\right)\right] \qquad a.s. \\ &= E\left[\log\left(\mathbf{b}^t\mathbf{X}(1)\right)\right] \end{aligned} \tag{43}$$

i.e. if $E\left[\log\left(\mathbf{b}^t\mathbf{X}(1)\right)\right] < 0$ *then* $\exists\varepsilon > 0$ *such that for almost all* $\omega \in \Omega$ $w(n) \leq -\varepsilon < 0$ *from a certain step and from that step on*

$$0 \leq s_0 \exp\left(nw(n)\right) \leq s_0 \exp\left(-n\varepsilon\right) \to 0 \qquad a.s. \tag{44}$$

Similarly if $E\left[\log\left(\mathbf{b}^t\mathbf{X}(1)\right)\right] > 0$

$$s_0 \exp\left(nw(n)\right) \to \infty \qquad a.s. \tag{45}$$

We can conclude that in the special case of a stationary ergodic process and constant rebalanced portfolios, $E\left[\log\left(\mathbf{b}^t\mathbf{X}(1)\right)\right]$ not $E\left[\mathbf{b}^t\mathbf{X}(1)\right]$ *determines the rate of growth.*

Assume that the initial fortune $S(0) \equiv 1$ such that accumulated wealth after n periods is

$$S(n) = \prod_{k=1}^{n} \mathbf{b}(k)^t \mathbf{X}(k) \tag{46}$$

In the previous example we discovered that the expected log return played an essential role in growth maximization, so it seems like a good idea to define a log-optimum.

[16] Hakansson and Ziemba [10] uses an example like this

Definition 5 *A non-anticipating portfolio* $\mathbf{b}^{\star}(k) \in \mathcal{F}(k-1)$ *is called a* log-optimum *for period* k *if*

$$E_{k-1}\left[\log\left((\mathbf{b}^{\star}(k))^{t}\,\mathbf{X}(k)\right)\right] = \max_{\substack{\mathbf{b}(k)\in\mathcal{F}(k-1)\\ \mathbf{b}(k)\in B}} E_{k-1}\left[\log\left(\mathbf{b}(k)^{t}\,\mathbf{X}(k)\right)\right] \quad (47)$$

$W^{\star}(k)$ *is the expected log return of* $\mathbf{b}^{\star}(k)$

$$W^{\star}(k) \equiv E\left[\log\left((\mathbf{b}^{\star}(k))^{t}\,\mathbf{X}(k)\right)\right] \quad (48)$$

Since we are operating on a compact set the maximum is well defined. However, we can not be sure that $\mathbf{b}^{\star}$ is uniquely defined. In the following we will suppress $\mathbf{b} \in B$ when we maximize. Because of Kelly's work in growth optimality a log-optimum strategy is also known as a Kelly strategy[17].

4.2.1 Properties of Log-optimum Wealth

We are chasing a strategy that accumulates more wealth than any other non-anticipating strategy. The example in the previous section leads us to believe that the log-optimum strategy could have this desirable quality. The following theorem states that for any point in time the expected ratio of the wealth accumulated by a non-anticipating strategy and the log-optimum strategy is less than one. Furthermore, the ratio is a supermartingale which means that we would expect this ratio to decrease as time goes by. We also show that the growth rate of the log-optimum strategy is at least as good asymptotically as any non-anticipating strategy. This is why $\mathbf{b}^{\star}$ also is termed the growth-optimal strategy first pointed out by Breiman [5].

Theorem 3 [18] *Let* $\{\mathbf{b}^{\star}(k)\}_{k=1}^{n}$ *be a log-optimum strategy with compounded wealth*

$$S^{\star}(n) = \prod_{k=1}^{n} \mathbf{b}^{\star}(k)^{t}\mathbf{X}(k) \quad (49)$$

and $\{\mathbf{b}(k)\}_{k=1}^{n}$ *a competing non-anticipating strategy with compounded wealth*

$$S(n) = \prod_{k=1}^{n} \mathbf{b}^{t}(k)\mathbf{X}(k)$$

Then $\left(\frac{S(n)}{S^{\star}(n)}, \mathcal{F}(n)\right)$ *is a nonnegative supermartingale converging almost surely to a random variable* Y *where* $E[Y] \leq 1$*. Furthermore* $E\left[\frac{S(n)}{S^{\star}(n)}\right] \leq 1$ *for every* n *and*

$$\limsup_{n\to\infty}\left(\frac{1}{n}\log\left(\frac{S(n)}{S^{\star}(n)}\right)\right) \leq 0 \quad a.s. \quad (50)$$

