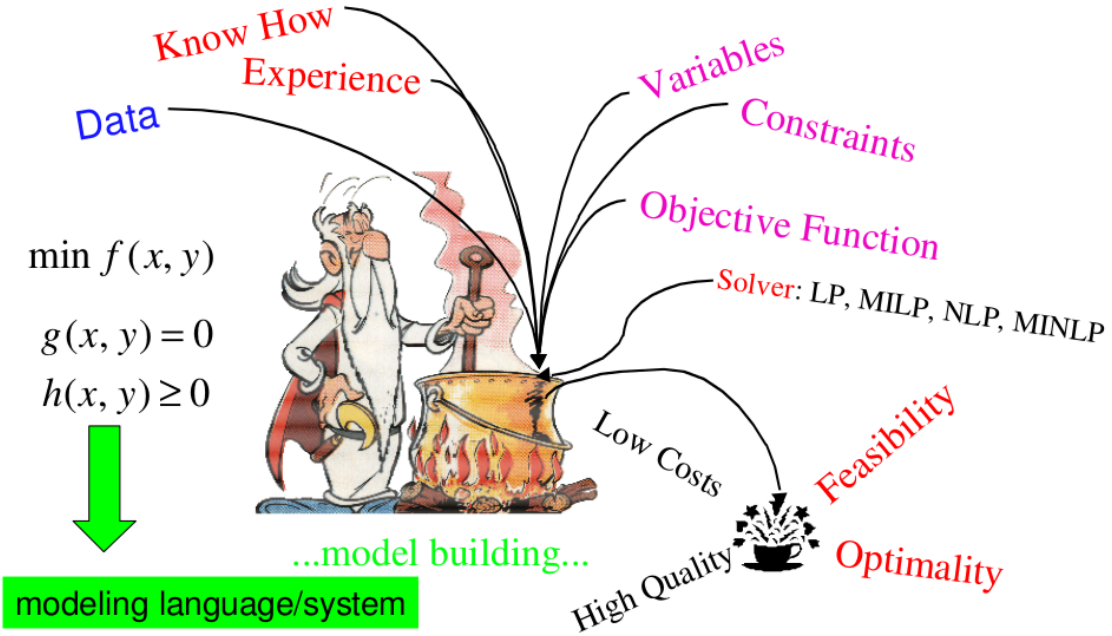

Optimization Intro

MTA 3007 Notebook

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MAT 3007 Optimization-Intro (2020-21 Summer)

Lecture 1 Introduction

- Terminology:

1) Euclidean inner product: $\langle x, y \rangle = x^T y$

2) Euclidean norm of a vector/point $\|x\|_2 = \|x\| = \sqrt{x^T x}$

3) $g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{bmatrix}$, then $g(x) \leq 0$ means $g_i(x) \leq 0, \forall i = 1, 2, \cdots m$

4) Feasible point: a decision satisfying all constraints

Feasible set/region Ω — a set of all feasible pts

5) Optimal solution: a feasible point / decision variable that attains an objective value, as good as other feasible pts.

(may be more than 1)

Optimal value: the objective value of any optimal solution.

(only 1)

6) Formal definition of extrema:

Local minimizer/maximizer: $x^* \in \Omega$ & $\exists\, \varepsilon > 0$ s.t. $f(x) \geq f(x^*)$ / $f(x) \leq f(x^*)$ for all $x \in \Omega \cap B_\varepsilon(x^*)$ ($B_\varepsilon(y) := \{x \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$)

strict local ~ (change $\geq, \leq$ to $>, <$)

(radius, origin)

global minimizer/maximizer: $x^* \in \Omega$ & $\nexists$ it holds that $f(x) \geq f(x^*)$ / $f(x) \leq f(x^*)$ for all $x \in \Omega$

global minimizer/maximizer $\equiv$ global solution $\equiv$ optimal solution

- Facts & Properties

Optimization & Statistics are two foundations of machine learning.

Modeling — transfer primitive questions into optimization problems.

The feasible set can be empty (called infeasible/degenerated)

The optimal value may be unbounded

$\Longrightarrow$ Classification:

- Infeasible
- Feasible; finite optimal value; not attainable
- Feasible; finite optimal value; attainable
- Feasible; unbounded optimal value.

Classifications of Optimization Problems:

- Unconstrained v.s. Constrained
- Linear (LP) v.s. Non-linear (NLP)
- Integer / Discrete (IP) v.s. Continuous

• Real example analyses:

: Definition (math def of optimization problem):

$\min_x / \max_x f(x)$ subject to $x \in \Omega$

or

$\min_x / \max_x f(x)$ subject to $g_i(x) \le 0, \forall i = 1, 2 \cdots m$, $h_j(x) = 0, \forall j = 1, 2 \cdots p$

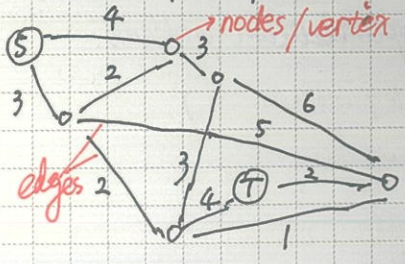
($\min_x f(x) = -\max_x -f(x)$ same)

(Unconstrained: $\Omega = \mathbb{R}^n$ or $m = p = 0$)

- Decision ~ decision variables: unknown & needs choosing
- Objective ~ objective functions
- Constrains ~ constraint functions: equalities & inequalities

↑ 3 main factors

eg. 1 (Shortest path problem)



Find shortest path from S to T

V: set of nodes

$E \subset V \times V$: set of existing edges

w_{ij} expresses the distance from i to j

Choice: $x_{ij} = \begin{cases} 1, & (i,j) \text{ edge is chosen} \\ 0, & \text{otherwise} \end{cases}$

optimization model:

$$\text{minimize}_{\{x_{ij}\}} \sum_{(i,j)\in E} w_{ij} x_{ij}$$

subject to $x_{ij} \in \{0,1\}, \forall (i,j) \in E$

constraints for a path $\leftarrow$ $\begin{cases} \sum_j x_{sj} = 1 = \sum_j x_{jt} & \text{(for the pinnacle)} \\ \sum_j x_{ij} = \sum_j x_{ji}, \forall i \neq s,t \end{cases}$

eg.2, For the same diagram in 1. find the smallest set of vertices that touch every edge.

$$x_i = \begin{cases} 1, & \text{node } i \text{ is chosen} \\ 0, & \text{otherwise} \end{cases}$$

optimization model:

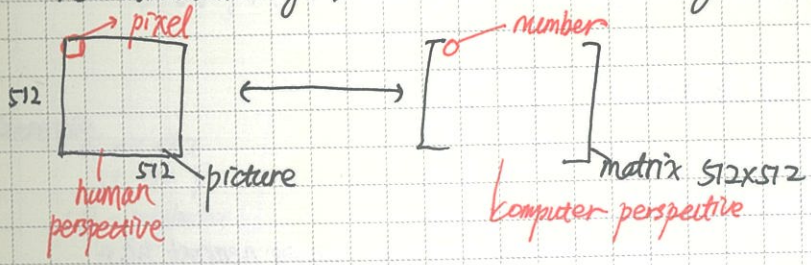
$$\text{minimize}_{\{x_i\}} \sum_i x_i \quad \text{(size of set)}$$

subject to $x_i + x_j \geq 1$, for $(i,j) \in E$ $\longleftrightarrow$ the requirement of question

$x_i \in \{0,1\}, \forall i \in V$

eg.3 (removing scratches in a picture)

Decision — the whole image; objective — remove scratches and obtain a "nicer" reconstructed image; Constraints — not change undamaged ones.



Lecture 2. Application & Linear Programming

- Machine Learning.

To extract important patterns, understand data & use it to make predictions or decisions.

Classification & regression are two fundamental tasks in machine learning

Support Vector Machine (SVM) is one of the most important techs for classification.

eg. (for classification)

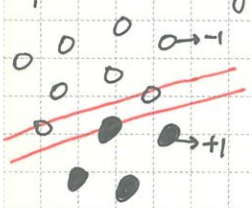
Learning to predict if one applicant should be approved a credit card.

	age	gender	salary	citizenship	years in job	approve
app 1	2.5	1	10	3	1 → x_i	+1
app 2	2.8	0	8	6	5 (a row (ith) containing a person's data)	+1
app 3	1.6	0	0	2	0	-1
⋮	⋮	⋮	⋮	⋮	⋮	⋮
app 5	3	1	8	2	1	-1

data matrix — representing by (X)　　Label — (y)

(label indicates supervision)

For all training samples: row, label i (draw as (x_i, y_i))



Idea: learn a function $f: \mathbb{R}^n \to \{-1, 1\}$ base on training samples

1) Linear classification: $f(x) = x^T w + b$ s.t. $f(x_i) y_i \geq 1, \forall i$

transfer ⟹ minimize w, b (0) → feasibility problem

subject to $y_i(x_i^T w + b) \geq 1, \forall i$

To make the line be the best, (not naive)

SVM → ① find support vectors (above & below):

(draw parallel line/plane, passing nearest pt.)

② let two line/planes be $x^T w + b = 1$ & $x^T w + b = -1$

(by normalize & choosing x, b)

③ maximize: $\frac{2}{\|w\|}$ (distance) → margin

better case ⟹ maximize $\frac{2}{\|w\|}$

w, b

subject to $y_i(x_i^T w + b) \geq 1, \forall i$

transfer ⟹ minimize $\frac{1}{2}\|w^2\|$

w, b

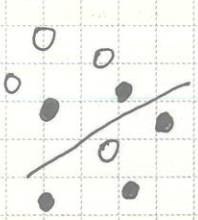
subject to $y_i(x_i^T w + b) \geq 1, \forall i$

non-linear, constrained, continuous

(use $\|w\|^2$ because $w \mapsto \|w\|$ is not differentiable at 0)

maximum-margin hyperplane

2) Non-linear-separable data



Define Hinge-loss $\max\{0, 1 - y_i(x_i^T w + b)\}$

— For points on the true side, 0

— For points on the wrong side, the farther, the larger.

Final Edition $\underset{\omega, b}{\text{minimize}} \frac{\lambda}{2}\|\omega\|^2 + \sum_{i=1}^{m} \max\{0, 1-y_i(x_i^T\omega+b)\}$ ← soft-margin SVM

$\lambda > 0$

ω, b arbitrary, λ – chosen to balance the margin & misclassification (weighted sum)

a special case — when $\lambda = 0$, $t_i = \max\{0, 1-y_i(x_i^T\omega+b)\}$ (linear programming)

transfer ⟹ $\underset{\omega, b, t}{\text{minimize}} \sum_i t_i$

subject to $t_i = (1-y_i(x_i^T\omega+b))^+$, $\forall i$

↳ can be written as $t_i \geq (1-y_i(x_i^T\omega+b))^+$

$\Leftrightarrow t_i \geq 0$, $t_i \geq (1-y_i(x_i^T\omega+b))$, $\forall i$

- Linear Programming & Standard Form

LP / linear optimization problem: where the objective function and all constraint functions are linear in the decision variables (independent from others)

1) Compact Way of impression:

minimize c^Tx

subject to $A_1x \geq b$ $\left(A_1 = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}\right)$

$A_2x \leq d$ e.g. (i.e. $-a_i^Tx \geq b_i$)

$A_3x = e$ (use matrices to express)

$x_i \geq 0$ $\forall i \in N_1$

$x_i \leq 0$ $\forall i \in N_2$

x_i free $\forall i \in N_3$

[Generally, $\min c^Tx$ s.t. $x \in P$ (P – polyhedron) ▲]

2) Standard Form: (of LP)

minimize c^Tx

$x \in \mathbb{R}^n$

subject to $Ax = b$, $x \geq 0$ (A – $m \times n$ matrix ($m < n$), $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $m < n$)

↳ infinite solutions

Transfer techs:

① max → min $\max c^Tx \Leftrightarrow -\min -c^Tx$

② $Ax \leq b$ or $Ax \geq b$ $Ax + s = b$, $s \geq 0$ & $Ax - s = b$, $s \geq 0$ (s – slack variables)

③ $x_i \leq 0$, define $y_i = -x_i$, $y_i \geq 0$ (replace $x_i \leq 0$) (every place $x_i \leftarrow -y_i$)

④ x_i free, $x_i = x_i^+ - x_i^-$ (use two variables to substitute one)

$x_i^+, x_i^- \geq 0$

☆ Standard form

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

$x \geq 0$ → all variables!

Examples:

eg1. $\max_{x_1, x_2} \; x_1 + 2x_2$

subject to $x_1 \leq 100$
$2x_2 \leq 200$
$x_1 + x_2 \leq 150$
$x_1, x_2 \geq 0$

compact way → standard form

$\min_{x_1, x_2, s_1, s_2, s_3} \; -x_1 - 2x_2$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$(A \quad x + s = b)$

$x_1, x_2 \geq 0$

eg2. $\min_{w, b, t} \; \sum_i t_i$

subject to $y_i(x_i^T w + b) + t_i \geq 1$

$t_i \geq 0$

standard form →

$\min_{w, w^+, w^-, b^+, b^-, t, s} \; \sum_i t_i$

subject to $y_i(x_i^T w^+ - x_i^T w^- + b^+ - b^-) + t_i - s_i = 1$

$w^+, w^-, b^+, b^- \geq 0$

$t_i, s_i \geq 0$

- MATLAB Code:

CVX — similarly express to math

("cvx_begin" ~ "cvx_end")

pay attention to
- "sum(sum(W .* X))"
- "sum(X(1, :)) = 1"
- "sum(X(i, :)) - sum(X(:, i)) = 0"

(some high-level language)

Lecture 3 Linear Programming Modeling

- Modeling Transformation

ΔQ1: Air Traffic Control in OR field. (maximin problem)

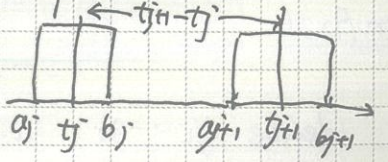
Flights land in the order 1, 2, ~ n, time interval $[a_i, b_i]$ $\forall i = 1, 2, \sim n$

Objective — maximize the minimum seperation time

$\max_t \min \{t_{j+1} - t_j\}$ (transform!)

subject to $a_j \leq t_j \leq b_j \quad \forall j$

$t_j \leq t_{j+1} \quad \forall j \leq n-1$



Trick: $\Delta = = \min_{j = 1, 2, \sim n-1} \{t_{j+1} - t_j\}$

LP: maximize Δ
Δ, t
subject to $t_{j+1} - t_j \geq \Delta \quad \forall j = 1, 2, \dots, n-1$
$b_j \geq t_j \geq a_j \quad \forall j$
$t_{j+1} - t_j \geq 0 \quad \forall j \leq n-1$

Generalize: maximin problem & minimax problem

$\max_x \min_{i=1,2,\dots,n} \{c_i^T x + d_i\}$ & $\min_x \max_{i=1,\dots,n} \{c_i^T x + d_i\}$
subject to $Ax = b$, $x \geq 0$ — subject to $Ax = b$, $x \geq 0$

no general way to do with $\sum |x_i|$ problem (Q_2) ← general tricks → has LP forms (piece-wise linear, convex form)

$y = \max / \min \{c_i^T x + d_i\}$
$c_i^T x + d_i \geq / \leq y$

ΔQ_2: Minimize Absolute Values

$\min_x \sum_{i=1}^n |x_i|$ subject to $Ax = b$ (basis pursuit in signal processing)
$t_i = |x_i|$ (trick) →
$\min_{x,t} \sum_{i=1}^n t_i$ subject to $t_i \geq x_i$, $t_i \geq -x_i$, $Ax = b$

(for $\max \sum |c_i^T x + d_i|$ → non-convexity, non-LP!)

Generalize: Lemma — a modeling tool

$\min_x \sum_{i=1}^n f_i(x)$ subject to $x \in \Omega$
$\Longleftrightarrow$
$\min_{x,t} \sum_{i=1}^n t_i$ subject to $t_i \geq f_i(x) \ \forall i$, $x \in \Omega$

(same optimal value)

AQ_3: Linear Fractional Programming (LFP)

$\min_x \frac{c^T x + d}{e^T x + f}$ (assume $e^T x + f \geq 0$ for all $Ax \leq b$)
subject to $Ax \leq b$

Trick: define $y = \frac{x}{e^T x + f}$ $z = \frac{1}{e^T x + f}$

LP: $\min_{y,z} c^T y + dz$
subject to $Ay - bz \leq 0$
$e^T y + fz = 1$
$z \geq 0$

(whole proof process: substitution, (y, z) is feasible with $z \neq 0$, $x = \frac{y}{z}$ is feasible, (vice versa)
optimal value of LP ≥ ~ of origin ~ of origi
⇒ The must be equivalent ≥ not
note: when $z = 0$, $x_0 + ty$ all feasible, (x_0 is feasible))

- Graphically solving LP

Definitions:

1) **Polyhedron**: a set which can be written as $\{x \in \mathbb{R}^n \mid Ax \geq b\}$ with $A \in \mathbb{R}^{m \times n}$ ($m \times n$ matrix), $b \in \mathbb{R}^m$

Corollary: $Ax = b, x \geq 0$ is a polyhedron (standard form)

reason $\begin{cases} Ax = b \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} Ax \geq b \\ -Ax \geq -b \\ Ix \geq 0 \end{cases} \Leftrightarrow \begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$ → polyhedron

($\hat{A}$, $\hat{b}$)

2) **Convex Set**: a set $S \subseteq \mathbb{R}^n$, for any $x, y \in S$, $\exists \lambda \in [0,1]$ $\lambda x + (1-\lambda) y \in S$

(Affine Set) ↪ $\lambda \in \mathbb{R}$

3) **Convex Combination**: $\forall x_1, x_2, \cdots x_n$ & $\lambda_1, \lambda_2 \cdots \lambda_n \geq 0$ satisfying $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{i=1}^{n} \lambda_i x_i$ — convex combination of $x_1, x_2 \cdots x_n$

4) **Extreme point**: $x \in$ polyhedron P, s.t. we cannot find two elements $y, z \in P$ with $y, z \neq x$ & a scalar $\lambda \in [0,1]$, satisfying $x = \lambda y + (1-\lambda) z$

(vertex/corner of a polyhedron)

△ Finding extreme point in LP

$$\begin{aligned} \text{minimize } & c^T x \\ x \\ \text{subject to } & Ax = b \\ & x \geq 0, \end{aligned}$$

$x \in \mathbb{R}^n$. A — $m \times n$ matrix with $m < n$

$b \in \mathbb{R}^m$

☆ generally assumpt that A has full row rank m.

(Otherwise, either can be redundand or empty)

Basic Solution:

x is called a basic solution of P if & only if:

(1) $Ax = b$; (2) $\exists$ indices $B(1), B(2), \cdots B(m)$ s.t.

$A_B = [A_{B(1)}\ A_{B(2)} \cdots A_{B(m)}]$ has all independent columns & $x_i = 0$,

$\forall i = B(1), B(2), \cdots B(m)$. (*note: A_i — ith column of A)

$x_B = A_B^{-1} b$ (invertible)

$B = \{B(1), B(2) \cdots, B(m)\}$ — basic indices

A_B — basic submatrix of A

☆ No more than m non-zeros could one have in basic solution (m constraints)

at most $\binom{n}{m} = \frac{n!}{m!(n-m)!} < +\infty$ basic solutions

Basic Feasible Solutions (BFS):

A basic solution x s.t. $x \geq 0$ can serve as a BFS.

▲ **Theorem:** For the standard LP polyhedron, $P := \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

x is an extreme point of $P \iff x$ is a BFS of P

Fundamental LP Theorem:

consider a LP in standard form, assume A — full row rank

(1) If the feasible set is non-empty, $\exists$ (there exists) a BFS.

(2) If there's an optimal solution, $\exists$ an optimal solution also BFS.

(Just look among BFSs to find optimal solution if existing)

If LP with m constraints has an optimal solution, $\exists$ (must) an optimal solution with no more than m entries. (positive)

• Tutorial — more modeling examples:

L_p-norm of x: $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$

exe 1: $\min_x \|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$ (ask for LP form)

(☆) $\|x\|_\infty = \lim_{p \to \infty} (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \in \lim_{p \to \infty} \max_{i=1,2,\cdots,n} \{|x_i|\}^{p \cdot \frac{1}{p}}$, $\lim_{p \to \infty} \left(n^{\frac{1}{p}} \max_{i=1,2,\cdots,n} \{|x_i|\}^{p \cdot \frac{1}{p}}\right)$

$= \lim \cdots$ (by squeezing theorem) $= \max_{i=1,2,\cdots,n} \{|x_i|\}$

idea: $s_i = |x_i|$ (substitution & transform L_p-norm)

$t = \max_{i=1,\cdots,m} \{|Ax_i - b|\}$

minimize $1^T s$

$x, s \in \mathbb{R}$

subject to $-s \leq x \leq s$

$\Rightarrow \begin{bmatrix} t \\ \vdots \\ t \end{bmatrix} \leq Ax - b \leq \begin{bmatrix} t \\ \vdots \\ t \end{bmatrix}$

$x = x^+ - x^-$

$\min_{x^+, x^-} 1^T x^+ + 1^T x^-$

subject to $\begin{bmatrix} t \\ \vdots \\ t \end{bmatrix} \leq Ax^+ - Ax^- - b \leq \begin{bmatrix} t \\ \vdots \\ t \end{bmatrix}$

$x^-, x^+ \geq 0$

(understand, give $|x_i|$, either x_i^+ or x_i^- should be zero to minimize $1^T x^+ + 1^T x^-$)

exe 2:

↳ find implicit x

$\min_x C^T x$

subject to $Ax = b$

Solutions: use the background knowledge $c = A^T\lambda + c'$ (complement, othogonal) — $A^T\lambda$: in row space of A $C(A^T)$ (given λ); c': give c' → in the nullspace of (A) $N(A)$

(i) if $b \notin C(A)$, not feasible / infeasible

(ii) if $b \in C(A)$ with $c' = 0$, then

$c^T x = \lambda^T A x = \lambda^T b$ is a specific #.

(iii) if $b \in C(A)$ with $\lambda = 0$, then $c = c'$, note that for $x = x_0 - tc'$ feasible for all t, $c^T x = c^T x_0 - t c^T c$, $\|c\|_2 > 0$ can be $-\infty$

(iv) $\lambda \neq 0$, same with (iii) (unbounded)

optimal value $p^* = \begin{cases} \infty, & b \notin C(A) \\ \lambda^T b, & b \in C(A), c = A^T\lambda \\ -\infty, & \text{elsewhere} \end{cases}$

exe 3. How to express projection of a vector P on plane S.

Projection $\Rightarrow$ minimize the distance from P to S

$$\begin{aligned} &\text{minimize}_{x} \ \|P - x\|^2 \\ &\text{subject to} \ \ x \in S \end{aligned}$$

exe 4. (Knapsack Problem)

Totally N numbers of goods, with weight $1, 2, \dots N$ (w_i) & value $1, 2, \dots N$ (v_i), The package can pack at most weight W, how to get more value.

