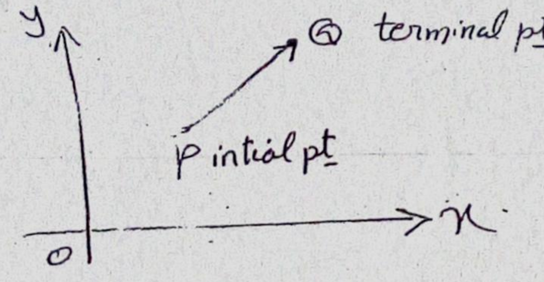
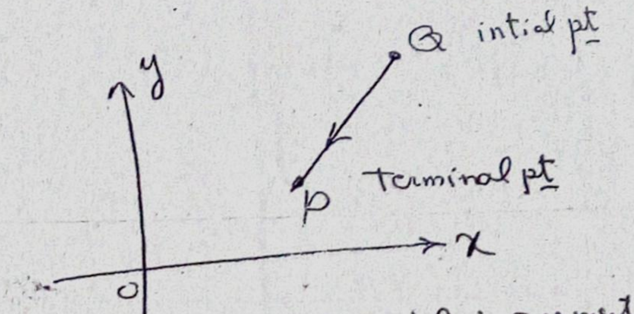
Vectors in the plane





The directed Line segment (vector) $\overrightarrow{PQ}$

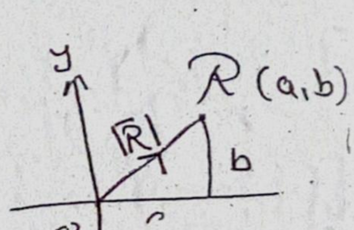
The directed line segment (vector) $\overrightarrow{QP}$.

→ A vector $\underline{V}$ in the xy-plane is an ordered pair (a, b),

The numbers a and b are called the Component of $\underline{V}$.

* The zero vector is the vecter (0,0) and is denated by $\underline{0}$.
* Two vectors are equal if their corresponding components are equal. That is $(a, b) = (c, b)$ if $a = c$ & $b = d$.

* Magnitude of a vector:



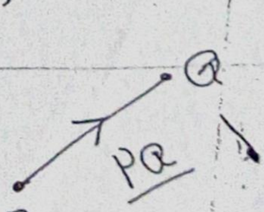
The vector $\overline{OR}$ is denated by $\underline{R}$.

morovere, the magnitude or length of a vecter is given by:

$$|\underline{R}| = \sqrt{a^2+b^2}.$$

* If ~~the~~ the pts $P = (a, b)$ and $Q = (c, d)$ then, the vecter $\underline{PQ} = Q - P = (c,d) - (a,b) = (c-a, d$

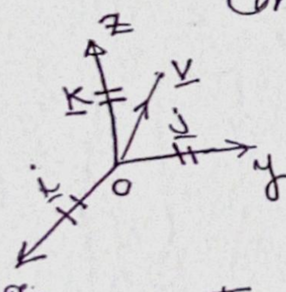
the coordinates of Q



Similarly: the vectors in the space $\mathbb{R}$ is given by (a, b, c) and the numbers a, b, c are components of the vector.

The length $= \sqrt{a^2 + b^2 + c^2}$.

* There are three special vectors in $\mathbb{R}^3$ (space) that allow us to represent other vectors in $\mathbb{R}^3$ in a convenient way. we denote by the vector:



$(1,0,0)$ by $\underline{i}$ & $(0,1,0)$ by $\underline{j}$ & $(0,0,1)$ by $\underline{k}$

Then, If $\underline{V} = (a, b, c)$ is any vector in $\mathbb{R}^3$; then

$$\underline{V} = (a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$= a\underline{i} + b\underline{j} + c\underline{k}$$

Addition and scalar multiplication of a vector:

let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ & $\underline{V} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ and α be a scalar: then:

(1) $\underline{u} \pm \underline{V} = (a_1 \pm a_2)\,\underline{i} + (b_1 \pm b_2)\,\underline{j} + (c_1 \pm c_2)\,\underline{k}$

