Hand-In 2

Continuous Time Finance 2 (Finkont2)

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March 19, 2024

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Note

The symbols Φ and ϕ represent the cumulative distribution function and the probability density function of the standard normal distribution, respectively.

1 Caplets Hedged

1.a

We demonstrate that the valuation of caplets is tantamount to pricing put options—and, by extension, call options—on zero-coupon bonds, irrespective of the chosen model. A caplet, being a type of interest rate cap option, endows its holder the right, albeit not the obligation, to receive a payment should the strike rate surpass a pre-established threshold known as the "cap rate." Acquiring caplets serves as a hedge against the risk of rising interest rates, ensuring that the holder never pays more than a predetermined rate on a loan subject to a floating interest rate. The interval $\mathbb{T} = [0, T]$ is subdivided into equidistant points $0 = T_0, T_1, \ldots, T_n = T$. Following the notation in Björk (2009), we simplify without loss of generality by setting the principle amount of money $K = 1$ and defining the duration of a basic time interval as $\delta = T_i - T_{i-1}$, with the cap rate represented by R. We proceed under the assumption that the floating interest rate, which underpins the cap, corresponds to the LIBOR spot rate $L(T_{i-1}, T_i)$ for the interval $[T_{i-1}, T_i]$ and cap rate R.

By Björk (2009) section 26.8 equation (26.62) caplet i can be written as a contingent claim, paid at time T_i by

$$\mathcal{X}_i = \delta(L - R)^+, \tag{1.1}$$

where the LIBOR spot rate from $[T_{i-1}, T_i]$ is given as

$$L(T_{i-1}, T_i) = -\frac{p(T_{i-1}, T_i) - 1}{\delta p(T_{i-1}, T_i)}, \tag{1.2}$$

by definition 19.2 in Björk (2020). Substituting (1.2) into (1.1) we achieve the expression

$$\begin{aligned}
\mathcal{X}_i &= \delta(L - R)^+ \\
&= \delta\left(-\frac{p(T_{i-1}, T_i) - 1}{\delta p(T_{i-1}, T_i)} - R\right)^+ \\
&= \delta\left(\frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} - R\right)^+ \\
&= \left(\frac{1}{p(T_{i-1}, T_i)} - (1 + \delta R)\right)^+ \\
&= \frac{1 + \delta R}{p(T_{i-1}, T_i)}\left(\frac{1}{1 + \delta R} - p(T_{i-1}, T_i)\right)^+ \\
&= \frac{R^\star}{p(T_{i-1}, T_i)}\left(\frac{1}{R^\star} - p(T_{i-1}, T_i)\right)^+ \qquad (R^\star = 1 + \delta R).
\end{aligned}$$

We then apply multiplication by $p(T_{i-1}, T_i)$ to determine the T_{i-1}-price.

$$p(T_{i-1}, T_i)\frac{R^\star}{p(T_{i-1}, T_i)}\left(\frac{1}{R^\star} - p(T_{i-1}, T_i)\right)^+ = R^\star\left(\frac{1}{R^\star} - p(T_{i-1}, T_i)\right). \tag{1.3}$$

From proposition 15.12 in Björk (2020) we now have the price of some T-claim $\mathcal{X}$ - specifically a caplet in this case - at time T_i is given by

$$\begin{aligned}
\Pi_{T_i}(\mathcal{X}_i) &= p(T_{i-1}, T_i) E^{\mathbb{Q}^{T_i}} (\mathcal{X}_i \mid \mathcal{F}_{T_i}) \\
&= p(T_{i-1}, T_i) E^{\mathbb{Q}^{T_i}} \left(\underbrace{\frac{R^\star}{p(T_{i-1}, T_i)} \left(\frac{1}{R^\star} - p(T_{i-1}, T_i) \right)^+}_{\mathcal{F}_{T_i}-\text{measurable}} \mid \mathcal{F}_{T_i} \right) \\
&= p(T_{i-1}, T_i) \frac{R^\star}{p(T_{i-1}, T_i)} \left(\frac{1}{R^\star} - p(T_{i-1}, T_i) \right)^+ \\
&= R^\star \left(\frac{1}{R^\star} - p(T_{i-1}, T_i) \right)^+,
\end{aligned}$$

where $E^{\mathbb{Q}^T}$ is the expectation under the T-forward measure $\mathbb{Q}^T$ for a fixed T which is defined as the martingale measure for the numeraire process $p(t, T)$.

Utilizing the Law of Total Expectation, combined with a subsequent application of Proposition 15.12, we can now calculate the price of a caplet for any $t < T_{i-1}$

$$\begin{aligned}
\Pi_t(\mathcal{X}_i) &= p(t, T_i) E^{\mathbb{Q}^{T_i}} (\mathcal{X}_i \mid \mathcal{F}_t) \\
&= p(t, T_i) E^{\mathbb{Q}^{T_i}} (E^{\mathbb{Q}^{T_i}} (\mathcal{X}_i \mid \mathcal{F}_{T_{i-1}}) \mid \mathcal{F}_t) \\
&= p(t, T_i) E^{\mathbb{Q}^{T_i}} \left(R^\star \left(\frac{1}{R^\star} - p(T_{i-1}, T_i) \right)^+ \mid \mathcal{F}_t \right) \\
&= R^\star p(t, T_i) E^{\mathbb{Q}^{T_i}} \left(\left(\frac{1}{R^\star} - p(T_{i-1}, T_i) \right)^+ \mid \mathcal{F}_t \right).
\end{aligned}$$

Thus, it is observed that a caplet can be equated to $R^\star$ put options on an underlying T_i-bond, where the option's exercise date is set at T_{i-1}, and the exercise price is $\frac{1}{R^\star}$. Therefore, a complete cap contract is effectively conceptualized as a collection of put options (and hence call options by parity).

It should be highlighted that if (1.3) is not satisfied, arbitrage opportunities emerge. This is because an investor could allocate their entire capital to a bank deposit or allocate all funds to purchasing zero-coupon bonds at time T_{i-1}, resulting in a free lunch and no one should eat for free...

1.b

In order to develop a self-financing, replicating strategy for a call option on a zero-coupon bond within the Hull-White model, it's imperative to first identify the hedging instrument capable of mimicking the call option's payoff. As suggested in the hint, two zero-coupon bonds with maturities at T_1 and T_2 emerge as the logical selections. Constructing a replicating strategy necessitates determining the appropriate allocation ratios for these hedging instruments at any point in time. This can be achieved by drawing parallels between the call option pricing formula within the Hull-White model and the formula used in the Black-Scholes model. Such a comparison enables an educated estimation (guess) of the positions in the two zero-coupon bonds.

In the Hull-White model, the pricing of a call option is detailed in proposition 21.10 Björk (2020). However, we adjust the formulas to account for $t \leq T_1$ as opposed to $t = 0$ which is possible because of time homogeneity

$$
\begin{aligned}
c(t, T_1, T_2, K) &= p(t, T_2)\Phi(d_1) - Kp(t, T_2)\Phi(d_2), \\
d_2 &= \frac{\log\left(\frac{p(t,T_2)}{Kp(t,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \\
d_1 &= d_2 + \sqrt{\Sigma^2}, \\
\Sigma^2 &= \frac{\sigma^2}{2a^3}\left(1 - e^{-2a(T_1 - t)}\right)\left(1 - e^{-a(T_2 - T_1)}\right)^2.
\end{aligned}
$$

We use the usual Black-Scholes formulas given in proposition 7.13 Björk (2020), namely (with superscript "BS" to differ from the Hull-White equations):

$$
\begin{aligned}
Call^{\text{BS}}(t, S_t, K, T) &= s\Phi(d_1^{\text{BS}}) - e^{-r(T-t)}K\Phi(d_2^{\text{BS}}), \\
d_1^{\text{BS}} &= \frac{1}{\sigma_{\text{BS}}\sqrt{T-t}}\left(\log\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma_{\text{BS}}^2\right)(T-t)\right), \\
d_2^{\text{BS}} &= d_1^{\text{BS}} - \sigma_{\text{BS}}\sqrt{T-t}.
\end{aligned}
$$

It is quite clear that there is a big resemblance between ZCB's in the Hull-White model and stock- and strike-prices in the Black-Scholes model.

We therefore make a qualified guess as hinted that the position in the T_1-ZCB should be proportional to $-K\Phi(d_2)$ and the position in the T_2-ZCB proportional to $\Phi(d_1)$

We can verify this guess by applying Euler's homogeneous function theorem Sydsæter and Hammond (2005), p. 432.

