Exponential Properties

Here, we present and prove four key properties of an exponential random variable.

Theorem. The exponential probability density function:

$$f(x) = \frac{1}{\theta} e^{-x/\theta}$$

for $x \geq 0$ and $\theta > 0$ is a valid probability density function.

Proof.

Theorem. The moment generating function of an exponential random variable X with parameter θ is:

$$M(t) = \frac{1}{1-\theta t}$$

for $t < 1/\theta$.

Proof. The moment generating function is by definition:

$$M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \left(\frac{1}{\theta}\right) e^{-x/\theta} dx$$

Simplifying and rewriting the integral as a limit, we have:

$$M(t) = \frac{1}{\theta} \lim_{b \to \infty} \int_0^b e^{x(t-1/\theta)} dx$$

Integrating, we have:

$$M(t) = \frac{1}{\theta} \lim_{b \to \infty} \left[\frac{1}{t-1/\theta} e^{x(t-1/\theta)} \right]_{x=0}^{x=b}$$

Evaluating at $x = 0$ and $x = b$, we have:

$$M(t) = \frac{1}{\theta} \lim_{b \to \infty} \left[\frac{1}{t-1/\theta} e^{b(t-1/\theta)} - \frac{1}{t-1/\theta} \right] = \frac{1}{\theta} \lim_{b \to \infty} \left\{ \left(\frac{1}{t-1/\theta} \right) e^{b(t-1/\theta)} \right\} - \frac{1}{t-1/\theta}$$

Now, the limit approaches 0 provided $t - 1/\theta < 0$, that is, provided $t < 1/\theta$, and so we have:

$$M(t) = \frac{1}{\theta} \left(0 - \frac{1}{t-1/\theta} \right)$$

Simplifying more:

$$M(t) = \frac{1}{\theta} \left(-\frac{1}{\frac{\theta t - 1}{\theta}} \right) = \frac{1}{\theta} \left(-\frac{\theta}{\theta t - 1} \right) = -\frac{1}{\theta t - 1}$$

and finally:

$$M(t) = \frac{1}{1-\theta t}$$

provided $t < 1/\theta$, as was to be proved.

Theorem. The mean of an exponential random variable X with parameter θ is:

$$\mu = E(X) = \theta$$

Proof.

Theorem. The variance of an exponential random variable X with parameter θ is:

$$\sigma^2 = Var(X) = \theta^2$$

Proof.