1 Number Systems

Decimal

1	5	7	
100	10	1	weights

$$1\times100 + 5\times10 + 7\times1 = 157$$

base-10 number system

Binary

Binary is a *base-2* number system.

$$\begin{array}{ccccc} 0 & 1 & 0 & 1 & 1 \\ \hline 16 & 8 & 4 & 2 & 1 \end{array} \text{ weights}$$

$0\times16 + 1\times8 + 0\times4 + 1\times2 + 1\times1 = 11$ dec

$8 + 2 + 1 = 11$ decimal.

We call a sequence of eight bits a *byte*.

To *complement* a bit means to flip it.

Hexadecimal

Hexadecimal (or hex for short) is the *base-16* number system.

16 symbols: 0 to 9 and A, B, C, D, E, and F

$$\begin{array}{ccc} 2 & C & 5 \\ \hline 256 & 16 & 1 \end{array}$$ weights (in decimal)

$$2\times256 + C\times16 + 5\times1$$

$$2\times256+12\times16+5\times1 = 512+192+5 = 709$$

Decimal	Binary	Hex
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A (or a)
11	1011	B (or b)
12	1100	C (or c)
13	1101	D (or d)
14	1110	E (or e)
15	1111	F (or f)

Rule: Adding a 0 on the right side of a positional whole number multiplies its value by its base.

Numbering Bits

bit 7 bit 0

`10101010`

bit 3

most significant bit
least significant bit

Adding Positional Numbers

decimal addition:

```
     1   carries
    157
  + 238
    ---
    395
```

binary addition:

```
     11   carries
    0011
  + 0011
    ----
    0110
```

hex addition:

```
    1    carries
    1B
  + 37
    --
    52
```

Representing Negative Numbers

```
  0011 = +3     Sign-magnitude
+ 1011 = -3
  1110 = -6
```

Carry out of leftmost column

```
1 111   carries
  1111
+ 0001
  0000
```

Rule: Adding 1 to a binary number with a fixed number of bits all of which are 1 results in all zeros.

```
  0011 = +3
+ 1100 = +3 with each bit flipped
  1111

  1100 = +3  with each bit flipped
+ 0001  add 1
  1101  Is this -3?
```

Is 1101 the representation of -3 that we want? Let's add it to +3 to see if it gives zero:

```
1 111      carries
  0011 = +3
+ 1101     Is this -3?
  0000
```

Rule: To negate a binary number in the two's complement system, flip its bits and add 1.

Rule: In the two's complement system, all 1's represents -1.

Signed and Unsigned Numbers

Unsigned	Value	Signed	Value
0000	0	1000	-8
0001	1	1001	-7
0010	2	1010	-6
0011	3	1011	-5
0100	4	1100	-4
0101	5	1101	-3
0110	6	1110	-2
0111	7	1111	-1
1000	8	0000	0
1001	9	0001	1
1010	10	0010	2
1011	11	0011	3
1100	12	0100	4
1101	13	0101	5
1110	14	0110	6
1111	15	0111	7

sign bit

Range of Numbers

n	Unsigned	Signed
3	0 to 7	-4 to 3
4	0 to 15	-8 to 7
5	0 to 31	-16 to 15
6	0 to 63	-32 to 31
7	0 to 127	-64 to 63
8	0 to 255	-128 to 127
9	0 to 511	-256 to 255
10	0 to 1023	-512 to 511
12	0 to 4095	-2048 to 2047
16	0 to 65535	-32768 to 32767
k	0 to 2^{k}-1	-2^{k-1} to 2^{k-1} -1

Powers of 2

n	2^n
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024 (aka 1K)
11	2,048 (aka 2K)
12	4,096 (aka 4K)
15	32,768 (aka 32K)
16	65,536 (aka 64K)
20	1,048,576 (aka 1M)
30	1,073,741,824 (aka 1G)

