Pricing Under the Heston Model Using Discretization Simulation Schemes

Seminar: Asset Prices and Financial Markets

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Contents

1 Financial Models

We shortly present Stochastic Volatility Models (SVM's) and some result for the specific model, namely the Heston model. As the purpose of this paper is to price options using simulation, an in depth exploration and derivation of the results will not be made as finding the pricing formula under the Heston model alone is a long tedious road.

1.1 Introduction to Stochastic Volatility Models

In practice, financial markets exhibit features such as volatility clustering as already described in Mandelbrot (1963), where periods of high volatility are followed by high volatility, and periods of low volatility are followed by low volatility. Additionally, the assumption of constant volatility fails to capture the leverage effect, where volatility tends to increase as asset prices decrease. These limitations of the Black-Scholes model motivate the need for stochastic volatility models.

Stochastic volatility models assume that the volatility of the underlying asset is itself a random process. This allows for a more accurate representation of market dynamics. The general form of a stochastic volatility model can be expressed as:

$$\begin{aligned} dS_t &= a(S_t, v_t, t)dt + b(S_t, v_t, t)dW_{1,t}^{\mathbb{P}}, \quad S(0) > 0; \\ dv_t &= g(v_t, t)dt + h(v_t, t)dW_{2,t}^{\mathbb{P}}, \quad v(0) > 0, \end{aligned}$$

where $dW_{1,t}^{\mathbb{P}}$ and $dW_{2,t}^{\mathbb{P}}$ are two correlated Wiener processes, $dW_{1,t}^{\mathbb{P}}dW_{2,t}^{\mathbb{P}} := d\langle W_{1,t}^{\mathbb{P}}, W_{2,t}^{\mathbb{P}}\rangle(t) = \rho dt \in (-1,1)$, and the functions a, b, g and h are such that the model is well defined. v_t represents the stochastic variance of the asset price. By allowing volatility to change over time in a random manner, these models can better capture the empirical characteristics of asset returns.

However, due to the additional risk in stochastic variance, that is, $dW_{1,t}^{\mathbb{P}} \neq dW_{2,t}^{\mathbb{P}}$, the financial markets with a risk-free asset and a svm are incomplete. This implies that there is an infinite amount of equivalent Martingale measures $\mathbb{Q}$. We can however characterize each equivalent Martingale measure by Girsanov's theorem.

1.2 The Heston Model

The Heston Model, introduced by Steven Heston in 1993, is a popular stochastic volatility model used in financial mathematics for pricing derivatives where the dynamics follow a square-root process. Unlike the Black-Scholes model, which assumes constant volatility, the Heston model incorporates stochastic volatility, making it a bivariate model, allowing it to better capture the empirical features of financial markets such as volatility clustering and the leverage effect. In short, it is a SVM.

The Heston model describes the evolution of the underlying asset price S_t and its variance v_t under the probability measure $\mathbb{P}$ by the following system of stochastic differential equations:

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t}^{\mathbb{P}}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t}^{\mathbb{P}} \end{aligned} \tag{1.1}$$

where:

- S_t is the price of the asset at time t.

- v_t is the instantaneous variance of the asset price at time t.
- μ is the drift rate of the asset.
- κ is the rate at which the variance reverts to its long-term mean θ.
- θ is the long-term mean variance.
- σ is the volatility of the variance process.
- $W_t^{\mathbb{P}}$ and $W_t^{\mathbb{P}}$ are two Wiener processes with correlation ρ.

The correlation ρ between the two Wiener processes is a critical parameter, influencing the dynamics of the asset and its variance. Specifically, $dW_{1,t}^{\mathbb{P}}$ and $W_{2,t}^{\mathbb{P}}$ satisfy

$$dW_{1,t}^{\mathbb{P}} W_{2,t}^{\mathbb{P}} = \rho dt.$$

The Heston model is widely appreciated for several key properties:

- **Stochastic Volatility**: Unlike constant volatility models, the Heston model captures the stochastic nature of volatility observed in real markets.
- **Mean Reversion**: The variance v_t reverts to a long-term mean θ, which aligns with observed market behavior.
- **Leverage Effect**: The negative correlation ρ can model the leverage effect, where asset prices and volatility are inversely related.

As the dynamics does involve the square of the variance, $\sqrt{v_t}$, a neat condition known as the Feller condition Feller (1951):

Theorem 1.1. *The square-root process cannot reach negative values if Feller's condition*

$$2\kappa\theta \geq \sigma$$

is satisfied. For

$$2\kappa\theta < \sigma$$

the origin is accesible and strongly reflective

However, note that this theorem is very rarely satisfied in pratice, or rather in a empirical setting, and even if satisfied might still produce negative variances under discretization of the continuous time process Van Haastrecht and Pelsser (2010).

Just as in most other pricing models, we wish to describe the processes under some risk-neutral probability measure $\tilde{\mathbb{Q}}$, which is also known as the equivalent Martingale measure (EMM). It is therefore the risk-neutral processes that should be used in the further pricing. In simpler models (such as Black-Scholes), under $\tilde{\mathbb{Q}}$, the shift of the probability measure from $\mathbb{P}$ to $\mathbb{Q}$ for the underlying asset (generally applicable to a GBM) occurres simply by changing the drift from μ to r.

