Notes and Exercises

Finance 2: Dynamic Portfolio Choice (Fin2)

Youssef Raad (zfw568)

April 22, 2024

Contents

1 Week 1

Lecture Notes:

- $v \geq 0$ ("v is non-negative") means that all v's coordinates are non-negative: $\forall i: \ v_i \geq 0$.
- $v > 0$ ("v is positive") means that $v \geq 0$ and that at least one coordinate is strictly positive: $\forall i: \ v_i$ and $\exists i: \ .v_i > 0$ (or equivalently $v \geq 0$ and $v \neq 0$)
- $v \gg 0$ ("v is strictly positive") (or rather $v \in \mathbb{R}^N_{++}$).

Definition 1. A financial market consists of a pair (π, C) where $\pi \in \mathbb{R}^N$ and C is an $N \times T$-matrix.

The interpretation is as follows: By paying the price π_i at date 0 one is entitled to a stream of payments $(c_{i,1}, \ldots, c_{i,T})$ at dates $1, \ldots, T$. Negative components are interpreted as amounts that the owner of the security has to pay. These N payment streams can be bought and sold generally combined; we use the concept of a portfolio to handle this.

Definition 2. A portfolio θ is an element of $\mathbb{R}^N$. The payment stream generated by θ is $C^\top\theta \in \mathbb{R}^T$. The price of the portfolio θ at date 0 is $\pi \cdot \theta = \pi^\top\theta = \theta^\top\pi$. The coordinate θ_i is interpreted as the number of units of security i that we buy at time 0.

- Negative coordinate: short position
- Positive coordinate: long position

To summerize:

- Price of portfolio at date $t = 0$ is $\pi \cdot \theta = \pi^\top\theta = \theta^\top\pi$
- Payment stream $(c_{i,1}, \ldots, c_{i,T})$ at dates $1, \ldots, T$ (denoted $C_{i,t}$)
- Payment stream generated by θ is $C^\top\theta \in \mathbb{R}^T$
- Time t payment of the portfolio is:

$$(C^\top\theta)_t = \sum_{i=1}^{N} C_{i,t}\theta_i = \sum_{i=1}^{N} \theta_i\pi_i$$

Definition 3. A portfolio θ is considered an *arbitrage opportunity* if the following inequality holds:

$$\begin{pmatrix} -\pi \cdot \theta \\ C^\top\theta \end{pmatrix} > 0$$

- $-\pi \cdot \theta > 0$ – The cost of the portfolio is negative.
- $C^\top\theta > 0$ – The payment stream of the portfolio θ is positive.

Weak or Type 1 Arbitrage: $\quad \pi \cdot \theta = 0, \quad C^\top\theta > 0$

Strong or Type 2 Arbitrage: $\quad \pi \cdot \theta < 0, \quad C^\top\theta \geq 0$

(First Fundamental) Theorem 1. The security market (π, C) is arbitrage-free if and only if there exists a strictly positive vector (of discount factors or discount vector) $d \in \mathbb{R}^T_{++}$ such that $\pi = Cd$. (i.e Current market price of each security (π) is equal to the present value of its expected future cash flows.)

Proof. $\Rightarrow$: Assume that d is a discount vector, and suppose contrarily that θ is an arbitrage. If θ is a type 2 arbitrage, we have $\pi^\top \theta < 0$. Because $\pi = Cd$, we also have that $\pi^\top \theta = d^\top C^\top \theta$. But as $d \gg 0$ and $C^\top \theta \geq 0$, the strict negativity is a contradiction. If θ is type 1 arbitrage, we know that $C\theta > 0$, and since $d \gg 0$ similar reasoning again leads to a contradiction. $\square$

1. **Case 1:** If C is an invertible quadratic matrix (i.e., the number of securities is equal to the number of time-points, and all securities are linearly independent), then we can directly find a unique discount vector candidate as:

$$d = C^{-1}\pi$$

   This scenario guarantees a unique solution assuming the matrix conditions hold.

2. **Case 2:** If $N > T$ (more securities than time-points) and we can identify an invertible $T \times T$ submatrix of C by selecting appropriate rows, there also exists a unique candidate for the discount vector. However, this solution requires additional verification as follows:

   Check that all $N - T$ remaining equations in $Cd = \pi$ hold true under this solution.

   Ensuring no contradictions arise in these extra conditions is essential to confirm the absence of arbitrage.

3. **Case 3:** When $N < T$ (more payment dates than assets), the analysis becomes more complex and requires additional consideration, which will be discussed at the end of this section. This scenario often involves underdetermined systems, which may have multiple solutions or none, demanding a deeper investigation into the conditions.

Definition 5. We say that a payment vector $y \in \mathbb{R}^T$ can be replicated if there exists a $\theta \in \mathbb{R}^N$ such that $C^\top \theta = y$. We say that the security market is complete if every $y \in \mathbb{R}^T$ can be replicated. (note: says nothing about the costs of such operation)

Second Fundamental Theorem 2. Assume that (π, C) is arbitrage-free. Then the market is complete if and only if there is a unique vector of discount factors.

The First Fundamental Theorem says that in any arbitrage-free market we may write

$$\pi_i = \sum_{t=1}^{T} C_{i,t} d_t$$

for all i and for some discount vector d.

In words, today's price of a bond is the sum of its discounted future payments. This means:

- We are only allowed to add or compare payments occurring at different dates if we properly discount them. Receiving \$1000 in 10 years does not have the same value to us today as receiving \$1000 today.
- The same discount vector must be used for all bonds. So, receiving \$1000 in 5 years has the same value irrespective of which bond it comes from. If it seems strange to the reader that anyone would treat payments differently depending on whence they came, then this section has fulfilled an important purpose.

2.7.2: Arbitrage-free price intervals in incomplete models Let us look at the linear programming problem $(U-P)$ for upper-primal:

$$\begin{aligned} \min_{\theta\in\mathbb{R}^N} \quad & \theta^\top \pi \\ \text{subject to} \quad & C^\top \theta \geq x, \\ & \theta \text{ free.} \end{aligned}$$

We assume that $(U-P)$ has a finite solution (boundedness comes from the absence of arbitrage; infeasibility would make things uninteresting), which we denote by $\theta_{*,U}$ and let $\pi_{*,U}$ be the associated optimal value. Solving $(U-P)$ can be interpreted as superreplicating x as cheaply as possible.

By exactly similar reasoning, solving the following problem:

$$\begin{aligned} \max_{\theta\in\mathbb{R}^N} \quad & \theta^\top \pi \\ \text{subject to} \quad & C^\top \theta \leq x, \\ & \theta \text{ free.} \quad (L-P) \end{aligned}$$

gives us a maximal lower bound, $\pi_{*,L}$, on the arbitrage-free price of x.

Hence, if the x-claim cannot be replicated, then the x-extended model is arbitrage-free if and only if

$$\pi_x \in \left]\pi_L^*, \pi_U^*\right[.$$

(replace with a single point if x-claim can be replicated which is not really what we are looking for anyways)

Using the duality table again, we see that the dual of $(U-P)$ is:

$$\begin{aligned} \max_{d\in\mathbb{R}^T} \quad & d^\top x \\ \text{subject to} \quad & Cd = \pi, \\ & d \geq 0. \quad (U-D) \end{aligned}$$

This (via the duality theorem) tells us that we can find the upper no-arbitrage bound by finding the discount vector that maximizes the value of the newly introduced claim.

Lecture notes: D is the pay-off matrix and π the time-0 prices. A portfolio θ is some vector in $\mathbb{R}^N$ where every i (entry) is the units of an security/asset. Lastly, we have a payoff vector $x \in \mathbb{R}^S$.

Question: Can we replicate this payoff vector x? i.e $\theta \in \mathbb{R}^N$, $D^\top\theta = x$? (Arbitrage-free price of x is $\pi^\top\theta$)

Incomplete model: Some assets can't be replicated (more states than assets $S > N$)

Answer:

1) we can look for super-replicating portfolios:

$$\mathbb{R}^S \ni D^\top\theta \geq x$$

2) We can look for *cheap* super-replicating portfolios: i.e

$$\begin{aligned} \min_{\theta \in \mathbb{R}^N} \quad & \pi^\top \theta \\ \text{subject to} \quad & D^\top \theta \geq x, \\ & \theta \text{ free.} \end{aligned}$$

Notation: U for upper bound, $*$ for optimal solution. θ_0^*, $\pi_u^* = \pi^\top \theta_U^*$.

Assumptions: Finite: else arbitrage, Feasible: else whats the point.

Result and "proof" is just from the book, but worse:

Result: $\pi_U^\star$ is a minimum upper bound on the arbitrage free time 0 price of a contract that pays x at time 1 (:time 0 price of x in short $=: \pi_x$).

Proof. We show both the minimal and the upper bound part:

Upper bound: If $\pi_x > \pi_U^\star$, then i would buy θ_U^* and I would sell x which yields a time 0 cash-flow $\pi_x - \pi^\top \theta_U^* > 0 \Rightarrow$ time 1 gives $D^\top \theta_U^* - x > 0$ which means that an arbitrage is present since the price of the asset is larger than the optimal value.

Minimal: If $\pi < \pi_U^*$, then no arbitrage with short position in x exists. Assume for contradiction that $(\theta_{ARB}, -1)$ (WLOG we can assume it is a position of -1 in x as scaling doesn't change an arbitrage). Because, by definition of arbitrage, we have $D^\top \theta_{ARB} - x \geq 0$ i.e θ_{ARB} satisfies the constraints in the above (LP) problem. Now assume again for contradiction that $\pi_x < \pi_U^*$. This implies $0 \geq \pi^\top \theta_{ARB} - \pi_x > \pi^\top \theta_{ARB} - \pi_0^*$. First inequality follows from arbitrage, second from $\pi_x < \pi_U^*$. This now implies $\pi^\top \theta_{ARB} < \pi_U$ (i.e we find something that is smaller (more optimal) than the optimal one: contradiction and we're done). $\square$

Cross currency betting arbitrage:

Odds: 1.25 on remain, 4.80 on leave would yield payoff £$1.25x$ and £$4.8x$, respectively.

Betting £1/1.25 and £1/4.80 would payback exactly 1 but yields a price of 1.0083. However, if we pay in dollars we get the payoff in dollars.

It seems reasonable to assume that the outcome: Remain, Leave, relates to the exchange rate between dollar/pound (\$/£): number of dollars needed to buy 1 pound.

