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FinKont2: Hand-in Exercise #3

Answers must be submitted via Absalon by 23:59 CEST on Sunday, April 14, 2024. For your convenience, we put a ⋆ near the text that states what you need to prove/calculate/etc.

1 Characteristic functions in the Heston model

In this section, we will complete the proof of the Heston's famous option-pricing formula. In the lectures, we have shown that the PDE for the price $\mathcal{V}(t, S, v)$ of a call option with strike K and the underlying asset S can be written in terms of variables $t, x = \ln(S), v$. After that, we have used the following ansatz:

$$\mathcal{V}(t,x,v) = e^x Q_1(x,v,t) - Ke^{-r(T-t)} Q_2(x,v,t) \tag{1}$$

and obtained two PDEs for functions Q_1 and Q_2. We could not solve that PDEs in closed-form, but verified that they have a convenient probabilistic representation. In particular, they can be seen as

$$Q_j(x,v,t;\ln(K)) = \mathbb{P}_j\left(x(T) \geq \ln(K) | x(t) = x, v(t) = v\right), \quad j = 1,2, \tag{2}$$

for some probability measures $\mathbb{P}_j$, $j = 1,2$ under which the processes of interest have the following "adjusted" SDEs:

$$\begin{aligned} dx(t) &= (r + u_j v(t))dt + \sqrt{v(t)}dW_1^{\mathbb{P}_j}(t); \\ dv(t) &= (a - b_j v(t))dt + \sigma\sqrt{v(t)}dW_2^{\mathbb{P}_j}(t). \end{aligned} \tag{3}$$

Afterwards, we have shown that $Q_j(x,v,t;\ln(K))$ can be calculated using the conditional characteristic functions $\Psi^j_{x(T)}(x,v,t;u) := \mathbb{E}^{\mathbb{P}_j}\left[e^{i\cdot u\cdot x(T)} | x(t) = x, v(t) = v\right]$, which in their turn solve the following PDEs:

$$0 = \frac{\partial \Psi_j}{\partial t} + (r + u_j v)\frac{\partial \Psi_j}{\partial x} + \frac{1}{2}v\frac{\partial^2 \Psi_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 \Psi_j}{\partial v \partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \Psi_j}{\partial v^2} + (a - b_j v)\frac{\partial \Psi_j}{\partial v}; \tag{4}$$

$$e^{i\cdot u\cdot x} = \Psi^j_{x(T)}(x,v,T;u) \quad \forall (x,v) \in \mathbb{R}\times(0,+\infty), \tag{5}$$

where we write in the PDE $\Psi_j := \Psi^j_{x(T)}(x,v,t;u)$ to shorten notation, $j = 1,2$.

The task now is to solve this PDE and obtain the final formula for the conditional characteristic function, which was shown in the lectures.

1.1 Changing variables and guessing the form of solution

⋆ Change variables $\tau = T - t$ in (5) and (4). Show that the resulting PDEs and terminal conditions in terms of variables τ, x, v are given by:

$$0 = -\frac{\partial \Psi_j}{\partial \tau} + (r + u_j v)\frac{\partial \Psi_j}{\partial x} + \frac{1}{2}v\frac{\partial^2 \Psi_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 \Psi_j}{\partial v \partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \Psi_j}{\partial v^2} + (a - b_j v)\frac{\partial \Psi_j}{\partial v}; \quad (6)$$

$$e^{i\cdot u\cdot x} = \Psi^j_{x(T)}(x, v, 0; u) \quad \forall (x, v) \in \mathbb{R} \times (0, +\infty), \quad (7)$$

Let us assume (do "educated guessing") that the solution to (6) & (7) has the following form:

$$\Psi^j_{x(T)}(x, v, \tau; u) = e^{C_j(\tau;u) + D_j(\tau;u)\cdot v + i\cdot u\cdot x}, \quad j = 1, 2. \quad (8)$$

⋆ Insert the ansatz (the guessed solution) (8) into (6) & (7) and show that the resulting PDE is given by:

$$\Psi_j \underbrace{\left\{-\frac{\partial C_j}{\partial \tau} + aD_j + rui\right\}}_{(*)} + \Psi_j v \underbrace{\left\{-\frac{\partial D_j}{\partial \tau} + u_j ui - \frac{1}{2}u^2 + \rho\sigma ui D_j + \frac{1}{2}\sigma^2 D_j^2 - b_j D_j\right\}}_{(**)} = 0\,.$$

1.2 Transforming the PDE to a system of ODEs and solving it

⋆ Argue why the PDE that you obtained at the end of Subsection 1.1 is equivalent to the system of two ordinary differential equations (*) and (**). Write down the respective conditions for C_j and D_j at $\tau = 0$.

