Continuous-time finance 2

Stochastic volatility models and Fourier methods in option pricing

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Introducing myself

- Born and raised in Kyiv, Ukraine
- Higher education:
  - B.Sc. in System Analysis (= 70% maths + 30% computer science) in Ukraine
  - M.Sc. in Mathematical Finance and Actuarial Science in Germany
  - PhD in Mathematical Finance $\cap$ Actuarial Science in Germany
- Since 2023/01 postdoctoral researcher in the Insurance & Economics Section at UCPH
- Hobbies: traveling, hiking, reading about capital markets
- Fun fact: join at https://www.menti.com/ with code 75942899

Plan for the next 4 lectures

- Lecture on 19 March 2024, big picture about stochastic volatility models (SVMs):
  - recap of Black-Scholes-Merton model and motivation for the extension to stoch. vol.
  - overview of various SVMs
- Lectures on 21 March 2024 (*online*) and on 2 April 2024 (*on-campus*), Heston model:
  - motivation, definition and properties of the Heston model
  - pricing of options on underlying assets driven by the Heston model
- Lecture on 4 April 2024, Fourier methods in option pricing:
  - recap of characteristic functions and motivation for using it in pricing
  - Carr-Madan technique and its applications

Overview of lectures

Overview of stochastic volatility models

Heston model

By the end of this lecture, you will be able to:

1. Recall the Black-Scholes-Merton (BSM) model and its pros and cons
2. Explain the motivation for modelling volatility in a stochastic way
3. List various stochastic volatility models (SVMs)
4. Discuss advantages and disadvantages of each SVM from Point 3

Recap of the BSM model

- Classical Black-Scholes-Merton model defined on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is given by:

$$dB(t) = rB(t)dt, \quad B(0) = 1;$$
$$dS(t) = \mu S(t)dt + \sigma S(t)dW_1^{\mathbb{P}}(t), \quad S(0) > 0,$$

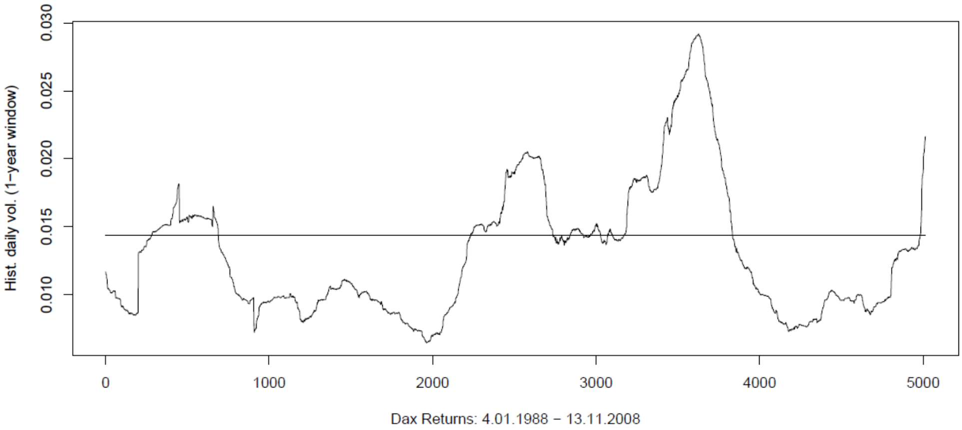
  where $\mu \in \mathbb{R}$, $r \in \mathbb{R}$, $\sigma \in \mathbb{R}_{\geq 0}$ are constants.

- Advantages – analytical beauty and tractability, economic interpretability
- Disadvantages – model assumptions oversimplify the reality, e.g.:
  - constant volatility (but recall the markets during COVID19 outbreak)
  - constant interest rate (but recall the current monetary policy)
  - log-returns are normally distributed (but empirical returns have fat tails)

Why should the assumption of constant volatility be relaxed?

- Reasons:
  1. volatility varies over time
  2. volatility clusters are omnipresent
  3. volatility process and stock price process may have jumps
  4. ...
- Stochastic volatility models (SVMs) address those issues and explain:
  - volatility variation over time
  - volatility clustering
  - negative leverage effect (when stock price $\downarrow$, then volatility often $\uparrow$)

Estimated realized volatility of DAX (fig. taken from Zagst (2023))



Implied vola for options with diff. strikes (Alexander (2008))

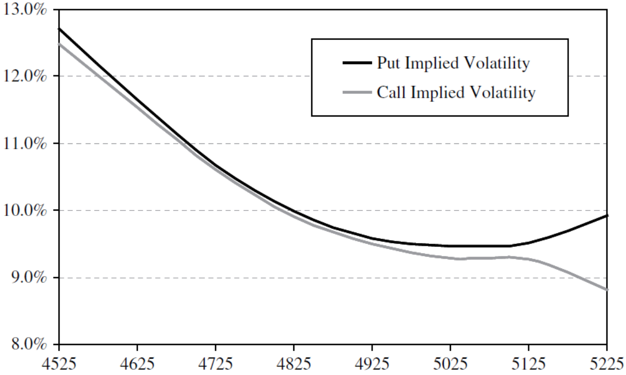


Figure III.4.2 Implied volatility skew of March 2005 FTSE 100 index future options

Implied vola for options with diff. T (fig. from Alexander (2008))

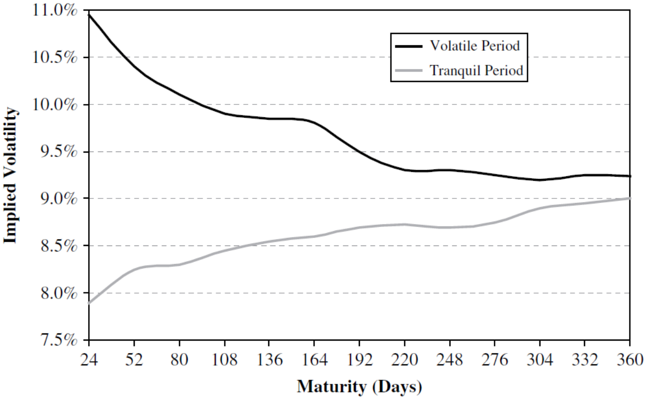
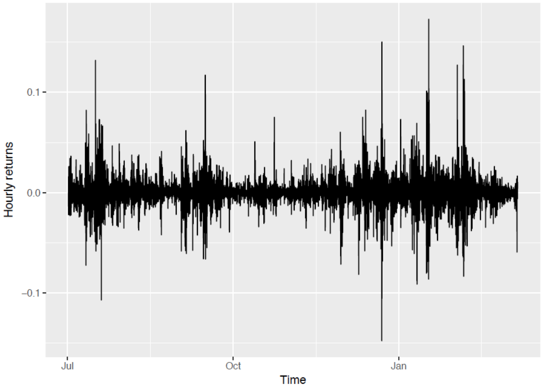
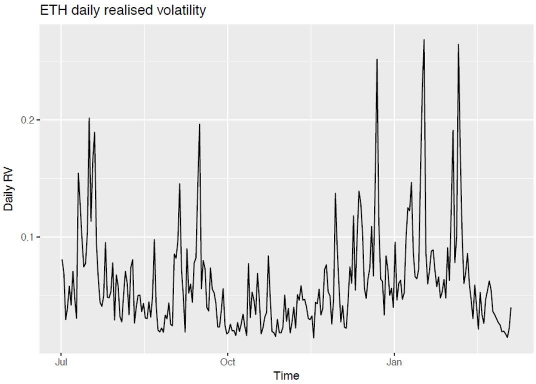


Figure III.4.5 Equity implied volatility term structures

Motivating example for crypto assets



(a) Discrete hourly returns of ETHEREUM



(b) Realised daily volatility

Why should we use SVMs?