Assume that at June 23rd the (\$/£)-exchange rate was 1.45 (which is observable). Now assuming that it depends on the outcome (ignoring noise), then at outcome remain the exchange rate goes to 1.5, leaves it goes to 1.3. Intuitively, remain means we need more dollars to buy 1 pound (pound strengthen relative to the dollar), and vice versa.

Why do we care?: Idea is that if we bet on remain and bet in pounds we get a double effect: the pound gets stronger against the dollar and also we remained, and vice versa for the dollar and leave.

Problems with this idea?: We have to know the exact exchange rate at the date of termination but a solution could be by looking at contracts in the stock market solely: longer paper by Rolf.

See Excel for an example with numbers (beware residual is in the opposite currency and thus we will end up with a strategy that cost θ, as desired). Max-min gives the best worst case scenario. Use solver to find the result, that is, a positive payoff for both states: i,e a cross-currency betting arbitrage. Use Rolf's results further down in the file to show that there is indeed a arbitrage present just from theory: We can create arbitrage in one *and/or* the other currency by the unequivalent conditions in the paper. In our example we have values:

$$c = 0.008\bar{3}, \quad u = \frac{1.50}{1.45}, \quad d = \frac{1.30}{1.45}.$$

Condition (A) in the file is for borrowing in pounds and (B) is borrowing in dollars.

Note for exam: Talk about either no arbitrage price intervals in incomplete models or cross-currency betting arbitrage in the exam. Questions regarding statistical arbitrage might be asked but shouldn't be in the 4-5 slides. Read yourself does however imply that it will be on the exam, unlike most other courses.

2 Week 2

Lecture Notes As usual, a 1-period model and π is the time-0 prices and the payment stream matrix is D, i.e $(\pi, D) \in \mathbb{R}^N \times R^{N\times S}$. The agent's problem is given by

$$\begin{aligned} \max_{\theta} \quad & U(e + D^\top \theta) \\ \text{s.t.} \quad & \pi^\top \theta \leq \underbrace{0}_{\text{WLOG}}, \end{aligned}$$

where:

- Smooth (:for differentation), strictly increasing in each coordinate (:pref. more to less), concave (:risk-aversion) in each coordinate utility function $U : \mathbb{R}^S_+ \to R$ maxes our wealth at time-1,
- θ our portoflio chose at time-0 subject to the constraint $\pi^\top \theta \leq 0$ (:price of portfolio),
- e is endovement (i.e some salary) either at time-0 or time-1.

(note: the "WLOG" follows because there is no utility from time-0 wealth/consumption (or we could invest everything into e)).

We introduce a mild regularity assumption: $\exists \theta_0 : \ D^T\theta_0 > 0$ (i.e atleast one state is positive but all states are non-negative.)

State-Price Utility Theorem: Marginal utility (:gradient of the utility evaluated) at optimum is proportional to a state-price vector (theorem 3 *write it here* in Lando). The proportionalality constant is positive.

Proof. For any portfolio $\theta \in \mathbb{R}^N$ we can look at, for some α to scale, $e + \alpha D^\top \theta$. We can then look at $U(\underbrace{c^*}_{e+\alpha D^\top\theta} + \alpha D^\top \theta) := g_\theta(\alpha)$. We look at $\pi^\top \theta = 0$. By optimality of c^*, $\alpha = 0$ is an optimum of

g_θ. This means by differentiality wrt to α we have that

$$g'_\theta(0)\underbrace{U(\nabla U(c^*))^\top}_{1\times S}\underbrace{D^\top}_{S\times N}\underbrace{\theta}_{N\times 1}=0.$$

we have shown that any vector that is orthogonal (:dot product =0) to π, is also orthogonal to $D\nabla U(c^*)$.

We claimed that this can only happen if π and $D\nabla U$ are proportional (: $\mu\pi = D\nabla U(c^*)$, $\mu \in \mathbb{R}$.). If they are not proportional we can find a vector ξ such that ξ is orthogonal to π, $\xi^\top\pi = 0$, such that $D\nabla U = m\pi + \xi$. This is a contradiction. Assume $\mu \in \mathbb{R}$. We now have that $\xi^\top D\nabla U(c^*) - \mu\xi^\top\pi + \xi^\top\xi \neq 0$. We also have that

$$\mu\pi = D\nabla U(c^*) \iff \underbrace{\pi = D(\nabla U(c^*)/\mu)}_{\text{(part of) definition of being a state-price-vector}}$$

□

Implications/usages? Often the utility is a function u where we have a set of posibilities $(p_1,\dots,p_S)$ and therefore has a expected utility representation

$$U(c) = \sum_{i=1}^{S} p_i u(c_i).$$

In this special case we note that the coordinates of the state-price vector satisfy for some λ (:constant of proportionalality) is

$$\Psi_i = \lambda p_i u'(c_i^*), \quad i = 1,\dots,S.$$

(marginal utility is positive but decreasing because of concavity of the utility. Marginal utility is therefore high in a state where consumption is low.)

The martingale method of portfolio optimization: Assume:

complete: then the state-price vector Ψ is unique.

incomplete:, then $U(c) = \sum_{i=1}^{S} p_i u(c_i)$. The unique solution is $D\Psi = \pi$ and we again have the coordinates of the state-price vector $\Psi_i = \lambda p_i u'(c_i^*), \quad i = 1,\dots,S.$. u' then has a inverse function $(u')^{-1}$ (by smoothness $\Rightarrow$ continuous). We then have (:technique name: (Pliska's) martingale method)

$$c_i^* = (u')^{-1}(\Psi_i/\lambda p_i)$$

(related to zero level pricing* reference in log-file).

Why does this relate to absence of arbitrage? Proposition 10

Proposition 10: If there exists a portfolio θ_0 with $D^\top\theta_0 > 0$ then the agent can find a solution to be maximization problem if and only if (π, D) is arbitrage free.

Remarks on the state price utility theorem:

- Intuition: "Bad states of nature have high state prices; a risk-averse agent is willing to pay (more than the expected) loss for insurance"
- Example with numbers: Exercise 1.4 and this spreadsheet and example 20 Lando and Poulsen.
- Optimality in complete models; the martingale method. We will see more of this later in multi-period models; the Pascucci and Runggaldier book.
- Using the state-prices determined from a presumably representative agent's optimum to price new assets in incomplete models. This is sometimes called indifference pricing or zero-level pricing in Luenberger (2002); it finds the price at which the agent will demand exactly 0 units of the new asset. (It is the same rationale that makes CAPM a pricing model.) This gives a single price instead of the typically quite wide no - arbitrage bounds produced via sub/super - replication and linear programming (as in Exercise 1.3). That combined with Carr and Madan (2001) and some empirical option data could make for an interesting master thesis.

self study - Betting against beta:

Brush up CAPM: Assumptions:

- n risky assets
- μ_i the rate of return of the i'th asset and we let $\mu_0 = r_0$ denote the riskless rate of return with μ_0 <global minimum variance portfolio.

Expected return on asset or portfolio is expressed as a function of β (:a measurement of its volatility of returns relative to the entire market). As tangency portfolio is efficient we have:

$$E(r_i) - \mu_0 = \beta_{i,tan}(E(r_{tan}) - \mu_0),$$

where:

$$\beta_{i,tan} = \frac{Cov(r_i, r_{tan})}{\sigma_{tan}^2}.$$

to derive CAPM, note Tangency portfolio=Market portfolio.

Market portfolio "the average of the stock market": Assume that the initial supply of risky asset j at time 0 has a value of P_0^j. (So P_0^j is the number of shares outstanding times the price per share.) The market portfolio of risky assets then has portfolio weights given as

$$w_j = \frac{P_0^j}{\sum_{i=1}^n P_0^i}$$

Assume wealths $W_i(0)$ to invest for K agents. Let ϕ_i denote the fraction of his wealth that agent i has invested in the tangency portfolio. By summing over all agents we get the total value of asset j:

$$\begin{aligned}\text{Total value of asset } j &= \sum_{i=1}^{K} \phi_i W_i(0) x_{\text{tan}}(j) \\ &= x_{\text{tan}}(j) \times \text{Total value of all risky assets,}\end{aligned}$$

where we have used that market clearing condition that all risky assets must be held by the agents.

The main economic assumption is that agents are mean-variance optimizers so that two fund separation applies. Hence we may as well write the market portfolio in equation (3.23). This is the CAPM:

$$E(r_i) - \mu_0 = \beta_{i,m} \underbrace{(E(r_m) - \mu_0)}_{\text{market premium}},$$

where:

$$\beta_{i,m} = \frac{Cov(r_i, r_m)}{\sigma_m^2}.$$

Note: that the type of risk for which agents receive excess returns are those that are correlated with the market. The intuition is as follows: If an asset pays off a lot when the economy is wealthy (i.e. when the return of the market is high) that asset contributes wealth in states where the marginal utility of receiving extra wealth is small. Hence agents are not willing to pay very much for such an asset at time 0. Therefore, the asset has a high return. The opposite situation is also natural at least if one ever considered buying insurance: An asset which moves opposite the market has a high pay off in states where marginal utility of receiving extra wealth is high. Agents are willing to pay a lot for that at time 0 and therefore the asset has a low return.

Note: the conceptual difference between the Study of optimality: in the former case we solve first order conditions for (a specific agent's) portfolio

Study of equilibrium; we solve for initial prices such that demand (agents' optimal portfolios) equals (exogenously fixed) supply. Note also that our analysis of equilibrium is simple, static, partial; we don't say how we get to equilibrium, nor study if it is in some sense stable.

Back to Beta: Equilibrium means looking at "when and how" (optimal) demand equals (the fixed) supply. Think of the supply as the number of stocks in the different companies.

The single index model is standard empirical regression framework, where thing are written up with stochastic variables

$$r_i - r_f = \alpha_i + \beta_{i,M}(r_M - r_f) + \varepsilon_i$$

- $\alpha := (E(r_i) - r_f) - \beta_{i,M}(E(r_M) - r_f)$.
- β has covariance-variance form from before.
- $(E(r_i) - r_f)$ viewed as function of β is called security market line.

