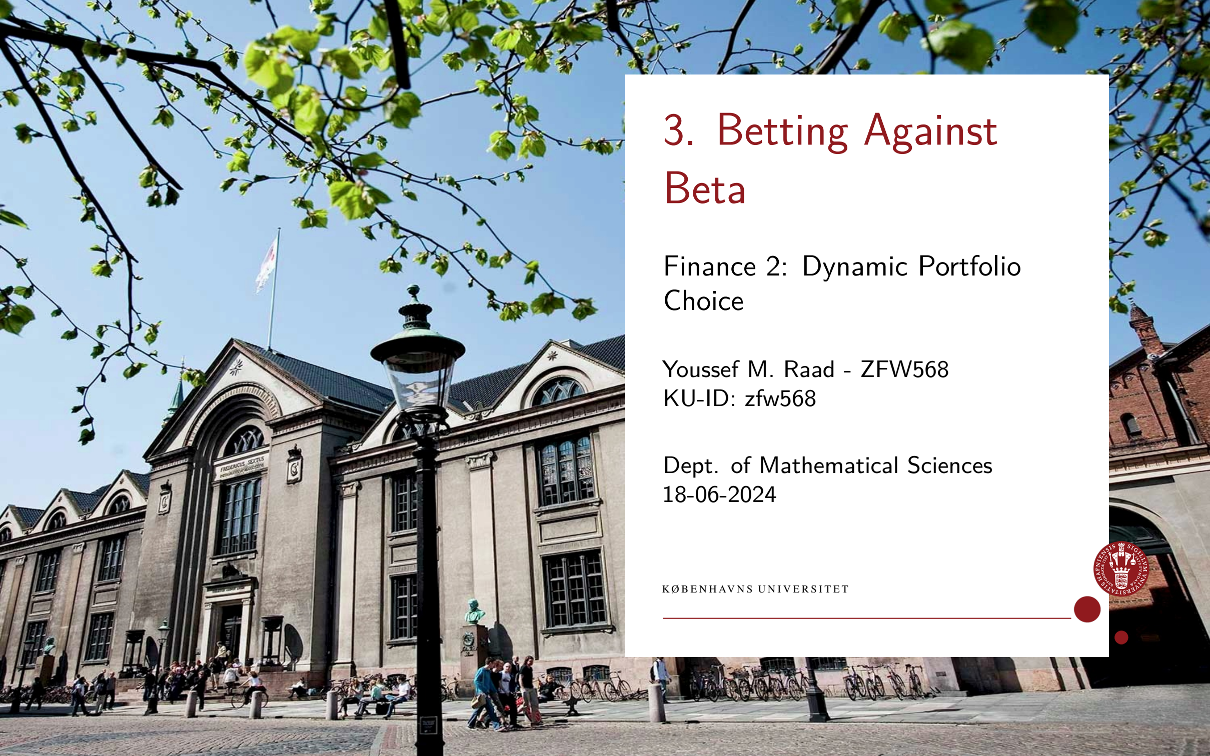


3. Betting Against Beta

Finance 2: Dynamic Portfolio Choice

Youssef M. Raad - ZFW568
KU-ID: zfw568

Dept. of Mathematical Sciences

18-06-2024

KØBENHAVNS UNIVERSITET

Overivew

1) Motivation by CAPM

2) Betting Against Beta
   a) Model and Equilibrium
   b) Proposition 1 ((i) and (ii)) - High Beta is Low Alpha
   c) Proposition 2 - How to Bet Against Beta?

3) Extra Slides

Motivation - Standard CAPM

Standard CAPM states that the expected excess return of a risky asset is proportional to the expected excess return of the market portfolio. The constant of proportionality is given by the beta:

$$\mathbb{E}[r_i] - r_f = \underbrace{\frac{cov(r_i, r_m)}{Var(r_m)}}_{\beta_{i,m}} \underbrace{(\mathbb{E}[r_m] - r_f)}_{\text{Market premium}}.$$

- The premise of CAPM is that all agents invest in the portfolio with the highest expected excess return per unit of risk (Sharpe ratio) and leverage or deleverage this portfolio accordingly to suit risk preferences.
- With no constraints investor can leverage a "normal" portfolio (lower-risk) if efficient and achieve a better trade-off between risk and expected return than the "aggressive" portfolio (higher-risk).
- Typically agents are constrained in the leverage they are able to take $\longrightarrow$ overweight in risky securities instead of using leverage $\longrightarrow$ risky high-beta assets bid up by constrained investors!
- Investors tilting towards high-beta assets indicates that risky high-beta assets require lower risk-adjusted returns than low-beta assets, which require leverage!

