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KU-MATH

FinKont2: Hand-In Exercise #2

Answers (individually composed – but feel free to discuss in groups) must be handed in via Absalon no later than 23:59 CET on Tuesday March 19, 2024. There are 10 equal-weighted (number.letter)-named questions.

Caplets Hedged

1.a
Read section 26.8 in the 3rd edition of Björk's book ('red Björk'). (For some reason this quite useful section seems to have dropped out of the 4th edition of the book, 'green Björk'.) Show that pricing caplets is equivalent to pricing put options (and hence call options) on zero-coupon bonds – irrespective of which dynamic model is used.

1.b
Derive a self-financing, replicating strategy for a call option on a zero-coupon bond in the Hull-White model. Hint: In the notation of green Björk's Proposition 21.10, the T_1 and T_2 zero-coupon bonds are natural hedge instruments. Look at equation (21.55) and compare it to the Black-Scholes-formula to make a *qualified guess* of what the positions in the two zero-coupon bonds should be at any given time. Finally, verify carefully; one or two applications of Euler's homogeneous function theorem may save you a lot of calculations.

Options on Coupon-Bearing Bonds

The trick in the exercise was first used in Jamshidian (1989), "An exact bond option pricing formula", *Journal of Finance*, vol. 44(1), pp 205-209. (Warning: Papers by Jamshidian are notoriuosly difficult to read.)

We consider the Vasicek model, and let A and B denote the functions such that $P(t,T) = \exp(A(t,T) - B(t,T)r(t))$. We now look at a *coupon bond* that makes deterministic positive payments $\alpha_1, \ldots, \alpha_N$ at dates $T_1, \ldots, T_N$. Clearly the price of this coupon bond is

$$\pi^C(t) = \sum_{i|T_i>t} \alpha_i P(t, T_i).$$

(It is strict inequality, ">", to keep in line with prices always being ex-dividend.) The last ingredient we need is a (positive) strike-K, expiry-T European call-option on the coupon bond.

2.a
Show that there exists a unique $r^* \in \mathbb{R}$ such that $\pi^C(T) \geq K$ if and only if $r(T) \leq r^*$.

Define the *adjusted strikes* via $K_i = \exp(A(T, T_i) - B(T, T_i)r^*)$.

2.b
Show that the pay-off of the call can be written as

$$(\pi^C(T) - K)^+ = \sum_{i|T_i > T} \alpha_i (P(T, T_i) - K_i)^+,$$

and explain how (given results known from for instance Björk) this leads to a closed-form (up to knowledge of r^*) expression for the price of the call on the coupon bond.

2.c
Suppose the current short rate is $r_0 = 0.02$, and that the (Q-)parameters (in our " drift $= \kappa(\theta^Q - x)$"-notation) of the Vasicek model are $\theta^Q = 0.05$, $\kappa = 0.1$, and $\sigma = 0.015$. Consider an option with $K = 4.5$, $T = 1$, $N = 5$, $T_i = i + 1$, and $\alpha_i = 1$ for all i. Calculate the time-0 price of the call. (Numbers, please; this will involve solving an equation numerically.)

Linking local and implied volatility with Duprie

We consider a 0-dividend stock in a 0-interest rate world. By $\sigma(S, t)$ we denote the local volatility function (so $dS_t = \sigma(S_t, t) S_t dW_t^Q$). By $\widehat{\sigma}(K, T)$ we denote the time 0 (Black-Scholes) implied volatility of a strike K, expiry T call option. We suppress arguments where it causes no confusion/ambiguity.

3.a
Verify Dupire's formula/equation (i.e. equation (1.6) in Gatheral's book, equation (*5) in this column applied to a local volatility model) for

(i) the Black-Scholes model,

(ii) the Bachelier model.

This means: Calculate the left-hand sides (easy) and the right-hand sides (requires some of formulas for Greeks; feel free to look these up somewhere – but do give a reference; the Breeden-Litzenberger formula (3rd displayed equation on page 10 in Gatheral) stating that the second derivative of the call price wrt. strike is the risk-neutral density of S_T is also handy) and see that they match.

3.b
On page 46 in these slides from a previous version of FinKont2, Antoine Savine says that we can rewrite the right-hand side of the Dupire equation in terms of (Black-Scholes) implied volatility (with subscripts denoting partial derivatives):

> •As a direct function of implied volatilities, we prove by differentiation of the Black-Scholes formula that:
>
> ➤ Since $C(K,T) = BS(K,T,\hat{\sigma}(K,T))$
>
> ➤ We find C_T and C_{KK} by differentiation of Black-Scholes and, with zero rates and dividends, Dupire's formula becomes:
>
> $$\sigma^2 = \frac{\hat{\sigma}^2 + 2T\hat{\sigma}\hat{\sigma}_T}{\left(1 + Kd_1\hat{\sigma}_K\sqrt{T}\right)^2 + \hat{\sigma}K^2T\left(\hat{\sigma}_{KK} - d_1\hat{\sigma}_K^{\,2}\sqrt{T}\right)}$$

Verify this. (Use the chain rule and more formulas and relations for Greeks – plus some tedious algebra. Gatheral does the same on page 11-13 in different notation – arguably algebraically smarter, but I think less intuitive.)

3.c
Read on page 19-21 in these slides by Antoine Savine about the so-called Dupire zero-sigma formula. (Gatheral does something similar on page 26-31. And I do it on page 7-8 here.) It is a seemingly complicated re-writing af the fundamental theorem of derivative trading in the case of a local volatility model. But there is method to the madness: Explain how Antoine arrives at the (crude) approximation

$$\widehat{\sigma}(K) \approx \sqrt{\frac{\sigma^2(S_0) + \sigma^2(K)}{2}}.$$

Show that under this approximation, the at-the-money (so $K = S_0$) implied volatility is the same as local volatility (at S_0), while the slope of the implied volatility curve at-the-money is half of the slope of the local volatility function (i.e. formulas 1. and 2. in the red box on Antoine's page 21). Explain how this leads to a(n even cruder) linear approximation to implied volatility in terms of local volatility.

3.d
Explain how (in the first slide set referenced; the one from 3.b) Antoine derives the approximation

$$\widehat{\sigma}(K) \approx \frac{\ln(S_0/K)}{\int_K^{S_0} \frac{1}{u\sigma(u)} du}.$$

(The thing in the denominator – viewed as s function of S_0 – is called the Lamperti transform and has a tendency to show up. If we have a diffusion process and apply its Lamperti transform to it, we get by the Ito formula a new process that simply has a 1 in front dW in its dynamics – which can be convenient.)

2.e

Investigate numerically how the various approximations perform in the Bachelier model. (Use the naive approximation $\widehat{\sigma}(K,T) = \sigma(K,T)$ as benchmark.)