In addition to the risk source for the underlying asset, SVM's, such as Heston, also includes a risk source arising from stochastic volatility. To bear the additional risk, a risk-averse investor would expect to obtain a risk premium. This means that the risk premium λ is deducted from the drift in the variance process, meaning we can specify a EMM by $\tilde{\mathbb{Q}}(\lambda)$ for some $\lambda \in \mathbb{R}$. However, following Gatheral (2011), assume that the process where the model is calibrated to option prices precisely provides the risk-neutral process, so the price λ for bearing additional risk due to volatility risk is set to 0. However, in general, the model under some $\tilde{\mathbb{Q}}(\lambda)$-measure can now be summarized as (by Heston (1993)):

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{v(t)}S(t)dW_{1,t}^{\tilde{\mathbb{Q}}} \\ dv(t) &= \tilde{\kappa}(\tilde{\theta} - v(t))dt + \sigma\sqrt{v(t)}dW_{2,t}^{\tilde{\mathbb{Q}}} \\ dW_{1,t}^{\tilde{\mathbb{Q}}}dW_{2,t}^{\tilde{\mathbb{Q}}} &= \rho dt \end{aligned}$$

where

$$\tilde{\kappa} = \kappa + \lambda \quad \text{and} \quad \tilde{\theta} = \frac{\kappa\theta}{\kappa + \lambda}.$$

Since we assumed $\lambda = 0$, the pricing measure can be derived from European option prices, the choice of statistical measure is irrelevant to us which simplifies exactly to our model in equation (1.1).

Heston (1993) shows that for $x = \ln(S_T)$ and some $u \in \mathbb{R}$ the characteristic function (too long and out of the scope of this paper) is given by

$$\Psi_j(x, v, \tau; u) = e^{C_j(\tau;u) + D_j(\tau;u)\cdot v + i\cdot u\cdot x}, \quad j = 1, 2,$$

where $\tau = T - t$ is the time to maturity. The parameters a, b_j, u_j for $j = 1, 2$ are

$$u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa + \lambda,$$

and

$$\begin{aligned} C_j(\tau; u) &= rui\tau + \frac{a}{\sigma^2}\left((b_j - \rho\sigma ui + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\cdot\tau}}{1 - g_j}\right)\right), \\ D_j(\tau; u) &= \frac{b_j - \rho\sigma ui + d_j}{\sigma^2} \cdot \frac{1 - e^{d_j\cdot\tau}}{1 - g_j e^{d_j\cdot\tau}}, \\ g_j &= \frac{b_j - \rho\sigma ui + d_j}{b_j - \rho\sigma ui - d_j}, \\ d_j &= \sqrt{(\rho\sigma ui - b_j)^2 - \sigma^2(2u_j ui - u^2)}, \end{aligned}$$

where $C_j(\tau; u)$ and $D_j(\tau; u)$ are complex-valued functions that solve a system of Riccati differential equations. Let $k = \ln K$, Heston then derives then $Q_j = P_j(x(T) \geq k \mid x(t) = x, v(t) = v)$ for $j = 1, 2$, the conditional probability that the call option ends in-the-money

$$Q_j(x, v, \tau; k) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left\{\frac{e^{-iuk}\Psi_j(x, v, \tau; u)}{iu}\right\}du,$$

yielding the pricing formula by a inverse Fourier transformation

$$C(t) = S_t Q_1(x, v, \tau; k) - Ke^{-r(\tau)}Q_2(x, v, \tau; k) \tag{1.1}$$

However from Albrecher et al. (2006), we know that Heston's original formula is highly unstable for certain subsets of the parameter space. Albrecher et al. (2006) (and more) propose simply reversing the sign of d in Heston's original formula which in turn yields a pricing formula that is stable over the entire domain. This is possible as the complex root, d, can assume two distinct values, which differ only by a sign change. Kendall and Stuart (1977) demonstrate that the integral converges quite rapidly which can be seen in Figure 1 for increasing upper bounds of integration. However, the provided code implementation does allow for integration over the entirety of the positive real numbers.. The code used to very precisely estimate the exact call price is seen in Appendix A.

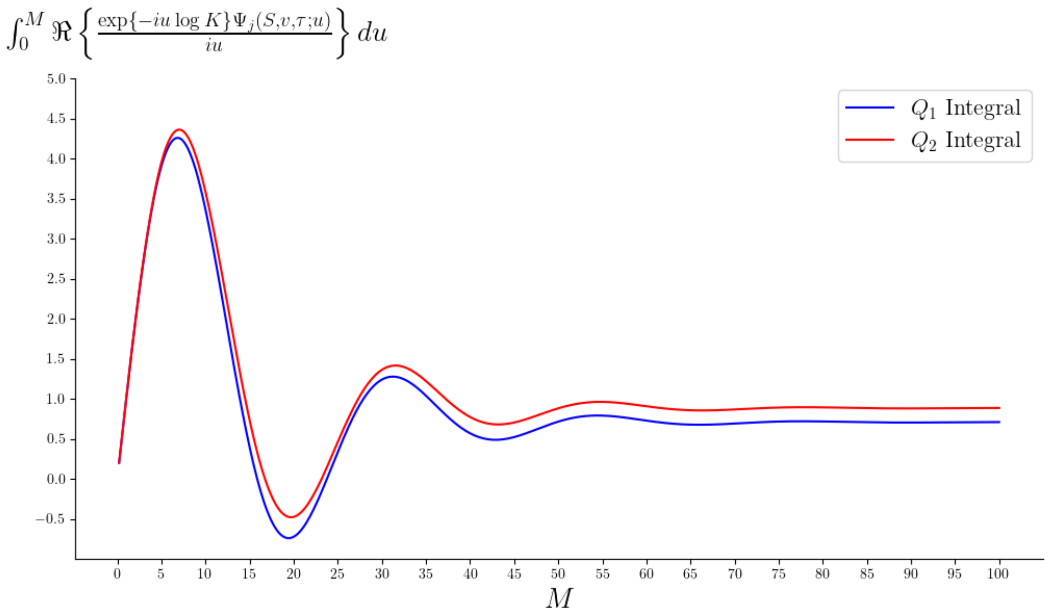

Figure 1: Fourier integral convergence properties illustrated by increasing the upper limit of the integral (shown along the x-axis and integral value on the y-axis). The fixed parameters used in the computation are set to: $\kappa = 2$, $\theta = 0.02$, $\rho = -0.5$, $\sigma = 0.3$, $\tau = 0.5$, $r = 0$, $v_0 = 0.01$, $S = 100$, and $K = 80$.