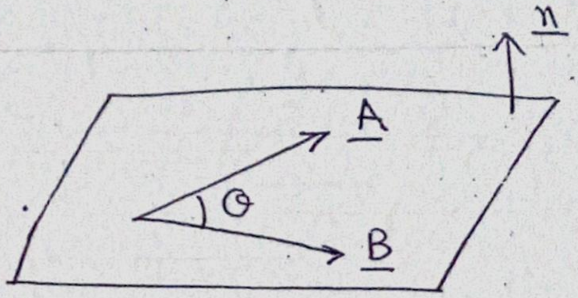
(2) $\alpha \underline{u} = \alpha a_1\,\underline{i} + \alpha b_1\,\underline{j} + \alpha c_1\,\underline{k}$

(3) $-\underline{u} = (-1)\,\underline{u} = -a_1\underline{i} - b_1\underline{j} - c_1\underline{k}$.

The product of 2 vectors $\begin{cases} \to \text{Dot product} \\ \to \text{cross product.} \end{cases}$

If $\underline{A} = A_1 \underline{i} + A_2 \underline{j} + A_3 \underline{K}$ &

$\underline{B} = B_1 \underline{i} + B_2 \underline{j} + B_3 \underline{K}$



are two vectors in $\mathbb{R}^3$ and

θ be the angle betn them,

and $\underline{n}$ be the unit vector $\perp$ on the plane including them.

* The inner product (dot product) is defined by:

$$\underline{A} \cdot \underline{B} = \begin{cases} |A|\,|B| \cos\theta \\ \\ A_1 B_1 + A_2 B_2 + A_3 B_3 \quad (\text{scalar value}) \end{cases}$$

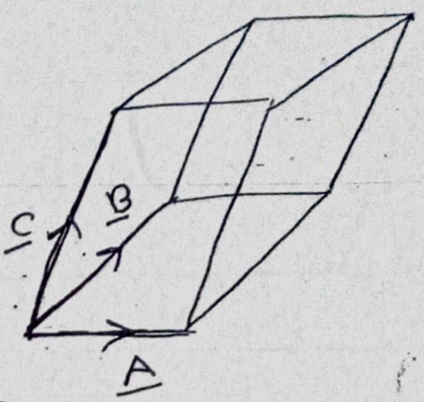
* The cross product is given by:

$$\underline{A} \times \underline{B} = \begin{cases} \begin{vmatrix} \underline{i} & \underline{j} & \underline{K} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\ \\ (|A|\,|B| \sin\theta)\, \underline{n} \end{cases} \quad (\text{vector})$$

Triple product: If $\underline{C} = C_1 \underline{i} + C_2 \underline{j} + C_3 \underline{k}$; then:

$$* \quad \underline{A} \cdot (\underline{B} \times \underline{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

The volume of parallelepiped



$$* \quad \underline{A} \times (\underline{B} \times \underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$$

→ Application of product of the vectors:

let $\underline{A}$, $\underline{B}$ and $\underline{C}$ are three vectors in $\mathbb{R}^3$:

* (1) The unit vector having the same direction as $\underline{A}$ is given by: $\hat{a} = \frac{1}{|\underline{A}|} \underline{A}$:

* (2) The two vectors $\underline{A}$ & $\underline{B}$ are orthogonal if:-

$$\underline{A} \cdot \underline{B} = 0$$

* (3) The two vectors $\underline{A}$ & $\underline{B}$ are parallel if:-

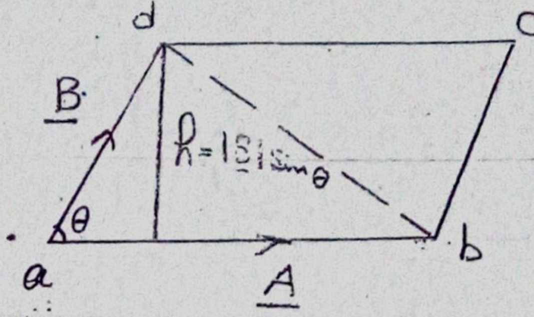
$$\underline{A} \times \underline{B} = 0$$

