Chapter 4

BIVARIATE (JOINT) DISTRIBUTIONS

4.1 Introduction

In this chapter we shall be concerned with the bivariate or joint distributions, that is, with situations where we are interested at the same time in a pair of r.v.'s defined over a joint sample space.

If X and Y are discrete r.v.'s, we write the probability that X will take on the value x and Y will take on the value y as

$$h(x,y) = P(X=x, Y=y)$$

Thus $P(X=x,Y=y)$ is the probability of the intersection of the events $\{X=x\}$ and $\{Y=y\}$.

Definition 4.1

If X and Y are discrete r.v.'s with possible pairs of values

$$(x_i, y_j)\ ,\quad i = 1, 2, \ldots ;\ j = 1, 2, \ldots$$

the function **h(x,y)** defined by

$$h(x,y) = P(X=x, Y=y) \quad \text{for} \quad x = x_1, x_2, \ldots ;\ y = y_1, y_2, \ldots$$

is called the joint p.m.f. (joint probability mass function) of X and Y.

The table containing the possible values of X and Y together with their joint probabilities (see table 4.1) is called the **joint probability distribution**.

Y \ X	x_1	x_2	x_3	...
y_1	$h(x_1, y_1)$	$h(x_2, y_1)$	$h(x_3, y_1)$	...
y_2	$h(x_1, y_2)$	$h(x_2, y_2)$	$h(x_3, y_2)$	...
y_3	$h(x_1, y_3)$	$h(x_2, y_3)$	$h(x_3, y_3)$	...
..	...	...	...	...
..	...	...	...	...

Table 4.1 The Joint Probability distribution of X and Y

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Definition 4.2

A bivariate function h(x,y) can serve as a joint p.m.f. of a pair of r.v.'s X and Y iff it satisfies the conditions

1. $h(x,y) \geq 0$ for all x and y
2. $\sum_i \sum_j h(x_i, y_j) = 1$, where the double summation extends over all possible pairs (x_i, y_j) of (X, Y).

Example 4.1

Determine the value of the constant k so that the following function,

$$h(x,y) = k\,x\,y \quad \text{for} \quad x = 1, 2, 3\ ;\ y = 1, 2, 3$$

can serve as a joint p.m.f. of two r.v.'s X and Y, then find $P(X + Y \geq 4)$.

Solution

To satisfy the first condition of theorem 4.1, the constant k must be nonnegative, and to satisfy the second condition, we must have

$$\sum_{x=1}^{3} \sum_{y=1}^{3} k\,x\,y = 1$$

i.e $\{k + 2k + 3k + 2k + 4k + 6k + 3k + 6k + 9k\} = 1$

Thus, we must have $36k = 1$ and therefore $k = 1/36$. Now,

$$\begin{aligned} P(X + Y \geq 4) &= 1 - P(X + Y < 4) \\ &= 1 - \{h(1, 1) + h(1, 2) + h(2, 1)\} \\ &= 1 - \{1/36 + 2/36 + 2/36\} \\ &= 31/36. \end{aligned}$$

Example 4.2

Two textbooks are selected at random from a shelf contains 3 statistics texts, 2 mathematics texts and 3 arts texts. If X is the number of statistics texts and Y the number of mathematics texts actually chosen, find the joint distribution of X and Y.

Solution

All possible pairs of X and Y are:

$$(0, 0)\,,\ (0,1)\,,\ (1,0)\,,\ (1,1)\,,\ (0,2) \text{ and } (2,0)$$

To find the probability associated with (1,0), for example, observe that we are concerned with the event of choosing 1 of the 3 statistics texts, none of the 2 math. texts and hence 1 of the 3 physics texts. The number of ways in which this can be done is

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$$\binom{3}{1}\binom{2}{0}\binom{3}{1}=9$$

and the total number of ways in which 2 of the 8 texts can be chosen is;

$$\binom{8}{2}=28$$

Since these the probability are all equally likely by virtue of the assumptions that the selection is random, it follows that the probability associated with (1,0) is

$$h(1,0) = P(X=1\ ,\ Y=0) = 9/28$$

Similarly, the probability associated with (1,1) is

$$h(1,1)=P(X=1,Y=1)=\frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{\binom{8}{2}}=\frac{3}{14}$$

