# 4. The ESG-Efficient Frontier

Finance 2: Dynamic Portfolio Choice

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# Overview

# Model Setup and ESG

Addresses the problem of one-period equilibrium by considering a one-period model with time points $t \in \{0, 1\}$, and:

- $n$ stocks and 1 risk-free asset with risk-free rate $r^f$.
- Excess rates of return $r = (r^1, ..., r^n)^\top$ - random variables with mean vector $\mu \in \mathbb{R}^n$ and positive definite covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$.
- Portfolio weights denoted by $x = (x^1, ..., x^n)^\top \in \mathbb{R}^n$ - so investor buys $x^i W$ of security $i$, where:
- Each investor has initial wealth $W$ and future wealth $\hat{W} = W(1 + r^f + x^\top r)$.
- Each stock has an ESG-score denoted by $s \in \mathbb{R}^n$. But what is it...?
  - E: Environmental - Often measured by carbon emissions, i.e. low is good.
  - S: Social/societal - Masured by "sin stocks" - Rolf , e.g. alcohol, tobacco, and fire arms.
  - G: Governance - How well do you run your business? Here measured by accruals where few accruals is associated with a high G-score (not yet recorded transaction in companies financial statement).

# Investor's Problem

Three types of investors:

- Type U: Maximize unconditional mean-variance utility, i.e. solves standard Markowitz problem:

$$\left[\hat{\mu} = \mathbb{E}[r],\ \hat{\Sigma} = \mathsf{Var}[r]\right].$$

- Type A: Driven by mean-variance preferences, but use ESG-scores to update views on risk and expected return $\rightarrow$ Maximize the conditional mean-variance utility and thus exploits the benefits of the ESG-scores, i.e. solves standard Markowitz problem conditional on ESG information:

$$\left[\hat{\mu} = \mathbb{E}[r \mid s],\ \hat{\Sigma} = \mathsf{Var}[r \mid s]\right].$$

- Type M: Seeks optimal trade-off between a high expected return, low risk, and high average ESG-score:

$$\left[\hat{\mu} = \mathbb{E}[r \mid \bar{s}],\ \hat{\Sigma} = \mathsf{Var}[r \mid \bar{s}] \text{ and ESG preferences}\right].$$

**In short:** We focus on type M as:

- Type U and A: Standard Markowitz problem.
- Type M: seeks trade-off between high expected return, low risk and high average ESG score.

# Solving Motivated Investor's Problem (1/4)

Let $\bar{\gamma}$: Absolute risk-aversion and $\gamma = \bar{\gamma}W$: Relative risk-aversion (remember: $W$ intitial wealth).

Let $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ denote the ESG preference function, and where $\bar{s} = \frac{x^\top s}{x^\top \mathbb{1}}$ is the average ESG-score. For the set of feasible portfolios $X = \{x \in \mathbb{R}^n \mid x^\top \mathbb{1} > 0\}$ (all long-based portfolios), the utility maximization problem is:

$$\underbrace{\max_{x \in X} \left( x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x \right.}_{\text{standard Markowitz}} + \left. \underbrace{f\left(\frac{x^\top s}{x^\top 1}\right)}_{\text{ESG-pref. func.}} \right).$$

**Observations for the above problem:**

- Note: Risk-free asset is ESG-neutral, i.e. no utility from investing in riskfree asset.
- Enjoys expected rate of rerturn $\to$ (More utility).
- Dislike variance $\to$ (Less utility).
- **Idea for solution:** In standard mean-variance $\to$ Investor combines tangency portfolio (:maximizes Sharp ratio) + risk-free security. So generalize that: Set up constraints on $\sigma$ and $\bar{s}$ $\to$ Relax constraints by maximizing over $\sigma$ $\to$ over $\bar{s}$, i.e. first choose the best portfolio given a level of risk $\sigma$ and an ESG-score $\bar{s}$ and then maximizing over $\sigma$ and then later over $\bar{s}$.

# Solving Motivated Investor's Problem (2/4)

Rewrite utility maximization problem:

$$\max_{\bar{s}} \left\{ \max_{\sigma} \left\{ \begin{array}{c} \max_{x \in X} \left\{ \left( x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x + f(\bar{s}) \right) \right\} \\ \text{s.t.} \quad \bar{s} = \frac{x}{x^\top \mathbb{1}}, \quad \sigma^2 = x^\top \Sigma x \end{array} \right\} \right\}$$

Innermost problem: Choosing the portfolio w/ highest Sharpe-ratio for given $s$.