(use $c_i \in \{0, 1\}$ for choice & not choosing)

$$\begin{aligned} &\text{maximize}_{c} \ \sum_{i=1}^{N} c_i v_i \\ &\text{subject to} \ \sum_{i=1}^{N} c_i w_i \leq W \\ &c_i \in \{0, 1\}, \ \forall i = 1, 2, \dots N \end{aligned}$$

Lecture 4 The Simplex Method

- IDEA: proceeding from one BFS to a neighboring one (standard LP form) to continuously improve (increase/reduce) the value

• Neighboring / Adjacent Basic Solutions:

Two basic solutions are neighboring/adjacent if they differ by exactly one basic index. (one basic index differs ⇔ one non-basic index differs)

Efficient way to check "all" neighbors — (avoid $m(n-m)$ choice & every time taking inverse)

1) First, rewrite $Ax=b$ as $[A_B\ A_N]\begin{bmatrix}x_B\\x_N\end{bmatrix}=b$ (re-arrange these columns)

2) Finding a random number in non-basic set N, j, wanting to increase x_j to get the neighbors. Move $x_j \to x_j+\theta d\ (\tilde{x})$, $\theta \geq 0$

d satisfies: 1) $d_j=1$; 2) $d_{j'}=0$, $\forall j' \in$ Basic "feasible" set

3) $A\tilde{x}=b$, therefore, $Ad=0 \Rightarrow A_B d_B + A_N d_N = 0$ (A_B: full rank; A_j)

d_B (specific) $=-A_B^{-1}A_j$ $d=\begin{bmatrix}d_B\\d_N\end{bmatrix}$, where d_N has $d_j=1$, all others zero.

d is called — j^{th} basic direction

3) Making $\tilde{x}$ feasible, $\tilde{x}=x+\theta d \geq 0$, $x \geq 0$

(non-basic variable i, $x_i=0$. basic variable j, small θ satisfying $x+\theta d \geq 0$)

↳ these contain just the case $x_B > 0$ (strictly)

when $x_B \not> 0$ (some entries $=0$), degeneracy

• Reduced Costs:

1) rewrite $c^T=[c_B^T\ c_N^T]$ (same as above), then c^Td

2) $c^Td=[c_B^T\ c_N^T]\begin{bmatrix}d_B\\d_N\end{bmatrix}=c_B^Td_B+c_N^Td_N=\boxed{c_j-c_B^TA_B^{-1}A_j=:\bar{c}_j}$ reduced cost

$\Rightarrow$ {positive $\bar{c}_j$ — increase; do not want to go

negative $\bar{c}_j$ — decrease; go in that direction (indicators of where to go)

Extension: j in B ($j=B(i)$), $\bar{c}_{B(i)}=c_{B(i)}-c_B^TA_B^{-1}A_{B(i)}=0$ ($A_B^{-1}A_{B(i)}=A_i$)

Stopping Point / Criterion:

Consider a basic solution x associated with the basic $B(1), B(2), \cdots B(m)$, $\bar{c}$ — the corresponding vector of reduced costs. If $\bar{c} \geq 0$, then x must be optimal.

- Choosing a stepsize (θ) (Change of Basis)

Want θ to be as large as possible — reduce more

★ $\theta^* = \max\{\theta \geq 0 \mid x + \theta d \geq 0\}$ → can get

$\theta^* = \min_{\{i: d_i < 0\}} -\frac{x_i}{d_i}$ → $\begin{cases} d \geq 0, \theta = \infty, \text{unbounded} \\ \text{some } d_i < 0, \text{this equation} \end{cases}$

New basis $\tilde{x} = x + \theta^* d$ (someone $B(\ell)$ out & j in)

$x_j > 0$, $x_{B(\ell)} = 0$

- The whole Process of simplex Method

Step 1. Reduced Costs c → can we continue? which way to choose?

Step 2. jth basic directions → given jth, how to go? (d, θ^*)

Step 3. Move on to $x + \theta^* d$ & then repeat.

↑ make the whole process complete

- Additions: degenerate cases & Choosing strategy

1) Theorem — properties of $\tilde{x}$: Let x be a non-degenerate BFS ($x_B > 0$) with basic indices B, and let $y = x + \theta^* d$ be generated by simplex iteration. Then, y is a basic feasible solution. ($A_{\hat{B}}$ is full rank) (with new indices)

2) Theorem — convergence: assume that the feasible set is non-empty & every BFS is non-degenerate. Then, the simplex method terminates after a finite # iterations. with the following options:

1) Stop with a BFS as optimal solution; 2) $d \geq 0$, $c^T d < 0$, optimal value $-\infty$

3) Treat with degeneracy:

Still change basic index from i to j, with $\theta^* = 0$ to next interation

Although objective value does not change, basis B → basis $\hat{B}$, d_B changes

⇒ $\min_{\{i: d_i < 0\}} -\frac{x_i}{d_i}$ may change ⇒ finite times then to non-degeneracy.

4) Pivot Rules (avoid cycle in degeneracy, efficiently)

☆ choosing j to go → direction, enter

(Most common) – Smallest Index Rule (smallest j s.t. $\bar{c}_j < 0$)

Most negative Rule (smallest $\bar{c}_j < 0$)

Most Decrement Rule (smallest $\theta^* \bar{c}_j < 0$)

☆ Choosing θ^* (one index to leave) → leave

Smallest Index Rule { smallest j s.t. $\theta^* = -\frac{x_i}{d_j}$, when $-\frac{x_i}{d_j}$ is one of the minimum }

at degenerate cases, $\geq k$ planes intersect in k dimension (at a point)

Theorem – Bland's Rule:

Use smallest rules (index) for choosing both the entering basis & the exiting basis,

No cycle will occur! — stop within finite times

Lecture 5 The Simplex Tableau

- Proof of property 1 in Lecture 4:

$m = \text{rank}(I) = \text{rank}(A_B^{-1} A_B) \leq \min\{\text{rank}(A_B^{-1}), \underset{\parallel}{\text{rank}}(A_B)\} \leq m$ ($\text{rank}(A_B) = m$)

$\Rightarrow \text{rank}(A_B^{-1}) = m$.

Because $\text{rank}(A_B^{-1} A_{\hat{B}}) = \text{rank}\left[\sum_{i \in B \setminus \{B(l)\}} A_B^{-1} A_i + \underbrace{A_B^{-1} A_j}_{-d_B \to \text{at } l\text{th row}}\right] = m$

Then $m = \text{rank}(A_B^{-1} A_{\hat{B}}) \leq \min\{\text{rank}(\underset{l\text{th}}{A_B^{-1}}), \text{rank}(A_{\hat{B}})\} \leq m$

$\Rightarrow \text{rank}(A_{\hat{B}}) = m$

- Finding Initial BFS:

1) Directly finding way: (adding slack variables (totally m amounts))

(Because the matrix which slack variables face is I) $\Rightarrow$ BFS choosing all variables

2) Two-phase Method (use simplex method for two times) ← more common one

① minimize $\underset{x,y}{\mathbb{1}^T y}$ subject to $Ax + y = b$, $x, y \geq 0$ ⟸ auxiliary problem

② Standard LP form

Explanation: (with the loss of generality, assume $b \geq 0$, otherwise change the row $-A_i^T x = -b_i$)

Initial BFS for auxiliary problem: $y = b$, $x = 0$ (written as $\underbrace{[A \ \ I]}_{\tilde{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\tilde{x}} = b$)

(BFS, m lines)

Apply simplex method to find the optimal solution

☆ Theorem: Feasibility

The original problem is feasible if & only if the optimal value of the auxiliary problem is 0.

{ Optimal value > 0 ⇒ cannot find x s.t. $Ax=b$ ($Ax<b$)

Optimal value = 0 ⇒ $Ax^*=b$, $x^*\geq 0$ & BFS x^* has new x_B^*

For degenerate case (x^* has less than m positive entries),
pick other columns to supplement, get an $m\times m$ matrix with rank $=m$

3) The Big-M Method

(tranform to another auxiliary)

$$\underset{x,y}{\text{minimize}} \quad c^Tx + M\cdot \mathbb{1}^Ty$$
$$\text{subject to} \quad Ax+y=b$$
$$x,y\geq 0$$

Explanation: BFS (initial) = ($y=b\geq 0$)

M should be large enough to avoid $y>0$ case (infeasible)
∞ (unbounded)

$y=0$ (feasible), optimal value $=c^Tx$

• Simplex Tableau

(avoid A_B^{-1}, simply solving by hand)

$c^T - c_B^TA_B^{-1}A$	$-c_B^TA_B^{-1}b$ → current "optimal value"
$A_B^{-1}A$	$A_B^{-1}\cdot b$ → current BFS

reduced costs ← all-ds' ($d=-A_B^{-1}A$)

$A_B^{-1}b = A_B^{-1}Ax = A_B^{-1}[A_B\ A_N]\begin{bmatrix}x_B\\x_N\end{bmatrix} = Ix_B = x_B$

The simplex

0_m^T	$c_N^T - c_B^T A_B^{-1} A_N$	$-c_B^T x_B$
I_m	$A_B^{-1} A_N$	x_B

← canonical form

☆ How to use simplex tableau:

(reduced costs — incoming basic index j — θ^* & outgoing $B(l)$ — new basis)

pivoting — transform between two canonical forms

eg. choose; $I = A_B^{-1} A_B$; outgoing column index; (production plan in lecture 1)

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

pivot; pivot row; pivot column; $A_B^{-1} A_N$; I

$\Rightarrow x_B = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 200 \\ 150 \end{bmatrix}$

(Bland's Rule)

choosing θ^* (Minimal Ratio Test / MRT)

☆ $\theta^* = \min_i \left\{ \frac{\bar{b}_i}{\bar{A}_{ij}} \mid A_{ij} > 0 \right\}$ (CBFS) ($\bar{b}_i \to x_i$, $\bar{A}_{ij} \to -d_i$)

(original $\theta^* = \min_{d_i > 0,\, i \in B} \frac{x_i}{d_i}$)

☆ Unbounded — $\bar{A}_{ij} \le 0$ for all i in one j, ($d \ge 0$)

Steps — Elimination: ☆ pivot = 1, other entries = 0 in the column. (updating the tableau)

Reason: $QA_B^{-1} = A_B^{-1}$

* $Q \Leftrightarrow$ do elimination

↳ $\bar{c}_{B(l)} = 0$ & all others 1, 0 ...

B	0	-2	1	0	0	100
1	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	-1	0	1	50

first step

$\Rightarrow$

B	0	0	-1	0	2	200
1	1	0	1	0	0	100
2	0	0	2	1	-2	100
4	0	1	-1	0	1	50

second step

$\Rightarrow$ ... until $c \ge 0$ or $d \ge 0$

Lecture 6 Duality Theory (I)

- Complexity Theory

complexity, big-Oh Notation, how to compute

(omitted, refer to 2020-21 Term 2 CSC1001 Notes)

Polynomial-Time Algorithm: (some)

1) Gassian Elimination $O(n^3)$

2) Fast method for matrix inversion $O(n^{2.3727})$

3) Naive method of sorting $O(n^2)$

Non-polynomial-Time Algorithm: (some)

1) Enumeration method for traveling salesman problem $O(n!)$

2) Dynamic Programming for TSP $O(2^n n^2)$

Given list of cities & distances between each two, asking for shortest route, visiting every city exactly once & return to the primitive one.

☆ Definition:

P — a problem, for which ∃ a polynomial-time algorithm. (solvable)

NP-hard problems — a class of problems with currently no polynomial-time algorithm to solve

⟹ LP is a polynomial-time solvable problem.

Ellipsoid Method (theoretically good, practically bad to manipulate) — "worst" case — contradict — Simplex Method (theoretically bad, practically good to use)

Both good (practice & theory): interior-point Method

- Duality For Linear Programming

(transform) → primal problem

$$\text{minimize}_x \; c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0$$

$$\iff \text{minimize}_x \; c^T x + \max_{y \in \mathbb{R}^m} y^T(Ax-b) \quad \text{subject to} \quad x \geq 0$$

$$\max_y y^T(Ax-b) = \begin{cases} Ax \neq b, & \infty \text{ (give up)} \\ Ax = b, & 0 \text{ (reason)} \end{cases} \quad \text{for some } y$$

$$\iff \min_x \max_y \; c^T x + y^T(Ax-b) \quad \text{subject to} \quad x \geq 0$$

$$\iff \text{maximize}_y \; b^T y + \min_{x \geq 0} x^T(c - A^T y)$$

(reason later)

$\Leftrightarrow$ maximize $b^T y$ (reason) $\begin{cases} A^T y > c, & -\infty \text{ (give up)} \\ A^T y \le c, & 0 \le x \ge 0 \end{cases}$

subject to $A^T y \le c$

↳ dual problem

y – dual variables . P(primal) & D(dual) have the same optimal value.

Lagrange (saddle point)

Nash equilibrium $L = L(x,p) = c^T x + p^T(Ax - b)$

$L(x^*, p) \le L(x^*, p^*) \le L(x, p^*)$

buyer ↓ seller

x^*, p^* – optimal value

☆ Dual Problem of all kinds of LP form (can be proved use tricks shown above)

P		D
$\min_x c^T x$		$\max_y b^T y$
subject to $a_i^T x \ge b_i$, $i \in M_1$	variables ⇔ constraints	$y_i \ge 0$, $i \in M_1$
$a_i^T x \le b_i$, $i \in M_2$		$y_i \le 0$, $i \in M_2$
$a_i^T x = b_i$, $i \in M_3$		y_i free, $i \in M_3$
$x_j \ge 0$, $j \in N_1$	variables ⇔ constraints	$A_j^T y \le c_j$, $j \in N_1$
$x_j \le 0$, $j \in N_2$		$A_j^T y \ge c_j$, $j \in N_2$
x_j free, $j \in N_3$		$A_j^T y = c_j$, $j \in N_3$
(a_i^T ⇒ row i of A)		(A_j^T ⇒ A_j = column j of A)

Simple form

P	$\min c^T x$	$\max b^T y$	D
constraints	$\ge b_i$	≥ 0	variables
	$\le b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\le c_j$	constraints
	≤ 0	$\ge c_j$	
	free	$= c_j$	

(eg. Soft SVM) margin

minimize $\sum_{i=1}^m t_i$

w, b, t

subject to $y_i(x_i^T w + b) + t_i \ge 1$, $\forall i = 1, 2, \dots m$

$t_i \ge 0$

"b" $= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \to m$-dimension

$c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{matrix} \in \mathbb{R}^n \\ \in \mathbb{R} \\ \in \mathbb{R}^m \end{matrix} \to (m+n+1)$-dimension

$A = \begin{bmatrix} y_1 x_1^T & y_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_m x_m^T & y_m & 0 & 0_{m\times 1} \end{bmatrix} \begin{bmatrix} w \\ b \\ t \end{bmatrix}$ (block II → y, block III → I_m)

block I → $\mathrm{diag}(y) X^T$, with $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

$= [\mathrm{diag}(y) X^T \; y \; I_m] \begin{bmatrix} w \\ b \\ t \end{bmatrix}$

Dual problem (of P)

maximize $\sum_{i=1}^m u_i$

u

subject to $X \cdot \mathrm{diag}(y) \cdot u = 0$

$y^T u = 0$

$0 \le u \le 1$

} seperately written way

• Weak Duality

☆ Theorem (duality under transformations):
Transform an LP to an equivalent one will get equivalent two problems.

☆ Theorem (doubly duality):
Doubly transform the primal problem (i.e. transform the dual to its dual) will get the identity (primal one).

☆ Weak Duality theorem:
If x, y are feasible to corresponding problem respectively, then we have $b^Ty \leq c^Tx$

(any dual feasible solution will give a lower bound on primal optimal value.
any primal feasible solution will give an upper bound on dual optimal value.)

proof: $b^Ty = (Ax)^Ty = x^T(A^Ty) \leq x^Tc = c^Tx$

corollary I: primal problem { unbounded ⟹ infeasible; infeasible ⟸ unbounded } dual problem

corollary II: x, y – feasible pts (let be) of the primal & dual problem respectively. If $c^Tx = b^Ty$, then x, y must be optimal solutions, correspondingly.

From corollary II: sufficient optimality conditions (3): $Ax = b, x \geq 0$; $A^Ty \leq c$; $c^Tx = b^Ty$ ⟹ optimal solution/value

(Unify finding linear opt & linear feasible pt. — same hard theoretically)

Lecture 7 Duality Theory (II)

- Strong Duality (Theory)

DOESNOT ALWAYS HOLD!!!
It holds when duality gap = 0.

In the case of LP, it always works

If the linear program has an optimal solution, so does its dual & the optimal values of the two are equal.

(i.e. the optimal conditions are "exactly when" conditions
$\begin{cases} Ax^* = b,\ x^* \geq 0 \\ A^T y^* \leq c \\ b^T y^* = c^T x^* \end{cases}$ $\Leftrightarrow$ x^*, y^* optimal solution (exactly when))

Solving LPs $\Leftrightarrow$ solving linear system
☆ (optimization $\Leftrightarrow$ solving systems)

Proof: (under simplex method)

suppose x^* is an optimal solution & BFS. (Fundamental LP)

construct $y^T = C_B^T A_B^{-1}$ (namely, $y = (A_B^T)^{-1} C_B$)

$\begin{cases} b^T y = y^T b = C_B^T A_B^{-1} b = C_B^T A_B^{-1} A_B x_B = C_B^T x_B = C^T x^* \quad (\text{optimal } y \text{ because of the weak duality.}) \\ A^T y \leq c \text{ because of } y^T A - C^T = C_B^T A_B^{-1} A - C^T = -\bar{C} \leq 0 \rightarrow \text{feasible} \end{cases}$

$y^T = C_B^T A_B^{-1}$ → dual optimal solution

☆ All (nearly) LP alg. (simplex, interior pt., ellipsoid) have corresponding y^T/y
(primal & dual can be solved simultaneously)

Possible relations:

P \ D	Finite opt.	Unbounded	Infeasible
Finite opt.	V	—	—
Unbounded	—	—	V
Infeasible	—	V	V

1) If both primal & dual are feasible, they have bounded opt. values.
2) Their opt. values same by strong duality.

- Complementarity Conditions

Let x, y be feasible pts of the primal & dual problem, respectively.

Then, x & y are optimal solutions exactly when

$x_i \cdot (c_i - A_i^T y) = 0, \ \forall i = 1, 2, \cdots n$ (or $\Leftrightarrow$ $\begin{cases} 1)\ x_i > 0,\ c_i = A_i^T y \\ 2)\ x_i = 0,\ c_i > A_i^T y \\ 3)\ \text{both zeros/equals} \end{cases}$)

proof: by strong duality, optimal solutions x & y

$\Leftrightarrow 0 = -b^T y + c^T x \Leftrightarrow -(Ax)^T y + c^T x = 0 \Leftrightarrow -x^T A^T y + x^T c = 0$

$\Leftrightarrow x^T (c - A^T y) = 0 \Leftrightarrow \begin{cases} x_i \geq 0,\ c_i \geq A_i^T y \\ \sum_{i=1}^{m} x_i (c_i - A_i^T y) = 0 \end{cases} \Leftrightarrow \text{all } x_i \cdot (c_i - A_i^T y) = 0$

Complementarity conditions:
- common (as shown above)
- ~ slackness ~ : $A^Ty+s=c,\ s\geq 0$ & $x_i\cdot s_i=0,\ \forall i$
- general ~ : $y_i(a_i^T-b_i)=0,\ \forall i$ & $x_j(A_j^Ty-c_j)=0,\ \forall j$

↳ "the primal-dual transformation form".

Then, we can get an equivalent set of optimal conditions

$$\begin{cases} Ax=b,\ x\geq 0 & \text{(or } x \text{ is feasible)} \\ A^Ty\leq c, & \text{(or } y \text{ is feasible)} \\ x_i(A_i^Ty-c_i)=0,\ \forall i, & \text{(or the complementarity conditions hold)} \end{cases} \quad \star$$

can be used to solve some problems

(By the way, $A_B^Ty=c_B \Longrightarrow y=(A_B^T)^{-1}c_B$
$x_N(A_N^Ty-c_N)=0$ (because $x_N=0$))

- Duality via Simplex Tableau

If the simplex tableau is got by adding m slack variables, namely

c^T	0_m	0
A	I_m	b

, then, after achieving optimal form, y^{*T}

$c^T-c_B^TA_B^{-1}A$	$-c_B^TA_B^{-1}$	$-c_B^TA_B^{-1}b$
$A_B^{-1}A$	A_B^{-1}	$A_B^{-1}b$

original x_B & $A_B^{-1}A_B$

(correspond $[A\ \vdots\ I]$
$c^T-c_B^TA_B^{-1}A,\ 0^T-c_B^TA_B^{-1}I_m$
(by elimination))

- Modeling Examples — application with dual

Exp 1 (production planning)

	Steel	Iron	Copper	Profit
alloy I	1	0	1	\$1
alloy II	0	2	1	\$2
Resources	100	200	150	

$$\begin{aligned} \underset{x_1,x_2}{\text{maximize}} \quad & x_1+2x_2 \\ \text{subject to} \quad & x_1\leq 100 \\ & 2x_2\leq 200 \\ & x_1+x_2\leq 150 \\ & x_1,x_2\geq 0 \end{aligned}$$

Transform to dual problem:

$$\begin{aligned} \text{minimize}_{p_1,p_2,p_3} \quad & 100p_1 + 200p_2 + 150p_3 \\ \text{subject to} \quad & p_1 + p_2 \geq 1 \\ & 2p_2 + p_3 \geq 2 \\ & p_1, p_2, p_3 \geq 0 \end{aligned}$$

physical meaning: at price p, selling resources.

From perspective of buyers { ① minimize total cost → objective function; ② the selling price ≥ profit → constraints. Otherwise the company (selling) will stop.

(Meaningful question)

Exp 2 (Multi-Firm Alliance)

Suppose firm 1, 2, 3, ..., m, making same set of products

alliance — they put all resources in a pool & use them jointly.

$$\Rightarrow \begin{aligned} \text{maximize}_x \quad & V^i = c^T x \\ \text{subject to} \quad & Ax \leq b_i \quad \text{(A: consumption matrix)} \\ & x \geq 0 \end{aligned}$$

for every company without alliance

$$\Rightarrow \begin{aligned} \text{maximize}_x \quad & V^S = c^T x \\ \text{subject to} \quad & Ax \leq \textstyle\sum_{i \in S} b_i \\ & x \geq 0 \end{aligned}$$

for every subset of company. ($S \subseteq \{1, 2, \dots, m\}$)

Then the original LP becomes: (determined by x & b)

$$\begin{aligned} \text{maximize}_{x,z} \quad & V^* = c^T x \\ \text{subject to} \quad & Ax \leq \sum_{i=1}^{m} b_i \\ & x \geq 0 \end{aligned}$$

(allocation constraints) &
$$\sum_{i=1}^{m} z_i = V^*$$
$$\sum_{i \in S} z_i \geq V^S \quad \forall S \subseteq \{1, 2, \dots, m\}$$ totally 2^m inequalities.

→ Otherwise they will leave

Dual: $\begin{aligned} \text{minimize}_y \quad & \left(\sum_{i=1}^{m} b_i\right)^T y \\ \text{subject to} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$

set $z_i = b_i^T y^*_{\text{opt.}}$ then $\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$ is the core of grand alliance

Check: ① By strong duality $\sum_{i=1}^{m} z_i = V^*$

② By weak duality $\sum_{i \in S} z_i \geq V^S$

Lecture 8 Sensitivity Analysis

- Modeling examples (cont'd)

Exp 3 (alternative systems)

CQ: how to verify $A^Ty \le c$ does NOT have solutions?
(prove non-existence)

Answer: If $\bar{P} := \{x \mid Ax=0, x \ge 0, c^Tx < 0\}$ is NOT empty, then there must be no solution to $A^Ty \le c$.

(Reason: duality theory

(primal) $\min c^Tx$ subject to $Ax=0$, $x \ge 0$ (dual) $\max 0^Ty$ subject to $A^Ty \le c$)

infeasible ⇔ unbounded
unbounded ⇔ infeasible

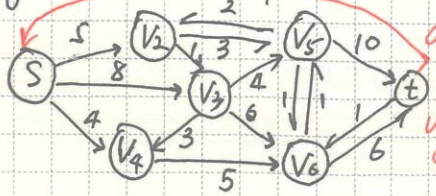
or weak duality $0^Ty \le c^Tx$ (any x)

(If x satisfied, any ax, $a>0$ satisfied, then c^Tx unbounded.)

Exp 4 (Maximum Flow Problem)

> Given a directed, weighted graph $G=(V,E)$ & a pair of nodes s & t.
> There is an edge capacity on each edge c_{ij}.

