We find the expressions for $C(\tau;\phi)$ and $D(\tau;\phi)$, which are part of the solution to the characteristic functions, are derived in detail. The characteristic functions for the probabilities P_j must satisfy the following PDE according to equation (OBS):

$$-\frac{\partial f_j}{\partial \tau} + (r + u_j v)\frac{\partial f_j}{\partial x} + \frac{1}{2}v\frac{\partial^2 f_j}{\partial x^2} + (a - b_j v)\frac{\partial f_j}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f_j}{\partial v^2} + \rho\sigma v\frac{\partial^2 f_j}{\partial x \partial v} = 0, \tag{B.1}$$

The boundary condition for the characteristic functions was $f_j(x, v, \tau; \phi) = e^{i\phi x}$, and the following solution form is guessed:

$$f_j(x, v, \tau; \phi) = e^{C(\tau;\phi) + D(\tau;\phi)v + i\phi x}.$$

The relevant derivatives of the characteristic functions are found. The arguments of $C(\tau;\phi)$ and $D(\tau;\phi)$ are omitted for clarity:

$$\begin{aligned}
\frac{\partial f_j}{\partial x} &= i\phi e^{C+Dv+i\phi x}, & \frac{\partial^2 f_j}{\partial x^2} &= -\phi^2 e^{C+Dv+i\phi x}, \\
\frac{\partial f_j}{\partial v} &= D e^{C+Dv+i\phi x}, & \frac{\partial^2 f_j}{\partial v^2} &= D^2 e^{C+Dv+i\phi x}, \\
\frac{\partial f_j}{\partial \tau} &= \left(\frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial \tau}v\right) e^{C+Dv+i\phi x}, & \frac{\partial^2 f_j}{\partial x \partial v} &= i\phi D e^{C+Dv+i\phi x}.
\end{aligned}$$

The derivatives are substituted into (B.1), where $e^{C+Dv+i\phi x}$ is factored out from all terms, and the terms are sorted into those with and without v:

$$\left(-\frac{\partial C}{\partial \tau} + ri\phi + aD\right) + \left(-\frac{\partial D}{\partial \tau} + u_j i\phi - \frac{1}{2}\phi^2 - b_j D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\phi D\right)v = 0. \tag{B.2}$$

The PDE in (B.2) is only satisfied if both parentheses equal zero. If one considers each parenthesis separately, they consist of two ordinary differential equations, which can be solved individually. The first parenthesis involves both terms with C and D, while the last parenthesis only involves terms with D, so the ordinary differential equation in the last parenthesis is solved first:

$$\frac{\partial D}{\partial \tau} = u_j i\phi - \frac{1}{2}\phi^2 - (b_j - \rho\sigma i\phi)D + \frac{1}{2}\sigma^2 D^2$$

The differential equation can be rewritten in the form:

$$\frac{\partial D}{\partial \tau} = \alpha - \beta D + \gamma D^2, \tag{B.3}$$

where

$$\alpha = u_j i\phi - \frac{1}{2}\phi^2, \quad \beta = b_j - \rho\sigma i\phi, \quad \gamma = \frac{1}{2}\sigma^2. \tag{B.4}$$

When a differential equation is written in the form of (B.3), the solution can be expressed as:

$$D = \frac{-\varphi'(\tau)}{\gamma\varphi(\tau)}. \tag{B.5}$$

Differentiating (B.5) gives:

$$\frac{\partial D}{\partial \tau} = \frac{\varphi'(\tau)^2 - \varphi''(\tau)\varphi(\tau)}{\gamma\varphi(\tau)^2}. \tag{B.6}$$

One can substitute (B.5) and (B.6) into (B.3), which gives:

$$\begin{gathered}
\frac{\varphi'(\tau)^2 - \varphi''(\tau)\varphi(\tau)}{\gamma\varphi(\tau)^2} = \alpha - \beta\frac{-\varphi'(\tau)}{\gamma\varphi(\tau)} + \gamma\left(\frac{-\varphi'(\tau)}{\gamma\varphi(\tau)}\right)^2 \\
\Longleftrightarrow \\
\frac{\varphi'(\tau)^2 - \varphi''(\tau)\varphi(\tau)}{\gamma\varphi(\tau)^2} = \alpha + \beta\frac{\varphi'(\tau)}{\gamma\varphi(\tau)} + \frac{\varphi'(\tau)^2}{\gamma\varphi(\tau)^2} \\
\Longleftrightarrow \\
\varphi'(\tau)^2 - \varphi''(\tau)\varphi(\tau) = \alpha\gamma\varphi(\tau)^2 + \beta\varphi'(\tau)\varphi(\tau) + \varphi'(\tau)^2 \\
\Longleftrightarrow
\end{gathered}$$

$$\varphi''(\tau) + \beta\varphi'(\tau) + \alpha\gamma\varphi(\tau) = 0. \tag{B.7}$$

Equation (B.7) is a second-order homogeneous differential equation, where the following solution form is guessed:

$$\varphi(\tau) = Ae^{\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau} + Be^{\left(-\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau}. \tag{B.8}$$

If the solution is differentiated, one gets:

$$\begin{aligned}
\varphi'(\tau) = &\left(-\frac{1}{2}\beta + \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}\right) Ae^{\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau} \\
&+ \left(-\frac{1}{2}\beta - \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}\right) Be^{\left(-\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau}.
\end{aligned} \tag{B.9}$$