(**) is a so-called Ricatti ordinary differential equation (Ricatti ODE). To solve it, ⋆ do the following substitution

$$D_j(\tau; u) = -\frac{\frac{\partial E_j(\tau;u)}{\partial \tau}}{\frac{\sigma^2}{2}E_j(\tau; u)} \quad (9)$$

and ⋆ show that the resulting ODE for E_j is given by:

$$\frac{\partial^2 E_j}{\partial \tau^2} - (\rho\sigma ui - b_j)\frac{\partial E_j}{\partial \tau} + \frac{\sigma^2}{2}\left(-\frac{1}{2}u^2 + u_j ui\right)E_j = 0.$$

The ODE for E_j is a linear ODE of the second order. A general solution to it can be written as:

$$E_j(\tau; u) = A_j e^{x_{j,+}\cdot\tau} + B_j e^{x_{j,-}\cdot\tau}, \quad (10)$$

with complex-valued coefficients:

$$x_{j,\pm} = \frac{\rho\sigma ui - b_j \pm d_j}{2} \quad \text{and} \quad d_j = \sqrt{(\rho\sigma ui - b_j)^2 - \sigma^2(2u_j ui - u^2)} \quad (11)$$

Note that $x_{j,+} - x_{j,-} = d_j$. Define a coefficient:

$$g_j := \frac{x_{j,-}}{x_{j,+}} \tag{12}$$

$\star$ Using the substitution (9), the general form (10) of E_j and the initial condition for D_j, argue that the following initial conditions for E_j must hold:

$$E_j(0;u) = A_j + B_j \quad \text{and} \quad \frac{\partial E_j(\tau;u)}{\partial \tau}|_{\tau=0} = x_{j,+}A_j + x_{j,-}B_j. \tag{13}$$

$\star$ Using those conditions, show that:

$$A_j = \frac{g_j E_j(0;u)}{g_j - 1} \quad \text{and} \quad B_j = -\frac{E_j(0;u)}{g_j - 1} \tag{14}$$

$\star$ and, thus:

$$E_j(\tau;u) = \frac{E_j(0;u)}{g_j - 1}\left(g_j e^{x_{j,+}\cdot\tau} - e^{x_{j,-}\cdot\tau}\right). \tag{15}$$

$\star$ Insert (15) into (9), simplify the expression step-by-step and show that:

$$D_j(\tau;u) = \frac{b_j - \rho\sigma ui + d_j}{\sigma^2} \cdot \frac{1 - e^{d_j\cdot\tau}}{1 - g_j e^{d_j\cdot\tau}} \tag{16}$$

$\star$ Using the ODE for C_j and (16), show step-by-step that:

$$C_j(\tau;u) = rui\tau + \frac{a}{\sigma^2}\left((b_j - \rho\sigma ui + d_j)\,\tau - 2\ln\left(\frac{1 - g_j e^{d_j\cdot\tau}}{1 - g_j}\right)\right) \tag{17}$$

You have shown that the characteristic function of the logarithmic stock price in the Heston model is given by (8) with C_j given by (17) and D_j given by (16).

1.3 Numerical implementation of characteristic functions

$\star$ Write a function *characteristicFunctionHeston* that:

- computes the characteristic function $\Psi_j = \Psi^j_{x(T)}(x, v, \tau; u)$ as per original formula of Heston (see also lecture slide 65)
- takes as input parameters $u, S_t, v_t, r, \kappa, \theta, \sigma, \rho, \lambda, \tau, j$, where $\tau = T - t$ is the time to maturity of an option
- returns as output: the value of the characteristic function evaluated at a specific u

$\star$ Compute the characteristic functions Ψ_1 and Ψ_2 for $u \in [-20, 20]$ with the following values of other arguments (in addition to the respective $j \in \{1, 2\}$): $S_t = 100, v_t = 0.06, r = 0.05, \kappa = 1, \theta = 0.06, \sigma = 0.3, \rho = -0.5, \lambda = 0.01, \tau = 1$. Report the respective values of the characteristic functions by plotting the real part & the imaginary part of $\Psi_1(u)$ for $u \in [-20, 20]$ and the real part & the imaginary part of $\Psi_2(u)$ for $u \in [-20, 20]$.