Q: largest flow sent from s to t subject to the capacity



assume from t to s ⇒ closed (natural constraints: with capacity flowing balance)

A: maximize x, Δ Δ → flow from t to s | total flows leaving s

subject to $\sum_{j=(j,i)\in E} x_{ji} - \sum_{j=(i,j)\in E} x_{ij} = 0 \quad \forall i \ne s,t, i \in V$ (balance)

$$\sum_{j=(j,s)\in E} x_{js} + \Delta - \sum_{j=(s,j)\in E} x_{sj} = 0$$

$$\sum_{j=(j,t)\in E} x_{jt} - \left(\sum_{j=(t,j)\in E} x_{tj} + \Delta\right) = 0$$

$$0 \le x_{ji} \le c_{ji} \quad \forall (j,i) \in E$$

Take the dual problem as follows,

derive

$$\sum_{(i,j)\in E}(-w_{ij}x_{ij}+z_{ij}(x_{ij}-c_{ij}))+\sum_{k\in V\setminus\{s,t\}}y_k\Big(\sum_{(i,k)\in E}x_{ik}-\sum_{(k,j)\in E}x_{kj}\Big)\leq 0 \quad \text{with } z_{ij}\geq 0 \text{ \& } w_{ij}\geq 0$$

$\Leftrightarrow \sum_{(i,j)\in E} M_{ij}(z,w,y) \leq \sum_{(i,j)\in E} z_{ij}c_{ij}$ (every item = (···)x_{ij})

want this to be the primal (optimal) objective value

i.e. want $\begin{cases} M_{si}=z_{si}-w_{si}+x_i=1 \\ M_{ij}=z_{ij}-w_{ij}+x_j-x_i=0 \\ M_{jt}=z_{jt}-w_{jt}-x_j=0 \end{cases}$

$\Rightarrow$ minimize $\sum_{(i,j)\in E} z_{ij}c_{ij}$ over $z\in\mathbb{R}^{|E|}, x\in\mathbb{R}^{|V|}$

subject to $z_{ij}\geq x_i-x_j$ (with $w_{ij}\geq 0$)

$x_s=1$ & $x_t=0$ / $x_s-x_t=1$

$z_{ij}\geq 0$

suppose $x_i\in\{0,1\}$

If $x_i=1$, $x_j=0$, $z_{ij}=x_i-x_j=1$

If $x_i\leq x_j$, $z_{ij}=0$

a cut S — all $x_i=1$

S' — all $x_j=0$

"$\sum_{i\in S, j\in S'} c_{ij}$" min-cut problem

minimize the weights of a cut

("the tightest bottle neck of the network")

- Sensitivity Analysis

CQ: when inputs change (i.e. A/b/c changes in standard form), how does the optimal value change?

☆ Local sensitivity (V - optimal value, with some inputs fixed)

{ If the dual has a unique optimal solution y^*, then $\nabla V(b)=y^*$ (gradient)

If the primal has a unique optimal solution x^*, then $\nabla V(c)=x^*$

holds for { ① Standard form ② with inequality constraints ($Ax\leq b$)

a way to judge uniqueness:

"In simplex tableau, $\bar{c}_i=0, \forall i\in B$; $\bar{c}_j>0, \forall j\in N$

Local sensitivity with changes:

Under the conditions above, with SMALL change (in b/c)

the change of objective value $\Delta V=\Delta b^T y^*$ or $\Delta c^T x^*$

Note: under inactive constraints, ΔV can be zero (because of $y_i^*=0$ / $x_i^*=0$)

☆ Global Sensitivity.

CQ: Under what scale of change can the optimal basis hold?

1) Change b: $b \to b+\Delta b$

(i) reduced costs: $\bar{c} = C^T - C_B^T A_B^{-1} A$ independent of b

(ii) Feasible? — $\tilde{x}_B = A_B^{-1}(b+\Delta b) = x_B^* + A_B^{-1}\Delta b$ $\begin{cases} \geq 0, & B \text{ — still opt. basis} \\ \text{otherwise,} & \text{restart.} \end{cases}$ $V(\tilde{b}) = V^* + \Delta b^T y^*$ $(y = (A_B^{-1})^T C_B)$

Real operation: write Δb as λe_i, with e_i having the i^{th} entry $= 1$. (some cases)

check whether $x_B^* + \lambda A_B^{-1} e_i \geq 0$

2) Change C: $C \to C+\Delta C$

(i) Feasible? — $\tilde{x}_B = A_B^{-1} b$ independent of C.

(ii) reduced costs: $\bar{c} = C^T - C_B^T A_B^{-1} A = \begin{cases} 0, & \text{for } B \\ C_N^T - C_B^T A_B^{-1} A_N, & \text{for } N \end{cases}$

$$\Rightarrow \begin{cases} \Delta C = \lambda e_j \text{ (same as above)} \\ j \in B: \ \bar{C} = C_B^T - C_N^T A_B^{-1} A_N - \lambda e_j^T A_B^{-1} A_N = r_N^T - \lambda e_j^T A_B^{-1} A_N \begin{cases} \geq 0, & \text{still opt. basis} \\ \text{otherwise,} & \text{restart} \end{cases} \\ j \in N: \ \bar{C} = C_B^T + \lambda e_j^T - C_B^T A_B^{-1} A_N = r_N^T + \lambda e_j^T \begin{cases} \geq 0, & \text{still opt. basis} \\ \text{otherwise,} & \text{restart} \end{cases} \end{cases}$$

(Note: sign of λ will change for "max" prob.)

What if the change outside the range?

change C — reduced cost renew

change b — first transfer to the dual — reduced cost for the dual renew

3) Change A:

- change in A_j, $j \in N$, then compute $\bar{c}_j$ $\begin{cases} \geq 0, & \text{optimal solution still} \\ \text{otherwise, update for the } j\text{th column} \end{cases}$
- change in A_j, $j \in B$, then $x_B = A_B^{-1} b$ changes, no simple way.

- Some additions:

1) proof of "extreme pts $\Leftrightarrow$ BFS":

BFS $\Longrightarrow$ extreme points (argue by contradicts & definition)

extreme pts $\Longrightarrow$ BFS (argue by contradicts & construction)

↳ If not BFS, then at least more than n entries > 0 (otherwise, BFS)

By the definition of linear (in)dependence, $\exists\, \alpha \neq 0$ s.t. $\alpha_1 A_{B(1)} + \cdots + \alpha_k A_{B(k)} = 0$

Construct two feasible pt. $x^- = x_B^* - \varepsilon\alpha$ & $x^+ = x_B^* + \varepsilon\alpha$ (with small enough $\varepsilon > 0$)
(basis part)

then $x^* = \frac{1}{2}x^- + \frac{1}{2}x^+$, contradictory with extreme points.

2) proof of "the fundamental LP theorem":

(a kind of induction method)

(i) a solution/feasible → at least a BFS: every time reduce at least one entry to zero.

with linear (in)dependence, $\exists \alpha \neq 0$, s.t. $\sum_{i=1}^{k} \alpha_i A_{B(i)} = 0$ with at least 1 $\alpha_i > 0$

construct $\tilde{x}_B = x_B - \varepsilon\alpha$, with $\varepsilon = \min\left\{\alpha_i > 0 \,\middle|\, \frac{x_{B(i)}}{\alpha_i}\right\}$ until it's a BFS.

(ii) an optimal solution → an optimal solution also BFS: same as part (i)

the same α ($* c_B^T \alpha_B$ must be zero, otherwise, either $x_B + \varepsilon\alpha$ or $x_B - \varepsilon\alpha$ with $\varepsilon > 0$ is better.) Repeat the step until it's a BFS.
($\alpha_B = \alpha$)

3) small techs when using simplex tableau

whether in two-phase method or slacks cases,

A_B^{-1} can be gotten by $A_B^{-1} R = M$ ← in simplex tableau

M: diagonal matrix which can be chosen from original A

$A_B^{-1} = M D^{-1}$

Lecture 9 The Interior Point Method

- The uniform expression of LP algorithms

Optimal conditions for LP
1) Primal feasibility
2) Dual feasibility
3) Complementarity conditions

$\Rightarrow$ Simplex Method
Maintain 1) & 3), find 2) when optimal
($x_i(c_i - A_i^T y_i)$, $y = (A_B^{-1})^T c_B$)
($(A_B^{-1})^T c_B$)

Dual-Simplex Method
Maintain 2) & 3), find 1) when optimal

The Interior-point Method: Maintain 1) & 2), find 3) when optimal
(characteristics: start from the interior point of the feasible set)

- IDEA & Properties of Interior Point Method

High-level idea: find x, y

$$\begin{cases} Ax = b,\ x \geq 0 \\ A^T y + s = c,\ s \geq 0 \\ x_i s_i \leq \mu_i,\ \forall i \end{cases}$$

$\mu \geq 0$ — complementarity gap
keep decreasing μ — a solution for LP

Complexity $\approx O(n^{3.5})$

Several variants (primal-dual for this class)

☆ Theorem (Quality of Solutions)
The interior point method will always find the opt. solution with the maximum possible # non-zeros. (high-rank solutions)

Software choice: MATLAB — linprog to choose method
CVX — interior pt. method
Excel — simplex method

- Tutorial — a theorem of alternative

(Gordan's Theorem) (a) $Ax > 0$ & (b) $A^T y = 0,\ y \geq 0$
exactly one is feasible. ($a \geqq b \Leftrightarrow a \geq b$ but $a \neq b$)

$Ax>0$ feasible $\Leftrightarrow$ $Ax\geq 1$ feasible (*Note: $Ax>0 \not\Rightarrow Ax\geq 1$)
(when $Ax=s$, $A\frac{x}{s}=1$)

(P) min 0^Tx | (D) max $\mathbb{1}^Ty$
subject to $Ax\geq 1$ | subject to $A^Ty=0$
$(y\neq 0)$ → Change this constraint to maintain feasibility
or transform to $y\geq 0$ ← use this for the duality

(P) feasible $\Rightarrow$ (D) feasible with $\mathbb{1}^Ty^*=0 \Rightarrow y^*=0 \Rightarrow A^Ty=0,\ y\geq 0$ infeasible
$Ax>0$ feasible

$Ax>0$ infeasible $\Rightarrow$ (P) infeasible $\xrightarrow{\text{by strong/weak duality}}$ (D) unbounded $\Rightarrow y\geq 0,\ A^Ty=0$ feasible
But (D) feasible ($y=0$)

(Same techs can be used for "Farka's Lemma"(a) $Ax=b,\ x\geq 0$; (b) $A^Ty\geq 0,\ b^Ty<0$)
exactly one holds

- Techniques: (more exercise)

① Use duality theory to transform a robust LP to a common one.

maximize c^Tx
subject to $a^Tx\leq b+u^Tx,\ \forall u\in\{u\mid Au\leq w\}$
$x\geq 0$

"tech":
for u^Tx, we have $a^Tx\leq b+$ minimize$_u$ $u^Tx=x^Tu$ $\{u\mid Au\leq w\}$ → duality theory → maximize w^Ty subject to $A^Ty=x$, $y\leq 0$
parts of the question

Answer
$\Rightarrow$ maximize$_{x,y}$ c^Tx
subject to $a^Tx\leq b+w^Ty$
$A^Ty=x$
$x,y\geq 0$

② (Duality in Chebyshev approximation)

Consider minimizing $\|Ax-b\|_\infty$ over all $x\in\mathbb{R}^n$, let v be the value of optimal cost.

(a) Let p be any vector satisfying $\sum_{i=1}^m|p_i|\leq 1$ & $p^TA=0^T$. Show that $p^Tb\leq v$

(b) Show that the optimal cost in this problem is v.

maximize p^Tb
subject to $p^TA=0^T$
$\sum_{i=1}^m|p_i|\leq 1$

Sol. (a) $-v\mathbb{1}\leq Ax^*-b\leq v\mathbb{1}$, therefore $p^T(Ax^*-b)\leq v\sum_{i=1}^m|p_i|$
& $p^T(Ax^*-b)\geq -v\sum_{i=1}^m|p_i|$ ($x^*\Rightarrow$ the optimal solution)
$\therefore p^Tb\leq v\sum_{i=1}^m|p_i|\leq v$

(b) ☆ By observation, we can suppose that the orginal LP & the LP in (b) are primal-dual.

(check the assumption/hypothesis = (a) – weak duality, min/max relationship; strong duality "=")

proof: the primal $\underset{x\in\mathbb{R}^n}{\text{minimize}} \|Ax-b\|_\infty$

$\Leftrightarrow$ $\underset{x,t}{\text{minimize}}\ t$ subject to $Ax-b \geq -t\mathbb{1}$, $Ax-b \leq t\mathbb{1}$, $t\geq 0$ (see tutorial-1) $\xrightarrow{\text{transform}}$ $\underset{}{\text{minimize}} \begin{bmatrix}0^T & 1\end{bmatrix}\begin{bmatrix}x\\ t\end{bmatrix}$ subject to $\begin{bmatrix}A & \mathbb{1}\end{bmatrix}\begin{bmatrix}x\\ t\end{bmatrix} \geq b$ → "$\geq$", $\begin{bmatrix}A & -\mathbb{1}\end{bmatrix}\begin{bmatrix}x\\ t\end{bmatrix} \leq b$ → "$\leq$", x free, $t\geq 0$ → "$=$"

Take the dual

$\Rightarrow$ $\underset{}{\text{maximize}} \begin{bmatrix}b^T & b^T\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}$ subject to $\begin{bmatrix}A^T & A^T\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix} = 0$, $\begin{bmatrix}\mathbb{1}^T & -\mathbb{1}^T\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix} \leq 1$, $y\geq 0,\ z\leq 0$

(|p| → every entry in abs(p) in MATLAB)

☆ substitution (1): $y = \frac{1}{2}(|p|+p) \geq 0$, $z = \frac{1}{2}(-|p|+p) \leq 0$

$\Rightarrow$ $\underset{}{\text{maximize}}\ p^Tb$ subject to $A^Tp = 0$ (i.e. $p^TA = 0^T$), $\sum_{i=1}^m |p_i| \leq 1$

Lecture 10 NLP – Intro

- Background Knowledge:

Gradient: $f(x): \mathbb{R}^n \to \mathbb{R}^1$ (x is an $n\times 1$ column)

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)$ – an $n\times 1$ column vector

First-order Taylor Expansion:

$f(x+td) = f(x) + t\nabla f(x)^T d + o(t)$ ($t\to 0$)

($x, d\in\mathbb{R}^n,\ t\in\mathbb{R}$)

(compared with when $x\in\mathbb{R}^2$)

Hessian Matrix of f (when f is twice partial differentiable):

$\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij}$ (an $n\times n$ matrix with $a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$)

Second-order Taylor Expansion:

$f(x+td) = f(x) + t\nabla f(x)^T d + \frac{1}{2}t^2 d^T \nabla^2 f(x) d + o(t^2)$ ($t\to 0$)

($x, d\in\mathbb{R}^n,\ t\in\mathbb{R}$)

• Optimality Conditions for Unconstrained NLP

☆ First-Order Necessary Conditions (FONC)

If x^* is a local minimizer of the unconstrained problem $\min_{x\in\mathbb{R}^n} f(x)$, then we must have $\nabla f(x^*)=0$ ← critical point / stationary point

Proof. Consider the First-order Taylor Expansion

$$f(x+td) = f(x) + t\nabla f(\tilde{x})d + o(t),\ t\to 0$$

Suppose $\nabla f(x)\neq 0$, then let $d=-\nabla f(x)$

$$f(x+td) = f(x) - t\left(\|\nabla f(x)\|^2 - \frac{o(t)}{t}\right),\ t\to 0$$

so $\lim_{t\to 0} f(x+td) = \lim_{t\to 0} f(x) - t\left(\|\nabla f(x)\|^2 - \frac{o(t)}{t}\right) \leq f(x)$ contradictory!

☆ Example. Least-Square approximation (how to get $A^TA\hat{x}=A^Tb$ in linear algebra)

Given m points $\{(x_{i1}, x_{i2}, \dots, x_{in}, y_i)\}$ $i=1,2,\dots m$

$(m>n)$ want $\beta\in\mathbb{R}^n$ such that $y\approx X\beta$ (with Least-Square)

$$\text{minimize}_{\beta} \sum_{i=1}^{m}\left(y_i - \sum_{j=1}^{n}\beta_j x_{ij}\right) = \|y-X\beta\|^2$$

$$= y^Ty - 2y^TX\beta + X\beta\beta^TX^T = \beta^TX^TX\beta$$

Then, by FONC, we have $\nabla f(\beta) = -2X^Ty + 2X^TX\beta = 0$

(if $f(x)=c^Tx$, $\nabla f(x)=c$; if $f(x)=x^TMx$, $\nabla f(x) = 2Mx = 2M^Tx$)

$\Rightarrow X^TX\beta = X^Ty$ (Least-square fitting with line) (candidates of local minimizer)

☆ Second-Order Necessary Conditions (SONC)

If x^* is a local minimizer of f, then it holds that:

(1) $\nabla f(x^*)=0$; (2) For all $d\in\mathbb{R}^n$: $d^T\nabla^2 f(x^*)d\geq 0$ (or $\nabla^2 f(x^*)$ is PSD)

(Semidefiniteness: We call a (symmetric) matrix A positive/negative semidefinite (PSD/NSD) iff $\forall x$, we have $x^TAx\geq 0 (\leq 0)$)

Proof: Consider the Second-order Taylor Expansion

$$f(x+td) = f(x) + t\nabla f(\tilde{x})d + \frac{1}{2}t^2 d^T\nabla^2 f(x)d + o(t^2),\ t\to 0$$

$$\overset{\text{FONC}}{=} f(x) + \frac{1}{2}t^2 d^T\nabla^2 f(x)d + o(t^2)$$

$$= f(x) + \frac{t^2}{2}d^T\nabla^2 f(x)d + o\left(\frac{t^2}{t^2}\right),\ t\to 0$$

If $d^T\nabla^2 f(x)d \geq 0$ does not hold, (i.e.) $d^T\nabla^2 f(x)d < 0$,

we have $f(x+td) = f(x) + \frac{1}{2}t^2\left(d^T\nabla^2 f(x)d + \underbrace{\frac{o(t^2)}{t^2}}_{\to 0}\right) \leq f(x)$

contradictory!

Additions.

☆ Positive Semidefinite Matrices. (PSD)

1) If a matrix A is not symmetric, use $\frac{1}{2}(A+A^T)$ to define.

(Because $x^TAx = \frac{1}{2}x^TAx + \frac{1}{2}\underbrace{x^TA^Tx}_{=x^T(x^TA^T)^T=x^TAx} = \frac{1}{2}x^T(A+A^T)x$)

2) A symmetric matrix is PSD iff all the eigenvalues are non-negative.

3) For any matrix A, A^TA is PSD. (Because $x^TA^TAx = \|Ax\|^2 \geq 0 \ \forall x$)

(If $f(x)=x^TMx$ with symmetric M, then $\nabla^2 f(x) = 2M$.)

☆ Second-order Sufficient Conditions (SOSC)

If x^* satisfies: (with twice partial differential function f)

(1) $\nabla f(x^*)=0$; (2) For all $d \in \mathbb{R}^n \setminus \{0\}$: $d^T\nabla^2 f(x^*)d > 0$ (or $\nabla^2 f(x^*)$ is PD)

(Definite matrices:
We call a symmetric matrix A positive/negative definite (PD/ND)
iff $\forall x \neq 0$: $x^TAx > 0 / < 0$)

Same as above: A symmetric matrix is PD. $\Leftrightarrow$ all its eigenvalues positive

Proof. Lemma (Rayleigh Quotient)

Let $A \in \mathbb{R}^{n\times n}$ & symmetric, then

$$\lambda_{min}(A)\|x\|^2 \leq x^TAx \leq \lambda_{max}(A)\|x\|^2, \quad \forall x \in \mathbb{R}^n$$

(λ_{min}: the smallest eigenvalue of A; λ_{max}: the largest eigenvalue of A)

By Taylor's Expansion

$$f(x^*+h) = f(x^*) + \frac{1}{2}h^T\nabla^2 f(x^*)h + o(\|h\|^2) \quad h \to 0$$

$$\Rightarrow f(x^*+h) \geq f(x^*) + \frac{1}{2}\underbrace{\mu\|h\|^2}_{h^T\nabla^2 f(x^*)h \geq \mu\|h\|^2} + o(\|h\|^2)$$

(μ: smallest EV)

$\Rightarrow f(x^*+h) \geq f(x^*) + \|h\|^2\left(\frac{\mu}{2} + \frac{o(\|h\|^2)}{\|h\|^2}\right)$, $\|h\| \to 0$

So $f(x^*+h) > f(x^*)$ (Because $\lim_{\|h\|\to 0} \frac{o(\|h\|^2)}{\|h\|^2} = 0$)

☆ For maximization:

FONC → $\nabla f(x^*) = 0$

SONC → NSD → all EVs ≤ 0

SOSC → ND → all EVs < 0

If $\nabla^2 f(x^*)$ is indefinite, with $\nabla f(x^*) = 0$, x^* is a saddle point.

Lecture 11 Optimality Conditions for Constrained Problems

- Existence of Solutions

☆ (Weierstrass Theorem) (Extreme value)

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function & Let $\Omega \subset \mathbb{R}^n$ be bounded, closed & non-empty set. Then, f attains a global maximum and minimum on Ω.

Def: (1) closed: For every convergent sequence x_k with $x_k \in \Omega, \forall k$, then $\lim_{k\to\infty} x_k \in \Omega$ (condition: $\lim_{k\to\infty} x_k = x$ (convergent))

(2) bounded: There is a $B > 0$ with $\|x\| < B$, $\forall x \in \Omega$ (no ifinity ones)

(3) Compact (a closed & bounded set)

☆ Coercivity (for unconstrianed problems):

A continuous function $f: \mathbb{R}^n \to \mathbb{R}$ is coercive if

$$\lim_{\|x\|\to\infty} f(x) = +\infty$$

(eg. x^2, x^4, $|x|$ — coercive
x, e^x, x^3 — non-coercive)

(Trick — finding suitable lower bound for suffiently large x)

★ Theorem (coerciuty & existence of solutions)

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous & coercive function. Then, $\forall \alpha > 0$, the level set $L_{\leq\alpha} := \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ is compact & f has at least one global minimizer.

- Optimality Conditions for Constrained problems

Def.

(1) Feasible Direction

Given $x \in \Omega$, we call d a feasible direction at x ($d \in \mathbb{R}^n$), if $\exists\, \bar{t} > 0$ ($t \in \mathbb{R}$) s.t. $x + td \in \Omega$, $\forall\, 0 < t < \bar{t}$

(eg. (i) $\Omega = \{x \mid Ax = b\}$, then all d's: $\{d \mid Ad = 0\}$
(ii) $\Omega = \{x \mid Ax \geq b\}$, then all d's: $\{d \mid a_i^T d \geq 0 \ \forall i \text{ s.t. } a_i^T x = b\}$)

(inactive $a_i^T x > b$, sufficiently small $\bar{t}$ can satisfy)

(2) Descent Direction

Let f be continuously differentiable. Then, d is called a descent direction at x iff $\nabla f(x)^T d < 0$.

(If d is a descent direction at x, then $\exists\, \bar{r} > 0$, s.t. $f(x + rd) < f(x)$, $\forall\, 0 < r < \bar{r}$)

(by Taylor Expansion)

☆ FONC for Constrained problem

Let x^* be a local minimum of $\min_{x \in \Omega} f(x)$, then for any feasible direction d at x^*, we must have $\nabla f(x^*)^T d \geq 0$.

(Proof: $f(x^* + td) = f(x^*) + t\,\nabla f(x^*)^T d + o(t)$, $t \to 0$ ($t < \bar{t}$)
assume $\nabla f(x^*)^T d < 0$, then $f(x^* + td) < f(x^*)$ contradict)

With (1) & (2), FONC can be written as $S_\Omega(x^*) \cap S_0(x^*) = \emptyset$

(S_Ω: set of feasible directions; S_0: set of descent directions)

☆

- General Optimal Conditions

(General Non-linear optimization Problem) (Implicit open constraints can be omitted in Lagrangian function)

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & f(x) \\ \text{subject to } & g_i(x) \leq 0, \ \forall i = 1, 2, \cdots m \\ & h_j(x) = 0, \ \forall j = 1, 2, \cdots p \end{aligned}$$

The feasible set $\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}$

Def. **Active Set** at a point $x \in \Omega$, the set $A(x) :=$
$\{i \mid g_i(x) = 0\}$ denotes the set of active constrains.