First, we see that $c(t)$ is homogeneous in the T_1-ZCB $p(t, T_1)$ and the T_2-ZCB $p(t, T_2)$ of degree 1 by

$$
\begin{aligned}
\lambda c(\lambda p(t, T_1, \lambda p(t, T_1)) &= \lambda p(t, T_2)\Phi(d_1) - \lambda Kp(t, T_1)\Phi(d_2) \\
&= \lambda c(p(t, T_1, p(t, T_1)).
\end{aligned}
$$

A application of Euler's homogeneous function theorem to $c(t)$ yields

$$c(t) = p(t,T_2)\frac{\partial c(t)}{\partial p(t,T_2)} + p(t,T_1)\frac{\partial c(t)}{\partial p(t,T_1)}.$$

Substituting in the definition of $c(t)$ yields

$$p(t,T_2)\Phi(d_1) - Kp(t,T_2)\Phi(d_2) = p(t,T_2)\frac{\partial c(t)}{\partial p(t,T_2)} + p(t,T_1)\frac{\partial c(t)}{\partial p(t,T_1)},$$

implying that

$$\begin{cases} p(t,T_2)\Phi(d_1) = p(t,T_2)\frac{\partial c(t)}{\partial p(t,T_2)} \\ -Kp(t,T_2)\Phi(d_2) = p(t,T_1)\frac{\partial c(t)}{\partial p(t,T_1)} \end{cases}$$
$$\Longleftrightarrow$$
$$\begin{cases} \Phi(d_1) = \frac{\partial c(t)}{\partial p(t,T_2)} \\ -K\Phi(d_2) = \frac{\partial c(t)}{\partial p(t,T_1)} \end{cases}.$$

The position in the two ZCB's is thus: buying $\Phi(d_1)$-ZCB's with maturity T_2 whilst shorting $K\Phi(d_2)$-ZCB's with maturity T_1. Bearing this in mind, we are able to construct a self-financing replicating portfolio as

$$h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} \underbrace{-K\Phi(d_2)}_{\text{units ZCB's with maturity } T_1} \\ \underbrace{\Phi(d_1)}_{\text{units ZCB's with maturity } T_2} \end{pmatrix}.$$

By definition, the value process of $h(t)$ is

$$\begin{aligned} V^h(t) &= h_1(t)p(t,T_1) + h_2(t)p(t,T_2) \\ &= p(t,T_2)\Phi(d_1) - p(t,T_1)K\Phi(d_2) \\ &:= c(t), \end{aligned} \tag{1.4}$$

which is the call-option price formula in the Hull-White model. This means that our portfolio does give the same pay-off as the call-option in the Hull-White model. What is left is to show the self-financing condition (definition 6.10 Björk (2020)) by examining the dynamics of our value process, i.e

$$\begin{aligned} dV^h(t) &= h_1(t)dp(t,T_1) + h_2(t)dp(t,t_2) \\ &= dp(t,T_2)\Phi(d_1) - dp(t,T_1)K\Phi(d_2). \end{aligned} \tag{1.5}$$

By Proposition 21.3 in Björk (2020) and the affine term structure in Hull-White the bond prices are given by

$$p(t,T_i) = e^{A(t,T_i) - B(t,T_i)r_t}, \quad i \in \{1,2\}.$$

By Ito's formula proposition 4.12 we see that

$$dp(t,T_i) = -B(t,T_i)p(t,T_i)dr_t + \frac{1}{2}B^2(t,T_i)p(t,T_i)(dr_t)^2, \tag{1.6}$$

and from Björk (2020) equation (21.9) that the $\mathbb{Q}$-dynamics of r_t in Hull–White (extended Vasicek) are given as

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t^{\mathbb{Q}}, \quad a(t) > 0. \tag{1.7}$$

Substituting equation (1.7) into equation (1.6) and using the multiplication table to find $(dr_t)^2 = ((\theta_t - ar_t)dt + \sigma dW_t^{\mathbb{Q}})^2 = \sigma^2 dt$, we derive the $\mathbb{Q}$-dynamics of $p(t,T_i)$ as

$$\begin{aligned} dp(t,T_i) &= -B(t,T_i)p(t,T_i)((\theta_t - ar_t)dt + \sigma dW_t^{\mathbb{Q}}) \\ &\quad + \frac{1}{2}B^2(t,T_i)p(t,T_i)((\theta_t - ar_t)dt + \sigma dW_t^{\mathbb{Q}})^2 \\ &= -B(t,T_i)p(t,T_i)((\theta_t - ar_t)dt + \sigma dW_t^{\mathbb{Q}}) \\ &\quad + \frac{1}{2}B^2(t,T_i)p(t,T_i)\sigma^2 dt \\ &\underbrace{=}_{*} \left(\frac{1}{2}B^2(t,T_i)\sigma^2 dt - B(t,T_i)(\theta_t - ar_t)\right)p(t,T_i)dt \\ &\quad - \sigma B(t,T_i)p(t,T_i)dW_t^{\mathbb{Q}} \\ &= r_t p(t,T_i)dt - \sigma B(t,T_i)p(t,T_i)dW_t^{\mathbb{Q}}. \end{aligned} \tag{1.8}$$

where $*$ follows from the fact that the local rate of return is equal to the short rate under the risk neutral (martingale) measure $\mathbb{Q}$ Björk (2020), p. 299. Substituting equation (1.8) into equation (1.5) yields

$$\begin{aligned} dV^h(t) &= \Phi(d_1)(r_t p(t,T_2)dt - \sigma B(t,T_2)p(t,T_2)dW_t^{\mathbb{Q}}) \\ &\quad - (r_t p(t,T_1)dt - \sigma B(t,T_1)p(t,T_1)dW_t^{\mathbb{Q}})K\Phi(d_2) \\ &= r_t \left(\underbrace{p(t,T_2)\Phi(d_1) - p(t,T_1)K\Phi(d_2)}_{:=c(t)}\right)dt \\ &\quad + (K\Phi(d_2)\sigma B(t,T_1)p(t,T_1) - \Phi(d_1)\sigma B(t,T_2)p(t,T_2))dW_t^{\mathbb{Q}} \\ &= r_t c(t)dt + (K\Phi(d_2)\sigma B(t,T_1)p(t,T_1) - \Phi(d_1)\sigma B(t,T_2)p(t,T_2))dW_t^{\mathbb{Q}}. \end{aligned}$$

Consider again the self-financing condition given in (1.5) and the equation (1.4) which stated $V^h(t) = c(t)$. Now, hedging works under any probability measure (and thus for any drift rate: Poulsen (2024), p. 25), which means we need to examine the stochastic diffusion of the price process, i.e., the $dW_t^{\mathbb{Q}}$-term as hedging should work along every trajectory. In other words, hedging works under any drift rate (as the local rate of return must be r_t under $\mathbb{Q}$ Björk (2020), p. 299) which implies we are only interested in the equality of diffusion terms.

Therefore, consider

$$\begin{aligned} dc(t) &= [\ldots]dt + \frac{\partial c(t)}{\partial p(t,T_1)}dp(t,T_1) + \frac{\partial c(t)}{\partial p(t,T_2)}dp(t,T_2) \\ &= [\ldots]dt + (-K\Phi(d_2))(r_t p(t,T_1)dt - \sigma B(t,T_1)p(t,T_1)dW_t^{\mathbb{Q}}) \\ &\quad + (\Phi(d_1))(r_t p(t,T_2)dt - \sigma B(t,T_2)p(t,T_2)dW_t^{\mathbb{Q}}) \\ &= r_t c(t)dt + (K\Phi(d_2)\sigma B(t,T_1)p(t,T_1) - \Phi(d_1)\sigma B(t,T_2)p(t,T_2))dW_t^{\mathbb{Q}}. \end{aligned}$$

It is now seen (specifically) that the diffusion term of $dc(t)$ matches that of $dV^h(t)$ exactly.

