Bayes' rule

Bayes' rule, named after the English mathematician Thomas Bayes, is a rule for computing conditional probabilities.

The rule

A formal statement of Bayes' rule follows.

Proposition Let and be two events. Denote their probabilities by and and suppose that both and . Denote by the conditional probability of given and by the conditional probability of given . Bayes' rule states that

Proof

The following example shows how Bayes' rule can be applied in a practical situation.

Example An HIV test gives a positive result with probability 98% when the patient is indeed affected by HIV, while it gives a negative result with 99% probability when the patient is not affected by HIV. If a patient is drawn at random from a population in which 0,1% of individuals are affected by HIV and he is found positive, what is the probability that he is indeed affected by HIV? In probabilistic terms, what we know about this problem can be formalized as follows:

$$P(positive|HIV) = 0.98$$
$$P(positive|NO\ HIV) = 1 - 0.99 = 0.01$$
$$P(HIV) = 0.001$$
$$P(NO\ HIV) = 1 - 0.001 = 0.999$$

Furthermore, the unconditional probability of being found positive can be derived using the law of total probability:

$$\begin{aligned} P(positive) &= P(positive|HIV)P(HIV) + P(positive|NO\ HIV)P(NO\ HIV) \\ &= 0.98 \cdot 0.001 + 0.01 \cdot 0.999 \\ &= 0.00098 + 0.00999 \\ &= 0.01097 \end{aligned}$$

Therefore, Bayes' rule gives

$$\begin{aligned} P(HIV|positive) &= \frac{P(positive|HIV)P(HIV)}{P(positive)} \\ &= \frac{0.98 \cdot 0.001}{0.01097} \\ &= \frac{0.00098}{0.01097} \simeq 0.08933 \end{aligned}$$

Therefore, even if the test is conditionally very accurate, the unconditional probability of being affected by HIV when found positive is less than 10 per cent!

Terminology

The quantities involved in Bayes' rule

often take the following names:

1. is called **prior probability** or, simply, **prior**;
2. is called **conditional probability** or **likelihood**;
3. is called **marginal probability**;
4. is called **posterior probability** or, simply, **posterior**.

Solved exercises

Below you can find some exercises with explained solutions.

Exercise 1

There are two urns containing colored balls. The first urn contains 50 red balls and 50 blue balls. The second urn contains 30 red balls and 70 blue balls. One of the two urns is randomly chosen (both urns have probability of being chosen) and then a ball is drawn at random from one of the two urns. If a red ball is drawn, what is the probability that it comes from the first urn?

Solution

In probabilistic terms, what we know about this problem can be formalized as follows:

The unconditional probability of drawing a red ball can be derived using the law of total probability:

$$\begin{aligned} P(\text{red}) &= P(\text{red}\mid\text{urn 1})P(\text{urn 1}) + P(\text{red}\mid\text{urn 2})P(\text{urn 2}) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{3}{10}\cdot\frac{1}{2} \\ &= \frac{1}{4} + \frac{3}{20} \\ &= \frac{5+3}{20} = \frac{2}{5} \end{aligned}$$

By using Bayes' rule, we obtain

Exercise 2

An economics consulting firm has created a model to predict recessions. The model predicts a recession with probability 80% when a recession is indeed coming and with probability 10% when no recession is coming. The unconditional probability of falling into a recession is 20%. If the model predicts a recession, what is the probability that a recession will indeed come?

Solution

What we know about this problem can be formalized as follows:

$$\begin{aligned}
&P(\text{rec. pred.} \mid \text{rec. coming}) = \frac{8}{10} \\
&P(\text{rec. pred.} \mid \text{rec. not coming}) = \frac{1}{10} \\
&P(\text{rec. coming}) = \frac{2}{10} \\
&P(\text{rec. not coming}) = 1 - P(\text{rec. coming}) = 1 - \frac{2}{10} = \frac{8}{10}
\end{aligned}$$

The unconditional probability of predicting a recession can be derived by using the law of total probability:

$$\begin{aligned}
P(\text{rec. pred.}) &= P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming}) \\
&\quad + P(\text{rec. pred.} \mid \text{rec. not coming})P(\text{rec. not coming}) \\
&= \frac{8}{10} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{8}{10} \\
&= \frac{24}{100}
\end{aligned}$$

Bayes' rule implies

$$\begin{aligned}
P(\text{rec. coming} \mid \text{rec. pred.}) &= \frac{P(\text{rec. pred.} \mid \text{rec. coming})P(\text{rec. coming})}{P(\text{rec. pred.})} \\
&= \frac{\frac{8}{10} \cdot \frac{2}{10}}{\frac{24}{100}} \\
&= \frac{16}{100} \cdot \frac{100}{24} = \frac{2}{3}
\end{aligned}$$

Exercise 3

Alice has two coins in her pocket, a fair coin (head on one side and tail on the other side) and a two-headed coin. She picks one at random from her pocket, tosses it and obtains head. What is the probability that she flipped the fair coin?

