Exponential distribution

How much time will elapse before an earthquake occurs in a given region? How long do we need to wait before a customer enters our shop? How long will it take before a call center receives the next phone call? How long will a piece of machinery work without breaking down?

Questions such as these are often answered in probabilistic terms using the exponential distribution.

All these questions concern the time we need to wait before a given event occurs. If this waiting time is unknown, it is often appropriate to think of it as a random variable having an exponential distribution. Roughly speaking, the time we need to wait before an event occurs has an exponential distribution if the probability that the event occurs during a certain time interval is proportional to the length of that time interval. More precisely, has an exponential distribution if the conditional probability

is approximately proportional to the length of the time interval comprised between the times and , for any time instant . In most practical situations this property is very realistic and this is the reason why the exponential distribution is so widely used to model waiting times.

The exponential distribution is also related to the Poisson distribution. When the event can occur more than once and the time elapsed between two successive occurrences is exponentially distributed and independent of previous occurrences, the number of occurrences of the event within a given unit of time has a Poisson distribution (see the lecture entitled Poisson distribution for a more detailed explanation and an intuitive graphical representation of this fact).

Definition

The exponential distribution is characterized as follows.

Definition Let be an absolutely continuous random variable. Let its support be the set of positive real numbers:

Let . We say that has an **exponential distribution** with parameter if its probability density function is

The parameter is called **rate parameter**.

A random variable having an exponential distribution is also called an exponential random variable.

The following is a proof that is a legitimate probability density function.

Proof

To better understand the exponential distribution, you can have a look at its density plots.

The rate parameter and its interpretation

We have mentioned that the probability that the event occurs between two dates and is proportional to (conditional on the information that it has not occurred before). The rate parameter is the constant of proportionality:

where is an infinitesimal of higher order than (i.e. a function of that goes to zero more quickly than does).

The above proportionality condition is also sufficient to completely characterize the exponential distribution.

Proposition The proportionality condition

is satisfied only if has an exponential distribution.

Proof

The conditional probability can be written as

$$P(t < X \leq t + \Delta t | X > t) = \frac{P(t < X \leq t + \Delta t, X > t)}{P(X > t)} = \frac{P(t < X \leq t + \Delta t)}{P(X > t)}$$

Denote by the distribution function of , that is,

and by its survival function:

Then,

Dividing both sides by , we obtain

where is a quantity that tends to when tends to . Taking limits on both sides, we obtain

or, by the definition of derivative:

This differential equation is easily solved by using the chain rule:

Taking the integral from to of both sides, we get

and

or

But (because cannot take negative values) implies

Exponentiating both sides, we obtain

Therefore,

or

But the density function is the first derivative of the distribution function:

and the rightmost term is the density of an exponential random variable. Therefore, the proportionality condition is satisfied only if is an exponential random variable

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Expected value

The expected value of an exponential random variable is

Proof

Variance

The variance of an exponential random variable is

Proof

Moment generating function

The moment generating function of an exponential random variable is defined for any :

Proof

Characteristic function

The characteristic function of an exponential random variable is

Proof

Distribution function

The distribution function of an exponential random variable is

Proof

More details

In the following subsections you can find more details about the exponential distribution.

Memoryless property

One of the most important properties of the exponential distribution is the **memoryless property**:

for any .

Proof

 is the time we need to wait before a certain event occurs. The above property says that the probability that the event happens during a time interval of length is independent of how much time has already elapsed ( ) without the event happening.

The sum of exponential random variables is a Gamma random variable

Suppose , , ..., are mutually independent random variables having exponential distribution with parameter . Define

Then, the sum is a Gamma random variable with parameters and .

