Beta distribution

The Beta distribution is a continuous probability distribution having two parameters. One of its most common uses is to model one's uncertainty about the probability of success of an experiment.

Suppose a probabilistic experiment can have only two outcomes, either success, with probability , or failure, with probability . Suppose also that is unknown and all its possible values are deemed equally likely. This uncertainty can be described by assigning to a uniform distribution on the interval . This is appropriate because , being a probability, can take only values between and ; furthermore, the uniform distribution assigns equal probability density to all points in the interval, which reflects the fact that no possible value of is, a priori, deemed more likely than all the others. Now, suppose that we perform independent repetitions of the experiment and we observe successes and failures. After performing the experiments, we naturally want to know how we should revise the distribution initially assigned to , in order to properly take into account the information provided by the observed outcomes. In other words, we want to calculate the conditional distribution of , conditional on the number of successes and failures we have observed. The result of this calculation is a Beta distribution. In particular, the conditional distribution of , conditional on having observed successes out of trials, is a Beta distribution with parameters and .

Definition

The Beta distribution is characterized as follows.

Definition Let be an absolutely continuous random variable. Let its support be the unit interval:

Let . We say that has a **Beta distribution** with shape parameters and if its probability density function is where is the Beta function.

A random variable having a Beta distribution is also called a Beta random variable.

The following is a proof that is a legitimate probability density function.

Proof

Expected value

The expected value of a Beta random variable is

Proof

Variance

The variance of a Beta random variable is

Proof

Higher moments

The -th moment of a Beta random variable is

Proof

Moment generating function

The moment generating function of a Beta random variable is defined for any and it is

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{B(\alpha + k, \beta)}{B(\alpha, \beta)} = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \prod_{n=0}^{k-1} \frac{\alpha + n}{\alpha + \beta + n}$$

Proof

The above formula for the moment generating function might seem impractical to compute, because it involves an infinite sum as well as products whose number of terms increase indefinitely. However, the function

is a function, called Confluent hypergeometric function of the first kind, that has been extensively studied in many branches of mathematics. Its properties are well-known and efficient algorithms for its computation are available in most software packages for scientific computation.

Characteristic function

The characteristic function of a Beta random variable is

$$\varphi_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \frac{B(\alpha + k, \beta)}{B(\alpha, \beta)} = 1 + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \prod_{n=0}^{k-1} \frac{\alpha + n}{\alpha + \beta + n}$$

Proof

Comments made about the moment generating function, including those about the computation of the Confluent hypergeometric function, apply also to the characteristic function, which is identical to the mgf except for the fact that is replaced with .

Distribution function

The distribution function of a Beta random variable is where the function

is called incomplete Beta function and is usually computed by means of specialized computer algorithms.

Proof

More details

In the following subsections you can find more details about the Beta distribution.

Relation to the uniform distribution

The following proposition states the relation between the Beta and the uniform distributions.

Proposition A Beta distribution with parameters and is a uniform distribution on the interval .

Proof

Relation to the binomial distribution

The following proposition states the relation between the Beta and the binomial distributions.

Proposition Suppose is a random variable having a Beta distribution with parameters and . Let be another random variable such that its distribution conditional on is a binomial distribution with parameters and . Then, the conditional distribution of given is a Beta distribution with parameters and .

Proof

By combining this proposition and the previous one, we obtain the following corollary.

Proposition Suppose is a random variable having a uniform distribution. Let be another random variable such that its distribution conditional on is a binomial distribution with parameters and . Then, the conditional distribution of given is a Beta distribution with parameters and .

This proposition constitutes a formal statement of what we said in the introduction of this lecture in order to motivate the Beta distribution. Remember that the number of successes obtained in independent repetitions of a random experiment having probability of success is a binomial random variable with parameters and . According to the proposition above, when the probability of success is a priori unknown and all possible values of are deemed equally likely (they have a uniform distribution), observing the outcome of the experiments leads us to revise the distribution assigned to , and the result of this revision is a Beta distribution.

Solved exercises

Below you can find some exercises with explained solutions.

Exercise 1

A production plant produces items that have a probability of being defective. The plant manager does not know , but from past experience she expects this probability to be equal to . Furthermore, she quantifies her uncertainty about by attaching a standard deviation of to her estimate. After consulting with an expert in statistics, the manager decides to use a Beta distribution to model her uncertainty about . How should she set the two parameters of the distribution in order to match her priors about the expected value and the standard deviation of ?