Continuing this way, we obtain the values shown in the following table,

y \ x	0	1	2
0	3/28	9/28	3/28
1	3/14	3/14	0
2	1/28	0	0

Actually, it is generally preferable to represent probabilities such as these by means of a formula. In other words, it is preferable to express the probabilities by means of a function with the values

$$h(x,y) = P(X = x\ ,\ Y = y)$$

for any pair of values (x,y) within the range of the r.v.'s X and Y. In this example, clearly we can write

$$h(x,y)=P(X=x\,,Y=y)=\frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}, \quad \text{where } x=0,1,2\,;\,y=0,1,2 \text{ and } 0\le x+y\le 2.$$

Definition 4.3

A bivariate function f(x,y), defined over the xy-plane, is called a joint probability

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density function (joint **p.d.f.**) of the continuous r.v.'s X and Y iff

$$P[(X,Y)\in A]=\iint_A f(x,y)\,dx\,dy$$

for any region A in the xy-plane.

Definition 4.4

A bivariate function f(x,y) can serve as a joint p.d.f. of a pair of continuous r.v.'s X and Y iff it satisfies the conditions

1. $f(x,y)\geq 0$ for all x and y
2. $\int_{-\infty}^{\infty}\int f(x,y)\,dx\,dy=1$

Example 4.3

Two electronic components of a missile system work in harmony for the success of the total system. Let X and Y denote the life in hours of the two components. The joint density of X and Y is

$$f(x,y)=\begin{cases} k(x^2+y^2) & , \quad 0\leq x\leq 1,\ 0\leq y\leq 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

(a) Determine the value of the constant k.

(b) Compute $P(0<X<0.5,\ 0<Y<0.5)$.

Solution

(a) Since f(x,y) is a joint p.d.f, then it must satisfy $\int_{-\infty}^{\infty}\int f(x,y)\,dx\,dy=1$, i.e.

$$\int_0^1\int_0^1 k(x^2+y^2)\,dx\,dy=1\Rightarrow\int_0^1\left[\int_0^1 k(x^2+y^2)\,dx\right]dy=1$$

$$k\int_0^1\left[(x^3/3+xy^2)\big|_0^1\right]dy=1\Rightarrow k\int_0^1\left[(1/3+y^2)\right]dy=1$$

$$k\left[(y/3+y^3/3)\right]_0^1\,dy=1\ \Rightarrow k(2/3)=1\Rightarrow k=3/2$$

(b)

$$P(0<X<½,\ 0<Y<½)=\int_0^{\frac12}\int_0^{\frac12}\tfrac{3}{2}(x^2+y^2)\,dx\,dy=\tfrac{3}{2}\int_0^{\frac12}\left[(x^3/3+xy^2)\big|_0^{\frac12}\right]dy$$

$$=\tfrac{3}{2}\left[\tfrac{1}{24}y+\tfrac{1}{2}y^3/3\right]_0^{\frac12}=\tfrac{1}{16}$$

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4.2 Marginal Distributions

If X and Y are discrete r.v.'s with joint p.m.f. h(x,y), the p.m.f. , called the **marginal p.m.f. of X** is given by

$$g_X(x_i) = \sum_j h(x_i, y_j) \quad , \quad i = 1, 2, \ldots$$

Correspondingly, the following function

$$g_Y(y_j) = \sum_i h(x_i, y_j) \quad , \quad j = 1, 2, \ldots$$

is called the **marginal p.m.f. of Y.**

Example 4.4

Referring to example 4.2, we have derived the joint p.m.f. of the r.v.'s X and Y, the number of statistics texts and the number of mathematics tests included among texts drawn at random from a shelf containing 3 statistics texts, 2 mathematics texts and 3 arts texts. Find the probability distributions of X alone and that of Y alone.