See example of theory and pratice below of what we call betting against beta i.e the theoretical line is under the empirical line:

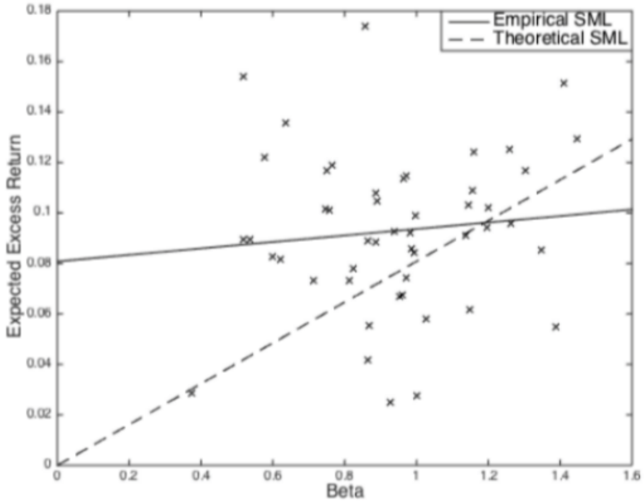


Figure 3.8: The empirical security market line computed on behalf of 49 industrial portfolios (French 1995-2015). On the same graph the theoretical "CAPM" counterpart has been plotted.

The paper Betting Against Beta: A betting against beta (BAB) factor is a portfolio that holds low-beta assets, leveraged to a beta of one, and that shorts high-beta assets, de-leveraged to a beta of one.

For instance, the BAB factor for US stocks achieves a zero beta by holding 1.4 of low-beta stocks and shortselling 0.7 of high-beta stocks, with offsetting positions in the risk-free asset to make it self-financing (: A portfolio is self-financing if there is no external infusion or withdrawal of money. The purchase of a new asset must be financed by the sale of an old one.).

Model predicts that BAB factors have a positive average return and that the return is increasing in the ex ante tightness of constraints and in the spread in betas between high- and low-beta securities (Proposition 2).

Theory on betting against beta: Assumptions:

- Overlapping-generations (OLG) economy: agents $i = 1, \ldots, I$ born in period t with wealth w_t^i and live for two periods.
- Agents trade securities $s = 1, \ldots, S,$, where s pays dividends δ_t^s and has x^{*s} shares outstanding (:actively held by stockholder) (all exogenous).
- Each time period t young agents choose a portfolio of shares $x = (x^1, \ldots, x^S)'$, investing the rest of their wealth at the risk-free return r^f, to maximize their utility

$$\max\, x'(E_t(P_{t+1} + \delta_{t+1}) - (1 + r^f)P_t) - \frac{\gamma^i}{2} x'\Omega_t x,$$

  where,

  - P_t is the vector of prices at time t
  - Ω_t variance-covariance matrix of $P_{t+1} + \delta_{t+1}$
  - γ^i is agent i's risk aversion.
  - Agent i is subject to the portfolio constraint:

$$m_t^i \sum_s x^S P_t^S \leq W_t^i.$$

    Some multiple m_t^i of the total dollars invested (the sum of the number of shares x^S times their prices P^S) must be less than the agent's wealth. The investment constraint depends on the agent i. For instance, some agents simply cannot use leverage, which is captured by $m_t^i = 1$.

    Some agents need to hold cash (:insurance for claims, mutual fund for daily redemptions) which is captured by $m^i > 1$ (for example $m_t^i = 1/(1 - 0.20) = 1.25$ represents an agent who must hold 20% of her wealth in cash).

    Some agent faces a margin requirement of 50%, then his multiplier is $m^i = 0.5$. With this margin requirement, the agent can invest in assets worth twice his wealth at most. A smaller margin requirement m^i naturally means that the agent can take greater positions

We are interested in the properties of the competitive equilibrium in which the total demand equals the supply

$$\sum_i x^i = x^*$$

To derive equilibrium, consider the first order condition for agent i

$$0 = E_t(P_{t+1} + \delta_{t+1}) - (1 + r^f)P_t - \gamma^i \Omega x^i - \Psi_t^i P_t,$$

where Ψ_t^i is the Lagrange multiplier of the portfolio constraint. Solving for x^i gives the optimal position:

$$x^i = \frac{1}{\gamma^i}\Omega^{-1}(E_t(P_{t+1} + \delta_{t+1}) - (1 + r^f + \Psi_t^i)P_t).$$

The equilibrium condition now follows from summing over these (i) positions:

$$x^* = \frac{1}{\gamma}\Omega^{-1}(E_t(P_{t+1} + \delta_{t+1}) - (1 + r^f + \Psi_t)P_t),$$

where the aggregate risk aversion γ is defined by $1/\gamma = \sum_i 1/\gamma^i$ and $\Psi_t = \sum_i (\gamma/\gamma^i)\Psi_t^i$ the weighted average Lagrange multiplier.(The coefficients γ/γ^i sum to one by definition of the aggregate risk aversion γ. The equilibrium price can then be computed:

$$P_t = \frac{E_t(P_{t+1} + \delta_{t+1}) - \gamma\Omega x^*}{1 + r^f + \Psi_t}$$

Now, using:

- Return of any security $r_{t+1}^i = (P_{t+1}^i + \delta_{t+i}^i)/P_t^i - 1$
- Return on the market r_{t+1}^M
- $\beta_t^s = cov_t(r_{t+1}^s, r_{t+1}^M)/var_t(r_{t+1}^M)$

we obtain proposition 1:

Proposition 1 (high beta is low alpha):

1. The equilibrium reqired return for any security s is

$$E_t(r_{t+1}^s) = r^f + \Psi_t + \beta_t^s \lambda_t$$

   where the risk premium is $\lambda_t = E_t(r_{t+1}^M)r^f - \Psi_t$ and Ψ_t is the average Lagrange multiplier, measuring the tightness of funding constraints.

2. A security's alpha (:excess returns earned above the benchmark return when adjusted for risk) with respect to the market is

$$\alpha_t^s = \Psi_t(1 - \beta_t^s)$$

   The alpha decreases in the beta, β_t^s

3. For an efficient portfolio, the Sharpe ratio is highest for an efficient portfolio with a beta less than one and decreases in β_t^s for higher betas and increases for lower betas

Tighter portfolio constraints (large Ψ_t) flatten the seecurity market line by increasing the intercept and decreasing the slope λ_t (: the risk premium).

Whereas the standard CAPM implies that the intercept of the security market line is r^f, the intercept here is increased by binding funding constraints (through the weighted average of the agents' Lagrange multipliers).

Why do zero-beta assets require returns in excess of the risk-free rate?

The answer has two parts. *need explaining*:

- Constrained agents prefer to invest their limited capital in riskier assets with higher expected return.
- Unconstrained agents do invest considerable amounts in zero-beta assets so, from their perspective, the risk of these assets is not idiosyncratic (: the inherent risk), as additional exposure to such assets would increase the risk of their portfolio.

Hence, in equilibrium, zero-beta risky assets must offer higher returns than the risk-free rate.

Assets that have zero covariance to the Tobin (1958) "tangency portfolio" held by an unconstrained agent do earn the risk-free rate, but the tangency portfolio is not the market portfolio in our equilibrium. The market portfolio is the weighted average of all investors' portfolios, i.e., an average of the tangency portfolio held by unconstrained investors and riskier portfolios held by constrained investors.

Hence, the market portfolio has higher risk (:as constrainted investors have more risk) and expected return than the tangency portfolio, but a lower Sharpe ratio.

The portfolio constraints further imply a lower slope λ_t of the security market line, i.e., a lower compensation for a marginal increase in systematic risk. The slope is lower because constrained agents need high unleveraged returns and are therefore willing to accept less compensation for higher risk.

Consider a factor that goes:

- Long low-beta assets
  - w_L is the relative portfolio weights of a portfolio of low-beta assets wth return $r^L_{t+1} = w'_L r_{t+1}$ with $\beta^L_t < \beta^H_t$.
- Short high-beta assets
  - w_H is the relative portfolio weights of a portfolio of high-beta assets wth return $r^H_{t+1} = w'_L r_{t+1}$ with $\beta^H_t > \beta^L_t$.

We then construct a betting against beta (BAB) factor as

$$r^{BAB}_{t+1} = \frac{1}{\beta^L_t}(r^L_{t+1} - r^f) - \frac{1}{\beta^H_t}(r^H_{t+1} - r^f)$$

this portfolio is market-neutral; that is, it has a beta of zero. The long side has been leveraged to a beta of one, and the short side has been de-leveraged to a beta of one. The model has several predictions regarding the BAB factor.

Proposition 2 (positive expected return of BAB): The expected excess return of the self-financing BAB factor is positive

$$E_t(r_{t+1}^{BAB}) = \frac{\beta_t^H - \beta_t^L}{\beta_t^L \beta_t^H}\Psi_t \geq 0$$

and increasing in the ex ante beta spread $\frac{\beta_t^H - \beta_t^L}{\beta_t^L \beta_t^H}$ and funding tightness Ψ_t

Proposition 2 shows that a market-neutral BAB portfolio that is long leveraged low-beta securities and short higherbeta securities earns a positive expected return on average. The size of the expected return depends on the spread in the betas and how binding the portfolio constraints are in the market, as captured by the average of the Lagrange multipliers Ψ_t.

Skriv proof til prop. 1 og 2 ind her

Proof: Proposition 1: The equilibrium prices were

$$P_t = \frac{E_t(P_{t+1} + \delta_{t+1}) - \gamma\Omega x^*}{1 + r^f + \Psi_t}$$

Rearranging yields

3 Week 3

Lecture Notes:

ESG-paper: Proposition 1 *right of graph of panel A* (2+3) regarding the Sharp-ratio-ESG-frontier.

ESG-score: (some number s, usually 60-70). If one were to care (we don't):

- E(nvironmental): A measure of how "ethically" (polution, unionize etc.) correct a firm is (Low: good, High: bad).
- S(ocial): usually "sin stocks" refer to alcohol, tobacco and firearms.
- G(overnance): accruals (fewer: higher governance score).