This means that the portfolio h is now shown to be self-financing and replicating strategy for the call option on a ZCB in the Hull-White model as both conditions are fulfilled. This strategy is to, at any time $t \leq T_1$, hold $h_2(t)$ units of a zero coupon bond maturing at time T_2 and $h_1(t)$ units of a zero coupon bond maturing at time T_1

$$h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} \underbrace{-K\Phi(d_2)}_{\text{units ZCB's with maturity } T_1} \\ \underbrace{\Phi(d_1)}_{\text{units ZCB's with maturity } T_2} \end{pmatrix}.$$

2 Options on Coupon-Bearing Bonds

2.a

We use some intermediate observations to prove the bi-implication. From proposition 21.4 (The Vasicek term structure) Björk (2020) we have that

$$P(t,T) = e^{A(t,T)-B(t,T)r_t}.$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right),$$

and

$$A(t,T) = \frac{(B(t,T)-T+t)\left(ab - \frac{1}{2}\sigma^2\right)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a},$$

where $B(t,T) > 0$ and $a > 0$. Now, consider a price on a coupon bond in the beforementioned Vasicek model given by

$$\begin{aligned} \pi^C(T) &= \sum_{i|T_i>T} \alpha_i P(T,T_i) \\ &= \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r_T}, \end{aligned}$$

where $\alpha_i > 0$ for $i \in 1,\dots,N$. This means that the mapping

$$\mathbb{R} \ni r_T \mapsto \alpha_i e^{A(T,T_i)-B(T,T_i)r_T} \in [0,\infty),$$

is a sequence of monotone decreasing functions for $i \in 1,\dots,N$ as $B(T,T_i) > 0$ and $\alpha_i > 0$.

Consequently

$$\mathbb{R} \ni r_T \mapsto \pi^C(T) = \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r_T},$$

where $\alpha_i > 0$ for $i \in 1,\dots,N$ is a monotone decreasing function, again, because $B(T,T_i) > 0$.

$\Rightarrow$: We observed that $r_T \mapsto \pi^C(T)$ is a monotone function, implying that for all strikes-K there is a unique $r^* \in \mathbb{R}$ such that $\pi^C(T) = K$, by the simple property of monotonicity.

$\Leftarrow$: The coupon bond price is

$$\begin{aligned} \pi^C(T) &= \sum_{i|T_i>T} \alpha_i P(T, T_i) \\ &= \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r_T} \\ &\underbrace{\geq}_{*} \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r^*} \\ &= K, \end{aligned}$$

where $*$ is partly by assumption, that is $r_T \leq r^*$ and that the mapping $r_T \mapsto \pi^C(T)$ is monotone decreasing.

As both directions of the bi-implication have been shown we can conclude that there exists a unique r^* such that $\pi^C(T) \geq K$ if and only if $r(T) \leq r^*$

2.b

Define

$$K_i = e^{A(T,T_i)-B(T,T_i)r^*},$$

where r^* - from 2.a - is the unique solution to the equation

$$\begin{aligned} K &= \sum_{i|T_i>T} \alpha_i K_i \\ &= \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r^*} \\ &= \sum_{i|T_i>T} \alpha_i P_{r^*}(T,T_i), \end{aligned}$$

where $P_{r^*}(T,T_i) = e^{A(T,T_i)-B(T,T_i)r^*}$. The pay-off of the call-option on coupon bonds using the result from 2.a - namely - that $\pi^C(T) \geq K$ if and only if $r(T) \leq r^*$ combined with proposition 21.4 (The Vasicek term structure) Björk (2020) yields

$$\begin{aligned} (\pi^C(T) - K)^+ &= \left(\sum_{i|T_i>T} \alpha_i P(T,T_i) - \sum_{i|T_i>T} \alpha_i P_{r^*}(T,T_i) \right)^+ \\ &= \left(\sum_{i|T_i>T} \alpha_i \left(P(T,T_i) - P_{r^*}(T,T_i)\right) \right)^+ \\ &\underbrace{=}_{2.a} \sum_{i|T_i>T} \alpha_i \left(P(T,T_i) - P_{r^*}(T,T_i)\right) \mathbb{1}_{r_T \leq r^*} \\ &= \sum_{i|T_i>T} \alpha_i \left(P(T,T_i) - P_{r^*}(T,T_i)\right)^+ \\ &= \sum_{i|T_i>T} \alpha_i \left(P(T,T_i) - \underbrace{e^{A(T,T_i)-B(T,T_i)r^*}}_{:=K_i} \right)^+ \\ &= \sum_{i|T_i>T} \alpha_i \left(P(T,T_i) - K_i\right)^+ . \end{aligned} \tag{2.1}$$

We now argue how (2.1) can be used to derive a closed form expression expression for the call price using results from Björk (2020).

Let $E^{\mathbb{Q}^T}$ be the expectation under the T-forward measure $\mathbb{Q}^T$ for a fixed T which is defined as the martingale measure for the numeraire process $P(t,T)$.

This leads to a closed-form (up to knowledge of r^*) expression for the price of the call on the coupon bond by equation (2.1) and proposition 15.12 Björk (2020)

$$\begin{aligned}
\Pi_t(X) &= Call^C(t) \\
&= P(t,T)E^{\mathbb{Q}^T}(X \mid \mathcal{F}_t) \\
&= P(t,T)E^{\mathbb{Q}^T}\left((\pi^C(T)-K)^+ \mid \mathcal{F}_t\right) \\
&= P(t,T)E^{\mathbb{Q}^T}\left(\left(\sum_{i|T_i>T}\alpha_i P(T,T_i)-K\right)^+ \mid \mathcal{F}_t\right) \\
&= P(t,T)E^{\mathbb{Q}^T}\left(\sum_{i|T_i>T}\alpha_i\left(P(T,T_i)-K_i\right)^+ \mid \mathcal{F}_t\right) \\
&\underbrace{=}_{*} \sum_{i|T_i>T}\alpha_i \underbrace{P(t,T)E^{\mathbb{Q}^T}\left((P(T,T_i)-K_i)^+ \mid \mathcal{F}_t\right)}_{:=Call(t)} \\
&\underbrace{=}_{**} \sum_{i|T_i>T}\alpha_i Call(t,K_i,T,T_i),
\end{aligned}$$

where $*$ follows from the property of linearity of the expectation and the fact that α_i is a constant for all $i \in 1,\dots,N$. $**$ follows from the fact that the underbraced expression in the former equation is exactly the Vasicek price of a call-option - on the underlying maturity-T_i ZCB's with adjusted strikes K_i, and expiry $T < T_i$ - at time t. In other words: we have now shown that a call-option on a portfolio of maturity-T_i ZCB's as the underlying is exactly equivalent to a portfolio of call-options with adjusted strikes, K_i and scaled with deterministic positive payments, $\alpha_i > 0$.

2.c

All the code can be viewed in Appendix A. The excercise builds on the just derived result from 2.b

$$Call^C(0) = \sum_{i|T>T_i} \alpha_i Call(0, K_i, T_i, T). \tag{2.2}$$

Equation (2.2) states that the call price of a European expiry-T call-option with strike-K can be calculated from a portfolio (:the sum) of Vasicek call prices of an option of maturity-T_i ZCB's as the underlying asset with adjusted strikes K_i and expiry $T < T_i$ multiplied by the deterministic payments, $\alpha_i > 0$. We are told that $\alpha_i = 1$ for all i which simplifies to just summing over the Vasicek ZCB calls to find the time-0 price of the call. The calculations will be done as such:

Description	**Symbol**	**Value**
Short rate	r_0	0.02
Long-term mean	$\theta^{\mathbb{Q}}$	0.05
Speed of reversion	κ	0.1
Volatility	σ	0.015
Strike Price	K	4.5
Time to Maturity	T	1
Number of periods	N	5
Periods	T_i	$i+1$
Payments	α_i	1

Table 1: Parameters of the Vasicek model and option details.

Firstly, we find the critical value, r^*. This will be done by numerical rootsearch using the `uniroot`-function in `R` for the expression

$$\begin{aligned}
\pi^C(T) &= \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r^*} \\
&\underbrace{=}_{\alpha_i=1} \sum_{i|T_i>T} e^{A(T,T_i)-B(T,T_i)r^*} \\
&= K \\
&= 4.5,
\end{aligned}$$

which yields a value of

$$r^* = 0.03329596.$$

Variations may arise from choosing different parameters for the numeric search algorithm.

Secondly, we find the adjusted strikes, K_i, by solving the equation

$$\begin{aligned} K_i &= P_{r^*}(T, T_i) \\ &= e^{A(T,T_i) - B(T,T_i) r^*}, \end{aligned}$$

which yields results seen in Table 2.

T_i	$T_i = 1$	$T_i = 2$	$T_i = 3$	$T_i = 4$	$T_i = 5$
K_i	0.9665046	0.9328958	0.8995211	0.8666389	0.8344366

Table 2: Values of K_i for different values of T_i.

Notice that

$$K = \sum_{i|T>T_i} \alpha_i K_i = 4.499997 \approx 4.5.$$

The adjusted strikes-K_i do approximately equal the strike-K. The `uniroot` function in `R` finds a root (zero) of a function within a specified interval to a certain tolerance. This means that the value of r^* we achieve is an approximation within the tolerance limits of the algorithm, not an exact mathematical solution. This approximation can introduce a small error in subsequent calculations and thus the minor deviation from $K = 4.5$.

