Hand-In 1

Continuous Time Finance 2 (Finkont2)

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Contents

Note

All references to Theorems, Propositions, Lemmas and Definition are from Björk (2020).

1 The Bachelier Model

1.a

To find the arbitrage-free price, we want to use risk neutral valuation as required for the arbitrage-free time-t price. To do this, we first have to find the $\mathbb{Q}$-dynamics. Define ... in front of dt as $\mu(S_t, t)$, achieving the $\mathbb{P}$-dynamics:

$$dS_t = \mu(S_t, t)dt + \sigma dW_t.$$

Let $L_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$ be a likelihood process and use Girsanov Thm. 12.3 to change measure in the dynamics to the risk neutral measure $\mathbb{Q}$

$$\begin{aligned} dW_t &= \varphi_t^\star dt + dW_t^{\mathbb{Q}} \\ &\Rightarrow dS_t = \mu(S_t, t)dt + \sigma(\varphi_t^\star dt + dW_t^{\mathbb{Q}}) \\ &= (\mu(S_t, t) + \sigma\varphi_t^\star)dt + \sigma dW_t^{\mathbb{Q}}. \end{aligned}$$

Define $Z_t = S_t\beta_t$ where β_t has dynamics $d\beta_t = r\beta_t dt$ where r is as usual the risk free rate. By Ito's formula

$$\begin{aligned} dZ_t \underbrace{=}_{\text{Thm. 4.11}} &\beta_t dS_t - S_t d\beta_t + (dS_t)(d\beta_t) \\ &= \beta_t((\mu(S_t, t) + \sigma\varphi_t^\star)dt + \sigma dW_t^{\mathbb{Q}}) - S_t \cdot r\beta_t dt \\ &= \beta_t(\mu(S_t, t) + \sigma\varphi_t^\star - S_t \cdot r)dt + \beta_t\sigma dW_t^{\mathbb{Q}}. \end{aligned}$$

Z_t is a martingale if the dt-term is 0 by Lem. 4.10 which is the case if Girsanov kernel is given by

$$\varphi^\star = \frac{S_t r - \mu(S_t, t)}{\sigma}.$$

Assuming now that the Novikov condition (Lem. 12.5) is indeed satisfied it is implied that the $\mathbb{Q}$-dynamics for the underlying in the Bachelier model is given exactly by

$$dS_t = rS_t dt + \sigma dW_t^{\mathbb{Q}}.$$

or simply, we assume that, under the risk-neutral measure $\mathbb{Q}$, the stock process satisfies an SDE given by

$$dS_t = rS_t dt + \sigma dW_t^{\mathbb{Q}}.$$

As per assumption $r = 0$ the $\mathbb{Q}$-dynamics of the underlying is given by

$$dS_t = \sigma dW_t^{\mathbb{Q}}.$$

At expiry, using the results from section 5.2 in Björk (2020) the underlying is then given by and distributed normally

$$
\begin{aligned}
S_T &= S_t + \int_t^T \sigma dW_s^{\mathbb{Q}} \\
&\underbrace{\sim}_{\text{Lem. 4.18}} \mathcal{N}\left(S_t,\ \int_t^T \sigma ds\right) \\
&= \mathcal{N}\left(S_t,\ \sigma^2(T-t)\right),
\end{aligned}
$$

which implies

$$
(S_T - K \mid \mathcal{F}_t) \sim \mathcal{N}(S_t - K, \sigma^2(T-t)).
$$

Having the distributional properties of $(S_T - K \mid \mathcal{F}_t)$ it is now possible for us to compute the arbitrage-free time-t price using the hint for the expectation with $\mu = S_t - K$, $\sigma = \sigma\sqrt{T-t}$, $h = \infty$ and $l = 0$. Lastly, we use the well known properties of the normal cumulative distribution function, namely that $\Phi(-x) = 1 - \Phi(x)$ and $\phi(-x) = \phi(x)$ and the limit behavior of $\Phi(\infty) \to 1$ and $\phi(\infty) \to 0$

$$
\begin{aligned}
\pi_t^{\text{Call, Bach}} &\underbrace{=}_{\text{Thm. 11.19}} E_t^{\mathbb{Q}}((S_T - K)^+) \\
&= E_t^{\mathbb{Q}}((S_T - K)\mathbb{1}_{0 \leq S_T - K \leq \infty}) \\
&= (S_t - K)\left(\Phi\left(\frac{\infty - (S_t - K)}{\sigma\sqrt{T-t}}\right) - \Phi\left(\frac{0 - (S_t - K)}{\sigma\sqrt{T-t}}\right)\right) \\
&\quad + \sigma\sqrt{T-t}\left(\phi\left(\frac{0 - (S_t - K)}{\sigma\sqrt{T-t}}\right) - \phi\left(\frac{\infty - (S_t - K)}{\sigma\sqrt{T-t}}\right)\right) \\
&= (S_t - K)\left(\Phi(\infty) - \Phi\left(\frac{-(S_t - K)}{\sigma\sqrt{T-t}}\right)\right) + \sigma\sqrt{T-t}\left(\phi\left(\frac{-(S_t - K)}{\sigma\sqrt{T-t}}\right) - \phi(\infty)\right) \\
&= (S_t - K)\left(1 - \left(1 - \Phi\left(\frac{(S_t - K}{\sigma\sqrt{T-t}}\right)\right)\right) + \sigma\sqrt{T-t}\left(\phi\left(\frac{(S_t - K)}{\sigma\sqrt{T-t}}\right) - 0\right) \\
&= (S_t - K)\Phi\left(\frac{(S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}\phi\left(\frac{(S_t - K)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

with Φ and ϕ denoting, respectively, the standard normal distribution and density function and $\{0 \leq S_T - K\} \Rightarrow \{0 \leq S_T - K \leq \infty\}$. Note that the mathematical notation is a bit naive but nevertheless is feasible.

The Δ-hedge ratio can now be computed using the just found arbitrage-free time-t price. First, note that, by the chain rule

$$
\frac{d}{dx}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{d\phi}{dz}\frac{dz}{dx}, \quad z = \frac{x-\mu}{\sigma}.
$$

Using $\frac{dz}{dx} = \frac{1}{\sigma}$ and $\frac{d}{dz}\phi(z) = -z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ we now see that

$$\begin{aligned}
\frac{d}{dx}\phi\left(\frac{x-\mu}{\sigma}\right) &= \left(-z \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}}\right) \cdot \frac{1}{\sigma} \\
&= -\frac{x-\mu}{\sigma^2} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-\left(\frac{x-\mu}{\sigma}\right)^2}{2}} \\
&= -\frac{x-\mu}{\sigma^2} \phi\left(\frac{x-\mu}{\sigma}\right),
\end{aligned}$$

or simply

$$\phi'(x) = -x\phi(x).$$

Thus, by the product- and chain-rule, substituting in the above derivative for ϕ and using $\Phi'(x) = \phi(x)$

$$\begin{aligned}
\Delta_t^{\text{Call, Bach}} &= \frac{\partial \pi_t^{\text{Call, Bach}}}{\partial S_t} \\
&= \left((S_t - K)\, \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t} \cdot \phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \right)' \\
&= 1 \cdot \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + (S_t - K)\left(\frac{1}{\sigma\sqrt{T-t}}\right) \Phi'\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \\
&\quad + \sigma\sqrt{T-t}\left(\frac{1}{\sigma\sqrt{T-t}}\right) \phi'\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) + \left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) - \left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{S_t - K}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

1.b

All the code can be viewed in Appendix A. When we usually discuss implied volatility we talk about some observed call option prices on the market. However, "the market" is now the Bachelier model.

