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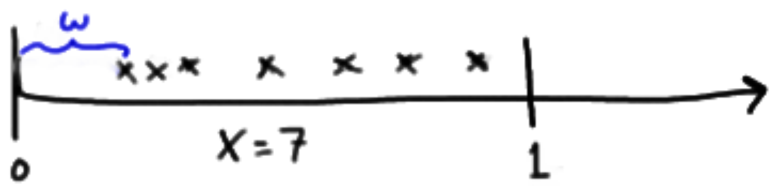
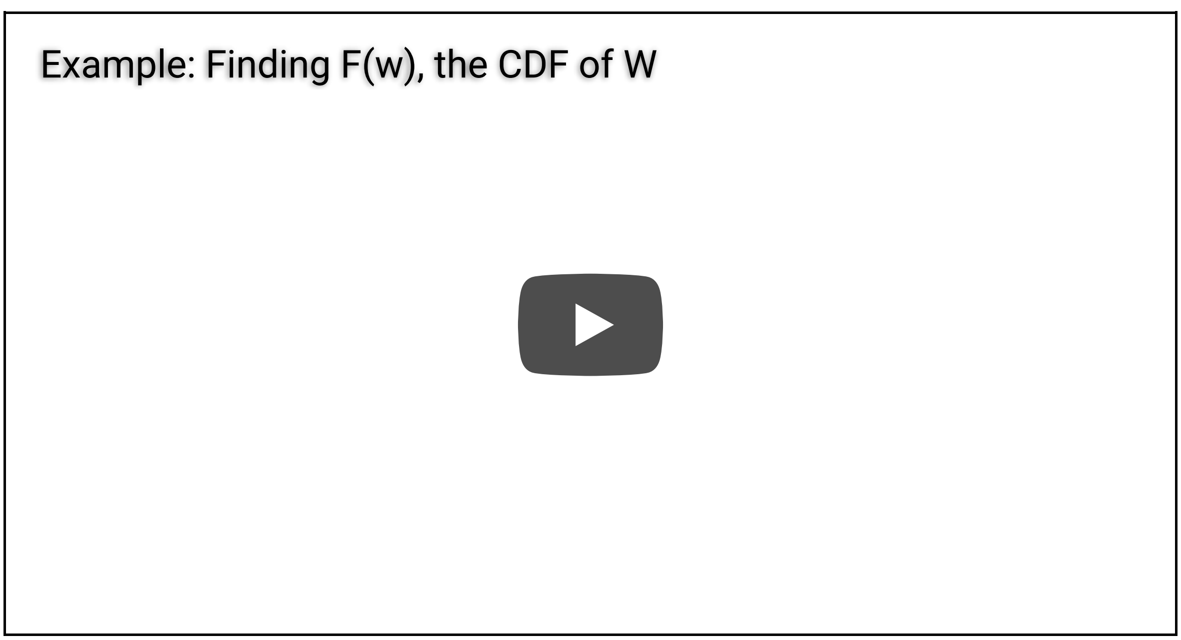
Exponential Distributions

Example

Suppose *X,* following an (approximate) Poisson process, equals the number of customers arriving at a bank in an interval of length 1. If λ, the mean number of customers arriving in an interval of length 1, is 6, say, then we might observe something like this:



In this particular representation, seven (7) customers arrived in the unit interval. Previously, our focus would have been on the discrete random variable *X*, the number of customers arriving. As the picture suggests, however, we could alternatively be interested in the continuous random variable *W*, the waiting time until the *first* customer arrives. Let's push this a bit further to see if we can find *F*(*w*), the cumulative distribution function of *W*:



Now, to find the probability density function $f(w)$, all we need to do is differentiate $F(w)$. Doing so, we get:

$$f(w) = F'(w) = -e^{-\lambda w}(-\lambda) = \lambda e^{-\lambda w}$$

for $0 < w < \infty$. Typically, though we "**reparameterize**" before defining the "official" probability density function. If λ (the Greek letter "lambda") equals the mean number of events in an interval, and θ (the Greek letter "theta") equals the mean waiting time until the first customer arrives, then:

$$\theta = \frac{1}{\lambda} \text{ and } \lambda = \frac{1}{\theta}$$

For example, suppose the mean number of customers to arrive at a bank in a 1-hour interval is 10. Then, the average (waiting) time until the first customer is 1/10 of an hour, or 6 minutes.

Let's now formally define the probability density function we have just derived.

Definition. The continuous random variable X follows an **exponential distribution** if its probability density function is:

$$f(x) = \frac{1}{\theta} e^{-x/\theta}$$

for $\theta > 0$ and $x \geq 0$.

Because there are an infinite number of possible constants θ, there are an infinite number of possible exponential distributions. That's why this page is called Exponential Distributions (with an s!) and not Exponential Distribution (with no s!).