Solution

From the results of example 4.2 and the associated joint probability distribution table, the marginal probability distribution tables of X and Y are given by

x	0	1	2
$g_X(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$g_Y(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

When X and Y are continuous r.v.'s, the p.m.f.'s are replaced by p.d.f.'s, the summations are replaced by integrals and we get

Definition 4.3

If X and Y are continuous r.v.'s with joint p.d.f. f(x,y), the function given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad , \quad -\infty < x < \infty.$$

for every possible value x of X, is called the **marginal p.d.f. of X.** Correspondingly, the

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marginal p.d.f. of Y is given by

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx \quad , \quad -\infty < y < \infty.$$

Example 4.5

Consider the joint p.d.f. given in example 4.3,

$$f(x,y)=\begin{cases} \frac{3}{2}(x^2+y^2) & , \quad 0\le x\le 1,\ 0\le y\le 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

find the marginal p.d.f.'s of X and Y.

Solution

The marginal p.d.f. of X is given by,

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\begin{cases} \int_0^1 \frac{3}{2}(x^2+y^2)\,dy & 0<x<1 \\ 0 & \text{otherwise} \end{cases}=\begin{cases} \frac{3}{2}(x^2+\frac{1}{3}) & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Since f(x,y) is symmetric in x and y, then, the marginal p.d.f. of Y is given by,

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dx=\begin{cases} \int_0^1 \frac{3}{2}(x^2+y^2)\,dx & 0<y<1 \\ 0 & \text{otherwise} \end{cases}=\begin{cases} \frac{3}{2}(y^2+\frac{1}{3}) & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

4.3 Conditional Distributions

As we know, the conditional probability of event A given that event B has been occurred is

$$P(A|B)=\frac{P(A\cap B)}{P(B)} \quad , \quad P(B)>0$$

Suppose now that A and B are the events $X=x_i$ and $Y=y_j$, so that we can write,

$$P(X=x_i\,|\,Y=y_j)=\frac{P(X=x_i,\ Y=y_j)}{P(Y=y_j)}=\frac{h(x_i,y_j)}{g_Y(y_j)}, \quad g_Y(y_j)>0$$

Denoting this conditional probability by $f(x_i|y_j)$ to indicate that x_i is a variable and y_j is fixed.

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Definition 4.4

If X and Y are discrete r.v.'s with joint p.m.f. h(x,y), the conditional p.m.f. of X given $Y = y_j$ is given by

$$f(x_i \mid y_j) = \frac{h(x_i, y_j)}{g_Y(y_j)} \quad , \quad g_Y(y_j) > 0.$$

for all possible values x_i of X, where $g_Y(y_j)$ is the marginal p.m.f. of Y evaluated at $y = y_j$. Similarly, the conditional p.m.f. of Y given $X = x_i$ is given by

$$f(y_j \mid x_i) = \frac{h(x_i, y_j)}{g_X(x_i)} \quad , \quad g_X(x_i) > 0.$$

for all possible values y_j of Y.

Example 4.6

With reference to examples 4.2 and 4.4, the conditional distribution of **X** given that $Y = 1$, is

$$f(0 \mid 1) = \frac{3/14}{3/7} = \frac{1}{2}, \qquad f(1 \mid 1) = \frac{3/14}{3/7} = \frac{1}{2} \qquad \text{and} \qquad f(2 \mid 1) = 0.$$

i.e

x	0	1	2
f(x|y=1))	$\frac{1}{2}$	$\frac{1}{2}$	0

When **X** and **Y** are continuous r.v.'s, the probability mass functions are replaced by the probability density functions, and we get

Definition 4.5

If X and Y are continuous r.v.'s with joint p.d.f. f(x,y), the conditional p.d.f. of X given $Y = y$ is given by

$$f(x \mid y) = \frac{f(x, y)}{f_Y(y)} \quad , \quad f_Y(y) > 0.$$

for all possible values x of X, where $f_Y(y)$ is the marginal p.d.f. of Y evaluated at $Y = y$. Similarly, the conditional p.d.f. of Y given $X = x$ is given by

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$$f(y \mid x) = \frac{f(x,y)}{f_X(x)} \quad , \quad f_X(x) > 0.$$

for all possible values y of Y.