Proof

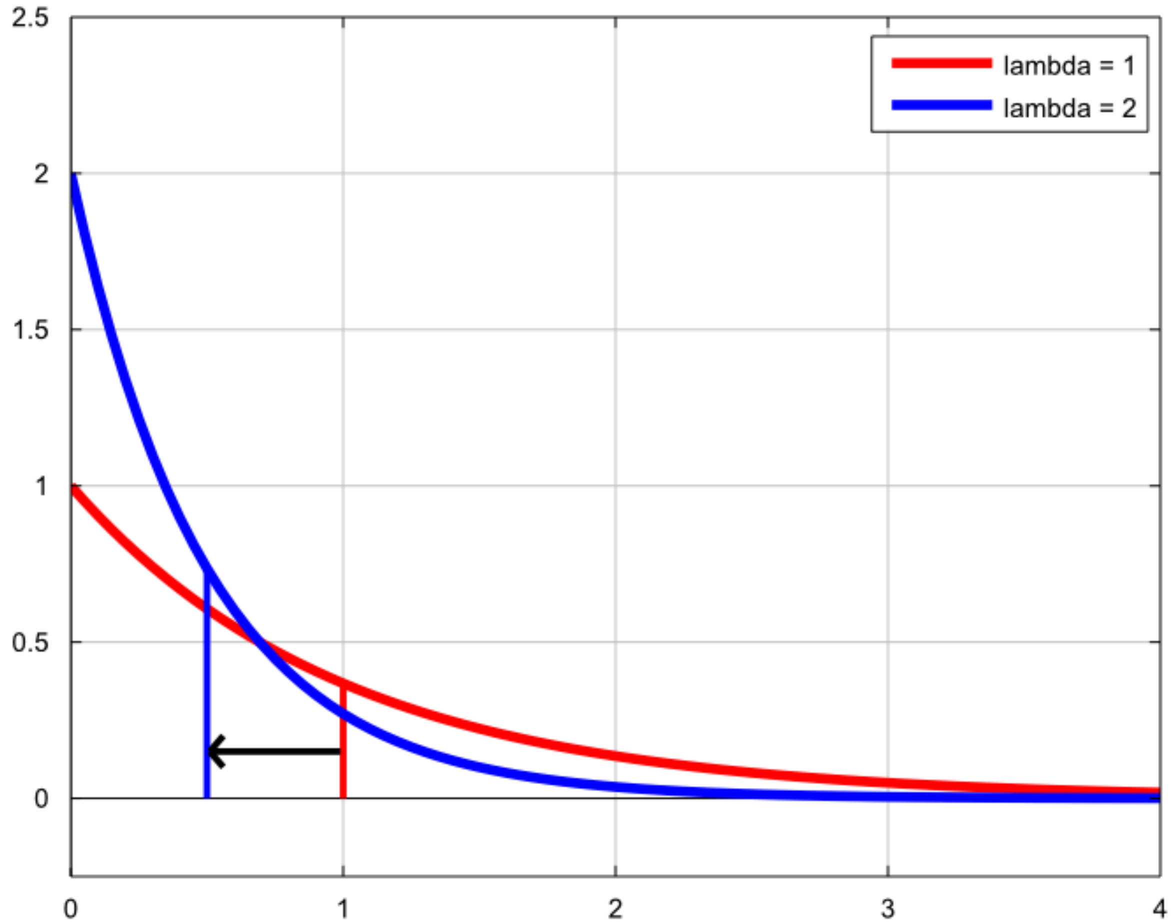
The random variable is also sometimes said to have an **Erlang distribution**. The Erlang distribution is just a special case of the Gamma distribution: a Gamma random variable is also an Erlang random variable when it can be written as a sum of exponential random variables.

Density plot

The next plot shows how the density of the exponential distribution changes by changing the rate parameter:

- the first graph (red line) is the probability density function of an exponential random variable with rate parameter ;
- the second graph (blue line) is the probability density function of an exponential random variable with rate parameter .

The thin vertical lines indicate the means of the two distributions. Note that, by increasing the rate parameter, we decrease the mean of the distribution from to .



Solved exercises

Below you can find some exercises with explained solutions.