Rewrite constraints: $\bar{s}$ and $\sigma^2$, respectively:

$$0 = x^\top (s - \bar{s}\mathbb{1}) = x^\top \tilde{s}, \quad \text{where } (s - \bar{s}\mathbb{1}) = \tilde{s}, \quad 0 = (\sigma^2 - x^\top \Sigma x).$$

Lagrangian, FOC and thus optimal portfolio $x$:

$$\mathcal{L}(x, \pi, \theta) = x^\top \mu - \frac{\gamma}{2}\sigma^2 + f(\bar{s}) + \pi(x^\top \tilde{s}) + \frac{\theta}{2}(\sigma^2 - x^\top \Sigma x),$$

$$\frac{\partial \mathcal{L}}{\partial x} = \mu + \pi \tilde{s} - \theta \Sigma x = 0 \quad \Longleftrightarrow \quad x = \frac{1}{\theta} \underbrace{\Sigma^{-1}(\mu + \pi \tilde{s})}_{(\dagger)},$$

where $(\dagger)$ is some vector not dependent on level of risk, $\sigma$!

# Solving Motivated Investor's Problem (3/4)

1st constraint: (i.e $0 = x^\top \tilde{s}$) gives first Lagrange multiplier:

$$\frac{\partial \mathcal{L}}{\partial \pi} = x^\top \tilde{s} = 0 \iff 0 = x^\top \tilde{s} = \underbrace{\frac{1}{\theta}}_{\text{some number}} (\mu + \pi \tilde{s})^\top \Sigma^{-1} \tilde{s}$$

$$\iff \pi = -\frac{\mu^\top \Sigma^{-1} \tilde{s}}{\tilde{s}^\top \Sigma^{-1} \tilde{s}}.$$

2nd constraint: (i.e $0 = (\sigma^2 - x^\top \Sigma x)$) gives second Lagrange multiplier (notation: $c_{xy} = x^\top \Sigma^{-1} y$):

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2}(\sigma^2 - x^\top \sigma x) = 0 \iff \sigma^2 = x^\top \Sigma x = \frac{1}{\theta^2}(\mu + \pi \underbrace{\tilde{s})^\top \Sigma^{-1} (\mu + \pi \tilde{s})}_{= x^\top \tilde{s} = 0 \text{ by 1st}} = \frac{1}{\theta^2} \mu^\top \Sigma^{-1} (\mu + \pi \tilde{s})$$

$$\iff \theta = \frac{1}{\sigma} \underbrace{\sqrt{\mu^\top \Sigma^{-1} (\mu + \pi \tilde{s})}}_{(\star)} = \frac{1}{\sigma} \underbrace{\sqrt{c_{\mu\mu} - \frac{(c_{s\mu} - c_{1\mu}\bar{s})^2}{c_{ss} - 2c_{1s}\bar{s} + c_{11}\bar{s}^2}}}_{(\star)},$$

where $(\star)$ is some number not dependent on the risk, $\sigma$!

$\Rightarrow$ Optimal porfolio $x$ can be written as $x = \sigma v = \sigma(\dagger)(\star)$ where $v$ depends only on the exogenous parameters and $\bar{s}$ (NOT risk, $\sigma$) as we saw in $(\dagger)$ and $(\star)$.

# Solving Motivated Investor's Problem (4/4)

Sharpe Ratio:

$$\mathsf{SR}(\bar{s}) = \frac{x^\top \mu}{\sigma} \quad \Longleftrightarrow \quad \mathsf{SR}(\bar{s})\sigma = x^\top \mu.$$

Substitute into the problem:

$$\max_{\bar{s}} \left\{ \max_{\sigma} \left\{ \mathsf{SR}(\bar{s})\sigma - \tfrac{\gamma}{2}\sigma^2 + f(\bar{s}) \right\} \right\}.$$