Motivation - Standard CAPM

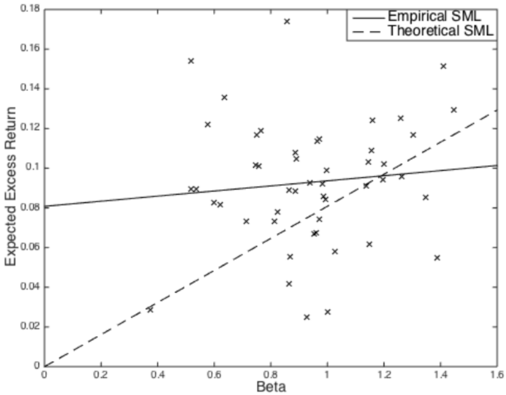


Figure: Theoretical CAPM and empirical SML from 49 industrial portfolios (French, 1995-2015) (Lando & Poulsen, 2023, p. 73).

- Security Market Line for U.S. stocks too flat relative to CAPM (Black, Jensen, and Scholes (1972)) (Black et al., 1972) (more empirics on Extra Slide 1 and 9).

How can we (as an unconstrained investor) exploit this tendency? The BAB factor!

Model - "Betting against beta"

Overlapping-generations (OLG) economy:

- Agents $i = 1, ..., I$ born each time period t and endowed with wealth W_t^i while living for two periods.
- Securities $s = 1, ..., S$ are traded at prices P_t, where security s pays dividends δ_t^s at time t and has x^{*S} number of shares outstanding.
- At each time period t, young agents choose a portfolio of shares $x = (x^1, ..., x^S)^\top$ with remainder in risk-free asset, risk-free return r^f.

Agent (i's) maximization problem is given by:

$$\max_{x \in \mathbb{R}^S} x^\top \left(\mathbb{E}_t \left(P_{t+1} + \delta_{t+1} \right) - \left(1 + r^f \right) P_t \right) - \frac{\gamma^i}{2} x^\top \Omega_t x, \quad \text{s.t.} \quad m_t^i \sum_s x^s P_t^s \leq W_t^i,$$

where Ω_t is the covariance matrix of $P_{t+1} + \delta_{t+1}$, γ^i is agent i's risk aversion, W_t^i denotes wealth and m_t^i represents leverage.

Different m's represents different types of agents:

- $m^i = 1$: No leverage.
- $m^i > 1$: No leverage and must have some wealth in cash.
- $m^i < 1$: Constraint leverage (i.e $m^i = 0.5 \sim$ twice his wealth).

Assume for simplicity that all securities have the same margin requirement.

Equilibrium (1/3)

Applying the Lagrangian we get the FOC (where the Lagrange multiplier of the portfolio constraint is ψ^i):

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \Leftrightarrow \quad 0 = \mathbb{E}_t\left[P_{t+1} + \delta_{t+1}\right] - (1 + r^f)P_t - \gamma^i \Omega x^i - \psi_t^i P_t$$

$$\Leftrightarrow \quad x^i = \frac{1}{\gamma^i}\Omega^{-1}\left(\mathbb{E}_t\left[P_{t+1} + \delta_{t+1}\right] - \left(1 + r^f + \psi_t^i\right) P_t\right)$$

Imposing that in equilibrium total demand equals supply, $\sum_i x^i = x^*$, by summation we achieve:

$$x^* = \frac{1}{\gamma}\Omega^{-1}\left(\mathbb{E}_t\left[P_{t+1} + \delta_{t+1}\right] - \left(1 + r^f + \psi_t\right) P_t\right),$$

where the aggregate risk aversion γ is defined by $\frac{1}{\gamma} = \sum_i \frac{1}{\gamma^i}$ and the weighted average Lagrange multiplier is $\psi_t = \sum_i \frac{\gamma}{\gamma^i}\psi_t^i$.