2 Option pricing in the context of the Heston model

In this exercise, you will compare different methods for pricing a call option on an underlying driven by the Heston model. In particular, you will use the Monte Carlo method, original Heston formula using Fourier transforms, and the Carr-Madan formula.

2.1 Monte Carlo and Euler discretization Scheme

Option pricing using Monte Carlo simulation means that we first simulate a large number $N > 0$ of artificial time series of the stock price and variance over the time period $[0, T]$. compute the option payoff for each path and, finally, obtain the option price by computing the average payoff of the option and discounting that quantity to the time when we price the option, e.g., $t = 0$

As you know from lectures, the generation of S and v time series for the Heston model may lead to negative value of v at some time points due to discretization errors. When this occurs, one can use the truncation scheme (i.e., using $\max\{0, v(t)\}$) or the reflection scheme (i.e., using $|v(t)|$).

$\star$ Write a function *generateHestonPathEulerDisc* that:

- simulates under an EMM one path of the price of a risky asset S with Heston model dynamics
- is based on the Euler discretization with on an equidistant time grid $\left\{t_i = \frac{iT}{n}, i = 0, \ldots, n\right\}$, where $n \in \mathbb{N}$ is the number of time points after $t = 0$
- uses full truncation to prevent the variance from being negative
- takes as input parameters: $S_0, v_0, r, \kappa, \theta, \sigma, \rho, \lambda, T, n$
- returns as output: a vector of values of S at time points in the grid, the number of times the variance equals 0

$\star$ Write a function *priceHestonCallViaEulerMC* that:

- simulates using you function *generateHestonPathEulerDisc* N paths of S
- computes based on simulated paths a Monte-Carlo (MC) estimate of the price of a European call option on the underlying asset S following a Heston model
- takes as input parameters $S_0, v_0, r, \kappa, \theta, \sigma, \rho, \lambda, T, n, N, K$, where K is the strike of a call option
- returns as output: the MC estimate of the option price, the standard deviation of the option payoff, the time for computing the MC estimate of the option price

$\star$ Price a call option using your *priceHestonCallViaEulerMC* function assuming the following values of the model parameters: $S_0 = 100, v_0 = 0.06, r = 0.05, \kappa = 1, \theta = 0.06, \sigma = 0.3, \rho = -0.5, \lambda = 0.01, T = 1, K = 100, T = 1, n = 100, N = 1000$.

$\star$ Report the output of the function call with the above-mentioned parameter values and comment on your results.

2.2 Monte Carlo and Milstein discretization scheme

⋆ Read Section 2 on "Milstein Scheme" in the file under the following hyperlink.

⋆ Using your code from Subsection 2.1, write two functions *generateHestonPathMilsteinDisc* and *priceHestonCallViaMilsteinMC*, which price a call option on S driven by the Heston model, have the same logic, input and output, but use MC method with Milstein discretization scheme.

⋆ Price a call option using your *priceHestonCallViaMilsteinMC* function assuming the following values of the model parameters: $S_0 = 100, v_0 = 0.06, r = 0.05, \kappa = 1, \theta = 0.06, \sigma = 0.3, \rho = -0.5, \lambda = 0.01, T = 1, K = 100, T = 1, n = 100, N = 1000$.

⋆ Report the output of the function call with the above-mentioned parameter values, comment on your results, and compare them with your results from Subsection 2.1.

2.3 Heston's original formula

⋆ Write a function *priceHestonCallViaOriginalFT* that:

- computes the price of a European call option on the underlying asset S following a Heston model (see lecture slides 65 and 66)
- may use the function *characteristicFunctionHeston* and functions for numerical integration, pre-implemented in the programming language you chose; your integration routine may truncate the integral from above at some sufficiently large $u_{max} > 0$, e.g., $u_{max} = 50$.
- takes as input parameters $S_t, v_t, r, \kappa, \theta, \sigma, \rho, \lambda, \tau,\ K$, and possibly u_{max} if you need to truncate the integrals from above
- returns as output: the price of the call option as per Heston's original formula

⋆ Price a call option using your *priceHestonCallViaOriginalFT* function assuming the following values of the model parameters: $S_0 = 100, v_0 = 0.06, r = 0.05, \kappa = 1, \theta = 0.06, \sigma = 0.3, \rho = -0.5, \lambda = 0.01, \tau = 1, K = 100$, and if needed $u_{max} = 50$. Measure the time for computing the price of this option.