4.4 Independence

When we are dealing with two or more r.v.'s, questions of independence are usually of great importance. As we know the two events A and B are said to be independent iff

$$P(A \cap B) = P(A) \cdot P(B)$$

Similarly, for the two r.v.'s X and Y we have,

Definition 4.4

If X and Y are two discrete r.v.'s with joint p.m.f. **h(x,y)** and marginal p.m.f.'s $g_X(x)$ and $g_Y(y)$ respectively, then X and Y are said to be **independent** iff

$$h(x_i, y_j) = g_X(x_i) \cdot g_Y(y_j) \quad , \text{for all i and j.}$$

More generally, the discrete r.v.'s $X_1, X_2, \dots, X_k$ are said to be independent iff

$$h(x_1, x_2, \dots, x_k) = g_{X_1}(x_1)\, g_{X_2}(x_2) \dots g_{X_k}(x_k) \quad \text{for all possible values } x_1, x_2, \dots, x_k.$$

Similarly, if **f(x , y)** is the joint p.d.f. of the continuous r.v.'s **X** and **Y** and $f_X(x)$ is the **marginal p.d.f. X** and $f_Y(y)$ is the **marginal p.d.f. Y**, then the r.v.'s **X** and **Y** are independent iff

$$f(x,y) = f_X(x) f_Y(y) \quad \forall \; x \& y$$

Example 4.7

The r.v.'s X and Y defined in examples 4.2 and 4.4 are dependent, since for example

$$h(0,0) = 3/28 \quad , g_X(0) = 5/12 \;\text{ and }\; g_Y(0) = 15/28$$

hence,

$$h(0,0) \neq g_X(0).g_Y(0)$$

Also the r.v.'s X and Y defined in examples 4.3 and 4.5 are dependent, since

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2 + y^2) & , \quad 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \quad \text{elsewhere} \end{cases} \neq f_X(x).f_Y(y)$$

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4.5 Expectations

The concept of a mathematical expectation can be easily extended to situations involving more than one r.v. For instance, if Z is the r.v. whose values are related to those of the two r.v.'s X and Y by means of the equation $z = g(x,y)$, we have the following definition;

Definition 4.6

If X and Y are discrete r.v.'s with joint p.m.f. h(x,y), then the expected value of g(X,Y) is,

$$E[g(X,Y)] = \sum_i \sum_j g(x_i, y_j) f(x_i, y_j)$$

Similarly, if X and Y are continuous r.v.'s with joint p.m.f. h(x,y), then the expected value of g(X,Y) is,

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

In particular, if $g(X,Y) = XY$, then

$$E[XY] = \begin{cases} \sum_i \sum_j x_i y_j f(x_i, y_j) & \text{, if X \& Y are discrete} \\ \iint_{-\infty}^{\infty} xy f(x,y)\, dx\, dy & \text{, if X \& Y are continuous} \end{cases}$$

Generalization of this definition to functions of any finite number of r.v.'s is straightforward.

Example 4.8

With reference to example 4.2, the expected value of XY is

$$E[XY] = \sum_x \sum_y xy\, h(x,y)$$

$$= 0*0*\frac{3}{28} + 1*0*\frac{9}{28} + 2*0*\frac{3}{14} + 0*1*\frac{3}{14} + 1*1*\frac{3}{14} + 2*1*0 + 0*2*\frac{1}{28} + 1*2*0 + 2*2*0$$

$$= \frac{3}{14}$$

Also, with reference to example 4.3, the expected value of XY is

$$E[XY] = \iint_{-\infty}^{\infty} xy f(x,y)\, dx\, dy = \int_0^1\int_0^1 xy.\tfrac{3}{2}(x^2 + y^2)\, dx\, dy = \frac{9}{16}$$

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4.5 Covariance and Correlation Coefficient

The **covariance** of the two r.v.'s X and Y, denoted by **cov(X,Y)** or σ_{xy} is defined as

$$\sigma_{xy} = \operatorname{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

The **correlation coefficient** of the r.v.'s X and Y, denoted by $\rho(X, Y)$ or ρ_{xy} is defined to be

$$\rho(X, Y) = \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Both the covariance and the correlation coefficient of the r.v.'s X and Y are measures of a linear relationship of X and Y in the following sense:

Cov(X , Y) will be positive when $(X - \mu_X)$ and $(Y - \mu_Y)$ tend to have the same sign with high probability; and cov(X , Y) will be negative when $(X - \mu_X)$ and $(Y - \mu_Y)$ tend to have the opposite signs with high probability. cov(X , Y) tends to measure the linear relationship of X and Y; however, its actual magnitude does not have much meaning since it depends on the variability of X and Y. The correlation coefficient removes, in a sense, the individual variability of each X and Y by dividing the covariance by the product of standard deviations , and thus the correlation coefficient is a better measure of the linear relationship of X and Y than is the covariance. Also, the correlation coefficient is unitless and, as we shall see below,

$$-1 \le \rho(X, Y) \le 1$$

Remark:

$$\operatorname{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$$

Proof.

$$\begin{aligned}\operatorname{cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY - \mu_X Y - X \mu_Y + \mu_X \mu_Y] \\ &= E[XY] - \mu_X E(Y) - E(X)\mu_Y + \mu_X \mu_Y \\ &= E[XY] - \mu_X \mu_Y\end{aligned}$$

Some Properties of Covariance

(1) $Cov(X, Y) = Cov(Y, X)$

(2) $Cov(X, X) = Var(X)$

(3) $Cov(X, a) = 0$, for any constant **a**.

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(4) For any constants a and b;

$$Cov(aX, by) = ab\,Cov(X, Y)$$

(5) If X_1, X_2 and Y are random variables, then

$$Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$$

This property can be generalized as follows;

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

(6) For any two r.v.'s X and Y we have

$$Var(X \pm Y) = Var(X) + Var(Y) \pm 2Cov(X, Y)$$

Example 4.9

With reference to examples 4.2, 4.4 and 4.10, we have

$$\mu_X = E[X] = \sum_x x\, g_x(x) = 0 \times \frac{10}{28} + 1 \times \frac{15}{28} + 2 \times \frac{3}{28} = \frac{21}{28} = \frac{3}{4}$$

$$\mu_Y = E[Y] = \sum_y y\, g_y(y) = 0 \times \frac{15}{28} + 1 \times \frac{3}{7} + 2 \times \frac{1}{28} = \frac{1}{2}$$

and

$$E[XY] = \sum_x \sum_y xy\, h(x, y) = \frac{3}{14}$$

Therefore,

$$cov(X, Y) = E[XY] - \mu_X \mu_Y = -9/56$$

In order to find the correlation coefficient, we have to calculate, first, the standard deviations σ_x and σ_y.

$$E[X^2] = \sum_x x^2 g_X(x) = 0^2 \times \frac{5}{14} + 1^2 \times \frac{15}{28} + 2^2 \times \frac{3}{28} = \frac{27}{28}$$

Hence,

$$\sigma_X^2 = var(X) = E[X^2] - \mu_X^2 = \frac{27}{28} - \left(\frac{3}{4}\right)^2 = \frac{45}{112} \approx 0.4$$

Similarly (note that distribution of X and Y is symmetric i.e X and Y have the same mean and the same variance),

$$\sigma_y^2 = var(Y) = \frac{9}{28}$$

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Finally,
$$\rho(X,Y)=\frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}=\frac{-9/28}{\sqrt{\frac{45}{112}\cdot\frac{9}{28}}}=-\frac{1}{\sqrt{5}}\approx-0.447$$

Some Properties of Correlation Coefficient

(1) $\rho(Y,X)=\rho(X,Y)$

(2) $\rho(X\pm a, Y\pm b)=\rho(X,Y)$.

(3) $\rho(aX,bY)=\rho(X,Y)$

(4) $\rho(X,X)=1$

Theorem 4.3

If X and Y are independent, then

a- $E[XY]=E[X].E[Y]$

b- $\operatorname{cov}(X,Y)=0$

It follows also that
$$\rho(X,Y)=0$$

Definition 4. 6

The r.v.'s X and Y are defined to be **uncorrelated** iff,
$$\rho(X,Y)=0$$

Remark:

It is of interest to note that the *independence* of two r.v.'s implies a zero covariance (*uncorrelated*), but a zero covariance does not necessarily imply their independence.

4.6 Sums of Random Variables

It is interested and important to know means, variances and covariances of sums or linear combinations of n r.v.'s.