2 Option Pricing in the Heston Model Using Simulation

Consider an arbitrary set of discrete times $\mathscr{T} = \{t_i\}_{i=1}^N$. We need to generate random paths for the pair (S_t, v_t) for all $t \in \mathscr{T}$. This is important, for example, when pricing path-dependent securities where the payout depends on observations of S_t at specific dates. To develop this approach, we first address the question of how to generate a random sample of $(S_{t+\Delta}, v_{t+\Delta})$ given (S_t, v_t) for any increment Δ. Repeatedly applying this one-period scheme (where Δ may vary at each time in $\mathscr{T}$) will yield a complete path $(S_t, v_t)_{t\in\mathscr{T}}$. Below, we outline several techniques for updating S and v from time t to $t + \Delta$. But firstly, we introduce discretization schemes of continuous time processes, then specific schemes and lastly Monte Carlo methods. The implementations of the schemes in `Python` - the algorithms - is seen in Appendix B.

2.1 Discretization Schemes

Assume that the model is defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and that the stock price S_t is driven by the stochastic differential equation (SDE)

$$dS_t = \underbrace{\mu(S_t, t)}_{\text{drift}}\, dt + \underbrace{\sigma(S_t, t)}_{\text{diffusion}}\, dW_t^{\mathbb{P}}, \tag{2.1}$$

where $W_t^{\mathbb{P}}$ is a Wiener process. We simulate S_t over the time interval $[0, T]$, which we assume is discretized as $0 = t_0 < t_1 < \cdots < t_m = T$, with the time increments equally spaced with width Δ. Equally-spaced time increments are primarily used for notational convenience, allowing us to write $t_i - t_{i-1}$ simply as Δ.

Integrating (2.1) from t to $t + \Delta$ yields:

$$S_{t+\Delta} = S_t + \int_t^{t+\Delta} \mu(S_u, u)du + \int_t^{t+\Delta} \sigma(S_u, u)dW_u^{\mathbb{P}}. \tag{2.2}$$

Equation (2.2) is the starting point for any discretization scheme. At time t, the value of S_t is known, and we aim to obtain the next value $S_{t+\Delta}$, that is, the incremented value of S_t to $S_{t+\Delta}$. This is the bread and butter of both the simple discretization schemes: Euler and Milstein.

For the latter two schemes (or four if we count the *Martingale corrected* versions of the schemes) Truncated Gaussian (TG) and Quadratic-Exponential (QE) we need some intermediate results of the variance process involving its conditional distribution and moments. Andersen (2007) Assumes that $\mu = 0 = r$ when deriving his results, however, fear not: Assume that the forward price process for the stock at expiry-T is $F_t^T = S_t e^{r(T-t)}$. Assume now that we apply some discretization scheme to F_t^T with increments Δ and note that

$$\begin{aligned} S_{t+\Delta} &= F_{t+\Delta}^T e^{-r(T-(t+\Delta))} \\ &= F_t^T \exp(\ \square\ ) e^{-r(T-(t+\Delta))} \\ &= S_t e^{r(T-t)} \exp(\ \square\ ) e^{-r(T-(t+\Delta))} \\ &= S_t \exp(r\Delta +\ \square\ ), \end{aligned}$$

where $\square$ is exactly the terms in the discretization scheme where we started with $\mu = 0 = r$.

Furthermore, we need the following proposition (found in Cox et al. (1985)):
Proposition 2.1. *Let $F_{\chi^2_d}$ be the cumulative distribution function of the non-central chi-squared distribution with non-centrality parameter λ and d degrees of freedom, i.e.*

$$F_{\chi^2_d}(x; d, \lambda) = \sum_{i=0}^{\infty} e^{-\frac{\lambda}{2}} \frac{\left(\frac{\lambda}{2}\right)^i}{i!} \frac{\int_0^x t^{\frac{d}{2}+i-1} e^{-\frac{t}{2}} dt}{2^{\frac{d}{2}+i}\Gamma\left(i+\frac{d}{2}\right)}$$

where Γ is the gamma function. Let $t < T$. Conditional on v_t, v_T is distributed as $\frac{\sigma^2\left(1-e^{-\kappa(T-t)}\right)}{4\kappa}$ times a non-central chi-squared distributed random variable with $\frac{4\kappa\theta}{\sigma^2}$ degrees of freedom and non-centrality parameter $\frac{4\kappa e^{-\kappa(T-t)}}{\sigma^2\left(1-e^{-\kappa(T-t)}\right)} v_t$, i.e.

$$\mathbb{P}\left(v_T \leq x \mid v_t\right) = F_{\chi^2_d}\left(\frac{4\kappa}{\sigma^2 e^{-\kappa(T-t)}} x; \frac{4\kappa\theta}{\sigma^2}, \frac{4\kappa e^{-\kappa(T-t)}}{\sigma^2\left(1-e^{-\kappa(T-t)}\right)} v_t\right)$$

