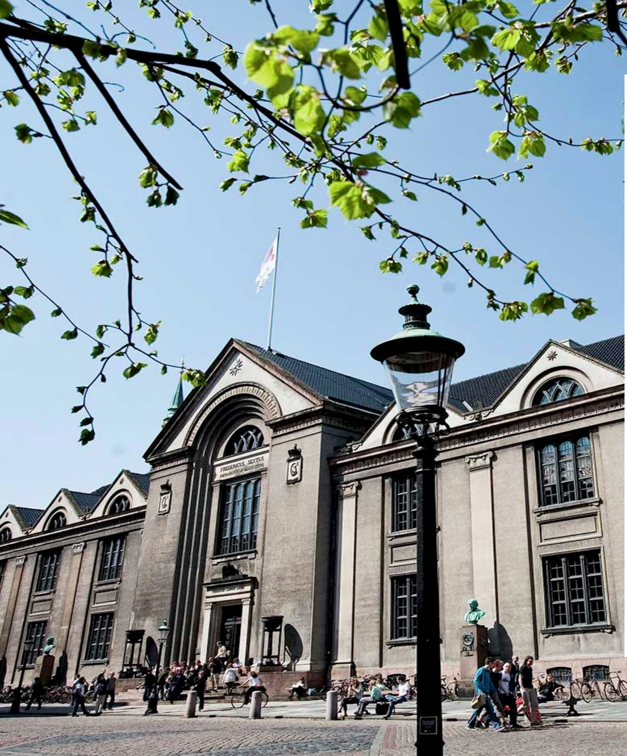

2. The State-Price Utility Theorem

Finance 2: Dynamic Portfolio Choice

Youssef M. Raad
KU-ID: zfw568

Dept. of Mathematical Sciences
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Overview

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   a) Utility Functions
   b) Agent's Problem
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   a) Proof
   b) Implications, Intuition and Usages
3) Extra Slides

Setup

One-period binomial model with N assets, a finite number of future states S and define the following (Lando & Poulsen, 2023, p. 10):

- Price vector: $\pi \in \mathbb{R}^N$.
- Matrix of payoffs for each given asset and each possible future state: $D \in \mathbb{R}^{N \times S}$.
- Portfolio of assets: $\theta \in \mathbb{R}^N$, and where θ_i is the amount invested in asset i.

Characterize the portfolio by:

- The price at time 0: $\pi^\top \theta \in \mathbb{R}$
- The payoff at time 1: $D^\top \theta \in \mathbb{R}^S$

Define the state-price vector by the following

Definition 19 (Lando & Poulsen, 2023)

$\psi \in \mathbb{R}^S_{++}$ (i.e. $\psi \gg 0$) is said to be a state-price vector for the system (π, D) if it satisfies:

$$\pi = D\psi.$$

Utility Functions

Utility function $U : \mathbb{R}^S_+ \to \mathbb{R}$ takes as input $e + D^\top \theta$, where

- $e = (e_1, ..., e_S) \in \mathbb{R}^S$ is the endowment and hence agent's random time-1 wealth.
- $D^\top \theta \in \mathbb{R}^S$ is the time-1 payoff of θ.
- $\Rightarrow$ Utility as a function of wealth - instead of a function of real goods - available in each state.

U is differentiable (i.e. "smooth"), strictly increasing in each coordinate, and concave, where

- Concavity $\sim$ Risk-aversion.

Mild regularity assumption: Existence of portfolio with a non-negative payoff which is strictly positive in at least one state, i.e:

$$\exists \theta_0 \ : \ D^\top \theta_0 > 0.$$

Agents Maximization Problem

A representative agent seeks to maximize utility by choosing a portfolio θ such that (Lando & Poulsen, 2023, p. 87):

$$\max_{\theta} U(\underbrace{e}_{\in \mathbb{R}^S} + \underbrace{D^\top \theta}_{\in \mathbb{R}^S})$$
$$\text{s.t } \underbrace{\pi^\top \theta}_{\in \mathbb{R}} \leq \underbrace{0}_{\text{WLOG}},$$

where $e = (e_1, ..., e_S) \in \mathbb{R}^S_+$ is the the random wealth/(endowment) the agent will have at time 1 and where we assume that $U : \mathbb{R}^S_+ \to \mathbb{R}$ is concave, differentiable and strictly increasing in each coordinate

WLOG: Utility from time-1 consumption only $\Rightarrow$ At time 0 invest all in θ_0 since $D^\top \theta_0 > 0 \Rightarrow$ New time-1 endowment $e' = e + D^\top \theta_0 \Rightarrow$ Does not change optimization problem itself - can always change initial decision $\Rightarrow$ Constraint $\pi^\top \theta \leq 0$ is WLOG.

