13. The Poisson Probability Distribution

The Poisson Distribution was developed by the French mathematician Simeon Denis Poisson in 1837.

The Poisson random variable satisfies the following conditions:

1. The number of successes in two disjoint time intervals is independent.
2. The probability of a success during a small time interval is proportional to the entire length of the time interval.

Apart from disjoint time intervals, the Poisson random variable also applies to **disjoint regions of space**.

Applications

- the number of deaths by horse kicking in the Prussian army (first application)
- birth defects and genetic mutations
- rare diseases (like Leukemia, but not AIDS because it is infectious and so not independent) - especially in legal cases
- car accidents
- traffic flow and ideal gap distance
- number of typing errors on a page
- hairs found in McDonald's hamburgers
- spread of an endangered animal in Africa
- failure of a machine in one month

The **probability distribution of a Poisson random variable** X representing the number of successes occurring in a given time interval or a specified region of space is given by the formula:

$$\boxed{P(X) = \frac{e^{-\mu}\mu^x}{x!}}$$

where

$x = 0, 1, 2, 3 \ldots$

$e = 2.71828$ (but use your calculator's e button)

$\mu =$ mean number of successes in the given time interval or region of space

Mean and Variance of Poisson Distribution

If μ is the average number of successes occurring in a given time interval or region in the Poisson distribution, then the mean and the variance of the Poisson distribution are both equal to μ.

$$E(X) = \mu$$

and

$$V(X) = \sigma^2 = \mu$$

Note: In a Poisson distribution, only **one** parameter, μ is needed to determine the probability of an event.

Example 1

A life insurance salesman sells on the average 3 life insurance policies per week. Use Poisson's law to calculate the probability that in a given week he will sell

a. Some policies

b. 2 or more policies but less than 5 policies.

c. Assuming that there are 5 working days per week, what is the probability that in a given day he will sell one policy?

Answer

Example 2

Twenty sheets of aluminum alloy were examined for surface flaws. The frequency of the number of sheets with a given number of flaws per sheet was as follows:

Number of flaws	Frequency
0	4
1	3
2	5
3	2
4	4
5	1
6	1

What is the probability of finding a sheet chosen at random which contains 3 or more surface flaws?

Answer

The total number of flaws is given by:

$$(0 \times 4) + (1 \times 3) + (2 \times 5) + (3 \times 2) + (4 \times 4) + (5 \times 1) + (6 \times 1) = 46$$

So the average number of flaws for the 20 sheets is given by:

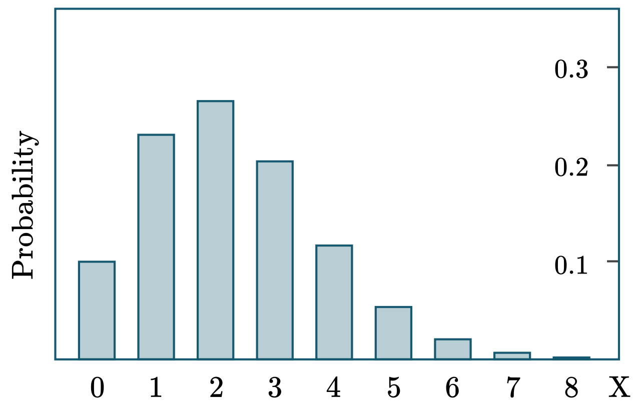
$$\mu = \frac{46}{20} = 2.3$$

The required probability is:

$$\begin{aligned}\text{Probability} &= P(X \geq 3) \\ &= 1 - (P(x_0) + P(x_1) + P(x_2)) \\ &= 1 - \left(\frac{e^{-2.3}2.3^0}{0!} + \frac{e^{-2.3}2.3^1}{1!} + \frac{e^{-2.3}2.3^2}{2!}\right) \\ &= 0.40396\end{aligned}$$

Histogram of Probabilities

We can see the predicted probabilities for each of "No flaws", "1 flaw", "2 flaws", etc on this histogram.