We prove a convenient corollary that follows simply from Proposition 2.1 as we now know the distribution of v_T given v_t:
Corollary 2.1. *Let $T > t$. Conditional on v_t, v_T has the following first two moments:*

$$\begin{aligned}
\mathbb{E}(v_T \mid v_t) &= \theta + (v_t - \theta)e^{-\kappa(T-t)} \\
\operatorname{Var}(v_T \mid v_t) &= \frac{v_t \varepsilon^2 e^{-\kappa(T-t)}}{\kappa}\left(1-e^{-\kappa(T-t)}\right) + \frac{\theta\varepsilon^2}{2\kappa}\left(1-e^{-\kappa(T-t)}\right)^2
\end{aligned}$$

Proof. Let Y be a non-central $\chi^2_d(\lambda)$ distributed random variable with d degrees of freedom and non-centrality parameter λ. then

$$\mathbb{E}[Y] = d + \lambda, \quad \text{and} \quad \operatorname{Var} = 2(d + 2\lambda)$$

The conditional mean and variance is then simply by Proposition 2.1 given as

$$\begin{aligned}
\mathbb{E}\left[v_T \mid v_t\right] &= \frac{\sigma^2\left(1-e^{-\kappa(T-t)}\right)}{4\kappa}\left(\frac{4\kappa\theta}{\sigma^2} + \frac{4\kappa e^{-\kappa(T-t)}}{\sigma^2\left(1-e^{-\kappa(T-t)}\right)} v_t\right) \\
&= \theta\left(1-e^{-\kappa(T-t)}\right) + v_t e^{-\kappa(T-t)},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Var}\left(v_T \mid v_t\right) &= \frac{\sigma^4\left(1-e^{-\kappa(T-t)}\right)^2}{8\kappa^2}\left(\frac{4\kappa\theta}{\sigma^2} + \frac{8\kappa e^{-\kappa(T-t)}}{\sigma^2\left(1-e^{-\kappa(T-t)}\right)} v_t\right) \\
&= \frac{\theta\sigma^2\left(1-e^{-\kappa(T-t)}\right)^2}{2\kappa} + \frac{v_t\sigma^2 e^{-\kappa(T-t)}}{\kappa}\left(1-e^{-\kappa(T-t)}\right).
\end{aligned}$$

□

Lastly, we have from Johnson et al. (1995):
Proposition 2.2. *The non-central chi-square distribution approaches a Gaussian distribution as the non-centrality parameter approaches ∞.*

From Proposition 2.1, we know that $v_{t+\Delta}$ is proportional to a non-central chi-square distribution with non-centrality parameter $v_t \cdot \frac{4\kappa e^{-\kappa((t+\Delta)-t)}}{\sigma^2\left(1-e^{-\kappa((t+\Delta)-t)}\right)}$, where the fraction is independent of v_t. This means, for sufficiently large v_t, a good proxy for $v_{t+\Delta}$ would be a Gaussian variable with the first two moments fitted to match those given in Corollary 2.1 by Proposition 2.2.

Next we consider the contrary, namely small v_t. The non-centrality parameter approaches zero, and the distribution of $v_{t+\Delta}$ becomes proportional to that of an ordinary (central) chi-square distribution with $4\kappa\theta/\sigma^2$ degrees of freedom Johnson et al. (1995). In other words
Proposition 2.3. *Let $\chi_d'^2(\lambda)$ denote a chi-squared random variable with d degrees of freedom and centrality parameter λ. For $d > 1$ the following representation is valid:*

$$\chi_d'^2(\lambda) = \chi_1^2(\lambda) + \chi_{d-1}'^2(\lambda)$$

We recall that the density of a central chi-square distribution with ν degrees of freedom is

$$f_{\chi^2}(x;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2} x^{\nu/2-1}.$$

For many cases of practical relevance, $4\kappa\theta/\sigma^2 \ll 2$, so the presence of the term $x^{\nu/2-1}$ in the above equation implies that, for small v_t, the density of $v_{t+\Delta}$ will be very large around 0. It should be clear that approximation of $v_{t+\Delta}$ with a Gaussian variable is typically not accurate when v_t is close to zero. Note that all the results described above can be generalized to $0 < s < t \leq T$.

2.1.1 Euler Scheme

The Monte Carlo Euler discretization scheme approximates the integrals using the left-point rule, implying that the deterministic integral of (3.2) is approximated as the product of the integrand at time t, and the integration range Δ:

$$\begin{aligned}\int_t^{t+\Delta} \mu(S_t,u)du &\approx \mu(S_t,t)\int_t^{t+\Delta} du \\ &= \mu(S_t,t)\Delta\end{aligned}$$

The left-point rule is a natural candidate as at time t the value of $\mu(S_t,t)$ is known (this is not the case for the right-point rule).

The stochastic integral is approximated in the exact same matter, i.e:

$$\begin{aligned}\int_t^{t+\Delta} \sigma(S_u,u)dW_u^{\mathbb{P}} &\approx \sigma(S_u,u)\int_t^{t+\Delta} \sigma(S_u,u)dW_u^{\mathbb{P}} \\ &= \sigma(S_u,u)(W_{t+\Delta}^{\mathbb{P}} - W_t^{\mathbb{P}}) \\ &= \sigma(S_u,u)\sqrt{\Delta}Z^{\mathbb{P}},\end{aligned}$$

because $W_{t+\Delta}^{\mathbb{P}} - W_t^{\mathbb{P}}$ and $\sqrt{\Delta}Z^{\mathbb{P}}$ are identically distributed, where we define as $Z^{\mathbb{P}}$ a standard normal random variable.