Having determined K_i and r^*, we can now calculate the Vasicek call option prices for each i, as presented in Table 3.

i	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$Call(0, K_i, T_i, T)$	0.01133672	0.02105457	0.02929436	0.03620085	0.04191637

Table 3: Values of $Call(0, K_i, T_i, T)$ for different values of i.
.

The values in Table 3 enables us to compute the price of a European call option on a portfolio of coupon-bearing bonds using `R` yielding the result

$$\begin{aligned} Call^C(0) &= \sum_{i|T>T_i} \alpha_i Call(0, K_i, T_i, T) \\ &= \sum_{i|T>T_i} Call(0, K_i, T_i, T \\ &= 0.1398029. \end{aligned}$$

3 Linking local and implied volatility w/ Duprie

3.a

We consider a 0-dividend and 0-interest rate world and proceed to show equation (1.6) (Dupire's formula) in Gatheral (2011)

$$\sigma_{local}^2 = \frac{\frac{\partial Call}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 Call}{\partial K^2}}, \tag{3.1}$$

in the Black-Scholes model and the Bachelier model in that order. To further avoid ambiguity I will denote the time-0 Black-Scholes implied volatility of a strike-K, expiry-T call option as $\hat{\sigma}(K,T)$ (sometimes the shorthand $\hat{\sigma}$ and the local volatility as $\sigma_{local}(S,t)$ (sometimes the shorthand σ_{local}).

Black-Scholes: The RHS is shown first. From proposition 7.13 Björk (2020) we have the 0-interest and 0-dividend rate price of a European call-option at $t = 0$

$$\begin{aligned} Call^{\mathrm{BS}} &= s\Phi(d_1) - K\Phi(d_2), \\ d_1 &= \frac{\log\left(\frac{s}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

As we are in a 0-dividend and 0-interest rate world we have from equation (1.4) in Gatheral (2011) that

$$\frac{\partial Call^{\mathrm{BS}}}{\partial T} = \frac{\sigma^2 K^2}{2} \frac{\partial^2 Call^{\mathrm{BS}}}{\partial K^2}. \tag{3.2}$$

By substitution of (3.2) into (3.1) (Dupire's formula) we see that

$$\begin{aligned} \sigma_{local}^2 &= \frac{\frac{\partial Call^{\mathrm{BS}}}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 Call^{\mathrm{BS}}}{\partial K^2}} \\ &= \frac{\frac{\sigma^2 K^2}{2} \frac{\partial^2 Call^{\mathrm{BS}}}{\partial K^2}}{\frac{1}{2}K^2 \frac{\partial^2 Call^{\mathrm{BS}}}{\partial K^2}} \\ &= \sigma^2 \frac{K^2 \frac{\partial Call^{\mathrm{BS}}}{\partial K^2}}{2} \frac{2}{K^2 \frac{\partial Call^{\mathrm{BS}}}{\partial K^2}} \\ &= \sigma^2. \end{aligned}$$

We now show the LHS which follows simply by observing that σ^2 is constant and thus

$$\begin{aligned} \sigma_{local}^2 &= E^{\mathbb{Q}}(\sigma^2(K,T) \mid S_T = K) \\ &= \sigma^2. \end{aligned}$$

Bachelier: The RHS is shown first. We have the call-option price from HandIn1.2024 Q1.a for $t = 0$ and 0-interest rate and 0-dividend

$$Call^{\mathrm{Bach}} = (S(t) - K)\Phi\left(\frac{S(t) - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S(t) - K}{\sigma\sqrt{T}}\right).$$

The derivative of the call-price wrt. to the strike-K is found by the product- and chain-rule using the usual derivative rules for the standard normal cumulative distribution function, namely $\Phi'(x) = \phi(x) \cdot x'$ and $\phi'(x) = -x\phi(x) \cdot x'$

$$\begin{aligned}
\frac{\partial Call^{\text{Bach}}}{\partial K} &= -\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right) + \phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)(S(t)-K)\left(-\frac{1}{\sigma\sqrt{T}}\right) \\
&\quad + \sigma\sqrt{T}\left(-\frac{S(t)-K}{\sigma\sqrt{T}}\right)\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\left(-\frac{1}{\sigma\sqrt{T}}\right) \\
&= -\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right) + \left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right) - \left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right) \\
&= -\Phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\partial^2 Call^{\text{Bach}}}{\partial K^2} &= -\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\left(-\frac{1}{\sigma\sqrt{T}}\right) \\
&= \frac{1}{\sigma\sqrt{T}}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right).
\end{aligned}$$

The derivative of the call-price wrt. to the expiry T is now found in similar fashion

$$\begin{aligned}
\frac{\partial Call^{\text{Bach}}}{\partial T} &= (S(t)-K)\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\left(-\frac{S(t)-K}{2\sigma T^{3/2}}\right) + \frac{\sigma}{2\sqrt{T}}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right) \\
&\quad + \sigma\sqrt{T}\left(-\frac{S(t)-K}{\sigma\sqrt{T}}\right)\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\left(-\frac{S(t)-K}{2\sigma T^{3/2}}\right) \\
&= \phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\left((S(t)-K)\left(-\frac{S(t)-K}{2\sigma T^{3/2}}\right) + \frac{\sigma}{2\sqrt{T}} - \sigma\sqrt{T}\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\left(-\frac{S(t)-K}{2\sigma T^{3/2}}\right)\right) \\
&= \phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\frac{\sigma}{2\sqrt{T}}.
\end{aligned}$$

We then have by (3.1) (Dupire's formula) that

$$\begin{aligned}
\sigma_{local}^2 &= \frac{\frac{\partial Call^{\text{Bach}}}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 Call^{\text{Bach}}}{\partial K^2}} \\
&= \frac{\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\frac{\sigma}{2\sqrt{T}}}{\frac{1}{2}K^2\frac{1}{\sigma\sqrt{T}}\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)} \\
&= \phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)\frac{\sigma}{2\sqrt{T}}\left(\frac{2\sigma\sqrt{T}}{K^2\phi\left(\frac{S(t)-K}{\sigma\sqrt{T}}\right)}\right) \\
&= \frac{\sigma^2}{K}.
\end{aligned}$$

We now show the LHS which again follows simply by

$$\begin{aligned}
\sigma^2_{local} &= E^{\mathbb{Q}}(\sigma^2(K,T) \mid S_T = K) \\
&= E^{\mathbb{Q}}\left(\left(\frac{\sigma}{S_t}\right)^2 \mid S_T = K\right) \\
&= E^{\mathbb{Q}}\left(\left(\frac{\sigma}{K}\right)^2\right) \\
&= \frac{\sigma^2}{K}.
\end{aligned}$$

3.b

We now wish to express the local volatility in terms of the Black-Scholes Implied volatility, implying that we have to rewrite the derivatives from (3.1) (Dupire's formula) as these are function of the implied volatility.

Firstly, if we observe some call price in the market, $Call_0$, at $t = 0$ then the Black-Scholes implied volatility is the solution to

$$Call_0(S_0, K, T) = Call_0^{\mathrm{BS}}(S_0, \hat{\sigma}(K, T), K, T)$$

where as usual the price of the 0-interest rate and 0-dividend Black-call price is given by proposition 7.13 Björk (2020), i.e

$$\begin{aligned} Call_0(S_0, K, T) &= Call_0^{\mathrm{BS}}(S_0, \hat{\sigma}(K, T), T) \\ &= S_0\Phi(d_1) - K\Phi(d_2), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + \frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}}. \\ d_2 &= d_1 - \hat{\sigma}\sqrt{T} \\ &= \frac{\log\left(\frac{S_0}{K}\right) - \frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}}. \end{aligned}$$

We now find the partial derivatives of d_i, $i \in \{1, 2\}$. See first that

$$\begin{aligned} d_2 &= d_1 - \hat{\sigma}\sqrt{T} \\ &\Rightarrow \\ \frac{\partial d_2}{\partial T} &= \frac{\partial d_1}{\partial T} - \hat{\sigma}_T\sqrt{T} - \hat{\sigma}\frac{1}{2\sqrt{T}}, \end{aligned}$$

and

$$\frac{\partial d_2}{\partial K} = \frac{\partial d_1}{\partial K} - \hat{\sigma}_K\sqrt{T}.$$

We calculate the partial derivative of the call price with respect to maturity-T. We will use shorthands such as f_{ix} to denote the partial derivative of a function f_i wrt. a variable x.