[17] The term Kelly criterion or Kelly strategy is used in Thorp [15]

[18] This theorem is called Asymptotic Optimality Principle in Algoet and Cover [1]

Proof.
Since

$$
\begin{aligned}
& E_n\left[\frac{S(n+1)}{S^\star(n+1)}\right] \\
= & E_n\left[\frac{S(n)}{S^\star(n)}\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}\right] \\
= & \frac{S(n)}{S^\star(n)}E_n\left[\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}\right]
\end{aligned}
\tag{51}
$$

we only need to prove that $E_n\left[\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}\right]\leq 1$.

Define $\mathbf{b}_\alpha \equiv (1-\alpha)\,\mathbf{b}^\star + \alpha\mathbf{b}$ for every $\mathbf{b}\in B$ and $\alpha\in]0,1[$. Since $\mathbf{b}^\star$ is the log-optimum the Kuhn-Tucker condition states

$$
\frac{d}{d\alpha}E_n\left[\log\left(\mathbf{b}_\alpha(n+1)^t\mathbf{X}(n+1)\right)\right]\Big|_{\alpha=0^+}\leq 0 \tag{52}
$$

i.e. the derivative is non positive in all directions. We will now evaluate the derivative [19]

$$
\begin{aligned}
& \frac{d}{d\alpha}E_n\left[\log\left(\mathbf{b}_\alpha(n+1)^t\mathbf{X}(n+1)\right)\right]\Big|_{\alpha=0^+} \\
= & \lim_{\alpha\searrow 0}\frac{E_n\left[\log\left(\mathbf{b}_\alpha(n+1)^t\mathbf{X}(n+1)\right)\right]-E_n\left[\log\left(\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)\right)\right]}{\alpha} \\
= & \lim_{\alpha\searrow 0}\frac{E_n\left[\log\left(\frac{\mathbf{b}_\alpha(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}\right)\right]}{\alpha} \\
= & \lim_{\alpha\searrow 0}\frac{E_n\left[\log\left(1-\alpha+\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}\alpha\right)\right]}{\alpha} \\
= & \lim_{\alpha\searrow 0}\frac{E_n\left[\log\left(1+\left(\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}-1\right)\alpha\right)\right]}{\alpha} \\
= & \lim_{\alpha\searrow 0}E_n\left[\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}-1\right]+\varepsilon(\alpha) \\
= & E_n\left[\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}-1\right] \\
= & E_n\left[\frac{\mathbf{b}(n+1)^t\mathbf{X}(n+1)}{\mathbf{b}^\star(n+1)^t\mathbf{X}(n+1)}\right]-1
\end{aligned}
$$

which is what we wanted. Since $\left(\frac{S(n)}{S^\star(n)},\mathcal{F}(n)\right)$ is a supermartingale we can state

$$
E\left[\frac{S(n)}{S^\star(n)}\right]\leq E\left[\frac{S(0)}{S^\star(0)}\right]=1 \qquad \text{for every } n \tag{53}
$$

[19] To evaluate the derivative we will use the Taylor expansion $\log(1+kx)=\log(1+k\cdot 0)+kx-\frac{k^2x^2}{(1+k\xi)^2}$, where $\xi\in[0,x]$ so $\frac{\log(1+kx)}{x}=k+\varepsilon(x)$, where $\varepsilon(x)\to 0$ for $x\to 0$.

Since $\frac{S(n)}{S^\star(n)} \geq 0$ we have a nonnegative supermartingale. In particular, $\frac{S(n)}{S^\star(n)}$ is bounded below and such a supermartingale has a limit Y a.s. by The Martingale Convergence Theorem. Furthermore, 0 closes the supermartingale, hence the limit Y closes the supermartingale

$$E_n[Y] \leq \frac{S(n)}{S^\star(n)} \quad \text{for every } n \tag{54}$$

in particular

$$E[Y] = E_0[Y] \leq \frac{S(0)}{S^\star(0)} = 1 \tag{55}$$

To prove (50) we wish to use the Borel-Cantelli lemma. By Chebychevs inequality and (53)

