Parametric Differentiation

Definition of parametric functions:

For a parameter t, the equations $x = g(t)$ and $y = h(t)$ are said to be the Parametric equation of the function $y = f(x)$ if the equation of the function can be deduced from the parametric equations by omitting the parameter t.

EX: the equations $x = 5\cos t$, $y = 5\sin t$ are the parametric equations of the Circle $x^2 + y^2 = 25$.

Parametric differentiation

If $x = g(t)$ and $y = h(t)$

$$x = g(t) \downarrow \frac{dx}{dx} \qquad y = h(t) \downarrow \frac{dy}{dt}$$

First derivative ① $\boxed{\frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}}$

Second derivative $\rightarrow$ ② $\boxed{\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dt} * \dfrac{dt}{dx}}$

Examples

$\boxed{1}$ Find y' for each of the following

① $x = 4\sin t \quad , \quad y = 2\cos 2t$

Solution

$\therefore \dfrac{dx}{dt} = 4\cos t \quad , \quad \dfrac{dy}{dt} = 2(-2\sin 2t) = -4\sin 2t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = (-\cancel{4}\sin 2t) \cdot \frac{1}{\cancel{4}\cos t} = \frac{-\sin 2t}{\cos t}$$

② $x = \sqrt{1+\sin t} \quad , \quad y = \sin^{-1}(\sqrt{t})$

Solution

$\therefore \dfrac{dx}{dt} = \dfrac{\cos t}{2\sqrt{1+\sin t}} \quad , \quad \dfrac{dy}{dt} = \dfrac{1}{\sqrt{1-(\sqrt{t})^2}} \cdot \dfrac{1}{2\sqrt{t}}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{1}{\cancel{2}\sqrt{t}\sqrt{1-t}} \cdot \frac{\cancel{2}\sqrt{1+\sin t}}{\cos t}$$

[2] Find y'' for the following:

$$x = \sin^{-1}(t) \quad , \quad y = \sqrt{1-t^2}$$

Solution

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \quad , \quad \frac{dy}{dt} = \frac{-2t}{2\sqrt{1-t^2}}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-t}{\cancel{\sqrt{1-t^2}}} \cdot \frac{\cancel{\sqrt{1-t^2}}}{1} = -t$$

$$\therefore y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dt} \cdot \frac{dt}{dx}$$

$$= (-1)\,\frac{\sqrt{1-t^2}}{1} = -\sqrt{1-t^2}$$

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[3] Prove the following relations:

① If $x = \cos\left(\frac{t}{t+1}\right)$, $y = \sin\left(\frac{t}{t+1}\right)$

Prove that $y^3 y'' + 1 = 0$

Proof

$$\because \sin^2\theta + \cos^2\theta = 1$$

(The same angle)

$$\therefore x^2 + y^2 = 1$$

$$2x + 2yy' = 0 \longrightarrow y' = \frac{-x}{y} \quad ①$$

$$2 + 2yy'' + y' \cdot 2y' = 0 \quad \div 2$$

$$1 + yy'' + y'^2 = 0$$

using ①

$$\therefore 1 + yy'' + \frac{x^2}{y^2} = 0 \quad * y^2$$

$$y^2 + y^3 y'' + x^2 = 0$$

$$\because x^2 + y^2 = 1$$

$$\therefore y^3 y'' + 1 = 0$$

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② If $x = t + \frac{1}{t}$, $y = t^2 + \frac{1}{t^2}$

show that

$$y'' = 2$$

Solution

$$\because \left(t + \frac{1}{t}\right)^2 = t^2 + \frac{1}{t^2} + 2$$

$$\therefore x^2 = y + 2$$

$$y = x^2 - 2$$

$$y' = 2x$$

$$y'' = 2$$

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③ If $x = \tan\left(\frac{t-1}{t+1}\right)$, $y = \sec\left(\frac{t-1}{t+1}\right)$