Equilibrium prices:

$$P_t = \frac{\mathbb{E}_t\left[P_{t+1} + \delta_{t+1}\right] - \gamma \Omega x^*}{1 + r^f + \psi_t}.$$

Equilibrium (2/3)

Rewrite in terms of return of any security, $r^i_{t+1} = \left(P^i_{t+1} + \delta^i_{t+1}\right)/P^i_t - 1$ (and define the vector: $e_s^\top$ extracts the s'th coordinate of Ωx^* $(S \times 1)$):

$$
\begin{aligned}
\mathbb{E}_t[r^s_{t+1}] &\equiv \frac{\mathbb{E}_t\left[P^s_{t+1} + \delta^s_{t+1}\right] - P^s_t}{P^s_t} \\
&= r^f + \psi_t + \frac{1}{P^s_t}\gamma e_s^\top \Omega x^* \qquad \text{[Substitute } P_t \text{ and extract using } e_s^\top \text{ to find } P^s_t\text{]} \\
&= r^f + \psi_t + \frac{1}{P^s_t}\gamma \underbrace{\operatorname{cov}_t\left(P^s_{t+1} + \delta^s_{t+1}, \left(P_{t+1} + \delta_{t+1}\right)^\top x^*\right)}_{:=\Omega \text{ and extract } s'\text{th-coord.}} \\
&\overset{\dagger}{=} r^f + \psi_t + \gamma \operatorname{cov}_t\left(r^s_{t+1}, r^M_{t+1}\right) P_t^\top x^*. \qquad \text{[Rewrite to returns]} \qquad (1)
\end{aligned}
$$

Trick: Multiply by market portfolio weights $w^s = \frac{x^{*s} P^s_t}{\sum_j x^{*j} P^j_t}$ and sum over all assets (constants "unaffected" since weights sum to 1):

$$
\begin{aligned}
\mathbb{E}_t\left[\sum_s w^s r^s_{t+1}\right] &= r^f + \psi_t + \gamma \operatorname{cov}_t\left(\sum_s w^s r^s_{t+1}, r^M_{t+1}\right) P_t^\top x^* \\
\Leftrightarrow \quad \mathbb{E}_t\left[r^M_{t+1}\right] &= r^f + \psi_t + \gamma \operatorname{Var}_t\left(r^M_{t+1}\right) P_t^\top x^*. \\
\Leftrightarrow \quad \gamma P_t x^* &= \frac{\lambda_t}{\operatorname{Var}_t\left(r^M_{t+1}\right)}, \qquad \underbrace{\lambda_t}_{\text{risk premium}} = \mathbb{E}_t\left[r^M_{t+1}\right] - r^f - \psi_t, \qquad (4)
\end{aligned}
$$

Equilibrium (3/3)

Substituting (4) into (1):

$$\begin{aligned}
\mathbb{E}_t\left[r^s_{t+1}\right] &= r^f + \psi_t + \text{cov}_t\left(r^s_{t+1}, r^M_{t+1}\right)\left(\frac{\mathbb{E}_t\left[r^M_{t+1}\right] - r^f - \psi_t}{\text{Var}_t\left(r^M_{t+1}\right)}\right) \\
&= r^f + \psi_t + \underbrace{\left(\frac{\text{cov}_t\left(r^s_{t+1}, r^M_{t+1}\right)}{\text{Var}_t\left(r^M_{t+1}\right)}\right)}_{:=\beta^s_t}\underbrace{\left(\mathbb{E}_t\left[r^M_{t+1}\right] - r^f - \psi_t\right)}_{:=\lambda_t} \\
&= r^f + \psi_t + \beta^s_t\left(\mathbb{E}_t\left[r^M_{t+1}\right] - r^f - \psi_t\right) \qquad (2) \\
\Leftrightarrow \mathbb{E}_t\left[r^s_{t+1}\right] - r^f &= \psi_t(1-\beta^s_t) + \beta^s_t\left(\mathbb{E}_t\left[r^M_{t+1}\right] - r^f\right) \\
\Leftrightarrow \mathbb{E}_t\left[r^s_{t+1}\right] - r^f &= \alpha^s_t + \beta^s_t\left(\mathbb{E}_t\left[r^M_{t+1}\right] - r^f\right), \qquad (3)
\end{aligned}$$

where $\alpha^s_t = \psi_t(1-\beta^s_t)$ is Jensen's alpha. This is Proposition 1 (next page)... Well partly, (iii) is not in.