The model setup:

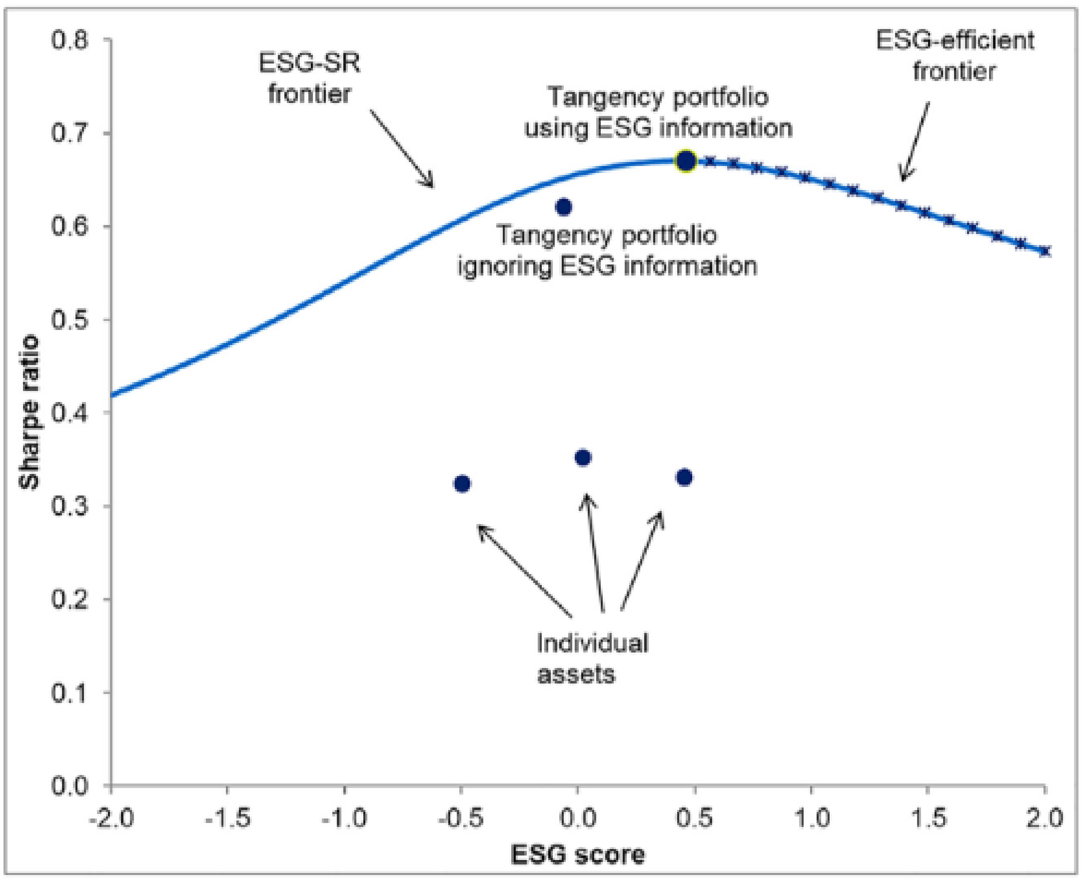
- n stocks and a risk free asset
- We consider excess rate of return of said assets, with mean vector $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$.
- 1 period, $t = 0, 1$
- x refers to portfolio weights
- Each stock has a ESG-score that we stack into a vector, $S \in \mathbb{R}^n$
- 3 types of investors:
  - U-type: Ignorant: ignorant to ESG-score, investors are unaware of ESG scores and simply seek to maximize their unconditional mean-variance utility
  - A-type: Aware: Use ESG scores to update their views on risk and expected return, investors also have mean-variance preferences, but they use assets' ESG scores to update their views on risk and expected return
  - M-type: Motivated: ESG is actively used, investors use ESG information and also have preferences for high ESG scores. In other words, M investors seek a portfolio with an optimal trade-off between a high expected return, low risk, and high average ESG score

Main findings: For each level of ESG, we compute the highest attainable Sharpe ratio (SR). We denote this connection between ESG scores and the highest SR by the **ESG-SR frontier**. They show that the investor's problem can be reduced to a trade-off between ESG and Sharpe ratio. In other words, risk and return can be summarized by the Sharpe ratio which is **independent of investor preferences**. Hence, an investment staff can first mechanically compute the frontier and then the investment board can choose a point on the frontier based on the board's preferences.

- (1) Theoretically, they show that an investor optimally chooses a portfolio on the ESG-efficient frontier.
- (2) The portfolios that span the frontier are all combinations of the risk-free asset, the tangency portfolio, the minimum-variance portfolio, and what they call the ESG-tangency portfolio (four-fund separation).
- (3) Investors are spread on the ESG-SR-frontier as:

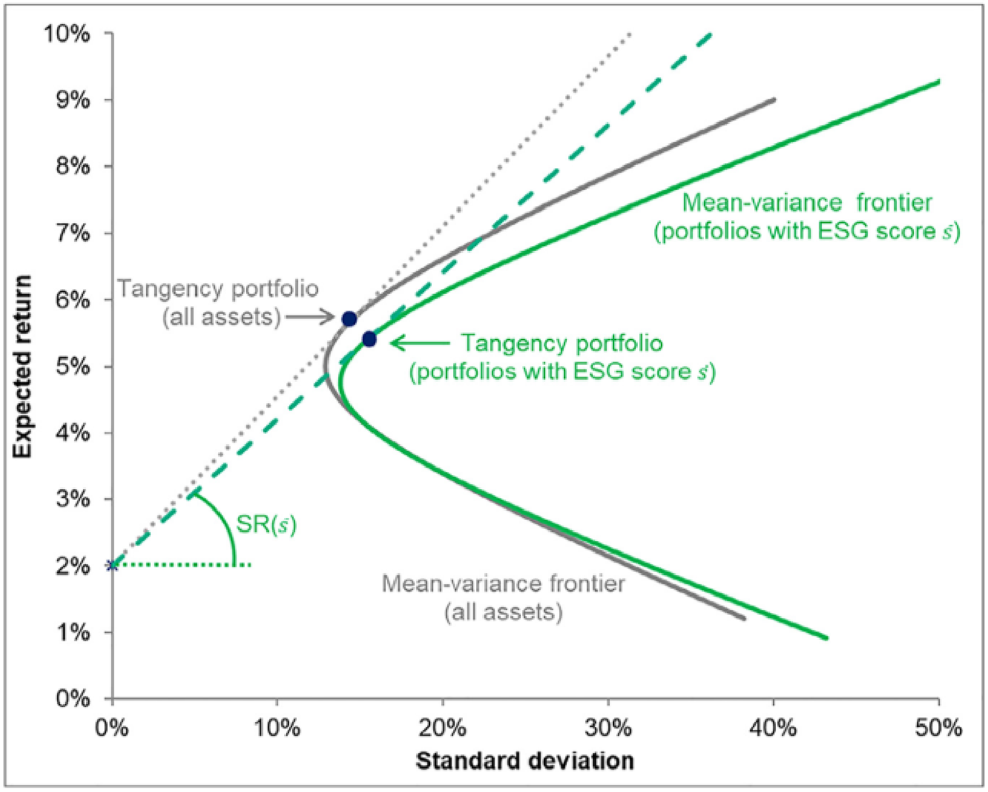
- U-Type: May choose a portfolio below the frontier, because they compute the tangency portfolio while ignoring the security information contained in ESG scores (they condition on less information)
- A-Type: Choose the portfolio with the highest SR: the tangency portfolio using ESG information
- M-Type: Preference for higher ESG, so they choose portfolios to the right of the tangency portfolio, on the ESG-efficient frontier. Choosing portfolios below or to the left of the efficient frontier is suboptimal because, in this case, the investor can improve one or both of the ESG score and the SR, without reducing the other

Panel A: ESG-efficient frontier



To understand why the ESG-SR frontier is humpshaped, consider first the tangency portfolio known from the standard mean-variance frontier, shown in Fig. 1, Panel B. This tangency portfolio has the highest SR among all portfolios, so its ESG score and SR define the peak in the ESG-SR frontier. Further, the ESG-SR frontier is humpshaped because restricting portfolios to have any ESG score other than that of the tangency portfolio must yield a lower maximum SR, as illustrated in Panel B.

Panel B: Mean-variance frontiers for all assets and portfolios with certain ESG score



The investor wants to solve

$$\max_{x \in X} \left(x'\mu - \frac{\gamma}{2} x'\Sigma x + f\left(\frac{x's}{x'1}\right) \right),$$

where γ is the relative risk-aversion. The argument in the function allows the risk-free asset to be ESG-neutral. Solution is in Appendix A4 in the paper or from the lecture notes below:

We look at the inner most problem of the solution:

$$x' \underbrace{(\bar{s}1 - s)}_{-\tilde{s}} = 0$$

The outer (second) constraint:

$$(\sigma^2 - x'\Sigma x) = 0$$

We now setup the Lagrange function:

$$\mathcal{L}(x, \pi, \theta) = x'\mu - \frac{\gamma}{2}\sigma^2 + f(\bar{s}) + \pi(x'\tilde{s}) + \frac{\theta}{2}(\sigma^2 - x'\Sigma x)$$

$$\Rightarrow$$

$$\frac{\partial \mathcal{L}}{\partial x} = \mu + \pi\bar{s} - \theta\Sigma x = 0$$

$$\iff$$

$$x = \frac{1}{\theta}\Sigma^{-1}(\mu + \pi\tilde{s})$$

1. constraint (equation 39 in the paper):

$$x'\tilde{s} = 0$$

$$\Longleftrightarrow$$

$$\underbrace{\frac{1}{\theta}}_{*} \cdot (\mu + \pi\tilde{s})'\Sigma^{-1}\tilde{s} = 0$$

$$\Longleftrightarrow$$

$$\pi = -\frac{\mu'\Sigma^{-1}\tilde{s}}{\tilde{s}'\Sigma^{-1}\tilde{s}}$$

$*$ is a number and can be omitted.

2. constraint:

$$\sigma^2 = \frac{1}{\theta^2}(\mu + \pi\tilde{s})'\Sigma^{-1}(\mu + \pi\tilde{s})$$

Using the first constraint, we can simplify as

$$\sigma^2 = \frac{1}{\theta^2}\mu'\Sigma^{-1}(\mu + \pi\tilde{s})$$

implying that the second Lagrange multiplier is

$$\theta = \frac{1}{\sigma}\sqrt{c_{\mu\mu} - \frac{(c_{s\mu} - c_{1\mu}\bar{s})^2}{c_{ss} - 2c_{1s}\bar{s} + c_{11}\bar{s}^2}}$$

This shows explicitly that we can write the optimal portfolio as

$$\hat{x} = \sigma v = \text{"some number"} \cdot \text{"some vector"},$$

where the vector v depends only on the exogenous parameters and $\bar{s}$, that is, not on σ.

Proposition 1 (ESG-SR trade-off): The investor should choose her average ESG score $\bar{s}$ to maximize the following function of the squared Sharpe ratio and the ESG preference function f:

$$\max_{\bar{s}} \left((SR(\bar{s}))^2 + 2\gamma f(\bar{s}) \right).$$

This proposition shows how investors optimally trade off ESG and Sharpe ratios. Not surprisingly, ESG affects the optimal portfolio choice, given that ESG is in the utility function, but the interesting result here is that we can analyze this trade-off using a part that depends only on securities [the ESG-SR frontier, $SR(\bar{s})$] and another part that depends only on preferences [$2\gamma f(\bar{s})$].