- To better reflect the non-deterministic nature of (realized) volatility
- To price options on (realized) volatility, e.g., volatility-sawps, moment swaps, etc.
- To develop volatility trading strategies
- To model the link between negative stock returns and high volatility:
  1. leverage effect: stock price $\downarrow \Rightarrow$ debt-to-equity value $\uparrow \Rightarrow$ equity is riskier $\Rightarrow$ stock vol. $\uparrow$
  2. risk aversion: stock price $\downarrow \Rightarrow$ investors re-balance their portfolios $\Rightarrow$ stock volatility $\uparrow$

$\Rightarrow$ modelling a correlation of stock price change and volatility change is needed

General form of continuous SV drift-diffusion model

- Under $\mathbb{P}$, continuous SV drift-diffusion models have the following structure:

$$dS(t) = a(S(t), v(t), t)dt + b(S(t), v(t), t)dW_1^{\mathbb{P}}(t), \quad S(0) > 0;$$
$$dv(t) = g(v(t), t)dt + h(v(t), t)dW_2^{\mathbb{P}}(t), \quad v(0) > 0,$$

  where $dW_1^{\mathbb{P}}$ and $dW_2^{\mathbb{P}}$ are two correlated Wiener processes and:
  - the functions a, b, g and h are such that the model is well-defined;
  - $dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t) := d\langle W_1^{\mathbb{P}}, W_2^{\mathbb{P}}\rangle(t) = \rho dt$ for $\rho \in (-1, 1)$
- Some SVMs are defined in terms of the volatility process $\sigma(t) = \sqrt{v(t)}$ for $\forall t \geq 0$
- Some SVMs are defined upfront under an equivalent martingale measure $\widetilde{\mathbb{Q}}$

Incompleteness of SVM models

- Due to the additional risk in stochastic variance ($W_2^{\mathbb{P}} \neq W_1^{\mathbb{P}}$):
  - financial markets with a risk-free asset and a SVM is **incomplete**
  - there are **infinitely many** equivalent martingale measures (EMMs) $\widetilde{\mathbb{Q}}$
- Each EMM can be characterized via Girsanov theorem
- Under $\widetilde{\mathbb{Q}}$, continuous SV drift-diffusion models have the following structure:

$$dS(t) = \widetilde{a}(S(t), v(t), t)dt + \widetilde{b}(S(t), v(t), t)dW_1^{\widetilde{\mathbb{Q}}}(t), \quad S(0) > 0;$$
$$dv(t) = \widetilde{g}(v(t), t)dt + \widetilde{h}(v(t), t)dW_2^{\widetilde{\mathbb{Q}}}(t), \quad v(0) > 0,$$

where $\widetilde{a}$, $\widetilde{b}$, $\widetilde{g}$, $\widetilde{h}$ are s.t. the model is well defined, $W_1^{\widetilde{\mathbb{Q}}}$, $W_2^{\widetilde{\mathbb{Q}}}$ are Wiener proc. under $\widetilde{\mathbb{Q}}$

Case with non-correlated S and v (I)

- Assume that $\widetilde{a}(S(t), v(t), t) = rS(t)$ and $\widetilde{b}(S(t), v(t), t) = \sqrt{v(t)}S(t)$, i.e.:

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{v(t)}S(t)dW_1^{\widetilde{\mathbb{Q}}}(t), \quad S(0) > 0; \\ dv(t) &= \widetilde{g}(v(t), t)dt + \widetilde{h}(v(t), t)dW_2^{\widetilde{\mathbb{Q}}}(t), \quad v(0) > 0 \end{aligned} \tag{1}$$

  and $dW_1^{\widetilde{\mathbb{Q}}}(t)dW_2^{\widetilde{\mathbb{Q}}}(t) = 0$.

- Denote by $\overline{\sigma}$ the average volatility:

$$\overline{\sigma} = \sqrt{\frac{1}{T}\int_0^T v(t)dt}$$

Case with non-correlated S and v (II)

- Denote the density function of $\overline{\sigma}$ by $f_{\overline{\sigma}}$
- Let $Call_{SVM}(0, S(0))$ be the $t = 0$ price of a call option on S with strike K
- If $f_{\overline{\sigma}}$ is known, then:

$$Call_{SVM}(0, S(0)) = \int_0^{+\infty} Call_{BSM}(0, S(0); \sigma) f_{\overline{\sigma}}(\sigma) d\sigma, \tag{2}$$

where $Call_{BSM}(0, S(0); \sigma)$ is the call price in the BSM model with t-depend. vol.

- Proof of (2) relies tower property and conditioning on sigma-algebra generated by v

Case with non-correlated S and v (II)

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  where $Call_{BSM}(0, S(0); \sigma)$ is the call price in the BSM model with t-depend. vol.
- Proof of (2) relies tower property and conditioning on sigma-algebra generated by v
- (2) can be generalized to other options

Overview (by no means exhaustive) of SVMs

- Hull–White model, Hull and White (1987)
- Stein–Stein model, Stein and Stein (1991)
- Heston model, Heston (1993)
- Barndorff-Nielsen–Shephard model, Barndorff-Nielsen and Shephard (2001)
- Stochastic Alpha-Beta-Rho model, Hagan et al. (2002)
- GARCH diffusion model, Alexander and Lazar (2005), Alexander and Lazar (2021)
- Markov switching models, Alexander (2008)

Hull–White model definition

- The model was introduced in Hull and White (1987) and is defined under $\widetilde{\mathbb{Q}}$ by:

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_1^{\widetilde{\mathbb{Q}}}(t), \quad S(0) > 0;$$
$$dv(t) = \mu_v v(t)dt + \xi v(t)dW_2^{\widetilde{\mathbb{Q}}}(t), \quad v(0) > 0,$$

  where:

  - μ_v is the instantaneous rate of change of v
  - ξ is the instantaneous volatility of v, also called vol of vol

- The model with $\mu_v = \alpha^2$ and $\xi = 2\alpha$ is also called the Hagan model

Hull–White model pros and cons

- Since v is a GBM, the following quantities can be easily calculated:

$$\mathbb{E}\left[\sigma(t)\right], \ \mathbb{E}\left[v(t)\right], \ \mathbb{V}ar\left[\sigma(t)\right], \ \mathbb{V}ar\left[v(t)\right]$$

- Advantages of the Hull–White model:
    - Simple to interpret and understand
- Disadvantages of the Hull–White model:
    - The are no jumps in volatility or stock price, thus, short-term options are underpriced
    - When simulating from calibrated models, the volatility tend to be too small
    - The volatility does not revert to its mean

Why mean reversion is important?

- Let us have a look at VIX on https://finance.yahoo.com/

Stein–Stein model definition

- The model was introduced in Stein and Stein (1991) and is defined under $\widetilde{\mathbb{Q}}$ by:

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW_1^{\widetilde{\mathbb{Q}}}(t), \quad S(0) > 0;$$
$$d\sigma(t) = \kappa\left(\theta - \sigma(t)\right)dt + \alpha dW_2^{\widetilde{\mathbb{Q}}}(t), \quad \sigma(0) > 0,$$

  where $dW_1^{\widetilde{\mathbb{Q}}}(t)dW_2^{\widetilde{\mathbb{Q}}}(t) = \rho dt$ with $\rho \in (-1, 1)$ and:
  - θ is the long-term average volatility
  - κ is the speed of mean reversion, i.e., the rate at which $\sigma(t)$ reverts to θ
  - α is the volatility of volatility process, i.e., so-called vol of vol
- σ has the same structure as the Vasicek model for short rate

Heuristic explanation about mean reversion

- Ignore the stochastic part and consider only drift:

$$d\sigma(t) = \kappa\left(\theta - \sigma(t)\right) dt$$

- This is a linear ordinary differential equation (ODE):

$$\sigma'(t) = \kappa\left(\theta - \sigma(t)\right)$$

- Solving it, we obtain:

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$$\sigma'(t) = \kappa\left(\theta - \sigma(t)\right)$$

- Solving it, we obtain:

$$\sigma(t) = \left(\sigma(0) - \theta\right) \exp\left(-\kappa t\right) + \theta \tag{3}$$

- Thus, $\lim_{t\uparrow+\infty} \sigma(t) = \theta$ is the long-term mean and κ is the mean-reversion speed

