12. The Binomial Probability Distribution

A **binomial** experiment is one that possesses the following properties:

1. The experiment consists of n repeated trials;
2. Each trial results in an outcome that may be classified as a **success** or a **failure** (hence the name, **binomial**);
3. The probability of a success, denoted by p, remains constant from trial to trial and repeated trials are independent.

The number of successes X in n trials of a binomial experiment is called a **binomial random variable**.

The probability distribution of the random variable X is called a **binomial distribution**, and is given by the formula:

$$\boxed{P(X) = C_x^n p^x q^{n-x}}$$

where

n = the number of trials

$x = 0, 1, 2, \ldots n$

p = the probability of success in a single trial

q = the probability of failure in a single trial

(i.e. $q = 1 - p$)

C_x^n is a combination

$P(X)$ gives the probability of successes in n binomial trials.

Mean and Variance of Binomial Distribution

If p is the probability of success and q is the probability of failure in a binomial trial, then the expected number of successes in r trials (i.e. the mean value of the binomial distribution) is

$$E(X) = \mu = np$$

The **variance** of the binomial distribution is

$$V(X) = \sigma^2 = npq$$

Note: In a binomial distribution, only **2** parameters, namely n and p, are needed to determine the probability.

Example 1

A die is tossed 3 times. What is the probability of

(a) No fives turning up?

(b) 1 five?

(c) 3 fives?

Answer

Image source

This is a **binomial** distribution because there are only 2 possible outcomes (we get a 5 or we don't).

Now, $n = 3$ for each part. Let X = number of fives appearing.

(a) Here, $x = 0$.

$$P(X=0) = C_x^n p^x q^{n-x} = C_0^3 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 = \frac{125}{216} = 0.5787$$

(b) Here, $x = 1$.

$$P(X=1) = C_x^n p^x q^{n-x} = C_1^3 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 = \frac{75}{216} = 0.34722$$

(c) Here, $x = 3$.

$$P(X=3) = C_x^n p^x q^{n-x} = C_3^3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 = \frac{1}{216} = 4.6296 \times 10^{-3}$$

Example 2

Hospital records show that of patients suffering from a certain disease, 75% die of it. What is the probability that of 6 randomly selected patients, 4 will recover?

Answer

This is a **binomial** distribution because there are only 2 outcomes (the patient dies, or does not).

Let X = number who recover.

Here, $n = 6$ and $x = 4$. Let $p = 0.25$ (success, that is, they live), $q = 0.75$ (failure, i.e. they die).

The probability that 4 will recover:

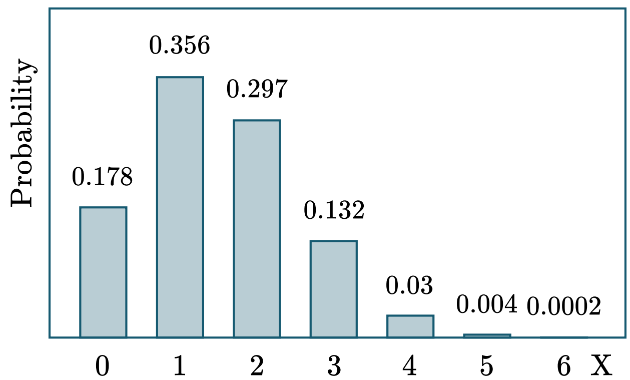
$$P(X) = C_x^n p^x q^{n-x} = C_4^6 (0.25)^4 (0.75)^2 = 15 \times 2.1973 \times 10^{-3} = 0.0329595$$

Histogram of this distribution:

We could calculate all the probabilities involved and we would get:

X	Probability
0	0.17798
1	0.35596
2	0.29663
3	0.13184
4	3.2959×10^{-2}
5	4.3945×10^{-3}
6	2.4414×10^{-4}

The histogram is as follows:



Histogram of the binomial distribution

It means that out of the 6 patients chosen, the probability that:

- **None of them** will recover is 0.17798,
- **One** will recover is 0.35596, and
- **All** 6 will recover is extremely small.

Example 3

In the old days, there was a probability of 0.8 of success in any attempt to make a telephone call. (This often depended on the importance of the person making the call, or the operator's curiosity!)

Calculate the probability of having 7 successes in 10 attempts.

Answer

Image source

Probability of success $p = 0.8$, so $q = 0.2$.

X = success in getting through.

Probability of 7 successes in 10 attempts:

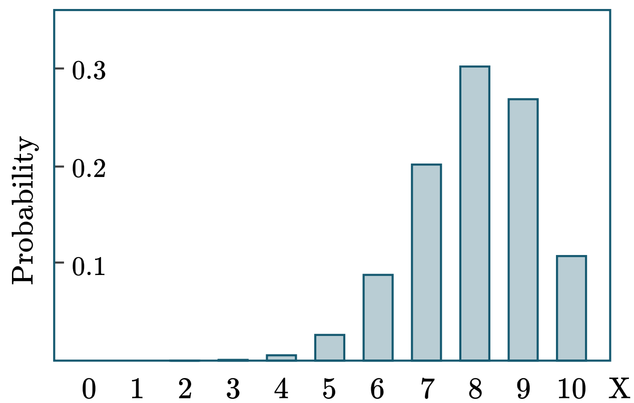
$$\begin{aligned}\text{Probability} &= P(X = 7) \\ &= C_7^{10}(0.8)^7(0.2)^{10-7} \\ &= 0.20133\end{aligned}$$

Histogram

We use the following function

$$C(10, x)(0.8)^x(0.2)^{10-x}$$

to obtain the probability histogram:



Histogram of the binomial distribution

Example 4

A (blindfolded) marksman finds that on the average he hits the target 4 times out of 5. If he fires 4 shots, what is the probability of

(a) more than 2 hits?