Converting Between Binary and Hex

1	1010	1110	0000	1100
↓	↓	↓	↓	↓
1	A	E	0	C

Converting Decimal to Binary

```
          remainders
               ↓
         0     1
10)      1     5        ↑
10)     15     7
10)    157       Start here and work up
```

```
       0   1
2)     1   0
2)     2   0
2)     4   1
2)     9   1
2)    19   1
2)    39   0
2)    78   1        ↑
2)   157     Start here and work up
```

Converting Fractions

```
   .153
   x 10
1  .530
   x 10
5  .300
   x 10
3.  000
```

whole parts are the digits

multiply fractional part of each product

Converting Binary Fraction

```
       .375
       x  2
  0   .750
      x  2
  1   .500
      x  2
  1   .000
```

.375 decimal = .011 binary

Infinite Length Equivalents

```
   .1
x  2
0 .2
x  2
0 .4      .2 repeats
x  2
0 .8
x  2
1 .6
x  2
1 .2
```

Zero and Sign Extension

Extend 11111111 to 16 bits

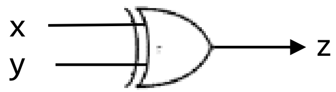
0000000011111111

1111111111111111

Rule: Always sign extend signed numbers.

Rule: Always zero extend unsigned numbers.

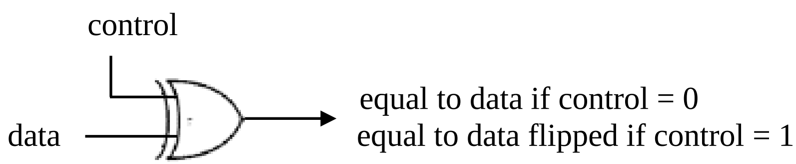
Exclusive OR Gate



Exclusive OR Gate

x	y	z
0	0	0
0	1	1
1	0	1
1	1	0

difference detecting gate



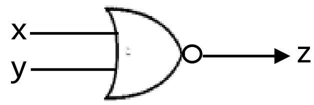
Control	Data	Output of exclusive OR
0	0	0
0	1	1
1	0	1
1	1	0

Data unaffected when control = 0

Data flipped when control = 1

Rule: If the control line of an exclusive OR is 0, the data passes through gate unchanged. If the control line is 1, the data is complemented.