* (4) If θ be the angle betn the vectors $\underline{A}$ & $\underline{B}$; then:

$$\cos\theta = (\underline{A} \cdot \underline{B}) * \frac{1}{|A||B|}$$

i) Finding the area of triangle (parallelogam)

area of the paralelograme =



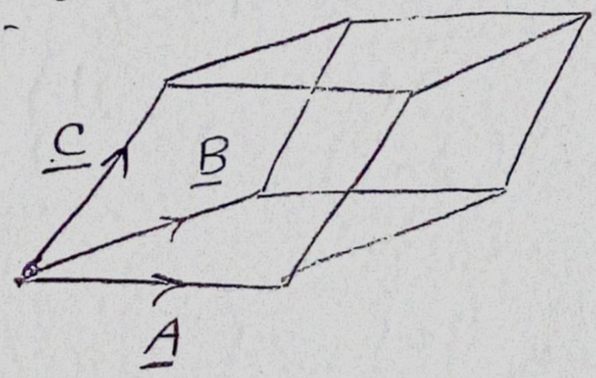
$$|\underline{A}| \cdot h = |\underline{A}||\underline{B}| \sin\theta$$

$$= |\underline{A} \times \underline{B}|$$

and hence the area of the triangle $= \frac{1}{2}|\underline{A} \times \underline{B}|$

where $\underline{A}$ & $\underline{B}$ are adjacent sides. #

(6) The volume of parallelpiped $\equiv V =$



$$|\underline{A} \cdot (\underline{B} \times \underline{C})|$$

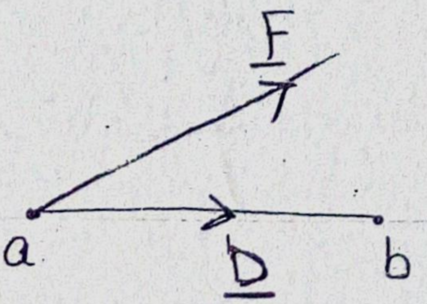
where $\underline{A}$, $\underline{B}$ & $\underline{C}$ are three adjacent edges.

(7) The three vectors $\underline{A}$, $\underline{B}$ & $\underline{C}$ are Coplaner

if $V = 0 \Rightarrow$ i.e. $\underline{A} \cdot (\underline{B} \times \underline{C}) = 0$

ii)

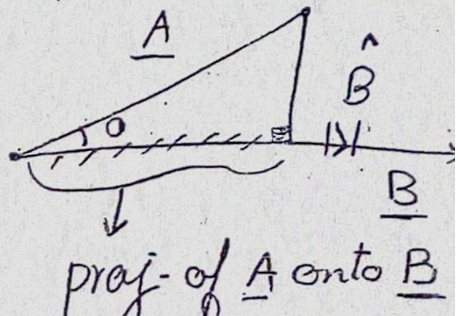
(8) The work done:



work done by the force $\underline{F}$ to move a body at the point a to the point b is

$$W = \underline{F} \cdot \underline{D} \text{ , where } \underline{D} = b - a$$

(9) Projection:



$$\text{The length of } \operatorname{proj}_{\underline{B}} \underline{A} = |\underline{A}| \cos\theta$$

$$= |\underline{A}| * \frac{(\underline{A} \cdot \underline{B})}{|\underline{A}||\underline{B}|}$$

$$= (\underline{A} \cdot \underline{B}) \frac{1}{|\underline{B}|} = (\underline{A} \cdot \hat{B})$$

$$\text{The projection vector} = \operatorname{proj}_{\underline{B}} \underline{A} = (\underline{A} \cdot \hat{B}) \hat{B}$$

& Component of $\underline{A}$ onto $\underline{B}$ $\equiv$ $\underline{A} \cdot \hat{B}$

(1) Find the components of the vector $\underline{v}$ with given initial point $p(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$.