Assembling the results yields the general form of the Monte Carlo Euler discretization scheme of (2.2):

$$S_{t+\Delta} = S_t + \mu(S_t,t)\Delta + \sigma(S_t,t)\sqrt{\Delta}Z^{\mathbb{P}}$$

We now proceed to apply Euler discretization to the Heston model given in equation (1.1). The stochastic differential equation of the variance is in integral form given by

$$v_{t+\Delta} = v_t + \int_t^{t+\Delta} \kappa(\theta - v_u)du + \int_t^{t+\Delta} \sigma\sqrt{v_u}dW_{2,u}^{\mathbb{P}}.$$

The Monte Carlo Euler discretization approximates the integrals by linearity using the left-point rule as

$$\int_t^{t+\Delta} \kappa(\theta - v_u)du \approx \kappa(\theta - v_t)\Delta,$$

and

$$\begin{aligned}\int_t^{t+\Delta} \sigma\sqrt{v_u}dW_{2,u}^{\mathbb{P}} &\approx \sigma\sqrt{v_t}(W_{2,t+\Delta}^{\mathbb{P}} - W_{2,t}^{\mathbb{P}}) \\ &= \sigma\sqrt{v_t\Delta}Z_v^{\mathbb{P}},\end{aligned}$$

as we do not know $v_{t+\Delta}$ at time t and where $Z_v^{\mathbb{P}}$ is a standard normal random variable. These results yield the process for the variance

$$v_{t+\Delta} = v_t + \kappa(\theta - v_t)\Delta + \sigma\sqrt{v_t\Delta}Z_v^{\mathbb{P}}$$

The stochastic differential equation of the stock price is in integral form given by

$$S_{t+\Delta} = S_t + \mu\int_t^{t+\Delta} S_u du + \int_t^{t+\Delta} \sqrt{v_u}S_u dW_{1,u}^{\mathbb{P}}.$$

The Monte Carlo Euler discretization approximates the integrals by linearity using the left-point rule as

$$\int_t^{t+\Delta} S_u du \approx S_t\Delta,$$

and

$$\begin{aligned}\int_t^{t+\Delta} \sqrt{v_u}S_u dW_{1,u}^{\mathbb{P}} &\approx \sqrt{v_t}S_t(W_{1,t+\Delta}^{\mathbb{P}} - W_{1,t}^{\mathbb{P}}) \\ &= \sqrt{v_t\Delta}S_t Z_s^{\mathbb{P}},\end{aligned}$$

where $Z_s^{\mathbb{P}}$ is a standard normal random variable with correlation ρ with the prievously defined random variable $Z_v^{\mathbb{P}}$. These results yield the process for the stock price

$$S_{t+\Delta} = S_t + \mu S_t\Delta + \sqrt{v_t\Delta}S_t Z_s^{\mathbb{P}}.$$

We simulate S_t over the time interval $[0, T]$ (to and including expiry), which we assume to be discretized as $0 = t_1 < t_2 < \ldots < t_m = T$, where the time increments are equally spaced with width Δ. For $n \in \mathbb{N}$ we use the Euler discretization on an equidistan time grid $\left\{t_i = \frac{iT}{n} \mid i = 0, \ldots, n\right\}$, where n is the number of time points after $t = 0$. We start with the initial values S_0 for the stock price and v_0 for the variance. To generate $Z_v^{\mathbb{P}}$ and $Z_s^{\mathbb{P}}$ with correlation ρ, we first generate two independent standard normal variables $Z_1^{\mathbb{P}}$ and $Z_2^{\mathbb{P}}$ and set $Z_v^{\mathbb{P}} = Z_1^{\mathbb{P}}$ and $Z_s^{\mathbb{P}} = \rho Z_1^{\mathbb{P}} + \sqrt{1-\rho^2}Z_2^{\mathbb{P}}$

by Cholesky decomposition. For an arbitrary but fixed λ (i.e no estimation of λ needed) we use the adjusments (as suggested in Heston (1993)):

$$\tilde{\kappa} = \kappa + \lambda \quad \text{and} \quad \tilde{\theta} = \frac{\kappa\theta}{\kappa + \lambda},$$

and thus by the fact that the drift under the risk neutral measure is equal to the risk-free rate

$$\begin{aligned} S_{t+\Delta} &= S_t + rS_t\Delta + \sqrt{v_t\Delta}S_t Z_s^{\tilde{\mathbb{Q}}}, \\ v_{t+\Delta} &= v_t + \tilde{\kappa}(\tilde{\theta} - v_t)\Delta + \sigma\sqrt{v_t\Delta}Z_v^{\tilde{\mathbb{Q}}}, \end{aligned}$$

yielding that the price under the equivalent martingale measure $\tilde{\mathbb{Q}}(\lambda)$ is unique, i.e choosing a different arbitrary $\lambda^\star \neq \lambda$ would yield another equivalent martingale measure $\tilde{\mathbb{Q}}(\lambda^\star)$ among the infinite number of measures.

Lastly, we note that there might occure negative v-values at some of the time points due to discretization errors. When this occurs we use a full truncation scheme, namely, $\max\{0, v_t\}$ but firstly count it towards our zero-variance-count for later reporting.