By assumption $r = 0$. We proceed to examine the implied volatility from the Black-Scholes model across different strikes, $K \in [50, 150]$, in the Bachelier model. We use parameters: $S(0) = 100$, $T = 1$ and remember that the Bachelier σ is nominal, thus $\sigma = 15$.

Using the call price formula found in 1.a for the Bachelier model we can calculate the observed market-call prices using the data given above against the solved for implied volatilities by numerical rootsearch of

$$\pi^{\text{Call, Black}}(\sigma_{Imp}, \ldots) = \pi^{\text{Call, Bach}}(\sigma, \ldots).$$

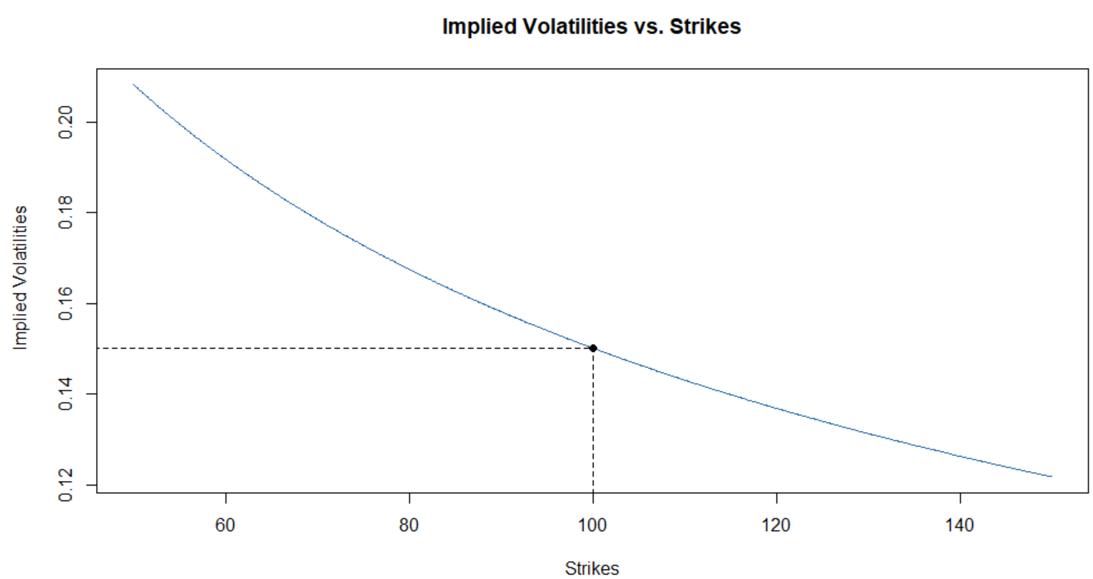

Figure 1: Implied volatilites (IV) vs. Strikes ($K \in [50, 150]$) in the Bachelier model using $S(0) = 100$, $T = 1$ and $\sigma = 15$.

We observe as expected that if $K = S(0) = 100$, then $\sigma_{Imp} = 0.15$, as the Black-Scholes volatility is not a nominal value but a percentage. It is clear that the implied volatility is inversely correlated to a certain extend, that is, the implied volatility is larger for lower strikes prices and lower for high strike prices. The implied volatility is also observed higher for in-the-money call-options and lower for in- and at-the-money call-options. i.e the implied volatility is increasing in the moneyness. This phenomena is what is usually referred to as *reverse skew* Kenton (2023). This is in the range of expectations as a constant relationship between implied volatility and strikes would happen only if our distribution assumption when calculating/(or observering) call-option prices were the same as when finding the implied volatility. To further examine this skew one has to examine the dynamics in both models in use. The Black-Scholes model includes S_t in the additive Gaussian noise term. This is not present in the Bachelier model as S_t is not present in that additive Gaussian noise term (as given in the assignment). The price of the underlying, the stock, is modeled by a arithmetic process in the Bachelier model as opposed to a geometric process in the Black-Scholes model.

In other words: The distribution of the underlying risky asset in the models are not equal as the dynamics of the underlying $S(t)$ is of different forms, namely the lack of S_t in the additive Gaussian noise term in the Bachelier model compared to the Black-Scholes model. The former having a underlying following the Gaussian distribution where as the latter has a underlying following the log-normal distribution.

This means that we have a false distribution assumption i.e the model is (obviously) not the right one because of the beforementioned reason which is causing the phenomena seen in figure 1.

1.c

Assume $r > 0$ which implies the $\mathbb{Q}$-dynamics found in 1.a to be

$$dS_t = rS_t dt + \sigma dW_t^{\mathbb{Q}}.$$

We then get the expression for the underlying at expiry-T and distribution using the results from section 5.2 in Björk (2020)

$$\begin{aligned}
S_T &= S_t e^{r(T-t)} + \int_t^T \sigma e^{r(T-s)} dW_s^{\mathbb{Q}} \\
&\underbrace{\sim}_{\text{Lem. 4.18}} \mathcal{N}\left(S_t e^{r(T-t)}, \int_t^T \sigma^2 e^{2r(T-s)} ds\right) \\
&= \mathcal{N}\left(S_t e^{r(T-t)}, \frac{\sigma^2}{2r}(e^{2r(T-t)} - 1)\right),
\end{aligned}$$

which implies

$$(S_T - K \mid \mathcal{F}_t) \sim \mathcal{N}\left(S_t e^{r(T-t)} - K, \frac{\sigma^2}{2r}(e^{2r(T-t)} - 1)\right).$$

Having the distributional properties of $(S_T - K \mid \mathcal{F}_t)$ it is now possible for us to compute the arbitrage-free time-t price using the exact same approach as in the description given in 1.a

$$\begin{aligned}
\pi_{t,r>0}^{\text{Call, Bach}}(t, S_t) &\underbrace{=}_{\text{Thm. 11.19}} e^{-r(T-t)} E_t^{\mathbb{Q}}((S_T - K)^+) \\
&= e^{-r(T-t)} E_t^{\mathbb{Q}}((S_T - K)\mathbb{1}_{0 \le S_T - K \le \infty}) \\
&= e^{-r(T-t)} \left(\left(e^{r(T-t)} S_t - K\right) \left(\Phi(\infty) + \Phi\left(\frac{0 - \left(e^{r(T-t)} S_t - K\right)}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}} \right) \right) \right. \\
&\quad \left. + \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)} \left(\phi\left(\frac{0 - \left(e^{r(T-t)} S_t - K\right)}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}} \right) + \phi(\infty) \right) \right) \\
&= \left(S_t - e^{-r(T-t)} K\right) \Phi\left(\frac{e^{r(T-t)} S_t - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}} \right) \\
&\quad + e^{-r(T-t)} \sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)} \phi\left(\frac{e^{r(T-t)} S_t - K}{\sqrt{\frac{\sigma^2}{2r}\left(e^{2r(T-t)} - 1\right)}} \right).
\end{aligned}$$

2 Quanto Hedging and The Kingdom of Denmark Put

2.a

The arbitrage free price of the guaranteed exchange rate put option is found by utilization of Prop. 7.13. Black–Scholes Formula where we are given the price of a European put option with strike price K and time of maturity T. Combining Prop. 7.13 with the put-call parity Prop. 10.2 to achieve the price of a general put option with strike K (in foreign currency)

$$\begin{aligned}
p(t,s) &\underbrace{=}_{\text{Prop. 10.2}} Ke^{-(r_J-\sigma_X^\top\sigma_J)(T-t)} + c(t,s) - s \\
&\underbrace{=}_{\text{Prop. 7.13}} Ke^{-(r_J-\sigma_X^\top\sigma_J)(T-t)} + s\Phi(d_1(s,t)) - e^{-(r_J-\sigma_X^\top\sigma_J)(T-t)}K\Phi(d_2(s,t)) - s \\
&= Ke^{-(r_J-\sigma_X^\top\sigma_J)(T-t)}(1-\Phi(d_2(s,t))) - s(1-\Phi(d_1(s,t))) \\
&= Ke^{-(r_J-\sigma_X^\top\sigma_J)(T-t)}\Phi(-d_2(s,t)) - s\Phi(-d_1(s,t)),
\end{aligned}$$

where we use multiple applications of $\Phi(-x) = 1-\Phi(x)$.