Next (inner) problem and FOC:

$$\max_{\sigma} \left\{ SR(\bar{s})\sigma - \tfrac{\gamma}{2}\sigma^2 + f(\bar{s}) \right\} \quad \Longleftrightarrow \quad \mathsf{SR}(\bar{s}) - 2\frac{\gamma}{2}\sigma = 0 \quad \Longleftrightarrow \quad \sigma = \frac{\mathsf{SR}(\bar{s})}{\gamma}.$$

Substitute $\sigma$, multiply by $2\gamma$ yields:

$$\max_{\bar{s}} \left\{ \tfrac{(\mathsf{SR}(\bar{s}))^2}{2\gamma} + f(\bar{s}) \right\} \Rightarrow \max_{\bar{s}} \left\{ \left(\mathsf{SR}(\bar{s})\right)^2 + 2\gamma f(\bar{s}) \right\},$$

which is proposition 1 (Pedersen et al., 2020, p. 6) (next page).

# Optimal ESG-SR Trade-off (Pedersen et al., 2020, p. 6)

**Proposition 1 - ESG-SR trade-off (Pedersen et al., 2020)**

*The (type M) investor should choose her average ESG score $\bar{s}$ to maximize the following function of the squared Sharpe ratio and the ESG preference function $f$:*

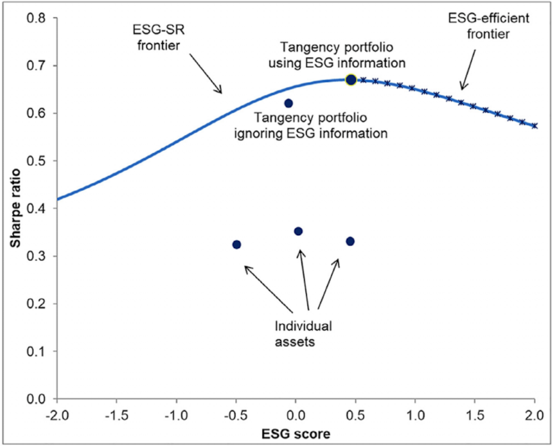
$$\max_{\bar{s}} \left\{ \left(\mathsf{SR}(\bar{s})\right)^2 + 2\gamma f(\bar{s}) \right\}.$$

**Observations (and some extra):**

- (Thank god) ESG affects the optimal portfolio choice.
- Tradeoff between Sharpe ratio ($\mathsf{SR}(\bar{s})$) and ESG preferences ($2\gamma f(\bar{s})$).
- Similar to Markowitz: ESG-SR frontier can be computed independent of preferences (security characteristics only) $\rightarrow$ Investor decide where to place herself afterwards depending on risk (Example on next page).
- If $f \equiv 0$ (no preferences) **but different** $\gamma$ (risk-aversion) for investors $\Rightarrow$ Investors should invest in same portfolio (same avg. ESG and Sharpe ratio) (More risk tolerant $\Rightarrow$ Larger fraction of wealth).
- If same $\gamma$ **but different** $f$ $\Rightarrow$ Further to the left or right on the ESG-efficient frontier.
- Changes within groups: Higher (relative) $\gamma$ (same $f$) $\Rightarrow$ More in risk-free asset and $\uparrow \gamma f(\bar{s})$ $\Rightarrow$ Pushed further right (more avg. ESG $\rightarrow$ "indirectly" enjoys ESG).

# Optimal ESG-SR Trade-off

*Panel A: ESG-efficient frontier*




- Type-M(otivated) investors choose portfolio on ESG-efficient frontier, i.e. preference for higher ESG: More preference $\Rightarrow$ further to the right, and vice versa.
- Type-A(ware) investors choose top-point of ESG-efficient frontier, i.e. highest SR (tang. port. using ESG info). Small preference for ESG $\Rightarrow$ Just to the right of tang. port using ESG info.
- Type-U(naware) investors choose tangency portfolio (or to the left) ignoring security information contained in ESG score, i.e. condition on less information.