For the boundary condition for the characteristic function to hold, the boundary condition $D(0) = 0$ must also hold. With this condition, it must also hold that $\varphi'(0) = 0$ for (B.5) to be satisfied:

$$\left(-\frac{1}{2}\beta + \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}\right) A + \left(-\frac{1}{2}\beta - \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}\right) B = 0 \Leftrightarrow B = \frac{\frac{1}{2}\beta - \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}}{-\frac{1}{2}\beta - \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}} A.$$

Substituting B into (B.8) and (B.9), which are then substituted into (B.5):

$$D = \frac{-\left(\left(-\frac{1}{2}\beta + \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}\right) Ae^{\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau} + \left(\frac{1}{2}\beta - \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}\right) Ae^{\left(-\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau}\right)}{\gamma\left(Ae^{\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau} + \frac{\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}}{-\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}}Ae^{\left(-\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau}\right)}.$$

We then multiply through by $2e^{-\left(\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau}$ in the numerator and denominator

$$D = \frac{\left(\beta - \sqrt{\beta^2 - 4\alpha\gamma}\right) - \left(\beta - \sqrt{\beta^2 - 4\alpha\gamma}\right) e^{\left(-\sqrt{\beta^2-4\alpha\gamma}\right)\tau}}{2\gamma\left(1 - \frac{\beta-\sqrt{\beta^2-4\alpha\gamma}}{\beta+\sqrt{\beta^2-4\alpha\gamma}} e^{\left(-\sqrt{\beta^2-4\alpha\gamma}\right)\tau}\right)}.$$

We define d and g for simplicity and insert the values α, β and γ from (B.4) in

$$\begin{aligned} d &\equiv \sqrt{\beta^2 - 4\alpha\gamma} \\ &= \sqrt{(b_j - \rho\sigma i\phi)^2 - \sigma^2 \left(2\phi u_j i - \phi^2\right)}, \end{aligned} \tag{B.10}$$

and

$$\begin{aligned} g &\equiv \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{\beta + \sqrt{\beta^2 - 4\alpha\gamma}} \\ &= \frac{b_j - \rho\sigma\phi i - d}{b_j - \rho\sigma\phi i + d}. \end{aligned} \tag{B.11}$$

The simplified solution for D can now be written as

$$D(\tau;\phi) = \frac{(b_j - \rho\sigma\phi i - d)\left(1 - e^{-d\tau}\right)}{\sigma^2\left(1 - g e^{-d\tau}\right)}.$$

We then proceed to find C using our newly found expression for D. As prievously stated, in (B.2), both parentheses have to equal 0 which implies

$$\begin{aligned} \frac{\partial C}{\partial \tau} = r\phi i + aD \Leftrightarrow C &= \int_0^\tau -r\phi i - aD(u)du \\ &= \int_0^\tau -r\phi i du - \int_0^\tau aD(u)du \\ &= -r\phi i\tau - a\int_0^\tau D(u)du. \end{aligned}$$

So it is $\int_0^\tau D(u)du$ that needs to be calculated. From (B.5) we know that D can be written as $D(\tau) = \frac{-\varphi'(\tau)}{\gamma\varphi(\tau)} = \frac{-1}{\gamma}\frac{\varphi'(\tau)}{\varphi(\tau)}$. It is commonly known that if $F(x) = \log(f(x))$, then $F'(x) = \frac{f'(x)}{f(x)} \Rightarrow \int \frac{f'(x)}{f(x)} = \log(f(x))$, which implies in our case that

$$\begin{aligned} \int_0^\tau D(u)du &= \frac{-1}{\gamma}\int_0^\tau \frac{\varphi'(u)}{\varphi(u)}du \\ &= \frac{-1}{\gamma}\left[\log(\varphi(u))\right]_{u=0}^{u=\tau} \\ &= \frac{-1}{\gamma}(\log(\varphi(\tau)) - \log(\varphi(0))) \\ &= \frac{-1}{\gamma}\left(\log\left(\frac{\varphi(\tau)}{\varphi(0)}\right)\right). \end{aligned}$$

$\varphi(\tau)$ is known from (B.8) as

$$\varphi(\tau) = e^{\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau} - \frac{\frac{1}{2}\beta - \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}}{\frac{1}{2}\beta + \sqrt{\frac{1}{4}\beta^2 - \alpha\gamma}} e^{\left(-\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau},$$

where the expresison for B is substituted in and A is now void as it is divided out as it is in both the denominator and numerator. The term $\frac{\frac{1}{2}\beta-\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}}{\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}}$ is known from (B.11), where it is defined as g. This leaves us with

$$\begin{aligned}
\int_0^t D(u)du &= \frac{-1}{\gamma}\left(\log\left(\frac{e^{\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau}\left(1-ge^{\left(-\sqrt{\beta^2-4\alpha\gamma}\right)t}\right)}{1-g}\right)\right) \\
&= \frac{-1}{\gamma}\left(\left(-\frac{1}{2}\beta+\sqrt{\frac{1}{4}\beta^2-\alpha\gamma}\right)\tau+\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right) \\
&= \frac{1}{\sigma^2}\left(\left(\beta-\sqrt{\beta^2-4\alpha\gamma}\right)\tau-2\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right) \\
&= \frac{1}{\sigma^2}\left(\left(b_j-\rho\sigma i\phi-d\right)\tau-2\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right).
\end{aligned}$$

The solution for C is thus

$$C(\tau;\phi) = ri\phi\tau + \frac{a}{\sigma^2}\left(\left(b_j-\rho\sigma i\phi-d\right)\tau-2\log\left(\frac{1-ge^{-d\tau}}{1-g}\right)\right).$$