High Beta Low Alpha

Proposition 1 - High Beta is Low Alpha (Frazzini & Pedersen, 2014)

(i) The equilibrium required return for any security s is (as given by (2) above)

$$\mathbb{E}_t\left[r_{t+1}^s\right] = r^f + \psi_t + \beta_t^s \lambda_t,$$

where the risk premium is $\lambda_t = \mathbb{E}_t\left[r_{t+1}^M\right] - r^f - \psi_t$ and ψ_t is the average Lagrange multiplier, measuring the tightness of funding constraints.

(ii) A security's alpha with respect to the market is $\alpha_t^s = \psi_t(1 - \beta_t^s)$ (as stated in (3) above). The alpha decreases in the beta, β_t^s.

- Constrained investors bid up high-beta assets, a high beta is associated with a low alpha $\Rightarrow$ Flatter SML than in standard CAPM.
- Tighter portfolio constraints $(\psi_t \uparrow)$ flattens the SML by increasing the intercept and decreasing the slope/risk premium $(\lambda_t \downarrow)$ (see (2)), i.e. a lower compensation for a marginal increase in systematic risk $\Rightarrow$ More constrained investors accept lower compensation and hold riskier assets.

How do We Bet Against Beta (1/2)?

(See Extra Slide 7: PnL if CAPM holds + market neutrality and Extra Slides for example by numbers)

Proposition 2 - Positive Expected Return of BAB (Frazzini & Pedersen, 2014)

The expected excess return of the self-financing BAB factor is positive (and increasing in the ex ante beta-spread and funding tightness)

$$\mathbb{E}_t\left[r_{t+1}^{BAB}\right] = \frac{\beta_t^H - \beta_t^L}{\beta_t^L \beta_t^H}\psi_t \geq 0.$$

Consider the BAB portfolio/factor/bet (L=Low, H=High and $\beta_t^L < \beta_t^H$):

1. Go long $\frac{1}{\beta^L}$ in low-beta stocks, with return r_{t+1}^L.
2. Go short $\frac{1}{\beta^H}$ in high-beta stocks, with return r_{t+1}^H.
3. Invest the difference $\frac{1}{\beta^H} - \frac{1}{\beta^L}$ at risk-free rate r^f.

The return (strictly speaking, not *rate* of return - costs 0 to initiate) on the BAB-bet is then:

$$\begin{aligned} r_{t+1}^{BAB} &= \frac{1}{\beta_t^L}\left(r_{t+1}^L - r^f\right) - \frac{1}{\beta_t^H}\left(r_{t+1}^H - r^f\right) \\ &= \frac{1}{\beta_t^L} r_{t+1}^L - \frac{1}{\beta_t^H} r_{t+1}^H + \underbrace{\left(\frac{1}{\beta_t^H} - \frac{1}{\beta_t^L}\right)}_{<0} r^f. \end{aligned}$$

How do We Bet Against Beta (2/2)?

Proposition 2 (positive expected return of BAB) (Frazzini & Pedersen (2013)

The expected excess return of the self-financing BAB factor is positive (and increasing in the ex ante beta-spread and funding tightness)

$$\mathbb{E}_t\left[r_{t+1}^{BAB}\right] = \frac{\beta_t^H - \beta_t^L}{\beta_t^L \beta_t^H}\psi_t \geq 0.$$

Proof: The excess return of the BAB factor is:

$$r_{t+1}^{BAB} = \frac{1}{\beta_t^L}\left(r_{t+1}^L - r^f\right) - \frac{1}{\beta_t^H}\left(r_{t+1}^H - r^f\right).$$

Taking expectation of the exess return of the BAB factor and applying (2)/proposition 1 (i) in † yields:

$$\begin{aligned}
\mathbb{E}_t\left[r_{t+1}^{BAB}\right] &= \frac{1}{\beta_t^L}\left(\mathbb{E}_t\left[r_{t+1}^L\right] - r^f\right) - \frac{1}{\beta_t^H}\left(\mathbb{E}_t\left[r_{t+1}^H\right] - r^f\right) \\
&\overset{\dagger}{=} \frac{1}{\beta_t^L}\left(\psi_t + \beta_t^L \lambda_t\right) - \frac{1}{\beta_t^H}\left(\psi_t + \beta_t^H \lambda_t\right) \\
&= \left(\frac{1}{\beta_t^L} - \frac{1}{\beta_t^H}\right)\psi_t \geq 0 \\
&\iff \left(\frac{\beta_t^H - \beta_t^L}{\beta_t^L \beta_t^H}\right)\psi_t \geq 0.
\end{aligned}$$

References

Black, F., Jensen, M. C., Scholes, M., et al. (1972). The capital asset pricing model: Some empirical tests.