(b) at least 3 misses?

Answer

Here, $n = 4$, $p = 0.8$, $q = 0.2$.

Let X = number of hits.

Let x_0 = no hits, $x_1 = 1$ hit, $x_2 = 2$ hits, etc.

(a) $P(X) = P(x_3) + P(x_4)$

$$\begin{aligned}&= C_3^4(0.8)^3(0.2)^1 + C_4^4(0.8)^4(0.2)^0 \\ &= 4(0.8)^3(0.2) + (0.8)^4\end{aligned}$$

$= 0.8192$

(b) 3 misses means 1 hit, and 4 misses means 0 hits.

$P(X) = P(x_1) + P(x_0)$

$$= C_1^4(0.8)^1(0.2)^3 + C_0^4(0.8)^0(0.2)^4$$

$$= 4(0.8)^1(0.2)^3 + (0.2)^4$$

$$= 0.0272$$

Example 5

The ratio of boys to girls at birth in Singapore is quite high at $1.09 : 1$.

What proportion of Singapore families with exactly 6 children will have at least 3 boys? (Ignore the probability of multiple births.)

[**Interesting and disturbing trivia:** In most countries the ratio of boys to girls is about $1.04 : 1$, but in China it is $1.15 : 1$.]

Answer

Image source

The probability of getting a boy is $\dfrac{1.09}{1.09 + 1.00} = 0.5215$

Let X = number of boys in the family.

Here,

$n = 6,$
$p = 0.5215,$
$q = 1 - 0.52153 = 0.4785$

When $x = 3$:

$$P(X) = C_x^n p^x q^{n-x} = C_3^6(0.5215)^3(0.4785)^3 = 0.31077$$

When $x = 4$:

$$P(X) = C_4^6(0.5215)^4(0.4785)^2 = 0.25402$$

When $x = 5$:

$$P(X) = C_5^6(0.5215)^5(0.4785)^1 = 0.11074$$

When $x = 6$:

$P(X) = C_6^6(0.5215)^6(0.4785)^0 = 2.0115 \times 10^{-2}$

So the probability of getting at least 3 boys is:

$$\text{Probability} = P(X \geq 3)$$

$$= 0.31077 + 0.25402 + 0.11074 + 2.0115 \times 10^{-2}$$

$$= 0.69565$$

NOTE: We could have calculated it like this:

$P(X \geq 3) = 1 - (P(x_0) + P(x_1) + P(x_2))$

Example 6

A manufacturer of metal pistons finds that on the average, 12% of his pistons are rejected because they are either oversize or undersize. What is the probability that a batch of 10 pistons will contain

(a) no more than 2 rejects? (b) at least 2 rejects?

Answer

Let $X =$ number of rejected pistons

(In this case, "success" means rejection!)

Here, $n = 10$, $p = 0.12$, $q = 0.88$.

(a)

No rejects. That is, when $x = 0$:

$P(X) = C_x^n p^x q^{n-x} = C_0^{10}(0.12)^0(0.88)^{10} = 0.2785$

One reject. That is, when $x = 1$

$P(X) = C_1^{10}(0.12)^1(0.88)^9 = 0.37977$

Two rejects. That is, when $x = 2$:

$P(X) = C_2^{10}(0.12)^2(0.88)^8 = 0.23304$

So the probability of getting no more than 2 rejects is:

$$\text{Probability} = P(X \leq 2)$$

$$= 0.2785 + 0.37977 + 0.23304$$

$$= 0.89131$$

(b) We could work out all the cases for $X = 2, 3, 4, \ldots, 10$, but it is much easier to proceed as follows:

$$\text{Probablity of at least 2 rejects}$$

$$= 1 - P(X \leq 1)$$

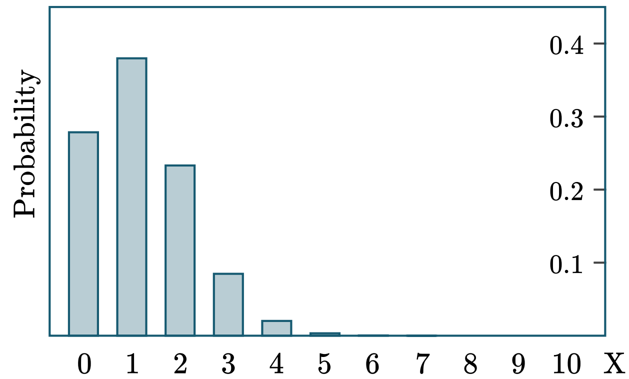
$$= 1 - (P(x_0) + P(x_1))$$

$$= 1 - (0.2785 + 0.37977)$$

$$= 0.34173$$

Histogram

Using the function $g(x) = C(10, x)(0.12)^x(0.88)^{10-x}$ and finding the values at $0, 1, 2, \ldots,$ gives us the histogram:



Histogram of the binomial distribution