$$\begin{aligned} & P\left(\frac{S(n)}{S^\star(n)} > \exp(n\varepsilon)\right) \\ \leq\ & \exp(-n\varepsilon) E\left[\frac{S(n)}{S^\star(n)}\right] \\ \leq\ & \exp(-n\varepsilon) \end{aligned} \tag{56}$$

so that

$$\begin{aligned} & \sum_{k=1}^{\infty} P\left(\frac{S(k)}{S^\star(k)} > \exp(k\varepsilon)\right) \\ \leq\ & \sum_{k=1}^{\infty} \exp(-k\varepsilon) \\ <\ & \infty \end{aligned} \tag{57}$$

and by the Borel-Cantelli lemma

$$\begin{aligned} P\left(\frac{S(n)}{S^\star(n)} > \exp(n\varepsilon) \text{ infinitely often}\right) &= 0 \\ \Leftrightarrow P\left(\frac{S(n)}{S^\star(n)} \leq \exp(n\varepsilon) \text{ from a certain step}\right) &= 1 \\ \Leftrightarrow P\left(\tfrac{1}{n}\log\left(\frac{S(n)}{S^\star(n)}\right) \leq \varepsilon \text{ from a certain step}\right) &= 1 \end{aligned} \tag{58}$$

Since $\varepsilon > 0$ was arbitrary chosen we conclude that

$$\limsup_n \tfrac{1}{n}\log\left(\frac{S(n)}{S^\star(n)}\right) \leq 0 \quad \text{a.s.} \tag{59}$$

■

4.2.2 Log-optimum Rate of Growth

We will now turn our attention to the growth rate of the log-optimum portfolio. We wish to find the asymptotic rate of growth of the log-optimum strategy. The more information available the better. Therefore, the best possible information set must be an infinite number of realized stock returns. Hence, the best possible growth rate must be a growth rate based on this information set. The theorem shows that the asymptotic rate of growth of the log-optimum strategy is equal to the highest growth rate based on the infinite past. For this purpose we define a two-sided process of stock returns $\{\mathbf{X}(k)\}_{k=-\infty}^{\infty}$. We are now able to define the maximum expected log return given the infinite past

$$\begin{aligned} W^{\star}(\infty) &\equiv W^{\star}(\mathbf{X}(0)|\,\mathbf{X}(-1),\mathbf{X}(-2),\ldots) \\ &\equiv \max_{\mathbf{b}(0)\in\sigma(\mathbf{X}(-1),\mathbf{X}(-2),\ldots)} E\left[\log\left(\mathbf{b}^{t}(0)\mathbf{X}(0)\right)\right] \end{aligned}$$

Theorem 4 [20]*If the stock return process $\{\mathbf{X}(k)\}_{k=-\infty}^{\infty}$ is stationary ergodic, then $S^{\star}(n)$ will grow exponentially fast almost surely with an asymptotic rate equal to the maximum expected log return given the infinite past*

$$\tfrac{1}{n}\log\left(S^{\star}(n)\right) \to W^{\star}(\mathbf{X}(0)|\,\mathbf{X}(-1),\mathbf{X}(-2),\ldots) \qquad a.s. \tag{60}$$

Proof.
In order to prove this theorem we would like to apply the ergodic theorem to the process $\left(\log\left(\mathbf{b}^{\star}(n)^{t}\,\mathbf{X}(n)\right)\right)_{n=1}^{\infty}$. However, this is not necessarily a stationary ergodic process since $\mathcal{F}(n-1)\subseteq\mathcal{F}(n)$ hence

$$\begin{aligned} & E\left[\log\left(\mathbf{b}^{\star}(n+1)^{t}\,\mathbf{X}(n+1)\right)\right] \\ = & \max_{\mathbf{b}(n+1)\in\mathcal{F}(n)} E\left[\log\left(\mathbf{b}^{t}(n+1)\,\mathbf{X}(n+1)\right)\right] \\ \geq & \max_{\mathbf{b}(n)\in\mathcal{F}(n-1)} E\left[\log\left(\mathbf{b}^{t}(n)\,\mathbf{X}(n)\right)\right] \\ = & E\left[\log\left(\mathbf{b}^{\star}(n)^{t}\,\mathbf{X}(n)\right)\right] \end{aligned} \tag{61}$$

That is we cannot be sure that the expectation remains constant as time goes by, in which case the process is not stationary ergodic. This is due to the fact that $\mathbf{b}^{\star}(n+1)$ is chosen with more information than $\mathbf{b}^{\star}(n)$, so we can state that information is not a bad thing.