Solution

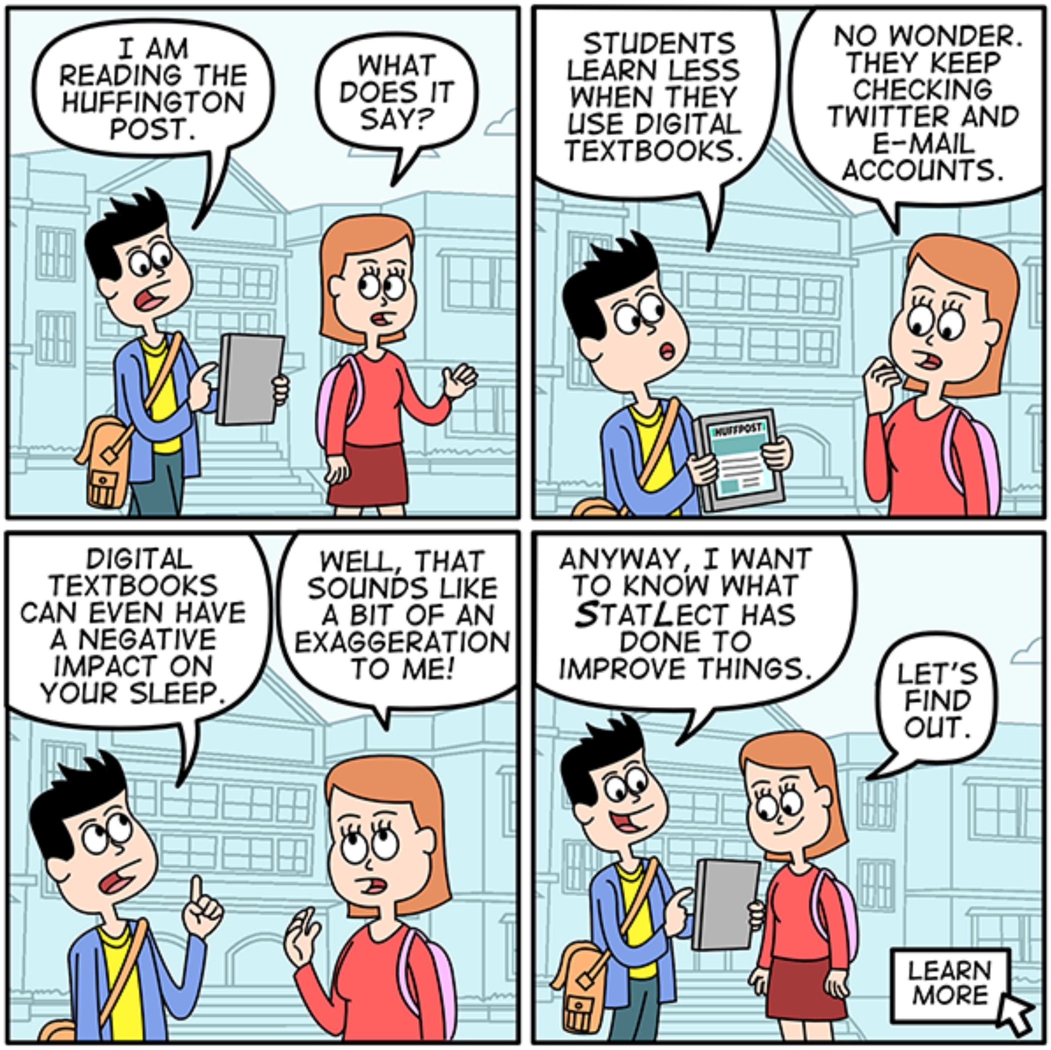
What we know about this problem can be formalized as follows:

The unconditional probability of obtaining head can be derived by using the law of total probability:

$$\begin{aligned}
P(\text{head}) &= P(\text{head}|\text{fair coin})P(\text{fair coin}) + P(\text{head}|\text{unfair coin})P(\text{unfair coin}) \\
&= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} \\
&= \frac{1}{4} + \frac{2}{4} = \frac{3}{4}
\end{aligned}$$

With Bayes' rule, we obtain

$$\begin{aligned}
P(\text{fair coin}|\text{head}) &= \frac{P(\text{head}|\text{fair coin})P(\text{fair coin})}{P(\text{head})} \\
&= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{3}{4}} \\
&= \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}
\end{aligned}$$



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