Find $|\underline{v}|$, where

(i) $P(1,0,0)$ & $Q(4,2,0)$

Soln: $\underline{V} = \underline{PQ} = Q - P = \langle 4-1, 2-0, 0-0 \rangle =$

$= \langle 3, 2, 0 \rangle = 3\underline{i} + 2\underline{j}$

$|\underline{V}| = \sqrt{9+4} = \sqrt{13}$ #

(2) Let $\underline{a} = 2\underline{i} - \underline{j} + 3\underline{k}$, $\underline{b} = \underline{i} + \underline{j} - \underline{k}$ & $\underline{c} = 4\underline{k}$.

Find:

(i) $3\underline{a} - 2\underline{b} + 4\underline{c} = (6\underline{i} - 3\underline{j} + 9\underline{k}) - (2\underline{i} + 2\underline{j} - 2\underline{k})$

$+ (16\underline{k}) = 4\underline{i} - 5\underline{j} + 27\underline{k}$

(ii) $3(\underline{b} - 2\underline{c}) = 3(\underline{i} + \underline{j} - 9\underline{k}) = 3\underline{i} + 3\underline{j} - 27\underline{k}$

(iii) $|\underline{a} + \underline{b}| = |3\underline{i} + 2\underline{k}| = \sqrt{9+4} = \sqrt{13}$

(iv) $|\underline{a}| + |\underline{b}| = \sqrt{14} + \sqrt{3}$

(v) $\dfrac{\underline{b}}{|\underline{b}|} = \dfrac{\underline{i} + \underline{j} - \underline{k}}{\sqrt{3}} = \dfrac{\underline{i}}{\sqrt{3}} + \dfrac{\underline{j}}{\sqrt{3}} - \dfrac{\underline{k}}{\sqrt{3}}$

i) $\underline{a} + (\underline{b} - \underline{c}) = 3\underline{i} - 2\underline{j}$ #

(3) In each case, find the resultant, its components and its magnitude:

(i) $\underline{p} = \langle 1,1,1 \rangle$, $\underline{q} = \langle 1,-2,3 \rangle$; $\underline{u} = \langle 2,3,1 \rangle$

Soln $\underline{R} = \underline{p} + \underline{q} + \underline{u} = \langle 4,2,5 \rangle$

$|\underline{R}| = \sqrt{16+4+25} = \sqrt{45}$ #

(4) Let $\underline{a} = \langle 2,1,3 \rangle$, $\underline{b} = \langle 1,0,-4 \rangle$ & $\underline{c} = \langle 3,-1,2 \rangle$, find:

(i) $3\underline{a} \cdot 2\underline{c} = \langle 6,3,9 \rangle \cdot \langle 6,-2,4 \rangle = 36 - 6 + 36 = 66$

(ii) $\underline{a} \cdot (\underline{b} + \underline{c}) = \langle 2,1,3 \rangle \cdot \langle 4,-1,-2 \rangle = 8 - 1 - 6 = 1$

(iii) $\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{a} = (2-12) + (3-8) + (6-1+6)$

$= -16$

(iv) $\underline{a} \wedge \underline{c} + \underline{c} \wedge \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 3 \\ 2 & -1 & 2 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$

$= 0$

$\underline{a} \wedge (3\underline{b} - 5\underline{c}) = \langle 2,1,3\rangle \wedge [\langle 3,0,-12\rangle + \langle -15,-5,10\rangle]$

$= \langle 2,1,3\rangle \wedge \langle -12,5,-22\rangle$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 3 \\ -12 & 5 & -22 \end{vmatrix} = -37\underline{i} + 8\underline{j} + 22\underline{k}$$

(vi) $\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 3 \\ 1 & 0 & -4 \end{vmatrix} = -4\underline{i} + 11\underline{j} - \underline{k}$

$(\underline{a} \wedge \underline{b}) \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 11 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 21\underline{i} + 5\underline{j} - 29\underline{k}$

(vii) $(\underline{a} \wedge \underline{b}) \cdot \underline{c} = \langle -4, 11, -1\rangle \cdot \langle 3,-1,2\rangle$

$= -12 - 11 - 2 = -25$ #.