The problem is that the number of elements in $\mathcal{F}(n)$ increases with n. Instead we will use a sandwich argument. Define

$$\mathcal{F}^{(k)}(n) \equiv \begin{cases} \sigma(\mathbf{X}(1),\ldots,\mathbf{X}(n)) & 1\leq n\leq k \\ \sigma(\mathbf{X}(n-k+1),\ldots,\mathbf{X}(n)) & k<n \end{cases} \tag{62}$$

$$\mathcal{F}^{(\infty)}(n) \equiv \sigma(\ldots,\mathbf{X}(-1),\mathbf{X}(0),\ldots,\mathbf{X}(n)) \tag{63}$$

such that $\mathcal{F}^{(k)}(n)\subseteq\mathcal{F}(n)\subseteq\mathcal{F}^{(\infty)}(n)$. Notice that $\mathcal{F}^{(k)}(k)=\mathcal{F}(k)$. We now have $\mathcal{F}^{(k)}(n)$ which contains k elements from a certain step and $\mathcal{F}^{(\infty)}(n)$ which contains an infinite number of elements for every n. That is both of them contain "the same number of elements" as n increases. Let $\mathbf{b}^{(k)}(n)$ be a $\mathcal{F}^{(k)}(n-1)$ measurable portfolio that achieves the

$$\max_{\mathbf{b}(n)\in\mathcal{F}^{(k)}(n-1)} E\left[\log\left(\mathbf{b}^{t}(n)\,\mathbf{X}(n)\right)\right]$$

[20]This theorem is called Asymptotic Equipartition Property in Algoet and Cover [1]

Similarly let $\mathbf{b}^{(\infty)}(n)$ be a portfolio that achieves the maximum taken over $\mathcal{F}^{(\infty)}(n-1)$. We also define the accumulated wealth as

$$S^{(k)}(n) \equiv \prod_{i=1}^{n} \mathbf{b}^{(k)}(i)^t \mathbf{X}(i) \tag{64}$$

$$S^{(\infty)}(n) \equiv \prod_{i=1}^{n} \mathbf{b}^{(\infty)}(i)^t \mathbf{X}(i) \tag{65}$$

Since the process $\{\mathbf{X}(k)\}_{k=-\infty}^{\infty}$ is stationary we have that

$$\begin{aligned} & \max_{\mathbf{b}(n)\in\mathcal{F}^{(\infty)}(n-1)} E\left[\log\left(\mathbf{b}^t(n)\mathbf{X}(n)\right)\right] \\ = & \max_{\mathbf{b}(0)\in\mathcal{F}^{(\infty)}(-1)} E\left[\log\left(\mathbf{b}^t(0)\mathbf{X}(0)\right)\right] \qquad \text{for every } n \end{aligned}$$

This means that we are facing the same problem in every period and our portfolio choice is independent of time. We can choose $\mathbf{b}^{(\infty)} \equiv \mathbf{b}^{(\infty)}(0)$ and the process $\left(\log\left(\left(\mathbf{b}^{(\infty)}\right)^t \mathbf{X}(n)\right)\right)_{n=1}^{\infty}$ is stationary ergodic by Jacobsen [11] theorem VII.4.8. We can now apply the ergodic theorem to this process

$$\tfrac{1}{n}\log S^{(\infty)}(n) \to E\left[\log\left(\left(\mathbf{b}^{(\infty)}\right)^t \mathbf{X}(0)\right)\right] \qquad \text{a.s.} \tag{66}$$