***In other words**, just like the standard Markowitz theory is powerful because the mean-variance frontier can be computed independent of preference parameters and then decisions about what portfolio to pick are based on risk aversion, the ESG-SR frontier can be computed independent of preferences and then the investor can decide in the end where on the frontier to place herself.*

Put differently, the ESG-SR frontier summarizes all security-relevant information. The investor's problem is to first place herself on the ESG-SR frontier and then decide on the amount of risk. This method works because investors care about the average ESG, which does not change when the investor chooses the risk level in the second step by choosing her cash holding. If investors care about total ESG, x_s, instead of average ESG, then the investor's problem cannot be summarized as the ESG-SR frontier, which also shows that our frontier results are not trivial.

We next characterize how the maximum Sharpe ratio depends on the ESG score. We use the notation $c_{ab} = \frac{a-1}{b} \in \mathbb{R}$ for any vectors $a, b \in \mathbb{R}^n$. **Proposition 2 (ESG-SR frontier):** The maximum Sharpe ratio, $SR(\bar{s})$, that can be achieved with an ESG score of $\bar{s}$ is given by:

$$SR(\bar{s}) = \sqrt{c_{\mu\mu} - \frac{(c_{s\mu} - \bar{s}c_{1\mu})^2}{c_{ss} - 2\bar{s}c_{1s} + \bar{s}^2 c_{11}}}.$$

The maximum Sharpe ratio across all portfolios is $SR(s^*) = \sqrt{c_{\mu\mu}}$, which is attained with an ESG score of $s^* = \frac{c_{s\mu}}{c_{1\mu}}$. Increasing the ESG score locally around s^* leads to nearly the same Sharpe ratio, $SR(s^* + \Delta) = SR(s^*) + o(\delta)$, because the first-order effect is zero, i.e., $\frac{dSR(s^*)}{ds} = 0$.

We next consider the nature of the optimal portfolio weights for an ESG-aware investor.

Proposition 3 (Four-Fund Separation): Given an average ESG score $\bar{s}$, the optimal portfolio is

$$x = \frac{1}{\gamma}\Sigma^{-1}(\mu + \pi(s - 1\bar{s})),$$

as long as $x'1 > 0$, where

$$\pi = \frac{c_{1\mu}\bar{s} - c_{s\mu}}{c_{ss} - 2c_{1s}\bar{s} + c_{11}\bar{s}^2}.$$

The optimal portfolio is therefore a combination of the risk-free asset, the tangency portfolio $\Sigma^{-1}\mu$, the minimum-variance portfolio $\Sigma^{-1}1$, and the ESG-tangency portfolio $\Sigma^{-1}s$. The optimal portfolio looks the same as the standard Markowitz solution, except that the expected excess returns μ have been adjusted. In other words, the optimal portfolio can be found as follows.

The investor first computes ESG-adjusted expected returns, $\mu + \pi\,(s - 1\bar{s})$, in the sense that each stock's expected excess return is increased if its ESG score s_i is above the desired average score $\bar{s}$ (i.e $s - 1\bar{s}$); otherwise, it is lowered. The amount of adjustment depends on the scaling parameter π, or the strength of the preference for ESG.

Next, the investor computes the optimal portfolio found in the standard way, that is, multiplying by $\frac{1}{\gamma}\Sigma^{-1}$. Therefore, all investors, regardless of their risk aversion and ESG preferences, should choose a combination of four portfolios (or funds): the risk-free asset, the standard tangency portfolio, the minimum variance portfolio, and the portfolio that we call the ESG-tangency portfolio. The ESG-tangency portfolio is the tangency portfolio if we replace the expected excess returns with the ESG scores.

Lecture Notes:

Dynamic/multiperiod models: The compat version chapter 5 L and P, 1+2.1 in P and R:

Probabilistic set-up:

- Filtered probability space: $(\Omega, \{\mathcal{F}_t\}_{t=0,1,\ldots,T'}, \mathbb{P})$

- Financial models where $(\{\rho_t\}, \{S_t\}, \{\delta_t\})$ are locally risk free rate, stock prices and dividend processes for stocks, respctively. (In the book: risk free rate is constant and dividend is 0)
- Portfolio $\sim$ trading strategies, i.e how many units of each stock i we are holding at time-t. Denoted as $\{\phi_t\}$, an $\mathbb{R}^n$-valued stochastic process adapted to $\{\mathcal{F}_t\}_{t=0,1,\ldots,T'}$

Value process of ϕ: stock prices $\times$ units of stock at time t:

$$V_t^\phi = \phi_t \cdot S_t$$

Self-financing: no external infusion of money to allow for the pruchase of new units of stocks, i.e no net cash flow. The self-financing condition (with consumption c_t) is thus

$$V_{t+1}^\phi = \phi_t \cdot (S_{t+1} + \delta_{t+1}) - c_t.$$

$\mathbb{Q}$: **Martingale/risk-neutral propbability measure:** equivalent if $\mathbb{Q} \sim \mathbb{P}$. Defined by: for all assets i, S_t and time points t:

$$S_t^i = \frac{1}{1+\rho_t} \mathbb{E}_t^{\mathbb{Q}}(S_{t+1}^i + \delta_{t+1}^i)$$

or by martingale measure:

$$m_t = \mathbb{E}_t(m_{t+1})$$

or by replication. Let X be and $\mathcal{F}_T$-measureable random variable. X can be replicated, if there exists a self-financing trading strategy from $0, T$, ϕ, such that

$$V_T^\phi = X$$

If any X can be replicated, we say that the model is complete.

Fundamental theorem of arbitrage pricing:

1. There is no arbitrage if and only if there exists a martingale measure
2. Assume the model (S, δ, ρ) is arbitrage-free. Then the market is complete if and only if the equivalent martingale measure is unique.

A special case/a workhose: The standard binomial model (and its parametrization)

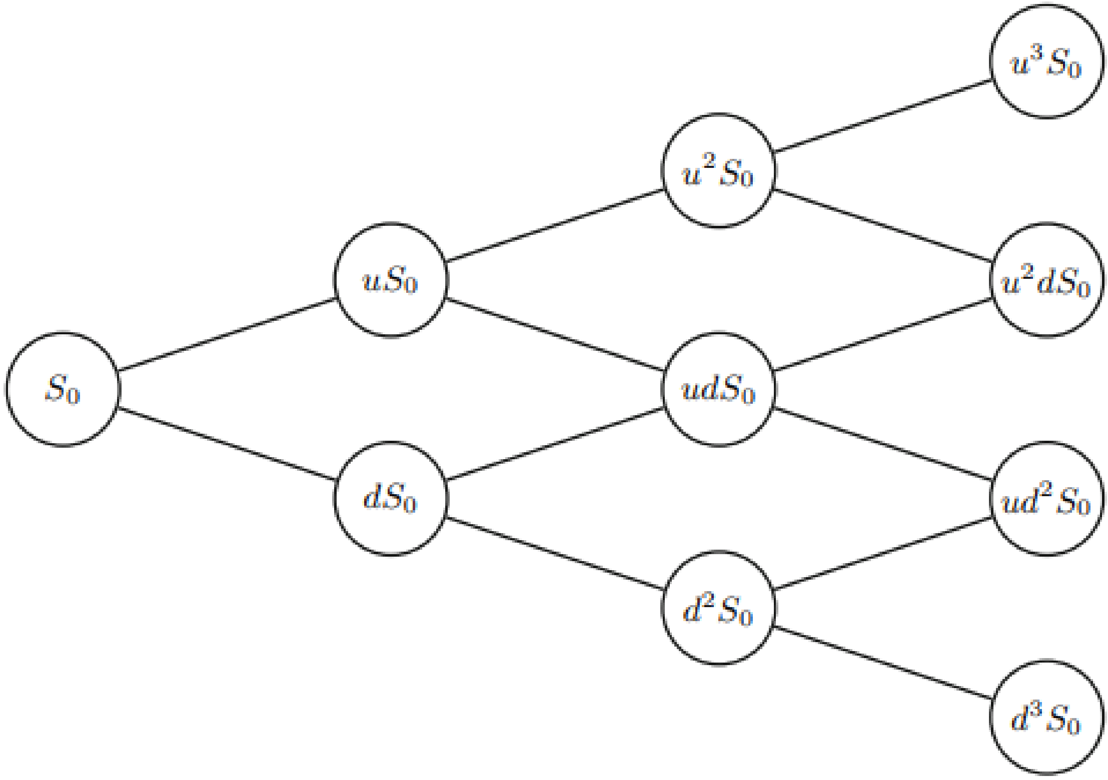


Fig. 1.1. Binomial tree with three periods

- No dividends
- Constant risk free rate $R = 1 + r$, $R_t = (1 + r)^t$
- Up and down movements are characterized by conditional probabilities p, $p - 1$.
- Conditional "up"-probability $q = \frac{R-d}{u-d}$ (follows from stock price is $S_t = \frac{1}{R}(quS_t + (1 - q)dS_t)$)
- Model is arbitrage free and complete (assuming $q \in (0, 1)$).

Parametrization (often used: not in this course) uses dt with $1/252$ (similar to finkont2).