Thus

$$\begin{aligned}
\frac{\partial Call_0}{\partial T} &= S_0\phi(d_1)d_{1T} - K\phi(d_2)d_{2T} \\
&\underbrace{=}_{*} S_0\phi(d_1)d_{1T} - \frac{S_0}{K}K\phi(d_1)d_{2T} \\
&= S_0\phi(d_1)d_{1T} - S_0\phi(d_1)d_{2T} \\
&= S_0\phi(d_1)\left(d_{1T} - d_{2T}\right) \\
&\underbrace{=}_{**} S_0\phi(d_1)\left(d_{1T} - \left(d_{1T} - \hat{\sigma}_T\sqrt{T} - \hat{\sigma}\frac{1}{2\sqrt{T}}\right)\right) \\
&= S_0\phi(d_1)\left(\hat{\sigma}\frac{1}{2\sqrt{T}} + \hat{\sigma}_T\sqrt{T}\right),
\end{aligned}$$

where $*$ follows from $\phi(d_2) = \frac{S_0}{K}\phi(d_1)$ which is simply from noticing the difference in exponents for $\phi(d_1)$ and $\phi(d_2)$ is exactly $\log(S_0/K)$. Now, taking the exponential yields the result. The equality $**$ follows from substitution of the partial derivative for d_{2T}.

We now find the partial derivative of the call price wrt. to the strike-K in the exact same manner

$$\begin{aligned}
\frac{\partial Call_0}{\partial K} &= S_0\phi(d_1)d_{1K} - \Phi(d_2) - K\phi(d_2)d_{2K} \\
&\underbrace{=}_{*} S_0\phi(d_1)d_{1K} - \Phi(d_2) - \frac{S_0}{K}K\phi(d_1)d_{2K} \\
&= S_0\phi(d_1)d_{1K} - \Phi(d_2) - S_0\phi(d_1)d_{2K} \\
&= S_0\phi(d_1)\left(d_{1K} - d_{2K}\right) - \Phi(d_2) \\
&\underbrace{=}_{**} S_0\phi(d_1)\left(d_{1K} - \left(d_{1K} - \hat{\sigma}_K\sqrt{T}\right)\right) - \Phi(d_2) \\
&= S_0\phi(d_1)\hat{\sigma}_K\sqrt{T} - \Phi(d_2)
\end{aligned}$$

where $*$ follows from $\phi(d_2) = \frac{S_0}{K}\phi(d_1)$ and $**$ from substitution of the partial derivative for d_{2K}.

Proceeding, we use the chain-rule combined with the fact that $\frac{\partial \phi(d_1)}{\partial K} = -d_1 \phi(d_1) \frac{\partial \phi(d_1)}{\partial K}$ yielding the partial derivative of d_1 wrt. the strike-K

$$
\begin{aligned}
d_{1K} &= \frac{\hat{\sigma}\sqrt{T}\left(\frac{K}{S_0}\left(\frac{-S_0}{K^2}\right) + \frac{2\hat{\sigma}}{2}T\hat{\sigma}_K\right) - \hat{\sigma}_K\sqrt{T}\left(\log\left(\frac{S_0}{K}\right) + \frac{1}{2}\hat{\sigma}^2 T\right)}{\hat{\sigma}^2 T} \\
&= \frac{-\frac{1}{K}\hat{\sigma}\sqrt{T} + \hat{\sigma}_K\hat{\sigma}^2 T\sqrt{T} - \hat{\sigma}_K\sqrt{T}\log\left(\frac{S_0}{K}\right) - \hat{\sigma}_K T\sqrt{T}\frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}^2 T} \\
&= \frac{-\frac{1}{K}\hat{\sigma}\sqrt{T} - \hat{\sigma}_K\sqrt{T}\log\left(\frac{S_0}{K}\right)}{\hat{\sigma}^2 T} + \frac{\frac{1}{2}\hat{\sigma}^2\hat{\sigma}_K T\sqrt{T}}{\hat{\sigma}^2 T} \\
&= \frac{-\frac{1}{K}\hat{\sigma}\sqrt{T} - \hat{\sigma}_K\sqrt{T}\log\left(\frac{S_0}{K}\right)}{\hat{\sigma}^2 T} + \frac{\hat{\sigma}_K\sqrt{T}}{2} \\
&= -\frac{1}{K\hat{\sigma}\sqrt{T}} - \frac{\hat{\sigma}_K\sqrt{T}\log\left(\frac{S_0}{K}\right)}{\hat{\sigma}^2 T} + \frac{\hat{\sigma}_K\sqrt{T}}{2} \\
&= \frac{1}{\hat{\sigma}\sqrt{T}}\left(-\frac{1}{K} - \frac{\hat{\sigma}_K\sqrt{T}\left(\log\left(\frac{S_0}{K}\right) - \frac{1}{2}\hat{\sigma}^2 T\right)}{\hat{\sigma}\sqrt{T}}\right) \\
&\underbrace{=}_{*} \frac{1}{\hat{\sigma}\sqrt{T}}\left(-\frac{1}{K} - d_2\sqrt{T}\hat{\sigma}_K\right) \\
&= \frac{1}{\hat{\sigma}\sqrt{T}}\left(-\frac{1}{K} - (d_1 - \hat{\sigma}\sqrt{T})\sqrt{T}\hat{\sigma}_K\right) \\
&= \frac{1}{\hat{\sigma}\sqrt{T}}\left(-\frac{1}{K} - d_1\hat{\sigma}_K\sqrt{T} + \hat{\sigma}_K\hat{\sigma}T\right),
\end{aligned}
$$

where $*$ follows from $d_2\sqrt{T}\hat{\sigma}_K = \frac{\hat{\sigma}_K\sqrt{T}\left(\log\left(\frac{S_0}{K}\right) - \frac{1}{2}\hat{\sigma}^2 T\right)}{\hat{\sigma}\sqrt{T}}$.

Lastly, we find the second partial derivative wrt. to the strike of the call price using the usual derivative rules for the standard normal cumulative distribution function and density used in Handin1 with the prievously found result for the first partial derivative

$$\begin{aligned}
\frac{\partial^2 Call_0}{\partial K^2} &= S_0\phi(d_1)(-d_1)d_{1K}\hat{\sigma}_K\sqrt{T} + \hat{\sigma}_{KK}\sqrt{T}S_0\phi(d_1) - \phi(d_2)d_{2K} \\
&= S_0\hat{\sigma}_K\sqrt{T}\phi(d_1)d_{1K}d_1 + \hat{\sigma}_{KK}\sqrt{T}S_0\phi(d1) - \phi(d_1)\frac{S_0}{K}\left(d_{1K} - \hat{\sigma}_K\sqrt{T}\right) \\
&= S_0\phi(d_1)\left(d_{1K}\left(-d_1\hat{\sigma}_K\sqrt{T} - \frac{1}{K}\right) + \hat{\sigma}_{KK}\sqrt{T} + \frac{1}{K}\hat{\sigma}_K\sqrt{T}\right) \\
&= S_0\phi(d_1)\left(\frac{1}{\hat{\sigma}\sqrt{T}}\left(-\frac{1}{K} - d_1\hat{\sigma}_K\sqrt{T} + \hat{\sigma}_K\hat{\sigma}T\right)\left(-d_1\hat{\sigma}_K\sqrt{T} - \frac{1}{K}\right) + \hat{\sigma}_{KK}\sqrt{T} + \frac{1}{K}\hat{\sigma}_K\sqrt{T}\right) \\
&= S_0\phi(d_1)\left(\frac{1}{\hat{\sigma}\sqrt{T}}\left(-d_1\hat{\sigma}_K\sqrt{T} - \frac{1}{K}\right)^2 + \left(\frac{\hat{\sigma}_K\hat{\sigma}_T\left(-d_1\hat{\sigma}_K\sqrt{T} - \frac{1}{K}\right)}{\hat{\sigma}\sqrt{T}}\right) + \hat{\sigma}_{KK}\sqrt{T} + \frac{1}{K}\hat{\sigma}_K\sqrt{T}\right) \\
&= \frac{S_0\phi(d_1)}{\hat{\sigma}\sqrt{T}}\left(\left(\frac{1}{K} + d_1\hat{\sigma}_K\sqrt{T}\right)^2 - d_1\hat{\sigma}_K T\sqrt{T}\hat{\sigma}_K\hat{\sigma} - \frac{\hat{\sigma}_K\hat{\sigma}T}{K} + \hat{\sigma}_{KK}\hat{\sigma}\sqrt{T}\sqrt{T} + \frac{\hat{\sigma}_K\hat{\sigma}\sqrt{T}\sqrt{T}}{K}\right) \\
&= \frac{S_0\phi(d_1)}{\hat{\sigma}\sqrt{T}}\left(\left(\frac{1}{K} + d_1\hat{\sigma}_K\sqrt{T}\right)^2 + T\hat{\sigma}(\hat{\sigma}_{KK} - d_1\sqrt{T}\hat{\sigma}_K^2)\right).
\end{aligned}$$