Simulations from the Stein–Stein model

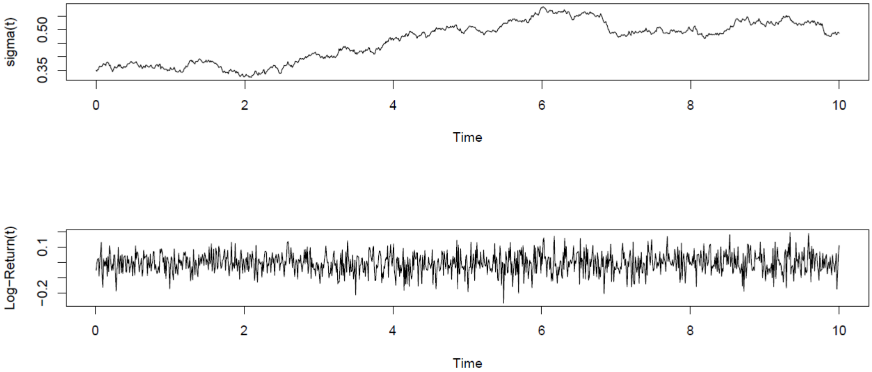


Figure: Log. returns and variance simulated from the Stein–Stein model (fig. taken from Zagst (2023))

Stein–Stein model pros and cons

- Advantages:
  - Analytical tractability and interpretability of the model
  - The characteristic function of $\ln(S(t))$ is known in closed form
- Disadvantages:
  - σ me become negative
  - The direction of the leverage effect can change (when $\sigma < 0$)
  - When simulating from calibrated models, the volatility tend to be too small
  - No jumps.

Heston model definition

- The model was introduced in Heston (1993) and is defined under $\mathbb{P}$ by:

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dW_1^{\mathbb{P}}(t), \quad S(0) > 0;$$
$$dv(t) = \kappa\left(\theta - v(t)\right)dt + \sigma\sqrt{v(t)}dW_2^{\mathbb{P}}(t), \quad v(0) > 0,$$

where $dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t) = \rho dt$ with $\rho \in (-1, 1)$ and:

  - θ is the long-term average variance
  - κ is the speed of mean reversion, i.e., the rate at which $v(t)$ reverts to θ
  - σ is the volatility of variance process, sometimes still called vol of vol

- v has the same structure as the Cox-Ingersoll-Ross model for short rate

Heston model remarks

- If the so-called Feller condition holds

$$\kappa\theta > \frac{1}{2}\sigma^2, \tag{4}$$

then the $v(t) > 0$ for $\forall t \geq 0$

- Heston assumes that under $\widetilde{\mathbb{Q}}$, the variance process is structurally the same:

$$dv(t) = \widetilde{\kappa}\left(\widetilde{\theta} - v(t)\right)dt + \widetilde{\sigma}\sqrt{v(t)}dW_2^{\widetilde{\mathbb{Q}}}(t), \quad v(0) > 0$$

- One of the most popular models in the industry for option pricing

Simulations from the Heston model

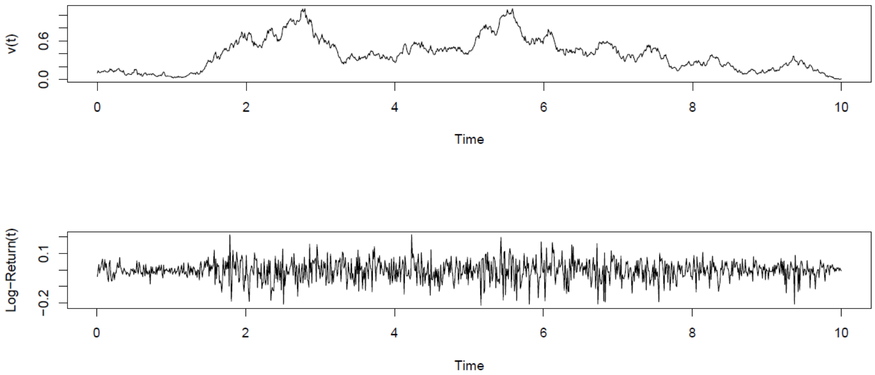


Figure: Log. returns and variance simulated from the Heston model (fig. taken from Zagst (2023))

Heston model pros and cons

- Advantages:
  - Analytical tractability and interpretability of the model
  - The characteristic function of $\ln(S(t))$ is known in closed form
  - Calibration of this model is simpler than that for other models
- Disadvantages:
  - Calibration of this model is still tricky
  - Simulation is tricky, as v may become negative even if (4) holds
  - No jumps in variance or in stock price

Barndorff-Nielsen–Shephard model definition

- This model was introduced in Barndorff-Nielsen and Shephard (2001)
- It is defined for $X(t) := \ln\left(S(t)/S(0)\right)$ by the following SDEs under $\mathbb{P}$:

$$
\begin{aligned}
dX(t) &= (\mu + \beta v(t))\, dt + \sqrt{v(t)} S(t) dW_1^{\mathbb{P}}(t) + \rho dZ\left(\lambda t\right), \quad X(0) = 0; \\
dv(t) &= -\lambda v(t) dt + dZ\left(\lambda t\right), \quad v(0) > 0,
\end{aligned}
$$

where Z is a Levy subordinator (non-decreasing stoch. proc. with independent stationary increments and $Z(0) = 0$) and

- μ and β are drift parameters of the return process
- λ adjusts the variance process
- $\rho < 0$ models the negative leverage effect

Simulations from the BNS model

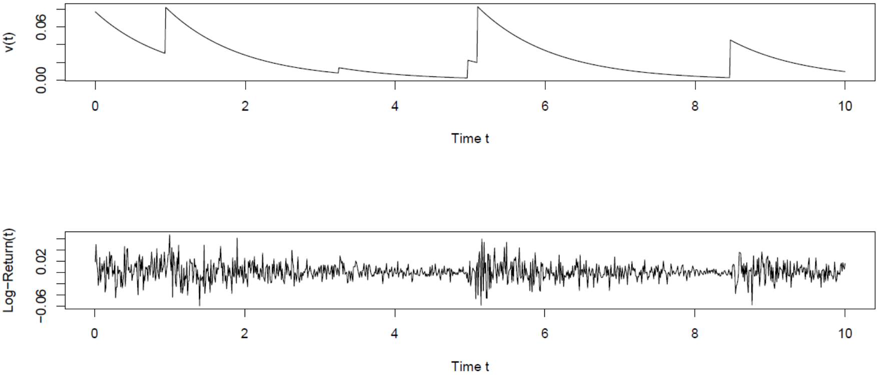


Figure: Log. returns and variance simulated from the BNS model (fig. taken from Zagst (2023))

Barndorff-Nielsen- Shephard pros and cons

- Advantages:
  - Jumps in volatility and in stock price are modelled
  - The characteristic function of $ln(S(t))$ is known
  - Supports negative leverage effect.
- Disadvantages:
  - Jumps in volatility and stock price necessarily coincide.
  - Only negative jumps in the stock price are modelled.

Stochastic $\alpha\beta\rho$ model definition

- Stochastic $\alpha\beta\rho$ model was proposed by Hagan et al. (2002) for forward contracts
- The model is given by the following SDEs:

$$dF(t) = \alpha(t)\,(F(t))^{\beta}\,dW_1^{\widetilde{\mathbb{Q}}}(t);$$
$$d\alpha(t) = \nu\alpha(t)dW_2^{\widetilde{\mathbb{Q}}}(t)$$

where $dW_1^{\widetilde{\mathbb{Q}}}(t)dW_2^{\widetilde{\mathbb{Q}}}(t) = \rho dt$ with $\rho \in (-1, 1)$ and:

  - β is the elasticity parameter
  - ν is the volatility of volatility process, i.e., so-called vol of vol

- Nice tutorials on SABR model can be found at https://www.youtube.com/@quantpie

Stochastic $\alpha\beta\rho$ model pros and cons

- Advantages:
  - Good approximations for the model implied volatility are known
  - Interpretability of the model
  - Relatively straightforward to calibrate.
- Disadvantages:
  - No mean reversion.
  - No jumps.
  - The model is not scale invariant, i.e., changes price if we change the monetary units

Weak GARCH diffusion model definition

- The model was derived in Alexander and Lazar (2005) $\Rightarrow$Alexander and Lazar (2021) as the continuous limit of the standard symmetric GARCH model
- The model is defined under $\mathbb{P}$ by:

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dW_1^{\mathbb{P}}(t), \quad S(0) > 0;$$
$$dv(t) = \kappa\left(\theta - v(t)\right)dt + \sigma v(t)dW_2^{\mathbb{P}}(t), \quad v(0) > 0,$$

where $dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t) = \rho dt$ with $\rho \in (-1, 1)$ and:

- θ is the long-term average variance
- κ is the speed of mean reversion
- σ is the volatility of variance process

GARCH diffusion model pros and cons

- Advantages:
  - The model has a firm micro-econometric foundation
  - Supports negative leverage effect.
- Disadvantages:
  - No simple method for model calibration
  - No jumps
- In Alexander (2008) jumps and Markov-switching are added to weak GARCH model

A simple Markov-switching SVM

- One can integrate different economy regimes using Markov chains (MCs)
- E.g., a 2-state Markov-switching model can be defined under $\widetilde{\mathbb{Q}}$ via:

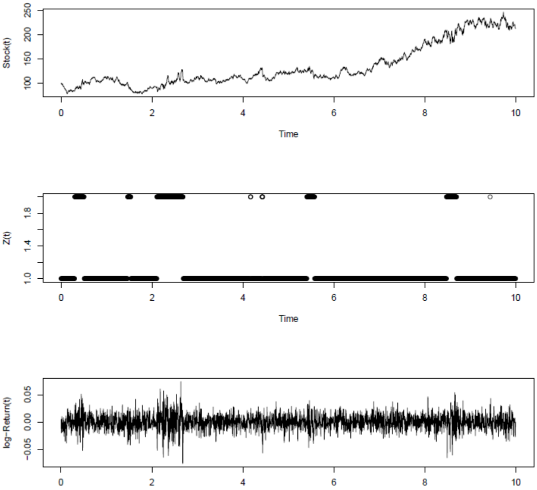
$$dS(t) = rS(t)dt + \sigma_{Z(t-)}S(t)dW_1^{\widetilde{\mathbb{Q}}}(t), \quad S(0) > 0$$

  where $Z(t) \in \{1, 2\}$ defines a 2-state MC in continuous time with $Z(0) = 1$ and intensity matrix

$$\begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$$

- The time between state changes has distribution func. $F(x) = 1 - \exp\left(-\lambda_{Z(t)}x\right)$
- Extension to more states is possible

Simulations from Markov-switching SVMs (fig. from Zagst (2023))



Markov-switching SVM pros and cons

- Advantages:
  - Understandable and interpretable
  - The characteristic function of $ln(S(t)/S(0))$ is known
- Disadvantages:
  - Jumps in stock price are absent
  - Underpricing of short-term options because of neglected stock price jump risk
  - Difficulties in calibration for more states

Intermediate summary

- SVM models relax the assumption of constant volatiltiy made in BSM
- SVM models allows to model the negative leverage effect, volatility clustering
- For Heston model $\exists$ efficient methods for pricing European options $\Rightarrow$ popularity
- Markov-switching can be added to many SVMs, but the resulting model becomes:
  - less analytically tractable
  - more difficult to calibrate

Overview of lectures

Overview of stochastic volatility models

Heston model

By the end of this lecture, you will be able to:

1. Define the Heston model
2. Discuss the impact of model parameters on the implied volatility surface
3. Derive the valuation PDE for puts/calls on $S(T)$ driven by Heston's model

Heston model definition

- The model was introduced in Heston (1993) and is defined under $\mathbb{P}$ by:

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dW_1^{\mathbb{P}}(t), \quad S(0) > 0;$$
$$dv(t) = \kappa\left(\theta - v(t)\right)dt + \sigma\sqrt{v(t)}dW_2^{\mathbb{P}}(t), \quad v(0) > 0,$$

  where $dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t) = \rho dt$ with $\rho \in (-1, 1)$ and:

  - θ is the long-term average variance
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  - σ is the volatility of variance process, sometimes still called vol of vol

- v has the same structure as the Cox-Ingersoll-Ross model for short rate

Representation with orthogonal Wiener processes

- The equivalent representation using *Cholesky decomposition*:

$$
\begin{aligned}
dS(t) &= \mu S(t)dt + \sqrt{v(t)}S(t)(\rho d\widehat{W}_1^{\mathbb{P}}(t) + \sqrt{1-\rho^2}d\widehat{W}_2^{\mathbb{P}}(t)), \quad S(0) > 0; \\
dv(t) &= \kappa\left(\theta - v(t)\right)dt + \sigma\sqrt{v(t)}d\widehat{W}_1^{\mathbb{P}}(t), \quad v(0) > 0,
\end{aligned}
$$

  where $d\widehat{W}_1^{\mathbb{P}}(t)d\widehat{W}_2^{\mathbb{P}}(t) = 0$, i.e., $\widehat{W}_1^{\mathbb{P}}$ and $\widehat{W}_2^{\mathbb{P}}$ are independent

- How to obtain the model dynamics under an EMM $\widetilde{\mathbb{Q}}$?

Representation with orthogonal Wiener processes

- The equivalent representation using *Cholesky decomposition*:

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)(\rho d\widehat{W}_1^{\mathbb{P}}(t) + \sqrt{1-\rho^2}d\widehat{W}_2^{\mathbb{P}}(t)), \quad S(0) > 0;$$
$$dv(t) = \kappa\left(\theta - v(t)\right)dt + \sigma\sqrt{v(t)}d\widehat{W}_1^{\mathbb{P}}(t), \quad v(0) > 0,$$

  where $d\widehat{W}_1^{\mathbb{P}}(t)d\widehat{W}_2^{\mathbb{P}}(t) = 0$, i.e., $\widehat{W}_1^{\mathbb{P}}$ and $\widehat{W}_2^{\mathbb{P}}$ are independent

- How to obtain the model dynamics under an EMM $\widetilde{\mathbb{Q}}$?
- Use Girsanov's Theorem

Reminder of the Girsanov's Theorem (fig. from Zagst (2023))

Theorem 2.109 (Girsanov's theorem)

Let $W = (W^{(1)}, \ldots, W^{(d)})$ be a d-dimensional BM on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. With $\gamma = \{\gamma_t\}_{t \geq 0}$, a suitable d-dimensional process, let

$$L_t := \exp\left\{ -\int_0^t \gamma_s' dW_s - \frac{1}{2}\int_0^t \|\gamma_s\|^2 \, ds \right\}.$$

Define

$$\tilde{W}_t^{(i)} := W_t^{(i)} + \int_0^t \gamma_s^{(i)} ds.$$

Under the equivalent probability measure $\tilde{\mathbb{Q}}$ with Radon-Nikodým derivative

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = L_T,$$

the process $\tilde{W} = (\tilde{W}^{(1)}, \ldots, \tilde{W}^{(d)})$ is a d-dimensional Brownian motion.

From $\mathbb{P}$ to $\widetilde{\mathbb{Q}}$

- We want to find $\gamma_1(t)$ and $\gamma_2(t)$ such that $dW_1^{\widetilde{\mathbb{Q}}}(t) = d\widehat{W}_1^{\mathbb{P}}(t) + \gamma_1(t)dt$ and $dW_2^{\widetilde{\mathbb{Q}}}(t) = d\widehat{W}_2^{\mathbb{P}}(t) + \gamma_2(t)dt$ lead to

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)(\rho dW_1^{\widetilde{\mathbb{Q}}}(t) + \sqrt{1-\rho^2}dW_2^{\widetilde{\mathbb{Q}}}(t))$$
$$dv(t) = smth \cdot dt + smth \cdot dW_1^{\widetilde{\mathbb{Q}}}(t)$$

- Heston (1993) suggests to consider only EMMs that preserve the structure v, i.e.:

$$dS(t) = rS(t)dt + \sqrt{v(t)}S(t)(\rho dW_1^{\widetilde{\mathbb{Q}}}(t) + \sqrt{1-\rho^2}dW_2^{\widetilde{\mathbb{Q}}}(t))$$
$$dv(t) = \widetilde{\kappa}\left(\widetilde{\theta} - v(t)\right)dt + \sigma\sqrt{v(t)}dW_1^{\widetilde{\mathbb{Q}}}(t)$$

where $\widetilde{\kappa} = \kappa + \lambda$, $\widetilde{\theta} = \kappa\theta/(\kappa+\lambda)$ (REQUIRED $\widetilde{\kappa} > 0$ and Feller condition under $\widetilde{\mathbb{Q}}$)