⋆ Report the output of the function call with the above-mentioned parameter values and the computation time. Comment on your results, comparing them with your results from Subsections 2.1 and 2.2.

2.4 Carr-Madan formula

⋆ Write a function *priceHestonCallViaCarrMadan* that:

- computes the price of a European call option on the underlying asset S following a Heston model as per Carr-Madan's approach
- may use the function *characteristicFunctionHeston* for evaluation the characteristic function (ensure that you correctly transition from $\mathbb{P}$ to $\widetilde{\mathbb{Q}}$ and map the model parameters under $\widetilde{\mathbb{Q}}$ to input arguments of your *characteristicFunctionHeston*)

- may use functions for numerical integration, pre-implemented in the programming language you chose; you may also code your own function for numerical integration, e.g., the one based on trapezoidal rule with an equidistant grid with the distance of $\Delta u = 0.01$ between the integration points; your integration routine may truncate the integral from above at some sufficiently large $u_{max} > 0$, e.g., $u_{max} = 50$.
- takes as input parameters $u_{max}, S_t, v_t, r, \kappa, \theta, \sigma, \rho, \lambda, \tau, \alpha$,
- returns as output: the price of the call option as per Carr-Madan's approach

$\star$ Price a call option using your *priceHestonCallViaCarrMadan* function assuming the following values of the model parameters: $S_0 = 100, v_0 = 0.06, r = 0.05, \kappa = 1, \theta = 0.06, \sigma = 0.3, \rho = -0.5, \lambda = 0.01, \tau = 1, K = 100, \alpha = 0.3$, and if needed $u_{max} = 50$. Measure the time for computing the price of this option.

$\star$ Report the output of the function call with the above-mentioned parameter values and the computation time. Comment on your results, comparing them with your results from Subsections 2.1, 2.2, and 2.3.

3 Asset allocation under the Heston model

In this exercise, we you will answer the following question (under some mathematical assumptions explained in the sequel). How to optimally invest money in a market with one risk-free asset S_0 and one risky asset S_1 driven by the Heston model?

Consider an investor who has initial capital $x_0 > 0$ and wants to invest it two assets during the time period $[0, T]$. The first asset is a risk-free asset S_0, e.g., a bank account with continuously compounding interest rate, which happens in reality on a daily or monthly basis, The second asset is a risky asset S_1, e.g., an exchange-traded fund (ETF) tracking S&P500 stock market index. In contrast to the lecture notes, we will parameterize the Heston model in a more convenient way, in which we set $\mu = r + \overline{\lambda} v(t)$ with $\overline{\lambda}$ being interpreted as a risk premium per unit of variance:

$$\begin{aligned}
dS_0(t) &= rS_0(t)dt,\ S_0(0) = 1,\ \text{a traded asset} \\
dS_1(t) &= \left(r + \overline{\lambda} v(t)\right) S_1(t)dt + \sqrt{v(t)} S_1(t) dW_1^{\mathbb{P}}(t),\ S_1(0) = s_1 > 0,\ \text{a traded asset} \\
dv(t) &= \kappa\left(\theta - v(t)\right)dt + \sigma\rho\sqrt{v(t)}\, dW_1^{\mathbb{P}}(t) + \sigma\sqrt{v(t)}\sqrt{1-\rho^2} dW_2^{\mathbb{P}}(t),\ v(0) > 0,\ \text{not traded}
\end{aligned}$$

where $W_1^{\mathbb{P}}$ and $W_2^{\mathbb{P}}$ are independent one-dimensional Wiener processes.

Assume that the Feller's condition $\kappa\theta > \frac{\sigma^2}{2}$ is satisfied, so that v is always positive. Let us denote by $\pi = (\pi(t))_{t\in[0,T]}$ the fraction of wealth invested the risky asset S_1 at time $t \in [0, T]$. π is called an investment strategy and is modelled by a continuous stochastic process that is integrable w.r.t.the return process $dS_1(t)/S_1(t)$.