# ESG-SR Frontier (Pedersen et al., 2020, p. 7)

**Proposition 2 - ESG-SR frontier (Pedersen & Fitzgibbons & Pomorski (2020))**

*The maximum Sharpe ratio, $SR(\bar{s})$, that can be achieved with an ESG-score of $\bar{s}$ is*

$$\mathrm{SR}(\bar{s}) = \sqrt{\mu^\top \Sigma^{-1}(\mu + \pi(s - 1\bar{s}))} = \sqrt{c_{\mu\mu} - \frac{\left(c_{s\mu} - \bar{s}c_{1\mu}\right)^2}{c_{ss} - 2c_{1s}\bar{s} + c_{11}\bar{s}^2}}.$$

*The maximum Sharpe ratio across all portfolios is $SR(s^*) = \sqrt{c_{\mu\mu}}$, which is attained with an ESG score of $s^* = \frac{c_{s\mu}}{c_{1\mu}}$. Increasing the ESG score locally around $s^*$ leads to nearly the same Sharpe ration, $SR(s^* + \Delta) = SR(s^*) + o(\Delta)$, because the first-order effect is zero, $\frac{dSR(s^*)}{ds} = 0$.*

# Four-fund Separation (Pedersen et al., 2020, p. 7)

**Proposition 3 - Four-fund separation (Pedersen & Fitzgibbons & Pomorski (2020))**

*Given an average ESG score $\bar{s}$, the optimal portfolio is*

$$x = \frac{1}{\gamma}\Sigma^{-1}(\mu + \pi(s - 1\bar{s}))$$

*as long as* $x^\top 1 > 0$, *where*

$$\pi = \frac{c_{1\mu}\bar{s} - c_{s\mu}}{c_{ss} - 2c_{1s}\bar{s} + c_{11}\bar{s}^2}.$$

*The optimal portfolio is therefore a combination of the riskfree asset, the tangency portfolio,* $\Sigma^{-1}\mu$, *the minimum-variance portfolio* $\Sigma^{-1}1$, *and the ESG-tangency portfolio,* $\Sigma^{-1}s$.

The construct is done as (next page):

# Closing Remarks

Following (Pedersen et al., 2020, p. 7): Optimal portfolio equivalent to standard Markowitz solution, but with adjusted expected excess return $\mu$. Optimal portfolio found by:

i) Investor computes ESG-adjusted expected returns, $\mu + \pi(s - 1\bar{s})$: each stock's expected excess return is increased if its ESG-score $s_i$ is above the desired average score $\bar{s}$; otherwise, it is lowered (scaling depends on $\pi$: if $\pi = 0 \rightarrow$ Traditional meanvariance optimization.)

ii) Investor computes the optimal portfolio via mutliplication by $\frac{1}{\gamma}\Sigma^{-1}$: $\frac{1}{\gamma}\Sigma^{-1}(\mu + \pi(s - 1\bar{s}))$ (proposition 3).

iii) All investors, regardless of their risk-aversion and ESG preferences, should choose a combination of four portfolios: The riskfree asset, the tangency portfolio, the minimum variance portfolio, and the ESG-tangency portfolio (proposition 3).

The optimal portfolio is therefore a combination of the riskfree asset, the tangency portfolio, $\Sigma^{-1}\mu$, the minimum-variance portfolio $\Sigma^{-1}1$, and the ESG-tangency portfolio, $\Sigma^{-1}s$.

# References

Pedersen, L. H., Fitzgibbons, S., & Pomorski, L. (2020). Responsible investing: The esg-efficient frontier. *Journal of financial economics*, *142*(2), 572–597.

# Extra Slide 1: ESG-CAPM

ESG-adjusted CAPM predict expected returns:

- Many type-U investors and High ESG predicts high expected returns

  $$\Rightarrow$$

  High-ESG stocks profitable and not bid up by type-U investors.

- Many type-A investors $\Rightarrow$ No connection between ESG and expected return (Aware investors).

  $$\Rightarrow$$

  Prices are bid up by type-A investors to reflect their expected profits, thus eliminating the connection between ESG and expected returns.

- Many type-M investors $\Rightarrow$ High ESG predicts low expected returns.

  $$\Rightarrow$$

  high-ESG stocks actually deliver low expected returns, because ESG-motivated investors are willing to accept a lower return for a higher ESG portfolio.