Now, the arbitrage-free time-t price of the guaranteed exchange rate put option can be expressed by

$$\begin{aligned}
F^{QP}(t,s) &\underbrace{=}_{\text{Thm. 7.11}} e^{-r_{US}(T-t)}E_t^{\mathbb{Q}}(Y_0(K-S_J(T))^+) \\
&\underbrace{=}_{*} e^{-r_{US}(T-t)}Y_0E_t^{\mathbb{Q}}((K-S_J(T))^+) \\
&= e^{-r_{US}(T-t)}Y_0e^{(r_J-\sigma_X^\top\sigma_J)(T-t)}\underbrace{e^{-(r_J-\sigma_X^\top\sigma_J)(T-t)}E_t^{\mathbb{Q}}((K-S_J(T))^+)}_{p(t,s)\text{ on } S_J}
\end{aligned}$$

where * follows by the fact that Y_0 is a constant. By pattern recognition, notice the underbraced European put option price on S_J with strike K has drift rate $r_j - \sigma_X^\top\sigma_J$ and volatility $||\sigma_J||$ and was achieved by simply multiplying the equation by a term and its inverse, a rewriting. Substituting in $p(t,S_J)$ - found at the start - we achieve the new expression for the arbitrage-free time-t price of the guaranteed exchange rate put option

$$\begin{aligned}
F^{QP}(t,s) &= e^{-r_{US}(T-t)}Y_0e^{(r_J-\sigma_X^\top\sigma_J)(T-t)}p(t,s)) \\
&= e^{-r_{US}(T-t)}Y_0e^{(r_J-\sigma_X^\top\sigma_J)(T-t)}(Ke^{-(r_J-\sigma_X^\top\sigma_J)(T-t)}\Phi(-d_2(t,s)) - s\Phi(-d_1(t,s))) \\
&= e^{-r_{US}(T-t)}Y_0(K\Phi(-d_2(t,s)) - e^{(r_J-\sigma_X^\top\sigma_J)(T-t)}s\Phi(-d_1(t,s))),
\end{aligned}$$

and substituting in the drift rate and volatility in the functions $d_1(t,s), d_2(t,s)$ yields the wanted result

$$\begin{aligned}
d_1(t,s) &= \frac{\log\left(\frac{s}{K}\right) + (r_J - \sigma_X^\top\sigma_J + \frac{1}{2}||\sigma_J||^2)(T-t)}{||\sigma_J||\sqrt{T-t}}, \\
d_2(t,s) &= d_1(t,s) - ||\sigma_J||\sqrt{T-t}.
\end{aligned}$$

We now proceed to finding $\Delta^{QP} = \frac{\partial F^{QP}(t,s)}{\partial s}$ using put-call parity and the non-reduced form of the found arbitrage-free time-t price of the guaraneed exchange rate put option

$$\begin{aligned}
F^{QP}(t,s) &= Y(0)e^{(r_J-\sigma_X^\top\sigma_J-r_{US})(T-t)}p(t,s) \\
&\underbrace{=}_{\text{Prop 10.2}} Y(0)e^{(r_J-\sigma_X^\top\sigma_J-r_{US})(T-t)}\left(Ke^{-r(T-t)}+c(t,s)-s\right) \\
&\Rightarrow \frac{\partial F^{QP}(t,s)}{\partial s} = Y(0)e^{(r_J-\sigma_X^\top\sigma_J-r_{US})(T-t)}(\Phi(d_1(t,s))-1) =: g(t,s),
\end{aligned}$$

where we used Excercise 1.2: *A standard result that is less obvious than you'd think* for the derivative of $c(s,t)$ wrt. s.

2.b

All the code can be viewed in Appendix A. We conduct the customary hedging experiment for a strategy in which we possess a certain quantity of foreign stock units, financed through a domestic bank account. The execution of the experiment involves adjusting our hedge in accordance with a predetermined time step. We use two different methods for the time-steps:

- Daily hedge on business days, i.e. 252 times T, because the time to maturity is $T = 2$.
- Once every hour for every business day, assuming that a business day is 8 hour long: $252 \cdot 8 \cdot T$.

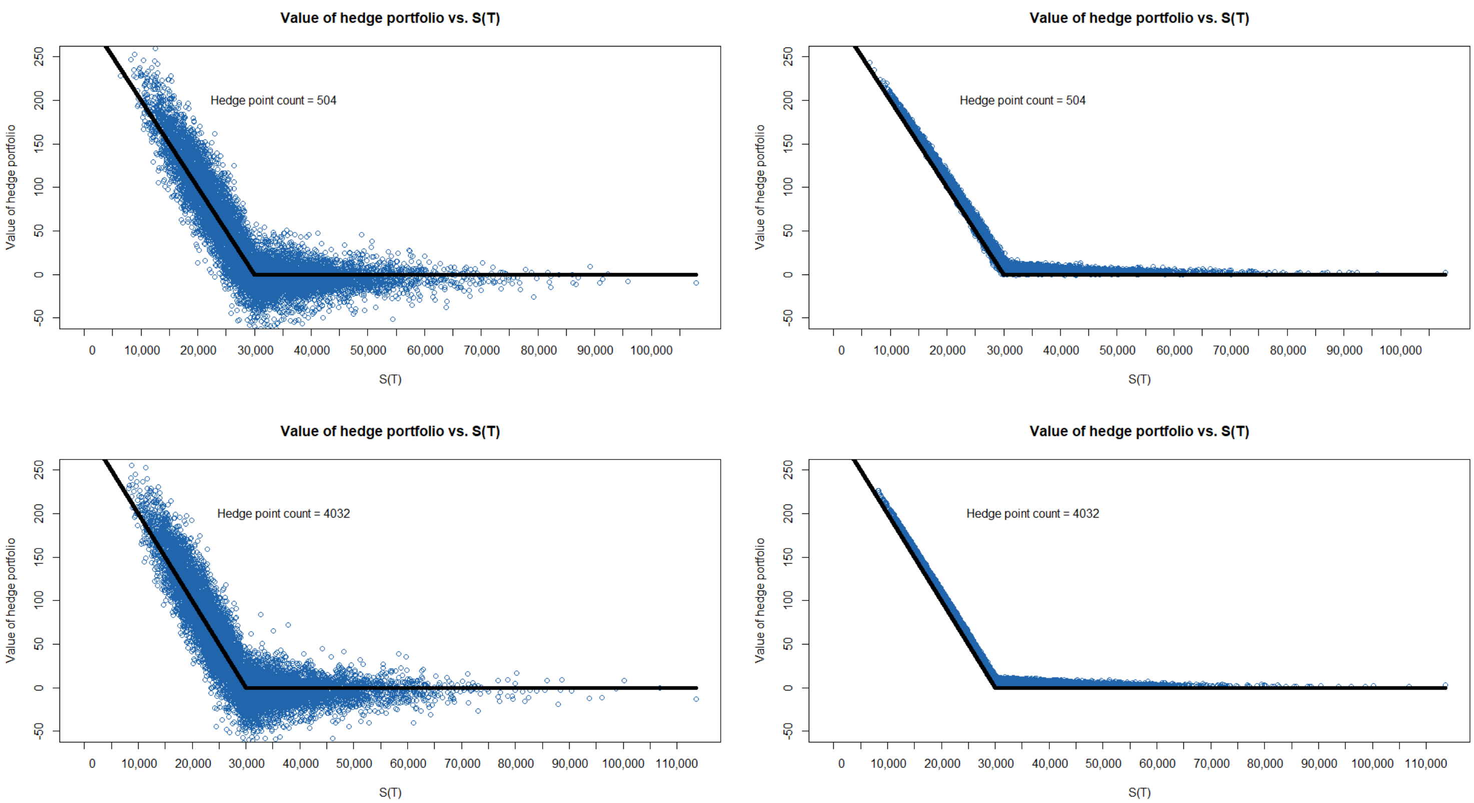

Figure 2: Hedge experiment using two different methods with daily and hourly hedging