Theorem 4.4

For r.v.'s $X_1, X_2, \ldots, X_n$,
$$E\left[\sum_{i=1}^{n} X_i\right]=\sum_{i=1}^{n} E(X_i)$$

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and

$$\operatorname{var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \operatorname{var}(X_i) + 2\sum_{i<j}\sum \operatorname{cov}(X_i, X_j)$$

For $n = 2$, we have

$$\operatorname{var}(X_1 \pm X_2) = \operatorname{var}(X_1) + \operatorname{var}(X_2) \pm 2\operatorname{cov}(X_1, X_2)$$

and

$$\operatorname{var}(a_1X_1 \pm a_2X_2) = a_1^2 \operatorname{var}(X_1) + a_2^2 \operatorname{var}(X_2) \pm 2a_1a_2 \operatorname{cov}(X_1, X_2)$$

Corollary 4.1

If the r.v.'s $X_1, X_2, \dots, X_n$ are uncorrelated or independent, then

$$\operatorname{var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \operatorname{var}(X_i)$$

Theorem 4.5

Let $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$ be two sets of r.v.'s, and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two sets of any constants, then

$$\operatorname{cov}\left[\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \operatorname{cov}(X_i, Y_j)$$

Example 4.10

If var(X) = 1, var(Y) = 5, var(Z) = 2, cov(X,Y) = -2, cov(X,Z) = -1, and Y and Z are independent, find **var(3X-Y+2Z)** and covariance of **U=X-2Y+3Z** and **V=-2X+3Y+4Z.**

Solution

$$\begin{aligned}
\operatorname{var}(3X-Y+2Z) &= 9\operatorname{var}(X) + \operatorname{var}(y) + 4\operatorname{var}(Z) + \\
&\quad 2\{-3\operatorname{cov}(X, Y) + 6\operatorname{cov}(X, Z) - 2\operatorname{cov}(Y, Z)\} \\
&= 9(1) + 5 + 4(2) + 2\{-3(-2) + 6(-1) + 0\} = 22.
\end{aligned}$$

$$\begin{aligned}
\operatorname{Cov}(U,V) &= \operatorname{cov}(X-2Y+3Z, -2X+3Y+4Z) \\
&= -2\operatorname{var}(X) + 3\operatorname{cov}(X, Y) + 4\operatorname{cov}(X, Z) \\
&\quad + 4\operatorname{cov}(X, Y) - 6\operatorname{var}(Y) - 8\operatorname{cov}(Y, Z) \\
&\quad - 6\operatorname{cov}(X, Z) + 9\operatorname{cov}(Y, Z) + 12\operatorname{var}(Z) = -14.
\end{aligned}$$

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EXERCISES

[1] A box contains 4 red chips, 3 white chips and 2 blue chips. A random sample of size 3 is drawn without replacement. Let X denotes the number of white chips in the sample and Y, the number of blue. Write down a formula for the joint p.m.f. of X and Y.

[2] If the joint p.m.f. of X and Y is given by

$$h(x, y) = c(x^2 + y^2) \quad , \quad x = -1, 0, 1, 3 \text{ and } y = -1, 2, 3.$$

Find the constant c, the marginal distributions of X and Y, cov(X,Y) and $\rho(X,Y)$.

[3] Determine the constant k so that

$$f(x,y) = \begin{cases} k\,x\,(x+y) & , \quad 0<x<1 \, , \, 0<y<2 \\ 0 & , \quad \text{o.w.} \end{cases}$$

can serve as a joint probability density function.

[4] If X and Y have the joint probability density function given by:

$$f(x, y) = \begin{cases} k & , \quad -1<x<1, \ 0<y<1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

Find: a- The value of k, b- Cov(3X-7,2Y+3).

[5] If the joint probability distribution of X and Y is given by the following table

X \ Y	-2	-1	2	4
1	0.1	0.2	0.0	0.1
2	0.0	0.1	0.1	0.2
3	0.1	0.0	0.1	0.0

Find: **P(Y>X) , COV(X,Y)** and **VAR(2X-5Y+7).**

[6] If $var(X_1)=5$, $var(X_2)=4$, $var(X_3)=7$, $cov(X_1,X_2)=3$, $cov(X_1,X_3)=-2$, and X_2 and X_3 are independent. Find the correlation coefficient of

$$Y = X_1 - 2X_2 + 3X_3 \text{ and } Z = -2X_1 + 3X_2 + 4X_3.$$

[7] Consider the joint density function

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$$f(x,y)=\begin{cases} k\dfrac{y}{x^4} & , \quad x>2,\ 0\le y\le 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

(a) Determine the value of the constant k.