Proposition (for theorem 3 proof) (Lando & Poulsen, 2023, p. 87):

Proposition 10 (Lando & Poulsen, 2023)

If there exists a portfolio θ_0 with $D^\top \theta_0 > 0$ then the agent can find a solution to the maximization problem if and only if the pair (π, D) is arbitrage-free.

Proof of the State-Price Utility Theorem (1/3) (Lando & Poulsen, 2023, p. 88)

Theorem 3 - The state-price utility theorem (Lando & Poulsen, 2023)

Assume that there exists a portfolio θ_0 with $D^\top \theta_0 > 0$. If there exists a solution θ^* to the maximisation problem and the associated optimal consumption is given by $c^* := e + D^\top \theta^* \gg 0$, then the gradient $\nabla U(c^*)$ is proportional to a state-price vector. The constant of proportionality is positive.

Proof (1): Proportionality: $c^* \gg 0$ so: for any $\theta \in \mathbb{R}^N$ we define:

$$g_\theta(\alpha) = U(c^* + \alpha D^\top \theta), \quad \alpha \in \mathbb{R},$$

where α is chosen such that $c^* + \alpha D^\top \theta \geq 0$.

Consider a zero-cost portfolio: θ with $\pi^\top \theta = 0$. By optimality of c^*: $g_\theta(\alpha)$ has optimum at $\alpha = 0$ i.e $g'_\theta(0) = 0$. The higher-dimensional chain rule yields:

$$\begin{aligned}
g'_\theta(\alpha) &= (\nabla U(c^* + \alpha D^\top \theta))^\top D^\top \theta \\
\Rightarrow g'_\theta(0) &= \underbrace{(\nabla U(c^*))^\top}_{1\times S} \underbrace{D^\top}_{S\times N} \underbrace{\theta}_{N\times 1} = 0, && (\alpha = 0) \\
\iff \theta^\top (D \nabla U(c^*)) &= 0. && \text{(transposing)}
\end{aligned}$$

Proof of the State-Price Utility Theorem (2/3) (Lando & Poulsen, 2023, p. 88)

Any θ with zero cost: $\pi^\top \theta = 0$ satisfies $\theta^\top D\nabla U(c^*) = 0 \Rightarrow$ Any portfolio that is orthogonal to π is also orthogonal to $D\nabla U(c^*) \Rightarrow \pi$ and $D\nabla U(c^*)$ are proportional ($\star$), i.e:

$$\exists \mu \in \mathbb{R}: \quad \mu\pi = D\nabla U(c^*)$$

Using definition 19 we see that $\pi = D\psi$ with $\psi = \frac{1}{\mu}\nabla U(c^*)$, showing that the gradient $\nabla U(c^*)$ is proportional to a state-price vector.

($\star$): For the purpose of contradiction: If not proportional, then there must exist some $\xi \in \mathbb{R}^N$ orthogonal to π and $\mu\pi + \xi = D\nabla U(c^*)$ (intuitively: ξ essentially captures the component of $D\nabla U$ that is orthogonal to π). But:

$$\underbrace{\xi^T D\nabla U(c^*)}_{=0} = \mu \underbrace{\xi^T \pi}_{=0} + \underbrace{\xi^T \xi}_{\neq 0} \neq 0$$

$\Rightarrow$ Contradiction, and so π and $D\nabla u(c^*)$ must be proportional.

Proof of the State-Price Utility Theorem (3/3) (Lando & Poulsen, 2023, p. 88)

Proof (2): Positivity: From the (mild regulatory) assumption, $D^\top \theta_0 > 0$ and from proposition 10 that $\pi^\top \theta_0 > 0$ or else arbitrage!

Since $c^* \gg 0$ and U is strictly increasing:

$$(\nabla U(c^*))^\top D^\top \theta_0 > 0,$$

just saw that:

$$\mu \underbrace{\pi^\top \theta_0}_{>0} = \underbrace{(\nabla U(c^*))^\top D^\top \theta_0}_{>0},$$

thus $\mu > 0$ follows.

Implications, Intuition and Usages (1/2)

(Extra Slide 4 for Excel)

- The state prices are proportional to marginal utility and actual probabilities, i.e. for every possible state, the state-price vector is a product of the probability of the state times the marginal utility in that state.
- Bad states (low consumption) have a high marginal utility (marginal utility is decreasing) $\Rightarrow$ High state prices, ψ_j for $j \in 1, \ldots, S$ (as $\psi = \frac{1}{\mu} U(c^*)$).
- Interpretation: Willingness to pay a high price for insurance that pays off in bad states, i.e. willing to pay a high price to avoid bad things (Concave utility function $\Rightarrow$ positive but decreasing marginal utility in c (i.e risk averse agent)).

Usages

Consider a utility function with an expected utility representation: Given a set of probabilities $(p_1, \ldots, p_s)$ and a function u:

$$U(c) = \sum_{j=1}^{S} p_j u(c_j).$$

We see from the state-price utility theorem that the state-price vector ψ solves $D\psi = \pi$ with coordinates (for some constant of proportionality $\mu \in \mathbb{R}$):

$$\psi_j = \mu p_j u'(c_j^*), \quad j \in 1, \ldots, S.$$

Complete: State-price vector is unique and thus (u is smooth and concave $\Rightarrow$ u' continuous and decreasing $\Rightarrow$ u' has inverse):

$$\psi_j = \mu p_j u'(c_j^*) \iff c_j^* = (u')^{-1}\left(\frac{\psi_j}{\mu p_j}\right), \quad j \in (1, \ldots, S).$$

Implications, Intuition and Usages (2/2)

In short: Explicit solution to a high-dimensional maximization problem in a one-dimensional way *Pliska's martingale method*.

Incomplete: (many ψ's) introduce a new asset x with no unique arbitrage-free price. The state-price utility theorem can now provide reasonable risk-neutral probabilities.

Let ψ_0^* be the state price induced by $\nabla U(c^*)$ associated with the optimal choice of portfolio for a (pressumably representative) certain agent. Then π_x^* is the (only!) arbitrage-free price of the new asset, where

$$\pi_x^* = \sum_j x_j \psi_0^*,$$

and where the representative agent demands 0 units of this new asset! This principle is known as **Zero-level Pricing**.

We can thus use the state-prices determined from a presumably representative agent's optimum to price new assets in incomplete models. This is possible due to the relation between the state prices and the risk-neutral probabilities.

Same rationale that makes CAPM a pricing model $\rightarrow$ we are pricing everything in relation to the given market portfolio/zero-demand level (Extra slide 5). This gives a single price instead of the typically quite wide no-arbitrage bounds produced via sub/super-replication and linear programming.

(Can be proved by introducing the new asset to the utility maximization problem and see the optimal portfolio in this extended model consists only of the "old" assets.)

References

Lando, D., & Poulsen, R. (2023). Finance 1 and beyond. *Department of Mathematical Sciences, University of Copenhagen.*

Extra Slide 1: WLOG? (1/2)

1. **Initial Setup**:
   - Initial wealth at time 0: x
   - Endowment at time 1: e
   - Prices of securities: π
   - Payoffs of securities: D
2. **Purchasing Initial Portfolio**:
   - Agent spends all initial wealth x to buy portfolio θ_0 where $D^\top \theta_0 > 0$ by mild regulatory assumption:

$$\pi^\top \theta_0 = x$$

3. **Redefining the Endowment**:
   - The new endowment at time 1 becomes:

$$e' = e + D^\top \theta_0$$

4. **New Optimization Problem**:
   - The agent's problem is now to maximize utility with the new endowment e':

$$\max_{\theta} \quad U(e' + D^\top \theta)$$
$$\text{subject to} \quad \pi^\top \theta \leq 0$$

Extra Slide 2: WLOG? (2/2)

Why is $<$ irrelevant?

If now we assume that there exists a security with a non-negative payoff which is strictly positive in at least one state (we just did), then, as U is increasing, we can set $\pi^\top\theta = 0$. The interpretation of this is that the agent sells endowment in some states to obtain more in other states. But no cash changes hands at time zero.

If constraint is $<$, then the feasible region is open (i.e does not include its boundary) $\Rightarrow$ No optimal solution. Can always find a better solution within the feasible region $\rightarrow$ We are interested in optimality (maximizing) and not just feasibility, hence it is irrelevant.

What if we want to include time 0-consumption also?

Extend U with a 0th coordinate of the form $u_0(e_0 - D^\top\theta)$ to allow for consumption at time 0. By the first question (prievous slide) the constraint:

$$\pi^\top\theta \leq e_0 - c_0,$$

can still be written WLOG as:

$$\pi^\top\theta' \leq 0$$

where c_0 is time 0 consumption.

Extra Slide 3: Existence, Uniqueness and Arbitrages

Theorem 1 (Lando & Poulsen, 2023)

The security market (π, C) is arbitrage-free if and only if there exists a strictly positive vector $d \in \mathbb{R}^T_{++}$ such that $\pi = Cd$

Theorem 2 (Lando & Poulsen, 2023)

Assume that (π, C) is arbitrage-free. Then the market is complete if and only if there is a unique vector of discount factors.

Define arbitrage as a portfolio θ (with time-0 portfolio price $\pi \cdot \theta$, time-1payment stream $D^\top \theta$) that satisfies:

- Strong arbitrage ("Printing Money": type 2): $\pi^\top \theta < 0, \quad D^\top \theta \geq 0.$
- Weak arbitrage ("Free lottery tickets": type 1): $\pi^\top \theta = 0, \quad D^\top \theta > 0.$
- Semi-Strong arbitrage ("All the lottery tickets for free"): $\pi^\top \theta = 0, \quad D^\top \theta >> 0.$
- Risk-free arbitrage: $(D^\top \theta)_j = c > 0$ for a non-positive price.

Extra Slide 4: Excel Example

Go to Github for Excel-file: orange fields = our choice.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	**Portfolio choice example**														
2															
3	Time 0-wealth, W(0)	200			**Investments**			**Utility**							
4	Initial stock price, S(0)	100			Fraction in stock	**0,7437556**		13,1853	<- E(u(C(1)))						
5	Interest rate	0,03			Fraction in bank	**0,2562444**		u(x) = (x^(1-RRA)-1)/(1-RRA)							
6	E^P(rate of return on stock)	-0,1625						RRA ~ relative risk aversion				0,7		0,3	
7	SD(aktie-afkastrate)	0,29182													
8	Risk-neutral up-prob'	0,4500													
9															
10	Time 1; the future				Consumption, C(1) (aka. time 1-wealth W(1))										
11	State number	p prob'	S(1)		c_i	u(c_i)	p_i*u(c_i)	p_i*u'(c_i)	p_i*u'(c_I)/SUM_j(p_j*u'(c_j))						
12	1	0,5	125		238,7252	13,8952	6,9476	0,0108	**0,4500**	<- Matches cell B8 @ optimum					
13	2	0,5	85		179,2248	12,4754	6,2377	0,0132	**0,5500**						
14															

- State-price utility theorem in action: The normalized, probability weighted marginal utilities, at the optimum matches the risk-neutral probabilities.
- A state with low consumption ("bad state") has a high marginal utility and thus high state prices, ψ_J.
- Oppositely, the state price associated with a high consumption ("good state") is lower.
- Risk-aversion: The more risk-averse the agent is, the less he invests in the stock.
- If we put in any short-sale or borrowing constraints, the state-price utility theorem no longer applies (if constraints bind).
- Good divergence: For $\gamma = 0$ the Solver diverges; an unrestricted risk-neutral agent will invest all in the risky asset with highest expected rate of return. Proposition 10 says that this should happen because this model has arbitrage.

Extra Slide 5: Relation to CAPM

The Capital Asset Pricing Model (CAPM) is an equilibrium argument explaining asset prices. It provides us the ability to explain expected rates of return on all assets or portfolio from the expected rate of return on the market portfolio and the covariance with the market portfolio.

$$\mathbb{E}[r_i] - r_f = \underbrace{\frac{\text{Cov}[r_i, r_m]}{\text{Var}[r_m]}}_{\beta_{i,m}} \Big(\mathbb{E}[r_m] - r_f\Big).$$

Agents receive excess returns from risk correlated with the market - cannot expect to be rewarded by buying risky assets uncorrelated with the market (unsystematic risk).

Interpretation:

- An asset moving with the market pays off a lot when the economy is wealthy (i.e. high market return, "good times"). This asset contributes wealth in states where the marginal utility of receiving extra wealth is small. This means that agents are not willing to pay very much for such an asset at time 0, as to why the asset has a high return.
- As asset moving opposite the market pays off a lot when the economy is not wealthy (i.e. low market return, "bad times"). This asset contributes wealth in states where the marginal utility of receiving extra wealth is high. This means that agents are indeed willing to pay a lot for such an asset at time 0, as to why the asset has a low return.

Altogether, CAPM provides some sort of pricing result $\rightarrow$ we are pricing everything in relation to the given market portfolio.