The results from the simulation experiment can be seen in figure 2 with the left graphs being the first experiment/strategy described in the assignment and the right graphs being the second one described. The black (ice)hockey-stick is the pay-off function of the guaranteed exchange rate put option and the coloured points being the hedge values. The right graphs are clearly a much better strategy using the position of currency in the foreign bank account as the strategy hedges/replicates the option a lot better in both time-steps and in the limit. The left graphs that does not include the former described element is clearly not a good hedging strategy for both time-steps, even in the limit. This is seen by huge derivation in regards to the value of the hedge points compared to the portfolio value.

2.c

We do a full-fledged, three-holdings continuous-time argument as "hinted" but suppressing notation a bit (such as (t) and (t_i)). We want to show that the second strategy is the replicating hedge for the guaranteed exchange rate put option. In other words we have to show the value process by construction is F^{QP}. Informally, that the rabbit falls into the hat at the end of calculations.

Consider the self-financing portfolio h defined by 3 separate holdings, h_i, $i \in [0,2]$:

- $h_0(t)$: The amount held in the domestic (US) bank account , B_{US}.
- $h_1(t)$: The amount of the foreign (J) stock held, $\tilde{S}_J(t) = X(t)S_J(t)$.
- $h_2(t)$: The amount held in the foreign (J) bank account, $\tilde{B}_J(t) = X(t)B_J(t)$.

The value function of the portfolio h is then by definition given by (and with following dynamics)

$$\begin{aligned} V^h(t) &\underbrace{=}_{\text{Def 6.10 (2)}} h_0(t)B_{US} + h_1(t)\tilde{S}_J + h_2(t)\tilde{B}_J \\ &\underbrace{\Rightarrow}_{\text{Def 6.10 (4)}} dV^h(t) = h_0(t)dB_{US}(t) + h_1(t)d\tilde{S}_J(t) + h_2(t)d\tilde{B}_J(t). \end{aligned}$$

From this point we have to find the dynamics of V^h and compare them to that of the pricing function. Doing the grunt work we see that:

The dynamics of B_{US} is given in the assignment.

The dynamics of $\tilde{B}_J(t)$ can now be found by

$$d\tilde{B}_J(t) \underbrace{=}_{\text{Lem. 18.1}} r_{US}\tilde{B}_J(t)dt + \tilde{B}_J(t)\sigma_X^\top dW(t).$$

The dynamics of $\tilde{S}_J(t)$ can now be found by Ito's product rule

$$\begin{aligned} d\tilde{S}_J(t) &= S_J dX(t) + X(t)dS_J(t) + (dX(t))(dS_J(t)) \\ &\underbrace{=}_{\text{Prop. 18.7}} S_J(t)(r_{US} - r_J)X(t)dt + X(t)\sigma_X^\top dW(t)) \\ &\quad + X(t)((r_J - \sigma_J\sigma_X^\top)S_J(t)dt + S_J(t)\sigma_J^\top dW(t)) + S_J(t)\sigma_J X(t)\sigma_X^\top dt \\ &= (r_{US} - r_J)S_J(t)X(t)dt + \sigma_X^\top S_J(t)X(t)dW(t) + (r_J - \sigma_J\sigma_X^\top)S_J(t)X(t)dt \\ &\quad + \sigma_J S_J(t)X(t)dW(t) + \sigma_J\sigma_X^\top S_J(t)X(t)dt \\ &= r_{US}\tilde{S}_J(t)dt + (\sigma_X^\top + \sigma_J)\tilde{S}_J(t)dW(t). \end{aligned}$$

By substitution of the dynamics of B_{US}, $\tilde{S}_J(t)$ and $\tilde{B}_J(t)$ into $dV^h(t)$ we see that we can rewrite it into the form of a stochastic differential equation

$$\begin{aligned}
dV^h(t) &= h_0(t)r_{US}B_{US}(t)dt + h_1(t)(r_{US}\tilde{S}_J(t)dt + (\sigma_X^\top + \sigma_J^\top)\tilde{S}_J(t)dW(t)) \\
&\quad + h_2(t)(r_{US}\tilde{B}_J(t)dt + \tilde{B}_J(t)\sigma_X^\top dW(t)) \\
&= (h_0(t)r_{US}B_{US}(t) + h_1(t)r_{US}\tilde{S}_J(t) + h_2(t)r_{US}\tilde{B}_J(t))dt \\
&\quad + (h_1(t)(\sigma_X^\top + \sigma_J^\top)\tilde{S}_J(t) + h_2(t)\tilde{B}_J(t)\sigma_X^\top)dW(t).
\end{aligned}$$

Lastly, we need the dynamics of $F^{QP}(t, S_J)$ which can be found by Ito's formula

$$\begin{aligned}
dF^{QP} &\underbrace{=}_{\text{Prop. 4.12}} \frac{\partial F^{QP}}{\partial t}dt + \frac{\partial F^{QP}}{\partial S_J(t)}dS_J(t) + \frac{1}{2}\cdot\frac{\partial^2 F^{QP}}{\partial S_J(t)^2}(dS_J(t))^2 \\
&= \frac{\partial F^{QP}}{\partial t}dt + \frac{\partial F^{QP}}{\partial S_J(t)}(S_J(t)(r_J - \sigma_X^\top\sigma_J)dt + S_J(t)\sigma_J^\top dW(t)) \\
&\quad + \frac{1}{2}\cdot\frac{\partial^2 F^{QP}}{\partial S_J(t)^2}S_J^2(t)\sigma_J^\top\sigma_J dt \\
&= \left(\frac{\partial F^{QP}}{\partial t} + \frac{\partial F^{QP}}{\partial S_J(t)}S_J(t)(r_J - \sigma_X^\top\sigma_J) + \frac{1}{2}\cdot\frac{\partial^2 F^{QP}}{\partial S_J(t)^2}S_J^2(t)\sigma_J^\top\sigma_J\right)dt \\
&\quad + \frac{\partial F^{QP}}{\partial S_J(t)}S_J(t)\sigma_J^\top dW(t).
\end{aligned}$$