Prove that

$$y'' = \frac{1}{y^3}$$

—— solution ——

$$\because 1 + \tan^2\theta = \sec^2\theta$$

$$\therefore 1 + x^2 = y^2 \quad ⊛$$

$$2x = 2yy' \quad \div 2$$

$$x = yy' \longrightarrow y' = \frac{x}{y} \quad ①$$

$$1 = yy'' + y'y'$$

$$1 = yy'' + y'^2$$

using ①

$$\therefore 1 = yy'' + \frac{x^2}{y^2} \quad * y^2$$

$$y^2 = y^3 y'' + x^2$$

$$y^2 - x^2 = y^3 y''$$

from ⊛ $\quad \because y^2 - x^2 = 1$

$$\therefore y^3 y'' = 1 \longrightarrow y'' = \frac{1}{y^3}$$

H.W

④ If $x = \frac{t+1}{t-1}$, $y = \left(\frac{t-1}{t+1}\right)^6$

Prove that $y'' = \frac{42}{x^8}$

Prove the following relations:

1) If $y = \tan(n \cos^{-1} x)$

Prove that (i) $y' = \dfrac{-n(1+y^2)}{\sqrt{1-x^2}}$

(ii) $(1-x^2)y'' - xy' - 2n^2 y(1+y^2) = 0$

Proof

$y' = \sec^2(n\cos^{-1}x) \cdot \dfrac{-n}{\sqrt{1-x^2}}$

$1+\tan^2 x = \sec^2 x$

$\therefore y' = \dfrac{-n(1+\tan^2(n\cos^{-1}x))}{\sqrt{1-x^2}}$

$y' = \dfrac{-n(1+y^2)}{\sqrt{1-x^2}}$ (i) $\quad \times \sqrt{1-x^2}$

$\sqrt{1-x^2}\, y' = -n(1+y^2)$

$\sqrt{1-x^2}\, y'' + y' \cdot \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = -n(2yy')$

$\times \sqrt{1-x^2}$

$(1-x^2)y'' - xy' = -2n\underbrace{yy'\sqrt{1-x^2}}$

from (i)

$\sqrt{1-x^2}\, y' = -n(1+y^2)$

$(1-x^2)y'' - xy' = -2ny \cdot (-n(1+y^2))$

$(1-x^2)y'' - xy' - 2n^2 y(1+y^2) = 0$

(ii)

$\boxed{2}$ If $y = (x + \sqrt{x^2-1})^4$

Prove that:

(i) $y' = \dfrac{4y}{\sqrt{x^2-1}}$

(ii) $(x^2-1)y'' + xy' - 16y = 0$

Proof

$\because y' = 4(x + \sqrt{x^2-1})^3 \cdot \left(1 + \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right)$

$y' = 4(x + \sqrt{x^2-1})^3 \left(\dfrac{\sqrt{x^2-1}}{\sqrt{x^2-1}} + \dfrac{x}{\sqrt{x^2-1}}\right)$

$y' = 4(x + \sqrt{x^2-1})^3 \left(\dfrac{x + \sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$

$y' = \dfrac{4(x + \sqrt{x^2-1})^4}{\sqrt{x^2-1}} = \dfrac{4y}{\sqrt{x^2-1}}$ (i) $\times \sqrt{x^2-1}$

$\sqrt{x^2-1}\, y' = 4y$

$\sqrt{x^2-1}\, y'' + y' \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}} = 4y'$ $\times \sqrt{x^2-1}$

$(x^2-1)y'' + xy' = 4\underbrace{\sqrt{x^2-1}\, y'}_{4y}$ From (i) $\sqrt{x^2-1}\, y' = 4y$

$\therefore (x^2-1)y'' + xy' - 16y = 0$

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$\boxed{3}$ If $y = \cos(\ln x) + \sin(\ln x)$