(b) Give the marginal density functions for both random variables and determine whether or not X and Y are independent..

(c) Find Cov(3X- 4Y+5 , 4X+5Y-2) and Var(3X - 4Y+5).

[8] Circle the correct answer from each of the following multiple choice questions

i. If the joint p.m.f. of X and Y is given by

$$h(x\,,y)=k(x^2+y^2) \quad , \quad x=-1,0,3 \ \text{ and } \ y=-2,1,3.$$

where k is a constant, then $P(X+Y\le 3)$ is

a. 53/72 b. 19/72 c. 11/18 d. None of the above.

ii. If the joint p.m.f. of X and Y is given by:

$$h(x\,,y)=kx(x+y) \quad , \quad x=1,2,3 \ \text{ and } \ y=-1,0,2.$$

where k is a constant, then $P(X+Y>2)$ is

a. 10/48 b. 25/48 c. 45/48 d. None of the above.

iii. If var(X)=5, var(Y)=4, and cov(X,Y)=3 then var(3X-2Y+5) is:

a. 22 b. 25 c. 43 d. None of the above.

iv. If the joint probability density function of X and Y is given by:

$$f(x,y)=\begin{cases} k & , \quad 0<x<1,\ -1<y<1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

then the value of the constant k is

a. 2 b. 1/2 c. 1/4 d. None of the above.

<+><+><+><+><+><+><+><+><+>

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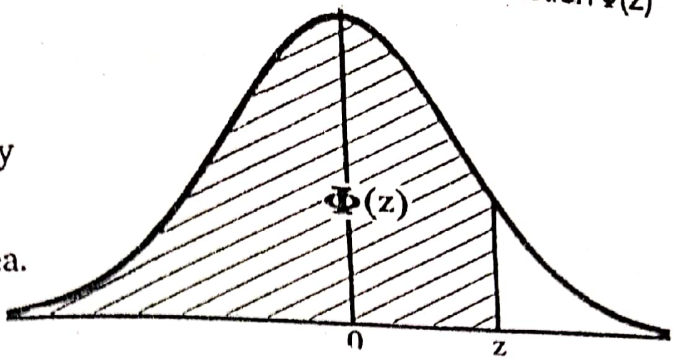
Table [1]. The *STANDARD NORMAL* Cumulative Distribution Function $\Phi(z)$