This means that we now have the dynamics of the bank-accounts, exchange-rate and Japanese stock dynamics, value process and price process under the US-martingale measure. By definition 8.1 - as stated before - we know the claim can be replicated if there exists a self-financing portfolio h such that

$$V_T^h = F^{QP}, \quad P - a.s.$$

Now, hedging works under any probability measure (and thus for any drift rate: Poulsen (a), p. 25), which means we need to examine the stochastic diffusion of the price process, i.e., the $dW(t)$-term as hedging should work along every trajectory. In short, have that h is a replication portfolio for the guaranteed exchange rate put option if the value process $V^h(t)$ of h has the same diffusion term as the price process F^{QP} as this would be equivalent to the fact that the value process and price process follow the same Brownian motion trajectory. In other words, when

$$\begin{aligned}
\frac{\partial F^{QP}}{\partial S_J(t)}S_J(t)\sigma_J^\top &= h_1(t)(\sigma_X^\top + \sigma_J^\top)\tilde{S}_J(t) + h_2(t)\tilde{B}_J\sigma_X^\top \\
&= h_1(t)(\sigma_X^\top + \sigma_J^\top)S_J(t)X(t) + h_2(t)B_J(t)X(t)\sigma_X^\top.
\end{aligned}$$

We now aim to calculate the hedging strategy for purchasing units of the foreign stock in such a way that won't change our value in USD irrespective of movement in the foreign currency, JPY. This hedge is possible if we equate terms containing the foreign stock price volatility (σ_J)

$$\begin{aligned}
\frac{\partial F^{QP}}{\partial S_J(t)} S_J(t)\sigma_J^\top(t) &= h_1(t)\sigma_J^\top(t)S_J(t)X(t) \\
\iff h_1(t) &= \frac{\partial F^{QP}}{\partial S_J(t_i)}\frac{1}{X(t_i)} \\
&= \frac{g(t_i, S_J(t_i))}{X(t_i)} \\
&= \Delta^{QP},
\end{aligned}$$

where Δ^{QP} is by definition from 2.b. This is our hedge when buying as we aim to maintain the portfolio's value in USD constant, irrespective of fluctuations in the JPY's value. This means that the strategy takes on the foreign currency risk of fluctuations as we derived that the guaranteed exchange rate put option indeed depends on the foreign currency risk. We thus need to hedge the currency risk by equating the terms containing exchange rate volatility (σ_X) which exactly means these terms will be equal to 0 in the dynamics of the value process when added and thus this will exterminate the foreign currency risk which means that the portfolio is locally risk-free in respect to the exchange rate volatility when

$$\begin{aligned}
& h_1(t_i)X(t_i)S_J(t_i)\sigma_X^\top + h_2(t_i)X(t_i)B_J(t_i)\sigma_X^\top = 0 \\
\iff & h_1(t_i)X(t_i)S_J(t_i)\sigma_X^\top = -h_2(t_i)X(t_i)B_J(t_i)\sigma_X^\top \\
\iff & h_2(t_i) = -\frac{h_1(t_i)S_J(t)}{B_J(t_i)} \\
= & -\Delta^{QP}\frac{S_J(t_i)}{B_J(t_i)}.
\end{aligned}$$

The amount $-\Delta^{QP}\frac{S_J}{B_J}$ is thus exactly the amount of units we need to deposit in the foreign bank account to exterminate the foreign currency risk. This hedge serves as our protection against exchange rate variability during the conversion back to USD. In executing a currency-carry trade, it's crucial to mitigate not only the potential fluctuation in the foreign stock's price but also the uncertainty surrounding exchange rate movements even when hedging a guaranteed exchange rate put option - which seems paradoxical at first but is a safeguard against secondary exposure. Our second strategy used in question 2.b, utilized both these aspects and thus resulted in a more effective hedge that actually replicates the guaranteed exchange rate put option because of the usage of the latter additional hedging method.

Substituting in the found values of h_1, h_2 into V^h shows that

$$\begin{aligned}
V^h(t) &= h_0(t)B_{US}(t) + h_1(t)\tilde{S}_J(t) + h_2(t)\tilde{B}_J(t) \\
&= h_0(t)B_{US}(t) + \Delta^{QP}\tilde{S}_J(t) - \Delta^{QP}\frac{S_J(t)}{B_J(t)}\tilde{B}_J(t) \\
&= h_0(t)B_{US}(t) + \Delta^{QP}\tilde{S}_J(t) - \Delta^{QP}\frac{S_J(t)}{B_J(t)}B_J(t)X(t) \\
&= h_0(t)B_{US}(t) + \Delta^{QP}\tilde{S}_J(t) - \Delta^{QP}S_J(t)X(t).
\end{aligned}$$

In other words our hedging strategy is: We hold Δ^{QP} of the foreign stock, whilst shorting a quantity $-\Delta^{QP} S_J$ of the foreign currency and then invest the remainder which is h_0 in the domestic bank account which makes the portfolio fulfill the self-financing condition. This method is shown to replicate the pay-off of the guaranteed exchange rate put option which is exactly given by the second strategy from question 2.b.

2.d

The story of the Kingdom of Denmark puts follows from a rather cleaver observation made by Goldman Sachs in the late 1980s to early 1990s that would later be known as the *Japanese equity bubble* Derman (2004), p. 209. During the late 1980s, as the Nikkei index reached unprecedented heights of ¥38,915.90. Goldman Sachs strategically purchased a significant number of puts, creating a vast insurance policy against a potential Nikkei decline. Numerous Japanese firms accessed capital markets to secure financing from investors. These companies offered their bondholders put options linked to the Nikkei index, effectively granting them a safeguard against potential declines in the Nikkei. Over the years, leading up to the crisis, some bondholders chose to retain their bonds while selling the associated put options for cash to various interested buyers. Goldman Sachs managed to buy these puts at a low cost, likely because the issuing companies were skeptical of a sustained downturn in the Nikkei amidst the booming Tokyo property values and rising Japanese equity markets Derman (2004), p. 209.

To grasp the narrative fully from this point forward, it's essential to comprehend the nature of The Nikkei index. The Nikkei is a price-weighted index meaning that the index is an average of the share price of the companies listed in the index. The Nikkei is of 225 Japanese stocks whose prices are quoted in yen. As the dynamics of The Nikkei index is reliant on each constituents performance, a decrease in one of the 225 companies would result in a hit proportional to that companies size and vice versa Chen (2022).

An American with a dollar-based currency would be able to invest in The Nikkei index but would then own the stock or some derivative, as usual, but denominated in yen Derman (2004), p. 210. The investor does however face two risks when investing in a foreign currency that have to be accounted for as one would imagine the performance of 225 companies to also be highly correlated to the yen:

1. The Nikkei index possesses the potential to experience a decline.
2. The yen is subject to potential depreciation against the dollar.

As Goldman Sachs had earlier acquired the insurance against a potential Nikkei decline it became a possibility to sell similar protection to public investors which is exactly what they did. In 1990 Goldman Sachs issued the so called *Kingdom of Denmark Nikkei put warrant* at a Nikkei level of ¥37,516.77 (just past the all time peak at the time) with expiration date in 1993 listed on the AMS (American Stock Exchange) Derman (2004), p. 209. Issuing just after reaching the all-time peak was a strategic decision, as it naturally led buyers to anticipate a further decline in value. But what does the pre-fix of Kingdom-of-Denmark actually imply? The Kingdom of Denmark were paid a fee by Goldman Sachs to guarantee that they would be liable for the put warrants in the case that Goldman Sachs would default. The size of the fee was $10M Poulsen (b), p. 17.