As all the puzzle pieces are assembled we can now substitute in the found derivatives into (3.1) (Dupire's formula) and see that by simplification

$$\begin{aligned}
\sigma_{local}^2 &= \frac{\frac{\partial Call}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 Call}{\partial K^2}} \\
&= \frac{2S_0\phi(d_1)\left(\hat{\sigma}_T\sqrt{T} + \frac{\hat{\sigma}}{2\sqrt{T}}\right)}{K^2\frac{S_0\phi(d_1)}{\hat{\sigma}\sqrt{T}}\left(\left(\frac{1}{K} + d_1\hat{\sigma}_K\sqrt{T}\right)^2 + T\hat{\sigma}\left(\hat{\sigma}_{KK} - d_1\sqrt{T}\hat{\sigma}_K^2\right)\right)} \\
&= \frac{2\hat{\sigma}_T\sqrt{T} + \frac{\hat{\sigma}}{\sqrt{T}}}{\frac{1}{\hat{\sigma}\sqrt{T}}\left(\left(1 + Kd_1\hat{\sigma}_K\sqrt{T}\right)^2 + K^2\hat{\sigma}T\left(\hat{\sigma}_{KK} - d_1\sqrt{T}\hat{\sigma}_K^2\right)\right)} \\
&= \frac{\hat{\sigma}^2 + 2T\hat{\sigma}\hat{\sigma}_T}{(1 + Kd_1\hat{\sigma}_K\sqrt{T})^2 + \hat{\sigma}K^2T(\hat{\sigma}_{KK} - d_1\hat{\sigma}_K^2\sqrt{T})},
\end{aligned}$$

as desired.

3.c

The approximation given by Savine of the local volatility that we have to derive is

$$\hat{\sigma}(K) \approx \sqrt{\frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(K)}{2}}.$$

Afterwards the implications:

1. The ATM ($S_0 = K$) implied volatility is the same as the local volatility:

$$\hat{\sigma}(K) \mid_{K=S_0} \approx \sigma_{local}(S_0)$$

2. The slope of the implied volatility curve ATM is half of the slope of the local volatility

$$\sigma(\hat{K}) \mid_{K=S_0} \approx \frac{\sigma'_{local}(S_0)}{2}$$

We will follow Savine's argumentation and cross the t's and dot the i's - and a bit more.

Observe that (3.1) (Dupire's formula) does not make any assumptions or conditions about the volatility of the underlying.

Therefore, assume as Savine (b), p. 16 that the local volatility $\mathbb{Q}$-dynamics are given as

$$\frac{dS_t}{S_t} = \sigma_{local}(S_t, t) dW_t^{\mathbb{Q}}$$
$$\Longleftrightarrow$$
$$dS_t = \sigma_{local}(S_t, t) S_t dW_t^{\mathbb{Q}}.$$

Furthermore, assume as Savine (b), p. 18 that the real world dynamics are parameterized by some given local volatility. We now model a trader hedging a call option using the Black-Scholes model in the real world using a implied volatility denoted by $\hat{\sigma}$. The idea is now: that to fit the market implied volatility we need an approximation as the closed-form formula of $\hat{\sigma}(T, K)$ as a function of $\sigma_{local}(S_t, K)$ isn't available.

So: Consider (as above) the setup using a general diffusion model and some non-stochastic volatility model such as the Black-Scholes with call price denoted by $Call^{\text{BS}}(S_t, t, T, K, \hat{\sigma})$ for fixed K, T where $\hat{\sigma}$ is some observed implied volatility used for the hedge.

Analogous to Savine (b), p. 19, by The Fundamental Theorem of Derivative Trading (FToDT) we have the Profit and Loss (P&L) over the time $\mathbb{T} = [0, T]$ with Greeks from the Black-Scholes model

$$\begin{aligned}\text{P\&L}(T) &= \int_0^T \vartheta_t dt + \frac{1}{2}\int_0^T \Gamma_t(\sigma^2_{local}(t) - \hat{\sigma}^2) dt \\ &\underbrace{=}_{*} \frac{1}{2}\int_0^T \Gamma_t S_t^2 (\sigma^2_{local}(S, t) - \hat{\sigma}^2) dt,\end{aligned}$$

where $*$ follows from $r = D = 0$ using by definition that

$$\Gamma_t = \frac{\partial^2}{\partial S_t^2} Call^{\text{BS}}(S_t, K, T, \hat{\sigma}, T) \quad \text{and} \quad \vartheta_t = \frac{\partial}{\partial t} Call^{\text{BS}}(S_t, K, T, \hat{\sigma}, T),$$

is the Black-Scholes Gamma and Vega, respectively, for some K, T and $\hat{\sigma}$ that are interlinked through the Black-Scholes PDE

$$\vartheta + \frac{1}{2}\hat{\sigma}^2 S_0^2 \Gamma_t = 0.$$

Proceeding, by assumption, the dynamics of the real world follows that of a local volatility model. Therefore, taking the expectation under $\mathbb{Q}$ of the P&L(T) yields

$$E^{\mathbb{Q}}(\text{P\&L}(T)) = \frac{1}{2} E^{\mathbb{Q}} \left(\int_0^T \Gamma_t S^2 (\sigma_{local}^2(S,t) - \hat{\sigma}^2) dt \right).$$

If $E^{\mathbb{Q}}(\text{P\&L}(T)) = 0$, then the option is priced "fairly" as in the sense that both models coincide. This will only happen if the local volatility is aligned to the implied volatility, or more specifically

$$E^{\mathbb{Q}}\left(\text{P\&L}(T)\right) = 0$$

$$\iff$$

$$E^{\mathbb{Q}} \left(\int_0^T \Gamma_t S^2 (\sigma_{local}^2(S,t) - \hat{\sigma}^2) dt \right) = 0$$

$$\iff$$

$$E^{\mathbb{Q}} \left(\int_0^T \hat{\sigma}^2 \Gamma_t S_t^2 dt \right) = E^{\mathbb{Q}} \left(\int_0^T \sigma_{local}^2(S,t) \Gamma_t S_t^2 dt \right)$$

$$\underbrace{\iff}_{*}$$

$$\hat{\sigma}^2 E^{\mathbb{Q}} \left(\int_0^T \Gamma_t S_t^2 dt \right) = E^{\mathbb{Q}} \left(\int_0^T \sigma_{local}^2(S,t) \Gamma_t S_t^2 dt \right)$$

$$\iff$$

$$\hat{\sigma}^2 = \frac{E^{\mathbb{Q}} \left(\int_0^T \Gamma_t S^2 \sigma_{local}^2(S,t) dt \right)}{E^{\mathbb{Q}} \left(\int_0^T \Gamma_t S^2 dt \right)}.$$

where $*$ follows from the fact that we used that the Black-Scholes implied volatility $\hat{\sigma}$ is constant. Let $\gamma(t, S_t)^{\mathbb{Q}}$ denote the time-t $\mathbb{Q}$-density of S_t. Using the Law of The Unconscious Statistician (LOTUS) Soch (2020) we achieve

$$\hat{\sigma}^2 = \frac{\int_0^T \int_{\mathbb{R}} \gamma(t, S_t)^{\mathbb{Q}} \Gamma_t S^2 \sigma_{local}^2(S,t) dS dt}{\int_0^T \int_{\mathbb{R}} \gamma(t, S_t)^{\mathbb{Q}} \Gamma_t S^2 dS dt}.$$

This refers to the zero-sigma formula, which can be seen as a counterpart to (3.1) (Dupire's formula). It calculates the implied volatility as an average of expected local variances across various spot prices and time periods. More specifically it states that the implied volatility is a quadratic weighted average of local volatilities and weights defined as a product of $\gamma(t, S_t)$ and the Black-Scholes Γ, meaning the weights are expressed as

$$w(S_t, t) = \gamma(S_t, t)^{\mathbb{Q}}\Gamma_t(S_t, t)S_t^2.$$

The weights are determined by the product of the probability density, γ which cannot be analytically determined in local volatility models, and the Black-Scholes Γ. This Γ is calculated within the Black-Scholes framework, where the implied volatility that results renders the formula implicit. The probability density γ peaks around the current spot price, while the Black-Scholes Γ peaks around the strike price at maturity. Therefore, the sigma-zero weights effectively create an average that bridges the current spot price and the strike price at maturity, setting aside other parts of the volatility surface which would be between $\sigma^2(K,T)$ and $\sigma^2(S_0, 0)$ (this can be seen in practice on Savine (b), p. 20). This gives us the rough approximation in question when considering time-homogeneous local volatility, implying that it depends on spot price (hence strike) alone

$$\hat{\sigma}^2(K) \approx \frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(K)}{2}$$
$$\iff$$
$$\hat{\sigma}(K) \approx \sqrt{\frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(K)}{2}}.$$