The wealth process $X^{x_0,\pi} = \left(X^{x_0,\pi}(t)\right)_{t\in[0,T]}$ (also known as the portfolio value process) under the real-world measure $\mathbb{P}$ evolves according to the following stochastic differential equation (SDE):

$$\begin{aligned}dX^{\pi}(t) &= (1-\pi(t))X^{\pi}(t)dS_0(t)/S(t) + \pi(t)X^{\pi}(t)dS_1(t)/S(t)\\ &= X^{\pi}(t)\left[\left(r+\pi(t)\overline{\lambda}v(t)\right)dt + \pi(t)\sqrt{v(t)}dW_1^{\mathbb{P}}(t)\right], \qquad X(0)=x_0>0.\end{aligned}$$

Let us also denote the set of all admissible investment strategies for the initial capital x_0 and the initial variance v_0 by $\mathcal{A}(x_0,v_0)$. Later, we will also consider sets $\mathcal{A}(t,x,v)$ of admissible investment strategies within the time period $[t,T]$ for which the time-t wealth of the investor equals x. We model the investor's preferences via a so-called power utility function $U(x)=x^p/p$, where $p\in(-\infty,0)\cup(0,1)$ is the risk-aversion parameter. In the context of expected utility theory, a rational investor solves the following asset-allocation problem:

$$\max_{\pi\in\mathcal{A}(x_0,v_0)} \mathbb{E}\left[U(X^{\pi}(T))\right] \tag{P}$$

3.1 Derivation of the Hamilton-Jacobi-Bellman PDE

In this section, we use *dynamic programming principle* (DPP) of stochastic control (also known as the *Bellman principle*) to **heuristically** derive a candidate optimal strategy π^* for the portfolio optimization problem (P).

In (P), the initial wealth x_0, the initial value v_0 of the variance, and the time T till the end of the investment horizon are fixed. The key idea of the Bellman principle is that it is easier to solve a dynamic version of this problem, where all of these variables are allowed to vary.

Let us consider the so-called value function $\mathcal{V}(t,x,v)$, which describes the maximal expected utility of terminal wealth that can be attained by trading optimally on $[t,T]$, given that the starting wealth at $t\in[0,T]$ is x and the respective value of the variance is v:

$$\mathcal{V}(t,x,v) = \max_{\pi\in\mathcal{A}(t,x,v)} \mathbb{E}\left[U(X^{\pi}(T))\right] \tag{18}$$

$\star$ Argue why $\mathcal{V}(t,x,v)$ satisfies the following terminal condition:

$$\mathcal{V}(T,x,v) = U(x). \tag{19}$$

Now you will derive the so-called Hamilton-Jacobi-Bellman (HJB) PDE for $\mathcal{V}$. The DPP states that:

$$\mathcal{V}(t,x,v) \geq \mathbb{E}\left[\mathcal{V}(t+h, X^{\pi}(t+h), v(t+h)) | X^{\pi}(t)=x, v(t)=v\right], \tag{20}$$

for any $h\in[0,T-t]$ and any admissible π.

$\star$ Consider any constant investment strategy π applied over a small time interval $[t,t+h]$. Assuming that the value function is smooth enough, derive via Ito's formula the representation of $\mathcal{V}(t+h,X^{\pi}(t+h),v(t+h))$, i.e.:

$$\mathcal{V}(t+h,X^{\pi}(t+h),v(t+h)) = \mathcal{V}(t,X^{\pi}(t),v(t)) + \int_t^{t+h}\ldots ds + \int_t^{t+h}\ldots dW_1^{\mathbb{P}}(s) + \int_t^{t+h}\ldots dW_2^{\mathbb{P}}(s) \tag{21}$$

$\star$ Insert (21) into (20) and move $\mathcal{V}(t,x,v)$ to the right-hand side of the inequality. Simplify the inequality under the assumption that the stochastic integrals with respect to the Wiener processes $W_1^{\mathbb{P}}$, $W_2^{\mathbb{P}}$ are martingales.

