In [1]:
```
#settings↔
```

```
Populating the interactive namespace from numpy and matplotlib
```

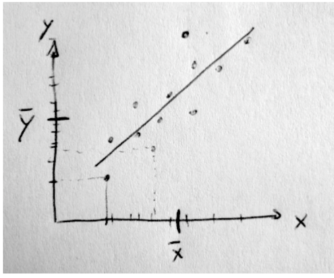
2.5 Linear Correlation

- So far: discrete variables (nominal)
- Now: continuous variables → offer additional modes of dependency analysis

Most simple Measure: *Linear Correlation Coefficient r*

Given:

- paired random variables X, and Y with sample $\{(x_i, y_i)\}_{i=1,\dots,N}$



- r is defined as:

$$r = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2}\sqrt{\sum_i (y_i - \bar{y})^2}}$$

where $\bar{x}, \bar{y}$ are the sample means of x resp. y

Geometric interpretation as cos(angle between vectors)

$$r = \cos\big(\angle(\hat{m}_x, \hat{m}_y)\big) = \frac{\vec{m}_x \cdot \vec{m}_y}{\|\vec{m}_x\|\|\vec{m}_y\|} = \hat{m}_x \cdot \hat{m}_y$$

with $\hat{m}_x, \hat{m}_y =$ vector of deviations of the mean

$$\vec{m}_x = ((x_1 - \bar{x}), \dots, (x_N - \bar{x}))^T$$

Thereby it is:

- perfect correlation:
$$r = +1 \Leftarrow \hat{m}_x = \hat{m}_y \Leftarrow \vec{m}_x, \vec{m}_y \text{ parallel}$$
- perfect negative correlation:
$$r = -1 \Leftarrow \hat{m}_x = -\hat{m}_y \Leftarrow \vec{m}_x, \vec{m}_y \text{ antiparallel}$$
- no correlation:
$$r = \ 0 \Leftarrow \hat{m}_x \text{ perpendicular/orthogonal on } \hat{m}_y$$

Attention: even if $r = 0$, the variables x, y can exhibit a *nonlinear* dependency!

Example :

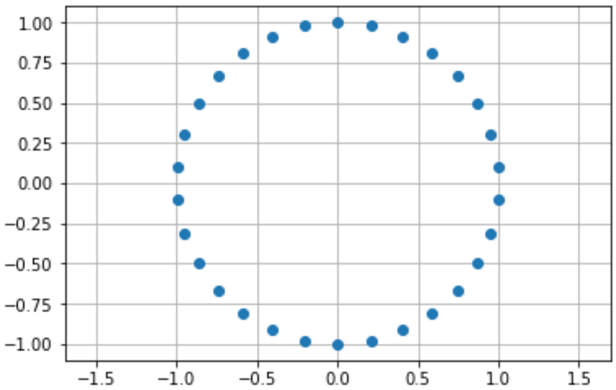
$x_k = \sin\left(\frac{2\pi}{N}k\right)$ and $y_k = \cos\left(\frac{2\pi}{N}k\right)$ for $k = 0, \ldots, (N-1)$.

Obviously $\bar{x} = \bar{y} = 0$

But we derive:

$$r \propto \sum_{k=0}^{N-1} x_k y_k$$
$$= \sum_{k=0}^{N-1} \sin\left(\frac{2\pi}{N}k\right) \cos\left(\frac{2\pi}{N}k\right)$$
$$r = \frac{1}{2}\sum_{k=0}^{N-1} \sin\left(\frac{4\pi}{N}k\right) = 0$$

In [12]:
```
# %pylab inline
N = 30
x = [sin(2*pi/N*k) for k in range(0,N)]
y = [cos(2*pi/N*k) for k in range(0,N)]
plot(x,y,"o")
axis('equal'); grid();
```



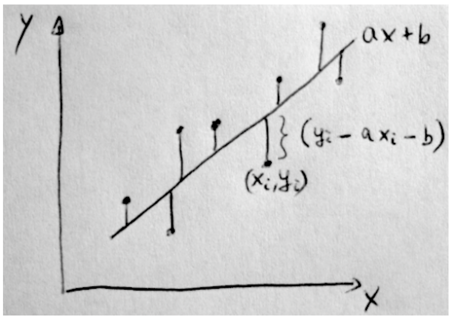
- yet obviously there is a *nonlinear* dependency $x_k^2 + y_k^2 = 1$!

Geometric Interpretation as Optimization Error

- Find the best linear description of typ $y(x_i) = f(x) = ax_i + b$
- optimal values (estimators) for a, b are the result of a minimization of

$$\chi^2(a,b) := \sum_{i=1}^{N} (y_i - ax_i - b)^2$$

w.r.t. the parameters a und b.



- Note: as exercise, calculate from $\frac{\partial\chi^2}{\partial a} = 0$ and $\frac{\partial\chi^2}{\partial b} = 0$ the optimal values for a, b and the minimal approximation error.}

With best fit parameters a, b, it is:

$$\chi^2_{\min} = (1 - r^2) \sum_{i=1}^{N} (y_i - \bar{y})^2$$

- i.e. $(1 - r^2)N \cdot \mathrm{Var}(y)$

2.5.1. Statistical Interpretation of r

r is a weak statistic: only with restriction, i.e. assumptions on the distribution of the data can we derive statistical statements about the rejectability of the null hypothesis.

Assumption A1

Let x, y be normal distributed and N large. Then it is

$$P(|r| > r_{\mathrm{beob}}) = \mathrm{erfc}\left(\frac{|r_{\mathrm{beob}}|\sqrt{N}}{\sqrt{2}}\right)$$

- We already know this function since $\Gamma(0.5, x^2) = \sqrt{\pi}\mathrm{erfc}(x)$ and $\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt$.

Assumption A2

Under the (often valid) assumption of a two-dimensional Gaussian distribution

$$p(x,y) \propto \exp\left(-\frac{1}{2}(a_{11}x^2 - 2a_{12}xy + a_{22}y^2)\right)$$

the derived statistic

$$t = r\sqrt{\frac{N-2}{1-r^2}}$$

is Student-t-distributed with $N - 2$ degrees of freedom (under valid H_0).

- Note that $(1 - A(\hat{t}, N-2))$ then delivers – analogue to the t-test – the probability of error that the null hypothesis is rejected although true.

In [ ]: