D. DUFFIE (1992) "DYNAMIC ASSET PRICING THEORY", 1ST ED, PRINCETON

NICE "PEDESTRIAN" PROOF OF THE BELLMAN EQN. IN SECTION 3E (p47-48)

AND IT HAS EXERCISES.

3 The Dynamic Programming Approach

THIS CHAPTER PRESENTS portfolio choice and asset pricing in the framework of dynamic programming, a technique for solving dynamic optimization problems with a recursive structure. The asset-pricing implications go little beyond those of the previous chapter, but there are computational advantages. After introducing the idea of dynamic programming in a deterministic setting, we review the basics of a finite-state Markov chain. The Bellman equation is shown to characterize optimality in a Markov setting. The first order condition for the Bellman equation, often called the "stochastic Euler equation," is then shown to characterize equilibrium security prices. This is done with additive utility in the main body of the chapter, and extended to more general recursive forms of utility in the exercises. The last sections of the chapter show the computation of arbitrage-free derivative security values in a Markov setting, including an application of Bellman's equation for optimal stopping to the valuation of American securities such as the American put option. An exercise presents algorithms for the numerical solution of term-structure derivative securities in a simple binomial setting.

A. The Bellman Approach

To get the basic idea, we start in the T-period setting of the previous chapter, with no securities except those permitting short-term riskless borrowing at any time t at the discount $d_t > 0$. The endowment process of a given agent is e. Given a consumption process c, it is convenient to define the agent's *wealth process* W^c by $W_0^c = 0$ and

$$W_t^c = \frac{W_{t-1}^c + e_{t-1} - c_{t-1}}{d_t}, \quad t \geq 1. \tag{1}$$

Given a utility function $U : L_+ \to \mathbb{R}$ on the set L of non-negative adapted processes, the agent's problem can be rewritten as

$$\sup_c U(c) \quad \text{subject to (1) and } c_T \leq W_T^c + e_T. \tag{2}$$

Dynamic programming is only convenient with special types of utility functions. One example is an additive utility function U, defined by

$$U(c) = E\left[\sum_{t=0}^{T} u_t(c_t)\right], \tag{3}$$

with $u_t : \mathbb{R}_+ \to \mathbb{R}$ strictly increasing and continuous for each t. Given this utility function, it is natural to consider the problem at any time t of maximizing the "remaining utility," given current wealth $W_t^c = w$. In order to keep things simple at first, we take the case in which there is no uncertainty, meaning that $\mathcal{F}_t = \{\Omega, \emptyset\}$ for all t. The maximum remaining utility at time t is then written, for each w in $\mathbb{R}$, as

$$V_t(w) = \sup_{c \in L_+} \sum_{s=t}^{T} u_s(c_s),$$

subject to $W_t^c = w$, the wealth dynamic (1), and $c_T \leq W_T^c + e_T$. If there is no budget-feasible consumption choice (because w is excessively negative), we write $V_t(w) = -\infty$.

Clearly $V_T(w) = u_T(w + e_T)$, $w \geq -e_T$, and it is shown as an exercise that, for $t < T$,

$$V_t(w) = \sup_{\bar{c} \in \mathbb{R}_+} u_t(\bar{c}) + V_{t+1}\left(\frac{w + e_t - \bar{c}}{d_t}\right), \tag{4}$$

the *Bellman equation.* It is also left as an exercise to show that an optimal consumption policy c is defined inductively by $c_t = C_t(W_t^c)$, where $C_t(w)$ denotes a solution to (4) for $t < T$, and where $C_T(w) = w + e_T$. From (4), the *value function* V_{t+1} thus summarizes all information regarding the "future" of the problem that is required for choice at time t.

B. First Order Conditions of the Bellman Equation

Throughout this section, we take the additive model (3) and assume in addition that, for each t, u_t is strictly concave and differentiable on $(0, \infty)$. Extending Exercise 2.2, there exists an optimal consumption policy c^*. We assume that c^* is strictly positive. Let W^* denote the wealth process associated with c^* by (1).

Lemma. *For any t, V_t is strictly concave and continuously differentiable at W_t^*, with $V_t'(W_t^*) = u_t'(c_t^*)$.*

Proof is left as Exercise 3.3, which gives a broad hint. The first order conditions for the Bellman equation (4) then imply, for any $t < T$, that the one-period discount is

$$d_t = \frac{u_{t+1}'(c_{t+1}^*)}{u_t'(c_t^*)}. \tag{5}$$

The same equation is easily derived from the general characterization of equilibrium security prices given by equation (2.9). More generally, the price $\Lambda_{t,\tau}$ at time t of a unit riskless bond maturing at any time $\tau > t$ is

$$\Lambda_{t,\tau} \equiv d_t\, d_{t+1} \cdots d_{\tau-1} = \frac{u_\tau'(c_\tau^*)}{u_t'(c_t^*)}, \tag{6}$$

which, naturally, is the marginal rate of substitution of consumption between the two dates.

Since the price of a coupon-bearing bond, the only kind of security in a deterministic setting, is merely the sum of the prices of its coupons and principal, (6) provides a complete characterization of security prices in this setting.

C. Markov Uncertainty

We start with the easiest kind of Markov uncertainty, a *time-homogeneous Markov chain.* Let the elements of a fixed set $Z = \{1, \ldots, k\}$ be known as *shocks.* For any shocks i and j, let $q_{ij} \in [0, 1]$ be thought of as the probability, for any t, that shock j occurs in period $t+1$ given that shock i occurs in period t. Of course, for each i, $q_{i1} + \cdots + q_{ik} = 1$. The $k \times k$ *transition matrix* q is thus a complete characterization of transition probabilities. This idea is formalized with the following construction of a probability space and filtration of tribes. It is enough to consider a state of the world as some particular sequence $(z_0, \ldots, z_T)$ of shocks that might occur. We therefore let $\Omega = Z^{T+1}$ and let $\mathcal{F}$ be the set of all subsets of Ω. For each t, let $X_t : \Omega \to Z$ (the random shock at time t) be the random variable defined by $X_t(z_0, \ldots, z_T) = z_t$. Finally, for each i in Z, let P_i be the probability measure on $(\Omega, \mathcal{F})$ uniquely defined by two conditions:

$$P_i(X_0 = i) = 1 \tag{7}$$

and, for all $t < T$,

$$P_i[X(t+1) = j \mid X(0),\, X(1),\, X(2), \ldots, X(t)] = q_{X(t),j}. \tag{8}$$

Relations (7) and (8) mean that, under probability measure P_i, X starts at i with probability 1 and has the transition probabilities previously described informally. In particular, (8) means that $X = \{X_0, \ldots, X_T\}$ is a Markov process: the conditional distribution of X_{t+1} given $X_0, \ldots, X_t$ depends only on X_t. To complete the formal picture, for each t, we let $\mathcal{F}_t$ be the tribe generated by $\{X_0, \ldots, X_t\}$, meaning that the information available at time t is that obtained by observing the shock process X until time t.

Lemma. *For any time t, let $f : Z^{T-t+1} \to \mathbb{R}$ be arbitrary. Then there exists a fixed function $g : Z \to \mathbb{R}$ such that, for any i in Z,*

$$E^i[f(X_t, \ldots, X_T) \mid \mathcal{F}_t] = E^i[f(X_t, \ldots, X_T) \mid X_t] = g(X_t),$$

where E^i denotes expectation under P_i.

This lemma gives the complete flavor of the Markov property.

D. Markov Asset Pricing

Taking the particular Markov source of uncertainty described in Section 3C, we now consider the prices of securities in a single- or representative-agent setting with additive utility of the form (3), where, for all t, u_t has a strictly positive derivative on $(0, \infty)$. Suppose, moreover, that for each t there are functions $f_t : Z \to \mathbb{R}^N$ and $g_t : Z \to \mathbb{R}$ such that the dividend is $\delta_t = f_t(X_t)$ and the endowment is $e_t = g_t(X_t)$. Then Lemma 3C and the general gradient solution (2.9) for equilibrium security prices imply the following characterization of the equilibrium security price process S. For each t there is a function $\mathcal{S}_t : Z \to \mathbb{R}^N$ such that $S_t = \mathcal{S}_t(X_t)$. In particular, for any initial shock i and any time $t < T$,

$$\mathcal{S}_t(X_t) = \frac{1}{\pi_t} E^i \left(\pi_{t+1} \left[f_{t+1}(X_{t+1}) + \mathcal{S}_{t+1}(X_{t+1}) \right] \,\middle|\, X_t \right), \tag{9}$$

where π is the state-price deflator given by $\pi_t = u_t'[g_t(X_t)]$. This has been called the *stochastic Euler equation* for security prices.

E. Security Pricing by Markov Control

We will demonstrate (9) once again, under stronger conditions, using instead Markov control methods. Suppose that $\{X_t\}$ is the shock process

already described. For notational simplicity, we take it that the transition matrix q is strictly positive and that, for all t,

- u_t is continuous, strictly concave, increasing, and differentiable on $(0, \infty)$;
- $e_t = g_t(X_t)$ for some $g_t : Z \to \mathbb{R}_{++}$; and
- $\delta_t = f_t(X_t)$ for some $f_t : Z \to \mathbb{R}^N_{++}$.

We assume, naturally, that $\mathcal{S}_t : Z \to \mathbb{R}^N_{++}$, $t < T$, and that there is no arbitrage. For each $t \le T$, consider the value function $V_t : Z \times \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$V_t(i, w) = \sup_{(c,\theta) \in L_+ \times \Theta} E \left[\sum_{j=t}^{T} u_j(c_j) \,\middle|\, X_t = i \right], \tag{10}$$

subject to

$$W_j^\theta = \theta_{j-1} \cdot \left[\mathcal{S}_j(X_j) + f(X_j) \right], \quad j > t; \quad W_t^\theta = w, \tag{11}$$

and

$$c_j + \theta_j \cdot \mathcal{S}_j(X_j) \le W_j^\theta + g(X_j), \quad t \le j \le T.$$

The conditional expectation in (10), which is well defined since $q \gg 0$, does not depend on the initial state X_0 according to Lemma 3C, so we abuse the notation by simply ignoring the initial state in this sort of expression. For sufficiently negative w, there is no solution, in which case we take $V_t(i, w) = -\infty$. For initial wealth $w = 0$ and time $t = 0$, (10) is equivalent to problem (2.4) with $S_j = \mathcal{S}_j(X_j)$ for all j.

We now define a sequence $F_0, \ldots, F_T$ of functions on $Z \times \mathbb{R}$ into $\overline{\mathbb{R}}$ that will eventually be shown to coincide with the value functions $V_0, \ldots, V_T$. We first define $F_{T+1} \equiv 0$. For $t \le T$, we let F_t be given by the Bellman equation

$$F_t(i, w) = \sup_{(\overline{\theta}, \overline{c}) \in \mathbb{R}^N \times \mathbb{R}_+} G_{it}(\overline{\theta}, \overline{c}) \quad \text{subject to } \overline{c} + \overline{\theta} \cdot \mathcal{S}_t(i) \le w + g(i), \tag{12}$$

where

$$G_{it}(\overline{\theta}, \overline{c}) = u_t(\overline{c}) + E \left[F_{t+1}\left(X_{t+1}, \overline{\theta} \cdot \left[\mathcal{S}_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1}) \right] \right) \,\middle|\, X_t = i \right].$$

The following technical conditions extend those of Lemma 3B, and have essentially the same proof.

Proposition. *For any i in Z and $t \leq T$, the function $F_t(i, \cdot) : \mathbb{R} \to \overline{\mathbb{R}}$, restricted to its domain of finiteness $\{w : F_t(i, w) > -\infty\}$, is strictly concave and increasing. If $(\overline{c}, \overline{\theta})$ solves (12) and $\overline{c} > 0$, then $F_t(i, \cdot)$ is continuously differentiable at w with derivative $F_{tw}(i, w) = u_t'(\overline{c})$.*

It can be shown as an exercise that, unless the constraint is infeasible, a solution to (12) always exists. In this case, for any i, t, and w, let $[\Phi_t(i, w), C_t(i, w)]$ denote a solution. We can then define the associated wealth process W^* recursively by $W_0^* = 0$ and

$$W_t^* = \Phi_{t-1}(X_{t-1}, W_{t-1}^*) \cdot [S_t(X_t) + f_t(X_t)], \quad t \geq 1.$$

Let (c^*, θ^*) be defined, at each t, by $c_t^* = C_t(X_t, W_t^*)$ and $\theta_t^* = \Phi_t(X_t, W_t^*)$. The fact that (c^*, θ^*) solves (10) for $t = 0$ and $w = 0$ can be shown as follows. Let (c, θ) be an arbitrary feasible policy. We have, for each t from the Bellman equation (12),

$$F_t(X_t, W_t^\theta) \geq u_t(c_t) + E\left[F_{t+1}\left(X_{t+1}, \theta_t \cdot [S_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})]\right) \,\middle|\, X_t\right].$$

Rearranging this inequality and applying the law of iterated expectations,

$$E[F_t(X_t, W_t^\theta)] - E\left[F_{t+1}\left(X_{t+1}, W_{t+1}^\theta\right)\right] \geq E[u_t(c_t)]. \tag{13}$$

Adding equation (13) from $t = 0$ to $t = T$ shows that $F_0(X_0, W_0) \geq U(c)$. Repeating the same calculations for the special policy $(c, \theta) = (c^*, \theta^*)$ allows us to replace the inequality in (13) with an equality, leaving $F_0(X_0, W_0) = U(c^*)$. This shows that $U(c^*) \geq U(c)$ for any feasible (θ, c), meaning that (θ^*, c^*) indeed solves equation (10) for $t = 0$. An optimal policy can thus be captured in *feedback* form in terms of the functions C_t and Φ_t, $t \leq T$. We also see that, for all $t \leq T$, $F_t = V_t$, so V_t inherits the properties of F given by the last proposition.

We can now recover the stochastic Euler equation (9) directly from the first order conditions to (12), rather than from the more general first order conditions developed in Chapter 2 based on the gradient of U.

Theorem. *A feasible policy (c^*, θ^*) with c^* strictly positive solves (10) for $t = 0$ and $w = 0$ if and only if, for all $t < T$,*

$$S_t(X_t) = \frac{1}{u_t'(c_t^*)} E\left[u_{t+1}'\left(c_{t+1}^*\right)[S_{t+1}(X_{t+1}) + f_{t+1}(X_{t+1})] \,|\, X_t\right]. \tag{14}$$

The theorem follows from the necessity and sufficiency of the first order conditions for (12), relying on the last proposition for the fact that $F_{t+1,w}(X_{t+1}, W_{t+1}^*) = u_{t+1}'(c_{t+1}^*)$.

In a single-agent model, we define a sequence $\{S_0, \ldots, S_T\}$ of security-price functions to be a *single-agent equilibrium* if $(e, 0)$ (no trade) solves (10) for $t = 0$, $w = 0$, and any initial shock i.

Corollary. *$\{S_0, \ldots, S_T\}$ is a single-agent equilibrium if and only if $S_T = 0$ and, for all $t < T$, the stochastic Euler equation (9) is satisfied.*

F. Arbitrage-Free Valuation in a Markov Setting

Taking the setting of Markov uncertainty described in Section 3C, but assuming no particular optimality properties or equilibrium, suppose that security prices and dividends are given, at each t, by functions S_t and f_t on Z into $\mathbb{R}^N$. It can be shown as an exercise that the absence of arbitrage is equivalent to the existence of a state-price deflator π given by $\pi_t = \psi_t(X_t)$ for some $\psi_t : Z \to \mathbb{R}$. With this, we have, for $0 < t \leq T$,

$$S_{t-1}(X_{t-1}) = \frac{1}{\psi_{t-1}(X_{t-1})} E\left(\psi_t(X_t)\,[f_t(X_t) + S_t(X_t)] \,|\, X_{t-1}\right). \tag{15}$$

In the special setting of Section 3D, for example, (9) tells us that we can take $\psi_t(i) = u_t'[g(i)]$.

Since $Z = \{1, \ldots, k\}$ for some integer k, we can abuse the notation by treating any function such as $S_t : Z \to \mathbb{R}$ interchangeably as a vector in $\mathbb{R}^k$ denoted S_t, with i-th element $S_t(i)$. In this sense, (15) can also be written

$$S_{t-1} = \Pi_{t-1}(f_t + S_t), \tag{16}$$

where Π_{t-1} is the $k \times k$ matrix with (i,j)-element $q_{ij}\psi_t(j)/\psi_{t-1}(i)$. For each t and $s > t$, we let $\Pi_{t,s} = \Pi_t \Pi_{t+1} \cdots \Pi_{s-1}$. Then (16) is equivalent to: For any t and $\tau > t$,

$$S_t = \Pi_{t,\tau} S_\tau + \sum_{s=t+1}^{\tau} \Pi_{t,s} f_s. \tag{17}$$

As an example, consider the "binomial" model of Exercise 2.1. We can let $Z = \{0, 1, \ldots, T\}$, with shock i having the interpretation: "There have so far occurred i 'up' returns on the stock." From the calculations in Exercise 2.1, it is apparent that, for any t, we may choose $\Pi_t = \Pi$, where

$$\begin{aligned}\Pi_{ij} &= \frac{p}{R}, \quad j = i+1,\\ &= \frac{1-p}{R}, \quad j = i,\\ &= 0, \quad \text{otherwise,}\end{aligned}$$

where $p = (R-D)/(U-D)$. For a given initial stock price x and any $i \in Z$, the stock-price process S of Exercise 2.1 can indeed be represented at each time t by $S_t : Z \to \mathbb{R}$, where $S_t(i) = xU^iD^{t-i}$.

We can recover the "binomial" option-pricing formula (2.16) by noting that the European call option with strike price K and expiration time τ may be treated as a security with dividends only at time τ given by the function $g : Z \to \mathbb{R}$, with $g(i) = [S_\tau(i) - K]^+$. From (17), the arbitrage-free value of the option at time t is $C_t^\tau = \Pi^{\tau-t}g$, where Π^t denotes the t-th power of Π. This same valuation formula applies to an arbitrary security paying a dividend at time τ defined by some payoff function $g : Z \to \mathbb{R}$.

G. Early Exercise and Optimal Stopping

In the setting of Section 3F, consider an "American" security, defined by some payoff functions $g_t : Z \to \mathbb{R}$, $t \in \{0, \dots, T\}$. As explained in Section 2.I, the security is a claim to the dividend $g_\tau(X_\tau)$ at any stopping time τ selected by the owner. Expiration of the security at some time $\overline{\tau}$ is handled by defining g_t to be zero for $t > \overline{\tau}$. Given the state-price deflator π defined by $\pi_t = \psi(X_t)$, as outlined in the previous section, the rational exercise problem (2.13) for the American security, with initial shock i, is given by

$$J_0(i) \equiv \sup_{\tau \in \mathcal{T}} \frac{1}{\psi_0(i)} E^i\left[\psi_\tau(X_\tau)g(X_\tau)\right], \tag{18}$$

where $\mathcal{T}$ is the set of stopping times. As explained in Section 2.I, if the American security is redundant and there is no arbitrage, then $J_0(i)$ is its cum-dividend value at time 0 with initial shock i. Provided the transition matrix q is strictly positive, the Bellman equation for (18) is

$$J_t(i) \equiv \max\left(g_t(i),\ \frac{1}{\psi_t(i)} E\left[\psi_{t+1}(X_{t+1})J_{t+1}(X_{t+1}) \middle| X_t = i\right]\right). \tag{19}$$

If q is not strictly positive, a slightly more complicated expression applies. It is left as an exercise to show that J_0 is indeed determined inductively,

backward in time from T, by (19) and $J_T = 0$. Moreover, (18) is solved by the stopping time

$$\tau^* = \min\left[t : J_t(X_t) = g_t(X_t)\right]. \tag{20}$$

In our alternate notation that treats J_t as a vector in $\mathbb{R}^k$, we can rewrite the Bellman equation (19) in the form

$$J_t = \max\left(g_t, \Pi_t J_{t+1}\right), \tag{21}$$

where, for any x and y in $\mathbb{R}^k$, $\max(x, y)$ denotes the vector in $\mathbb{R}^k$ that has $\max(x_i, y_i)$ as its i-th element. This form (21) of the Bellman equation applies even if q is not strictly positive.

Equation (21) leads to a simple recursive solution algorithm for the American put valuation problem of Exercise 2.1. Given an expiration time $\overline{\tau} < T$ and exercise price K, we have $J_T = 0$ and

$$J_t = \max\left[(K - S_t)^+,\ \Pi J_{t+1}\right], \tag{22}$$

or more explicitly: For any t and $i \leq t$,

$$J_t(i) = \max\left([K - S_t(i)]^+,\ \frac{pJ_{t+1}(i+1) + (1-p)J_{t+1}(i)}{R}\right), \tag{23}$$

where $S_t(i) = xU^iD^{t-1}$ and $p = (R-D)/(U-D)$.

More generally, consider an *American security* defined by dividend functions $h_0, \dots, h_T$ and exercise payoff functions $g_0, \dots, g_T$. For a given expiration time $\overline{\tau}$, we have $h_t = g_t = 0$, $t > \overline{\tau}$. The owner of the security chooses a stopping time τ at which to exercise, generating the dividend process δ^τ defined by

$$\begin{aligned}\delta_t^\tau &= h_t(x_t), \quad t < \tau,\\ &= g_t(X_t), \quad t = \tau,\\ &= 0, \quad t > \tau.\end{aligned}$$

Assuming that δ^τ is redundant for any exercise policy τ, the security's arbitrage-free cum-dividend value is defined recursively by $J_T = 0$ and the extension of (21):

$$J_t = \max\left(g_t,\ h_t + \Pi_t J_{t+1}\right). \tag{24}$$

Exercises

3.1 Prove the Bellman equation (4).

3.2 For each t and each w such that there exists a feasible policy, let $C_t(w)$ solve equation (4). Let W^* be determined by equation (1) with $c_{t-1} = C_{t-1}(W^*_{t-1})$ for $t > 0$. Show that an optimal policy c^* is given by $c^*_t = C_t(W^*_t)$, $t < T$, and $c^*_T = e_T + W^*_T$.

3.3 Prove Lemma 3B. Hint: If $f : \mathbb{R} \to \mathbb{R}$ is concave, then for each x there is a number β such that $\beta(x - y) \geq f(x) - f(y)$ for all x and y. If f is also differentiable at x, then $\beta = f'(x)$. If f is differentiable and strictly concave, then f is continuously differentiable. Let $w^* = W^*_t$. If $c^*_t > 0$, there is an interval $I = (\underline{w}, \overline{w}) \subset \mathbb{R}$ with $w^* \in I$ such that $v : I \to \mathbb{R}$ is well defined by

$$v(w) = u_t(c^*_t + w - w^*) + V_{t+1}(W^*_{t+1}).$$

Now use the differentiability of v, the definition of a derivative, and the fact that $v(w) \leq V_t(w)$ for all $w \in I$.

3.4 Prove equation (9).

3.5 Prove Proposition 3E.

3.6 Prove Theorem 3E and its Corollary.

3.7 Consider the case of securities in positive supply, which can be taken without loss of generality to be a supply of 1 each. Equilibrium in the context of Section 3E is thus redefined by: $\{\mathcal{S}_0, \dots, \mathcal{S}_T\}$ is an equilibrium if (c^*, θ^*) solves (10) at $t = 0$ and $w = \mathbf{1} \cdot [\mathcal{S}_0(X_0) + f_0(X_0)]$, where $\mathbf{1} = (1, \dots, 1)$ and, for all t, $\theta^*_t = \mathbf{1}$, and $c^*_t = g_t(X_t) + \mathbf{1} \cdot f_t(X_t)$. Demonstrate a new version of the stochastic Euler equation (9) that characterizes equilibrium in this case.

3.8 (Recursive Utility Revisited) The objective in this exercise is to extend the basic results of the chapter to the case of a recursive-utility function that generalizes additive utility. Rather than assuming a typical additive-utility function U of the form

$$U(c) = E\left[\sum_{t=0}^{T} \rho^t u(c_t)\right], \tag{25}$$

we adopt instead the more general recursive definition of utility given by $U(c) = Y_0$, where Y is a process defined by $Y_{T+1} = 0$ and, for any $t \leq T$,

$$Y_t = J\left(c_t, E_t[h(Y_{t+1})]\right), \tag{26}$$

where $J : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. This is the special case treated in Exercise 2.9 of what is known as recursive utility. (In an even more general recursive-utility model, the von Neumann-Morgenstern criterion $E[h(\cdot)]$ is replaced by a general functional on distributions, but we do not deal with this further generalization.) Note that the special case $J(q, w) = u(q) + \rho w$ and $h(y) = y$ gives us the additively separable criterion (25). The conventional additive utility has the disadvantage that the elasticity of intertemporal substitution (as measured in a deterministic setting) and relative risk aversion are fixed in terms of one another. The recursive criterion, however, allows one to examine the effects of varying risk aversion while holding fixed the utility's elasticity of intertemporal substitution in a deterministic setting.

(A) (Dynamic Programming) Provide an extension of the Bellman equation (12) for optimal portfolio and consumption choice, substituting the recursive utility for the additive utility. That is, state a revised Bellman equation and regularity conditions on the utility primitives (J, h) under which a solution to the Bellman equation implies that the associated feedback policies solving the Bellman equation generate optimal consumption and portfolio choice. (State a theorem with proof.) Also, include conditions under which there exists a solution to the Bellman equation. For simplicity, among your conditions you may wish to impose the assumptions that J and h are continuous and strictly increasing.

(B) (Asset Pricing Theory) Suppose that J and h are differentiable, increasing, and concave, with either h or J (or both) strictly concave. Provide any additional regularity conditions that you feel are called for in order to derive an analogue to the stochastic Euler equation (9) for security prices.

(C) (An Investment Problem) Let $G : Z \times \mathbb{R}_+ \to \mathbb{R}_+$ and consider the capital-stock investment problem defined by

$$\sup_{c \in L_+} U(c) \tag{27}$$

subject to $0 \leq c_t \leq K_t$ for all t, where $K_0, K_1, \dots$, is a capital-stock process defined by $K_t = G(X_t, K_{t-1} - c_{t-1})$, and where $X_0, \dots, X_T$ is the Markov process defined in Section 3C. The utility function U is the recursive function defined above in terms of (J, h). Provide reasonable conditions on (J, h, G) under which there exists a solution. State the Bellman equation.

(D) (Parametric Example) For this part, in order to obtain closed-form solutions, we depart from the assumption that the shock takes only a finite number of possible values, and replace this with a normality assumption. Solve the problem of part (C) in the following case:

(a) X is the real-valued shock process defined by $X_{t+1} = A + BX_t + \epsilon_{t+1}$, where A and B are scalars and $\epsilon_1, \epsilon_2, \ldots$ is an *i.i.d.* sequence of normally distributed random variables with $E(\epsilon_t) = 0$ and $\text{var}(\epsilon_t) = \sigma^2$.
(b) $G(x, a) = a^\gamma e^x$ for some $\gamma \in (0, 1)$.
(c) $J(q, w) = \log(q) + \rho \log(w^{1/\alpha})$ for some $\alpha \in (0, 1)$.
(d) $h(v) = e^{\alpha v}$ for $v \geq 0$.

Hint: You may wish to conjecture a solution to the value function of the form $V_t(x, k) = A_1(t)\log(k) + A_2(t)x + A_3(t)$, for time-dependent coefficients A_1, A_2, and A_3. This example is unlikely to satisfy the regularity conditions that you imposed in part (C).

(E) (Term Structure) For the consumption endowment process e defined by the solution to part (D), return to the setting of part (B), and calculate the price $\Lambda_{t,s}$ at time t of a pure discount bond paying one unit of consumption at time $s > t$. Note that α is a measure of risk tolerance that can be studied independently of the effects of intertemporal substitution in this model, since, for deterministic consumption processes, utility is independent of α, with $J[q, h(v)] = \log(q) + \rho \log(v)$. Does higher risk tolerance imply higher, lower, or an ambiguous change in short-term interest rates? (Justify your answer.)

3.9 Show equation (5) directly from equation (2.9).

3.10 Consider, as in the setup described in Section 3F, securities defined by the dividend-price pair (δ, S), where, for all t, there are functions f_t and S_t on Z into $\mathbb{R}^N$ such that $\delta_t = f_t(X_t)$ and $S_t = S_t(X_t)$. Show that there is no arbitrage if and only if there is a state-price deflator π such that, for each time t, $\pi_t = \psi_t(X_t)$ for some function $\psi_t : Z \to (0, \infty)$.

3.11 Verify that problem (18) is solved by the stopping time τ^* defined by (20), where J_t is defined by $J_T = 0$ and the Bellman equation (19).

3.12 (Binomial Term-Structure Algorithms) This exercise asks for a series of numerical solutions of term-structure valuation problems in a setting with binomial changes in short-term interest rates. In the setting of Section 3F, suppose that short-term riskless borrowing is possible at any time t

at the discount d_t. The one-period interest rate at time t is denoted r_t, and is given by its definition:

$$d_t = \frac{1}{1 + r_t}.$$

The underlying shock process X has the property that either $X_t = X_{t-1} + 1$ or $X_t = X_{t-1}$. That is, in each period, the new shock is the old shock plus a 0-1 binomial trial. An example is the binomial stock-option pricing model of Exercise 2.1, which is reconsidered in Section 3F. As opposed to that example, we do not necesarily assume here that interest rates are constant. Rather, we allow, at each time t, a function $\rho_t : Z \to \mathbb{R}$ such that $r_t = \rho_t(X_t)$. For simplicity, however, we take it that at any time t the pricing matrix Π_t defined in Section 3F is of the form

$$
\begin{aligned}
(\Pi_t)_{ij} &= \frac{p}{1 + \rho_t(i)}, \quad j = i + 1, \\
&= \frac{1 - p}{1 + \rho_t(i)}, \quad j = i, \\
&= 0, \quad \text{otherwise},
\end{aligned}
$$

where $p \in (0, 1)$ is the "risk-neutral" probability that $X_{t+1} - X_t = 1$. Literally, there is an equivalent martingale measure Q under which, for all t, we have

$$Q(X_{t+1} - X_t = 1 \mid X_0, \ldots, X_t) = p.$$

It may help to imagine the calculation of security prices at the nodes of the "tree" illustrated in Figure 3.1. The horizontal axis indicates the time periods; the vertical axis corresponds to the possible levels of the shock, assuming that $X_0 = 0$. At each time t and at each shock level i, the price of a given security at the (i, t)-node of the tree is given by a weighted sum of its value of the two succesor nodes $(i + 1, t + 1)$ and $(i, t + 1)$. Specifically,

$$S_t(i) = \frac{1}{1 + \rho_t(i)} \left[pS_{t+1}(i + 1) + (1 - p)S_{t+1}(i) \right].$$

Two typical models for the short rate are obtained by taking $p = 1/2$ and either

(a) the *Ho and Lee model*: For each t, $\rho_t(i) = a_t + b_t i$ for some constants a_t and b_t; or
(b) the *Black-Derman-Toy model*: For each t, $\rho_t(i) = a_t \exp(b_t i)$ for some constants a_t and b_t.

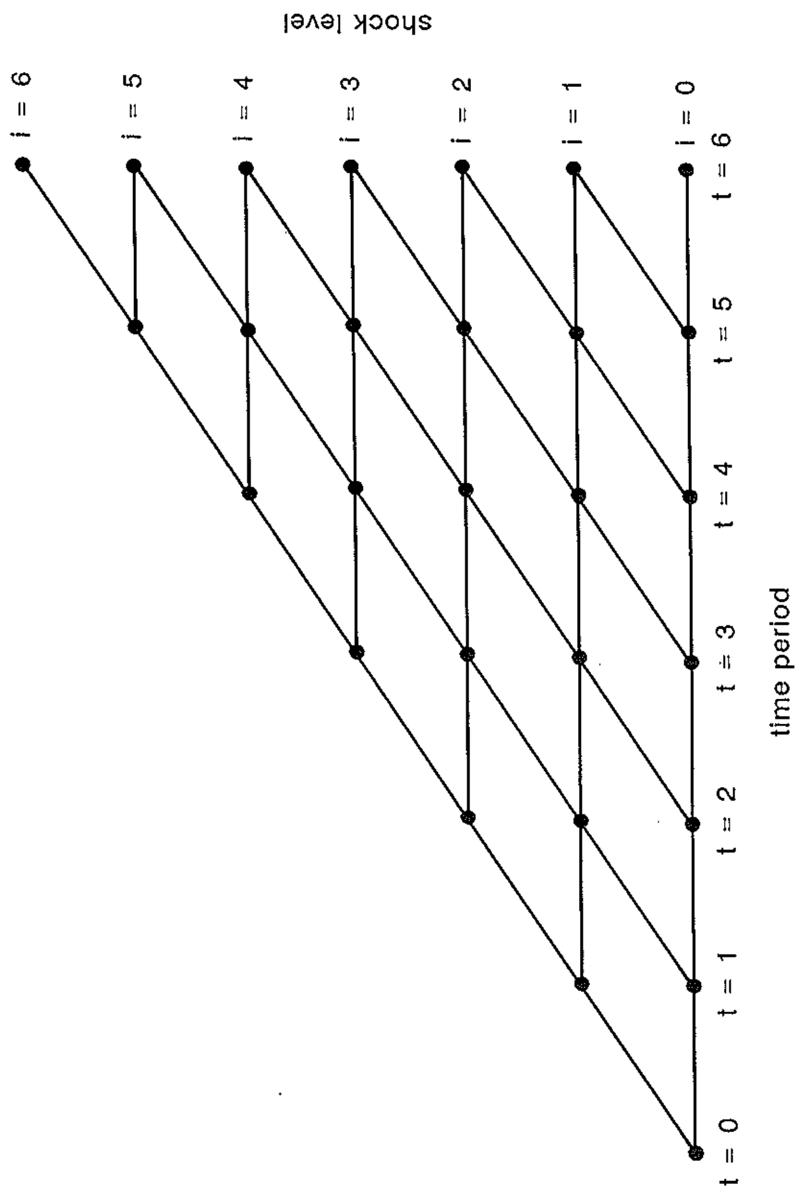

Figure 3.1. A Binomial Tree

(A) For both cases (a) and (b), prepare computer code to calculate the arbitrage-free price $\Lambda_{0,t}$ of a zero-coupon bond of any given maturity t, given the coefficients a_t and b_t for each t. Prepare an example taking $b_t = 0.01$ for all t and $a_0, a_1, \ldots, a_T$ such that $E^Q(r_t) = 0.01$ for all t. (These parameters are of a typical order of magnitude for monthly periods.) Solve for the price $\Lambda_{0,t}$ of a unit zero-coupon riskless bond maturing at time t, for all t in $\{1, \ldots, 50\}$.

(B) Consider, for any i and t, the price $\psi(i,t)$ at time 0 of a security that pays one unit of account at time t if and only if $X_t = i$.

Show that ψ can be calculated recursively by the forward difference equation

$$\psi(i, t+1) = \frac{\psi(i,t)}{2[1+\rho_t(i)]} + \frac{\psi(i-1,t)}{2[1+\rho_t(i-1)]}, \qquad 0 < i < t+1, \tag{28}$$

and, for $i = 0$ or $i = t+1$,

$$\psi(i, t+1) = \frac{\psi(i,t)}{2[1+\rho_t(i)]}. \tag{29}$$

The initial condition is $\psi(0,0) = 1$.

Knowledge of this "shock-price" function ψ is useful. For example, the arbitrage-free price at time 0 of a security that pays the dividend $f(X_t)$ at time t (and nothing otherwise) is given by $\sum_{i=0}^{t} \psi_t(i) f(i)$.

(C) In practice, the coefficients a_t and b_t are often fitted to match the initial term structure $\Lambda_{0,1}, \ldots, \Lambda_{0,T}$, given the "volatility" coefficients $b_0, \ldots, b_T$. The following algorithm has been suggested for this purpose, using the fact that $\Lambda_{0,t} = \sum_{i=0}^{t} \psi_t(i)$.

(a) Let $\psi(0,0) = 1$ and let $t = 1$.
(b) Fixing ψ_{t-1} and b_t, let $\lambda_t(a_t) = \sum_{i=0}^{t} \psi_t(i)$, where ψ_t is given by the forward difference equation (28)–(29). Only the dependence of the t-maturity zero-coupon bond price $\lambda_t(a_t)$ on a_t is notationally explicit. Since $\lambda_t(a_t)$ is strictly monotone in a_t, we can solve numerically for that coefficient a_t such that $\Lambda_{0,t} = \lambda_t(a_t)$. (A Newton-Raphson search will suffice.)
(c) Let t be increased by 1. Return to step (b) if $t \leq T$. Otherwise, stop.

Prepare computer code for this algorithm (a)–(b)–(c). Given $b_t = 0.01$ for all t, solve for a_t for all t, for both the Ho and Lee and the Black-Derman-Toy models, given an initial term structure that is given by $\Lambda_{0,t} = \alpha^t$, where $\alpha = 0.99$.

(D) Extend your code as necessary to give the price of American call options on coupon bonds of any given maturity. For the coefficients $a_0, \ldots, a_T$ that you determined from part (C), calculate the initial price of an American option on a bond that pays coupons of 0.011 each period until its maturity at time 20, at which time it pays 1 unit of account in addition to its coupon. The option has an exercise price of 1.00 and expiration at time 10. Do this for both the Ho and Lee model and the Black-Derman-Toy model.

Notes

Bellman's principle of optimality is due to Bellman (1957). Freedman (1983) covers the theory of Markov chains. For general treatments of dynamic programming in a discrete-time Markov setting, see Bertsekas (1976) and Bertsekas and Shreve (1978). The proof for Lemma 3B that is sketched in Exercise 3.3, on the differentiability of the value function, is from Benveniste and Scheinkman (1979), and easily extends to general state spaces; see, for example, Duffie (1988b) and Stokey and Lucas (1989). The *semi-group* pricing approach implicit in equation (17) is from Duffie and Garman (1985). Exercise 3.8, treating asset pricing with the recursive utility of Exercise 2.9, is extended to the infinite-horizon setting of Epstein and Zin (1989) in Exercise 4.12. See Epstein (1991) for more on recursive utility, and Streufert (1991a,b,c) for more on dynamic programming with a recursive-utility function.

The extensive exercise on binomial term-structure valuation algorithms is based almost entirely on Jamshidian (1991), who emphasizes the connection between the solution ψ of the difference equation (28)–(29) and *Green's function.* This connection is reconsidered in Chapters 7 and 10 for continuous-time applications. The two particular term-structure models appearing in this exercise are based, respectively, on Ho and Lee (1986) and Black, Derman, and Toy (1990). The parametric form shown here for the Ho and Lee model is slightly more general than the form actually appearing in Ho and Lee (1986). Most authors take the convention that X_{t+1} is $X_t + 1$ or $X_t - 1$, which generates a slightly different form for the same model. The two forms are equivalent after a change of the parameters. Continuous-time versions of these models are considered in Chapter 7. Chapter 10 also deals in more detail with algorithms designed to match the initial term structure. Exercise 10.5 demonstrates convergence, with a decreasing length of time period, of the discrete-time Black-Derman-Toy model to its continuous-time version. Jamshidian (1991) considers a larger class of examples.

4 The Infinite-Horizon Setting

THIS CHAPTER PRESENTS infinite-period analogues of the results of Chapters 2 and 3. Although it requires additional technicalities and produces few new insights, this setting is often deemed important for reasons of elegance or for serving the large-sample theory of econometrics, which calls for an unbounded number of observations. We start directly with a Markov dynamic programming extension of the finite-horizon results of Chapter 3, and only later consider the implications of no arbitrage or optimality for security prices without using the Markov assumption. Finally, we return to the stationary Markov setting to review briefly the large-sample approach to estimating asset pricing models. Only Sections 4A and 4B are essential; the remainder could be skipped on a first reading.

A. The Markov Dynamic Programming Solution

Suppose $X = \{X_1, X_2, \ldots\}$ is a time-homogeneous Markov chain of shocks valued in a finite set Z, defined exactly as in Section 3C, with the exception that there is an infinite number of time periods. Sources given in the Notes explain the existence of a probability space $(\Omega, \mathcal{F}, P_i)$, for each initial shock i, satisfying the defining properties $P_i(X_0 = i) = 1$ and

$$P_i(X_{t+1} = i \mid X_0, \ldots, X_t) = q_{X(t),j},$$

where q is the given transition matrix. As in Chapter 3, $\mathcal{F}_t$ denotes the tribe generated by $\{X_0, \ldots, X_t\}$. This is the first appearance in the book of a set Ω of states that need not be finite, but because there is only a finite number of events in $\mathcal{F}_t$ for each t, most of this chapter can be easily understood without referring to Appendix C for a review of general probability spaces.

Let L denote the space of sequences of random variables of the form $c = \{c_0, c_1, c_2, \ldots\}$ such that there is a constant k with the property that,