NOR Gate

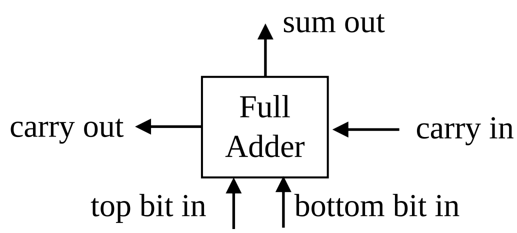


x	y	z
0	0	1
0	1	0
1	0	0
1	1	0

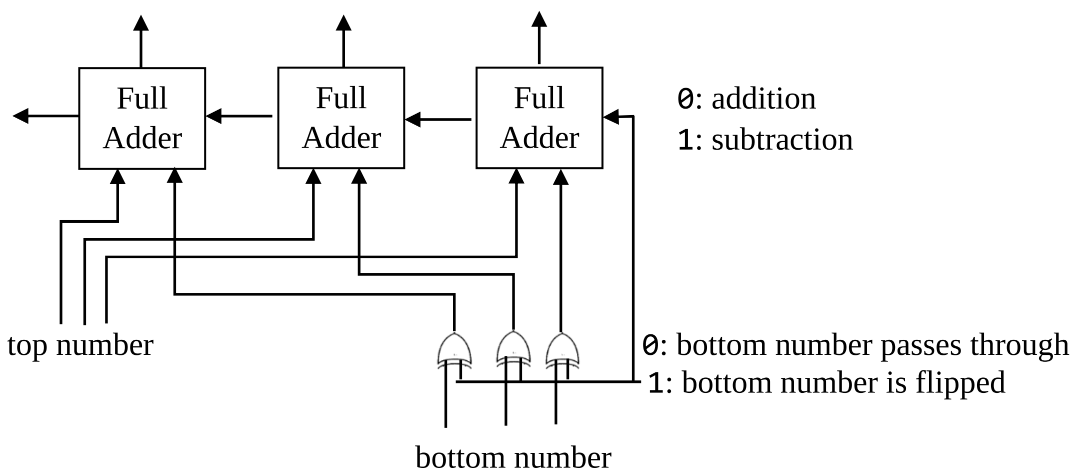
1 when all inputs = 0

zero detecting gate

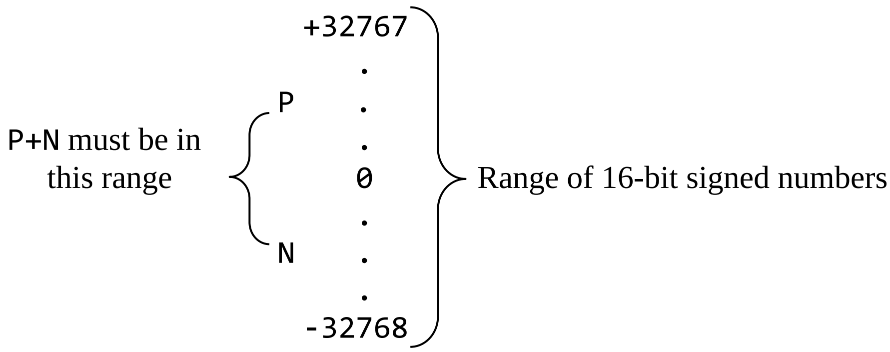
Addition and Subtraction



Three-bit Adder/Subtractor Circuit



Signed Overflow



Rule: If two signed numbers *with different signs* are added or two signed numbers *with the same sign* are subtracted, overflow cannot occur.

Overflow scenarios:

Adding two non-negative numbers

(a)

```
  0--------------
+ 0--------------
-----------------
  1--------------
```

Sign bit should be 0

Adding two negative numbers

(b)

```
  1--------------
+ 1--------------
-----------------
  0--------------
```

Sign bit should be 1

Here are all the possible scenarios for no overflow:

(c)

```
  0--------------
+ 0--------------
-----------------
  0--------------
```

(d)

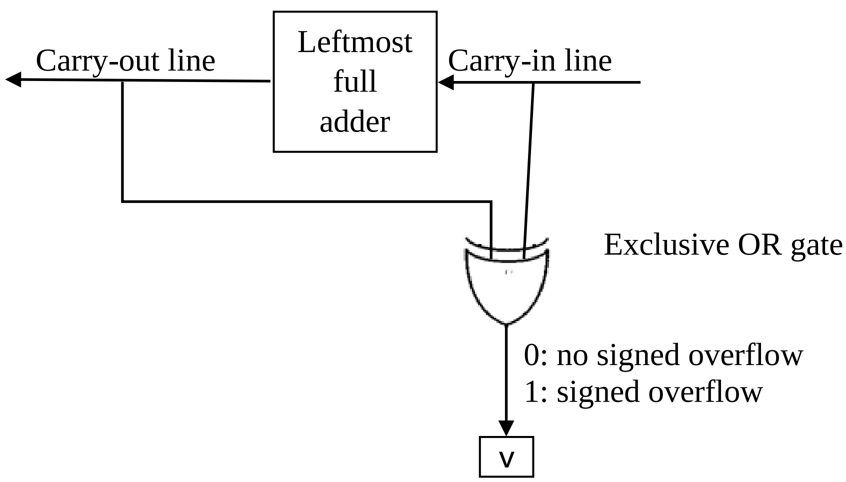
```
  1--------------
+ 1--------------
-----------------
  1--------------
```

(e)

```
  0--------------
+ 1--------------
-----------------
  0--------------
```

(f)

```
  0--------------
+ 1--------------
-----------------
  1--------------
```



Rule: In an addition or subtraction of signed numbers, overflow has occurred if the carry into the leftmost position does not match the carry out.

Rule: If overflow occurs during the addition of two signed numbers, the sign bit of the computed result is wrong.