$\star$ Divide the resulting inequality by h, let $h \downarrow 0$, and show that the resulting limit is given by:

$$0 \geq \mathcal{V}_t + \frac{1}{2}\sigma^2 v \mathcal{V}_{vv} + \kappa(\theta - v)\mathcal{V}_v + x(r + \pi\overline{\lambda}v)\mathcal{V}_x + \frac{1}{2}\pi^2 x^2 v \mathcal{V}_{xx} + \pi x \sigma v \rho \mathcal{V}_{xv}$$

Since the above inequality holds for *any* constant investment strategy π, it follows from the previous inequality that:

$$0 \geq \mathcal{V}_t + \frac{1}{2}\sigma^2 v \mathcal{V}_{vv} + \kappa(\theta - v)\mathcal{V}_v + \sup_{\pi}\left\{x(r + \pi\overline{\lambda}v)\mathcal{V}_x + \frac{1}{2}\pi^2 x^2 v \mathcal{V}_{xx} + \pi x \sigma v \rho \mathcal{V}_{xv}\right\}$$

$\star$ Repeat the above derivations and argue that for an optimal investment strategy π^* the following must hold:

$$0 = \mathcal{V}_t + \frac{1}{2}\sigma^2 v \mathcal{V}_{vv} + \kappa(\theta - v)\mathcal{V}_v + \sup_{\pi}\left\{\underbrace{x(r + \pi\overline{\lambda}v)\mathcal{V}_x + \frac{1}{2}\pi^2 x^2 v \mathcal{V}_{xx} + \pi x \sigma v \rho \mathcal{V}_{xv}}_{=:g(\pi)}\right\} \tag{22}$$

(22) is called the HJB PDE.

3.2 Solution to the Hamilton-Jacobi-Bellman PDE

$\star$ Assuming that $\mathcal{V}_{xx} < 0$, show that the function $g(\pi)$ in (22) has a unique (given arbitrary but fixed admissible t, x, v) maximizer that is given by:

$$\pi^*(t,x,v) = -\overline{\lambda}\frac{\mathcal{V}_x(t,x,v)}{x\mathcal{V}_{xx}(t,x,v)} - \sigma\rho\frac{\mathcal{V}_{xv}(t,x,v)}{x\mathcal{V}_{xx}(t,x,v)} \tag{23}$$

$\star$ Show that the insertion (23) into (22) and simplication leads to the following PDE:

$$0 = \mathcal{V}_t + \kappa\theta\mathcal{V}_v + xr\mathcal{V}_x + v\left(\frac{1}{2}\sigma^2\mathcal{V}_{vv} - \kappa\mathcal{V}_v - \frac{1}{2}\frac{\left(\overline{\lambda}\mathcal{V}_x + \sigma\rho\mathcal{V}_{xv}\right)^2}{\mathcal{V}_{xx}}\right). \tag{24}$$

Recall that the terminal condition is $\mathcal{V}(T,x,v) = U(x) = x^p/p$

To find the solution, use the separation ansatz

$$\mathcal{V}(t,x,v) = \frac{x^p}{p}h(t,v), \;\; h(T,v) = 1. \tag{25}$$

$\star$ Show that under the ansatz (25), the optimal investment strategy is :

$$\pi^*(t,x,v) = \frac{\overline{\lambda}}{1-p} + \frac{\sigma\rho}{1-p}\frac{h_v}{h} \tag{26}$$

$\star$ Insert the ansatz (25) into (24) and derive the following PDE fof h:

$$0 = h_t + \kappa\theta h_v + prh + v\left(\frac{1}{2}\sigma^2 h_{vv} - \kappa h_v + \frac{1}{2}\frac{p(\overline{\lambda}h + \sigma\rho h_v)^2}{(1-p)h}\right). \tag{27}$$

The structure implies that $h(t,v)$ is exponentially affine:

$$h(t,v) = \exp(a(\tau(t)) + b(\tau(t))v) =: h, \tag{28}$$

with time horizon $\tau(t) = T - t$ and, therefore, using boundary condition $h(T,z) = 1 \,\forall z > 0$, we get:

$$a(0) = a(\tau(T)) = 0, b(0) = b(\tau(T)) = 0.$$

$\star$ Insert the exponentially affine structure (28) of $h(t,v)$ into (27), rearrange the terms to emphasize the linearity in v, and show that the following holds:

$$\begin{aligned} 0 = &- a'(\tau)h + b(\tau)\kappa\theta h + prh + v\Big[- b'(\tau)h + b^2(\tau)\left(\frac{1}{2}\sigma^2 h + \frac{p\sigma^2\rho^2 h}{2(1-p)}\right) \\ &+ b(\tau)\left(-\kappa h + \frac{p\overline{\lambda}\sigma\rho h}{1-p}\right) + \frac{p\overline{\lambda}^2 h}{2(1-p)}\Big] \end{aligned}$$