Pascucci and Runggaldie (abstract setting): Uses slightly different notation (Ch. 1,2), i.e:

- like to work with rates of return rather than prices, $\mu_{t+1} = \frac{S_{t+1}\delta_{t+1} - S_t}{S_t}$
- Trading strategies α, β, c, i.e α =Vector of stock weights $d_i = \frac{\phi^i S^i}{V^\phi}$ (not absolute value), β=Represent amount in bond, determined after to make strategy self-financing (given c).
- α_t is chosen at $t - 1$.
- $\rho_t = r$ and $\delta = 0$

Maximization of expected utility (dynamic portfolio choice problems/investment consumption problem): Problems where we maximize over a set of self-financing strategies ϕ and some starting capital v:

$$\max_{\phi,\, V_0^{\phi}=v} \mathbb{E}\left(\sum_{t=0}^{T-1} \delta^t \mu_t(c_t) + \delta^T \mu_T(V_T^{\Phi})\right), \quad \delta \neq \text{dividend}.$$

Assume complete market (any $\mathcal{F}_T$-measurable X can be replicated) and no utility from intermediate consumption (i.e no utility from the terms $\mu_t(c_t)$). We then consider problems over ϕ, a set of self-financing strategies and $V_0^{\phi} = v$ is the initial wealth

$$\max_{\phi,\, V_0^{\phi}=v} E(u(V_T^{\phi}))$$

The discounted value process, $V_t^{\phi}/(1+r)^t$ of a self financing trading strategy is a $\mathbb{Q}$-martingale which implies that $V_0^{\phi} = v = \frac{1}{(1+r)}\mathbb{E}^{\mathbb{Q}}(V_T^{\phi})$. In other words, we can build the condition V_0^{ϕ} into the constraints using the above equality

4 Week 4

The Martingale Method (following P and R):

Complete market: terminal wealth

Consider the problem of maximization of expected utility of terminal wealth

$$\max_{\alpha} \mathbb{E}\left\{u(V_N^{(\alpha)})\right\}.$$

starting from an initial wealth v. The martingale method consists of three steps:

- **(P1)** recalling the Notation 2.4, determine the set of terminal values that can be reached by a self financing and predictable strategy

  $$\mathcal{V}_v = \left\{V \mid V = V_N^{(\alpha)},\ \alpha \text{ predictable},\ V_0^{(\alpha)} = v\right\}$$

  Solution: If market is arbitrage free and complete, there exists a unique Martingale Measure $\mathbb{Q}$. We then have the characterization:

  $$\mathcal{V}_v = \left\{V \mid \mathbb{E}^{\mathbb{Q}}[B_N^{-1}V] = v\right\}.$$

  for the set of terminal values that are reachable by means of a self-financing strategy with initial value v.

- **(P2)** determine the optimal terminal reachable value $\bar{V}_N$ namely the one that realizes the maximum:

  $$\max_{\alpha} \mathbb{E}\left\{u(V_N^{(\alpha)})\right\}.$$

  Solution: We use **Theorem 2.18.** In a market that is complete and free of arbitrage, consider the problem of maximization of the expected utility of terminal wealth given above starting from an initial capital $v \in \mathbb{R}_+$. Under the condition

  $$u'(I) = \mathbb{R}_+, \quad (2.30)$$

the optimal terminal value is given by

$$\bar{V}_N = \mathcal{I}(\lambda \tilde{L}) \quad (2.31)$$

where $\tilde{L} = B_N^{-1} L$ with $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$, being $\mathbb{Q}$ the martingale measure, and $\lambda \in \mathbb{R}$ is determined by the equation

$$\mathbb{E}^{\mathbb{P}}\left(\mathcal{I}\left(\lambda \tilde{L}\right)\tilde{L}\right) = v. \quad (2.32)$$

Equation (2.32) is called the "budget equation."

- **(P3)** determine a self-financing strategy $\bar{\alpha}$ such that

$$V_N^{(\bar{\alpha})} = \bar{V}_N$$

. **Solution:** Corresponds to a standard hedging problem (where the payoff to be replicated is the optimal terminal value $\bar{V}_N$.

Notice that the martingale method decomposes the original dynamic portfolio optimization problem into a static problem (the determination of the optimal terminal value) and a hedging problem (corresponding to a "martingale representation problem").

Write in example 2.19 and proof of theorem 2.18

Incomplete market: terminal wealth *NOT CURRICULUM*: Updated, its back in..

Solving P1:

In the case when the market is free of arbitrage and incomplete, the set of martingale measures is infinite. Observe first of all that, on the basis of Theorem 1.21, the set of terminal values that can be obtained starting from an initial wealth v has the following characterization:

$$\mathcal{V}_v = \left\{V \mid \mathbb{E}^{\mathbb{Q}}[B_N^{-1}V] = v, \text{ for each martingale measure } \mathbb{Q}\right\}.$$

Secondly, the family of martingale measures is the intersection of an affine space of $\mathbb{R}^M$ with the set of strictly positive probability measures.

$$\mathbb{R}_+^M = \{\mathbb{Q} = (\mathbb{Q}_1, \dots, \mathbb{Q}_M) \mid \mathbb{Q}_j > 0, \; j = 1, \dots, M\}$$

In particular, there exist measures $Q^{(1)}, \dots, Q^{(r)} \in \overline{\mathbb{R}_M^+}$ such that every martingale measure Q can be expressed as a linear combination of the form

$$\mathbb{Q} = a_1 \mathbb{Q}^{(1)} + \cdots + a_r \mathbb{Q}^{(r)},$$

in which the sum of the weights a_i is one. Consequently, we have

$$\mathcal{V}_v = \left\{V \mid \mathbb{E}^{\mathbb{Q}^{(j)}}\left[B_N^{-1}V\right] = v \text{ for } J = 1, \dots, r\right\}.$$

Solving P2:

Once the "extremal" measures $\mathbb{Q}^{(1)},\dots,\mathbb{Q}^{(r)}$ have been identified, the following result, which generalizes Theorem 2.18, solves the problem P2 of determining the optimal terminal reachable values $\tilde{V}$ when starting from an initial capital $v \in I$.

Theorem 2.21. Under the condition

$$u'(I) = \mathbb{R}_+, \quad (2.43)$$

the optimal terminal value is

$$\tilde{V}_N = \mathcal{I}\left(\sum_{j=1}^{r} \lambda_j \tilde{L}^{(j)}\right), \quad (2.44)$$

where $\tilde{L}^{(j)} = B_N^{-1} L^{(j)}$ with $L^{(j)} = \frac{d\mathbb{Q}^{(j)}}{d\mathbb{P}}$ and $\lambda_1,\dots,\lambda_r \in \mathbb{R}$ are determined from the system of budget equations

$$\mathbb{E}^{\mathbb{P}}\left[\mathcal{I}\left(\sum_{k=1}^{r} \lambda_k \tilde{L}^{(k)}\right)\tilde{L}^{(j)}\right] = v, \quad j = 1,\dots,r.$$

Consider a stochastic process $(V_n)_{n=0,\dots,N}$ on a finite probability space $(\Omega,\mathcal{F},\mathbb{P})$. This process could represent the value of a portfolio, and its evolution depends on a control process (such as an investment strategy or a consumption process). The evolution is described by the recursive relation:

$$V_k = G_k(V_{k-1},\mu_k;\eta_{k-1}(V_{k-1})), \tag{4.1}$$

for $k = 1,\dots,N$, where:

- $\mu_1,\dots,\mu_N$ are d-dimensional independent random variables, typically representing the risk factors influencing the asset prices in a discrete-time market.
- $\eta_0,\dots,\eta_N$ are generic functions $\eta_k : \mathbb{R} \to \mathbb{R}^\ell$, for $k = 0,\dots,N$ and $\ell \in \mathbb{N}$, termed as control functions or controls.
- $G_1,\dots,G_N$ are generic functions $G_k : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$, for $k = 1,\dots,N$.

To see this in use we need to use Proposition 2.8.

Proposition 2.8: The value of a self-financing strategy with consumption (α,β,C) is determined by the initial value $V_0 \in \mathbb{R}$ and by the processes $\pi_1,\dots,\pi_d$ and C according to the following recursive relation:

$$V_n = (V_{n-1} - C_{n-1})(1+r_n) + V_{n-1}\left(\sum_{i=1}^{d} \pi_n^i(\mu_n^i - r_n)\right).$$

Example 2.20 (see also 2.19): Consider a self-financing strategy with consumption and suppose that the processes of investment and consumption are functions of the portfolio value, namely that

$$\alpha_k = \alpha_k(V_{k-1}), \quad C_k = C_k(V_k), \quad k \geq 1.$$

Then, by Proposition 2.8, the value of the strategy satisfies the recursive relation (2.72) with

$$G_k(v,\mu_k;\eta_{k-1}) = v\left(1 + r_k + \sum_{i=1}^{d} \eta_{k-1}^i(\mu_k^i - r_k)\right) - (1+r_k)\eta_{k-1}^{d+1},$$

where η is the $(d+1)$-dimensional process, the components of which are the ratios invested in the risky assets as well as the consumption:

$$\eta_k = \begin{cases} (\pi^1_{k+1}, \dots, \pi^d_{k+1}, C_k), & \text{for } k = 0, \dots, N-1, \\ (0, \dots, 0, C_N), & \text{for } k = N. \end{cases}$$

Notation 2.31: Having fixed $v \in \mathbb{R}_+$ and $n \in \{0, 1, \dots, N-1\}$, denote by $(V_k^{n,v})_{k=n,\dots,N}$ the process defined by $V_n^{n,v} = v$ and, recursively, by (2.72) for $k > n$. Furthermore, put

$$U^{n,v}(\eta_n, \dots, \eta_N) = \mathbb{E}\left[\sum_{k=n}^{N} u_k\left(V_k^{n,v}, \eta_k\left(V_k^{n,v}\right)\right)\right], \quad (2.75)$$

where $u_0, \dots, u_N$ are given functions $u_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for $n = 0, \dots, N$.

We are interested in the optimization problem that consists in determining the supremum of $U^{0,v}(\eta_0, \dots, \eta_N)$ over the controls $\eta_0, \dots, \eta_N$, namely

$$\sup_{\eta_0, \dots, \eta_N} U^{0,v}(\eta_0, \dots, \eta_N). \quad (2.76)$$

We are also interested in determining, whenever they exist, the optimal controls that achieve this supremum.

The method of Dynamic Programming (in the sequel DP) to solve the optimization problem (2.76) is based on the idea that if a control is optimal over an entire sequence of periods, then it has to be optimal over each single period.

Theorem 2.32: For each $n = 0, \dots, N$, we have

$$\sup_{\eta_n, \dots, \eta_N} U^{n,v}(\eta_n, \dots, \eta_N) = W_n(v) \quad (2.77)$$

where $W_n(v)$ is defined recursively by

$$\begin{aligned} W_N(v) &= \sup_{\xi \in \mathbb{R}} u_N(v, \xi), \\ W_{n-1}(v) &= \sup_{\xi \in \mathbb{R}} \{u_{n-1}(v, \xi) + \mathbb{E}\left[W_n\left(G_n(v, \mu_n; \xi)\right)\right]\} \quad \text{for } n = N, \dots, 1. \quad (2.78) \end{aligned}$$

We point out that (2.78) leads to a recursive algorithm in which, at every step, we perform a standard optimization of a function of real variables. In particular, under suitable assumptions that guarantee that the supremum in (2.77) is attained (namely is actually a maximum), the algorithm also allows determining the optimal controls $\bar{\eta}_0, \dots, \bar{\eta}_N$. In fact, they result from the points where the functions to be maximized in (2.78) attain their suprema: more precisely, suppose that for each n there exists $\bar{\xi}_n \in \mathbb{R}$ which maximizes the function

$$\xi \mapsto u_{n-1}(v, \xi) + \mathbb{E}\left[W_n\left(G_n(v, \mu_n; \xi)\right)\right],$$

and notice that $\bar{\xi}_n$ depends implicitly on v; then the function $\bar{\eta}_{n-1}$ defined by

$$\bar{\eta}_{n-1}(v) = \bar{\xi}_n$$

is an optimal control.

