In [1]:
```
%pylab inline
import scipy, scipy.stats
pylab.rcParams['figure.figsize'] = (8.0, 4.0)
# from IPython.html.widgets import interact, fixed
# import ipywidgets
from __future__ import print_function
```

```
Populating the interactive namespace from numpy and matplotlib
```

2.6. Nonparametric Correlation

Basic Idea:

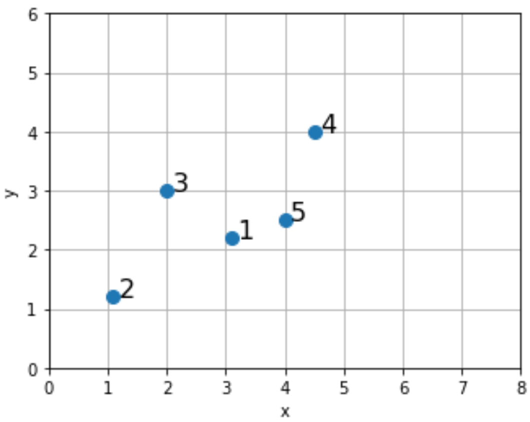
- Instead of using the values x_i
- use the rank place r_i of x_i
- in a sorted series $x_{r_1} \leq x_{r_2} \leq \ldots \leq x_{r_N}$

Example

i	x	y	r	s
1	3.1	2.2	3	2
2	1.1	1.2	1	1
3	2	3	2	4
4	4.5	4	5	5
5	4	2.5	4	3

In [2]:
```
x = array([[3.1, 2.2], [1.1,1.2], [2,3], [4.5,4], [4,2.5]])
plot(x[:,0], x[:,1], "o", markersize=8)
[text(el[0]+0.1,el[1], "%s" %(i+1), fontsize=16) for i, el in enumerate(x)]
axis([0,8,0,6]); xlabel("x"); ylabel("y"); grid(True); gca().set_aspect('equal')
x[:,1]
ios = x[:,1].argsort() # this is not s[i]. This is i[s]!!!
print(ios+1)
```

```
[2 1 5 3 4]
```



$\bar{r} = \bar{s} = 3, r_{sp} = \frac{0+4+(-1)+4}{10} = \frac{7}{10}$

Motivation:

- more generally applicable
- robustness: more robust against measurement errors
- better statistical interpretation

What do we do if $x_i = x_j$?

- these are called 'ties'
- jitter x_i, x_j, i.e. add a small random number
- this results in different rankings k and $k + 1$.
- assign to both data points the mean rank $k + 1/2$
- apply the analogue procedure for $q > 2$ equal values.

Ranks offer better statistical means of interpretation, because the distribution of ranks is better specified: typically it is a uniform distribution from 1 to N.

2.6.2. Spearman's Rank Correlation Coefficient

Let $\{r_i\}, \{s_i\}$ be the ranks of $\{x_i\}, \{y_i\}, i = 1, \ldots, N$. Then the Spearman Rank Correlation Coefficient is defined as

$$r_{\mathrm{SP}} = \frac{\sum_i (r_i - \bar{r})(s_i - \bar{s})}{\sqrt{\sum_i (r_i - \bar{r})^2} \cdot \sqrt{\sum_i (s_i - \bar{s})^2}}$$

- note the similiarity to the definition of the linear correlation coefficient r

Statistical Test:

- The null hypothesis H_0: x and y are linearly uncorrelated
- Statistic:

$$t = r_{\mathrm{SP}} \sqrt{\frac{N-2}{1-r_{\mathrm{SP}}^2}}$$

- Distribution of t under valid H_0 = Student-t distribution with $\nu = N - 2$ degrees of freedom
- This is independent of the actual distributions of x and y