Inactive set at a point $x \in \Omega$, the set $I(x) :=$
$\{i \mid g_i(x) < 0\}$ denotes the set of inactive constrains.

☆ **FONC for linearly constrained problem** $\left(\begin{array}{l} \min\limits_x f(x) \\ \text{subject to } Ax \geq b \end{array} \right)$

If x^* is a local minimum of the problem,
the $\exists$ some $y \in \mathbb{R}^m$ with $\boxed{y \geq 0;\ \nabla f(x^*) - A^T y = 0;\ y_i(a_i^T x^* - b_i) = 0\ \forall i}$
gradient of Lagrangian function

Proof.
$$\left(\begin{array}{l} \min\limits_d \nabla f(x^*)^T d \\ \text{subject to } a_i^T d \geq 0,\ \forall i \in A(x^*) \end{array} \right) \geq 0 \iff \text{FONC: } \begin{array}{l} \nabla f(x^*)^T d \geq 0 \\ a_i^T d \geq 0\ \forall i \in A(x^*) \end{array}$$

$$\Updownarrow$$

$$\left(\begin{array}{l} \min\limits_d \nabla f(x^*)^T d \\ \text{subject to } Cd \geq 0 \end{array} \right) \quad \text{with } C = \begin{bmatrix} -c_1^T- \\ -c_2^T- \\ \cdots \\ -c_n^T- \end{bmatrix}, \quad c_i^T = \begin{cases} a_i^T, & i \in A(x^*) \\ 0, & i \in I(x^*) \end{cases} \Bigg) \geq 0$$

Take the dual problem (D) $\max\ 0^T t = 0$
subject to $C^T t = \nabla f(x^*)$
$t \geq 0$

By weak duality
(P) $\geq 0 \iff$ (D) feasible, finite opt. 0
$\iff t \geq 0,\ A^T t = \nabla f(x^*),\ t_i(a_i^T x^* - b_i) = 0,\ \forall i$
complementarity, let $t_i = 0,\ \forall i \in I(x^*)$

★ Special case, change $Ax \geq b$ to $Ax = b$: (if x^* is a local minimum)
$\exists y \in \mathbb{R}^n$ with $A^T y = \nabla f(x^*)$

☆ Consider the non-linear program
$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \leq 0,\ \forall i = 1, 2, \cdots m \end{array}$$
(f, g continuous differentiable)

(for $h(x) = 0 \Leftrightarrow \begin{cases} h(x) \leq 0 \\ -h(x) \leq 0 \end{cases}$ substitute)

Lemma – No Feasible Descent for Inequality constraints
Let x^* be a local minimum of above, there does not exist a vector
$d \in \mathbb{R}^n$ s.t. $\nabla f(x^*)^T d < 0$ & $\nabla g_i(x^*)^T d < 0\ \forall i \in A(x^*)$

Why $\nabla g_i^T(x^*)d<0$ implies feasibility? ($i\in A(x^*)$)

Reason: Taylor's expansion $g_i(x^*+td)=g_i(x^*)+t\nabla g_i^T(x^*)d+o(t)_{t\to 0}$

$\nabla g_i^T(x^*)d<0$ indicates that $g_i(x^*+td)<g_i(x^*)=0$

Proof: assume $\exists d$ s.t. $\nabla f(x^*)^T d<0$, $\nabla g_i^T(x^*)d<0$, $\forall i\in A(x^*)$

$\Rightarrow f(x^*+td)<f(x^*)$ $t\to 0$ & $g_i(x^*+td)<g_i(x^*)=0$, $(t<\tilde{t})$

which means that x^*+td is a local minimum ($\exists$ small enough $\bar{t}$, $g_i(x^*+\bar{t}d)<0$, $i\in A(x^*)$)

contradictory!! ($f(x^*+td)<f(x^*)$, $t\in(0,\min\{\bar{t},\tilde{t}\})$

Lecture 12 KKT Conditions

- **Fritz-John Conditions** (FJ Conditions) — another form of FONC

Let x^* be a local minimum of $\left(\min_x f(x) \text{ subject to } g_i(x)\le 0, \forall i=1,\dots m\right)$, then $\exists\ \lambda_0, \lambda_1, \dots \lambda_m \ge 0$, and are not all zeros s.t. (λ: dual variables)

$$\lambda_0\nabla f(x^*)+\sum_{i=1}^m \lambda_i \nabla g_i(x^*)=0 \quad \text{(gradient of Lagrangian function)}$$

$$\lambda_i g_i(x^*)=0 \quad \forall i=1,2,\dots m \quad \text{(complementarity conditions)}$$

Simple Proof: By the lemma above,

$\nexists d$: $\nabla f(x^*)^T d<0$, $\nabla g_i^T(x^*)d<0$, $\forall i\in A(x^*)$

which means $\left(\min_{d,t} t \text{ subject to } \nabla f(x^*)^T d\le t,\ \nabla g_i(x^*)^T d\le t,\ \forall i\in A(x^*)\right)\ge 0$

Because t can be only chosen to be ≥ 0 to promise feasible of d.

$\Leftrightarrow \left(\min_{d,t} t \text{ subject to } \nabla f(x^*)^T d\le t,\ Cd\le t\mathbb{1}\right)\ge 0$

with $C=\begin{bmatrix}-c_1^T-\\ \dots\\ -c_m^T-\end{bmatrix}$, $c_i^T=\begin{cases}\nabla g_i(x^*)^T, & i\in A(x^*)\\ 0, & i\in I(x^*)\end{cases}$

Take the dual

$$\max_{\lambda_0,\lambda} 0 \quad \text{subject to} \quad \lambda_0\le 0,\ \lambda\le 0$$
$$\lambda_0\nabla f(x^*)+C^T\lambda=0$$
$$-\lambda_0-\mathbb{1}\lambda=1$$

By weak duality, the dual must be feasible.

use $-\lambda_0$, $-\lambda$ to substitute for λ_0, λ, respectively.

$$\begin{cases} \lambda_0 \geq 0, \lambda \geq 0 \iff \lambda_0, \lambda_1, \cdots \lambda_m \geq 0 \\ \lambda_0 \nabla f(x^*) + C^T \lambda = 0 \iff \begin{cases} \lambda_0 \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) = 0 \\ \lambda_i g_i(x^*) = 0, \ \forall i = 1, 2, \cdots m \ \text{(make } \lambda_i = 0, i \in \mathcal{Z}(x^*)) \end{cases} \\ \underbrace{\lambda_0 + \mathbb{1}\lambda = 1}_{\hookrightarrow \text{cannot be all zeros}} \iff \end{cases}$$

Defects of FJ conditions: λ_0 can be zero — may have too many pts not local minimum (when $\lambda_0 = 0$, no contacts with $\nabla f(x)$/$f(x)$.)

↓ make it more precise

- Karush-Kuhn-Tucker Conditions (KKT conditions)

  — another form of FONC

  Let x^* be a local minimum of $\left(\min_x f(x) \text{ subject to } g_i(x) \leq 0 \ \forall i = 1, \cdots m\right)$

  & suppose vectors $\{\nabla g_i(x^*) \mid i \in A(x^*)\}$ are linearly independent.

  Then, $\exists \lambda_1, \cdots \lambda_m \geq 0$ s.t. $\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) = 0$

  $\lambda_i g_i(x^*) = 0, \ \forall i = 1, \cdots, m$

  Proof. Use FJ conditions, $\lambda_0 \neq 0$ because of linear independence, then divide each side by λ_0.

- KKT Conditions (General Setting)

  Background: (1) general optimization problem form:

  $$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \ f(x) \\ &\text{subject to} \ g_i(x) \leq 0, \ \forall i = 1, 2, \cdots m \\ &\qquad\quad h_j(x) = 0, \ \forall j = 1, \cdots, p \end{aligned}$$

  (2) Lagrangian function

  $$L(x, \lambda, \mu) := f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^p \mu_j h_j(x)$$

  $$\left(f(x) + \lambda^T g(x) + \mu^T h(x)\right)$$

  (λ_i, μ_j — can be seen as dual variables)

☆ General KKT Conditions.

If x^* is a local minimizer & a constraint qualification holds, then $\exists\ \lambda$ & u s.t.

(1) Main Condition

$$\nabla L(x, \lambda, u) = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) = 0$$

(con x^*)

(2) Complementarity Conditions

$\lambda_i g_i(x^*) = 0,\ \forall i = 1, \cdots m$; $\mu_j h_j(x^*) = 0,\ \forall j = 1, \cdots p$

often omitted

(3) Primal Feasibility

$g_i(x^*) \leq 0,\ \forall i = 1, \cdots m$; $h_j(x^*) = 0,\ \forall j = 1, \cdots p$

(4) Dual Feasibility

$\lambda_i \geqslant 0,\ \forall i = 1, \cdots m$; μ_j free, $\forall j = 1, \cdots p$

often omitted

→ Constraint Qualifications.

1) Linearly Independent Constraint Qualification.

(LICQ) $\{\nabla g_i(x^*) \mid i \in A(x^*)\} \cup \{\nabla h_j(x^*) \mid j = 1, \cdots p\}$ are linearly independent

2) Others: ACQ, GCQ, MFCQ, PLICQ, Slater's condition.

→ Comments

(i) KKT is a kind of FONC, for general constrained optimization problems.

(ii) KKT point / stationary point = a point satisfies KKT conditions.

☆ Second-Order Conditions (KKT form) for constrained problem

1) SONC Let x^* be a regular point (satisfy LICQ) & local min, then

(1) The KKT conditions hold.

(2) $d^T \nabla^2_{xx} L(x^*, \lambda, \mu) d \geqslant 0,\ \forall d \in C(x^*)$

(PSD on the critical cone)

Comments:

(i) Hessian of the Lagrangian function

$$\nabla^2_{xx}\mathcal{L}(x^*,\lambda,\mu) = \nabla^2 f(x) + \sum_{i=1}^{m}\lambda_i \nabla^2 g_i(x) + \sum_{j=1}^{p}\mu_j \nabla^2 h_j(x)$$

(ii) Critical cone ($C(x^*)$):

$$C(x) := \{ d\in\mathbb{R}^n \mid \nabla f(x)^T d = 0;\ \nabla g_i(x)^T d \le 0,\ \forall i\in A(x);\ \nabla h_j(x)^T d = 0,\ \forall j \}$$

2) SOSC: Let x^* be a KKT point with multiplier λ, μ

(i.e.) (1) KKT conditions work.

(2) $d^T \nabla^2_{xx}\mathcal{L}(x^*,\lambda,\mu) d > 0,\ \forall d\in C(x^*)\setminus\{0\}$

(PD on the critical cone)

Then, x^* is a strict local minimizer

Lecture 13 Convexity

- Definition (convex & concave)

(1) Convex Function

A convex function f on a convex set Ω satisfies:

for every $x_1, x_2 \in \Omega$, and any $\lambda\in[0,1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

(2) Concave Function

A concave function f on a convex set Ω satisfies:

($-f$ is convex) or $f(\lambda x_1 + (1-\lambda)x_2) \ge \lambda f(x_1) + (1-\lambda) f(x_2)$

- Test: ① Convexity via Hessian / Second-order condition

Let f be twice differentiable, then f is convex (on $\mathbb{R}^n$) iff its Hessian Matrix is PSD (i.e.) $d^T\nabla^2 f(x) d \ge 0,\ \forall d\in\mathbb{R}^n,\ x\in\mathbb{R}^n$

(PD – strictly convex)

② First-order condition

If $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable, then f is convex on its convex domf iff

$$f(y) \ge f(x) + \nabla f(x)^T (y-x),\ \forall x, y \in \text{domf}$$

(> – strictly convex)

Proof: ② suppose $f(x)$ is convex,

$$f(\theta y+(1-\theta)x)\le \theta f(y)+(1-\theta)f(x),\quad \forall \theta\in[0,1]$$

$$\Leftrightarrow f(x+\theta(y-x))-f(x)\le \theta(f(y)-f(x))$$

$$\Rightarrow \lim_{\theta\to 0}\frac{f(x+\theta(y-x))-f(x)}{\theta}\le f(y)-f(x)$$

$\lim_{\theta\to 0}\frac{f(x+\theta(y-x))-f(x)}{\theta}$ $\xrightarrow{\text{definition}}$ $\nabla f^T(x)(y-x)$

Suppose first-order condition satisfied,

Let $z=\theta x+(1-\theta)y$, then $f(x)\ge f(z)+\nabla f^T(z)(x-z)$

$\forall\theta\in[0,1]$ $\quad f(y)\ge f(z)+\nabla f^T(z)(y-z)$

$\Rightarrow \theta f(x)+(1-\theta)f(y)\ge f(z)$ (sum)

① Suppose $f(x)$ is convex, by Taylor expansion

$$f(x+td)=f(x)+t\nabla f^T(x)d+\frac{1}{2}t^2d^T\nabla^2f(x)d+o(t^2)\quad t\to 0$$

By First-order condition $f(x+td)\ge f(x)+\nabla f^T(x)(td)$

$$\Rightarrow \frac{1}{2}t^2\left(d^T\nabla^2f(x)d+\frac{o(t^2)}{t^2}\right)\Big|_{t\to 0}\ge 0\Rightarrow d^T\nabla^2f(x)d\ge 0\quad(\nabla^2f(x)\succeq 0)$$

Suppose $\nabla^2f(x)\succeq 0$, then for any $x,y\in \text{dom}f$

$$f(y)=f(x)+\nabla f^T(x)(y-x)+\frac{1}{2}(y-x)^T\nabla^2f(x)(y-x)+o(\|y-x\|^2)$$

(Taylor's theorem) $\qquad \frac{1}{2}(y-x)^T\nabla^2f(z)(y-x)$

z between x,y $(z=\theta x+(1-\theta)y)$

$\Rightarrow f(y)\ge f(x)+\nabla f^T(x)(y-x)$, by First-order condition

- Operations that perseve convexity

1) Nonnegative weighted sums. If $a_1,\cdots,a_m\ge 0$, $f_1,\cdots,f_m$ are convex functions the $\sum_{i=1}^m a_if_i$ is still convex.

2) Composition with an affine mapping. Suppose $f:\mathbb{R}^m\to\mathbb{R}$ is convex, $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$ given, define $g:\mathbb{R}^n\to\mathbb{R}$, $g(x)=f(Ax+b)$ is convex.

3) Positive maximum & supremum: (piece-wise-convex)

If $f_1, \cdots, f_m$ are convex functions, then $f(x) = \max\{f_1(x), \cdots, f_m(x)\}$ is still convex. (Can be extended to infitely/uncountably many)

- Convexity & Optimal

(1) Local & Global minimize:

Let $f: \Omega \to \mathbb{R}$ be a convex function & $\Omega \subset \mathbb{R}^n$ be a convex set. Then any local minimizer of problem $\begin{pmatrix} \min\limits_x f(x) \\ \text{subject to } x \in \Omega \end{pmatrix}$ is also a global minimizer.

Proof: suppose x is not a global minimizer. $\exists\, y$ s.t. $f_0(y) < f_0(x)$, $\|y-x\|_2 > R$ (y feasible)

(because $f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z-x\|_2 \le R\}$

Construct $z = (1-\theta)x + \theta y$, $\theta = \frac{R}{2\|y-x\|_2}$, then $z = x + \theta(y-x)$

$\|z-x\|_2 = \theta\|y-x\|_2 = \frac{R}{2} < R$. by convexity of f_0

$f_0(z) \le (1-\theta) f_0(x) + \theta f_0(y) < f_0(x)$ contradict!

(2) Stationary & Global minimizer:

Let f be the convex & suppose $\Omega := \{x \mid g(x) \le 0,\ h(x) = 0\}$ is a convex set

Then the KKT conditions for $\begin{pmatrix} \min\limits_x f(x) \\ \text{subject to } x \in \Omega \end{pmatrix}$ are sufficient for global optimality (not necessary) ↳ because of LICQ

Proof: Firstly, consider the reduced case:

(Let $S \subset \mathbb{R}^n$ be an open convex set. Suppose that $f: \mathbb{R}^n \to \mathbb{R}$ is a convex on S & continuously differentiable at $\bar{x}$, then $\bar{x}$ is a global min iff $\nabla f(\bar{x}) = 0$)

⇒ proof of this proposition: (i) sufficient part, $\bar{x}$ is a global min ⇒ $\bar{x}$ is a local min

⇒ $\bar{x}$ satisfies FONC ⇒ $\nabla f(\bar{x}) = 0$

(ii) necessary part, by convexity & First-order conditions

$f(y) \ge f(\bar{x}) + \nabla f(\bar{x})(y-\bar{x}) = f(\bar{x})$, $\forall\, y$

Then, use the proposition

$$f(\bar{x}) \underset{\text{complementarity}}{=} f(\bar{x}) + \sum_{i=1}^m \lambda_i g_i(\bar{x}) + \sum_{j=1}^p \mu_j h_j(\bar{x}) \underset{\text{proposition}}{=} \min_x \{ f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^p \mu_j h_j(x) \}$$

$$(L(x, \bar{\lambda}, \bar{\mu}))$$

$$\le \min_{\substack{g_i(x) \le 0 \\ h_j(x) = 0}} \{ f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^p \mu_j h_j(x) \} \le \min_{x \in \Omega} f(x)$$

↓ local/global relation

(3) A Lemma — how to recognize some convex sets?

Let f be a convex/concave function, then for any c, the level set $L_{\le c} = \{x \mid f(x) \le c\}$ / $L_{\ge c} = \{x \mid f(x) \ge c\}$ is a convex set.

(Tricks: transformations of monotone / variable substitutions)

Lecture 14 Algorithms for Unconstrained Problems

- General ideas of algorithms:

The nature of opinization algorithms is iterative procedures.

{ starting from some point x_0, generate a sequence $\{x_k\}$
terminates when attach / satisfactory. every step $f(x_{k+1}) < f(x_k)$
hopefully, $\{x_k\}$ converges to a local min x^*

Def. (convergence). Let $\{x_k\}$ be a sequence of real vectors. Then $\{x_k\}$ converges to x^* iff $\forall \varepsilon > 0$, $\exists$ a pos. integer K s.t. $\|x_k - x^*\| < \varepsilon$, $\forall k \ge K$

- Algorithm for single variable problems:

☆ Bisection Method (binary search) — $g(x) = f'(x)$, then, need x^* s.t $g(x^*) = 0$

Start from x_l, x_r s.t. $g(x_l) < 0$ & $g(x_r) > 0$

(1) Define $x_m = \frac{1}{2}(x_l + x_r)$

(2) If $g(x_m) = 0$, output x_m

(3) Otherwise { $g(x_m) > 0$, let $x_r = x_m$; $g(x_m) < 0$ let $x_l = x_m$

(4) If $(x_r - x_l) < \varepsilon$, stop & output $\frac{1}{2}(x_r + x_l)$, otherwise iterate.

usage — find approx. stationary / critical points.

☆ Golden Section Method — do not need to use $f'(x)$

(unimodal — f has only one critical point)

Start from $[x_l, x_r]$ (suppose min in $[x_l, x_r]$). $\Phi \in (0, 0.5)$

(1) Set $x_l' = \Phi x_r + (1-\Phi) x_l$, $x_r' = (1-\Phi) x_r + \Phi x_l$ ($x_r' > x_l'$)

(2) { If $f(x_l') < f(x_r')$, which means $[x_l, x_r] \to [x_l, x_r']$ (condensed) $\Rightarrow$ let $x_r = x_r'$
If $f(x_l') \ge f(x_r')$, which means $[x_l, x_r] \to [x_l', x_r] \Rightarrow$ let $x_l = x_l'$

(3) If $x_r - x_l < \varepsilon$, output $\frac{1}{2}(x_l + x_r)$, otherwise iterate.

Q — why golden section? / what is Φ?

Suppose we update $x_r = x_{r'}$. we want x_e = new $x_{r'}$ for less calculation

$\Rightarrow \Phi x_r + (1-\Phi)x_e = \Phi x_e + (1-\Phi)\left((1-\Phi)x_r + \Phi x_e\right)$

$\begin{cases} (1-\Phi)^2 = \Phi \\ \Phi(2-\Phi) = 1-\Phi \end{cases} \Rightarrow \Phi^2 - 3\Phi + 1 = 0 \Rightarrow \boxed{\Phi = \frac{3-\sqrt{5}}{2}} \ (\Phi < 0.5)$

$\Rightarrow 1-\Phi = \frac{\sqrt{5}-1}{2} \approx 0.618$ golden section!

- High-dimensional problems — algorithms — Descent Methods

(i) Gradient descent Method:

⇒IDEA: find search direction → find step size & stopping criterion

aim $x_{k+1} = x_k + \alpha_k d_k$, $f(x_{k+1}) < f(x_k)$ — main idea for all descent methods

Descent Direction — $d \in \mathbb{R}^n$ s.t. $\nabla f(x)^T d < 0$ (to lower the function values) (gradient, steepest, Newtons)

⟹ to be definite, choose $-\nabla f(x)$ to be d, $\nabla f(x)^T(-\nabla f(x)) = -\|\nabla f(x)\|^2 \le 0$

⇒ (1) Initialization — select an initial point $x^0 \in \mathbb{R}^n$

(2) Pick a step size (constant / exact line search / backtracking) on the function

$\Phi(\alpha) = f(x_k - \alpha \nabla f(x_k))$, get α_k

(3) Set $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$, when $\|\nabla f(x^{k+1})\| < \varepsilon$ (tolerance) STOP. stopping criterion

otherwise iterate.

☆ Choose step size α_k:

WAY I: $\alpha_k = \bar{\alpha}$ (commonly used for the sake of convenience)

WAY II: EXACT LINE SEARCH choose α_k s.t $\alpha_k = \underset{\alpha \ge 0}{\operatorname{argmin}} f(x_k + \alpha d_k)$ → output argument which minimizes f

(because $\underset{\alpha \ge 0}{\operatorname{argmin}} f(x_k + \alpha d_k) := g(\alpha)$ is a single variable function)

⇒ { analytically get α / use golden section method

WAY III: BACKTRACKING / Armijo LINE SEARCH

Let $\alpha, \gamma \in (0,1)$, choose α_k in $\{1, \alpha, \alpha^2, \cdots\}$ (as the largest element) s.t $f(x_k + \alpha_k d_k) - f(x_k) \le \gamma \alpha_k \nabla f(x_k) d_k$ ← Armijo condition

$\alpha_k \to$ finitely many step, starting from 1.

"Feasibility": define $\Phi_k(\alpha) = f(x_k + \alpha d_k) - f(x_k) \Rightarrow \Phi_k'(\alpha) = \nabla f(x_k) d_k$

Armijo condition $\Leftrightarrow \Phi_k(\alpha) \le r\alpha \cdot \Phi_k'(0) \Leftrightarrow \frac{\Phi_k(\alpha)}{\alpha} \le r\Phi_k'(0)$, by MVT, $\exists \alpha \in (0,1)$

☆ Convergence & properties of GD Method:

Global Convergence: Fixed critical points independent of the chosen initial point

Accumulation Point: a point x is an accumulation point of $\{x_k\}$ if for every $\varepsilon > 0$, $\exists$ infinitely many numbers k with $x_k \in B_\varepsilon(x)$

Comments: 1) If x is an accumulation point of $\{x_k\}$, $\exists$ a subsequence $\{x_{k_l}\}$ that converges to x.

2) If $\{x_k\}$ converges to some $x \in \mathbb{R}^n$, x is the only unique accumulation point of $\{x_k\}$.

3) A bounded sequence always possess at least an accumulation point.

☆ Theorem: Global Convergence

Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, & let x_k be generated by gradient method for solving $\min_{x \in \mathbb{R}^n} f(x)$ (unconstrained) with exact / backtracking linear search. Then $f(x_k)$ is nonincreasing & every accumulation point of x_k is a stationary point of f.