We proceed to show that the ATM ($S_0 = K$) the implied volatility is the same as the local volatility at S_0. This is clearly seen by

$$\begin{aligned}
\hat{\sigma}(K) \mid_{K=S_0} &\approx \sqrt{\frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(K)}{2}} \\
&= \sqrt{\frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(S_0)}{2}} && (K = S_0) \\
&= \sqrt{\sigma^2_{local}(S_0)} \\
&= \sigma_{local}(S_0).
\end{aligned}$$

Lastly, we show that the slope of the implied volatility curve ATM ($S_0 = K$) is half of the slope of the local volatility function. Using the chain rule and using $S_0 = K$ we see that

$$\begin{aligned}
\hat{\sigma}'(K) \mid_{K=S_0} &= \frac{1}{2}\left(\frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(S_0)}{2}\right)^{-1/2} \frac{1}{2}(2\sigma_{local}(S_0)\sigma'_{local}(S_0)) \\
&= \frac{1}{2}\left(\sigma^2_{local}(S_0)\right)^{-1/2} \sigma_{local}(S_0)\sigma'_{local}(S_0) \\
&= \frac{\sigma_{local}(S_0)\sigma'_{local}(S_0)}{2\sigma_{local}(S_0)} \\
&= \frac{\sigma'_{local}(S_0)}{2}.
\end{aligned}$$

These two results exactly gives us the every crude and edgy new (linear) approximations as well, that is, if $K \approx S_0$, then

$$\hat{\sigma}(K) \approx \sigma_{local}(K) \quad \text{and} \quad \hat{\sigma}'(K) \approx \frac{\sigma'_{local}(K)}{2}.$$

These approximations are accurate ATM Savine (b), p. 21.

3.d

We have from 3.b that (3.1) (Dupire's formula) can be written as a function of (Black-Scholes) implied volatilities by

$$\begin{aligned}\sigma^2_{local} &= \frac{\hat{\sigma}^2 + 2T\hat{\sigma}\hat{\sigma}_T}{(1 + Kd_1\hat{\sigma}_K\sqrt{T})^2 + \hat{\sigma}K^2T(\hat{\sigma}_{KK} - d_1\hat{\sigma}^2_K\sqrt{T})},\\ d_1 &= \frac{\log(S_0/K)}{\hat{\sigma}\sqrt{T}} + \frac{\hat{\sigma}\sqrt{T}}{2}.\end{aligned}$$

By substitution of d_1 into σ^2_{local} we see that

$$\begin{aligned}\sigma^2_{local} &= \frac{\hat{\sigma}^2 + 2T\hat{\sigma}\hat{\sigma}_T}{\left(1 + K\left(\frac{\log(S_0/K)}{\hat{\sigma}\sqrt{T}} + \frac{\hat{\sigma}\sqrt{T}}{2}\right)\hat{\sigma}_K\sqrt{T}\right)^2 + \hat{\sigma}K^2T\left(\hat{\sigma}_{KK} - \left(\frac{\log(S_0/K)}{\hat{\sigma}\sqrt{T}} + \frac{\hat{\sigma}\sqrt{T}}{2}\right)\hat{\sigma}^2_K\sqrt{T}\right)}\\ &= \frac{\hat{\sigma}^2 + 2T\hat{\sigma}\hat{\sigma}_T}{\left(1 + K\left(\frac{\log(S_0/K)\hat{\sigma}_K}{\hat{\sigma}} + \frac{T\hat{\sigma}\hat{\sigma}_K}{2}\right)\right)^2 + \hat{\sigma}K^2T\left(\hat{\sigma}_{KK} - \left(\frac{\log(S_0/K)\hat{\sigma}^2_K}{\hat{\sigma}} + \frac{\hat{\sigma}^2_K\hat{\sigma}T}{2}\right)\right)}.\end{aligned}$$

We now preform a short expiry-T expansion, meaning an approximation dropping the expiry-T-terms

$$\begin{aligned}\sigma^2_{local} &\underbrace{\approx}_{T\to 0} \frac{\hat{\sigma}^2}{\left(1 + K\frac{\log(S_0/K)\hat{\sigma}_K}{\hat{\sigma}}\right)^2}\\ &\Rightarrow\\ \sigma_{local} &\approx \frac{\hat{\sigma}}{1 + \frac{1}{\hat{\sigma}}K\log(S_0/K)\hat{\sigma}_K} \qquad (3.3)\\ &\iff\\ \frac{1}{\sigma_{local}} &\approx \frac{1 + \frac{1}{\hat{\sigma}}K\log(S_0/K)\hat{\sigma}_K}{\hat{\sigma}}.\end{aligned}$$

Note, somehow analogously to Savine (a), p. 48, letting $K = S_0$ in (3.3), unwinding the shorthand but using approximations as the expansion in expiry-T is an approximation, we see that

$$\begin{aligned}\sigma_{local}(S_0) &\approx \frac{\hat{\sigma}(S_0)}{1 + \frac{1}{\hat{\sigma}(S_0)}K\log(S_0/S_0)\hat{\sigma}_K(S_0)}\\ &= \hat{\sigma}(S_0).\\ &\iff\\ \sigma_{local}(S_0) &\approx \hat{\sigma}(S_0)\\ &\iff\\ \frac{1}{\sigma_{local}(S_0)} &\approx \frac{1}{\hat{\sigma}(S_0)}.\end{aligned}$$

Returning to (3.3), we see that we can rewrite the expression as

$$\begin{aligned}\frac{1}{\sigma_{local}} &\approx \frac{1+\frac{1}{\hat{\sigma}}K\log(S_0/K)\hat{\sigma}_K}{\hat{\sigma}} \\ &\approx \frac{1}{\hat{\sigma}} + K\log\left(\frac{S_0}{K}\right)\frac{\hat{\sigma}_K}{\hat{\sigma}^2} \\ &\underbrace{\approx}_{*} \frac{1}{\hat{\sigma}} - K\log\left(\frac{S_0}{K}\right)\left(\frac{1}{\hat{\sigma}}\right)_K,\end{aligned}$$

where $*$ follows from the fact that $(1/\hat{\sigma})_K = (-1/\hat{\sigma}^2)\hat{\sigma}_K$ using the chain rule.

Define the function $f(K) = \frac{1}{\hat{\sigma}(K)}$ and thus

$$f'(K) = -\frac{\hat{\sigma}'(K)}{\hat{\sigma}^2(K)} = -\frac{\hat{\sigma}_K(K)}{\hat{\sigma}^2(K)}.$$

Then f satisfies the ODE given by

$$f(K) - K\log\left(\frac{S_0}{K}\right)f'(K) \approx \frac{1}{\sigma_{local}(K)}, \tag{3.4}$$

with boundary $\hat{\sigma}(S_0) = \sigma_{local}(S_0)$. This means that at the given boundary we achieve the expression

$$f(S_0) = \frac{1}{\hat{\sigma}(S_0)} = \frac{1}{\sigma_{local}(S_0)},$$

and thus the solution (see $\star$ for a Trial-By-Combat testing of the solution)

$$\begin{aligned}f(K) &= \frac{1}{\hat{\sigma}(K)} \\ &\approx \frac{1}{\log(S_0/K)}\int_K^{S_0}\frac{1}{u\sigma(u)}du \\ &\approx \frac{\int_K^{S_0}\frac{1}{u\sigma(u)}du}{\log(S_0/K)} \\ &\Rightarrow \\ \hat{\sigma}(K) &\approx \frac{\log(S_0/K)}{\int_K^{S_0}\frac{1}{u\sigma(u)}du},\end{aligned}$$

where the denominator is the Lamperti transform (as a function of S_0).