Exercise 1

Let be an exponential random variable with parameter . Compute the following probability:

Solution

First of all we can write the probability as using the fact that the probability that an absolutely continuous random variable takes on any specific value is equal to zero (see Absolutely continuous random variables and zero-probability events). Now, the probability can be written in terms of the distribution function of as

$$\begin{aligned} P(2 \leq X \leq 4) &= P(2 < X \leq 4) \\ &= F_X(4) - F_X(2) \\ &= [1 - \exp(-\ln(3) \cdot 4)] - [1 - \exp(-\ln(3) \cdot 2)] \\ &= \exp(-\ln(3) \cdot 2) - \exp(-\ln(3) \cdot 4) \\ &= 3^{-2} - 3^{-4} \end{aligned}$$

Exercise 2

Suppose the random variable has an exponential distribution with parameter . Compute the following probability:

Solution

This probability can be easily computed by using the distribution function of :

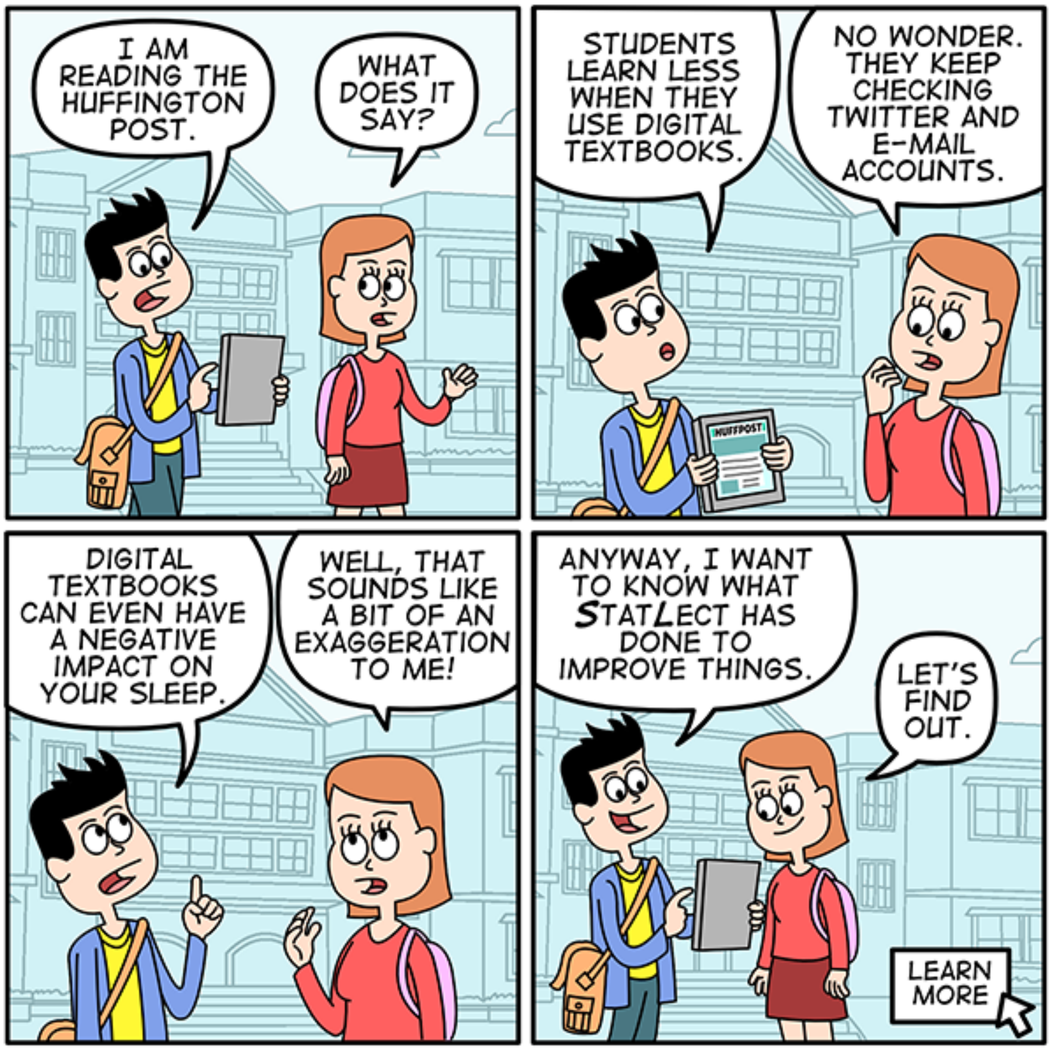
Exercise 3

What is the probability that a random variable is less than its expected value, if has an exponential distribution with parameter ?

Solution

The expected value of an exponential random variable with parameter is

The probability above can be computed by using the distribution function of :



The book