From $\mathbb{P}$ to $\widetilde{\mathbb{Q}}$

- Suggestion as per Heston (1993):

$$\gamma_1(t) = \frac{\lambda}{\sigma}\sqrt{v(t)}$$
$$\gamma_2(t) = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\mu - r}{\sqrt{v(t)}} - \frac{\rho\lambda}{\sigma}\sqrt{v(t)}\right)$$

- These special γ_1, γ_2 do not generate all possible EMMs, but:
  - lead to analytically tractable results for option pricing
  - have economic arguments that support such as choice
- The choice of λ is based on the best fit of model prices under $\widetilde{\mathbb{Q}}(\lambda)$ to market prices

Further practical insights into model parameters

- Let us look at some interactive plots from 5:02 to 9:19 here:

  https://www.youtube.com/watch?v=csZFUoE3uuA

Further practical insights into model parameters

- Let us look at some interactive plots from 5:02 to 9:19 here:

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- $v(0)$ and θ control the level of the implied volatility curve, the second moment,
- $\widetilde{\kappa}$ controls the speed of convergence of v to θ
- ρ control the skew of the implied volatility surface
- σ control the smile of the implied volatiltiy

How to derive the PDE for pricing options? (I)

- Way 1: construct a self-financing portfolio that hedges the value of an option.
- The portfolio consists of positions in:
  - bank (money market) account B
  - stock S
  - some other asset (e.g., option) that has exposure to v
- Way 2: use Feynman-Kac Theorem

Intended learning outcomes

1. Derive the PDE for pricing call/put options on $S(T)$ driven by HM
2. Apply the Change of Numéraire Th to simplify the pricing PDE
3. Use characteristic function of $\ln(S(T))$ to "solve" the pricing PDE
4. Discuss potential challenges (and remedies) of numerical applications of HM
5. Get an overview of assignments in Hand In 3

Reminder of strong solutions to SDEs (fig. from my dissertation)

Definition 2.2.6 (Strong solution of an SDE, Def. 3.34, p. 157, Korn (2014)). *If on* $(\Omega, \mathcal{F}, \mathbb{Q})$ *there exists an* m*-dimensional continuous process* $X = (X(t))_{t\geq 0}$ *with*

$$X(0) = x, \quad x \in \mathbb{R}^m \quad \textit{constant}, \tag{2.22}$$

$$X_i(t) = x_i + \int_0^t \mu_i(s, X(s))\, ds + \sum_{j=1}^n \int_0^t \sigma_{i,j}(s, X(s))\, dW_j^{\mathbb{Q}}(s) \tag{2.23}$$

$\mathbb{Q}$*-a.s. for all* $t \geq 0$, $i \in 1, \dots, m$, *such that*

$$\int_0^t \left(|\mu_i(s, X(s))| + \sum_{j=1}^n (\sigma_{i,j}(s, X(s)))^2 \right) ds < +\infty \tag{2.24}$$

$\mathbb{Q}$*-a.s. for all* $t \geq 0$, $i \in 1, \dots, m$, *holds, then* X *is called a strong solution of the SDE:*

$$dX(t) = \mu(t, X(t))\, dt + \sigma(t, X(t))\, dW^{\mathbb{Q}}(t); \tag{2.25}$$

$$X(0) = x \tag{2.26}$$

for given functions $\mu : [0, +\infty) \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : [0, +\infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

Reminder of Characteristic Operators (fig. from my dissertation)

Definition 2.2.7 (Characteristic operator, Def. 3.39, p. 164, Korn (2014)). *Let X be the unique solution of SDE (2.25) such that $\mu(t,x)$ as well as $\sigma(t,x)$ are continuous and satisfy the following conditions:*

$$||\mu(t,x)-\mu(t,y)|| + ||\sigma(t,x)-\sigma(t,y)|| \leq K||x-y|| \tag{2.27}$$

$$||\mu(t,x)||^2 + ||\sigma(t,x)||^2 \leq K^2\left(1+||x||^2\right) \tag{2.28}$$

for all $t \geq 0$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and a constant $K > 0$. For a twice continuously differentiable function $f : \mathbb{R}^D \to \mathbb{R}$, the operator $\mathcal{D}_t$ defined by

$$\left(\mathcal{D}_t f\right)(x) := \frac{1}{2}\sum_{i=1}^{m}\sum_{k=1}^{m} a_{i,k}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_k}(x) + \sum_{i=1}^{m}\mu_i(t,x)\frac{\partial f}{\partial x_i}(x) \tag{2.29}$$

with

$$a_{i,k}(t,x) := \sum_{j=1}^{n}\sigma_{i,j}(t,x)\sigma_{k,j}(t,x)$$

is called the characteristic operator for X.

Reminder of Feynman-Kac Theorem (fig. from my dissertation)

Theorem 2.2.8 (Feynman-Kac representation, Th. 3.41, p. 165, Korn (2014)). *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a twice continuously differentiable function that satisfies the following condition:*

$$|f(x)| \leq L\left(1 + ||x||^{2\lambda}\right) \quad \text{or} \quad f(x) \geq 0 \tag{2.30}$$

for $\lambda \geq 1$. Assume that there exists a solution $\mathcal{V}(t,x) : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ to the following Cauchy problem:

$$-\mathcal{V}_t + k\mathcal{V} = \mathcal{D}_t\mathcal{V} \quad \text{on} \quad [0,T) \times \mathbb{R}^m; \tag{2.31}$$

$$\mathcal{V}(T,x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^m, \tag{2.32}$$

where $k : [0,T] \times \mathbb{R}^m \to [0,+\infty)$ is a continuous function, $\mathcal{V}(t,x)$ is continuously differentiable w.r.t. t and twice continuously differentiable w.r.t. x, $\mathcal{D}_t$ is the characteristic operator for X that is a unique solution to SDE (2.25) with μ as well as σ being continuous and satisfying conditions (2.27) and (2.28) respectively. If also $\mathcal{V}(t,x)$ satisfies the following condition:

$$\max_{t\in[0,T]} |\mathcal{V}(t,x)| \leq M(1 + ||x||^{2\eta}) \quad \text{for} \quad M > 0, \eta \geq 1, \tag{2.33}$$

then $\mathcal{V}(t,x)$ has the following representation:

$$\mathcal{V}(t,x) = \mathbb{E}^{\mathbb{Q}}\left[f(X(T)) \cdot \exp\left(-\int_t^T k(s,X(s))ds\right) \middle| X(t) = x\right] \tag{2.34}$$

and is a unique solution to the Cauchy problem (2.31)-(2.32), which fulfills condition (2.33).

How to derive the PDE for pricing options? (II)

- Choose $\lambda \in \mathbb{R} \Rightarrow \widetilde{\mathbb{Q}} := \widetilde{\mathbb{Q}}(\lambda)$ becomes fixed, $W^{\widetilde{\mathbb{Q}}} := \left(W_1^{\widetilde{\mathbb{Q}}}, W_2^{\widetilde{\mathbb{Q}}}\right)^\top$ is a Wiener proc.
- Let $X := (S, v)^\top \Rightarrow dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW^{\widetilde{\mathbb{Q}}}(t)$ with:

$$\mu\left(t, (S, v)^\top\right) = \begin{pmatrix} rS \\ \kappa(\theta - v) - \lambda v \end{pmatrix} \quad \text{and} \quad \sigma\left(t, (S, v)^\top\right) = \begin{pmatrix} \sqrt{v} S \rho & \sqrt{v} S \sqrt{1-\rho^2} \\ \sigma\sqrt{v} & 0 \end{pmatrix}$$

- Applying FK Th for above X, option payoff $f(X(T))$, $k(t, X(t)) = r$, we get:

$$-\mathcal{V}_t + r\mathcal{V} = \frac{1}{2} v S^2 \mathcal{V}_{SS} + \rho\sigma v S \mathcal{V}_{vS} + \frac{1}{2}\sigma^2 v \mathcal{V}_{vv} + rS\mathcal{V}_S + (\kappa(\theta - v) - \lambda v)\, \mathcal{V}_v,$$

$$\mathcal{V}\left(T, (S, v)^\top\right) = f\left((S, v)^\top\right) \quad \forall (S, v) \in (0, +\infty)^2$$

where $\mathcal{V}\left(t, (S, v)^\top\right)$ is the option price (technical assumptions must be verified)

How to solve the HM PDE for a European call option? (I)

- Consider a European call with payoff $(S(T)-K)^+$, $S(T)$ follows HM
- Let $\mathcal{V}(t,S(t),v(t))$ be the value process of a call, $\mathcal{V}(T,S(T),v(T))=(S(T)-K)^+$
- To derive the call price $\mathcal{V}(t,S(t),v(t))$, we:
    1. Change variables $x:=\ln(S)$ to get a PDE for $\mathcal{V}(t,x,v)$
    2. Use the ansatz inspired from call option pricing in Black-Scholes-Merton model:
    $$\mathcal{V}(t,x,v)=e^x Q_1(x,v,t)-Ke^{-r(T-t)}Q_2(x,v,t), \tag{5}$$
    3. Show that $Q_1(x,v,t)$ and $Q_2(x,v,t)$ solve two separate PDEs with terminal conditions:
    $$Q_j(x,v,T;\ln(K))=\mathbb{1}_{\{x\geq\ln(K)\}},\quad j=1,2.$$

    PDEs for Q_1 and Q_2 are still too complicated for closed-form solution... BUT

How to solve the HM PDE for a European call option? (II)

... $Q_1 \& Q_2$ have a convenient probabilistic representation

4 Represent $Q_1 \& Q_2$ via integrals of char. functions $\Psi^j_{x(T)}$ of $x(T) = \ln(S(T))$, $j = 1, 2$

5 Derive the PDE for $\Psi^j_{x(T)}$, $j = 1, 2$, and obtain their *explicit analytical form*

6 Insert $\Psi^j_{x(T)}$, $j = 1, 2$ into the result of Step 4.

- We show now Steps 1–4 and 6, whereas Step 5 is part of Hand In 3
- In derivations below, we follow Section 1.7.3. in Desmettre and Korn (2018) [DK2018]
- All unreferenced pictures below have been taken from DK2018

Steps 1 and 2

- Step 1: changing variables $x := \ln(S)$, we obtain the following PDE:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\nu\frac{\partial V}{\partial x} + \frac{1}{2}\nu\frac{\partial^2 V}{\partial x^2} + \rho\sigma\nu\frac{\partial^2 V}{\partial \nu \partial x} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 V}{\partial \nu^2} + r\frac{\partial V}{\partial x} - rV$$
$$+ (\kappa(\theta - \nu) - \lambda\nu)\frac{\partial V}{\partial \nu} = 0\,,$$

  where terminal condition is $\mathcal{V}(T, x, v) = (e^x - K)^+ \quad \forall (x, v) \in \mathbb{R} \times (0, +\infty)$

- Step 2: ansatz (5) $[\mathcal{V}(t, x, v) = e^x Q_1(x, v, t) - Ke^{-r(T-t)}Q_2(x, v, t)]$ is inspired by:
  - $\mathbb{E}^{\widetilde{\mathbb{Q}}}\left(\frac{B(t)}{B(T)}(S(T)-K)^+|\mathcal{F}_t\right) = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left(\frac{B(t)S(T)}{B(T)}\mathbb{1}_{\{S(T)\geq K\}}|\mathcal{F}_t\right) - \mathbb{E}^{\widetilde{\mathbb{Q}}}\left(\frac{B(t)K}{B(T)}\mathbb{1}_{\{S(T)\geq K\}}|\mathcal{F}_t\right)$
  - Change of Numéraire Theorem

Step 2: Change of Numéraire Theorem (fig. from Zagst (2023))

Theorem 3.20 (Change of numéraire)

Let $N = \{N_t\}_{t\geq 0}$ be a numéraire such that $N/P^{(0)}$ is a $\tilde{\mathbb{Q}}$-martingale. Define the new measure $\mathbb{Q}^N$ via

$$\frac{d\mathbb{Q}^N}{d\tilde{\mathbb{Q}}}\bigg|\mathcal{F}_t := \eta_t = \frac{N_t}{P_t^{(0)}}\frac{P_0^{(0)}}{N_0}, \quad \frac{d\mathbb{Q}^N}{d\tilde{\mathbb{Q}}} := \eta_T.$$

Then, the processes $P^{(i)}/N$ are $\mathbb{Q}^N$-martingales. Moreover, for any $\tilde{\mathbb{Q}}$-attainable contingent claim D we have

$$P_t^{(0)}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[\frac{D}{P_T^{(0)}}\bigg|\mathcal{F}_t\right] = N_t\mathbb{E}_{\mathbb{Q}^N}\left[\frac{D}{N_T}\bigg|\mathcal{F}_t\right].$$

- In our application of this theorem in Step 2:
  - $P^{(0)}$ corresponds to our risk-free asset B
  - N corresponds to our risky asset S driven by the Heston model
  - D corresponds to our $S(T)$

Steps 2 and 3

- Inserting (5) from Step 2 into PDE for $\mathcal{V}(t,x,v)$, we get:

$$
e^x \underbrace{\left\{ \begin{array}{c} \frac{\partial Q_1}{\partial t} + \left(r + \frac{1}{2}\nu\right)\frac{\partial Q_1}{\partial x} + \frac{1}{2}\nu\frac{\partial^2 Q_1}{\partial x^2} + \rho\sigma\nu\frac{\partial^2 Q_1}{\partial \nu \partial x} \\ + \frac{1}{2}\sigma^2\nu\frac{\partial^2 Q_1}{\partial \nu^2} + (\rho\sigma\nu + \kappa\theta - \kappa\nu - \lambda\nu)\frac{\partial Q_1}{\partial \nu} \end{array} \right\}}_{(*)}
$$

$$
- Ke^{-r(T-t)} \underbrace{\left\{ \begin{array}{c} \frac{\partial Q_2}{\partial t} + \left(r - \frac{1}{2}\nu\right)\frac{\partial Q_2}{\partial x} + \frac{1}{2}\nu\frac{\partial^2 Q_2}{\partial x^2} + \rho\sigma\nu\frac{\partial^2 Q_2}{\partial \nu \partial x} \\ + \frac{1}{2}\sigma^2\nu\frac{\partial^2 Q_2}{\partial \nu^2} + (\kappa\theta - \kappa\nu - \lambda\nu)\frac{\partial Q_2}{\partial \nu} \end{array} \right\}}_{(**)} = 0
$$

- Due to separation of Q_1 and Q_2, this PDE is equiv. to a system of (*) and (**)

Step 3

$$\frac{\partial Q_j}{\partial t} + (r + u_j v)\frac{\partial Q_j}{\partial x} + \frac{1}{2}v\frac{\partial^2 Q_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 Q_j}{\partial v \partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 Q_j}{\partial v^2} + (a - b_j v)\frac{\partial Q_j}{\partial v} = 0$$

with

$$u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}, \quad a = \kappa\theta, \quad b_1 = \kappa + \lambda - \rho\sigma, \quad b_2 = \kappa + \lambda. \tag{6}$$

and terminal conditions:

$$Q_j(x, v, T; \ln(K)) = \mathbb{1}_{\{x \geq \ln(K)\}}, \quad j = 1, 2.$$

- Closed-form solution to above PDEs are not know, BUT there is a workaround

Probabilistic representation of Q_1 and Q_2 (Prop. 1.65 in DK2018)

- Let $x = (x(t))_{t\geq 0}$ and $v = (v(t))_{t\geq 0}$ have the following SDEs under some $\mathbb{P}_j$, $j = 1, 2$:

$$\begin{aligned} dx(t) &= (r + u_j v(t))dt + \sqrt{v(t)}dW_1^{\mathbb{P}_j}(t) \\ dv(t) &= (a - b_j v(t))dt + \sigma\sqrt{v(t)}dW_2^{\mathbb{P}_j}(t) \end{aligned} \tag{7}$$

  with $dW_1^{\mathbb{P}_j}(t)dW_2^{\mathbb{P}_j}(t) = \rho dt$ and parameters as on the previous slide.

- Then Q_j is the conditional probability that the call *option ends in-the-money (ITM)*:

$$Q_j(x, v, t; \ln(K)) = \mathbb{P}_j\left(x(T) \geq \ln(K) | x(t) = x, v(t) = v\right), \quad j = 1, 2, \tag{8}$$

  where $\mathbb{P}_1$ and $\mathbb{P}_2$ are some "adjusted" probability measures

- It is possible to derive characteristic functions of $x(T)$ and use them to compute Q_j

Characteristic functions and Fourier transforms (I)

- Let X be a random variable (RV) with a probability measure μ_X. Then:

$$\Psi_X(u) := \int_{-\infty}^{+\infty} e^{iux}\mu_X(dx) = \left(\int_{-\infty}^{+\infty} e^{iux} \underbrace{f_X(x)}_{\text{PDF if } \exists} dx\right) = \mathbb{E}\left[e^{iuX}\right] \tag{9}$$

  is the characteristic function (CF) of X (PDF = "probability density function").

- Let μ be a finite positive measure on $(\mathcal{B}(\mathbb{R}), \mathbb{R})$. Then:

$$\widehat{\mu}(u) := \int_{-\infty}^{+\infty} e^{iux}\mu(dx) \tag{10}$$

  is the Fourier transform (FT) of measure μ.

Characteristic functions and Fourier transforms (II)

- So CF of X is the FT of the probability measure μ_X induced by X:

$$\Psi_X(u) \overset{(9)}{=} \int_{-\infty}^{+\infty} e^{iux}\mu_X(dx) \overset{(10)}{=} \widehat{\mu}_X(u) \tag{11}$$

- Let $f : \mathbb{R} \to \mathbb{C}$ be integrable. Then:

$$\widehat{f}(u) = \int_{-\infty}^{+\infty} e^{iux} f(x)dx \quad \text{is the Fourier transform of } f \tag{12}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\widehat{f}(u)du \quad \text{is the inverse Fourier transform of } \widehat{f} \tag{13}$$

Fourier inversion formula

- Let X be an $\mathbb{R}$-valued RV with a cumulative distribution function F_X
- If F_X is continuous at $x = b$, then the *Fourier inversion formula* holds:

$$F_X(b) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \frac{1}{iu} \left(e^{iub} \underbrace{\widehat{\mu_X}(-u)}_{\underset{(11)}{=} \Psi_X(-u)} - e^{-iub} \underbrace{\widehat{\mu_X}(u)}_{\underset{(11)}{=} \Psi_X(u)} \right) du$$

- A consequence of the Fourier inversion formula:

$$\mathbb{P}(X \geq b) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \mathfrak{Re} \left(\frac{e^{-iub} \Psi_X(u)}{iu} \right) du, \tag{14}$$

where $\mathfrak{Re}(z)$ denotes the real part of $z \in \mathbb{C}$, i.e., for $z = v + iw$ we have $\mathfrak{Re}(z) = v$.

Using Fourier transforms for option pricing in Heston model (I)

- According to (8), we have:

$$Q_j(x, v, t; \ln(K)) = \mathbb{P}_j\left(x(T) \geq \ln(K) | x(t) = x, v(t) = v\right), \quad j = 1, 2,$$

where $\mathbb{P}_1$ and $\mathbb{P}_2$ are some "adjusted" probability measures, $x(T) = \ln(S(T))$

- The CF $\Psi^j_{x(T)}(x, v, t; u) := \mathbb{E}^{\mathbb{P}_j}\left[e^{i \cdot u \cdot x(T)} | x(t) = x, v(t) = v\right]$ satisfies the PDE:

$$0 = \frac{\partial \Psi_j}{\partial t} + (r + u_j v)\frac{\partial \Psi_j}{\partial x} + \frac{1}{2} v \frac{\partial^2 \Psi_j}{\partial x^2} + \rho \sigma v \frac{\partial^2 \Psi_j}{\partial v \partial x} + \frac{1}{2}\sigma^2 v \frac{\partial^2 \Psi_j}{\partial v^2} + (a - b_j v)\frac{\partial \Psi_j}{\partial v}; \tag{15}$$

$$e^{i \cdot u \cdot x} = \Psi^j_{x(T)}(x, v, T; u) \quad \forall (x, v) \in \mathbb{R} \times (0, +\infty), \tag{16}$$

where we write $\Psi_j := \Psi^j_{x(T)}(x, v, t; u)$, $j = 1, 2$.

Using Fourier transforms for option pricing in Heston model (II)

- Heston (1993) shows that (also an exercise in Hand In 3):

$$\Psi^j_{x(T)}(x, v, t; u) = e^{C_j(\tau;u)+D_j(\tau;u)\cdot v+i\cdot u\cdot x}, \quad j = 1, 2. \tag{17}$$

where $\tau = T - t$, parameters a, b_j, u_j for $j = 1, 2$ are given in (6) and

$$C_j(\tau; u) = rui\tau + \frac{a}{\sigma^2}\left(\left(b_j - \rho\sigma ui + d_j\right)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\cdot\tau}}{1 - g_j}\right)\right) \tag{18}$$

$$D_j(\tau; u) = \frac{b_j - \rho\sigma ui + d_j}{\sigma^2} \cdot \frac{1 - e^{d_j\cdot\tau}}{1 - g_j e^{d_j\cdot\tau}} \tag{19}$$

$$g_j = \frac{b_j - \rho\sigma ui + d_j}{b_j - \rho\sigma ui - d_j}$$

$$d_j = \sqrt{\left(\rho\sigma ui - b_j\right)^2 - \sigma^2\left(2u_j ui - u^2\right)}$$

Collecting the results for option pricing in Heston model

- In the Heston model, the price of a call under $\widetilde{\mathbb{Q}}$ corresponding to λ is given by:

$$Call(t) = S(t)Q_1(\ln(S(t)), v(t), t; \ln(K)) - Ke^{-r(T-t)}Q_2(\ln(S(t)), v(t), t; \ln(K))$$

  where:

$$Q_j(\ln(S(t)), v(t), t; \ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathfrak{Re}\left(\frac{e^{-iu\ln(K)}\Psi^j_{\ln(S(T))}(\ln(S(t)), v(t), t; u)}{iu}\right) du$$

  and the CFs $\Psi^j_{\ln(S(T))}(\ln(S(t)), v(t), t; u)$ are given by (17).

- Computation of the above integrals can be numerically tricky

Intended learning outcomes

1. Refresh in memory several key aspects of the Heston model
2. Discuss in detail the calibration of HM, its potential challenges and remedies
3. Provide feedback to Rolf and Yevhen regarding the FinKont2 course
4. Learn how to use the general Carr-Madan approach to option pricing
5. Apply Carr-Madan technique to price a call option
6. Discuss updates in Hand In 3

Calibration and the choice of an EMM $\widetilde{\mathbb{Q}}$ (I)

- Due to the incompleteness of the market, $\exists$ inf. many $\widetilde{\mathbb{Q}} \Rightarrow$ inf. many option prices
- For an arbitrary but fixed λ (parameterizing $\gamma_2(t)$ as per Heston), the price is unique
- For pricing a call, we need model parameters:

$$\Theta := (\kappa, \theta, \sigma, \rho, v(0), \lambda) \in (0, +\infty]^3 \times [-1, 1] \times (0, +\infty) \times \mathbb{R} =: \Xi$$

- For $Call^M$ market prices, $Call^H$ Heston-model prices dep. on Θ, we calibrate via:

$$\min_{\Theta \in \Xi} \sum_{i=1}^{N} w_i L(Call^M(K_i, T_i), Call^H(0, S(0), K_i, T_i))$$

where $w_1, \ldots, w_N > 0$ are weights, $L : (0, +\infty)^2 \to \mathbb{R}$ is loss func. (e.g., abs. error)

Calibration and the choice of an EMM $\widetilde{\mathbb{Q}}$ (II)

- When estimation of λ is not needed, HM is calibrated w.r.t $\widetilde{\Theta} = \left(\widetilde{\kappa}, \widetilde{\theta}, \sigma, \rho, v(0)\right)$
- In practice, HM may be recalibrated several times per day and sometimes via:

$$\min_{\Theta \in \Xi} \sum_{i=1}^{N} w_i L(Call^M(K_i, T_i), Call^H(0, S(0), K_i, T_i)) + \delta \|\Theta - \overline{\Theta}\|^2,$$

  where $\overline{\Theta}$ is the estimate Θ from previous calibration, $\delta > 0$ is a weight parameter
- Usage of different starting values in above optimization is helpful
- It is important to check obtained Θ^* of for plausibility!