Lipschiz Continuity: ∇f is Lipschiz Continuous over $\mathbb{R}^n$ means

$$\|\nabla f(x) - \nabla f(y)\| \le L \|x - y\|, \quad \forall (x, y) \in \mathbb{R}^n$$

where $L > 0$ called Lipschiz constant. The class of functions with Lipschiz gradient with constant L is denoted by $C_L^{(1)}(\mathbb{R}^n)$ ((1): continuously differentiable; L: Lipschiz constant) → Lipschiz continuous

(Comment: If f is twice differentiable, $f \in C_L^{(1)}(\mathbb{R}^n) \iff \|\nabla^2 f(x)\| \le L, \ \forall x \in \mathbb{R}^n$

spectral norm $= \sqrt{\lambda_{\max}(A^T A)}$ (largest singular value))

Linear Convergence: we say that x_k converges linearly with rate $\eta \in (0,1)$ to $x^* \in \mathbb{R}^n$, if $\exists \ell > 0$ s.t. $\|x_{k+1} - x^*\| \le \eta \|x_k - x^*\|, \ \forall k \ge \ell$

Why "linear"? → rate/order of convergence → { A sequence $\{x_n\}$ is said to have order of convergence $q \ge 1$ & rate of convergence μ if $\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^q} = \mu$

☆ Theorem: Rates for convex problems

Let $f \in C_L^{1,1}$, & suppose $\exists \mu > 0$ s.t. $\mu\|d\|^2 \le d^T \nabla f(x)\, d \quad (\forall d, \forall x)$ (strong convexity)

Let x_k be generated by the gradient method & let x^* be the solution of $\min_x f(x)$

Then x_k converges linearly to x^* with rate $\eta = 1 - \frac{M\mu}{2}$ (with $M = \begin{cases} \bar{\alpha}(1 - \frac{L\bar{\alpha}}{2}), & \text{constant stepsize } \bar{\alpha} \in (0, \frac{2}{L}) \\ \frac{1}{2L}, & \text{exact line search} \\ \gamma \min\{1, \frac{2\delta(1-\gamma)}{L}\}, & \text{backtracking} \end{cases}$)

it follows $f(x_k) - f(x^*) \le \eta^k (f(x_0) - f(x^*))$

$\|\nabla f(x_k)\| \le \sqrt{\frac{L}{\mu}}\eta^k \|\nabla f(x_0)\|$, $\|x_k - x^*\| \le \sqrt{\frac{L}{\mu}}\eta^k \|x^0 - x^*\|$

☆More properties: (GDMethod)

When using exact line search, the directions between the consecutive steps are perpendicular. (i.e.) $(d^{k+1})^T d_k = 0$ (proof: FONC $\nabla f(x_k + \alpha d_k)^T d_k = 0$, $\nabla f(x_k + \alpha d_k) = -d_{k+1}$)

(ii) Newton's Method

Features: { converge much faster than the gradient descent method.
required second-order informative
more sensitive to the initial point

① in $\mathbb{R}$: Critical point x^* means $g(x^*) = 0$ (with $g'(x) = f(x)$)

$\Rightarrow x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} = x_k - \frac{f'(x_k)}{f''(x_k)}$ a kind of descent method

(Given $f''(x) = g'(x) \neq 0$ at each step)

☆Theorem: Convergence of Newton's Method (one dimension)

If g is twice cont. differentiable & x^* is the root of g where $g'(x^*) = 0$

provided that $|x_0 - x^*|$ is sufficiently small, the sequence generated by Newton's method will satisfy $|x_{k+1} - x^*| \le C|x_k - x^*|^2$ (quadratic convergence)

with $C = \sup_x \frac{1}{2}\left|\frac{g''(x)}{g'(x)}\right|$ (converges much faster than gradient method)

Another intepretation of Newton's Method:

$f(x) \approx f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}f''(x_k)(x - x_k)^2$. (estimate by quadratic functions)

& get the argmin $\rightarrow x_{k+1} = x^k - \frac{f'(x_k)}{2 \cdot \frac{1}{2} f''(x_k)}$ (suppose $f''(x_k) > 0$)

② in $\mathbb{R}^n$: similar to $\mathbb{R}$ $x_{k+1} = x_k - \left(\nabla^2 f(x_k)\right)^{-1} \nabla f(x_k)$

search direction / Newton direction

When f is convex / $\nabla^2 f$ is pos. def, $-\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x) < 0$ is the search descent direction

How to ensure Newton's Method's convergence with any initial point?

backtracking line search — use α_k as a step size.

⇒ Complete steps of Newton's Method

(1) Initialization: select an initial point $x_0 \in \mathbb{R}^n$

(2) Compute the Newton's direction d_k which satisfies $\nabla^2 f(x_k) d_k = -\nabla f(x_k)$

(3) Choose a stepsize α_k by line search, calculating $x_{k+1} = x_k + \alpha_k d_k$

(4) If $\|\nabla f(x_{k+1})\| \leq \varepsilon$, STOP.

☆ Theorem: Convergence of Newton's Method (high/n dimension)

Let f be twice cont. diff. & x^* be the local minimizer of f. For some given $\varepsilon > 0$, assume that (i) f is strongly convex with constant m (i.e. $\nabla^2 f(x) \succeq mI$)

both on $B_\varepsilon(x^*)$: (ii) $\nabla^2 f$ is Lipschitz continuous for all $x, y \in B_\varepsilon(x^*)$ with constant L, (i.e.) $\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L\|x-y\|$ ⇔ $\nabla^2 f(x) - mI$ is pos. semidef.

If $\{x_k\}$ is generated by Newton's Method, with assumptions above, we have for $k = 0, 1, \cdots$ $\|x_{k+1} - x^*\| \leq \frac{L}{2m}\|x_k - x^*\|^2$ (quadratic convergence)

& if $\|x^0 - x^*\| \leq \frac{m}{L}\min\{1, \varepsilon\}$, we have $\|x_k - x^*\| \leq \frac{2m}{L}\left(\frac{1}{2}\right)^{2^k}$

- Comparison between gradient method & Newton's method

Gradient Descent Method { less memory space, computational easy; converges slowly, more iterations } > large-size problems

Newton's Method { converges fast, less iterations; more storage, computational cost } > small-size problems

Advanced: algorithm in between — Quasi-Newton Methods (approximate $\nabla^2 f(x)$)

algorithm for large-size ones — Stochastic gradient Methods

Lecture 15 Algorithms for Constrained Problems (Intro)

- Ideas for the algorithm

Recall gradient descent method, what if $x_{k+1} = x_k + \alpha_k d_k$ outside the feasible region?

$\Longrightarrow$ project it to the feasible region > projected gradient method

- Euclidean / Orthogonal Projection

1) Def. $P_C(x_0) := \arg\min_x \{ \|x - x_0\| \mid x \in C \}$ (projection of x_0 on to a set (convex) C)

It can be seen as a minimization programming with unique minimizer x^* (non-empty, closed & convex)

$$\begin{cases} \min_x \frac{1}{2}\|x - x_0\|^2 \\ \text{subject to } x \in C \end{cases}, \quad \text{write } x^* = P_C(x_0)$$

2) Special cases.

(i) projection on a polyhedron $\{x \mid Ax \le b\}$

$\Rightarrow$ consider special cases: 1) $C = \{x \mid a_i^T x = b\}$, then $P_C(x_0) = x_0 + \frac{(b - a_i^T x_0)}{a_i^T a_i} a_i$

(proof: KKT conditions)

2) $C = \{x \mid a_i^T x \le b\}$, then $P_C(x_0) = \begin{cases} x_0 + \frac{(b - a_i^T x_0)}{a_i^T a_i} a_i, & a_i^T x_0 > b \\ x_0, & a_i^T x_0 < b \end{cases}$

(proof: KKT conditions)

3) $C = \{x \mid \ell \le x \le u\}$ (box constraints)

$P_C(x_0)_k = \max\{\min\{x_0, u\}, \ell\}$ (k: the k^{th} item)

(ii) ball constraints

$C = \{x \in \mathbb{R}^n \mid \|x - m\| \le r\}$ given $m, r \in \mathbb{R}^n, \mathbb{R}$

$$P_C(x_0) = \begin{cases} x_0, & \text{if } \|x_0 - x_0\| \le r \\ m + \frac{r}{\|x_0 - m\|}\|x_0 - m\|, & \text{if } \|x - x_0\| > r \end{cases}$$

(Proof: KKT conditions
$(x - x_0) + \lambda(x - m) = 0$
$\lambda(\|x - m\| - r) = 0$
discuss $\lambda \neq 0$, $\lambda = 0 \Rightarrow$ get $\lambda \Rightarrow$ answer)

(iii) Matrix projection

(rank k matrices) $C = \{X \in \mathbb{R}^{m \times n} \mid \text{rank } X \le k\}$ with $k \le \min\{m, n\}$

For X_0, by SVD, $X_0 = \sum_{i=1}^{r} \delta_i u_i v_i^T$, with $r = \text{rank } X_0$

then $Y = \sum_{i=1}^{\min\{k, r\}} \delta_i u_i v_i^T$ is a projection of X_0 on C.

- Optimization Problems with Convex Constraints

☆ Theorem: FONC for Problems with convex constraints

Let f be cont. diff. on an open set that contains the convex, closed $\Omega \in \mathbb{R}^n$

Let $y^* \in \Omega$ be a local minimizer of $\left(\min_{y\in\Omega} f(y)\right)$, then $\nabla f(y^*)^T(y-y^*) \geq 0$, $\forall y \in \Omega$

(Proof. Recall FONC for constrained problem – $\forall$ feasible d $\nabla f(y^*)^T d \geq 0$

& d satisfies $\exists \bar{t}$, $y^* + td \in \Omega$, $t \in [0, \bar{t})$ ← can always do that

use $y - y^*$ to substitute d. $y^* + t(y-y^*) = ty + (1-t)y^* \in \Omega$, let $\bar{t} = 1$

(y in Ω)

$\Rightarrow$ every feasible d corresponds to a y in Ω (one-to-one mapping))

☆ Projection Theorem: let Ω be a non-empty, closed & convex set

then a point y^* is the projection of x onto Ω ($y^* = P_\Omega(x)$) iff

$$(y^* - x)^T(y - y^*) \geq 0, \quad \forall y \in \Omega$$

(Proof: $y^* = P_\Omega(x) \iff \nabla f(y^*) = y^* - x \iff \nabla f(y^*)^T(y-y^*) = 0$, $y \in \Omega$

($f(y) = \frac{1}{2}\|y-x\|^2$) { y^* minimizer (projection p)

(use convexity critial pt. $\Leftrightarrow$ minimizer) $\Leftarrow (y^* - x)^T(y-y^*) \geq 0$)

corollary:

(i) The mapping $P_\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continous with constant 1.

(Proof: $(P_\Omega(x) - x)^T(P_\Omega(y) - P_\Omega(x)) \geq 0$ (Because $P_\Omega(y) \in \Omega$)

$(P_\Omega(y) - y)^T(P_\Omega(x) - P_\Omega(y)) \geq 0$ (similarly)

Add up $\Rightarrow (P_\Omega(x) - P_\Omega(y))^T(P_\Omega(y) - P_\Omega(x) + (x-y)) \geq 0$

$\Rightarrow \|P_\Omega(x) - P_\Omega(y)\|^2 \leq (P_\Omega(x) - P_\Omega(y))^T(x-y) \leq \|P_\Omega(x) - P_\Omega(y)\| \|x-y\|$

$\Rightarrow \|P_\Omega(x) - P_\Omega(y)\| \leq \|x-y\|$ (Cauchy-Schwarz inequality))

(ii) The vector x^* is a critical point exactly when

$$x^* - P_\Omega(x^* - \lambda \nabla f(x^*)) = 0, \quad \forall \lambda > 0$$

(Proof: $\nabla f^T(x^*)(y-x^*) \geq 0$ $\underset{\lambda>0}{\Leftrightarrow}$ $\lambda \nabla f^T(x^*)(y-x^*) \geq 0$ $\forall y \in \Omega$

$\Leftrightarrow \left(x^* - (x^* - \lambda \nabla f(x^*))^T (y-x^*) \geq 0 \Leftrightarrow x^* = P_\Omega(x^* - \lambda \nabla f(x^*))\right)$

☆ The Projected Gradient Method:

1) Initilization: choose initial pt. $x_0 \in \Omega$ & $\alpha, \gamma \in (0,1)$

2) Select $\lambda_k > 0$, computing $\nabla f(x_k)$ & new direction $d_k = P_\Omega(x_k - \lambda_k \nabla f(x_k)) - x_k$

3) If $\|d_k\| \leq \lambda_k \varepsilon$, STOP, output x_k

4) Choose a maximal step, using backtracking line search with α, γ

Armijo condition $f(x_k + \alpha_k d_k) - f(x_k) \leq \gamma \alpha_k \nabla f(x_k)^T d_k$

5) Set $x_{k+1} = x_k + \alpha_k d_k$.

Comments: (i) $x_{k+1} = x_k + \alpha_k (P_\Omega(x_k - \lambda_k \nabla f(x_k)) - x_k) \in \Omega$ (in the line connected x_k, $P_\Omega(\cdot)$)

(ii) descent direction: $\nabla f(x_k)^T (P_\Omega(x_k - \lambda_k \nabla f(x_k)) - x_k) = \nabla f(x_k)^T (P_\Omega(x_k - \lambda_k \nabla f(x_k)) - P_\Omega(x_k))$

$= \frac{1}{\lambda_k} (x_k - (x_k - \lambda_k \nabla f(x_k)))^T (P_\Omega(x_k - \lambda_k \nabla f(x_k)) - P_\Omega(x_k))$

$\overset{\text{(last page)}}{\leq} -\frac{1}{\lambda_k} \|P_\Omega(x_k - \lambda_k \nabla f(x_k)) - P_\Omega(x_k)\|^2 = -\frac{1}{\lambda_k}\|d\|^2 < 0$ (promise descent)

(iii) Convergence: assume f is cont. diff., Ω is non-empty, convex, closed & stepsize λ_k bounded $0 < \underline{\lambda} \leq \lambda_k \leq \bar{\lambda}$

$\Rightarrow$ every accumulation point is a critical point. (of $\{x_k\}$)

Lecture 16 Integer Linear Programming (ILP)

- IDEA for solving integer programming.

1) General form: maximize $c^T x$ subject to $Ax = b$, $x \geq 0$, $x \in \mathbb{Z}^n$

(special case: binary integer programming: maximize $c^T x$ subject to $Ax = b$, $x_i \in \{0,1\}$ $\forall i$)

2) Relaxation: LP relaxation:

Aim - to get an approximation opt. value & solutions

☆ Strategy: For $x \in \mathbb{Z}^n$ — drop it directly

For $x_i \in \{0,1\}$ — $0 \le x_i \le 1$

Directly rounding the optimal solution to integers may not yield a good solution!

⟹ However, the usage of LP relaxation — Bound!

(i) For maximization IP, optimal value of relaxed LP provides an upper bound.

(ii) ~ minimization ~, ... a lower bound

Integrality gap $= |V^{IP} - V^{LP}|$.

$(0 \le V^{IP} - V^{rounding} \le V^{LP} - V^{rounding}$ (max case) when RHS = 0. $V^{IP} = V^{LP} = V^{rounding})$

Above is the ideal case, appearing in total unimodularity (TU) A with integer b

TU: a matrix A is totally unimodular if det of each sub-matrix is in $\{0, \pm 1\}$

A sufficient condition of TU: Let $A \in \mathbb{R}^{m \times n}$, then A is TU when

eg. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (B / C)

(i) every column of A has no more than 2 non-zero entries

(ii) every entry in A belongs to $\{0, \pm 1\}$

(iii) The rows can be partitioned into 2 disjoint B & C

s.t. ☆ every two non-zero entries in the same column belong to with same sign different sets.

☆ every two non-zero entries with different signs in the same column belong to same set the.

- **Branch-and-Bound Method**

1) IDEA: { Branching — divide the feasible region in-to smaller ones & solve them.
Bounding — Use bounds (by relaxation) to reduce some branches.

2) Procedure (max case)

(i) Branching: ☆ Solving the LP relaxation. (stop when getting integer ones)

☆ With optimal solution of LP x^*, if x_i^* (choose from small i) is fractional

⟹ branch them into $x_i \le \lfloor x_i^* \rfloor$; $x_i \ge \lceil x_i^* \rceil$ (2 branches)

(eliminate $\lfloor x_i^* \rfloor < x_i < \lceil x_i^* \rceil$ — no influence)

☆ Each of 2 problems, same method to solve, opt. solution & value y_1^*, v_1^*; y_2^*, v_2^*, compared to get opt. sol.

(DFS in practice) — space, bound & simpleness

(ii) Bounding Procedure: { Upper bound – the original LP relaxation's opt. value
Lower bound – every obj. value with integer feasible sol. in every branch.

When the opt. value of some LP relaxation of IP branches ≤ current lower bounds, (because of feasibility)
no need to consider these branches (discard)

3) Complexity: exponential, no polynomial-time algorithm in IP currently.

4) With softwares: { Gurobi can be used together with CVX for IPs
"intlinprog" function used in MATLAB for small size IPs.

(Another way – dynamic programming omitted)

Additional Topic – Duality Theory in General Convex cases

- The Lagrangian Dual Problems

Consider the Lagrangian problem in standard form

$$\begin{aligned} &\text{minimize } f_0(x) \\ &\text{subject to } f_i(x) \le 0,\ i=1,\cdots,m \\ &\qquad\quad h_j(x)=0,\ j=1,\cdots,p \end{aligned}$$

① Define the Lagrangian associated with the problem $L: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$L(x,\lambda,\mu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{j=1}^p \mu_j h_j(x)$$

with domain: $\mathrm{dom}\, L = D \times \mathbb{R}^m \times \mathbb{R}^p$

(λ_i, μ_j: dual variables / Lagrangian multipliers)

② Define the Lagrange dual function $g: \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as the minimum value of Lagrangian over x, for $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$

$$g(\lambda,\mu) = \inf_{x\in D} L(x,\lambda,\mu) = \inf_{x\in D}\left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{j=1}^p \mu_j h_j(x)\right)$$

(when the Lagrangian is unbounded below in x, $g(\lambda,\mu) = -\infty$)

↳ the dual function is concave even when Lagrangion is not convex. (affine mapping & pointwise minimum)

③ CQ: what's the (best lower bound) given $\lambda \ge 0$?

Lagrange Dual Problem

$$\begin{aligned} &\text{maximize } g(\lambda,\mu) \\ &\text{subject to } \lambda \ge 0 \end{aligned}$$

eg.1 (LP Dual form derived with Lagrangian)

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

Sol. ⇒ get the Lagrangian $L(x,\lambda,\mu) = c^T x + \lambda^T(-x) + \mu^T(Ax-b)$
$= -\mu^T b + (A^T\mu + c - \lambda)^T x$

The dual function $g(\lambda,\mu) = \begin{cases} -\mu^T b, & A^T\mu + c - \lambda = 0 \\ -\infty, & A^T\mu + c - \lambda \neq 0 \end{cases}$

We get the dual form $\left(\begin{array}{l} \text{maximize } -\mu^T b \\ \text{subject to } A^T\mu + c - \lambda = 0 \\ \qquad\qquad \lambda \geq 0 \end{array}\right) \overset{y=-\mu}{\Leftrightarrow} \left(\begin{array}{l} \text{maxmize } b^T y \\ \text{subject to } c \geq A^T y \end{array}\right)$

☆ eg.2 (The two-way partitioning problem)

Consider the (non-convex) problem

minimize $x^T W x$
subject to $x_i^2 = 1, \quad i = 1, \dots, n \qquad (W \in S^n)$

(Interpretation: two-way partitioning on a set $\{1,2,\dots,n\}$ (with n elements / anyset)
a feasible x corresponds to the partition $\{1,\dots,n\} = \{i \mid x_i = 1\} \cup \{j \mid x_j = -1\}$
The matrix coeffient W_{ij} – the cost of having elements i & j in the same partion,
$-W_{ij}$ – " different & objective fun – total cost.)

Sol. Get the Lagrangian $L(x,\lambda) = x^T W x + \sum_{i=1}^n \lambda_i (x_i^2 - 1)$
$= x^T (W + \mathrm{diag}(\lambda)) x - \mathbb{1}^T \lambda$

The dual function $g(\lambda) = \begin{cases} -\mathbb{1}^T\lambda, & W + \mathrm{diag}(\lambda) \geq 0 \\ -\infty, & W + \mathrm{diag}(\lambda) \text{ not pos. semi-def.} \end{cases}$

⇒ The dual problem $\left(\begin{array}{l} \text{maximize } -\mathbb{1}^T\lambda \\ \text{subject to } W + \mathrm{diag}(\lambda) \geq 0 \end{array}\right\}$ an SDP (semi-def program)

- Weak Duality & Strong Duality.

(1) Weak Duality: the optimal value of the Lagrange dual problem, denoted by d^*, is, by definition, the best lower bound on p^* obtained from the Lagrange dual function, then

☆ $d^* \leq p^*$ holds (even if non-convex) when d^*, p^* finite

The optimal duality gap $p^* - d^*$

(2) Strong duality: $d^* = p^*$, i.e. the optimal duality gap = zero

NOT hold in general !!!

For convex problems with constraint qualification (CQ), it works.

☆ Slater's condition: $\exists\, x \in \text{relint } \mathcal{D}$ such that x is strictly feasible

(i.e.) $f_i(x) < 0$; $h_j(x) = 0$, $\forall i, j$

($\text{relint } C = \{x \in C \mid B(x,r) \cap \text{aff } C \subseteq C, \text{ for some } r > 0\}$)

eg. (a non-convex quadratic problem with strong duality)

minimizing a nonconvex quadratic function over the unit ball / trust region problem.

$$\begin{array}{ll} \text{minimize} & x^TAx + 2b^Tx \\ \text{subject to} & x^Tx \le 1 \end{array}, \quad \text{where } A \in \mathbb{S}^n \text{ \& } A \neq 0,\ b \in \mathbb{R}^n$$

Sol. the Lagrangian $L(x,\lambda) = x^TAx + 2b^Tx + \lambda(x^Tx - 1)$

$= x^T(A+\lambda I)x + 2b^Tx - \lambda$

the dual function is given by $g(\lambda) = \begin{cases} -b^T(A+\lambda I)^\dagger b - \lambda, & b \in \mathcal{C}(A+\lambda I),\ A+\lambda I \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$ (pseudo-inverse)

$\Rightarrow$ the Lagrange dual problem is thus

$$\begin{array}{ll} \text{maximize} & -b^T(A+\lambda I)^\dagger b - \lambda \\ \text{subject to} & A+\lambda I \succeq 0,\ b \in \mathcal{C}(A+\lambda I) \end{array}$$

$\Longrightarrow$ diagonalization

$$\begin{array}{ll} \text{maximize} & -\dfrac{\sum_{i=1}^n |q_i^Tb|^2}{\lambda_i + \lambda} - \lambda \\ \text{subject to} & \lambda \ge \max\{0, -\lambda_{\min}(A)\} \end{array}, \quad \lambda \ge 0$$

(eigenvalues λ_i & eigenvectors q_i of A)

$\Rightarrow$ Conclusion: Strong duality holds for any opt. problem with quadratic objective func & one quadratic inequality, provided slater's condition holds.