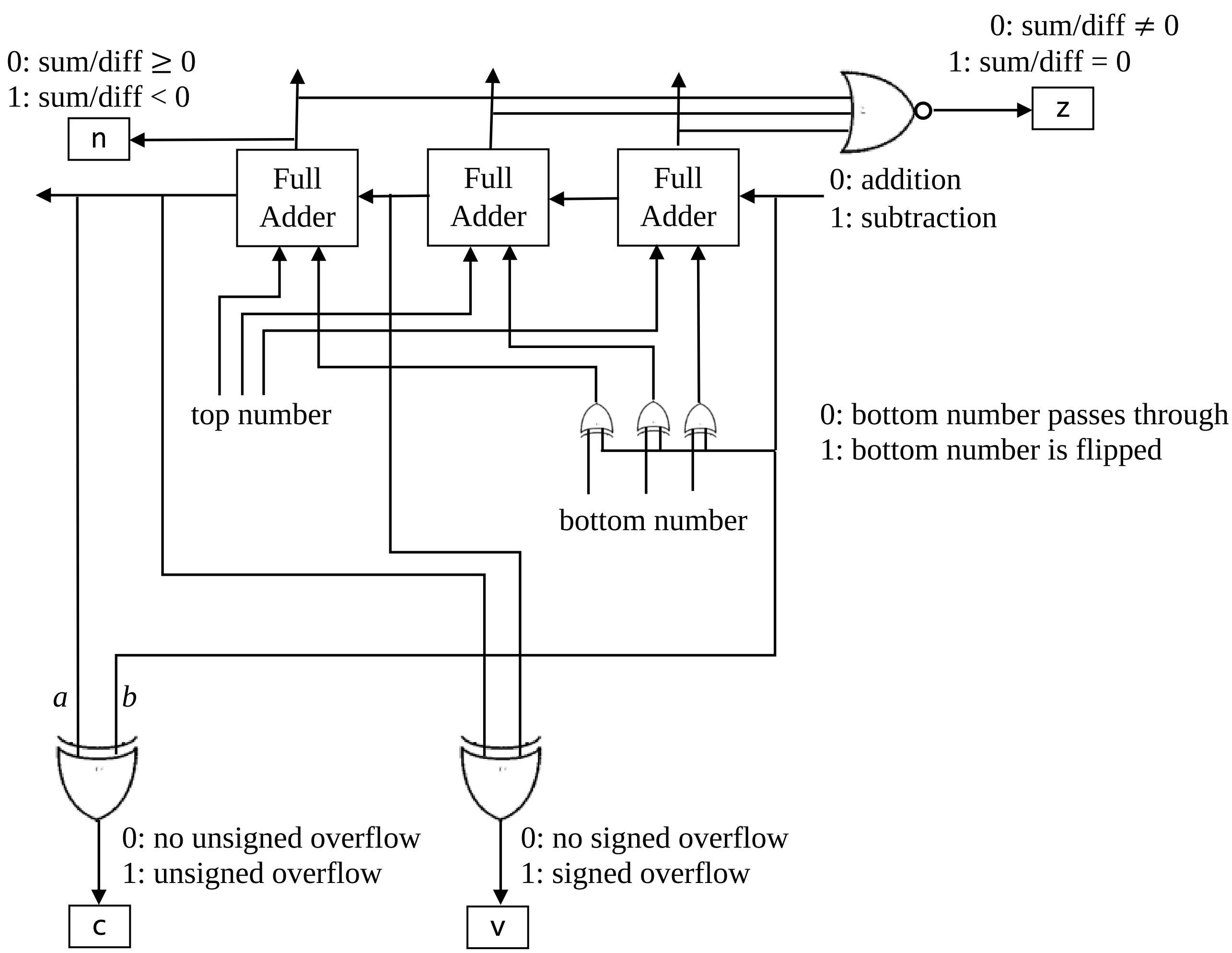
Unsigned Overflow

Carry out of leftmost position

```
1 111111111111111      carries
  1111111111111111 = 65535
+ 0000000000000001 = 1
  ----------------
  0000000000000000 = 0
```

Rule: In an addition of unsigned numbers, a carry out of the leftmost position indicates overflow. In a subtraction, no carry out of the leftmost position (or a borrow in if the borrow technique is used) indicates overflow.