Prove that $x^2y'' + xy' + y = 0$

——— Proof ———

$$y' = -\sin(\ln x) \cdot \frac{1}{x} + \cos(\ln x) \cdot \frac{1}{x} \quad \underline{\underline{\times x}}$$

$$xy' = -\sin(\ln x) + \cos(\ln x)$$

$$xy'' + y' = -\cos(\ln x) \cdot \frac{1}{x} - \sin(\ln x) \cdot \frac{1}{x} \quad \underline{\underline{\times x}}$$

$$x^2y'' + xy' = -y$$

$$x^2y'' + xy' + y = 0$$

$\boxed{4}$ If $x^y = e^{x-y}$

Prove that $y' = \dfrac{\ln x}{(1+\ln x)^2}$

——— Proof ———

Take ln to both sides

$$\therefore \ln x^y = \ln e^{x-y}$$

$$y \ln x = (x-y) \cancel{\ln e} \rightarrow 1$$

$$y \ln x = x - y$$

$$y \ln x + y = x$$

$$y(\ln x + 1) = x$$

$$y = \frac{x}{1+\ln x}$$

$$\therefore y' = \frac{(1+\ln x)(1) - \cancel{x}(\frac{1}{\cancel{x}})}{(1+\ln x)^2}$$

$$y' = \frac{\cancel{1}+\ln x - \cancel{1}}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

$\boxed{5}$ If $y = \cos x^{\cos x^{\cos x^{\cdots\infty}}}$

Prove that $y' = \dfrac{-y^2 \tan x}{1 - y\ln(\cos x)}$

proof

y Can be written as

$$y = (\cos x)^y$$

Take ln to both sides

$$\ln y = \ln(\cos x)^y$$

$$\ln y = y \ln(\cos x)$$

$$\frac{1}{y} y' = y \cdot \frac{-\sin x}{\cos x} + y' \ln(\cos x) \qquad \underline{xy}$$

$$y' = -y^2 \tan x + y y' \ln(\cos x)$$

$$y'(1 - y\ln(\cos x)) = -y^2 \tan x$$

$$y' = \frac{-y^2 \tan x}{1 - y\ln(\cos x)}$$

$\boxed{6}$ If $y = x^{x^{x^{\cdots\infty}}}$ prove that :

$$xy' = \frac{y^2}{1 - y\ln x}$$

Proof

y Can be written as

$y = x^y$ Take ln to both sides

$$\ln y = y \ln x$$

$$\frac{1}{y} y' = y \cdot \frac{1}{x} + y' \ln x \qquad \times xy$$

$$xy' = y^2 + xy y' \ln x$$

$$xy' - xyy' \ln x = y^2$$

$$xy'(1 - y\ln x) = y^2 \longrightarrow xy' = \frac{y^2}{1 - y\ln x}$$

$\boxed{7}$ If $y = \sqrt{\ln x + \sqrt{\ln x + \sqrt{\ln x + \cdots\infty}}}$

Prove that $y' = \dfrac{1}{x(2y-1)}$

Proof

y Can be written as

$$y = \sqrt{\ln x + y}$$

$$y' = \frac{\frac{1}{x} + y'}{2\sqrt{\ln x + y}} \qquad \times 2\sqrt{\ln x + y}$$