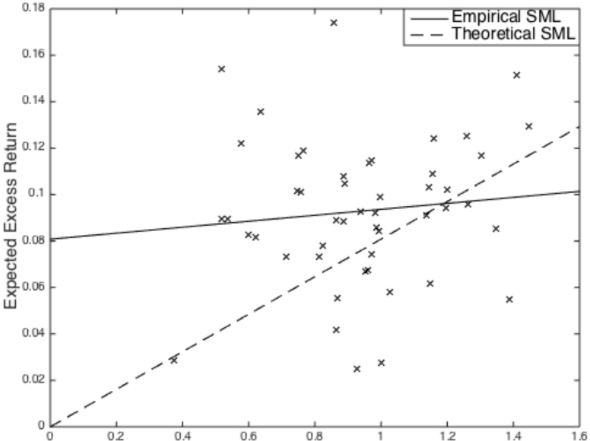
Frazzini, A., & Pedersen, L. H. (2014). Betting against beta. *Journal of financial economics, 111*(1), 1–25.

Lando, D., & Poulsen, R. (2023). Finance 1 and beyond. *Department of Mathematical Sciences, University of Copenhagen.*

Extra Slide 1: The empirical SML vs. Theorettical SML

The empirical security market line computed on behalf of 49 industrial portfolios (French 1995-2015). On the same graph the theoretical CAPM counterpart has been plotted.

- Asset above line $\iff$ positive Jensen's alpha $\iff$ undervalued wrt risk.
- Asset under line $\iff$ negative Jensen's alpha $\iff$ overvalued wrt risk.
- Clearly, systemic risk is under-compensated in the real world ($\uparrow \beta$ lower expected excess return compared to theoretical SML)



Extra Slide 2: Proposition 1 (iii)

Proposition 1 (Frazzini & Pedersen (2013))

(iii) For an efficient portfolio, the Sharpe ratio is highest for an efficient portfolio with a beta less than one and decreases in β_t^s for higher betas and increases for lower betas.

Proof. The Sharpe ratio increases in beta until the tangency portfolio is reached and decreases thereafter.

Hence, (iii) follows from the fact that the tangency portfolio has a beta less than one. This is true because the market portfolio is an average of the tangency portfolio (held by unconstrained agents) and riskier portfolios (held by constrained agents) so the market portfolio is riskier than the tangency portfolio.

Hence, the tangency portfolio must have a lower expected return and beta (strictly lower if and only if some agents are constrained).

In (very) short: Tangency portfolio held by unconstrained investors (market portfolio is riskier than tangency portoflio, because it has constrained- and unconstrained-investor's portfolios $\Rightarrow$ by definition of β (risk relative to market) it is < 1 as it is less risky.

Extra Slide 3: Example: Do I Really Understand the BAB-Factor (1/6)?

Excel-file avaliable on Github.

Suppose:

- Stock A has a time-0 price of $\$57$, a time-1 price $\$62$, and $\beta_A = 0.75$.
- Stock B has a time-0 price of $\$32$, a time-1 price $\$38$, and $\beta_B = 1.3$.
- Interest rate is 2%.

Want to make a $\$100$ BAB-bet. What do we do?

Note(ation): $A \sim L$ and $B \sim H$.

Rates of return for the two stocks:

$$r^A_{t+1} = \frac{\$62}{\$57} - 1 = 0.0877193, \quad r^B_{t+1} = \frac{\$38}{\$32} - 1 = 0.1875.$$

Low-beta-leg: As we want to make a $\$100$ bet, then in order to get a β of 1, the (long) leveraged investment in stock A must satisfy

$$\frac{\text{\$BAB-bet size}}{\beta_A} = \frac{\$100}{0.75} = \$133.333,$$

i.e. we invest $\$133.333$ in stock A, which is equivalent to trading (here buying) units of stock A

$$\frac{\text{Total \$ spent}}{\text{Price of 1 unit}} = \frac{\$133.333}{\$57} = 2.339.$$

Extra Slide 4: Example: Do I Really Understand the BAB-Factor (2/6)?