2.1.2 Milstein Scheme

We assume the coefficients $\mu(S_t), \sigma(S_t)$ only depend on S and not direcctly on t and is described the SDE

$$dS_t = \mu(S_t)\Delta + \sigma(S_t)dW_t^{\mathbb{P}} \tag{2.3}$$

or in integral notation with the shorthand $\mu_t = \mu(S_t), \sigma_t = \sigma(S_t)$

$$S_{t+\Delta} = S_t + \int_t^{t+\Delta} \mu_s ds + \int_t^{t+\Delta} \sigma_s dW_s^{\mathbb{P}}.$$

By Itō's lemma, we find the expansion of μ_t, σ_t

$$\begin{aligned} d\mu_t &= \left(\mu_t'\mu_t + \frac{1}{2}\mu_t''\sigma_t^2\right)\Delta + (\mu_t'\sigma_t)dW_t^{\mathbb{P}} \\ d\sigma_t &= \left(\sigma_t'\mu_t + \frac{1}{2}\sigma_t''\sigma_t^2\right)\Delta + (\sigma_t'\sigma_t)dW_t^{\mathbb{P}} \end{aligned}$$

At time s where $t < s < t + \Delta$ the integral form of the coefficients is

$$\begin{aligned} \mu_s &= \mu_t + \int_t^s \left(\mu_u'\mu_u + \frac{1}{2}\mu_u''\sigma_u^2\right)du + \int_t^s (\mu_u'\sigma_u)dW_u^{\mathbb{P}} \\ \sigma_s &= \sigma_t + \int_t^s \left(\sigma_u'\mu_u + \frac{1}{2}\sigma_u''\sigma_u^2\right)du + \int_t^s (\sigma_u'\sigma_u)dW_u^{\mathbb{P}} \end{aligned}$$

Substituting the integral form expanded expressions for the coefficients into (2.3) yields

$$\begin{aligned} S_{t+\Delta} = S_t &+ \int_t^{t+\Delta}\left(\mu_t + \int_t^s \left(\mu_u'\mu_u + \frac{1}{2}\mu_u''\sigma_u^2\right)du + \int_t^s (\mu_u'\sigma_u)dW_u^{\mathbb{P}}\right)ds \\ &+ \int_t^{t+\Delta}\left(\sigma_t + \int_t^s \left(\sigma_u'\mu_u + \frac{1}{2}\sigma_u''\sigma_u^2\right)du + \int_t^s (\sigma_u'\sigma_u)dW_u^{\mathbb{P}}\right)dW_s^{\mathbb{P}}. \end{aligned}$$

We ignore terms higher than order one, meaning, we ignore $dsdu = \mathcal{O}\left((\Delta)^2\right)$ and $dsdW_u^{\mathbb{P}} = \mathcal{O}\left((\Delta)^{3/2}\right)$ and thus

$$S_{t+\Delta} = S_t + \mu_t \int_t^{t+\Delta} + \sigma_t \int_t^{t+\Delta} dW_s^{\mathbb{P}} + \underbrace{\int_t^{t+\Delta}\int_t^s (\sigma_u'\sigma_u)dW_u^{\mathbb{P}}dW_s^{\mathbb{P}}}_{(*)}. \tag{2.4}$$

Applying Euler discretization from subsection (2.1.2) to $(*)$ in equation (2.4) yields

$$\begin{aligned} \int_t^{t+\Delta}\int_t^s (\sigma_u'\sigma_u)dW_u^{\mathbb{P}}dW_s^{\mathbb{P}} &\approx \sigma_t'\sigma_t \int_t^{t+\Delta}\int_t^s dW_u^{\mathbb{P}}dW_s^{\mathbb{P}} \\ &= \sigma_t'\sigma_t \int_t^{t+\Delta}(W_s^{\mathbb{P}} - W_t^{\mathbb{P}})dW_s^{\mathbb{P}} \\ &= \sigma_t'\sigma_t \left(\int_t^{t+\Delta} W_s^{\mathbb{P}}dW_s^{\mathbb{P}} - W_t^{\mathbb{P}}W_{t+\Delta}^{\mathbb{P}} + \left(W_t^{\mathbb{P}}\right)^2\right) \end{aligned}$$

Define $dY_t = W_t^{\mathbb{P}} dW_t^{\mathbb{P}}$. Using Itō's lemma we see that $Y_t = \frac{1}{2}\left(W_t^{\mathbb{P}}\right)^2 - \frac{1}{2}t$ as $\frac{\partial Y}{\partial t} = -\frac{1}{2}$, $\frac{\partial Y_t}{\partial W_t^{\mathbb{P}}} = W_t^{\mathbb{P}}$ and $\frac{\partial^2 Y_t}{\partial \left(W_t^{\mathbb{P}}\right)^2} = 1$, such that

$$dY_t = \left(-\frac{1}{2} + 0 + \frac{1}{2} \cdot 1 \cdot 1\right)\Delta + (W_t^{\mathbb{P}} \cdot 1) dW_t^{\mathbb{P}} = W_t^{\mathbb{P}} dW_t^{\mathbb{P}}$$

This means we can further rewrite the term $(*)$ in equation (2.4) as

$$\int_t^{t+\Delta} \int_s^t \sigma_u' \sigma_u dW_u^{\mathbb{P}} dW_s^{\mathbb{P}} \approx \frac{1}{2} \sigma_u' \sigma_u \left[\underbrace{(W_{t+\Delta} - W_t)}_{(**)}^2 - \Delta \right],$$

where $(**)$ is equal in distribution to $\sqrt{\Delta} Z$ where Z is distributed as standard normal. Combinding our rewritings of $(*)$ in equation (2.4) we achieve the general form of Milstein discretization scheme

$$S_{t+\Delta} = S_t + \mu_t \Delta + \sigma - t\sqrt{\Delta} Z + \frac{1}{2} \sigma_t' \sigma_t \Delta \left(Z^2 - 1\right). \tag{2.5}$$

We now proceed to apply Milstein discretization to the Heston model given in equation (1.1). Given a value v_t we proceed to $v_{t+\Delta}$ by

$$v_{t+\Delta} = v_t + \tilde{\kappa}(\tilde{\theta} - v_t)\Delta + \sigma\sqrt{v_t \Delta} Z_v^{\tilde{\mathbb{Q}}} + \frac{1}{4}\sigma^2 \Delta \left(\left(Z_v^2\right)^{\tilde{\mathbb{Q}}} - 1 \right),$$

and given a value S_t we proceed to $S_{t+\Delta}$ by

$$S_{t+\Delta} = S_t + r S_t \Delta + \sqrt{v_t \Delta} S_t Z_s^{\tilde{\mathbb{Q}}} + \frac{1}{4} S_t^2 \Delta \left((Z_s^{\tilde{\mathbb{Q}}})^2 - 1 \right).$$

We generate the random variables, under $\tilde{\mathbb{Q}}(\lambda)$, with correlation ρ exactly as described in subsection 2.1.1 and use the altered $\tilde{\kappa}, \tilde{\theta}$.