$$2y'\underbrace{\sqrt{\ln x + y}}_{y} = \frac{1}{x} + y'$$

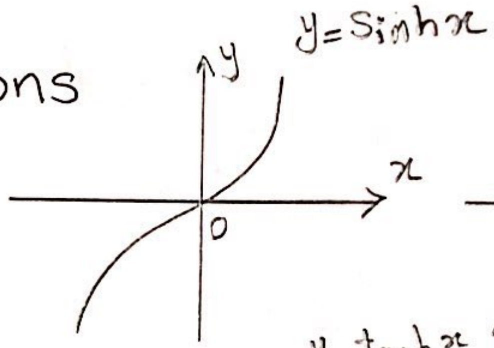
$$2y'y = \frac{1}{x} + y'$$

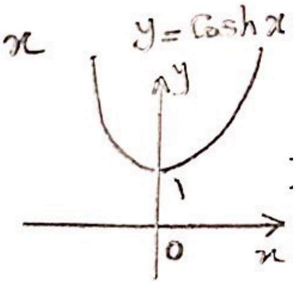
$$2y'y - y' = \frac{1}{x}$$

$$y'(2y-1) = \frac{1}{x}$$

$$y' = \frac{1}{x(2y-1)}$$

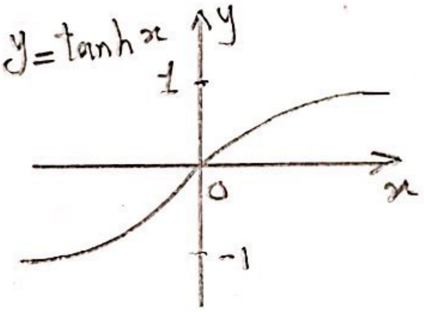
Hyperbolic Functions





$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$



$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

Inverse Hyperbolic Functions

The inverse hyperbolic functions which denoted by:

$$\sinh^{-1} x \;,\; \cosh^{-1} x \;,\; \tanh^{-1} x$$

where
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

For example , $y=\sinh^{-1} x$

then $x=\sinh y=\frac{e^{y}-e^{-y}}{2}$ $\quad * 2$

$$2x=e^{y}-e^{-y} \quad * e^{y}$$

$$ax^{2}+bx+c=0$$
$$x=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a}$$

$$e^{2y}-2xe^{y}-1=0$$

$$(e^{y})^{2}-2x(e^{y})-1=0$$

$$e^{y}=\frac{2x\pm\sqrt{4x^{2}-4\times 1\times -1}}{2}$$

$$e^{y}=\frac{2x\pm 2\sqrt{x^{2}+1}}{2}=\frac{\cancel{2}(x\pm\sqrt{x^{2}+1})}{\cancel{2}}$$

$$e^{y}=x\pm\sqrt{x^{2}+1}$$, take ln to both sides

The left hand side being positive,

the right hand side should also be positive

$$\therefore y=\ln(x+\sqrt{x^{2}+1}) \quad , \ln e=1$$

$$\therefore \sinh^{-1} x=\ln(x+\sqrt{x^{2}+1})$$

In the same way, we get

$\cosh^{-1} x$ and $\tanh^{-1} x$

Derivative of Hyperbolic Functions:

Let $u \equiv u(x)$, u is a function of x

① $y = \sinh(u) \longrightarrow y' = \cosh(u) \cdot u'$

② $y = \cosh(u) \longrightarrow y' = \sinh(u) \cdot u'$

③ $y = \tanh(u) \longrightarrow y' = \operatorname{sech}^2(u) \cdot u'$

④ $y = \operatorname{sech}(u) \longrightarrow y' = -\operatorname{sech}(u) \tanh(u) \cdot u'$

⑤ $y = \operatorname{Cosech}(u) \longrightarrow y' = -\operatorname{Cosh}(u) \operatorname{Coth}(u) \cdot u'$

⑥ $y = \operatorname{Coth}(u) \longrightarrow y' = -\operatorname{Cosech}^2(u) \cdot u'$

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⑦ $y = \sinh^{-1}(u) \longrightarrow y' = \dfrac{1}{\sqrt{1+(u)^2}} \dfrac{du}{dx}$

⑧ $y = \cosh^{-1}(u) \longrightarrow y' = \dfrac{1}{\sqrt{(u)^2-1}} \dfrac{du}{dx}$

⑨ $y = \tanh^{-1}(u) \longrightarrow y' = \dfrac{1}{1-(u)^2} \dfrac{du}{dx}$

Examples

Find y' for the following functions :-

① $y = x^3 e^{\cosh^2 x^3}$

$$y' = x^3 e^{\cosh^2 x^3} \cdot 2\cosh(x^3)\sinh(x^3) \cdot (3x^2) + 3x^2 e^{\cosh^2(x^3)} .$$