Solution

We know that the expected value of a Beta random variable with parameters and is

while its variance is

The two parameters need to be set in such a way that

This is accomplished by finding a solution to the following system of two equations in two unknowns:

where for notational convenience we have set and . The first equation gives

or

By substituting this into the second equation, we get

or

Then we divide the numerator and denominator on the left-hand side by :

By computing the products, we get

By taking the reciprocals of both sides, we have

By multiplying both sides by , we obtain

Thus the value of is

and the value of is

By plugging our numerical values into the two formulae, we obtain

Exercise 2

After choosing the parameters of the Beta distribution so as to represent her priors about the probability of producing a defective item (see previous exercise), the plant manager now wants to update her priors by observing new data. She decides to inspect a production lot of 100 items, and she finds that 3 of the items in the lot are defective. How should she change the parameters of the Beta distribution in order to take this new information into account?

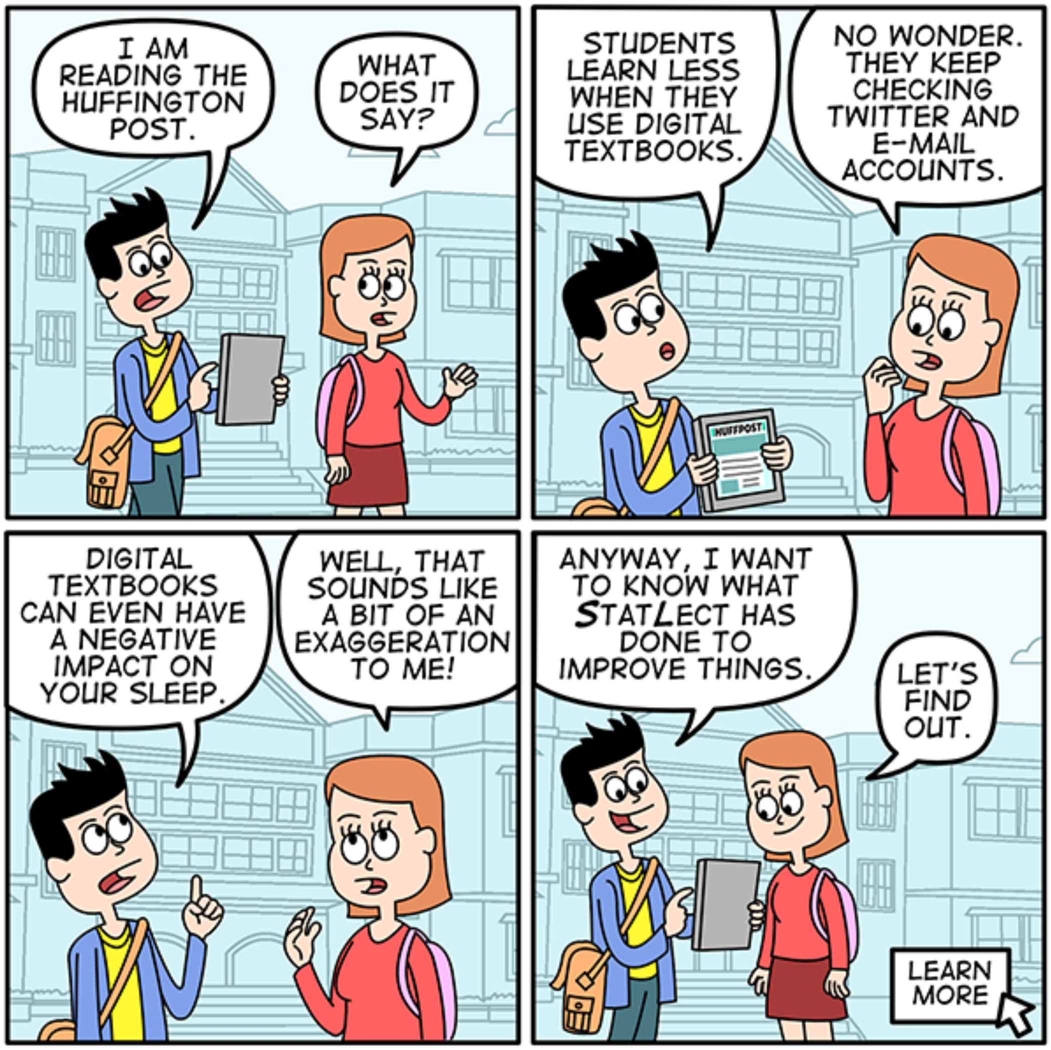
Solution

Under the hypothesis that the items are produced independently of each other, the result of the inspection is a binomial random variable with parameters and . But updating a Beta distribution based on the outcome of a binomial random variable gives as a result another Beta distribution. Moreover, the two parameters and of the updated Beta distribution are

Exercise 3

After updating the parameters of the Beta distribution (see previous exercise), the plant manager wants to compute again the expected value and the standard deviation of the probability of finding a defective item. Can you help her?

Solution



The book

Most of the learning materials found on this website are now available in a traditional textbook format.