Alternative Correlation Measures

$$D = \sum_{i=1}^{N} (r_i - s_i)^2$$

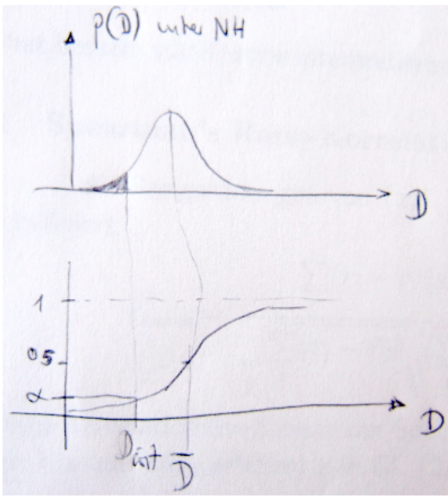
- Sum of the squared rank deviations
- is 0, if the rank places are equally sorted in both features

If there are no ties, the following connection with $r_{\backslash\text{tiny SP}}$ can be found:

$$r_{SP} = 1 - \frac{6D}{N^3 - N}$$

There is no direct test for D: assuming true H_0, D is approximately normally distributed

- with mean $\bar{D} \approx \frac{1}{6}(N^3 - N)$
- and variance $\mathrm{Var}(D) \approx \dfrac{(N-1)N^2(N+1)^2}{36}$
- different than in previous tests here the statistics can also be smaller than a critical level for the null hypothesis to be rejectable.
- if a D_{crit} is surpassed, the features x and y are anticorrelated, i.e. x increases while y decreases.



2.6.3. Kendall's τ

Instead of using ranks, here we have an even more coarse reduction:

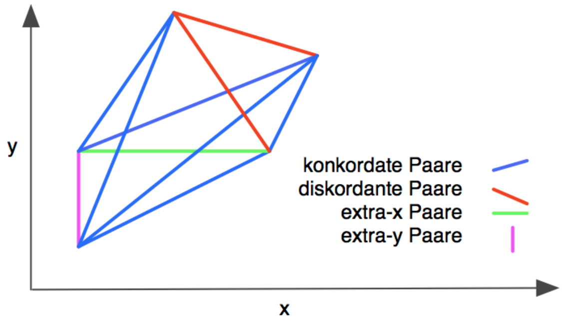
- higher rank
- equal rank
- lower rank

The advantage is, that no sorting is required

- since ranks are monotonous with feature values, $r_i < r_j$ is identical to $x_i < x_j$.

Point of departure:

- N data points (paired features) (x_i, y_i)



Regard all $\frac{1}{2}N(N-1)$ pairs (i, j) and count the following for quantities:

name	definition	meaning
concordant	nr. of pairs with $(x_i - x_j)(y_i - y_j) > 0$	positive slope
discordant	nr. of pairs with $(x_i - x_j)(y_i - y_j) < 0$	negative slope
extra-x	nr. of pairs with $x_i \neq x_j \wedge y_i = y_j$	horizontal
extra-y	nr. of pairs with $x_i = x_j \wedge y_i \neq y_j$	vertical

Statistical Test:

- Null hypothesis H_0: x and y are uncorrelated
- Statistics:

$$\tau = \frac{\text{con} - \text{dis}}{\sqrt{\text{con} + \text{dis} + \text{extra-y}} \cdot \sqrt{\text{con} + \text{dis} + \text{extra-x}}}$$

- Note that $-1 \leq \tau \leq 1$ and
  - $\tau = 1 \Leftrightarrow$ only con $> 0 \Leftrightarrow x, y$ have same order
  - $\tau = -1 \Leftrightarrow$ only dis $> 0 \Leftrightarrow x, y$ have opposite order
- **Remark:**
  - double points (i.e. exactly identical coordinates) affect the counters nonetheless, so they cannot just be eliminated
  - as pair they count neither for extra-x nor extra-y.
- Distribution of τ under true H_0:

$$\tau \sim N\left(\mu = 0, \sigma^2 = \frac{4N + 10}{9N(N+1)}\right)$$

- Note that $N(\mu, \sigma^2)$ = Normal distribution with mean μ and variance σ^2
- The method is particularly suitable if there are very few alternative values. In this case r_{SP} would be dominated by mean ranks.
- Complexity: $o(N^2)$ while r_{SP} scales with $o(N \log(N))$
  - so could be problematic with large data sets

```
In [ ]:
```