(5). (i) show that $\underline{a} = \langle 1,-1,2\rangle$, $\underline{b} = \langle 0,4,2\rangle$ and $\underline{c} = \langle -10,-2,4\rangle$ are orthogonal.

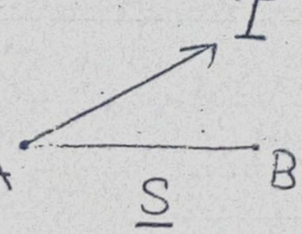
→ Soln: $\underline{a} \cdot \underline{b} = 0 - 4 + 4 = 0$, $\underline{a} \perp \underline{b}$

$\underline{a} \cdot \underline{c} = -10 + 2 + 8 = 0$ $\quad \underline{a} \perp \underline{c}$

$\underline{b} \cdot \underline{c} = -8 + 8 = 0$ $\quad \underline{c} \perp \underline{b}$

then $\underline{a}$, b and $\underline{c}$ are orthogona.

) Find the work done by a force $\underline{P} = \langle 2, 1, 0 \rangle$ acting on a body if it is displaced from a point $A(0,0,0)$ to a point $B(0,1,0)$



Soln:

$$\underline{S} = \underline{AB} = B - A = (0,1,0) - (0,0,0) = \langle 0, 1, 0 \rangle$$

$$w = \underline{P} \cdot \underline{S} = \langle 2,1,0 \rangle \cdot \langle 0,1,0 \rangle = 1$$

(iii) Find the component of $\underline{a} = \langle 1, 1, 2 \rangle$ in the direction of $\underline{b} = \langle 2, 1, 2 \rangle$.

Soln:

$$\text{Comp}_{\underline{b}}\, \underline{a} = \underline{a} \cdot \hat{b},$$

$$\hat{b} = \frac{\underline{b}}{|\underline{b}|} = \frac{\langle 2,1,2 \rangle}{\sqrt{9}}$$

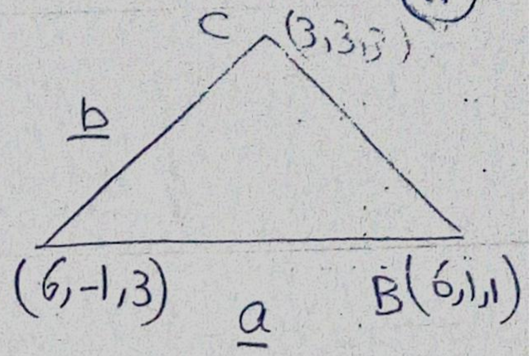
$$\therefore \text{Comp}_{\underline{b}}\, \underline{a} = \langle 1,1,2 \rangle \cdot \frac{\langle 2,1,2 \rangle}{3}$$

$$= \frac{1}{3}(2+1+4) = 7/3$$

(iv) Find the area of the triangle whose vertices are $(6,-1,3)$; $(5,1,1)$ and $(3,3,3)$

$\underline{c} = \underline{AB} = <0,2,-2>$

$\underline{b} = \underline{BC} = <-3,2,2>$



$\Delta ABC = \frac{1}{2}|\underline{a} \wedge \underline{b}|$

$$\underline{a} \wedge \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & -2 \\ -3 & 2 & 2 \end{vmatrix} = 8\underline{i} + 6\underline{j} + 6\underline{k}$$

$\therefore |\underline{a} \wedge \underline{b}| = \sqrt{64+36+36} = \sqrt{136}$

$\therefore$ area of $\Delta ABC = \frac{1}{2}\sqrt{136}$ #.

(V) Show whether the three vectors are coplaner

$\underset{\underline{a}}{<4,-8,3>}, \underset{\underline{b}}{<7,2,5>}, \underset{\underline{c}}{<19-52,8>}$.

Soln:

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{a} \cdot \begin{vmatrix} \underline{i} & \underline{j} & k \\ 7 & 2 & 5 \\ 10 & -52 & 8 \end{vmatrix} =$$

$<4,-8,3> \cdot <276,-6,-384>$

$= 1104 + 48 - 1152 = 0$

then the three vectors are coplaner

#