For every k and for every $n > k$ we similarly have that

$$\begin{aligned} & \max_{\mathbf{b}(n)\in\mathcal{F}^{(k)}(n-1)} E\left[\log\left(\mathbf{b}^t(n)\mathbf{X}(n)\right)\right] \\ = & \max_{\mathbf{b}\in\sigma(\mathbf{X}(-k),\ldots,\mathbf{X}(-1))} E\left[\log\left(\mathbf{b}^t\mathbf{X}(0)\right)\right] \qquad \text{for every } n \end{aligned}$$

such that we can choose $\mathbf{b}^{(k)} \equiv \mathbf{b}^{(k)}(k+1)$ which is independent of time so that the process $\left(\log\left(\left(\mathbf{b}^{(k)}\right)^t \mathbf{X}(n)\right)\right)_{n=k+1}^{\infty}$ is stationary ergodic, and

$$\begin{aligned} & \tfrac{1}{n}\log S^{(k)}(n) \\ = & \tfrac{1}{n}\sum_{i=1}^{n} \log\left(\left(\mathbf{b}^{(k)}\right)^t \mathbf{X}(i)\right) & (67) \\ = & \tfrac{1}{n}\log S^{\star}(k) + \tfrac{1}{n}\sum_{i=k+1}^{n} \log\left(\left(\mathbf{b}^{(k)}\right)^t \mathbf{X}(i)\right) \\ \to & E\left[\log\left(\left(\mathbf{b}^{(k)}\right)^t \mathbf{X}(0)\right)\right] \qquad \text{a.s. for } n\to\infty \\ = & E\left[\log\left(\left(\mathbf{b}^{(k)}\right)^t \mathbf{X}(k+1)\right)\right] \\ = & \max_{\mathbf{b}(k+1)\in\mathcal{F}^{(k)}(k)} E\left[\log\left(\mathbf{b}^t(k+1)\mathbf{X}(k+1)\right)\right] & (68) \\ = & \max_{\mathbf{b}(k+1)\in\mathcal{F}(k)} E\left[\log\left(\mathbf{b}^t(k+1)\mathbf{X}(k+1)\right)\right] & (69) \\ = & E\left[\log\left(\mathbf{b}^{\star}(k+1)^t\mathbf{X}(k+1)\right)\right] \\ = & W^{\star}(k+1) & (70) \end{aligned}$$

Furthermore, since $\mathbf{b}^{(k)}(n) \in \mathcal{F}^{(k)}(n) \subseteq \mathcal{F}(n)$ and $\mathbf{b}^{\star}(n) \in \mathcal{F}(n) \subseteq \mathcal{F}^{(\infty)}(n)$ we have from the previous theorem that

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n}\log\left(\frac{S^{(k)}(n)}{S^{\star}(n)}\right) &\leq& \; 0 \quad \text{a.s.} \qquad (71)\\
\Leftrightarrow\; &\lim_{n\to\infty} \tfrac{1}{n}\log\left(S^{(k)}(n)\right) + \limsup_{n\to\infty}\left(-\tfrac{1}{n}\log\left(S^{\star}(n)\right)\right) &\leq& \; 0 \quad \text{a.s.}\\
\Leftrightarrow\; &\lim_{n\to\infty} \tfrac{1}{n}\log\left(S^{(k)}(n)\right) - \liminf_{n\to\infty}\left(\tfrac{1}{n}\log\left(S^{\star}(n)\right)\right) &\leq& \; 0 \quad \text{a.s.}
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} \tfrac{1}{n}\log\left(\frac{S^{\star}(n)}{S^{(\infty)}(n)}\right) &\leq& \; 0 \quad \text{a.s.} \qquad (72)\\
\Leftrightarrow\; &\limsup_{n\to\infty}\left(\tfrac{1}{n}\log\left(S^{\star}(n)\right)\right) &\leq& \; \lim_{n\to\infty}\tfrac{1}{n}\log\left(S^{(\infty)}(n)\right) \quad \text{a.s.}
\end{aligned}$$

hence

$$\begin{aligned}
W^{\star}(k+1) &= \lim_{n\to\infty}\tfrac{1}{n}\log S^{(k)}(n)\\
&\leq \liminf_{n\to\infty}\tfrac{1}{n}\log S^{\star}(n)\\
&\leq \limsup_{n\to\infty}\tfrac{1}{n}\log S^{\star}(n)\\
&\leq \lim_{n\to\infty}\tfrac{1}{n}\log S^{(\infty)}(n)\\
&= W^{\star}(\infty) \qquad (73)
\end{aligned}$$

Now we only need to see that $W^{\star}(k) \nearrow W^{\star}(\infty)$. We have already seen that $W^{\star}(\infty) \geq W^{\star}(k+1) \geq W^{\star}(k)$ so that $\{W^{\star}(k)\}_{k=1}^{\infty}$ has a limit.