This requires a bank position of

$$\text{Total \$ spent} - \text{\$ available} = \$133.333 - \$100 = -\$33.333 < 0,$$

i.e. this requires borrowing money.

High-beta-leg: As we want to make a $\$100$ bet, then in order to get a β of 1, the (short) de-leveraged investment in stock B must satisfy

$$-\frac{\text{\$BAB-bet size}}{\beta_A} = -\frac{\$100}{1.3} = -\$76.923,$$

i.e. we invest $-\$76.923$ in stock B, which is equivalent to trading (here selling) units of stock B

$$\frac{\text{Total \$ spent}}{\text{Price of 1 unit}} = \frac{-\$76.923}{\$32} = -2.404.$$

This requires a bank position of

$$\text{Total \$ spent} - \text{\$ available} = -\$76.923 - \$100 = -\$23.077 < 0,$$

i.e. this requires borrowing money.

Extra Slide 5: Example: Do I Really Understand the BAB-Factor (3/6)?

The value of the portfolio at time 1 ($= r_{t+1}^{\text{BAB}}$) is:

- From stock A:

$$\text{Number of units traded} \times \$ \text{ time-1 price} = 2.339 \times \$62 = 145.03.$$

- From stock B:

$$\text{Number of units traded} \times \$ \text{ time-1 price} = -2.404 \times \$38 = -91.35.$$

- From bank:

$$\$\Big(\text{Position in bank associated with stock A} + \text{Position in bank associated with stock B}\Big) \times (1 + r^f)$$
$$= \$\Big[-33.333 + (-23.077)\Big] \times (1 + 0.02) = -\$57.54.$$

- In total, PnL:

$$\$\Big(\text{PnL from A} + \text{PnL from B} + \text{PnL from bank}\Big) = \$\Big[145.03 + (-91.35) + (-57.54)\Big] = -\$3.855,$$

i.e. a loss of 3 dollars and 86 cents.

Extra Slide 6: Example: Do I Really Understand the BAB-Factor (4/6)?

Note: This is in fact the same as equation 9 (Frazzini & Pedersen, 2014), namely

$$
\begin{aligned}
r_{t+1}^{\text{BAB}} &= \frac{1}{\beta_t^A}\left[r_{t+1}^A - r^f\right] - \frac{1}{\beta_t^B}\left[r_{t+1}^B - r^f\right] \\
&= \frac{1}{0.75}\left[0.1875 - 0.02\right] - \frac{1}{1.3}\left[0.0877193 - 0.02\right] = -0.038554,
\end{aligned}
$$

which matches perfectly with out calculation, because equation 9 (Frazzini & Pedersen, 2014) is the return (PnL) on a $\$1$ bet, and so we can simply multiply r_{t+1}^{BAB} by a factor 100, which effectively makes it a PnL of a $\$100$ bet.

Extra Slide 7: Example: Do I Really Understand the BAB-Factor (5/6)?

The covariance between the BAB-return and return of the the market portfolio is

$$\mathrm{Cov}\left[r^{\mathrm{BAB}}, r^M\right] = \frac{1}{\beta_A}\mathrm{Cov}(r^A - r^f, r^M) - \frac{1}{\beta_A}\mathrm{Cov}(r^B - r^f, r^M) \qquad = \frac{1}{\beta_A}\underbrace{\mathrm{Cov}(r^A, r^M)}_{=\beta_A \mathrm{Var}[r^M]} - \frac{1}{\beta_A}\underbrace{\mathrm{Cov}(r^B, r^M)}_{=\beta_B \mathrm{Var}[r^M]}$$

$$= \frac{\beta_A}{\beta_A}\mathrm{Var}[r^M] - \frac{\beta_B}{\beta_B}\mathrm{Var}[r^M] = 0,$$

i.e. market neutrality.