$$\Phi(z) = P(Z \le z)$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

= The shaded area.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	z
.0	.500 00	.503 99	.507 98	.511 97	.515 95	.519 94	.523 92	.527 90	.531 88	.535 86	.0
.1	.539 83	.543 79	.547 76	.551.72	.555 67	.559 62	.563 56	.567 49	.571 42	.575 34	.1
.2	.579 26	.583 17	.587 06	.590 95	.594 83	.598 71	.602 57	.606 42	.610 26	.614 09	.2
.3	.617 91	.621 72	.625 51	.629 30	.633 07	.636 83	.640 58	.644 31	.648 03	.651 73	.3
.4	.655 42	.659 10	.662 76	.666 40	.670 03	.673 64	.677 24	.680 82	.684 38	.687 93	.4
.5	.691 46	.694 97	.698 47	.701 94	.705 40	.708 84	.712 26	.715 66	.719 04	.722 40	.5
.6	.725 75	.729 07	.732 37	.735 65	.738 91	.742 15	.745 37	.748 57	.751 75	.754 90	.6
.7	.758 03	.761 15	.764 24	.767 30	.770 35	.773 37	.776 37	.779 35	.782 30	.785 23	.7
.8	.788 14	.791 03	.793 89	.796 73	.799 54	.802 34	.805 10	.807 85	.810 57	.813 27	.8
.9	.815 94	.818 59	.821 21	.823 81	.826 39	.828 94	.831 47	.833 97	.836 46	.838 91	.9
1.0	.841 34	.843 75	.846 13	.848 49	.850 83	.853 14	.855 43	.857 69	.859 93	.862 14	1.0
1.1	.864 33	.866 50	.868 64	.870 76	.872 85	.874 93	.876 97	.879 00	.881 00	.882 97	1.1
1.2	.884 93	.886 86	.888 77	.890 65	.892 51	.894 35	.896 16	.897 96	.899 73	.901 47	1.2
1.3	.903 20	.904 90	.906 58	.908 24	.909 88	.911 49	.913 08	.914 65	.916 21	.917 73	1.3
1.4	.919 24	.920 73	.922 19	.923 64	.925 06	.926 47	.927 85	.929 22	.930 56	.931 89	1.4
1.5	.933 19	.934 48	.935 74	.936 99	.938 22	.939 43	.940 62	.941 79	.942 95	.944 08	1.5
1.6	.945 20	.946 30	.947.38	.948 45	.949 50	.950 53	.951 54	.952 54	.953 52	.954 48	1.6
1.7	.955 43	.956 37	.957 28	.958 18	.959 07	.959 94	.960 80	.961 64	.962 46	.963 27	1.7
1.8	.964 07	.964 85	.965 62	.966 37	.967 11	.967 84	.968 56	.969 26	.969 95	.970 62	1.8
1.9	.971 28	.971 93	.972 57	.973 20	.973 81	.974 41	.975 00	.975 58	.976 15	.976 70	1.9
2.0	.977 25	.977 78	.978 31	.978 82	.979 32	.979 82	.980 30	.980 77	.981 24	.981 69	2.0
2.1	.982 14	.982 57	.983 00	.983 41	.983 82	.984 22	.984 61	.985 00	.985 37	.985 74	2.1
2.2	.986 10	.986 45	.986 79	.987 13	.987 45	.987 78	.988 09	.988 40	.988 70	.988 99	2.2
2.3	.989 28	.989 56	.989 83	.990 10	.990 36	.990 61	.990 86	.991 11	.991 34	.991 58	2.3
2.4	.991 80	.992 02	.992 24	.992 45	.992 66	.992 86	.993 05	.993 24	.993 43	.993 61	2.4
2.5	.993 79	.993 96	.994 13	.994 30	.994 46	.994 61	.994 77	.994 92	.995 06	.995 20	2.5
2.6	.995 34	.995 47	.995 60	.995 73	.995 85	.995 98	.996 09	.996 21	.996 32	.996 43	2.6
2.7	.996 53	.996 64	.996 74	.996 83	.996 93	.997 02	.997 11	.997 20	.997 28	.997 36	2.7
2.8	.997 44	.997 52	.997 60	.997 67	.997 74	.997 81	.997 88	.997 95	.998 01	.998 07	2.8
2.9	.998 13	.998 19	.998 25	.998 31	.998 36	.998 41	.998 46	.998 51	.998 56	.998 61	2.9
3.0	.998 65	.998 69	.998 74	.998 78	.998 82	.998 86	.998 89	.998 93	.998 97	.999 00	3.0
3.1	.999 03	.999 06	.999 10	.999 13	.999 16	.999 18	.999 21	.999 24	.999 26	.999 29	3.1
3.2	.999 31	.999 34	.999 36	.999 38	.999 40	.999 42	.999 44	.999 46	.999 48	.999 50	3.2
3.3	.999 52	.999 53	.999 55	.999 57	.999 58	.999 60	.999 61	.999 62	.999 64	.999 65	3.3
3.4	.999 66	.999 68	.999 69	.999 70	.999 71	.999 72	.999 73	.999 74	.999 75	.999 76	3.4
3.5	.999 77	.999 78	.999 78	.999 79	.999 80	.999 81	.999 81	.999 82	.999 83	.999 83	3.5
3.6	.999 84	.999 85	.999 85	.999 86	.999 86	.999 87	.999 87	.999 88	.999 88	.999 89	3.6
3.7	.999 89	.999 90	.999 90	.999 90	.999 91	.999 91	.999 92	.999 92	.999 92	.999 92	3.7
3.8	.999 93	.999 93	.999 93	.999 94	.999 94	.999 94	.999 94	.999 95	.999 95	.999 95	3.8
3.9	.999 95	.999 95	.999 96	.999 96	.999 96	.999 96	.999 96	.999 96	.999 97	.999 97	3.9

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