Histogram of the Poisson distribution

The histogram was obtained by graphing the following function for integer values of x only.

$$\frac{e^{-2.3}2.3^x}{x!}$$

Example 3

If electricity power failures occur according to a Poisson distribution with an average of 3 failures every twenty weeks, calculate the probability that there will not be more than one failure during a particular week.

Answer

The average number of failures per week is: $\mu = \frac{3}{20} = 0.15$

"Not more than one failure" means we need to include the probabilities for "0 failures" plus "1 failure".

$$P(x_0) + P(x_1) = \frac{e^{-0.15}0.15^0}{0!} + \frac{e^{-0.15}0.15^1}{1!} = 0.98981$$

Example 4

Vehicles pass through a junction on a busy road at an average rate of 300 per hour.

a. Find the probability that none passes in a given minute.
b. What is the expected number passing in two minutes?
c. Find the probability that this expected number actually pass through in a given two-minute period.

Answer

The average number of cars per minute is: $\mu = \frac{300}{60} = 5$

(a) $P(x_0) = \frac{e^{-5}5^0}{0!} = 6.7379 \times 10^{-3}$

(b) Expected number each 2 minutes = $E(X) = 5 \times 2 = 10$

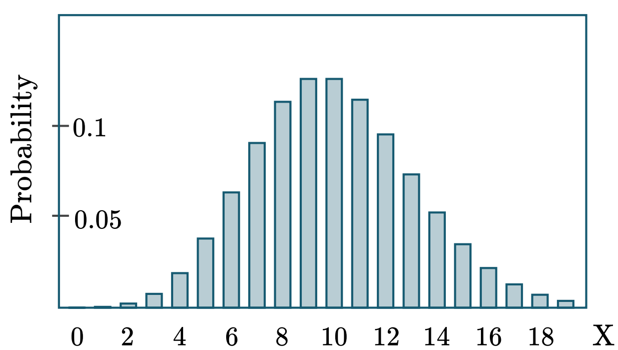
(c) Now, with $\mu = 10$, we have: $P(x_{10}) = \frac{e^{-10}10^{10}}{10!} = 0.12511$

Histogram of Probabilities

Based on the function

$$P(X) = \frac{e^{-10}10^x}{x!}$$

we can plot a histogram of the probabilities for the number of cars for each 2 minute period:



Histogram of the Poisson distribution

Example 5

A company makes electric motors. The probability an electric motor is defective is 0.01. What is the probability that a sample of 300 electric motors will contain exactly 5 defective motors?

Answer

The average number of defectives in 300 motors is $\mu = 0.01 \times 300 = 3$

The probability of getting 5 defectives is:

$$P(X) = \frac{e^{-3}3^5}{5!} = 0.10082$$

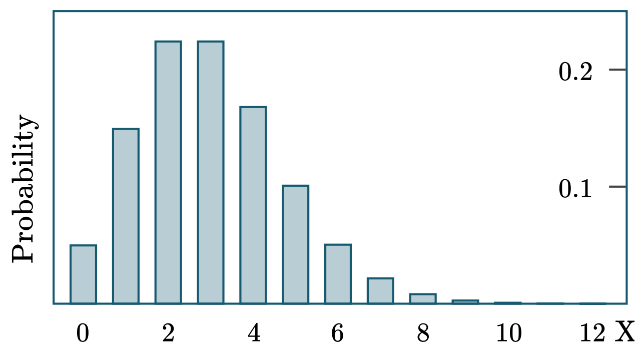
NOTE: This problem looks similar to a binomial distribution problem, that we met in the last section.

If we do it using binomial, with $n = 300$, $x = 5$, $p = 0.01$ and $q = 0.99$, we get:

$$P(X = 5) = \text{C}(300{,}5)(0.01)^5(0.99)^{295} = 0.10099$$

We see that the result is very similar. We can use binomial distribution to approximate Poisson distribution (and vice-versa) under certain circumstances.

Histogram of Probabilities



Histogram of the Poisson distribution