$$\begin{aligned}
&E\left[\log\left((\mathbf{b}^{\infty}(0))^{t}\,\mathbf{X}(0)\right)\right]\\
=\; &\max_{\mathbf{b}(0)\in\mathcal{F}^{(\infty)}(-1)} E\left[\log\left(\mathbf{b}(0)^{t}\,\mathbf{X}(0)\right)\right]
\end{aligned}$$

that is $\mathbf{b}^{\infty}(0) \in \mathcal{F}^{(\infty)}(-1) = \cup_{k=1}^{\infty}\mathcal{F}(-k)$ where

$$\mathcal{F}(-k) = \sigma\left(\mathbf{X}(-k), \ldots, \mathbf{X}(-1)\right) \quad \text{for } k > 0$$

Hence there exists a $N > 0$ such that $\mathbf{b}^{\infty}(0) \in \mathcal{F}(-N)$ and

$$\begin{aligned}
&W^{\star}(\infty)\\
=\; &E\left[\log\left((\mathbf{b}^{\infty}(0))^{t}\,\mathbf{X}(0)\right)\right]\\
=\; &\max_{\mathbf{b}(0)\in\mathcal{F}(-N)} E\left[\log\left(\mathbf{b}^{t}(0)\,\mathbf{X}(0)\right)\right]\\
\leq\; &\max_{\mathbf{b}(0)\in\mathcal{F}(-N-1)} E\left[\log\left(\mathbf{b}^{t}(0)\,\mathbf{X}(0)\right)\right]\\
=\; &\max_{\mathbf{b}(N+2)\in\mathcal{F}(N+1)} E\left[\log\left(\mathbf{b}^{t}(N+2)\,\mathbf{X}(N+2)\right)\right]\\
=\; &W^{\star}(N+2)
\end{aligned}$$

Hence $W^\star(k) = W^\star(\infty)$ from a certain step. ∎

A stationary ergodic process can be thought of as an asymptotically independent process that is the present is independent of the distant past. In light of this we can state that an increase in growth rate (we have left out the time indices on $\mathbf{b}$ for short)

$$\begin{aligned} & W^\star(k+1) - W^\star(k) \\ = & \max_{\mathbf{b}\in\mathcal{F}(-k)} E\left[\log\left(\mathbf{b}^t X(0)\right)\right] - \max_{\mathbf{b}\in\mathcal{F}(-k+1)} E\left[\log\left(\mathbf{b}^t X(0)\right)\right] \end{aligned} \tag{74}$$

becomes small as time passes since the extra information $X(-k)$ is almost independent of $X(0)$ when k becomes large. We can also see that the limit must be

$$\max_{\mathbf{b}(0)\in\mathcal{F}^{(\infty)}(-1)} E\left[\log\left(\mathbf{b}^t(0)\, X(0)\right)\right]$$

since the information $X(-k), X(-k-1), ...$ is of almost no use to us for large k because of the asymptotic independence.

Remark 1 *Let $\mathbf{X}(1), \mathbf{X}(2), ...$ be independent, identically distributed random variables. Then they are in particular stationary ergodic and theorem 4 applies. Since $\mathbf{X}(1), \mathbf{X}(2), ...$ are independent we do not get any relevant information from the past so each period we are facing the same problem, hence $\mathbf{b}^\star$ is a constant rebalanced portfolio.*

Remark 2 *Assume that the stock return process is stationary ergodic and we have access to information of the infinite past. Then we know from the proof of theorem 4 that $\mathbf{b}^\star$ is a constant rebalanced portfolio. However, we do not always need all that information to get a constant rebalanced portfolio. If we possess q periods of information and the market is q'th order Markov, i.e. information older than $q+1$ periods is irrelevant then $\mathbf{b}^\star$ is a constant rebalanced portfolio.*

Example 1 (Continued) *We will now take another look at the example from the introduction and let a fair coin $\left(p = \frac{1}{2}\right)$ decide the outcome of the two stocks.*

$$\begin{aligned} X_1 &= \begin{cases} 0.9 & \text{in case of heads} \\ 1.2 & \text{in case of tails} \end{cases} \\ X_2 &= \begin{cases} 0.4 & \text{in case of heads} \\ 2.0 & \text{in case of tails} \end{cases} \end{aligned}$$

In Section 3.5 we saw that a myopic policy is optimal for the logarithmic utility function so we do not have to worry about the horizon.