2.1.3 Truncated Gaussian Scheme

2.1.4 Truncated Gaussian Scheme: Martingale Corrected

2.1.5 Quadratic-Exponential Scheme

2.1.6 Quadratic-Exponential Scheme: Martingale Corrected

2.1.7 Monte Carlo Simulation for Pricing

Our goal is to find a solution to the Heston process, by using the schemes presented above together with Monte Carlo simulation. The solution of the SDE will (hopefully) converge towards the real value. By simulating the process a large number of times, the value will eventually converge towards the real value, $C(0)$. The basic idea of Monte Carlo is to approximate an integral by taking the average of some sequence of simulated paths.

For example, say we wanted to evaluate the following integral/expectation:

$$I = \mathbb{E}[\Phi(X)] = \int \Phi(X) f(X) dx,$$

where $X \in \mathbb{R}^d$, $\Phi : \mathbb{R}^d \to \mathbb{R}$, and f is the probability density function of X. $I = \mathbb{E}[\Phi(X)]$ can then be approximated in the following way:

1. Draw N values $X_1, \ldots, X_N$ i.i.d. from f.
2. The integral can then be evaluated as

$$I \approx \frac{1}{N} \sum_{i=1}^{N} \Phi(X_i).$$

So, to compute an estimate of the European call option price, $\hat{C}(0) = \mathbb{E}_0^{\mathbb{Q}}\Big((\hat{S}_T - K)^+\Big)$, we use Monte Carlo methods. Specifically, for some given discretization scheme of S_t (denote this discretization by $\hat{S}_t$), we draw N independent samples of $\hat{S}_T^{(1)}, \hat{S}_T^{(2)}, \ldots, \hat{S}_T^{(N)}$ using an equidistant time-grid $\left\{t_i = \frac{iT}{n},\ i = 0, \ldots, n\right\}$ with fixed step Δ where $n \in \mathbb{N}$, then $\hat{C}(0)$ is estimated in a Monte Carlo fashion as

$$\hat{C}(0) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\hat{S}_T^{(i)} - K\right)^+.$$

The right-hand side of this equation is a random variable with mean $\hat{C}(0)$ and a standard deviation ("Monte Carlo error") of order $\mathcal{O}(N^{-1/2})$ by the Central Limit Theorem. Using a sufficiently high number N of samples, we can keep the standard deviation low and obtain a high-accuracy estimate for $\hat{C}(0)$ by the Law of Large Numbers.

3 Numerical Tests

4 Discussion

5 Conclusion

A Appendix: Code

GitHub repository holding implementation of every simulation scheme and exact call option calculations.

B Appendix: Scheme Algorithms

We outline the algorithm implementation in `Python`. This does not include in depth methodology to gain efficieny and supplementary functions such as pricing and zero variance count. We omit notation of which measure we are under as that is described excessively in the theoretical parts of the paper.

Algorithm 1 Euler scheme (`generateHestonPathEulerDisc`)

Inputs: $S_0, v_0, r, \kappa, \theta, \sigma, \rho, T, n$
Set $S_0 = S_0$
Set $v_0 = v_0$
Generate $\mathbf{Z}_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
Generate $\mathbf{Z}_2 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
Set $\mathbf{Z}_v = \mathbf{Z}_1$
Set $\mathbf{Z}_s = \rho \mathbf{Z}_1 + \sqrt{1-\rho^2}\mathbf{Z}_2$
for $i = 1$ to n **do**
$\quad v_{t+i} = v_t^+ + \kappa(\theta - v_t^+)\Delta + \sigma\sqrt{v_t^+\Delta}Z_v^{(i)}$
$\quad S_{t+i} = S_t + rS_t\Delta + \sqrt{v_t^+\Delta}S_t Z_s^{(i)}$
end for
return S_n

Algorithm 2 Milstein scheme (`generateHestonPathMilsteinDisc`)

Inputs: $S_0, v_0, r, \kappa, \theta, \sigma, \rho, T, n$
Set $S_0 = S_0$
Set $v_0 = v_0$
Generate $\mathbf{Z}_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
Generate $\mathbf{Z}_2 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
Set $\mathbf{Z}_v = \mathbf{Z}_1$
Set $\mathbf{Z}_s = \rho \mathbf{Z}_1 + \sqrt{1-\rho^2}\mathbf{Z}_2$
for $i = 1$ to n **do**
$\quad v_{t+i} = v_t^+ + \kappa(\theta - v_t)\Delta + \sigma\sqrt{v_t^+\Delta}Z_v^{(i)} + \frac{1}{4}\sigma^2\Delta\left(\left(Z_s^{(i)}\right)^2 - 1\right)$
$\quad S_{t+i} = S_t + rS_t\Delta + \sqrt{v_t^+\Delta}S_t Z_s^{(i)} + \frac{1}{4}S_t^2\Delta\left(\left(Z_s^{(i)}\right)^2 - 1\right)$
end for
return S_n