② $y = (1 + \sin^{-1}\sqrt{x})^{\sinh x}$

Take ln to both sides

$$\ln y = \sinh x \ln(1 + \sin^{-1}\sqrt{x})$$

$$\frac{1}{y} y' = \sinh x \cdot \frac{1}{1 + \sin^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} + \cosh x \ln(1 + \sin^{-1}\sqrt{x})$$

$$y' = y\left[\frac{\sinh x}{2\sqrt{x}\sqrt{1-x}\,(1 + \sin^{-1}\sqrt{x})} + \cosh x \ln(1 + \sin^{-1}\sqrt{x})\right]$$

③ $y = \sqrt[3]{\sinh^{-1} x + x^2} + \ln(\sqrt{\tanh 3x})$

$$y = (\sinh^{-1} x + x^2)^{1/3} + \frac{1}{2}\ln(\tanh 3x)$$

$$y' = \frac{1}{3}(\sinh^{-1}x + x^2)^{-2/3} \cdot \left(\frac{1}{\sqrt{1+x^2}} + 2x\right)$$

$$+ \frac{1}{2} \frac{3\,\mathrm{sech}^2(3x)}{\tanh(3x)}$$

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④ $y = \tanh^{-1}(\sqrt{\sec x})$

$$y' = \frac{1}{1-(\sqrt{\sec x})^2} \cdot \frac{\sec x \tan x}{2\sqrt{\sec x}}$$

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⑤ $y = x^{\sin^{-1}x} + (x^3 + \sinh x)^x$

Let $u = x^{\sin^{-1}x}$ & $v = (x^3 + \sinh x)^x$

$\therefore y = u + v$

$y' = u' + v'$

$\ln u = \sin^{-1}x \ln x$

$$\frac{1}{u}u' = \sin^{-1}x \cdot \frac{1}{x} + \frac{\ln x}{\sqrt{1-x^2}}$$

$$u' = u\left[\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right]$$

$\ln v = x \ln(x^3 + \sinh x)$

$$\frac{1}{v}v' = x \cdot \frac{3x^2 + \cosh x}{x^3 + \sinh x} + \ln(x^3 + \sinh x)$$

$$v' = v\left[\frac{3x^3 + x\cosh x}{x^3 + \sinh x} + \ln(x^3 + \sinh x)\right]$$

$$\therefore y' = x^{\sin^{-1}x}\left[\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right] + (x^3 + \sinh x)^x \cdot \left[\frac{3x^3 + x\cosh x}{x^3 + \sinh x} + \ln(x^3 + \sinh x)\right]$$

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⑥ $y = \dfrac{(x+1)^2 e^{\operatorname{sech} x}}{\sqrt{x^3-1}}$

Take ln to both sides

$\ln y = 2\ln(x+1) + \operatorname{sech} x \, \cancel{\ln e}^{\,1} - \frac{1}{2}\ln(x^3-1)$

$\frac{1}{y} y' = \frac{2}{x+1} - \operatorname{sech} x \tanh x - \frac{1}{2}\frac{3x^2}{x^3-1}$

$y' = y\left[\frac{2}{x+1} - \operatorname{sech} x \tanh x - \frac{1}{2}\frac{3x^2}{x^3-1}\right]$

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⑦ If $e^{4x^2+y} + \cosh^{-1}(4x^2+y) = 5$

prove that $y'' = -8$

—— Solution ——

$e^{4x^2+y} \cdot (8x+y') + \frac{1}{\sqrt{(4x^2+y)^2-1}} \cdot (8x+y') = 0$

$(8x+y')\left[e^{4x^2+y} + \frac{1}{\sqrt{(4x^2+y)^2-1}}\right] = 0$

$\neq 0$

$\therefore 8x + y' = 0$

$y' = -8x$

$y'' = -8$

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