If CAPM holds for $i \in \{A, B\}$

$$\mathbb{E}[r^i] - r^f = \beta_i\Big(\mathbb{E}[r^M] - r^f\Big),$$

and so the expected PnL on the BAB-factor is

$$\mathbb{E}[r^{\mathrm{BAB}}] = \frac{1}{\beta_A}\mathbb{E}[r^A - r^f] - \frac{1}{\beta_B}\mathbb{E}[r^B - r^f] = \frac{1}{\beta_A}\underbrace{\Big(\mathbb{E}[r^A] - r^f\Big)}_{=\beta_A\left(\mathbb{E}[r^M] - r^f\right)} - \frac{1}{\beta_B}\underbrace{\Big(\mathbb{E}[r^B] - r^f\Big)}_{=\beta_B\left(\mathbb{E}[r^M] - r^f\right)}$$

$$= \frac{\beta_A}{\beta_A}\Big(\mathbb{E}[r^M] - r^f\Big) - \frac{\beta_B}{\beta_B}\Big(\mathbb{E}[r^M] - r^f\Big) = 0.$$

Extra Slide 8: Example: Do I Really Understand the BAB-Factor (6/6)?

Note for stock A

$$\frac{\$ \text{ Invested in stock A} \times \beta_A}{\$\text{BAB-bet size}} = \frac{\$133.333 \times 0.75}{\$100} = 1,$$

equivalently for stock B

$$-\frac{\$ \text{ Invested in stock B} \times \beta_B}{\$\text{BAB-bet size}} = -\frac{-\$76.923 \times 1.3}{\$100} = 1,$$

which shows (in a pedestrian way) that the low β-portfolio (long position, assets), as well as the high β-portfolio (short position, liabilities), has a β of 1.

Extra Slide 9: Empirical results

Because constrained investors bid up high-beta assets, high beta is associated with low alpha which explains the "flatter" Security Market Line than compared to that of the CAPM!! This is also supported empirically:

Portfolio	P1 (low beta)	P2	P3	P4	P5	P6	P7	P8	P9	P10 (high beta)	BAB
Excess return	**0.91**	**0.98**	**1.00**	**1.03**	**1.05**	**1.10**	**1.05**	**1.08**	**1.06**	**0.97**	**0.70**
	(6.37)	(5.73)	(5.16)	(4.88)	(4.49)	(4.37)	(3.84)	(3.74)	(3.27)	(2.55)	(7.12)
CAPM alpha	**0.52**	**0.48**	**0.42**	**0.39**	**0.34**	**0.34**	0.22	0.21	0.10	−0.10	**0.73**
	(6.30)	(5.99)	(4.91)	(4.43)	(3.51)	(3.20)	(1.94)	(1.72)	(0.67)	(−0.48)	(7.44)
Three-factor alpha	**0.40**	**0.35**	**0.26**	**0.21**	**0.13**	0.11	−0.03	−0.06	**−0.22**	**−0.49**	**0.73**
	(6.25)	(5.95)	(4.76)	(4.13)	(2.49)	(1.94)	(−0.59)	(−1.02)	(−2.81)	(−3.68)	(7.39)
Four-factor alpha	**0.40**	**0.37**	**0.30**	**0.25**	**0.18**	**0.20**	0.09	0.11	0.01	−0.13	**0.55**
	(6.05)	(6.13)	(5.36)	(4.92)	(3.27)	(3.63)	(1.63)	(1.94)	(0.12)	(−1.01)	(5.59)
Five-factor alpha	**0.37**	**0.37**	**0.33**	**0.30**	**0.17**	**0.20**	0.11	0.14	0.02	0.00	**0.55**
	(4.54)	(4.66)	(4.50)	(4.40)	(2.44)	(2.71)	(1.40)	(1.65)	(0.21)	(−0.01)	(4.09)
Beta (ex ante)	0.64	0.79	0.88	0.97	1.05	1.12	1.21	1.31	1.44	1.70	0.00
Beta (realized)	0.67	0.87	1.00	1.10	1.22	1.32	1.42	1.51	1.66	1.85	−0.06
Volatility	15.70	18.70	21.11	23.10	25.56	27.58	29.81	31.58	35.52	41.68	10.75
Sharpe ratio	0.70	0.63	0.57	0.54	0.49	0.48	0.42	0.41	0.36	0.28	0.78

Consistent with Proposition 1 (Frazzini & Pedersen, 2014) and Black (Black et al., 1972), the alphas decline almost monotonically from the low-beta to the high-beta portfolios $\Rightarrow$ Flatter SML.

Note also that the Sharpe ratios decline monotonically from low-beta to high-beta portfolios.