Three-bit Adder/Subtractor Circuit with n, z, c, and v Flags



Floating Point

hidden bit $\qquad$ exponent

$$1.1\underbrace{10101}_{\text{fractional part}} \times 2^3$$ (which equals 13.25 decimal)

- One bit: the sign bit (0 for zero or positive, 1 for negative)
- Eight bits: the exponent *to which 127 is added*
- 23 bits: the fractional part, zero extended on the right

10101000000000000000000.

Thus, the 32-bit floating-point representation of 13.25 decimal is

$$\overset{\text{sign}}{0}\ \underbrace{10000010}_{\text{adjusted exponent}}\ \underbrace{10101000000000000000000}_{\text{fractional part}}$$

Special Representations

Exponent field	Fractional part	Value
all 0's	all 0's	0.0
all 0's	nonzero	denormal number (exponent = -126, hidden bit 0)
all 1's	all 0's	infinity
all 1's	nonzero	NaN (Not a Number)

Denormal Numbers

Smallest positive magnitude that can be represented by a regular floating-point number is

0 00000001 00000000000000000000000

which equals

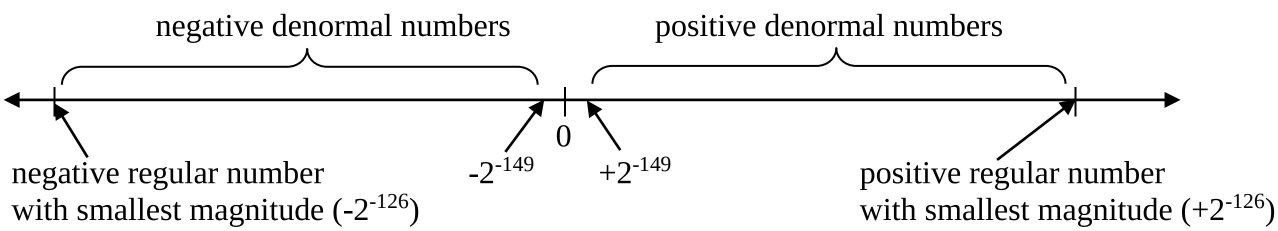
$1.00000000000000000000000 \times 2^{-126}$

hidden bit

0 00000000 11111111111111111111111 =
$0.11111111111111111111111 \times 2^{-126}$

which is just under 2^{-126}, down to

0 00000000 00000000000000000000001 =
$0.00000000000000000000001 \times 2^{-126}$



Floating-point numbers (`float` and `double` in C) are *only an approximation of the set of real numbers in mathematics*

```c
// ex0101.c  Finite precision of floating point numbers
#include <stdio.h>
int main()
{
   double sum = 0.0;
   int i;
   for (i = 1; i <= 10; i++)
      sum = sum + 0.1;
   printf("%.17f\n", sum);    // sum is not 1.0
   return 0;
}
```

```c
// ex0102.c  Floating-point errors
#include <stdio.h>
int main()
{
   double x = 1E70;    // 10^70
   double y = 1E5;     // 10^5
   if (x == x + y)
      printf("y must be zero\n");  // displayed but y is not zero
   else
      printf("y must be nonzero\n");
   return 0;
}
```

Observation: Numbers can act like zero when added to numbers with significantly larger magnitudes.

```c
// ex0103.c  Lack of associativity in floating-point computations
#include <stdio.h>
int main(void)
{
   double sum = 0.0;
   int i;
   for (i = 1; i <= 100000000; i++)
      sum = sum + 1.0/i;
   printf("%.17f\n", sum);

   sum = 0.0;
   for (i = 100000000; i >= 1; i--)         // better for loop
      sum = sum + 1.0/i;
   printf("%.17f\n", sum);
   return 0;
}
```

Observation: Addition of floating-point numbers is not associative (i.e., order of evaluation matters).