Summing up, the method of DP leads to a deterministic algorithm in which, at every step, we determine (by backwards recursion), the optimal value and the optimal controls by means of a standard scalar maximization procedure.

Example 2.33 (Maximization of Expected Utility of Terminal Wealth): The value of a self-financing strategy is defined recursively by

$$V_k = G_k(V_{k-1}, \mu_k; \pi_k) = V_{k-1}\left(1 + r_k + \sum_{i=1}^{d} \pi_{i,k}(\mu_{i,k} - r_k)\right).$$

We have

$$U^{n,v}(\pi_{n+1}, \ldots, \pi_N) = \mathbb{E}\left[u\left(V_N^{n,v}\right)\right],$$

and, by Theorem 2.32,

$$\sup_{\pi_{n+1},\ldots,\pi_N} \mathbb{E}\left[u\left(V_N^{n,v}\right)\right] = W_n(v)$$

where

$$\begin{aligned} W_N(v) &= u(v), \\ W_{n-1}(v) &= \sup_{\bar{\pi}_n \in \mathbb{R}^d} \mathbb{E}\left[W_n\left(G_n(v, \mu_n; \bar{\pi}_n)\right)\right], \quad \text{for } n = N, \ldots, 1. \end{aligned} \tag{2.81}$$

Lecture Notes: Dynamic optimal portfolio problems with the Martingale method. We use the Martingale method to sovle a specific example: log-utility from terminal consumption in the standard binomial model (section 2.4.1 in R and P: done in the linked Excel-file).

Base case:

$$\max_{V_0^\phi = v} \mathbb{E}^{\mathbb{P}}(u(V_N^\phi))$$

where ϕ is a self-financing strategy which is an adapted proces. The static optimization problem (P2) is then solved by theorem 2.18.

again, remember to write in example 2.19 in notes: rewritings don't matter

Example 2.19: The Radon-Nikodym derivative of $\mathbb{Q}$ wrt $\mathbb{P}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathbb{Q}(\{\omega\})}{\mathbb{P}(\{\omega\})}$$

where $\{\omega\}$ corresponds to different paths. Ultimately, we only have to track up and down movement counts. the probability for a up move is p, down move $1-p$. The probability for some path is thus:

$$p^{\#\text{up-count}}(1-p)^{\#\text{down-count}}$$

Section 2.4.1 (we omitted 2.4.2 for some reason): Note that

$$u(x) = \ln(x), \quad u'(x) = \frac{1}{x}, \quad \text{solve for } x: \; y = \frac{1}{x} \iff x = \frac{1}{y}$$

We then get the expression for $\bar{V}_N$ using 2.18 (beware of inverse of Radon-Nikodym derivative) to find the solution to the static optimization problem.

The optimal value of the expected utility is then found by using $\ln(\bar{V}_N)$ and the linearity of the expectation operator.

Excel-file for binomial model: Some things to remember:

- Movements given by: $u, d = e^{\alpha \pm \sigma}$
- Random variable: $\nu_N = \#\text{up-count}$
- Optimal (terminal) wealth: $\bar{V}_N = v(1+r)^N \left(\frac{p}{q}\right)^{\nu_N} \left(\frac{1-p}{1-q}\right)^{N-\nu_N}$

To find optimal strategy, we solve in Excel numerically the replication equations), i.e we see (approximately as Rolf is lazy) from Excel under the optimal wealther (with $v = 500$ not 100):

$$\begin{aligned} \bar{V}_N &\approx 1320, 406 \\ S_n &\approx 909, 223, \end{aligned}$$

we then construct a portfolio of stocks a and b in bank such that (using equalities to ease notation)

$$\begin{aligned} 406a + (1.04)b &= 1320 \\ 223a + (1.04)b &= 909, \end{aligned}$$

which gives

$$a = \frac{1320 - 909}{406 - 223}$$
$$b = \frac{1}{1.04}\left(1320 - 406a\right).$$

Value of the replicating portfolio is then

$$\begin{aligned} 287a + b = 1035 \\ &= \frac{1}{R}\mathbb{E}^{\mathbb{Q}}_{N-1}\left(\bar{V}_N\right) \\ &= \frac{1}{1.04}\mathbb{E}^{\mathbb{Q}}_{N-1}\left(\bar{V}_N\right) \end{aligned}$$

(this is the same as in the book in 2.4.1 but using a concrete example, i.e the data from Excel.)

Important note in Excel file: there is a pattern with the optimal stock position in relative weight. Hence, the optimal portfolio has the particular structure that the weight in the stock is the same in all nodes.

- Not a proof but this is always the case (see 2.4.1 for proof)...
- We never assumed the weights were constant. The solution method was general. Any adapted strategy could have been used. This is a property (in asset allocation course we will see that it is sometimes worht looking for such *fixed fraction* portfolios).
- Is it true only for log-utility and binomial-model? No. Much more general result is actually that it does not depend on the investment horizon.

Note: Exam question 7 about consumption based CAPM is deleted. For martingale method question excel file and the above is a good example. **Remember** to say - if asked - that the relative portfolio weight is constant but that it is **not** easy to see why (referer to section 2.4.1 but use the first part of 2.4.1 in the presentation).

Bellman equation:

CRRA: utility functions of the form

$$u(x) = \frac{x^{1-a} - 1}{1 - a}.$$

If $RRA = a = 1$ we achieve log-utility by L'Hopital.

Example:

- $v = 100$
- $S_0 = 100$
- Assume 50% are kept in the stock (i.e at $t = 0$ 1/2 stock as price is 100)
- $t = 1$: Suppose $S_1 = 200$, then $w_1 = \frac{1}{2} \cdot 200 + 50 = 150$. $\frac{\frac{1}{2} \cdot 200}{150} = 2/3$ of wealth in the stock. We must hold $x \cdot 200 = \frac{1}{2} \cdot 150 \Rightarrow x = \frac{75}{200} < 1/2$.

i.e we must adjust the portfolio dynamically.

Martingale method: complete market, intermediate consumption:

$$\max_{\alpha,C} \mathbb{E}\left[\sum_{n=0}^{N} u_n(C_n)\right]$$

Incomplete model: i.e the trinomial-model (see book). Martingale method does not work well in incomplete models because we could can't replicate every claim. (see P and R: 2.2.2 for theoretical fix where we optimize over $\bar{V}_N$'s for which $\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(\bar{V}_N) = v$ for all Martin2gale measures $\mathbb{Q}$. If it holds for all the measures, then the strategy in question can be replicated. In theorey it is thus possible but in pratice not so much).

Theorem 2.32 (Bellman equation (2.78)): For each $n = 0, \ldots, N$, we have

$$\sup_{\eta_n,\ldots,\eta_N} U^{n,v}(\eta_n, \ldots, \eta_N) = W_n(v) \quad (2.77)$$

where $W_n(v)$ is defined recursively by (the Bellman equation 2.78)

$$\begin{aligned} W_N(v) &= \sup_{\xi\in\mathbb{R}} u_N(v, \xi), \\ W_{n-1}(v) &= \sup_{\xi\in\mathbb{R}} \{u_{n-1}(v, \xi) + \mathbb{E}\left[W_n\left(G_n(v, \mu_n; \xi)\right)\right]\} \quad \text{for } n = N, \ldots, 1. \end{aligned}$$

Suppose we are at time $n-1$; we then have to choose the two quantities: Consumption and investment (this is ξ).

The abstract problem is then localized. The optimal value and optimal controls are determined by backwards recursion by a standard maximization procedure.

(see example 2.4.2 for usage of the dynamic programming application opposed to the martingale method last time: remark 2.38)

5 Week 5

Lecture notes: Further remarks on dynamic optimal portfolios from last week: RRA (constant relative risk aversion):

$$\underbrace{RRA}_{\gamma} = -\frac{xu''}{u'}$$

$$\Rightarrow$$

$$u(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}$$

If $\gamma = 1 \sim \ln x$ by L'hopital (however, usually 2-5, reasonable 1-10 (and also 1/2 to achieve a square-root fct.)).

Note for exam: say log case is special :). Exams are on 18-20.

Myopic ("near-sighted"): Portfolio does not depend on investment horizon (i.e fixed fraction).

If $u(x)$ is a utility function and some positive number $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ we can construct $\tilde{u}(x) = \alpha u(x) + \beta$ representing the exact same preferences, i.e indifference between some x_1, x_2 is preserved (also known as *affine invariance*). The optimal values are not directly comparable with other x's unless we are dealing with absolute terms such as: profit, cost, etc.

Dynamic trading gains: The dynamic trading strategy is better than any of the by and hold strategies (Figure 1). We can find the certainty equivalent (when are we indifferent) by solving the equation

$$W_0(v \times (1-f)) = G(v; \hat{\pi}_0)$$

i.e how much of my wealth am I willing to give up to trade dynamically. The solution will be independent on v as the $\ln(v)$-terms cancel out.

Power utility and standard binomial model: p. 18 for solution.

Horizon dependece of investment strategy: "More stock to young investor" but:

- No: We saw that in certain models that there is NO horizon dependence for the optimal strategy (for example binomial had a myopic optimal port. strategy).
- Yes: Human capital (:present value of future wages) could be comparable to a bond (in other words, not so risky). In this sense an education can be seen as human capital and thus more of the financial wealth should be invested in stocks.
- No: Human capital is not like a bond, it is risky but in a different way than stocks, i.e we do not observe the same fluctuations for our salary but if a certain emergency might arise it could be detrimental, so more risky: unemployment for months, lower salary after unemployment, whereas stocks are relatively uncorrelated between periods.

Conclusion: it is not obvious whatsoever if you should have more in risky assets.