# Extra Slide 2: Utility of Type-M Investor

- Let $\bar{\gamma}$: Absolute risk-aversion and $\gamma = \bar{\gamma} W$: Relative risk-aversion.
- Let $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ denote the ESG preference function.
- Type M investor seeks to maximize utility $U$ over wealth, $W$, and average ESG-score, $\bar{s} = \frac{x^\top s}{x^\top 1}$, given the extended mean-variance framework

$$
\begin{aligned}
U &= \mathbb{E}\left[\hat{W} \mid s\right] - \frac{\bar{\gamma}}{2}\mathrm{var}\left[\hat{W} \mid s\right] + Wf(\bar{s}) \\
&= W\left(1 + r^f + x^\top \mu\right) - \frac{\bar{\gamma}}{2} W^2 x^\top \Sigma x + Wf\left(\frac{x^\top s}{x^\top 1}\right) \\
&= W\left(1 + r^f + x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x + f\left(\frac{x^\top s}{x^\top 1}\right)\right).
\end{aligned}
$$

# Extra Slide 3: Remarks for Proposition 1 (1/2)

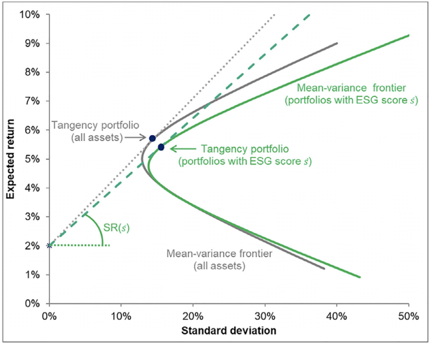
- One term regarding securities and another regarding preferences $\Rightarrow$ ESG-SR frontier can be computed independent of preferences (security characteristics only) and then the investor can decide where on the frontier to put herself.
- $\rightarrow$ Analogy to standard Markowitz theory: Mean-variance frontier can be computed independent of preference parameters and then decisions about what portfolio to pick are based on risk aversion.
- First investor decides where to be on the ESG-SR frontier, and then the level/amount of risk $\rightarrow$ Works with average ESG, which does not change with risk level.

# Extra Slide 4: Remarks for Proposition 1 (2/2)

- Risk-Aversion: For $f \equiv 0$ all investors should hold the same portfolio of risky assets (i.e. same SR and $\bar{s}$), but a higher risk aversion implies a larger fraction of wealth in this portfolio.
- ESG-Preference: For $\gamma$ constant then investors with stronger ESG-preferences should buy a portfolio with lower SR but higher $\bar{s}$.
- Interaction effects: For investors equally motivated by ESG, then investors with higher risk aversion invests more in the riskfree asset AND tilts portfolio toward higher ESG and lower SR.
  - $\rightarrow$ SR less important for higher risk-aversion, so ESG becomes more important relatively.
- More generally, observing an investor's portfolio of risky assets and its placement on the ESG-SR frontier is revelatory about $\gamma f(\bar{s})$; observing the investor's cash position (or leverage), about the risk aversion $\gamma$.

# Extra Slide 5: Remarks for Proposition 1 (EXTRA)

*Panel B: Mean-variance frontiers for all assets and portfolios with certain ESG score*




- Tangency portfolio from standard mean-variance portfolio has higheste SR among all portfolios, so its ESG score and SR define the peak in the ESG-SR fontier.
- The ESG-SR frontier is hump-shaped because restricting portfolios to have any ESG score other than that of the standard tangency portfolio must yield a lower maximum SR.
    1. $\Rightarrow$ ESG-tangency portfolio is juuuuust to the right of tangency portfolio.

# Extra Slide 6: Proof of Proposition 2

The maximum Sharpe ratio for a given ESG score $\bar{s}$ is the Sharpe ratio of the optimal portfolio given in the proof of Proposition 1:

$$\mathrm{SR}(\bar{s}) = \frac{\mu^\top x}{\sigma} = \frac{\mu^\top \Sigma^{-1}(\mu + \pi\tilde{s})}{\sigma\theta} = \frac{\sigma^2\theta^2}{\sigma\theta} = \sigma\theta,$$

which is true from the last two equation in the proof of Prop. 1, and so

$$= \sigma\frac{1}{\sigma}\sqrt{\mu^\top \Sigma^{-1}(\mu + \pi\tilde{s})} = \sqrt{\mu^\top \Sigma^{-1}(\mu + \pi(s - 1\bar{s}))} = \sqrt{c_{\mu\mu} - \frac{\left(c_{s\mu} - \bar{s}c_{1\mu}\right)^2}{c_{ss} - 2c_{1s}\bar{s} + c_{11}\bar{s}^2}}.$$

- Max Sharpe-ratio is attained by the tangency portfolio, which is proportional to $\Sigma^{-1}\mu$.
- This portfolio has the ESG-score and Sharpe ratio stated in the proposition.

