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Fin2: Exercises for Wednesday May 20, 2020

Exercise 5.1 (Constant relative risk aversion + standard binomial model.) Extend the Bellman-equation analysis in Pascucci & Rungaldier's section 2.4.2 to solve for an agent who has constant relative risk-aversion a, i.e. $u(x) = \frac{x^{1-a}-1}{1-a}$. Note that by positive affine invariance we may for computational convienience rewrite this as $u(x) = x^b$.

Exercise 5.2 (Merton $\sim$ tangent $\sim$ growth/log optimal: same-same.) Consider a T-period, n-asset model with rates of returns r that have mean vector μ and covariance matrix Σ and are *iid* in time.

(1) Argue that the optimal risky fractions for an investor (who cares only about terminal wealth) with constant relative risk-aversion a are approximately given by the so-called Merton-portfolio

$$\pi = \frac{1}{a}\Sigma^{-1}(\mu - r_0\mathbf{1}),$$

where r_0 is the risk-free interest rate. Hint: This has two parts. First, a Bellman-equation argument (using the form of the utility function) similar in structure to what we did on Wednesday May 13 to show that the optimal portfolio choice problem does indeed boil down to finding such a vector of portfolio weights. Second, some Taylor-approximantions applied to the resulting 1-period/static optimization problem .
2) Investigate numerically (i.e. choose some parameter values) how close this approximate weight solution is to the exact (but un-intuitive) weight solution from Exercise 5.1 in the standard binomial case. (You can also use the portfolio choice spreadsheet from Exercise Sheet #1.)
(3) Dig up your Finance 1-notes (or similar) an see that $\Sigma^{-1}(\mu - r_0\mathbf{1})$ is "very related" to the tangent portfolio from mean-variance/Markowitz/CAPM-analysis.
(4) Look in Claus Munk (2008) "Dynamic Asset Allocation" (around Theorem 5.2 in page 74) to find confirmation that the result is exact in continuous models (i.e. with very small time-steps), and that we can equip the investor with utility from intermediate consumption without ruining the result.
(5) Consider an investor who consumes nothing and invests the fixed fractions of his wealth in the risky assets according to the vector π in each period. His

wealth after t periods (if he starts with 1) is

$$V(t;\pi) = V(t-1;\pi)\left((1-\pi^\top\mathbf{1})(1+r_0) + \pi^\top(r+\mathbf{1})\right) =: \prod_{i=1}^{t} Y(i;\pi)$$

where the suitably defined Y's are *iid*, and $Y(\pi)$ denotes their common distribution. The growth rate of his wealth after t periods, $G(t)$, is defined via $V(t;\pi) = \exp(tG(t;\pi))$ i.e.

$$G(t;\pi) = \frac{1}{t}\sum_{i=1}^{t} \ln Y(t;\pi).$$

Argue that $G(t;\pi) \to \mathbf{E}(\ln Y(\pi))$ almost surely as $t \to \infty$. Thus, a natural strategy is to chose the π that maximizes this. Show that, to a second order Taylor approximation, this is solved by

$$\pi = \Sigma^{-1}(\mu - r_0\mathbf{1}),$$

This is why this portfolio is also sometimes referred to as the growth optimal portfolio. And why we can think of a growth rate maximizing investor as one that has log-utility.

Exercise 5.3 (A static model with transaction costs. This is at best on the border of what was covered at the lectures, but ...)
Go through Example 3 in the Pedersen & Garleanu (2013) paper. The example shows that the predictability of returns affects optimal portfolio choice in a way that it is simply not possible to capture in a static, or one-period, framework.