6 Week 6

Definition 2.1.1. A random variable ν taking values in $\{0, 1, 2, \ldots, N\}$ is a stopping time if, for any $n \in \{0, 1, 2, \ldots, N\}$,

$$\{\nu = n\} \in \mathcal{F}_n$$

Intuitively, this condition means that the "decision" of whether to stop at time n must be based only on the information present at time n, not on any future information.

or equivalently that ν is a stopping time if and only if

$$\{v \leq n\} \in \mathcal{F}_n$$

Sequence stopped at a stopping time: Let $(X_n)_{0\leq n\leq N}$ be a sequence adapted to the filtration $(\mathcal{F}_n)_{0\leq n\leq N}$ and let ν be a stopping time. The sequence stopped at time ν is defined as

$$X_n^{\nu(\omega)} = X_{\min(\nu(\omega), n(\omega))}$$

i.e on the set $\nu = j$ we have

$$X_n^\nu = \begin{cases} X_j & \text{if } j \leq n \\ X_n & \text{if } j > n. \end{cases}$$

(if we stop at $\nu = j \leq n$ the sequence is stopped so $X_n^\nu = X_j$. If the sequence is not stopped $\nu = j > n$ it holds that $X_n^\nu = X_n$).

onedrive fucker, manglende noter, week forskudt

Lecture notes:

Optimal Stopping (section 2.1-2.3): Setup: Finite filtered probability space $(\Omega, (\mathcal{F}_n)_{0,\ldots,N}, \mathbb{P})$ (however the prob. measure will be the martingale measure $\mathbb{Q}$).

Stopping time: A random variable τ taking random values in $\{0, 1, 2, \ldots, N\}$ is a stopping time if, for any $n \in \{0, 1, 2, \ldots, N\}$,

$$\{\omega \mid \tau = n\} = \{\tau = n\} \in \mathcal{F}_n$$

Intuitively, this condition means that the "decision" of whether to stop at time n must be based only on the information present at time n, not on any future information (or that $\{\tau = n\}$ is $\mathcal{F}_n$-measurable)

Observations about stopping times:

- We could have used $\{\tau \leq n\}\, \mathcal{F}_n$ for all n. Why? $\{\tau \leq n\} = \{\tau = 0\} \cup \{\tau = 1\} \ldots \cup \{\tau = n\}$ i.e every subset is $\mathcal{F}_n$-measurable (for cont. time often).
- "First time that a certain thing happens" is a stopping time. $\{X_n\}$ is an adapted process, $A \in \mathcal{F}$, $\tau = \min\{n \mid X_n \in A\}$ and N if X_n does not enter the set A. Then τ is a stopping time $\{\nu = n\} = \{X_0 \notin A\} \cap \{X_1 \notin A\} \ldots \cap \{X_n \in A\} \in \mathcal{F}_n$.
- $\{\tau \geq n\} \in \mathcal{F}_{n-1}$ i.e it happens at day n or later. If it **haven't** happened at time $n-1$ then it must happen at time n or later if it ever happens. Proof: $\{\tau \geq n\} = (\underbrace{\{\tau \leq n-1\}}_{\mathcal{F}_{n-1}\text{-measureable}})^C \in \mathcal{F}_{n-1}$.

Stopped processes Need adapted process $\{X_n\}$ and a stopping time τ. We can then define the stopped process X^τ via $X_n^\tau(\omega) = X_{\min\{\tau(\omega),n\}}(\omega)$.

Proposition 2.1.4 in L and L ("Stopped martingales are martingales"): X^τ is an adapted process. If X is a (super)martingale, then so is X^τ.

Proof: see L and L. Essentially the sum is telescoping, converging to 1.

Let $\{Z_n\}$ be some adapted process (i.e the payoff) from exercising an american option. Define the **Snell envelope** of Z as:

$$U_n = \begin{cases} U_N = Z_N & n = N \\ U_n = \max\{Z_n, E(U_{n+1} \mid \mathcal{F}_n)\}, & n = 0, \dots, N-1. \end{cases}$$

(U is a supermartingale as it is the maximum of the cond. exp and Z_n, i.e $U \geq Z$ (the smallest dominating supermartingale Z)).

Consider: Define $\tau^* = \min\{n \mid U_n = Z_n\}$ which will happen sooner or later; the quantity is well-defined as a stopping time (ν_0 in the book). From proposition 2.1.5 (*tjek om rigtig*) U^{τ^*} is a martingale (NOT just a supermartingale:).

Proof: See L and L.

Snell-envelope: recap: smallest supermartingale that dominates $\{Z_t\}$. We then defined a stopping time to be the first time the Snell-envelope has Z_n as a maximum: $\tau^* = \min\{n \mid U_n = Z_n\}$. We proved that the stopped Snell-envelope U^{τ^*} is a martingale.

τ^* is called an optimal stopping time. The reason for the name is that τ^* solves the following optimization problem over the set of all stopping times:

$$\max_{\tau \in J_{0,N}} E(Z_\tau), \quad \text{(Optimal stopping problem)}$$

i.e Z is some payoff.

Proof: For any stopping time τ, we know that the stopped Snell-envelope U^τ is a supermartingale, i.e that

$$U_0 = U_0^\tau \geq E(U_N^\tau) = E(U_{\min(\tau,N)}) = E(U_\tau) \geq E(Z_\tau).$$

For τ^*, U^{τ^*} is a martingale and then by the martingale property:

$$U_0 = U_0^{\tau^*} = E(U_N^{\tau^*}) = E(U_{\min(\tau,N)}) = E(U_{\tau^*}) = E(Z_{\tau^*}),$$

hence τ^* solves the maximization problem. In general we can define the Snell-envelope. So: we need to find some stopping time that makes the Snell-envelope a martingale.

Doob decomposition: Any supermartingale U (not the Snell-envelope: but will be used for it) can be written uniquely as

$$U = M - A$$

where

$$M = \text{Martingale} \quad \text{and} \quad A = \text{non-decreasing predictable process.}$$

American options: Setup:

- Complete (unique martingale measure $\mathbb{Q}$) financial multiperiod model
- $\{S_t\}$: Price of some underlying asset ("stock")
- $\{r_t\}$: (locally) risk-free rate

We then look at option with payoff function g (i.e call: $g(x) = (x-K)^+$ or put $g(x) = (K-x)^+$).

European version: Pays $g(S_T)$ to hold at expiry time T. Can not be exercised before expiry.

American version: Pays $g(S_\tau)$ to holder at time τ where τ is a stopping time chosen by the holder.

Common snse/intuition: The arbitrage free time-t price of the American option is:

$$\pi_t^{\mathrm{AMR}} = \max\left(g(S_t), \frac{1}{1+r_t} E_t^{\mathbb{Q}}\left(\pi_{t+1}^{\mathrm{AMR}}\right)\right)$$

or in words: what we get if exercise (intrinsic value) or discounted $\mathbb{Q}$-expected value of the next period (continuation value).

Comments:

- American option price doesn't satisfy local characterization. The price is not always equal to the discounted $\mathbb{Q}$-expected value of the next period (as the max can be $g(S_t)$). not a contingent claim but solely based on the market/nature.
- π_t^{AMR} is the price at time-t given not exercised before.
- Assuming for a moment that the interest rate is $r_t = 0$, then we see that $\left\{\pi_t^{\mathrm{AMR}}\right\}$ is the Snell-envelope of $\{g(S_t)\}$.
- Will prove that:

  $$\pi_t^{\mathrm{AMR}} = \max\left(\underbrace{g(S_t)}_{\text{intrinsic value}}, \underbrace{\frac{1}{1+r_t} E_t^{\mathbb{Q}}\left(\pi_{t+1}^{\mathrm{AMR}}\right)}_{\text{continuation value}}\right),$$

  is the arbitrage free price by showing if larger $\Rightarrow$ arbitrage and if smaller $\Rightarrow$ aribtrage using optimal stopping theory (See Rolf's paper for proof). For $>$: exercise at stopping time τ. We must pay $g(S_\tau) \leq Z_\tau$. Value of the replicating portfolio is by Doob with $Z = M - A$:

  $$M_\tau = Z_\tau + \underbrace{A_\tau}_{\geq 0} \geq Z_\tau \geq g(S_\tau),$$

  i.e something to spare.

7 Week 7

Lecture notes: Recap: we proved:

$$\pi_t^{\mathrm{AMR}} = \max\left(\underbrace{g(S_t)}_{\text{intrinsic value}}, \underbrace{\frac{1}{1+r_t} E_t^{\mathbb{Q}}\left(\pi_{t+1}^{\mathrm{AMR}}\right)}_{\text{continuation value}}\right),$$

i.e what we saw in fin1 except it did not satisfy local characterization (see above comments). eg:

- Put-option, $g(x) = (K - X)^+$, pricing in a binomial model (see and translate R-code to python. Note R-code is made to also handle Bermodan-options ("semi"-fixed exercise))
- **Question at exam**: if $r > 0$ and $\delta = 0$, then I should hold an american call option ($\Rightarrow Call^{\text{AMR}} = Call^{\text{EU}}$). **Proof.** Suppose $t < T$ and $S_t > K$ (i.e in the money). Then

$$Call_t^{\text{AMR}} \geq Call_t^{\text{EU}} \geq Call_t^{\text{EU}} - Call_t^{\text{EU}} \underbrace{=}_{\text{put/call-parity}} S_t - e^{-r(T-t)}K > \underbrace{S_t - K}_{\text{from exercising}},$$

  i.e "worth more alive than dead" (compare first to last inequality).

- Note the price is given for a conditional expectation at time-t. `temp<-...q` in R-code. Suppose we have simulated some stock price paths (i.e handin3) and ask **How not** to price (say) american put option:

$$Put_{t_i}^{AMR} \max\left\{(K - S_{t,i})^+, e^{-r} Put_{t+1,i}^{AMR}\right\}$$

  This is wrong as we should calculate the conditional $\mathbb{Q}$-expected value of the put at time $t+1$. Without $\mathbb{E}_t^{\mathbb{Q}}$ we are not calculating the expected value (i.e looking into the future).

How to: Longstaff & Schwartz: Trick: Regress across paths (unlike above). The regression at time 3, 2 etc.-tables has discounting numbers given from the interest rate being $r = 0.06$ and thus $e^{-r} = e^{-0.06} = 0.9417$. In the table they should have discounted for 2 periods in the time 1 tables but as the vlaue is 0 they haven't.

Regression form:

$$Y = X\beta + \text{noise}, \quad \hat{\beta}^{\text{OLS}} = (X^\top X)^{-1} X^\top Y$$

For optimal early exercise decision at time x-table we exercise if the continuation is larger than the exercise and vice versa.

Technical: Can't exercise at time-0 in paper but we wouldn't anyways.

Reenactments: Optimal stopping time in American options:

Question: What if we bought the self financing port. that replicated the optimally stopped snell envelope?

If counterparty was "slow" at exercising and got lucky we "start over" - so to speak.

Reenactments: Bellman-equation

Question: