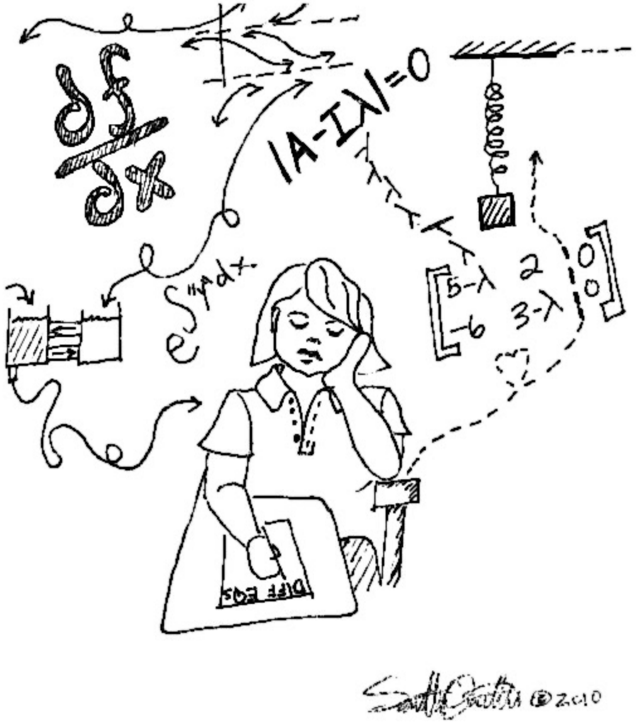

Ordinary Differential Equations

MAT 2002 Notebook

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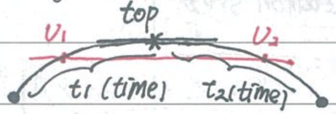
Date: 2022 Jan 12

MAT 2002 Odinary Differential Equations (ODE)

- Introductions & Motivations

☆ example 1) In the basketball game, consider 2 players & a ball-passing process.

(want to compare t_1 & t_2)



$v_1 > v_2$ for any height (understand)

" $t_1 < t_2$ " — calculation by ODEs (quantify) energy loss

☆ example 2) In the coffee shop, consider two cups of coffee A & B, with (A + cream) immediately, (B + cream) in 5 minutes.

(want to compare temperatures A & B) " A = B " (cream in room temperature)

☆ example 3) Tacoma Narrows Bridge (resonance)

Design of TMD (tuned mass damage)

- First-order differential equation (DE)

1) General Form: $F(t, y, y') = 0$ or $F(t, y(t), y'(t)) = 0$

(general first order DE)

Another First-order DE form: $y' = f(t, y)$ (get y' out)

2) First-order linear equation $\boxed{y' + a(t)y = b(t)}$ inhomogeneous / non-homogeneous

special cases: i) $a \equiv 0$, $y = \int b(t)dt$

ii) $b \equiv 0$, $\frac{y'}{y} = -a(t) \Rightarrow y = Ce^{-\int a(t)dt}$, $C \in \mathbb{R}$

homogeneous (possibly, $y = 0$)

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Another more convincing way: $e^{\int a(t)dt} y' + a(t)e^{\int a(t)dt} y = 0$ (integrating factor: $e^{\int a(t)dt}$)

$\Rightarrow (e^{\int a(t)dt} \cdot y)' = 0$, meaning that $y = Ce^{-\int a(t)dt}$, $C \in \mathbb{R}$

($y' + a(t)y = 0$ homogeneous)

$\Rightarrow$ Modify as $\boxed{y = C(t)e^{-\int a(t)dt}}$, choose $C(t)$ s.t. (inhomogeneous) "Variation of Parameter"

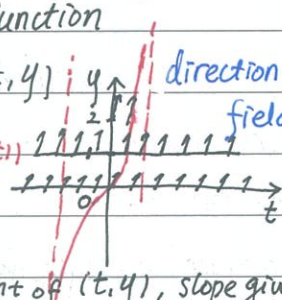
$C'(t)e^{-\int a(t)dt} - C(t)a(t)e^{-\int a(t)dt} + a(t)C(t)e^{-\int a(t)dt} = b(t)$

$\Leftrightarrow C'(t) = b(t)e^{\int a(t)dt}$.

Sol. for inhomogeneous equation $y(t) = e^{-\int a(t)dt}\left(\int b(t)e^{\int a(t)dt}\right)$

3) Equation (DE) + initial value $\longrightarrow$ function

4) Geometric Significance of $y' = f(t, y)$ — slope of $(t, y(t))$



direction / slope field

(e.g. $y' = 1 + y^2$ & $y(0) = 0$) $\rightarrow y = \tan t$

For non-linear equation, the solution may not be everywhere

(At any point of (t, y), slope given by $y' = f(t, y)$)

5) Separable Equations: (with the form) $\frac{dy}{dt} = \frac{g(t)}{f(y)}$

(solve — $\int f(y)dy = \int g(t)dt$)

(e.g. 1 Malthus Equation: $p(t)$ — population of a species

r = birth rate − death rate. Then, ($\frac{dp}{dt} = rp$)

$\Rightarrow \boxed{p(t) = p(0)e^{rt}}$ (not reasonable because $p(t) \to \infty$) only under unbdd resources / small population

(e.g. 2 Logistic Equation (see MAT3300 Notes on previous pages)

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($\frac{dp}{dt} = rp(1-\frac{p}{k})$) (k — carrying capacity) natural growth rate

Sol. $\frac{dp}{dt} = p(r-\frac{rp}{k}) = p(a-bp) \Rightarrow \int \frac{dp}{p(a-bp)} = \int dt$

$\Rightarrow \frac{1}{a}(\int \frac{dp}{p} + \int \frac{dbp}{a-bp}) = \int dt \Rightarrow \frac{1}{a}\ln|p| - \frac{1}{a}\ln|a-bp| = t$

$\therefore at + C = \ln\frac{p}{|a-bp|}$

$$\boxed{\begin{cases} \text{Case I: } p(0) < \frac{a}{b}, & \frac{p}{a-bp} = \frac{p(0)}{a-bp(0)}e^{at} \\ \text{Case II: } p(0) > \frac{a}{b}, & \frac{p}{bp-a} = \frac{p(0)}{bp(0)-a}e^{at} \\ \text{Case III: } p(0) = \frac{a}{b}, & p(t) \equiv \frac{a}{b} \text{ (unique solution)} \end{cases}}$$

(case I. (e.g.) $p = \frac{ap_0}{bp_0 + (a-bp_0)e^{-a(t-t_0)}}$)

(e.g.3 A commercial fishery is estimated to have carrying capacity 10000 kg of certain kind of fish. Suppose the annual growth rate of the total fish population (measured by weight) is governed by the logistic equation $p' = p(1-\frac{p}{10000})$.

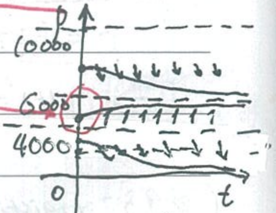
Q: Suppose after waiting for a certain time t, the owner decides to harvest 2400 kg of fish annually at a constant rate. Find the minimum t. (Initially 2000 kg total)

Sol. $\Rightarrow p = 2500e^t/(1+\frac{1}{4}e^t) = \frac{10000}{4e^{-t}+1}$ (initial population)

Equation after harvesting $p' = p(1-\frac{p}{10000}) - 2400$

$\Rightarrow \frac{dp}{dt} = -\frac{1}{10000}(p-4000)(p-6000)$

wait at least $p \geq 4000$



$4000 = \frac{10000}{4e^{-t}+1}$, $t \doteq 1$ year (1 year is enough.))