- Saddle-point interpretation

(1) max-min characterization

consider the more general case $\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \quad \forall i = 1, \dots, n \end{array}$

we know $\boxed{p^* = \inf_{x \in \mathcal{D}} \sup_{\lambda \ge 0} L(x,\lambda)}$ (because $\sup_{\lambda \ge 0} L(x,\lambda) = \begin{cases} f_0(x), & f_i(x) \le 0,\ i = 1, \dots, n \\ +\infty, & \text{otherwise} \end{cases}$)

☆ $\boxed{d^* = \sup_{\lambda \ge 0} \inf_{x \in \mathcal{D}} L(x,\lambda)}$ (by the definition)

weak duality says $\sup_{\lambda\geq 0} \inf_{x} L(x,\lambda) \leq \inf_{x} \sup_{\lambda\geq 0} L(x,\lambda)$

(In reality, it works for every function $f(x,\lambda)$,

because $f(\tilde{x},\lambda) \geq \inf_x f(x,\lambda)$, for any $\tilde{x}$, given λ

$\Rightarrow \sup_{\lambda\geq 0} f(\tilde{x},\lambda) \geq \sup_{\lambda\geq 0} \inf_x f(x,\lambda)$, then let $\tilde{x} = \arg\min_x (\sup_{\lambda\geq 0} f(x,\lambda))$

$\Rightarrow \boxed{\inf_x \sup_{\lambda\geq 0} f(x,\lambda) \geq \sup_{\lambda\geq 0} \inf_x f(x,\lambda)}$) max-min inequality

strong duality says $\sup_{\lambda\geq 0} \inf_x L(x,\lambda) = \inf_x \sup_{\lambda\geq 0} L(x,\lambda)$

(2) Saddle-point interpretation

we refer to a pair $(\hat{w},\hat{z}) \in W \times Z$ as a saddle point of f

if $f(\hat{w},z) \leq f(\hat{w},\hat{z}) \leq f(w,\hat{z})$, $\forall w \in W$, $z \in Z$

$\Rightarrow f(\hat{w},\hat{z}) = \inf_{w\in W} f(w,\hat{z})$ (= $\sup_{z\in Z} f(\hat{w},z)$) $\Longrightarrow$ strong duality holds! (implies)

$\Rightarrow$ (x^* & λ^* are primal & dual optimal pts with strong duality exactly when (x^*,λ^*) is a saddle-point of the Lagrangian $L(x,\lambda)$)

Explanation:

$$\sup_{z\in Z} f(\hat{w},z) \geq \inf_{w\in W} \sup_{z\in Z} f(w,z) \quad \leftarrow \text{"Weak Duality"}$$

$$\| \qquad\qquad \vee$$

$$f(\hat{w},\hat{z}) = \inf_{w\in W} f(w,\hat{z}) \leq \sup_{z\in Z} \inf_{w\in W} f(w,z)$$

$\Rightarrow$ sup inf = inf sup, when applied in x & λ, we can get that strong duality holds for $L(x,\lambda)$, $\lambda \geq 0$ & $x \in R$

Distributed Optimization & Stastical Learning via ADMM

(2021-22 Term 1 additional topic)

Part 1 precursors

- The conjugate Function

(1) Let $f: \mathbb{R}^n \to \mathbb{R}$, define the function $f^*: \mathbb{R}^n \to \mathbb{R}$ as the conjugate of f.

where $f^*(y) := \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$.

$\mathrm{dom} f^*$: all $y \in \mathbb{R}^n$ s.t. $\sup_{x \in \mathrm{dom} f} (y^T x - f(x)) < \infty$

(2) Examples:

(Log-determinant) consider $f(X) = \log\det X^{-1}$ on $\mathbb{S}^n_{++}$ (pos. def)

For general matrix $\langle X, Y \rangle = \mathrm{tr}(X^T Y)$

For square $= \mathrm{tr}(XY)$

The conjugate of f - defined as $f^*(Y) = \sup_{X \succ 0} (\mathrm{tr}(YX) + \log\det X)$ (defined as inner product of Y, X ($\langle X, Y \rangle$) on $\mathbb{S}^n$)

$\mathrm{dom} f^* = -\mathbb{S}^n_{++}$ because if $Y \not\succeq 0$, then

$\exists$ an eigenvector v of Y, with $\|v\| = 1$, and corresponding eigenvalue $\lambda \geq 0$

Take $X = I + tvv^T \succ 0$

$\mathrm{tr}(YX) + \log\det X = \mathrm{tr}(Y) + t\lambda + \log\det(I + tvv^T) = \mathrm{tr}(Y) + t\lambda + \log(1+t)$

$\to \infty$ as $t \to \infty$.

If $Y \prec 0$, find the maxizing X by gradient

$\nabla_X(\mathrm{tr}(YX) + \log\det X) = Y + X^{-1} = 0$, yield that $Y = -X^{-1}$.

(3) Basic Properties

Frenchel's inequality $f(x) + f^*(y) \geq x^T y$.

($f(x) + f^*(y) = f(x) + \sup_{x \in \mathrm{dom} f}(y^T x - f(x)) \geq y^T x$)

"=" holds when $\sup_{x \in \mathrm{dom} f}(y^T x - f(x)) = \max_{x \in \mathrm{dom} f}(y^T x - f(x))$

Legendre Transform: if f is convex, differentiable with $\mathrm{dom} f = \mathbb{R}^n$

then any maximizer x^* satisfies $y = \nabla f(x^*)$

Let $z \in \mathbb{R}^n$, $y = \nabla f(z)$,

$\Rightarrow f^*(y) = z^T \nabla f(z) - f(z)$ (in this case)

• Dual Decomposition

(1) Dual Ascent (like gradient descent)

consider the equality-constrained convex problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } Ax=b \end{aligned} \qquad (*)$$

with $x\in\mathbb{R}^n$, $A\in\mathbb{R}^{m\times n}$. $f:\mathbb{R}^n\to\mathbb{R}$ is convex.

We get the Lagrangian $L(x,y)=f(x)+y^T(Ax-b)$

Dual Func. $g(y)=\inf_x f(x)+y^T(Ax-b)=-f^*(-A^Ty)-b^Ty$

$\Rightarrow$ Dual Problem $\underset{y\in\mathbb{R}^m}{\text{maximize}}\ g(y)$

★ Assume strong duality holds $\Rightarrow x^*=\arg\min_x L(x,y^*)$

(provided only one minimizer of $L(x,y)$)

Iterations of dual ascent method

$$x^{k+1}:=\arg\min_x L(x,y^k)$$
$$y^{k+1}:=y^k+\alpha^k(Ax^{k+1}-b)$$ evaluate $\nabla g(y)$ as

where $\alpha^k>0$ is the step size, $k/k+1$ – iteration counter

with appropriate choice of α^k, $g(y^{k+1})>g(y^k)$

then y^k converges to optimal sol. when $k\to\infty$.

(2) Dual Decomposition

(The major benefit of dual ascent)

(i) A Decentralized Algorithm.

Suppose, for instanse, objective f is seperable, which means

$$f(x)=\sum_{i=1}^N f_i(x_i),\quad \text{where } x=(x_1,\cdots,x_N)\ \&\ x_i\in\mathbb{R}^{n_i}$$

are subvectors of x.

Partitioning the matrix A as $[A_1\cdots A_N]$

so that $Ax=\sum_{i=1}^N A_ix_i$

The Lagrangian can be written as

$$L(x,y)=\sum_{i=1}^{N}L_i(x_i,y)=\sum_{i=1}^{N}\left(f_i(x_i)+y^TA_ix_i-\frac{1}{N}y^Tb\right)$$

(ii) The Iterations of dual decomposition

$$x_i^{k+1}:=\underset{x_i}{\operatorname{argmin}}\,L_i(x_i,y^k) \quad \text{— broadcast}$$
$$y^{k+1}:=y^k+\alpha^k(Ax^{k+1}-b) \quad \text{— gather}$$

> 2 steps

- Augmented Lagrangian & the Method of Multipliers

(1) Aim: Bring robustness to the dual ascent, yielding convergence without assumptions like strict convexity / fitness of f.

(2) The augmented Lagrangian for (*) is

$$L_\rho(x,y)=f(x)+y^T(Ax-b)+\rho/2\,\|Ax-b\|_2^2$$

(Namely $L_0(x,y)+\rho/2\|Ax-b\|_2^2$, $Ax-b=0$ for feasible x)

(subject to $Ax=b$)

(3) The Iterations

$$x^{k+1}:=\underset{x}{\operatorname{argmin}}\,L_\rho(x,y^k)$$
$$y^{k+1}:=y^k+\rho(Ax^{k+1}-b)$$

hard to seperate because of $\|Ax-b\|_2^2$

method of multipliers

Part II Alternating Direction Method of Multipliers (ADMM)

- Algorithm

(1) Aim: Blend the decomposability of dual ascent with the superior convergence properties of the method of multipliers.

Solve the problem in the form

$$\underset{x,z}{\text{minimize}}\ f(x)+g(z) \quad (x\in\mathbb{R}^n,\ z\in\mathbb{R}^m)$$
$$\text{subject to } Ax+Bz=c \quad (A\in\mathbb{R}^{p\times n},\ B\in\mathbb{R}^{p\times m})$$

(Assume f, g are both convex)

The augmented Lagrangian $L_\rho(x,z,y)=f(x)+g(z)+y^T(Ax+Bz-c)+\frac{\rho}{2}\|Ax+Bz-c\|_2^2$.

(2) The Iteration steps:

(of ADMM)

$$x^{k+1} := \arg\min_x L_\rho(x, z^k, y^k)$$
$$z^{k+1} := \arg\min_z L_\rho(x^{k+1}, z, y^k)$$
$$y^{k+1} := y^k + \rho(Ax^{k+1} + Bz^{k+1} - c)$$

updated in an alternating fashion

Scaled Form Let residual $r = Ax + Bz - c$ → primal residual

$y^T r + \frac{\rho}{2}\|r\|_2^2 = \frac{\rho}{2}\|r + \frac{1}{\rho}y\|_2^2 - \frac{1}{2\rho}\|y\|_2^2$, Let $u = \frac{1}{\rho}y$

then $y^T r + \frac{\rho}{2}\|r\|_2^2 = \frac{\rho}{2}\|r + u\|_2^2 - \frac{\rho}{2}\|u\|_2^2$.

$$x^{k+1} := \arg\min_x (f(x) + \frac{\rho}{2}\|Ax + Bz^k - c + u^k\|_2^2)$$
$$z^{k+1} := \arg\min_z (g(z) + \frac{\rho}{2}\|Ax^{k+1} + Bz - c + u^k\|_2^2)$$
$$u^{k+1} := u^k + r^{k+1} \ (\text{i.e., } Ax^{k+1} + Bz^{k+1} - c)$$

- Optimality Conditions & Stopping Criterion

(i) Necessary & Sufficient optimality conditions

primal feasibility: $Ax^* + Bz^* - c = 0$ (1)

dual feasibility: $0 \in \partial f(x^*) + A^T y^*$ (2)

$0 \in \partial f(z^*) + B^T y^*$ (3)

(Addition. "∂" denotes the subdifferential operator

(Defn) $\partial f(x) := \{ g_x \mid f(y) \geq f(x) + g_x^T(y - x), \forall y \}$ $f(x)$ does NOT need to be differentiable at x.)

(ii) Because z^{k+1} minimize $L_\rho(x^{k+1}, z, y^k) \Rightarrow 0 \in \partial f(z^{k+1}) + B^T y^{k+1}$

which means (3) holds for all $(x^{k+1}, z^{k+1}, y^{k+1})$, $\forall k \in \mathbb{N}$

Similarly $s^{k+1} = \rho A^T B(z^{k+1} - z^k) \in \partial f(x^{k+1}) + A^T y^{k+1}$

[dual residual] ($s^{k+1} \to 0$ & $r^{k+1} \to 0$ as ADMM iterates)

(iii) Stopping Criteria

when r^k, s^k are both small, the objective suboptimality must be small

A reasonable termination criterion: $\|r^k\|_2 \leq \varepsilon^{pri}$; $\|s^k\| \leq \varepsilon^{dual}$

Chosen way: (absolute & relative criterion)

ε^{pri} (tolerance for the primal conditions)
$= \sqrt{p}\,\varepsilon^{abs} + \varepsilon^{rel} \max\{\|Ax^k\|_2, \|Bz^k\|_2, \|c\|_2\}$

ε^{dual} (tolerance for the dual conditions)
$= \sqrt{n}\,\varepsilon^{abs} + \varepsilon^{rel}\|A^T y^k\|_2$

ε^{abs} = absolute tolerance
ε^{rel} = relative tolerance
p, n = ℓ_2 norms are in $\mathbb{R}^p$, $\mathbb{R}^n$ (respectively)

- Proof for convergence

2 assumptions: (1) $f: \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ & $g: \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ are closed, proper & convex.

(closed: epigraph $\text{epi}f = \{(x,t)\in\mathbb{R}^n\times\mathbb{R} \mid f(x)\le t\}$ is closed.
or sublevel sets $\{x\in\text{dom}f \mid f(x)\le\alpha\}$ are closed $\forall\alpha\in\mathbb{R}$)

(proper convex function: epigraph ($\text{epi}f = \{(x,t)\in\mathbb{R}^n\times\mathbb{R} \mid f(x)\le t\}$ is non-empty & contains no vertical lines
(i.e., $f(x) < +\infty$ for at least one x & $f(x) > -\infty$ for all x)

criterion of proper: $f(x)$ is proper iff $\text{dom}f$ is convex, non-empty & restriction of f to C is finite.

(2) Unaugmented Lagrangian $L_0 = f(x) + g(z) + y^T(Ax+Bz-c)$ has a saddle point (i.e., $\exists(x^*, z^*, y^*)$ not necessarily unique.
s.t. $L_0(x^*, z^*, y) \le L_0(x^*, z^*, y^*) \le L_0(x, z, y^*)$, $\forall x, y, z$ ...

Need to show
$$\begin{cases} p^* - p^{k+1} \le y^{*T} r^{k+1} & (A.1) \quad (\text{If } r^{k+1}\to 0,\ p^{k+1}\to p^*) \\ p^{k+1} - p^* \le -(y^{k+1})^T r^{k+1} - \rho(B(z^{k+1}-z^k))^T(-r^{k+1} + B(z^{k+1}-z^*)) & (A.2) \\ V^{k+1} \le V^k - \rho\|r^{k+1}\|_2^2 - \rho\|B(z^{k+1}-z^k)\|_2^2 & (A.3.) \ (\Rightarrow r^{k+1}\to 0,\ B(z^{k+1}-z^k)\to 0,\ \Rightarrow s^k\to 0) \end{cases}$$
where $V^k := \frac{1}{\rho}\|y^k - y^*\|_2^2 + \rho\|B(z^k - z^*)\|_2^2$
is a Lyapunov Function for the algorithm.

(proof.)
Since $L_0(x^*, z^*, y^*) \le L_0(x^{k+1}, z^{k+1}, y^*)$
saddle point property $\Rightarrow p^* \le p^{k+1} + y^{*T} r^{k+1}$ (which is A.1)
Since x^{k+1} minimize $L_\rho(x, z^k, y^k)$, $0 \in \partial_x L_\rho(x^{k+1}, z^k, y^k)$ (closed, proper, convex $\Rightarrow$ subdifferential)
$= \partial f(x^{k+1}) + A^T y^k + \rho A^T(Ax^{k+1} + Bz^k - c)$

$y^{k+1} = y^k + \rho r^{k+1} \Rightarrow$

$0 \in \partial f(x^{k+1}) + A^T(y^{k+1} - \rho B(z^{k+1} - z^k))$

$\therefore x^{k+1}$ minimizes $f(x) + (y^{k+1} - \rho B(z^{k+1} - z^k))^T Ax$

Similarly, z^{k+1} minimizes $g(z) + (y^{k+1})^T Bz$

$$\therefore \begin{cases} f(x^{k+1}) + (y^{k+1} - \rho B(z^{k+1} - z^k))^T A x^{k+1} \le f(x^*) + (y^{k+1} - \rho B(z^{k+1} - z^k))^T A x^* \\ g(z^{k+1}) + (y^{k+1})^T B z^{k+1} \le g(z^*) + (y^{k+1})^T B z^* \end{cases}$$

Add the two together, $p^{k+1} + (y^{k+1})^T r^{k+1} - (\rho B(z^{k+1} - z^k))^T A x^{k+1} \le p^* - (\rho B(z^{k+1} - z^k))^T A x^*$ (which is A.2)

Adding A.1 & A.2, regrouping terms (& multiply by 2)

$$2(y^{k+1} - y^*)^T r^{k+1} - 2\rho(B(z^{k+1} - z^k))^T r^{k+1} + 2\rho(B(z^{k+1} - z^k))^T (B(z^{k+1} - z^*)) \le 0$$

substitute $y^{k+1} = y^k + \rho r^{k+1}$, $r^{k+1} = \frac{1}{\rho}(y^{k+1} - y^k)$, rewrite the 1st term

$$2(y^{k+1} - y^*)^T r^{k+1} = \frac{1}{\rho}\|y^{k+1} - y^*\|_2^2 + \rho\|r^{k+1}\|_2^2 - \frac{1}{\rho}\|y^k - y^*\|_2^2$$

& $-2\rho(B(z^{k+1} - z^k))^T r^{k+1} + 2\rho(B(z^{k+1} - z^k))^T B(z^{k+1} - z^*) + \rho\|r^{k+1}\|_2^2$ $(z^{k+1} - z^* = z^{k+1} - z^k + z^k - z^*)$

$$= \rho\|r^{k+1} - B(z^{k+1} - z^k)\|_2^2 + \rho\|B(z^{k+1} - z^k)\|_2^2 + 2\rho(B(z^{k+1} - z^k))^T (B(z^k - z^*))$$

substitute $z^{k+1} - z^k = (z^{k+1} - z^*) - (z^k - z^*)$ in the last 2 terms

$$\Rightarrow V^k - V^{k+1} \ge \rho\|r^{k+1} - B(z^{k+1} - z^k)\|_2^2 \ge \rho\|r^{k+1}\|_2^2 + \rho\|B(z^{k+1} - z^k)\|_2^2$$

(Because z^{k+1} minimizes $g(z) + (y^{k+1})^T Bz$, z^k minimizes $g(z) + (y^k)^T Bz$

$$\Rightarrow (y^{k+1} - y^k)^T B(z^{k+1} - z^k) = \rho(r^{k+1})^T B(z^{k+1} - z^k) \le 0.)$$

Part III Consensus & Distributed Model Fitting

- Background knowledge

① soft thresholding consider $f(x) = \lambda\|x\|_1$ with $\lambda > 0$ & $A = I$ (in ADMM)

In this case x_i-updation is $x_i^+ := \underset{x_i}{\operatorname{argmin}}(\lambda|x_i| + \rho/2(x_i - v_i)^2)$

The solution is $x_i^+ = S_{\lambda/\rho}(v_i)$, where $S_\kappa(a) = \begin{cases} a - \kappa, & a > \kappa \\ 0, & |a| \le \kappa \\ a + \kappa, & a < -\kappa \end{cases}$

(i.e., $S_\kappa(a) = \left(1 - \frac{\kappa}{|a|}\right)_+ a$, $\forall a \ne 0$)

(Sol. sub-gradient method)

② proximity operator (of f with penalty ρ) (denoted as $\text{prox}_{f,\rho}(v)$)

$$\text{prox}_{f,\rho}(v) := \underset{x}{\operatorname{argmin}}(f(x) + \rho/2\|x - v\|_2^2)$$

• Global variable consensus optimization

① general case: minimize $f(x)=\sum_{i=1}^{n} f_i(x)$, $x\in\mathbb{R}^n$, $f_i:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ are convex ↘ can add some constraints

↓ rewrite

minimize $f(x)=\sum_{i=1}^{n} f_i(x_i)$

subject to $x_i - z = 0$, $i=1,\dots,N$

$\xrightarrow[\text{Form}]{\text{ADMM}}$

$$x_i^{k+1} := \arg\min_{x_i}\left(f(x_i) + y_i^{kT}(x_i - \bar{x}^k) + \frac{\rho}{2}\|x_i - \bar{x}^k\|_2^2\right)$$
$$y_i^{k+1} := y_i^k + \rho(x_i^{k+1} - \bar{x}^{k+1})$$

② with regularization

minimize $f(x)+g(z)=\sum_{i=1}^{n} f_i(x_i)+g(z)$

subject to $x_i - z = 0$, $i=1,\dots,N$

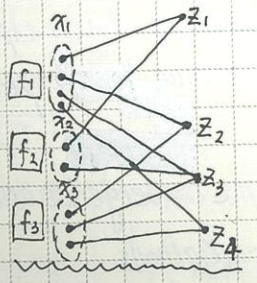
$\xrightarrow[\text{Form}]{\text{ADMM}}$

$$x_i^{k+1} := \arg\min_{x_i}\left(f_i(x_i) + \frac{\rho}{2}\|x_i - z^k + u_i^k\|_2^2\right)$$
$$z^{k+1} := \arg\min_{z}\left(g(z) + \frac{N\rho}{2}\|z - \bar{x}^{k+1} - u^k\|_2^2\right)$$
$$u_i^{k+1} := u_i^k + x_i^{k+1} - z_i^{k+1}$$

• General Form consensus optimization

① Each component of each local variable corresponds to some global variable component z_g.

(i.e., $\exists$ a mapping $g = G(i,j)$ s.t. $(x_i)_j = z_{G(i,j)}$)



(a case of general consensus)

Let $\hat{z}_i \in \mathbb{R}^{n_i}$, with $(\hat{z}_i)_j = z_{G(i,j)}$ ⇒ (minimize $\sum_{i=1}^{N} f_i(x_i)$ subject to $x_i - \hat{z}_i = 0$, $i=1,\dots,N$)

$\xrightarrow[\text{Form}]{\text{ADMM}}$

$$x_i^{k+1} := \arg\min_{x_i}\left(f(x_i) + y_i^{kT}x_i + \frac{\rho}{2}\|x_i - \hat{z}_i\|_2^2\right)$$
$$z_g^{k+1} := \frac{\sum_{G(i,j)=g}\left[(x_i^{k+1})_j + \frac{1}{\rho}(y_i^k)_j\right]}{\sum_{G(i,j)=g}\mathbb{1}} \quad \text{(indicator function)}$$

avg = 0 ⇒ $= \frac{1}{k_g}\sum_{G(i,j)=g}(x_i^{k+1})_j$

$$y_i^{k+1} := y_i^k + \rho(x_i^{k+1} - \hat{z}_i^{k+1})$$

② Sharing:

minimize $\sum_{i=1}^{N} f_i(x_i) + g\left(\sum_{i=1}^{N} x_i\right)$ ($x_i \in \mathbb{R}^{n_i}$, $i=1,\dots,n$; $\sum x_i = z_i$)

$\xrightarrow[\text{Form}]{\text{ADMM}}$ Iterations ($x_i - z_i = 0$)

$$x_i^{k+1} := \arg\min_{x_i}\left(f_i(x_i) + \frac{\rho}{2}\|x_i - x_i^k + \bar{x}^k - \bar{z}^k + u\|_2^2\right)$$
$$\bar{z}^{k+1} := \arg\min_{z}\left(g(N\bar{z}) + \frac{N\rho}{2}\|\bar{z} - u^k - \bar{x}^{k+1}\|_2^2\right)$$
$$u^{k+1} := u^k + \bar{x}^{k+1} - \bar{z}^{k+1}$$

• Distribute Model Fitting

1) General Convex model fitting problem

minimize $_{x\in\mathbb{R}^n}$ $l(Ax-b) + r(x)$ (regularization function)

(l: loss function)

Assume additive $l = \sum_{i=1}^{m} l_i(a_i^T x - b)$, common regularizations: Tikhonov ~ $= \lambda\|x\|_2^2$; lasso penalty $= \lambda\|x\|_1$

2) Examples: (supervised learning)

① Regression: with the form $b_i = a_i^T x + v_i$

(a_i – i^{th} feature vector, v_i – measurement noises (indep. with log-concave densities p_i)), then $l_i(w) = -\log p_i(-w)$.

$r = 0$, MLE of x under noise model p

r_i – negative log prior density of x, MAP estimation

② Classification: $q_i \in \{-1, 1\}$ & $q_i = \text{sign}(p_i^T w + v)$

(Find weight vector $w \in \mathbb{R}^{n-1}$ & offset $v \in \mathbb{R}$) Then, $l_i = l_i(q_i(p_i^T w + v))$ ($q_i(p_i^T w + v)$: margin)

This can be modeled as $\frac{1}{m}\sum_{i=1}^{m} l_i(q_i(p_i^T w + v)) + r^{wt}(w)$ penalized erm (empirical risk minimization)

- l_i – hinge loss $(1-\mu_i)_+$ + l_2 penalty $\Rightarrow$ SVM
- l_i – logistic loss $\log(1+\exp(-\mu_i)) \Rightarrow$ logistic regression
- l_i – exponential loss $\exp(-\mu_i) \Rightarrow$ boosting

☆

Additions (some convergence analyses & introductions)

- First-Order Methods

1) Gradient Descent Method: (convergence analyses) "∇f"

) Assume that f is convex, differentiable & Lipschitz continuous with const $L > 0$.

Then, we get, with fixed step size $t \le 1/L$, then $f(x^k) - f^* \le \frac{\|x^0 - x^*\|_2^2}{2tk}$

(proof.) By 1st-order condition of convexity,

$$f^* = f(x^*) \ge f(x^l) + \nabla f(x^l)^T (x^* - x^l) \quad ①$$

Because of $C_L^{1,1}$, we get $\nabla^2 f \le LI$ (if $f \in C^2$), then it's easy to get that $f(x^{l+1}) \le f(x^l) + \nabla f^T(x^l)(x^{l+1} - x^l) + \frac{L}{2}\|x^{l+1} - x^l\|_2^2 = f(x^l) - t(1 - \frac{tL}{2})\|\nabla f(x^l)\|_2^2$ ②

combine ① & ②, we have $f(x^{l+1}) - f^* \le \nabla f(x^l)^T (x^l - x^*) - t(1 - \frac{Lt}{2})\|\nabla f(x^l)\|_2^2$

($\nabla f(x^l) = \frac{1}{t}(x^{l+1} - x^l)$, $\|\nabla f(x^l)\|_2^2 = \frac{1}{t}(x^l - x^{l+1})$)

$\Rightarrow f(x^{l+1}) - f^* \le \frac{1}{t}\|x^l - x^*\|_2^2 - \frac{1}{t}(x^{l+1} - x^*)^T(x^l - x^*) - \frac{1}{t}(1 - \frac{Lt}{2})\|x^l - x^{l+1}\|_2^2$

($\le t/2$ because $t \le 1/L$.)

$\therefore f(x^{l+1}) - f^* \le \frac{1}{2t}(\|x^l - x^*\|_2^2 - \|x^{l+1} - x^*\|_2^2)$ ③

Add ③ from $l = 0$ to ∞, we get $\sum_{l=0}^{\infty} f(x^{l+1}) - f^* \le \frac{1}{2t}\|x^0 - x^*\|_2^2$ ($\sum$ converges)

$\therefore f(x^l) - f(x^*) \to 0$ as $l \to \infty$, which means that it converges

Because $f(x^l) - f(x^*)$ decreases (according to ②), we get

$$k[f(x^k) - f(x^*)] \le \sum_{l=1}^{k} f(x^l) - f^* \le \sum_{l=1}^{\infty} f(x^l) - f^* \le \frac{1}{2t}\|x^0 - x^*\|_2^2. \quad \square$$

We say gradient descent has convergence rate $O(1/k)$ (or $O(1/\varepsilon)$)

That is, it finds ε-suboptimal point ($f^{k+1}-f^* \le \varepsilon$) in $O(1/\varepsilon)$ iterations.

(For backtracking, with t replaced by γ/L, we can get similar results)

(ii) Assume [strong convexity] holds (with all other properties same as (i)), $\nabla^2 f(x) \succeq mI$

Then, gradient descent with fixed $t \le \frac{2}{m+L}$ or with backtracking line search ($t>0$)

satisfies $\boxed{f(x^k)-f^* \le \delta^k \frac{L}{2}\|x^0-x^*\|_2^2}$, where $\delta \in (0,1)$.

(proof.) Similar to (i), we get $f(x^{k+1})-f^* \le f(x^k)-f^* - t(1-\frac{tL}{2})\|\nabla f(x^k)\|_2^2$

By Polyak-Lojasieurcz inequality, $\|\nabla f(x^k)\|_2^2 \ge 2m[f(x^k)-f^*]$

$\Rightarrow f(x^{k+1})-f^* \le \underbrace{[1-2mt(1-\frac{tL}{2})]}_{\ge 1-\frac{m}{L}>0,\ <1-2mt\cdot\frac{m}{m+L}<1} [f(x^k)-f^*] \triangleq \delta[f(x^k)-f^*]$, $\forall k \in \mathbb{N}$

$\therefore f(x^k)-f^* \le \delta^k[f(x^0)-f(x^*)] \le \delta^k \cdot \frac{L}{2}\|x^0-x^*\|_2^2$. □

(For backtracking, let $\delta = 1-\min\{2m\alpha, \frac{2\gamma\alpha m}{L}\}$, for exact, $\delta = 1-\frac{m}{L}$)

We say rate under strong convexity is $O(\delta^k)$ (or $O(\log(\frac{1}{\varepsilon}))$), that is,

it finds ε-suboptimal point in $O(\log\frac{1}{\varepsilon})$ iterations (Also called linear convergence)

☆ First-order Method: iterative, update x^k in $x^0 + \mathrm{span}\{\nabla f(x^0), \cdots, \nabla f(x^{k-1})\}$

(i) (Thm - Nesterov) $\forall k \le \frac{n-1}{2}$ (n - problem dimension), & any starting point x^0, $\exists$ a function in the problem class s.t. any 1st-order method satisfies $f(x^k)-f^* \ge \frac{3L}{32}\frac{\|x^0-x^*\|_2^2}{(k+1)^2}$.

From this we can know that the best rate for 1st-order method is $O(1/\sqrt{\varepsilon})$.

(over class of differentiable functions with $C_L^{1,1}$)

(ii) For non-convex condition, E.g. gradient-descent with fixed step size $t \le 1/L$:

(easily to show) $\min_{i=0,\cdots,k}\|\nabla f(x^i)\|_2 \le \sqrt{\frac{2(f(x^0)-f^*)}{t(k+1)}}$ $O(1/\varepsilon^2)$

2) Sub-gradient Descent Method (For Non-differentiable Functions):

(i) Operations: Every time let $x^k = x^{k-1} - t_k \cdot g^{k-1}$, where g^{k-1} is a sub-gradient at x^{k-1}

Because sub-gradient method is NOT necessarily a descent one, we take [$f(x^k_{best}) = \min_{i=0,\cdots,k} f(x^i)$].

(ii) Step-size choise: (rule) must be pre-specifred, NOT adaptively computed.

{ Fixed step size $t_k = t$, all $k = 1, 2, \cdots$
{ Diminishing step-size $\sum_{k=1}^{\infty} t_k^2 < \infty$; $\sum_{k=1}^{\infty} t_k = \infty$ (e.g. harmonic series, $t_k = 1/k$)

(iii) Convergence analysis:

Assume f is convex, & Lipschitz continuous with const $G > 0$ (i.e. $|f(x) - f(y)| \le G\|x - y\|_2$)

Then, with { fixed step size t, $\lim_{k\to\infty} f(x^k_{best}) - f^* \le \frac{Gt^2}{2}$
{ diminishing step size, $\lim_{k\to\infty} f(x^k_{best}) = f^*$

(proof.) $\|x^{k+1} - x^*\|_2^2 = \|x^k - x^* - t_k g^k\|_2^2 = \|x^k - x^*\|_2^2 - 2t_k g^{kT}(x^k - x^*) + t_k^2\|g^k\|_2^2$

Because of convexity, $f^* - f(x^k) \ge g^{kT}(x^* - x^k)$, thus $\|x^{k+1} - x^*\|_2^2 \le \|x^k - x^*\|_2^2$

$-2t_k[f(x^k) - f^*] + t_k^2\|g^k\|_2^2$.

$\therefore \|x^{k+1} - x^*\|_2^2 \le \|x^0 - x^*\|_2^2 - 2\sum_{\ell=0}^{k} t_\ell[f(x^\ell) - f^*] + \sum_{\ell=0}^{\infty} t_\ell^2\|g^\ell\|_2^2$, $\forall k \in \mathbb{N}$ (*)

Because $\|x^{k+1} - x^*\|_2^2 \ge 0$, $f(x^\ell) - f^* \ge f(x^k_{best}) - f^*$ & $\|g^\ell\|_2^2 \le G^2$, let $\|x^0 - x^*\|_2 = R > 0$

we get $0 \le R^2 - 2[f(x^k_{best}) - f^*]\sum_{\ell=0}^{k} t_\ell + G^2\sum_{\ell=0}^{k} t_\ell^2$. $\forall k \in \mathbb{N}$

{ Fixed step size: $f(x^k_{best}) - f^* \le \frac{R^2 + G^2\sum_{\ell=0}^{k} t_\ell^2}{2\sum_{\ell=0}^{k} t_\ell} = \frac{R^2}{2kt} + \frac{G^2 t}{2}$, $\forall k \in \mathbb{N}$
{ Diminishing step size: $\lim_{k\to\infty} f(x^k_{best}) - f^* \le \frac{R^2 + G^2\sum_{\ell=0}^{\infty} t_\ell^2}{2\sum_{\ell=0}^{\infty} t_\ell} = 0$.

(Note.) Convergence rate: $\frac{R^2}{2kt} \le \frac{\varepsilon}{2}$ & $\frac{G^2 t}{2} \le \frac{\varepsilon}{2} \Rightarrow k = \frac{R^2}{t\varepsilon} = \frac{R^2G^2}{\varepsilon^2}$.

$O(\frac{1}{\varepsilon^2})$ (compared with gradient method)

★ Polyak Step-size $t_k = \frac{f(x^{k-1}) - f^*}{\|g^{k-1}\|_2^2}$ (when f^* is in hand)

(motivated by minimize RHS of (*))

(iv) Non-smooth first-order Method: iterative, update x^k in

$x^0 + \text{span}\{g^0, \cdots, g^{k-1}\}$

(Thm - Nesternov) $\forall k \le n-1$ & any starting point x^0, $\exists$ a function in the problem class s.t. any non-smooth 1st-order methods s.t. $f(x^k) - f^* \ge \frac{RG}{2(1+\sqrt{k+1})}$

(Cannot do better than $O(\frac{1}{\varepsilon^2})$)

3) Proximal Gradient Descent:

A method between gradient-descent & sub-gradient methods, applied to special-structured functions.

(i) Suppose $f(x) = g(x) + h(x)$ (Decomposition)

g: convex, differentiable; h: convex (probably) Not-differentiable

We try to apply gradient-descent on "g", because $\tilde{x}^+ = x - t\nabla g(x) = \underset{z}{\operatorname{argmin}} \{ g(x) + \nabla g(x)^T(z-x) + \frac{1}{2t}\|z-x\|_2^2 \}$, approximate "$g$" by this quadratic function (For the next step). Thus, we need $x^+ = \underset{z}{\operatorname{argmin}} \{ \frac{1}{2t}\|z-(x-t\nabla g(x))\|_2^2 + h(z) \}$

$\triangleq \operatorname{prox}_{h,t}[x - t\nabla g(x)]$ (proximity operator, see in previous pages)

$\therefore$ Every step let $x^{k+1} = \operatorname{prox}_{h,1/t_{k+1}}(x^k - t_{k+1}\nabla g(x^k)) \triangleq x^k - t_{k+1} G_{t_{k+1}}(x^k)$,

where $G_{t_{k+1}}(x^k) = \frac{1}{t_{k+1}}[x^k - \operatorname{prox}_{h,1/t_{k+1}}(x - t_{k+1}\nabla g(x^k))]$.

(ii) Convergence Analysis:

(Thm) Assume $f = g + h$ with properties above, moreover, ∇g is [Lipschitz continuous with constant $L > 0$] & [$\operatorname{prox}_{h,1/t}(x)$ can be evaluated.] Then, with fixed stepsize $t \le 1/L$ we have $f(x^k) - f^* \le \frac{\|x^0 - x^*\|_2^2}{2tk}$ (same as gradient descent!)

(proof.) Similarly to proofs for gradient descent. $C_L^{1,1}$ for $g \Rightarrow g(x^{k+1}) - g(x^k) \le -t\nabla g(x^k)^T G_t(x^k) + \frac{Lt^2}{2}\|G_t(x^k)\|_2^2$. By defn of proximity operator, we get $0 \in \partial h(x^{k+1}) + [x^{k+1} - x^k + t\nabla g(x^k)]\frac{1}{t} \Rightarrow G_t(x^k) - \nabla g(x^k) \in \partial h(x^{k+1})$, by convexity of h,

$\therefore f(x^{k+1}) - f(x^k) = (g+h)(x^{k+1}) - (g+h)(x^k) \le (\frac{Lt^2}{2} - t)\|G_t(x^k)\|_2^2 \le -\frac{t}{2}\|G_t(x^k)\|_2^2$. ("$Lt \le 1$") (Descent!)

In fact, we can find an even stronger inequality, i.e.,

$f(x^{k+1}) - f(z) = (g+h)(x^{k+1}) - (g+h)(z) \le g(x^k) - t\nabla g(x^k)^T G_t(x^k) + \frac{Lt^2}{2}\|G_t(x^k)\|_2^2 - g(z)$

$+ h(x^{k+1}) - h(z) \le \frac{Lt^2}{2}\|G_t(x^k)\|_2^2 - t\|G_t(x^k)\|_2^2 + G_t(x^k)^T(x^k - z)$. ($\le$ ... convexity; convexity)

Take $z = x^*$, it's easy to get that $f(x^{k+1}) - f^* \le G_t^T(x^k)(x^k - x^*) - \frac{t}{2}\|G_t(x^k)\|_2^2$

$\Rightarrow f(x^{k+1}) - f^* \le \frac{1}{2t}(\|x^k - x^*\|_2^2 - \|x^{k+1} - x^*\|_2^2)$, similarly to gradient-descent ....

iii) Accelaration: by accelarating proximal gradient descent, it's possible to achieve the optimal $O(1/\sqrt{\varepsilon})$ convergence rate. (4 IDEAs from Nesterov)

One of the ideas: $x^{(0)} = x^{(-1)} \in \mathbb{R}^n$;

$$\begin{cases} v = x^{k-1} + \frac{k-2}{k+1}(x^{k-1} - x^{k-2}) & (*) \\ x^{(k)} = \text{prox}_{h,1/t}(v^{k-1} - t_k \nabla g(v^{k-1})) \end{cases}$$

(*) carries some momentum from previous iterations.

Thm) With the same assumptions as prox-gradient, accelarated proximal gradient method with fixed step $t \leq 1/L$ satisfies $f(x^k) - f^* \leq \frac{2\|x^0 - x^*\|_2^2}{t(k+1)^2}$ ($O(1/\sqrt{\varepsilon})$)

DDA 4300 Optimization (II) (PS & ML), CUHKSZ

(2022-23, Spring)

- Math Formulations of Optimization

1) $\min_{x} f(x)$ subject to $x \in \mathcal{X}$ (P)

Classifications:
- unconstrained → $\mathcal{X} = \mathbb{R}^n$ (whole set)
- linear / non-linear → both objective & constraints are affine / otherwise
- convex → f - convex function & $\mathcal{X}$ - convex set
- conic linear → linear + $\vec{x}$ in a cone
- (mixed) integer → some of variables are restricted to integer
- Stochastic → f - expected function with random parameters
- fixed-point / min-max → multiple agents with zero-sum objectives

2) Structured Optimization: Conic Linear Programmings (CLP)

$\min_{\vec{x} \in \mathbb{R}^n} \vec{c}^T \vec{x}$ subject to $A\vec{x} = \vec{b}$, $\vec{x} \in K$ ($A \in \mathbb{R}^{m \times n}$, K is a cone)

$\vec{x} \in K \Rightarrow c\vec{x} \in K, \forall c \geq 0$ (assume at origin) (top)

LP-cone

- Linear Programming (LP): non-negative constant cone ($\vec{x} \geq 0$)
- Second-order Cone Programming (SOCP): second-order cone ($x_1 \geq \|\vec{x}(2:n)\|$)
- Semi-definite Cone Programming (SDP): semidefinite matrix cone (e.g. $\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0$) s.psd.

(Dual Question is introduced) Different Cones give different solns.

3) Facility Location Problem: minimize $\sum_j \|\vec{y} - \vec{c}_j\|_p$ ($p=1$ Manhaton distance)

decision of facility; location of client

($\Leftrightarrow$ minimize $\sum_j \delta_j$ subject to $\vec{y} + \vec{x}_j = \vec{c}_j$, $\|\vec{x}_j\|_p \leq \delta_j$, $\forall j$) relaxation to be a cone

p-order conic linear programming (POCP) p-linear cone

4) Sparse Linear Regression:

minimize $\|\vec{x}\|_1$ (LASSO) $\Leftrightarrow$ relaxation of feasible set (or $\min \sum_{i=1}^n y_i$ s.t. $A\vec{x}=\vec{b}$, $-\vec{y}\le\vec{x}\le\vec{y}$ ← linear!)

s.t. $A\vec{x}=\vec{b}$

$\min \sum_{i=1}^n x_{1,i}+x_{2,i}$

s.t. $A(\vec{x}_1-\vec{x}_2)=\vec{b}$, $\vec{x}_1\ge 0$, $\vec{x}_2\ge 0$)

(proof: equivalence of those 3)

☆ Better: minimize $\|\vec{x}\|_p$ ($0<p<1$) → NOT a norm

subject to $A\vec{x}=\vec{b}$

(or add penalty minimize $\|A\vec{x}-b\|_2^2+\beta\|\vec{x}\|_p^p$) unconstrained, non-convex

(Note that previously minimize $\|\vec{x}\|_0=|\{j \mid x_j\neq 0\}|$ is the aim)

subject to $A\vec{x}=\vec{b}$

5) Data Classification: support vector machine & logistic regression

(i) A binary-classification SVM: (red data $\vec{a}_i\in\mathbb{R}^d$, $i=1,\dots,n_1$

blue data $\vec{b}_j\in\mathbb{R}^d$, $j=1,\dots,n_2$)

W.T.F (want to find) a hyper-plane, slope $\vec{x}$ and intersect x_0

to seperate 2 classes

⇒ subject to $\vec{a}_i^T\vec{x}+x_0\ge 1$, $\forall i$

$\vec{b}_j^T\vec{x}+x_0\le -1$, $\forall j$

come from >0 / <0 in optimization $\ge\varepsilon$ / $\le-\varepsilon$ scaling ≥ 1 / ≤ -1

If strict seperation is impossible: minimize β → minimize error size

s.t. $\vec{a}_i^T\vec{x}+x_0+\beta\ge 1$, $\forall i$

$\vec{b}_j^T\vec{x}+x_0-\beta\le -1$, $\forall j$

$\beta\ge 0$

To make it strongly convex → unique soln

change the objective to $\beta+\mu\|\vec{x}\|^2$

(QP)

(ii) Ellipsoidal Seperation: Find a form $\vec{y}^TX\vec{y}+\vec{y}^T\vec{x}+x_0=0$ (intersect, Hessian, slope)

minimize $\mathrm{tr}(X)+\|\vec{x}\|^2$ → scaling

s.t. $\vec{a}_i^TX\vec{a}_i+\vec{a}_i^T\vec{x}+x_0\ge 1$, $\forall i$ (can add β as loss, too)

$\vec{b}_j^TX\vec{b}_j+\vec{b}_j^T\vec{x}+x_0\le -1$, $\forall j$

(SDP) $X\succeq 0$ (pos. semi-def)

(iii) Logistic Regression: use function $\frac{e^{\vec{y}^T\vec{x}+x_0}}{1+e^{\vec{y}^T\vec{x}+x_0}}$ for seperation

$\frac{e^{\vec{a}_i^T\vec{x}+x_0}}{1+e^{\vec{a}_i^T\vec{x}+x_0}}=\begin{cases}1, & \text{if } \vec{a}_i\in C\\ 0, & \text{if } \vec{a}_i\notin C\end{cases}$ Use MLE: to maximize $\prod_{\vec{a}_i\in C}\left(\frac{e^{\vec{a}_i^T\vec{x}+x_0}}{1+e^{\vec{a}_i^T\vec{x}+x_0}}\right)\prod_{\vec{a}_i\notin C}\left(\frac{1}{1+e^{\vec{a}_i^T\vec{x}+x_0}}\right)$

want — Likelihood functn

6) Portfolio Management

(i) minimize $\vec{x}^TV\vec{x}$ → proportion on every asset

s.t. $\vec{r}^T\vec{x}\ge\mu$

$\vec{e}^T\vec{x}=1$ ($\vec{e}=\vec{1}$)

(Quadratic)

(ii) Robust cases: $\min \max_i \vec{x}^TV_i\vec{x}$

s.t. $\min_i \vec{r}_i^T\vec{x}\ge\mu$

$\vec{1}^T\vec{x}=1$.)

Since V_i, $\vec{r}_i$ can be estimated under various senarios.

Rewrite the robust portfolio Management

$\Leftrightarrow$ minimize α

s.t. $\vec{r_i}^T \vec{x} \geq \mu, \ \forall i$

$\vec{x}^T V_i \vec{x} \leq \alpha^2, \ \forall i$

$\vec{e}^T \vec{x} = 1.$

(QCQP) quadratically constrained quadratic problem

$V_i = R_i R_i^T$

$\Leftrightarrow$

$y_i = R_i x_i$

minimize α

s.t. $\vec{r_i}^T \vec{x} \geq \mu, \ \vec{y_i} - R_i \vec{x_i} = 0, \ \forall i$

$\| y_i \| \leq \alpha, \ \forall i$

$\vec{e}^T \vec{x} = 1$

(SOCP)

(iii) Portfolio Selection

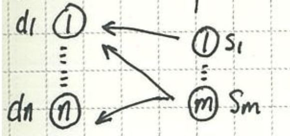
minimize $\vec{x}^T V \vec{x}$

s.t. $\vec{r}^T \vec{x} \geq \mu$

$\vec{e}^T \vec{x} = 1$

$0 \leq \vec{x} \leq \vec{y}, \ \vec{e}^T \vec{y} \leq k, \ \vec{y} \in \{0,1\}^n$ → mixed-integer quadratic programming

7) The transportation problem



minimize $\sum_{ij} c_{ij} x_{ij}$ (c_{ij} → cost, x_{ij} → amount of transport)

s.t. $\sum_j x_{ij} = s_i, \ \forall i$

$\sum_i x_{ij} = d_j, \ \forall j$

$x_{ij} \geq 0$

(can add some null supply points ⇒ m=n matching)

The minimal transportation cost is called Wasserstein Distance (WD) between 2 distributions $\vec{s}$ and $\vec{d}$ (some kind of "distance")

(After normalization, can be interpreted as 2 probability distributions)

(ii) The [Wasserstein Barycenter Problem] is to find a distribution s.t. sum of its WD to every set of distributions is minimized.

(e.g. minimize$_{\vec{s}}$ $WD_l(\vec{s}, \vec{d_l}) + WD_m(\vec{s}, \vec{d_m}) + WD_r(\vec{s}, \vec{d_r})$

s.t. $\sum_i s_i = s, \ s_i \geq 0, \ \forall i$)

different senarios to estimate, every with prob. $\frac{1}{3}$ — Stochastic optimization

"Hierachy" "optimal of optimal" — kind of "probability center"

a high-level decision! > change/reduce to simple case ← write WD_l as min ~ s.t. ~ ...) — LP-formulation

Application: "consencus center of image" (Image: distribution on pixels) > "statistical median" (← min WD) (NOT simple algebric mean!)

(iii) More applications: Combinatorial Auction

To buy a combination $(1, 1, \cdots 0, \cdots 0, 1, \cdots)$ (say, b_i) (win - give 1 and lose - give 0) with some cost c_i, and quantity limit q_i, $\forall i$

"option market"

The aim is to decide the fill ordered $\vec{x}=(x_i)$: (with quantity $0 \le x_i \le q_i$ being sold).

for designer

Profit $= \sum_i c_i x_i - p_j \sum_i b_{ij} x_i$ (p_j: probability)

if No "p_j" in hand → risk-averse → maximize the worst case

↳ maximize $\sum_i c_i x_i - \max_j (\sum_i b_{ij} x_i)$ s.t. $\vec{0} \le \vec{x} \le \vec{q}$

Formulation to LP: maximize $\vec{c}^T \vec{x} - y$ (auxiliary variables)

s.t. $B^T \vec{x} - y \le 0$

$\vec{0} \le \vec{x} \le \vec{q}$

Dual to the LP: dual variable $\vec{y}$ represents the concensus probability of winners!

customers decide cost to buy & also quantities (data-driven from "$\vec{c}, B, \vec{q}$" — from customers)

and maket developer decide how many to sell. (kind of confidence of bet!)