Algorithm 3 Truncated Gaussian scheme (`generateHestonPathTGDisc`)

Inputs: $S_0, v_0, r, \kappa, \theta, \sigma, \rho, T, n, \gamma_1, \gamma_2, \alpha$
Set $S_0 = S_0$
Set $v_0 = v_0$
Set $\gamma_1 = \gamma_1 \in (0,1)$
Set $\gamma_2 = 1 - \gamma_1 \in (0,1)$
Set $\alpha = \alpha$ around 4 or 5
Compute $K_0 = \frac{-\rho\kappa\theta}{\sigma}\Delta$
Compute $K_1 = \gamma_1\Delta\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) - \frac{\rho}{\sigma}$
Compute $K_2 = \gamma_2\Delta\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) + \frac{\rho}{\sigma}$
Compute $K_3 = \gamma_1\Delta(1-\rho^2)$
Compute $K_4 = \gamma_2\Delta(1-\rho^2)$
Generate $\mathbf{Z}_s \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
Generate $\mathbf{U}_v \sim \mathcal{U}(0,1)^n$
Set $\mathbf{Z}_v = \Phi^{-1}(\mathbf{U}_v)$ where Φ^{-1} is the inverse CDF of the standard normal distribution
for $i = 1$ to n **do**
  Compute $m = \theta + (v_t - \theta)e^{-\kappa\Delta}$
  Compute $s^2 = \frac{v_t\sigma^2 e^{-\kappa\Delta}}{\kappa}\left(1 - e^{-\kappa\Delta}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa\Delta}\right)$
  Set $\psi = \frac{s^2}{m^2}$
  if $\psi^{-1/2} > \alpha$ **then**
    Set $\mu = m$
    Set $\sigma = s$
    $v_{t+i} = \left(\mu + \sigma Z_v^{(i)}\right)$
  else
    Compute $f_\mu(\psi) = \frac{r(\psi)}{\phi(r(\psi)) + r(\psi)\Phi(r(\psi))}$
    Compute $f_\sigma(\psi) = \frac{\psi^{-1/2}}{\phi(r(\psi)) + r(\psi)\Phi(r(\psi))}$
    Set $\mu = f_\mu(\psi) \cdot m$
    Set $\sigma = f_\sigma(\psi) \cdot s$
    $v_{t+i} = \left(\mu + \sigma Z_v^{(i)}\right)$
  end if
  $S_{t+i} = S_t \exp\left(r \cdot \Delta + K_0 + K_1 \cdot v_t\right) \exp\left(K_2 \cdot v_{t+\Delta} + \sqrt{K_3 \cdot v_t + K_4 \cdot v_{t+\Delta}} \cdot Z_s^{(i)}\right)$
end for
return S_n

Algorithm 4 Quadratic-Exponential scheme (`generateHestonPathQEDisc`)

Inputs: $S_0, v_0, r, \kappa, \theta, \sigma, \rho, T, n, \gamma_1, \gamma_2, \alpha$
Set $S_0 = S_0$
Set $v_0 = v_0$
Set $\gamma_1 = \gamma_1 \in (0,1)$
Set $\gamma_2 = 1 - \gamma_1 \in (0,1)$
Set $\psi_C = \psi_C \in [1,2]$
Compute $K_0 = \frac{-\rho\kappa\theta}{\sigma}\Delta$
Compute $K_1 = \gamma_1\Delta\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) - \frac{\rho}{\sigma}$
Compute $K_2 = \gamma_2\Delta\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) + \frac{\rho}{\sigma}$
Compute $K_3 = \gamma_1\Delta(1-\rho^2)$
Compute $K_4 = \gamma_2\Delta(1-\rho^2)$
Generate $\mathbf{Z}_s \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
Generate $\mathbf{U}_v \sim \mathcal{U}(0,1)^n$
Set $\mathbf{Z}_v = \Phi^{-1}(\mathbf{U}_v)$ where Φ^{-1} is the inverse CDF of the standard normal distribution
for $i = 1$ to n **do**
  Compute $m = \theta + (v_t - \theta)e^{-\kappa\Delta}$
  Compute $s^2 = \frac{v_t\sigma^2 e^{-\kappa\Delta}}{\kappa}\left(1 - e^{-\kappa\Delta}\right) + \frac{\theta\sigma^2}{2\kappa}\left(1 - e^{-\kappa\Delta}\right)$
  Set $\psi = \frac{s^2}{m^2}$
  if $\psi \le \psi_C$ **then**
    Set $b^2 = 2\psi^{-1} - 1 + \sqrt{2\psi^{-1}}\sqrt{2\psi^{-1} - 1} \ge 0$
    Set $a = \frac{m}{1+b^2}$
    $v_{t+i} = a\left(b + Z_v^{(i)}\right)^2$
  else
    Compute $p = \frac{\psi - 1}{\psi + 1} \in [0,1)$
    Compute $\beta = \frac{1-p}{m} > 0$
    if $0 \le U_v^{(i)} \le p$ **then**
      $v_{t+i} = 0$
    else
      $v_{t+i} = \beta^{-1}\ln\left(\frac{1-p}{1-U_v^{(i)}}\right)$
    end if
  end if
  $S_{t+i} = S_t \exp\left(r\cdot\Delta + K_0 + K_1\cdot v_t\right)\exp\left(K_2\cdot v_{t+\Delta} + \sqrt{K_3\cdot v_t + K_4\cdot v_{t+\Delta}}\cdot Z_s^{(i)}\right)$
end for
return S_n

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