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6) Orthogonal Trajectories: (a kind of applied problems)

(i) e.g. A family of parabolas $y = cx^2, c \in \mathbb{R}$

Q: To find the orthogonal trajectories of these curves

Sol. $-1 = (\frac{dy}{dx})\cdot 2cx = (\frac{dy}{dx})\cdot\frac{2x\cdot y}{x^2} = (\frac{dy}{dx})\cdot\frac{2y}{x}$ (given $x \neq 0$)

$\Rightarrow \int 2y\,dy + \int x\,dx = 0 \Rightarrow \frac{x^2}{2} + y^2 = k_{\geq 0}$ (a family of ellipse)

(ii) Consider $F(x,y,c)=0$, giving $F_x + F_y y' = 0$ (I.F.T)

$\Rightarrow$ Family of OT: $y' = \frac{F_y(x,y,c)}{F_x(x,y,c)}$ (c is gotten from $F(x,y,c)=0$)

7) Newton's Cooling Law

The rate of change of temperature $T(t)$ of a body immersed in a medium of constant temp A is proportional to $A - T(t)$

i.e. $\frac{dT}{dt} = k(A - T(t))$, $k > 0$ (constant)

(e.g.2) Assume T_x, T_o, room temp A, coffee T_c, cream T_m

ratio between $\frac{\text{cream}}{\text{coffee}}$: r

$|A - T| = ce^{-kt}$, $c > 0$

$$\boxed{\begin{cases} \text{(i) } T(0) > A, & T(t) = A + (T(0)-A)e^{-kt} \\ \text{(ii) } A = T(0), & T(t) = A \\ \text{(iii) } A > T(0), & T(t) = A - (A - T(0))e^{-kt} \end{cases}}$$

Back to e.g.2 $T_o(10) = A + (T_o(0) - A)e^{-10k}$
$= A + (\frac{T_c + rT_m}{1+r} - A)e^{-10k}$

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$\tilde{T}_x(10) = A + (\tilde{T}_x(0)-A)e^{-10k} = A + (T_c(0)-A)e^{-10k}$

$\therefore T_x(10) = \frac{\tilde{T}_x(10)+rT_m}{1+r} = \frac{A+(T_c-A)e^{-10k}+rT_m}{1+r}$

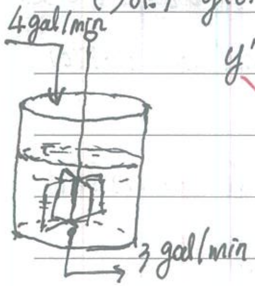
$T_x(10)-T_0(10) = \frac{1}{1+r}[rT_m - rA - e^{-10k}(rT_m-rA)]$

$= \frac{r}{1+r}(T_m-A)(1-e^{-10k})$

sign-determined

(e.g.1 A 120gal tank contains initially 90lb of salt dissolved in 90-gal of water. Brine containing 2 lb/gal of salt flows into the tank at 4 gal/min & the well-stirred mixture flows out at the rate of 3 gal/min. How much salt doest it contain when it's full.

(Sol.) $y(t)$ = the salt in tank ; volume. $90+t$ ($\rho = \frac{y}{90+t}$)



$y' = \text{inflow} - \text{outflow} = 2\ \text{lb/gal} \cdot 4\ \text{gal/min} - \frac{y}{90+t}\cdot 3\ \text{gal/min}$

$\Rightarrow y' = 8 - \frac{3y}{90+t}$, $y(0) = 90\text{lb}$

Solving the eqn, $y(t) = 2(90+t) - \frac{90^4}{(90+t)^3}$

when full, $y(30) = 2\times120 - \frac{90^4}{120^3} \approx 202$)

(e.g.2 In a naval exercise, a distroyer D is hunting a submarine S. Suppose D at (9,0) detects S at (0,0) & at the same time S also detects D. Assume S will drive immediately at full speed (15km/h) in a straight course of unknown direction. Which path should D follow to be certain of passing directly

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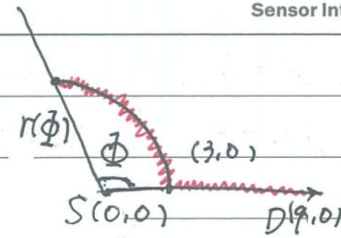
over S with speed 30km/h



$2r(\Phi) = 6 + \int_0^{\Phi} \sqrt{r^2(\theta)+r'^2(\theta)}\,d\theta$

Take derivatives $\Rightarrow$ $r'(\Phi) = \pm\frac{1}{\sqrt{3}}\, r(\Phi)$ $\Rightarrow r(\Phi) = 3e^{\frac{1}{\sqrt{3}}\Phi}$