Back to the risks one would face when investing in The Nikkei; as we would expect The Nikkei to decrease simultaneously with a decrease in the yen we would want some protection the risk of a depreciating currency Derman (2004), p. 210. The "normal" or non-exotic/vanilla foreign (Japanese) put option is an option that at time-T-expiration pays off

$$Y(T)(K - S_J(T))^+,$$

where $Y(T)$ is the exchange rate at time T and $+$ indicating we are considering the maximum between $Y(T)(K - S_J(T))$ or 0. However, the Kingdom of Denmark Nikkei put warrants carried a internalized protection against the depreciation (and actually a vulnerability in the case of appreciation) of the yen. More specifically, the protection against a weakening of the yen against the dollar was as such that the yen-dollar exchange rate was guaranteed in advance no matter the time-t exchange rate. This is exactly the type of option described in question 2 and hence the name: Guaranteed exchange rate put option or *quanto* for the lingo enjoyers

$$Y_0(K - S_J(T))^+.$$

Goldman Sachs engaged in a investing strategy by purchasing yen-based puts at a low cost and selling more expensive puts linked to the Kingdom of Denmark to eager investors and speculators. However, this strategy by Goldman Sachs posed a significant risk due to a mismatch in the pay-off's, where a decline in the Nikkei would result in payouts in yen for the options bought, but obligations to pay in dollars for the puts sold. This created a vulnerability where fluctuations in the dollar-yen exchange rate could potentially reduce or eliminate Goldman Sachs' profits. To mitigate this risk, Goldman Sachs needed to continuously hedge against any changes in the dollar-yen exchange rate affecting the value of both the options purchased and those sold Derman (2004), p. 211.

This leads us to the paradoxical nature of the guaranteed exchange rate put option. The Kingdom of Denmark Nikkei put's payout in dollars was independent of fluctuations in the dollar-yen exchange rate. However, in order to hedge the put, Goldman Sachs needed to mitigate its exposure to the Nikkei, necessitating a position in Nikkei futures. The valuation of these futures was contingent upon the Nikkei index measured in yen. Consequently, holding these Nikkei futures led to a secondary exposure to the yen, which also required hedging. It was this secondary hedging of the yen, aimed at safeguarding the primary hedge against the Nikkei, that introduced a dependence on the correlation Derman (2004), p. 216.

This observation is in complete concordance with the results we derived in question 2.b, which involved the hedging of a guaranteed exchange rate put option through two distinct strategies. Upon comparison, it was determined that the latter strategy offers superior performance. This is because - as Goldman Sachs also realized - When executing a currency-carry hedge, it's essential to mitigate not only the risk associated with fluctuations in the foreign stock's price but also the risk of changes in the exchange rate because of the secondary exposure because of the fact that the futures are contingent upon the price of the underlying in the foreign currency. This risk must be hedged; otherwise, we end up in a scenario where the pay-off from the guaranteed exchange rate put option cannot be replicated.

To gain a more accurate understanding of the mathematical aspects, let's re-examine the formula for the formula for the arbitrage-free time-t guaranteed exchange rate put option

$$F^{QP}(t, S_J) = e^{-r_{US}(T-t)} Y_0 (K\Phi(-d_2(t,s)) - e^{(r_J - \sigma_X^\top \sigma_J)(T-t)} s\Phi(-d_1(t,s))),$$

with

$$d_1(t,s) = \frac{\log\left(\frac{s}{K}\right) + (r_J - \sigma_X^\top \sigma_J + \frac{1}{2}||\sigma_J||^2)(T-t)}{||\sigma_J||\sqrt{T-t}},$$
$$d_2(t,s) = d_1(t,s) - \sqrt{T-t}||\sigma_J||.$$

Low and behold, as Goldman Sachs (or more specifically Piotr Karasinski Derman (2004), p. 211) had found, it is clear as day that the arbitrage-free time-t guaranteed exchange rate put option price depends upon σ_X, the volatility of the the exchange rate.

3 A Reflection Theorem

3.a

We apply Ito's formula on the process $\tilde{Z}_t := \frac{Z_t}{Z_0} = \frac{(S_t/H)^p}{(S_0/H)^p} = \left(\frac{S_t}{S_0}\right)^p = X_t^p$. First we compute the usual deterministic derivatives of the function $f(x,t) = x^p$

$$\frac{\partial f}{\partial x} = px^{p-1}, \quad \frac{\partial^2 f}{\partial x^2} = p(p-1)x^{p-2}, \quad \frac{\partial f}{\partial t} = 0$$

Using the multiplication table ($(dW_t)^2 = dt \Rightarrow d(X_t)^2 = X_t^2\sigma^2 dt$), substituting in the random variable X_t and $dX_t = \mu X_t dt + \sigma X_t dW_t$ we see that

$$\begin{aligned}
df(X_t,t) &\underbrace{=}_{\text{Prop. 4.12}} pX_t^{p-1}dX_t + \frac{1}{2}p(p-1)X_t^{p-2}X_t^2\sigma^2 dt \\
&= pX_t^{p-1}X_t(rdt + \sigma dW_t^{\mathbb{Q}}) + \frac{1}{2}p(p-1)X_t^p\sigma^2 dt \\
&= pX_t^p rdt + pX_t^p\sigma dW_t^{\mathbb{Q}} + \frac{1}{2}p(p-1)X_t^p\sigma^2 dt \\
&= X_t^p(pr + \frac{1}{2}p(p-1)\sigma^2)dt + pX_t^p\sigma dW_t^{\mathbb{Q}} \\
&\underbrace{=}_{*} pX_t^p\sigma dW_t^{\mathbb{Q}} \\
&\underbrace{\Rightarrow}_{**} d\tilde{Z}_t = p\tilde{Z}_t\sigma dW_t^{\mathbb{Q}},
\end{aligned}$$

where $*$ follows from substituting in $p := 1 - \frac{2r}{\sigma^2}$ and $**$ from substituting in $\tilde{Z}_t = X_t^p$. Hence, $\tilde{Z}_t$ is a martingale by lemma 4.10 as $\tilde{Z}_t$ has no dt-term. It now follows that

$$\begin{aligned}
\frac{Z_t}{Z_0} &= \left(\frac{S_t}{S_0}\right)^p \\
&\underbrace{=}_{\text{Prop. 5.2}} \frac{1}{S^p}S^p e^{\left(p\left(r-\frac{\sigma^2}{2}\right)t + \sigma p W_t^{\mathbb{Q}}\right)} \\
&= e^{\left(p\left(r-\frac{\sigma^2}{2}\right)t + \sigma p W_t^{\mathbb{Q}}\right)} > 0,
\end{aligned}$$

as $e^x > 0$ for $x \in \mathbb{R}$. Taking expectations of the martingale we notice that

$$\begin{aligned}
E_t^{\mathbb{Q}}(\tilde{Z}_T) &\underbrace{=}_{*} E_t^{\mathbb{Q}}(E_0^{\mathbb{Q}}(\tilde{Z}_T)) \\
&\underbrace{=}_{**} E_t^{\mathbb{Q}}(\tilde{Z}_0) \\
&= \tilde{Z}_0 \\
&= \frac{Z_0}{Z_0} \\
&= 1,
\end{aligned}$$

where $*$ follows from the Law of Total Expectation and $**$ from the Martingale property since $0 \leq T$. Hence, $Z(t)/Z(0)$ is a positive, mean-1 $\mathbb{Q}$-martingale and that that

$$\frac{d\mathbb{Q}^Z}{d\mathbb{Q}} = \frac{Z(T)}{Z(0)},$$

defines a probability measure $\mathbb{Q}^Z \sim \mathbb{Q}$.

Now, by (Abstract) Bayes formula (Prop. B.41), assume the existence of some random variable X on $(\Omega, \mathcal{F}, \mathbb{Q})$ (sufficiently integrable) and let $\mathbb{Q}^Z$ be another measure on the measurable space $(\Omega, \mathcal{F})$. With Radon-Nikodym derivative

$$L = \tilde{Z}_T = \frac{d\mathbb{Q}^Z}{\mathbb{Q}} \text{ on } \mathcal{F}$$

Assume further that $X \in L^1(\Omega, \mathcal{F}, \mathbb{Q}^Z)$ and that $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E^{\mathbb{Q}^z}(X \mid \mathcal{G}) = \frac{E^{\mathbb{Q}}(\tilde{Z}_T \cdot X \mid \mathcal{G})}{E^{\mathbb{Q}}(\tilde{Z}_T \mid \mathcal{G})}, \quad \mathbb{Q}^z - a.s.$$

or in the wanted form

$$\iff E^{\mathbb{Q}}(\tilde{Z}_T \cdot X \mid \mathcal{G}) = E^{\mathbb{Q}}(\tilde{Z}_T \mid \mathcal{G}) E^{\mathbb{Q}^z}(X \mid \mathcal{G}),$$

which then yields exactly the arbitrage-free time-t value

$$\begin{aligned}
\pi^{\hat{g}}(t) \underbrace{=}_{\text{Thm. 7.11}} & e^{-r(T-t)} E_t^{\mathbb{Q}}(\hat{g}(S_T)) \\
&= e^{-r(T-t)} E_t^{\mathbb{Q}}((S_T/H)^p g(H^2/S_T)) \\
&\underbrace{=}_{*} e^{-r(T-t)} E_t^{\mathbb{Q}} \left(Z_T g(H^2/S_T)\right) \\
&= e^{-r(T-t)} E_t^{\mathbb{Q}} \left(Z_0 \frac{Z_T}{Z_0} g(H^2/S_T)\right) \\
&\underbrace{=}_{**} Z_0 e^{-r(T-t)} E_t^{\mathbb{Q}} \left(\tilde{Z}_T g(H^2/S_T)\right) \\
&\underbrace{=}_{***} Z_0 e^{-r(T-t)} E_t^{\mathbb{Q}}(\tilde{Z}_T) E_t^{\mathbb{Q}^Z} \left(g(H^2/S_T)\right) \\
&\underbrace{=}_{****} Z_0 \tilde{Z}_t e^{-r(T-t)} E_t^{\mathbb{Q}^Z} \left(g(H^2/S_T)\right) \\
&\underbrace{=}_{*****} e^{-r(T-t)} \left(\frac{S_t}{H}\right)^p E_t^{\mathbb{Q}^Z} \left(g(H^2/S_T)\right),
\end{aligned}$$

where we used the following:

- $*$: Definition of Z_T.
- $**$: Expectation of a constant (Z_0) and definition of $\tilde{Z}_T$.
- $***$: Bayes theorem argument (above) with $X := g\left(\frac{H^2}{S_T}\right) \in L^1(\Omega, \mathcal{F}, \mathbb{Q}^Z)$.
- $****$: Martingale property.
- $*****$: Definition of $\tilde{Z}_t$ and Z_t.

3.b

From 3.a we have that $\tilde{Z}_t = Z_t/Z_0$ is a positive, mean-1 $\mathbb{Q}$-martingale and that that

$$\frac{d\mathbb{Q}^Z}{d\mathbb{Q}} = \frac{Z_T}{Z_0},$$

defines a probability measure $\mathbb{Q}^Z \sim \mathbb{Q}$ and $L_0 = \tilde{Z}_0 = \frac{Z_0}{Z_0} = 0$. Hence, we can utilize Girsanov's theorem (Thm. 12.3) where

$$L_T = \tilde{Z}_T := \frac{Z_T}{Z_0} = \frac{d\mathbb{Q}^Z}{d\mathbb{Q}}.$$

Now, given the $\mathbb{Q}$-dynamics of $\tilde{Z}_t$ from the previous question

$$d\tilde{Z}_t = \tilde{Z}_t p\sigma dW_t^{\mathbb{Q}},$$

by Girsanovs theorem we have that Girsanov's kernel is given by

$$d\tilde{Z}_t = \tilde{Z}_t \varphi^\star dW_t^{\mathbb{Q}} \Rightarrow \varphi^\star = p\sigma,$$

and then

$$dW_t^{\mathbb{Q}} = \varphi^\star dt + dW_t^{\mathbb{Q}^Z} \iff dW_t^{\mathbb{Q}^Z} = dW_t^{\mathbb{Q}} - \varphi^\star dt,$$

where $W^{\mathbb{Q}^Z}$ is a $\mathbb{Q}^Z$-Brownian motion.

We apply Ito to the process defined by $H^2/S(t)$. First we compute the usual deterministic derivatives of the function $f(x,t) = \frac{H^2}{x}$

$$\frac{\partial f}{\partial x} = -\frac{H^2}{x^2}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{2H^2}{x^3}, \quad \frac{\partial f}{\partial t} = 0$$

Substituting these derivatives, the random variable S_t and the $\mathbb{Q}$-dynamics of S_t under the BM-model into Ito's Lemma yields

$$\begin{aligned}
df(S_t,t) &\underbrace{=}_{\text{Thm. 4.11}} \left(0 + rS_t\left(-\frac{H^2}{S_t^2}\right) + \frac{1}{2}\sigma^2 S_t^2\left(\frac{2H^2}{S_t^3}\right)\right)dt + \sigma S_t\left(-\frac{H^2}{S^2}\right)dW_t^{\mathbb{Q}} \\
&= \frac{H^2}{S_t}(\sigma^2 - r)dt - \frac{H^2}{S_t}\sigma dW_t^{\mathbb{Q}} \\
&\underbrace{=}_{*} Y_t(\sigma^2 - r)dt - Y_t\sigma dW_t^{\mathbb{Q}} \\
&\underbrace{=}_{**} Y_t(\sigma^2 - r)dt - Y_t\sigma\left(\varphi^\star dt + dW_t^{\mathbb{Q}^Z}\right) \\
&\underbrace{=}_{***} Y_t(\sigma^2 - r - \sigma^2 p)dt + \sigma Y_t\left(-dW_t^{\mathbb{Q}^Z}\right) \\
&\underbrace{=}_{****} Y_t\left(\sigma^2 - r - \sigma^2\left(1 - \frac{2r}{\sigma^2}\right)\right)dt + \sigma Y_t\left(-dW_t^{\mathbb{Q}^Z}\right) \\
&= Y_t(\sigma^2 - r - \sigma^2 + 2r)dt + \sigma Y_t\left(-dW_t^{\mathbb{Q}^Z}\right) \\
&= Y_t r dt + \sigma Y_t\left(-dW_t^{\mathbb{Q}^Z}\right),
\end{aligned}$$

where $W^{\mathbb{Q}^Z}$ is a $\mathbb{Q}^Z$-Brownian motion. We used the following:

- $*$: Definition of Y_t.
- $**$: Girsanov result $dW_t^{\mathbb{Q}} = \varphi^\star dt + dW_t^{\mathbb{Q}^Z} \iff dW_t^{\mathbb{Q}^Z} = dW_t^{\mathbb{Q}} - \varphi^\star dt$.
- $***$: Definition of $\varphi^\star = p\sigma$ and reducing.
- $****$: Definition of $p = 1 - \frac{2r}{\sigma^2}$ and reducing.

3.c

Considering the behavior of Y_t under $\mathbb{Q}^Z$, identified as a Geometric Brownian Motion, alongside the behavior of S_t under $\mathbb{Q}$ in the Black-Scholes model, also described by a Geometric Brownian Motion, it becomes evident that their solutions are presented as follows

$$\begin{aligned}
Y_T &\underbrace{=}_{\text{Prop. 5.2}} Y_t e^{\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)-\sigma\left(W_T^{\mathbb{Q}^Z}-W_t^{\mathbb{Q}^Z}\right)\right)}, \\
S_T &\underbrace{=}_{\text{Prop. 5.2}} S_t e^{\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)-\sigma\left(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}}\right)\right)},
\end{aligned}$$

where $W_T^{\mathbb{Q}^Z} - W_t^{\mathbb{Q}^Z} \sim \mathcal{N}(0, T-t)$ and $W_T^{\mathbb{Q}} - W_t^{\mathbb{Q}} \sim \mathcal{N}(0, T-t)$ by definition. Observe that both $S_T \mid S_t$ and $Y_T \mid Y_t$ are log-normally distributed, or equivalently $\log S_T \mid S_t$ and $\log Y_T \mid Y_t$ are normally distributed

$$\begin{aligned}
\log Y_T \mid Y_t &\sim \mathcal{N}\left(\log Y_t + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right), \\
\log S_T \mid S_t &\sim \mathcal{N}\left(\log S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right).
\end{aligned}$$

Essentially, observe that $Y_t = \left(\frac{H^2}{S_t}\right)$, i.e a function of S_t at the starting point, t, which means that all the information contained at time-t for S_t is also given for Y_t. We thus now have that the conditional distribution $Y_T \mid Y_t$ under $\mathbb{Q}^Z$ is the same as the conditional distribution $S_T \mid S_t = Y_t$ under $\mathbb{Q}$ as they have to coincide at time t. It now follows mostly from substitution of definitions that

$$\begin{aligned}
E^{\mathbb{Q}^Z}\left(g\left(\frac{H^2}{S_T}\right) \mid \mathcal{F}_t\right) &= E^{\mathbb{Q}^Z}\left(g(Y_T) \mid \mathcal{F}_t\right) \\
&\underbrace{=}_{*} E^{\mathbb{Q}^Z}\left(g(Y_T) \mid Y_t = Y_t\right) \\
&\underbrace{=}_{**} E^{\mathbb{Q}}\left(g(S_T) \mid S_t = Y_t\right) \\
&= f(Y_t, t) \\
&= f\left(\frac{H^2}{S_t}, t\right),
\end{aligned}$$

where $*$ follows form the Markov property and $**$ from the above distribution-argument provided $S_t = Y_t$. Finally, using the above derivation and 3.a

$$\begin{aligned}
\pi_t^{\hat{g}} &= e^{-r(T-t)}\left(\frac{S_t}{H}\right)^p E^{\mathbb{Q}^Z}\left(g\left(\frac{H^2}{S_T}\right) \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)}\left(\frac{S_t}{H}\right)^p E^{\mathbb{Q}^Z}\left(g(Y_T) \mid \mathcal{F}_t\right) \\
&= e^{-r(T-t)}\left(\frac{S_t}{H}\right)^p f\left(\frac{H^2}{S_t}, t\right).
\end{aligned}$$

We now sketch how this result can be used for closed-form expressions for the barrier option.

A down-and-out option is one variant within the two main categories of knock-out barrier options, the other being an up-and-out option. Both are available in the formats of put and call options. In the context of barrier options, the pay-off and existence of the option are conditional on whether the underlying asset's price reaches a predetermined threshold, or rather, barrier.

Barrier options are divided into knock-out and knock-in types. A knock-out option becomes void if the price of the underlying asset hits a specific level (a barrier), which limits the holder's potential gains and reduces the risk for the issuer. In contrast, a knock-in option is initially valueless and only becomes active when the underlying asset's price reaches a designated level.

The key aspect of these options is that if the underlying asset's price reaches the barrier (we will denote this as $H = B$) at any point during the option's lifespan, the option is considered knocked out or terminated, and it will not be reinstated Scott (2022). We will focus on the Down-and-Out, or in short, DnO as the derivation of the closed-form expression for the price is done analogously using parity (and vice versa).

From now on notation from Poulsen (2006) will be employed. Define the DnO barrier option by

$$\text{DnO} := \nu(S_T)\mathbb{1}_{m(T)>B}, \quad m(T) = \min_{u\leq T} S_u.$$

Therefore, let $g(x) = \nu(x)\mathbb{1}_{x>B}$ and the B-reflected claim of g be given by $\hat{g}$. We now examine the so called h-claim with adjusted pay-off given exactly by $h = g - \hat{g}$. Previous results tell us that the arbitrage-free time-t value with barrier $B = H$ is for g and its B-reflected claim $\hat{g}$ given by

$$\pi^g(t) = e^{-r(T-t)}E_t^{\mathbb{Q}}(g(S_T)) = e^{-r(T-t)}f(S_t, t),$$

and

$$\pi^{\hat{g}}(t) = e^{-r(T-t)}\left(\frac{S_t}{B}\right)^p f\left(\frac{B^2}{S_t}\right).$$

The arbitrage-free time-t value of the h-claim for is thus

$$\begin{aligned}
\pi^h(t) &= \pi^g(t) - \pi^{\hat{g}}(t)\\
&= e^{-r(T-t)}f(S_t, t) - e^{-r(T-t)}\left(\frac{S_t}{B}\right)^p f\left(\frac{B^2}{S_t}\right)\\
&= e^{-r(T-t)}\left(f(S_t, t) - \left(\frac{S_t}{B}\right)^p f\left(\frac{B^2}{S_t}\right)\right).
\end{aligned}$$

By definition of $g(x)$ we have the simplification $g(x) = \nu(x) = h(x)$ for $x > B$ as $\mathbb{1}_{x>B} = 1$ yielding a payoff at expiry-T given by $g(S_T)$ for the h-claim if the barrier is not hit, i.e

$$\begin{aligned}
\pi^h(T) &= \pi^g(T) \\
&= e^{-r(T-T)} E_T^{\mathbb{Q}}(g(S_T)) \\
&= E_T^{\mathbb{Q}}(g(S_T)) \\
&= g(S_T).
\end{aligned}$$

Imagine now that S_t decreases as such that $S_t = B$ within the options lifespan which by definition would cause the h-claim to expire worthless and become void as the barrier is hit. We see this easily by substitution of $S_t = B$ into $\pi^h(t)$

$$\begin{aligned}
\pi^h(t) &= e^{-r(T-t)} \left(f(S_t, t) - \left(\frac{S_t}{B}\right)^p f\left(\frac{B^2}{S_t}, t\right)\right) \\
&= e^{-r(T-t)} \left(f(B, t) - \left(\frac{B}{B}\right)^p f\left(\frac{B^2}{B}, t\right)\right) \\
&= e^{-r(T-t)} \left(f(B,t) - f(B,t)\right) \\
&= e^{-r(T-t)} \left(f(S_t,t) - f(S_t,t)\right) \\
&= e^{r(T-t)} \cdot 0 \\
&= 0.
\end{aligned}$$

We have now shown the h-claim posses two properties depending on the barrier B:

- If $S_t > B$ until and including expiry T, then the h-claim expires with payoff $g(S_T)$.
- If $S_t = B$, then the h-claim has a value of 0.

This is exactly the same as the barrier option which means that

$$\pi^h(t) = e^{-r(T-t)} \left(f(S_t, t) - \left(\frac{S_t}{B}\right)^p f\left(\frac{B^2}{S_t}\right)\right),$$

is the arbitrage-free value of the barrier option at time-t which is the arbitrage-free value of the h-claim at time-t

However, this result is meaningless if the g-claim itself has not been valued but we are able to employ usual put- and call-option formulas from Björk (2020).

A Appendix: Code

GitHub profile with the code in both R and Python for practice. Updated and uploaded February 28, 2024.

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