$\star$ Argue why the previous equation is equivalent to a system of two ODEs for a and b:

$$a'(\tau) = \kappa\theta b(\tau) + pr; \tag{29}$$

$$\begin{aligned} b'(\tau) &= \frac{1}{2}\underbrace{\left(\sigma^2 + \frac{p\sigma^2\rho^2}{1-p}\right)}_{k_2} b^2(\tau) - \underbrace{\left(\kappa - \frac{p\overline{\lambda}\sigma\rho}{1-p}\right)}_{k_1} b(\tau) + \frac{1}{2}\underbrace{\frac{p\overline{\lambda}^2}{1-p}}_{k_0} \\ &= \frac{1}{2}k_2 b(\tau)^2 - k_1 b(\tau) + \frac{1}{2}k_0; \end{aligned} \tag{30}$$

and boundary conditions $a(0) = 0, b(0) = 0$ with constants k_0, k_1, k_2 that have to satisfy $k_1^2 - k_0k_2 > 0$.

According to [Liu and Muhle-Karbe, 2013] (see formulas (3.14) – (3.19) and mind the relation between our notation and their notation) and references therein, the solutions to (29) and (30) under the assumption $k_1^2 - k_0k_2 > 0$ are known:

$$a(\tau) = pr\tau + \frac{2\theta\kappa}{k_2}\ln\left(\frac{2k_3\exp\left(\frac{1}{2}(k_1+k_3)\tau\right)}{2k_3 + (k_1+k_3)\left(\exp(k_3\tau) - 1\right)}\right) \tag{31}$$

$$b(\tau) = k_0\frac{\exp(k_3\tau) - 1}{\exp(k_3\tau)(k_1+k_3) - k_1 + k_3}\ . \tag{32}$$

with $k_3 = \sqrt{k_1^2 - k_0k_2}$. You can use these solutions without deriving them.

$\star$ Show that the ansatz (25) indeed satisfies the first assumption in Subsection 3.2, i.e., $\mathcal{V}_{xx}(t,x,v) < 0$, and, thus, conclude that the candidate for the optimal investment strategy is indeed given by:

$$\pi^*(t) = \frac{\overline{\lambda}}{1-p} + \frac{\sigma\rho}{1-p}b(\tau(t)), \tag{33}$$

where $b(\tau(t))$ is given by (32) with $\tau = \tau(t) = T - t$.

3.3 Interpretation of the results

Due to the arguments in the previous subsection, (33) is only a candidate for the solution to (P). One would need to rigorously prove a so-called verification theorem that states that π^* in (33) is indeed the golbal solution to (P), i.e., there does not exist another investment strategy $\widehat{\pi}$ that yields a higher expected utility at the respective terminal wealth $X^{\widehat{\pi}}(T)$. This verification theorem has been proven in [Kraft, 2005] and [Liu and Muhle-Karbe, 2013]. The rigorous proof of the verification theorem is beyond the scope of Hand In 3. Instead, you are encouraged to learn about the economic aspects of the optimal investment strategy (33).

$\star$ Conduct a sensitivity analysis of the optimal investment strategy π^* with respect to the model parameters. In particular, generate plots of $\pi^*(t), t \in [0, T]$ for varying model parameters, insert them in your Hand In 3 solutions and interprete those plots from both the economic perspective, e.g., how the investment strategy behaves (e.g., more risky, less risky) when each model parameter increases/decreases. Note that from empirical observations of the financial markets $\rho < 0$ and $\overline{\lambda} > 0$.

References

[Kraft, 2005] Kraft, H. (2005). Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility. *Quantitative Finance*, 5(3):303–313.

[Liu and Muhle-Karbe, 2013] Liu, R. and Muhle-Karbe, J. (2013). Portfolio Choice with Stochastic Investment Opportunities: a User's Guide.