Calibration and the choice of an EMM $\widetilde{\mathbb{Q}}$ (III)

- Escobar and Gschnaidtner (2016) show that parameter recovery can be misleading
- Three factors that strongly influence the resulting calibrated parameters:
  1. the number of calibration instruments (i.e., option prices or implied volatilities)
  2. the loss function L and the weights w_i (see also Christoffersen and Jacobs (2004))
  3. the choice of initial values (esp. their distance to the true model parameters)
- Selected methods for improved calculation of option prices:
  - Schoutens et al. (2004) & Albrecher et al. (2006) rewriting (18) and (19)
  - Carr and Madan (1999) using a damping factor for faster decay of the integrand
- An overview of selected implementations of Heston model: *Rolf's file*

Little Heston trap and a way out

- When $\kappa\theta \neq m\sigma^2$ for some $m \in \mathbb{N}$, $\ln\left(\frac{1-g_j e^{d_j \cdot \tau}}{1-g_j}\right)$ as a func. of u has discontinuities
- To fix it, Albrecher et al. (2006) and others notice the relation:

$$\Psi(u) := \Psi_2(x, v, t; u) \Rightarrow \Psi_1(x, v, t; u) = \exp\left(-r\tau - x\right)\Psi(u - i) \tag{20}$$

...and derive an equivalent representation $\widetilde{\Psi}(u)$ of $\Psi(u)$:

$$\begin{aligned}\tilde{\Psi}(u) &= \exp(iu(\ln(S(t)) + r\tau)) \\ &\cdot \exp\left(\frac{\kappa\theta}{\sigma^2}\left((\kappa - \rho\sigma u i - d)\tau - 2\ln\left(\frac{1 - \tilde{g}e^{-d\tau}}{1 - \tilde{g}}\right)\right)\right) \\ &\cdot \exp\left(\frac{v(t)}{\sigma^2}(\kappa - \rho\sigma u i - d)\frac{1 - e^{-d\tau}}{1 - \tilde{g}e^{-d\tau}}\right)\end{aligned}$$

where $\widetilde{g} = 1/g$ (we omit indices $j = 2$ for readability)

Final comments about the Heston model

- Why is it so popular among SVM? It:
  1. has a rather simple formula for option pricing $\Rightarrow$ "easy" calibration
  2. matches well the implied volatility surface of traded options
  3. has many helpful generalizations
- Selected generalizations of the Heston model:
  1. stochastic interest rate (Heston 1993)
  2. addition of jumps (Bates 1996)
  3. time-dependent model parameters (Mikhailov-Nögel 2003)
- Further reading: Rouah (2013)

Option pricing based on Fourier transforms

- Disadvantages of option pricing using Monte Carlo (MC):
  1. slow
  2. option price (depends on the seed, num. simulations, etc.) may not be precise enough
  3. usage of MC in an optimization routine (e.g., for calibrating a model) is problematic
- Alternative – option pricing based on Fourier transforms (FT pricing for short)
  - is **independent of the model** (as long as we know CF of $\ln(S(T))$)
  - depends on the option payoff
- CFs under $\widetilde{\mathbb{Q}}$ are known for many SVMs and Levy models
- Often, models are calibrated using FT, then MC is used for pricing exotic options

Carr-Madan approach to FT pricing

- The approach was proposed in Carr and Madan (1999) and is especially fast
- It works for many option payoffs, including plain vanilla, digital, and power options
- E.g., consider a Eur. call option with maturity T, strike K, and the underlying asset S
- Denote $k := \ln(K)$, $Y(T) := \ln(S(T))$
- Assume that $Y(T)$ has a PDF $f_{Y(T)}$ under $\widetilde{\mathbb{Q}}$ and its CF $\Psi_{Y(T)}$ is known, i.e.:

$$\Psi_{Y(T)}(u) = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[e^{iuY(T)}\right] = \int_{-\infty}^{+\infty} e^{iuy} f_{Y(T)}(y)dy$$

Carr-Madan approach to FT pricing

- According to risk-neutral valuation principle, the price of the call equals:

$$C(k) := C(S_0, 0; k) = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[e^{-rT}\left(e^{Y(T)} - e^k\right)^+\right] = \int_k^{+\infty} e^{-rT}\left(e^y - e^k\right) f_{Y(T)}(y)dy$$

- Consider a call price modified by a factor $e^{\alpha k}$ for some $\alpha > 0$ (to be specified later):

$$c(k) := c(S_0, 0; k) := e^{\alpha k} C(S_0, 0; k) \Leftrightarrow C(k) = e^{-\alpha k} c(k) \tag{21}$$

- Modified call option price $c(k)$ can be seen as a function of k

Carr-Madan approach to FT pricing

- Technique of Carr-Madan:

  1. compute the FT $\widehat{c}$, making restrictions on α to ensure the existence of $\widehat{c}$
  2. in Step 1, change the order of integration, which allows to establish the link to $\Psi_{Y(T)}$
  3. use inverse FT to compute c from $\widehat{c}$, which, in its turn, is linked to $\Psi_{Y(T)}$
  4. simplify the result of Step 3 using the propreties of complex numbers

- For our call option example, Steps 1 & 2 yield:

$$\widehat{c}(u) = \frac{e^{-rT}\Psi_{Y(T)}(u-(1+\alpha)i)}{\alpha^2+\alpha-u^2+i(1+2\alpha)u}$$

Carr-Madan approach to FT pricing

- For our call option example, Steps 3 & 4 yield:

$$C(k) = \frac{e^{-\alpha k}}{\pi} \mathfrak{Re} \left(\int_0^{+\infty} e^{-iuk} \frac{e^{-rT} \Psi_{Y(T)}(u - (1+\alpha)i)}{\alpha^2 + \alpha - u^2 + i(1+2\alpha)u} \, du \right), \tag{22}$$

  where $i := \sqrt{-1}$ is the imaginary unit
- (22) has to be numerically computed
- $\alpha \in (\alpha_{\text{min}}, \alpha_{\text{max}})$ ensures the existence of $\widehat{c}$
  - $\alpha_{\text{min}} = 0$ prevents problems with the change of integration order in Steps 1 & 2
  - $\alpha_{\text{max}} = \sup_\alpha \left\{ \alpha : \mathbb{E}^{\widetilde{\mathbb{Q}}} \left[(S(T))^{1+\alpha} \right] < +\infty \right\}$ ensures existence of $\widehat{c}$

Impact of α and its the choice of its value

- The modification factor **dampens** the integrand and makes its less oscillating
- Various methodologies for choosing α:
  - Carr and Madan (1999) suggest to use $\alpha = \alpha^{\text{max}}/4$
  - Schoutens et al. (2004) use ad-hoc choices ranging from 0.75 to 25
  - Under some assumptions on the integrand (I) in (22), total variation of I is minimized by:

$$\alpha^* = \text{argmin}_{\alpha_{\text{min}},\alpha_{\text{max}}} \left(\alpha k + \frac{1}{2} I\right)$$

- Depending on the specific option payoff, it may be more convenient to use $e^{-\alpha k}$
- When (22) is truncated from above, α also controls the truncation error

For which option payoffs can Carr-Madan approach be used?

- The list of option payoffs with known FT includes:
  - plain vanilla calls and puts
  - power calls and puts
  - self-quanto calls and puts
  - digital calls and puts
- E.g., see Raible (2000) for Laplace transforms (generalization of FTs) for some payoffs

Advantages of Carr-Madan approach

- Only 1 integration scheme is required (e.g., vs 2 in Heston (1993))
- Numerator in the integrand of (22) is a quadratic function w.r.t. u
- Computational accuracy, provided that the damping factor α is appropriately chosen
- A fast approach to inverting FT is available (so-called Fast FT)
- Fast FT allows **simultaneous pricing** of multiple options with different strikes

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