"Information / Prediction Markets" — economics

8) Graph Realization & Sensor Network Location

Locate the place (coordinates) by using anchors & lights.

$\left.\begin{array}{l} \|\vec{x}_i - \vec{x}_j\|_2^2 = d_{ij}^2, \ \forall (i,j) \in N_x \\ \|\vec{a}_k - \vec{x}_j\|_2^2 = \hat{d}_{kj}^2, \ \forall (k,j) \in N_a \end{array}\right\}$ ($\vec{x}_i$: location of "i"; $\vec{a}_k$: anchor) → realize of G, with distances between nodes (N_x), and also vertexes & anchors (N_a) ($\mathbb{R}^d$)

↳ formulate

minimize$_{\vec{x}_i, \forall i}$ $\sum_{(i,j)\in N_x} (\|\vec{x}_i - \vec{x}_j\|^2 - d_{ij}^2) + \sum_{(k,j)\in N_a} (\|\vec{a}_k - \vec{x}_j\|_2^2 - \hat{d}_{kj}^2)$ non-linear, non-convex

matrix → do relaxation! (SOCP or SDP)

9) Stochastic Optimizations & Learning

(i) Modeling: minimize$_{\vec{x}\in X}$ $\mathbb{E}_{F(\xi)}[h(\vec{x}, \xi)]$ ~ some distributions; ξ: uncertainty

(ii) Learning with noises / distortions: (robustness)

(iii) Deep learning on NN (neural-network)

Input vec $\vec{x} \triangleq \vec{y}^0$

Output vec (at layer l) $\vec{y}^l$ → $y_j^\ell = \max\{0, w_{0j}^\ell + \sum_i w_{ij}^\ell y_i^{\ell-1}\}, \ \forall j, \ \ell = 1, \dots, L$ (w_{ij}^ℓ: weight at layer ℓ; ReLu)

Aim: use massive $\vec{x}$ & $\vec{y}^L$ (last-layer) to train reasonable w_{ij}^ℓ ($\vec{y}^L$: vector-mapping (of $\vec{x}$))

★ Net-work verification.

minimize$_{(\vec{x}, \vec{y}^L)}$ $\|\vec{x} - \vec{\hat{x}}\|^2$ → smallest distortion

s.t. $\vec{y}^L(\vec{x}) \in$ a convex region, outside of $\vec{y}^L(\vec{\hat{x}})$

Special case: linearly-constrained quadratic minimization

minimize$_{(\vec{x}, \vec{y}^L)}$ $\|\vec{x} - \vec{\hat{x}}\|^2 + \mu \sum_\ell \sum_j y_j^\ell (y_j^\ell - w_{0j}^\ell - \sum_i w_{ij}^\ell y_i^{\ell-1})$

s.t. $\vec{y}^L \in$ a convex polyhedron, outside $\vec{y}^L(\vec{\hat{x}})$

$y_j^\ell \ge w_{0j}^\ell + \sum_i w_{ij}^\ell y_i^{\ell-1}, \ y_j^\ell \ge 0 \ \ \forall j, \ell, \ \vec{y}^0 = \vec{x}$

10) Reinforcement Learning: Markov Decision / Game Processes (sequential decision-making; outcome partly random & partly in control)

(i) Defn: Given number of states, i, (2 people decide in turns, zero-sum game.)
with A_i – collection of actions, every action $j \in A_i$, is assigned with a cost c_{ij}, and a probability distribution $\vec{p}_j$ to transfer to all possible states at next time period.

A stationary policy $\vec{\pi} = (\pi_1, \cdots, \pi_m)$, specifies action $\pi_i \in A_i$, that the decision maker will take, also leading to a cost-to-go value for every i.

(ii) MDP modeling: (discount factor $r^k \in [0,1]$)

$$\min_{\pi} \sum_{t=0}^{\infty} r^t \mathbb{E}\left[C^{\pi_{i^t}}(i^t, i^{t+1}) \right]$$

(expected discounted sum; cost-to-go with π_{i^t})

Stochastic game (zero-sum):
one partition to minimize, the other to maximize

(iii) Solve MDP: choose the optimal policy s.t. the cost-to-go vector $\vec{y}^*$ satisfying

$$y_i^* = \min\{ c_j + r \vec{p}_j^T \vec{y}^*, \forall j \in A_i \}, \forall i$$

(saddle-point optimization)

optimal policy → $(\pi_i^* = \arg\min\{ c_j + r \vec{p}_j^T \vec{y}^*, \forall j \in A_i \}. \forall i.)$

In the game setting,
$$\begin{cases} y_i^* = \min\{ c_j + r \vec{p}_j^T \vec{y}^*, \forall j \in A_i \}. \forall i \in I^- \\ y_i^* = \max\{ c_j + r \vec{p}_j^T \vec{y}^*. \forall j \in A_i \}. \forall i \in I^+ \end{cases}$$

Reformulated as LP: maximize$_{\vec{y}}$ $\vec{e}^T \vec{y}$ → to make $\vec{y}$ be saddle

subject to
$$\begin{matrix} y_1 - r\vec{p}_j^T \vec{y} \\ \vdots \\ y_m - r\vec{p}_j^T \vec{y} \end{matrix} \begin{matrix} \leq \\ \vdots \\ \leq \end{matrix} \begin{matrix} c_j, & j \in A_1 \\ \vdots & \\ c_j, & j \in A_m \end{matrix}$$

(relaxation)

[Thm] when $\vec{y}^T \vec{e}$ is maximized, there must be at least 1 inequality constraint in A_i, which becomes equal, for every state i.
i.e. $\vec{y}$ is a fixed-point soln.

- Math Foundations of Optimization

1) Cone: C is a cone if $x \in C$ implies $\alpha x \in C$, $\forall$ scalar $\alpha > 0$

Dual of a cone ★ $C^* := \{ y \mid \langle x, y \rangle \geq 0, \forall x \in C \}$ (closed convex cone)

$(C^*)^*$ is the closure of convex hull of C

(e.g.s Polyhedral cone — $\{\vec{x} \mid A\vec{x} \le \vec{0}\}$
finite extreme points $\{A\vec{x} \mid \vec{x} \ge \vec{0}\}$ → conic combination

p-order cone $\{(t,\vec{x}) \mid t \ge \|\vec{x}\|_p\}$ dual p'-order cone $\{(t,\vec{x}) \mid t \ge \|\vec{x}\|_{p'}\}$

p, p' are dual exponent ($p^{-1}+p'^{-1}=1$) → proved by Hölder's inequality

Extreme points of convex set: points which cannot be expressed by convex combinations by other 2 points.)

2) Convex functions: $f\circ\phi$ is convex, if ϕ is convex & f is convex, non-decreasing

(e.g. $z(\vec{b}) := \min_{\vec{x}} f(\vec{x})$ s.t. $A\vec{x}=\vec{b}$, $\vec{x}\ge 0$ is convex, where f is convex (proof by defn $z(\vec{b}) \triangleq f(\vec{x}(\vec{b}))$ unique soln. & $f(\vec{x}(\vec{b}_1+\vec{b}_2)) \le f(\vec{x}(\vec{b}_1)+\vec{x}(\vec{b}_2))$ ↑ sub-additivity (inside f) & convexity of f

$\log(1+e^{\vec{a}^T\vec{x}+x_0})$ is convex.) ★ ↑composition

3) Carathéodory Theorem for generation of cone,

Given $A\in\mathbb{R}^{m\times n}$, let convex polyhedral cone $C=\{A\vec{x} \mid \vec{x}\ge\vec{0}\}$.

For any $\vec{b}\in C$, $\vec{b}=\sum_{i=1}^{d}\vec{a}_{ji}x_{ji}$, $x_{ji}\ge 0\ \forall i$, $\vec{a}_{j1},\cdots\vec{a}_{jd}$ are l.indep. from $\vec{a}_1,\cdots,\vec{a}_n$ — reduced/mean representation

↳ $\{\vec{x} \mid A\vec{x}=\vec{b},\ \vec{x}\ge\vec{0}\}$, basic soln $\vec{x}_B$ exists, due to above thm.

BFS must be an extreme or corner point, as a direct corollary from Carathéodory's.

Hyper-plane: $H \triangleq \{\vec{x} \mid \vec{a}^T\vec{x}=b\}$

★ Seperating hyperplane theorem: Let C be a closed, convex set. $\vec{b}$ is NOT in C. then

$\exists \vec{y}$ s.t. $\langle\vec{b},\vec{y}\rangle > \sup\langle\vec{x},\vec{y}\rangle$, $\forall\vec{x}\in C$.

($\Leftrightarrow$) $\exists H \triangleq \{\vec{z} \mid \langle\vec{y},\vec{z}\rangle = s\}$, s.t. $\vec{b}$ & C are seperated by H.

(proof.) omitted. Note that consider dual $(\{\alpha\vec{b} \mid \alpha>0\})^* \notin C^*$, otherwise $(C^*)^*$ contains $\vec{b}$, contradictory with closedness & convexity.

4) Farka's lemma: proved from seperating hyperplane theorem.

(Alternating system: $\{\vec{x} \mid A\vec{x}=\vec{b};\ \vec{x}\ge 0\}$ & $\{\vec{y} \mid A^T\vec{y}\le 0,\ \vec{b}^T\vec{y}>0\}$, only 1 (exactly) is non-empty)

Farka's lemma for general closed convex cones:

$\{\vec{x} \mid A\vec{x}=\vec{b},\ \vec{x}\in K\}$ & $\{\vec{y} \mid -A^T\vec{y}\in K^*,\ \vec{b}^T\vec{y}=0\}$ cannot be alternating system, generally. (Counter-e.g. (neither non-empty): consider $K=S_+^2$ (p.s.d, 2x2)

$A := (a_1, a_2)$, with $a_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $a_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$Ax=\begin{bmatrix}tr(a_1^Tx)\\tr(a_2^Tx)\end{bmatrix}$ (or $\begin{bmatrix}\langle a_1,x\rangle\\\langle a_2,x\rangle\end{bmatrix}$) and $b=\begin{pmatrix}0\\2\end{pmatrix}$

Note that, here $C:=\{Ax \mid x\in K\}$ is NOT closed.)

☆ Let K be a closed convex cone, if $\exists\ \vec{y}$ s.t. $-y^TA \in \text{int } K^*$, then $C := \{Ax \mid x \in K\}$ is a closed convex cone.

Consequently, $\{Ax = \vec{b},\ x \in K\}$ & $\{-\vec{y}^TA \in K^*,\ \vec{b}^T\vec{y} = 1 \text{ (or } > 0)\}$ are an alternative pair.

5) Dual of Conic LP:

(CL,P) minimize $\langle C, X\rangle$
subject to $\langle a_i, x\rangle = b_i,\ \forall i = 1, \dots, n$
$x \in K$ → closed, pointed & convex

(CL,D) maximize $\vec{b}^T\vec{y}$
subject to $\sum_{i=1}^{m} y_i a_i + s = c$
$s \in K^*$ ← dual slack; dual cone

☆ contain no straight line

(Theorem) The dual of the dual is the primal (in conic linear programming)

(i) Dual of transportation problem:

(P) minimize $\langle C, X\rangle = tr(C^TX)$
s.t. $\langle E_j, X\rangle = s_j$
$\langle E_i^T, X\rangle = d_i$
$X \in$ non-negative constant cone

(D) maximize $\vec{u}^T\vec{s} + \vec{v}^T\vec{d}$ (another way to charge)
$\vec{u}, \vec{v}$
s.t. $\sum_i u_i E_i + \sum_j v_j E_j^T \le C$
$(\Leftrightarrow u_i + v_j \le c_{ij},\ \forall i, j)$ no more than previous charge

Interpretation: charge u_i when sending 1 unit from i, & ... v_j ... accepting 1 ... from j

(ii) The dual of MDP-LP.

(D) minimize $\sum_i \sum_{j_k \in A_i} c_{j_k} x_{j_k}$ ← state action fluency / flux
$\vec{x}$
subject to $\sum_i \sum_{j_k \in A_i} (\vec{e_i} - \gamma \vec{P_{j_k}}) x_{j_k} = \vec{e}$
(flow in − flow out with price)
$x_{j_k} \ge 0,\ \forall i, j_k \in A_i$

expected PV of # times that in i, taking action j_k. (e.g. $1 + r + r^2$ ↑ period 1, ↑ period 2, ...)

(iii) Nash-Equilibrium: (2-person zero-sum game)

e.g., P (payoff matrix) $= \begin{bmatrix} +3 & -1 & -4 \\ -3 & +1 & +4 \end{bmatrix}$ (2 people, column & row players)

random strategies: a vector of probabilities (with states)

☆ Nash Equilibrium: pure strategy — the expected payout (dominating) ≥ payout for the other action; randomized strategy — the expected payout for each part → other's choice of the action (in the strategy must be the same), & these ≥ actions that are not part of the action.

"Column" strategy: maximize v s.t. $\vec{x}$ $v\vec{e} \leq P\vec{x}$, $\vec{e}^T\vec{x} = 1$, $\vec{x} \geq 0$ (decision) — due to Nash Equilibrium

"Row" strategy: minimize u s.t. $\vec{y}$ $u\vec{e} \geq P^T\vec{y}$, $\vec{e}^T\vec{y} = 1$, $\vec{y} \geq 0$ — dual equal to each other

6) Duality Theorems for CLP.

(Weak Duality) $\langle C, X\rangle - \vec{b}^T\vec{y} = \langle X, S\rangle \geq 0$ (for CLP), ★ Duality gap; $\forall$ feasible X of (CLP), and $(\vec{y}, S)$ of (CLD)

(Strong Duality) Let $X^* \in F_p$ and $(y^*, S^*) \in F_d$. Then, $\langle C, X^*\rangle = \vec{b}^T\vec{y}^*$, implies that X^* is optimal for (CLP) & $(\vec{y}^*, S^*)$ ··· for (CLD)

(proof.) Let x^* be a minimizer of (LP). Then, the following system: (for LP)

$$\begin{cases} A\vec{x}' - \vec{b}\tau = \vec{0} \\ \begin{bmatrix}\vec{x}' \\ \tau\end{bmatrix} \geq \vec{0} \\ \vec{c}^T\vec{x}' - (\vec{c}^T\vec{x}^*)\tau = -1 < 0 \end{cases}$$

has no solution, for otherwise $\vec{x}'/\tau$ is feasible ($\tau > 0$) for (LP) — $\left(\begin{matrix}\min \vec{c}^T\vec{x} \\ \text{s.t. } A\vec{x} = \vec{b} \\ \vec{x} \geq 0\end{matrix}\right)$ "$(\vec{x}', \tau)$"

$\vec{c}^T\vec{x}'/\tau < \vec{c}^T\vec{x}^*$, contradictory!

if $\tau = 0$, $\vec{x}' + \vec{x}^*$ is feasible for (LP), $\vec{c}^T(\vec{x}' + \vec{x}^*) = \vec{c}^T\vec{x}^* - 1 < \vec{c}^T\vec{x}^*$, contra!

∴ With Farka's lemma, $\exists\, \vec{y}^*$ s.t. $\vec{c} - A^T\vec{y}^* \geq 0$, $-\vec{c}^T\vec{x}^* + \vec{b}^T\vec{y}^* \geq 0$.

use weak-duality ⇒ ∃ such optimal $\vec{y}^*$.

For general CLP: ∃ duality gap in some cases, i.e., strong duality does NOT always hold.

e.g1. In general CLP, $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $a_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ & $\vec{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$K = S_+^3$ (SDP-cone)

(zero-duality gap but NOT attainable)

e.g2. $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $a_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ & $b_1 = 2$. $K = S_+^2$. ★ (The dual does NOT have interior)

(proof.) (i) Let F_p be non-empty & has an interior (condition), z^* be its infimum. ↖ 0-duality gap but may not attainable

Consider alternative system:

$$\begin{cases} Ax - \vec{b}\tau = \vec{0} \\ \langle C, X\rangle - z^*\tau < 0 \\ (x, \tau) \in K \times \mathbb{R}_+ \end{cases} \quad \& \quad \begin{cases} \vec{y}^T A + S = C \\ -\vec{b}^T\vec{y} + K = -z^* \\ (S, K) \in K^* \times \mathbb{R}_+ \end{cases}$$

↑ infeasible ↖ ∃ $\vec{y}^*$, S^* & use weak duality, done

(ii) WTS: ∃ $x \in F_p$ s.t. $\langle C, X\rangle = z^*$ (attainable infimum) ⇐ (given) F_d feasible, $\text{int}\, F_d \neq \phi$ ↑ 1-more condition

& z^* is the sup of (CLD) ↖ 0-duality gap & attainable

iii) Suppose $\mathcal{F}_d = \emptyset$. F_p is feasible & $\text{int}\, F_p \neq \emptyset$. Then, $\exists\ \bar{x} \in \text{int}\, K$, $\bar{\tau} > 0$

s.t. $A\bar{x} - \vec{b}\bar{\tau} = 0$, $(\bar{x}, \bar{\tau}) \in \text{int}(K \times \mathbb{R}_+)$.

$\forall\ z^*$, consider again the alternative pair above. the latter becomes infeasible

$\therefore$ The former has soln (x_{z^*}, τ_{z^*}), $\forall\ z^*$. If $\tau_{z^*} > 0$. $\langle C, x_{z^*}/\tau_{z^*} \rangle < z^*$, if $\tau = 0$
$\uparrow \forall z^*$

$\hat{x} + \alpha x_{z^*}$ is feasible if $\hat{x}$ is feasible, obj $\mapsto -\infty$ as $\alpha \mapsto \infty$ (unbdd)

Summary: (i) Both (CLP) & (CLD) feasible $\Rightarrow$ { one has interior — 0-duality gap; both have interior — attainable opt. soln with 0-duality gap }

(ii) One of (CLP), (CLD) infeasible, another is feasible & has interior $\Rightarrow$ it is unbdd.

Posible relations of SDP-LP:

P \ D	Finite opt.	Unbdd	Infeasible
Finite opt.	✓	—	✓
Unbdd	—	—	✓
Infeasible	✓	✓	✓

"new from LP"

Rules for taking dual of CLP.

(CLP) $\min \sum_k \langle C_k, x_k \rangle$ s.t. $\sum_k A_k x_k = \vec{b}$, $x_k \in K_k$, $\forall k$ $\quad\leftrightsquigarrow\quad$ (CLD) $\min \vec{b}^T \vec{y}$ s.t. $\vec{y}_k^T A_k + S_k = C_k$, $\forall k$, $S_k \in K_k^*$, $\forall k$

Optimality conditions: (for SDP)

$$\begin{cases} \langle X, S \rangle = \text{tr}(X^T S) = 0 \\ AX = \vec{b} \\ -\vec{y}^T A - S = -C \\ X, S \succcurlyeq 0 \end{cases} \Leftrightarrow (XS)_{ij} = 0,\ \forall i,j$$

(under SDP cone, could be proved)

☆ Equivalence of Convex Optimization & CLP:

(CO) $\min_{\vec{x}, \alpha} \alpha$ s.t. $C_0(\vec{x}) - \alpha \leq 0$, $C_k(\vec{x}) \leq 0$ $\forall k$ $\xrightarrow{\text{the same form}}$ $\min_{\vec{x}} \vec{c}^T \vec{x}$ s.t. $C_i(\vec{x}) \leq 0$, $\forall i$ $\xrightarrow{\text{CLP form}}$ $\min_{\tau, \vec{x}} \begin{bmatrix} 0 \\ \vec{c} \end{bmatrix}^T \begin{bmatrix} \tau \\ \vec{x} \end{bmatrix}$ s.t. $\begin{bmatrix} 1 \\ \vec{0} \end{bmatrix}^T \begin{bmatrix} \tau \\ \vec{x} \end{bmatrix} = 1$, $(\tau; \vec{x}) \in \bigcap_{k=1}^m K_k$

where $K_i := \overline{\{(\tau; \vec{x}) \mid \tau > 0 \ \&\ \tau C_i(\vec{x}/\tau) \leq 0\}}$ ← closure, ← cone

How to construct the dual cone: (of $\bigcap_i K_i$)

i.e. $\{(K,S) \mid K\tau + \vec{s}^T\vec{x} \geq 0, \forall (\tau;\vec{x}) \in \bigcap_i K_i\}$, w.l.o.g. let $\tau = 1$.

$\Leftrightarrow \{(K,S) \mid K + \vec{s}^T\vec{x} \geq 0, \forall \vec{x} \text{ s.t. } C_i(\vec{x}) \leq 0, \forall i\}$

consider $\phi(\vec{s}) := \inf\{\vec{s}^T\vec{x} \mid C_i(\vec{x}) \leq 0, \forall i\}$, the dual cone $K^* = \{(K;\vec{s}) \mid K + \phi(\vec{s}) \geq 0\}$

☆ Robust Optimization: application of duality

Robust Min-max model $\min_{\vec{x}} \max_{\vec{u}} \{\vec{u} \geq \vec{0}, \vec{u} \leq \vec{e}\} (\vec{c} + C\vec{u})^T\vec{x}$ (choice of adversary)

s.t. $A\vec{x} = b$; $\vec{x} \geq \vec{0}$ (choice of player)

(change adversary's decision to its dual → Robust Min-min model)

$\min_{\vec{x}} \min_{\vec{y}} \{\vec{y} \geq C^T\vec{x}, \vec{y} \geq \vec{0}\} \vec{c}^T\vec{x} + \vec{e}^T\vec{y}$

s.t. $A\vec{x} = b$, $\vec{x} \geq 0$ > put together, we get a pure opt. question (min…)

Combinatorial Auction: the meaning of dual

(D-CAP) $\max_{p,y} \vec{q}^T\vec{y}$ (decision / fraction; price information gaps)

s.t. $B\vec{p} + \vec{y} \geq \vec{c}$ ($\vec{p}$: state price (concensus probability))

$\vec{e}^T\vec{p} = 1$

$(\vec{p};\vec{y}) \geq \vec{0}$

☆ Online L-P: $\max_{\vec{x}} \sum_t \pi_t x_t$ (x_t: decision variable)

s.t. $\sum_{t=1}^n a_{it} x_t \leq b_i$, $\forall i$ (b_i: total amount) only know $(n, \vec{b})$ at the start.

$0 \leq x_t \leq 1$, $\forall t$ Data of x_t is revealed sequentially.

An algorithm A is c-competitive iff $\mathbb{E}_\sigma[\sum_{t=1}^n \pi_t x_t(\sigma, A)] \geq c \cdot opt(A,\pi)$, $\forall (A,\pi)$ (off-line)

Strategy: $x_t = 0$, $1 \leq t \leq \varepsilon n$, solve $\max_{\vec{x}} \sum_{t=1}^{\varepsilon n} \pi_t x_t$ s.t. $\sum_{t=1}^{\varepsilon n} a_{it} x_t \leq b_i$, $\forall i$; $0 \leq x_t \leq 1$, $\forall t$

⇒ get dual optimal sln: $\hat{\vec{p}}$

Determine, future $x_t = \begin{cases} 0, & \text{if } \pi_t \leq \hat{\vec{p}}\vec{a}_t \\ 1, & \text{otherwise} \end{cases}$

as long as $a_{it} x_t \leq b_i - \sum_{j=1}^{t-1} a_{ij} x_j$, $\forall i$, otherwise $x_t = 0$

7) Support-Size & Rank of CLP Solutions & Applications

(i) Let x^*, s^* be optimal solutions with 0-duality gap:

$|supp(x^*)| + |supp(s^*)| \leq n$ ← prob. size.

[Strict Complementarity] If (LP), (LD) are both feasible, ∃ a pair of strict complementarity solutions $x^* \in F_p$ & $(y^*, s^*) \in F_d$ s.t. $\langle x^*, s^* \rangle = 0$

& $|supp(x^*)| + |supp(s^*)| = n$