# Extra Slide 7: Proof of Proposition 3

From the proof of Prop. 1 we have $x = \frac{1}{\theta}\Sigma^{-1}(\mu + \pi\tilde{s})$.
From proofs of Prop. 1+2 we have $\theta = \frac{1}{\sigma}\mathsf{SR}(\bar{s})$ and $\sigma = \frac{\mathsf{SR}(\bar{s})}{\gamma}$, which implies

$$\theta = \frac{1}{\sigma}\mathsf{SR}(\bar{s}) = \frac{1}{\frac{\mathsf{SR}(\bar{s})}{\gamma}}\mathsf{SR}(\bar{s}) = \gamma.$$

Combining yields

$$x = \frac{1}{\theta}\Sigma^{-1}(\mu + \pi\tilde{s}) = \frac{1}{\gamma}\Sigma^{-1}(\mu + \pi(s - 1\bar{s})),$$

where we recall that $\tilde{s} = s - 1\bar{s}$.

# Extra Slide 8: ESG-Adjusted CAPM

ESG-adjusted CAPM predict expected returns:

- Many type-U investors $\Rightarrow$ High ESG predicts high expected returns.
  - High-ESG stocks profitable and not bid up by type-U investors.
- Many type-A investors $\Rightarrow$ No connection between ESG and expected return.
  - Prices are bid up by type-A investors to reflect their expected profits.
- Many type-M investors $\Rightarrow$ High ESG predicts low expected returns.
  - ESG-motivated investors are willing to accept a lower return for a higher ESG portfolio and thus bid up the price.

# Extra Slide 9: ESG-Adjusted CAPM

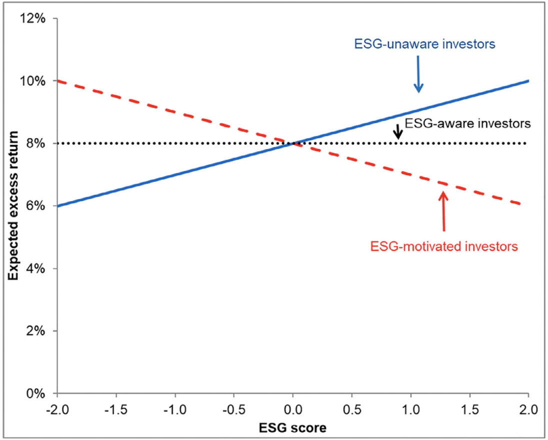
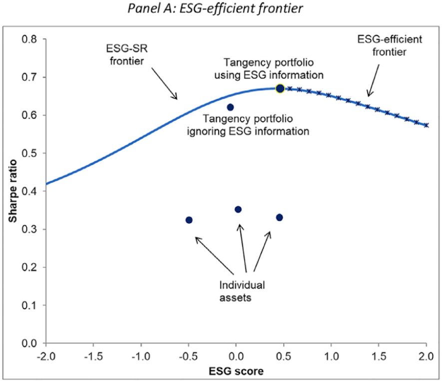
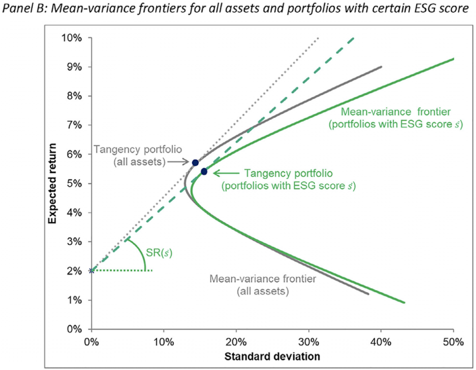



Figure: ESG-adjusted capital asset pricing model (ESG-CAPM).

# Extra Slide 10




- Type-M investors choose portfolio on ESG-efficient frontier.
- Type-A investors choose top-point of ESG-efficient frontier.
- Type-U investors choose tangency portfolio ignoring ESG information.