Let $\mathbf{b} = (b, 1-b)^t$ then we have

$$\begin{aligned} E\left[\log\left(\mathbf{b}^t\mathbf{X}\right)\right] &= 0.5\left(\log\left(0.9b + 0.4(1-b)\right) + \log\left(1.2b + 2(1-b)\right)\right) \\ &= 0.5\left(\log\left((0.5b+0.4)(-0.8b+2)\right)\right) \\ &= 0.5\left(\log\left(-0.4b^2 + 0.68b + 0.8\right)\right) \end{aligned}$$

that is $b^\star = \frac{0.68}{0.8} = 0.85$ which gives us the log return $W^\star(1) = 0.04263$. In figure 2 we can see the log return for all choices of b.

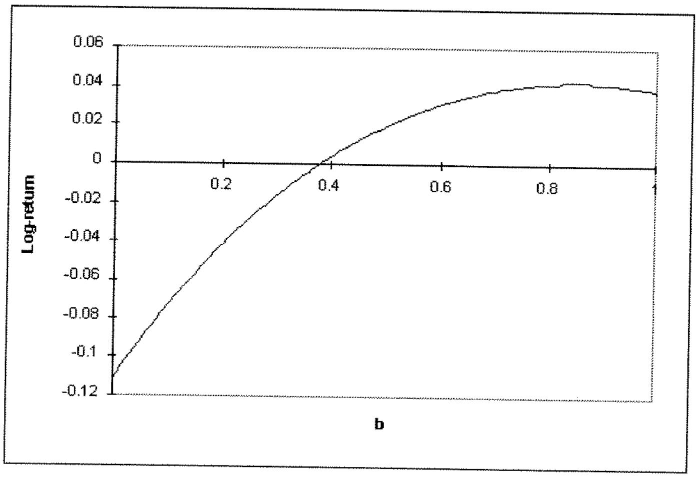

Figure 2: Log-return as a function of the proportion held in asset number 1.

Consider a different CRRA utility function

$$U(S) = \frac{1}{1-\alpha_2} S^{1-\alpha_2}$$

Since a myopic policy is optimal the MPP reduces to

$$\begin{aligned} & \max_{b \in \mathbb{R}} E\left[\frac{1}{1-\alpha_2} S^{1-\alpha_2}\right] \\ = \quad & \max_{b \in \mathbb{R}} \frac{1}{2(1-\alpha_2)} \left[(0.9b + 0.4(1-b))^{1-\alpha_2} + (1.2b + 2.0(1-b))^{1-\alpha_2}\right] \end{aligned}$$

For this to be well defined for every α_2 $-0.8 < b < 2.5$. Now, the first order condition is

$$\begin{aligned} 0.5\,(0.9b + 0.4(1-b))^{-\alpha_2} &= 0.8\,(1.2b + 2.0(1-b))^{-\alpha_2} \\ \Leftrightarrow \qquad b &= \frac{2 \cdot 0.8^{-\frac{1}{\alpha_2}} - 0.4 \cdot 0.5^{-\frac{1}{\alpha_2}}}{0.5^{1-\frac{1}{\alpha_2}} + 0.8^{1-\frac{1}{\alpha_2}}} \end{aligned} \tag{75}$$

In figure 3 we have plotted b as a function of α_2. The figure illustrates that for high risk aversion we wish to hold a larger portion of the safest asset X_1. For a risk aversion higher than 1.6 we wish to short the risky asset X_2. For $\alpha_2 \to \infty$ we can see from (75) that $b \to \frac{1.6}{1.3}$. Using $b = \frac{1.6}{1.3}$

$$\begin{aligned} bX_1 + (1-b)X_2 &= \begin{cases} \frac{1.6 \cdot 0.9 - 0.3 \cdot 0.4}{1.3} & \text{in case of heads} \\ \frac{1.6 \cdot 1.2 - 2.0 \cdot 0.3}{1.3} & \text{in case of tails} \end{cases} \\ &= \begin{cases} \frac{1.32}{1.3} & \text{in case of heads} \\ \frac{1.32}{1.3} & \text{in case of tails} \end{cases} \end{aligned}$$

corresponds to creating a riskless asset, which is the ultimate risk averse strategy.