Extra Slide 10: Remarks on Proposition 1 (2/2)

- Market portfolio equals tangency portfolio in standard CAPM (no constrained agents), now market portfolio has higher risk and expected return than the tangency portfolio:
  - Constrained agents prefer to invest limited capital in riskier assets with higher expected return.
  - Unconstrained agents invest considerable amounts in zero-beta assets which increases the risk of their portfolio.
  - $\Rightarrow$ In equilibrium, zero-beta risky assets must offer higher returns than the risk-free rate.
  - $\Rightarrow$ As constrained investors bid up high-beta assets, a high beta is associated with a low alpha.
- Tightening portfolio constraints imply lower λ_t, because constrained agents need high unleveraged returns and are, therefore, willing to accept less compensation for higher risk $\Rightarrow$ More constrained investors hold riskier assets.
- Intercept of SML $r^f + \psi_t$ (as opossed to r^f in standard CAPM), i.e. increased by binding funding constraint.
- Tighter portfolio constraints $(\psi_t \uparrow)$ flattens the SML by increasing the intercept and decreasing the slope/risk premium, $\lambda_t \downarrow$, i.e. a lower compensation for a marginal increase in systematic risk.

Extra Slide 11: Remarks on Proposition 1 (2/2)

- Market portfolio is a weighted average of all investors' portfolios, i.e. an average of the tangency portfolio held by unconstrained investors and riskier portfolios held by constrained investors $\Rightarrow$ Market portfolio has higher risk and expected return than the tangency portfolio, but a lower Sharpe-ratio.
- Because constrained investors bid up high-beta assets, high beta is associated with low alpha which explains the "flatter" Security Market Line than compared to that of the CAPM!!
- Tighter portfolio constraints ($\psi_t \uparrow$) flattens the SML by increasing the intercept and decreasing the slope of SML, $\lambda_t \downarrow$, i.e. a lower compensation for a marginal increase in systematic risk.
- λ_t lower becuase constrained agents need high unleveraged returns and are, therefore, willing to accept less compensation for higher risk.
- Standard CAPM: Intercept of SML is r^f.
- Proposition 1: Intercept is $r^f + \psi_t$, i.e. increased by binding funding constraint.
- In equilibrium, zero-beta risky assets must offer higher returns than the risk-free rate.

Extra Slide 12: †

Abuse bilinearity of the covariance and that the covariance of a r.v and constant is 0:

$$
\begin{aligned}
\mathbb{E}_t[r^s_{t+1}] &= r^f + \psi_t + \frac{1}{P^s_t}\gamma \operatorname{cov}_t\left(P^s_{t+1} + \delta^s_{t+1}, (P_{t+1} + \delta_{t+1})^\top x^*\right) \\
&= r^f + \psi_t + \frac{1}{P^s_t}\gamma \operatorname{cov}_t\left(\frac{P^s_{t+1} + \delta^s_{t+1}}{P^s_t} P^s_t, \frac{(P_{t+1} + \delta_{t+1})^\top}{P^\top_t} P^\top_t x^*\right) \\
&= r^f + \psi_t + \frac{1}{P^s_t} P^s_t \gamma \operatorname{cov}_t\left(\frac{P^s_{t+1} + \delta^s_{t+1}}{P^s_t}, \frac{(P_{t+1} + \delta_{t+1})^\top}{P^\top_t}\right) P^\top_t x^* \\
&= r^f + \psi_t + \gamma \operatorname{cov}_t\left(\underbrace{\frac{P^s_{t+1} + \delta^s_{t+1}}{P^s_t} - 1}_{=r^s_{t+1}} + 1, \underbrace{\frac{(P_{t+1} + \delta_{t+1})^\top}{P^\top_t} - 1}_{=r^M_{t+1}} + 1\right) P^\top_t x^* \\
&= r^f + \psi_t + \gamma \operatorname{cov}_t\left(r^s_{t+1} + 1, r^M_{t+1} + 1\right) P^\top_t x^* \\
&= r^f + \psi_t + \gamma \left[\operatorname{cov}_t\left(r^s_{t+1}, r^M_{t+1}\right) + \underbrace{\operatorname{cov}_t\left(r^s_{t+1}, 1\right)}_{=0} + \underbrace{\operatorname{cov}_t\left(1, r^M_{t+1}\right)}_{=0} + \underbrace{\operatorname{cov}_t\left(1, 1\right)}_{=0}\right] P^\top_t x^* \\
&= r^f + \psi_t + \gamma \operatorname{cov}_t\left(r^s_{t+1}, r^M_{t+1}\right) P^T_t x^*.
\end{aligned}
$$