We now try to see if the solution is the actual solution. By The Fundamental Theorem of Calculus the derivative of the Lamperti transformation is given as

$$\frac{\partial}{\partial K}\left(\int_K^{S_0}\frac{1}{u\sigma(u)}du\right) = -\frac{1}{K\sigma_{local}(K)},$$

implying by the product- and chain-rule

$$\begin{aligned}
f'(K) &= \frac{\partial}{\partial K}\left(\frac{1}{\hat{\sigma}(K)}\right) \\
&\approx \frac{\partial}{\partial K}\left(\frac{\int_K^{S_0}\frac{1}{u\sigma(u)}du}{\log(S_0/K)}\right) \\
&= \frac{-\frac{1}{K\sigma_{local}(K)}\log(S_0/K) - \left(\frac{1}{K}\right)\int_K^{S_0}\frac{1}{u\sigma(u)}du}{\log(S_0/K)^2} \\
&= \frac{-\frac{1}{K\sigma_{local}(K)}}{\log(S_0/K)} + \frac{\frac{1}{K}\int_K^{S_0}\frac{1}{u\sigma(u)}du}{(\log(S_0/K))^2}.
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) (the PDE) we see that it is indeed the solution as

$$\begin{aligned}
\frac{1}{\sigma_{local}(K)} &\approx f(K) - K\log\left(\frac{S_0}{K}\right)f'(K) \\
&\approx \frac{\int_K^{S_0}\frac{1}{u\sigma(u)}du}{\log(S_0/K)} - K\log\left(\frac{S_0}{K}\right)\left(\frac{-\frac{1}{K\sigma_{local}(K)}}{\log(S_0/K)} + \frac{\frac{1}{K}\int_K^{S_0}\frac{1}{u\sigma(u)}du}{(\log(S_0/K))^2}\right) \\
&= \frac{\int_K^{S_0}\frac{1}{u\sigma(u)}du}{\log(S_0/K)} + \frac{1}{\sigma_{local}(K)} - \frac{\int_K^{S_0}\frac{1}{u\sigma(u)}du}{\log(S_0/K)} \\
&= \frac{1}{\sigma_{local}(S_0)}.
\end{aligned}$$

This exactly means that $f(K)$ satisfies the ODE given by Savine (a), p. 48. The boundry follows exactly by (3.3) setting $K = S_0$

$$\begin{aligned}
\sigma_{local}(S_0) &\approx \frac{\hat{\sigma}(S_0)}{1 + \frac{1}{\hat{\sigma}(S_0)}K\log(S_0/S_0)\hat{\sigma}_K(S_0)} \\
&= \hat{\sigma}(S_0). \\
&\iff \\
\sigma_{local}(S_0) &\approx \hat{\sigma}(S_0) \\
&\iff \\
\frac{1}{\sigma_{local}(S_0)} &\approx \frac{1}{\hat{\sigma}(S_0)}.
\end{aligned}$$

However, for the solution, notice it is not defined at the point $K = S_0$ which is easily seen by

$$\begin{aligned}
f(S_0) &= \frac{1}{\hat{\sigma}(S_0)} \\
&= \frac{\int_{S_0}^{S_0}\frac{1}{u\sigma(u)du}}{\log(S_0/S_0)} \\
&= \frac{0}{0}.
\end{aligned}$$

3.e

All the code can be viewed in Appendix A. As stated in the assignment we do a investigation, numerically, and make observations. The experiment will be conducted as such:

Let the approximations and the benchmark be denoted as:

Approximation 1 (3.c): $$\hat{\sigma}(K) \approx \sqrt{\frac{\sigma^2_{local}(S_0) + \sigma^2_{local}(K)}{2}}.$$

Approximation 2 (3.d): $$\hat{\sigma}(K) \approx \frac{\log(S_0/K)}{\int_K^{S_0} \frac{1}{u\sigma_{local}(u)} du}.$$

Benchmark: $$\hat{\sigma}(K,T) = \sigma_{local}(K,T).$$

The parameters used will be mostly similar to handin1 and seen in Table 4:

Description	Symbol	Value
Initial Stock Price	S_0	100
Time to Maturity	T	1
Volatility	σ	15
Strike Price Interval	K	[50, 150]

Table 4: Parameters used in the numerical volatility experiment

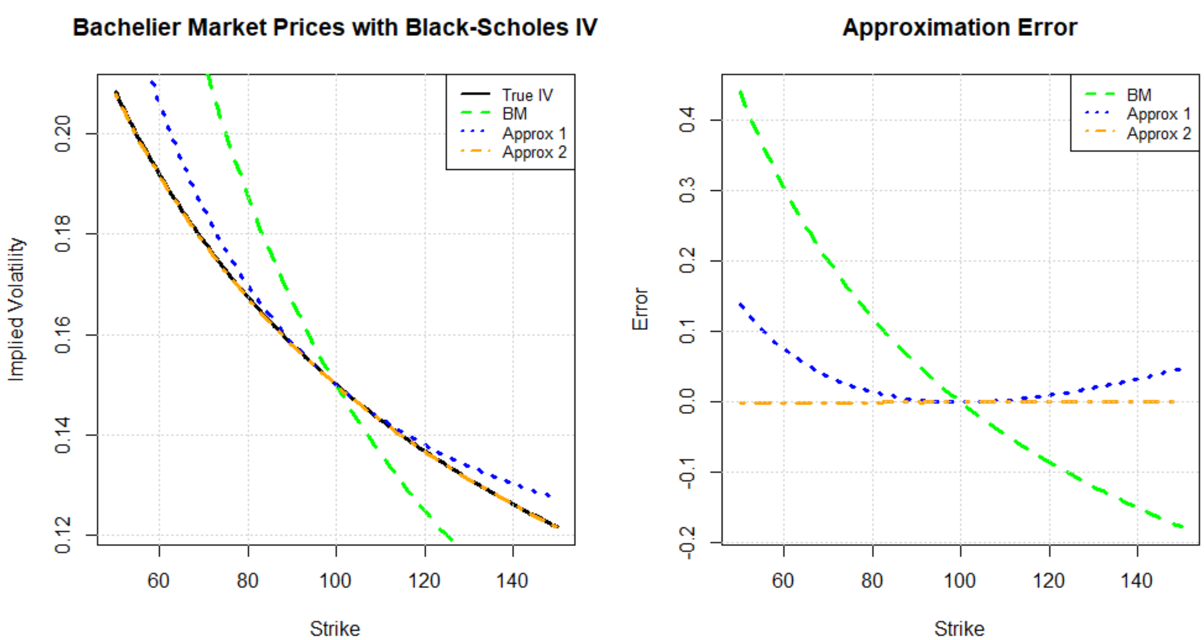

Figure 1: (L): Implied volatilites vs. Strikes. (R): Error for the three approximations.

Note that:

- Approximation 1 (3.c) is the zero-sigma approxmiation.
- Approximation 1 (3.d) is the Lamperi approximation.
- Benchmark is the naive approximation $\sigma(K, T)$.

In `R` we determine the Black-Scholes implied volatilities from a call option across strikes in the Bachelier model. The Black-Scholes implied volatilities will be our baseline or "true" implied volatilities which the approximations will be compared in relation to.

The left plot shows the "true" implied volatilities and the implied volatilities from our different approximations. Conversely, the graph on the right illustrates the discrepancy between each approximation and the actual implied volatility, i.e the error.

Clearly from examination of the left graph the best fitting approximation is Approximation 2 with Approximation 1 following closely and the Benchmark last. However, the true IV, Approximation 1/2 and the Benchmark all exhibit some (similar) volatility skew. All the models behave the same ATM ($S_0 = K = 100$).

Specifically, we see that for ITM and ATM the implied volatilities are relatively high. Conversely, for OTM strikes-K the implied volatility is lower. In other words, the implied volatility is increasing in moneyness.

However, the approximations are not acting exactly a like when we consider extremes og the ATM and OTM. Approximation 2 follows that of the true implied volatility perfectly. Approximation 1 does deviate more as we move to the extremes of the strikes-$K \in [50, 150]$. This tendency is also seen for the Benchmark approximation but in in a much more extreme sense leading to larger deviations from the true implied volatility.

The error graphs as expected displays the same tendency: Approximation 2 fitting perfectly with Approximation 1 following closely and the Benchmark last with an error of 0 ATM. However, we note that Approximation 2 is **not** defined ATM and thus would need some sort of interpolation, i.e the naive Benchmark approximation such that $\hat{\sigma}(K, T) = \sigma_{local}(K, T)$ when the strike-K is equal to that of S_0.

Approximation 1 is noted to be used with extreme care as it is a very rough approximation Savine (b), p. 21. This is well seen within the graphs as larger skew is maintained through any strike except near or ATM and worsening at the extremes of the interval $K \in [50, 150]$. This means that its usefulness is mostly limited to ATM volatilities and skew in time homogeneous local volatility models.

Approximation 2 (the Lamperti approximation) is therefore seen to be superior in this case and follows as explained in Savine (a), pp. 48-49. Namely, that the approximation is extremely accurate and that it works around and most differently from the other approximations - away-from-the-money (i.e not ATM).

A Appendix: Code

GitHub profile with the code in R.

B References

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