8) Uniqueness of solution

☆(Thm) If f & $\frac{\partial f}{\partial y}$ both are continuous in the neighborhood of (t_0, y_0), then (*) has at most 1 solution (in a rectangular region) R

$(*) \begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

(proof) Suppose y_1, y_2 are 2 sols of (*). $y_1' = f(t,y_1) \overset{\text{Integral}}{\Longrightarrow} y_1 = y_0 + \int_{t_0}^t f(s, y_1(s))\,ds$

similarly, $y_2 = y_0 + \int_{t_0}^t f(s, y_2(s))\,ds$

$\therefore |y_1(t) - y_2(t)| = \left|\int_{t_0}^t f(s,y_1(s)) - f(s,y_2(s))\,ds\right|$

$\leq \int_{t_0}^t |f(s,y_1(s)) - f(s,y_2(s))|\,ds$

$\overset{\text{MVT}}{=} \int_{t_0}^t \boxed{\left|\frac{\partial f}{\partial y}(s,\tilde{y}(s))\right|} \cdot |y_1(s)-y_2(s)|\,ds \leq M\int_{t_0}^t |y_1(s)-y_2(s)|\,ds$

bdd in a small neighborhood ("□")

$w(t) := |y_1(t) - y_2(t)|$

$w(t) \leq M \underbrace{\int_{t_0}^t w(s)\,ds}_{\text{set } Z(t)} \Rightarrow Z'(t) \leq M Z(t)$

$[e^{-Mt}Z(t)]' \leq 0 \overset{\text{decreasing}}{\Longrightarrow} e^{-Mt}Z(t) \leq e^{-Mt_0}\underbrace{Z(t_0)}_{=0}$

$\Rightarrow Z(t) \leq 0$ but $w(t) \geq 0 \Rightarrow Z(t) = w(t) = 0$.

(e.g. $\begin{cases} y' = \frac{3}{2} y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$, $y_1 \equiv 0$, $y_2 = \pm t^{\frac{3}{2}}$)

9) Existence of solution

Consider the initial value problem (IVP) $(D) \begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

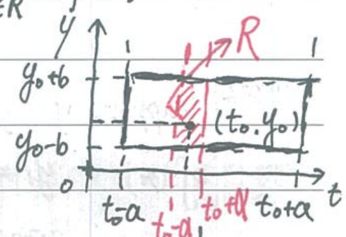
choose a rectangular region R,

f, f_y cts in R, thus bounded in R.

(Let $M = \max_{(t,y)\in R} |f(t,y)|$, $L = \max_{(t,y)\in R} |f_y(t,y)|$)

(proof.) Transform to an integral equation

(I) $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$



$y_0(t) = y_0$ (First investigate)

$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))\,ds$ (A better estimation)

$\downarrow$ continue this way

$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\,ds$, $\forall k \in \mathbb{N}$.

$t \in V_\alpha(t_0)$

$\alpha = \min\{a, \frac{b}{M}\}$

Sequence of function $\{y_i(t)\}_{i=0}^{\infty}$

Goal: To show this sequence is convergent to the solution.

$y_{n+1}(t) = y_0(t) + \sum_{k=0}^{n} (y_{k+1}(t) - y_k(t))$. Take limit, we get

$\lim_{n\to\infty} y_n(t) = y_0(t) + \sum_{k=0}^{\infty} (y_{k+1}(t) - y_k(t))$

Because $|y_{k+1}(t) - y_k(t)| = \left|\int_{t_0}^{t} (f(s, y_k(s)) - f(s, y_{k-1}(s)))\,ds\right| \leq$

(only consider $t > t_0$)

$\int_{t_0}^{t} |f(s, y_k(s)) - f(s, y_{k-1}(s))|\,ds \overset{MVT}{=\!=} \int_{t_0}^{t} |y_k(s) - y_{k-1}(s)||f_y(s, \hat{y}(s))|\,ds$

$\leq L \int_{t_0}^{t} |y_k(s) - y_{k-1}(s)|\,ds$

(Condition for the last "$\leq$": $(s, y_k(s))$, $(s, y_{k-1}(s))$ lie in R)

Reason: $|y_1(t) - y_0| \leq \int_{t_0}^{t} |f(s, y_0(s))|\,ds \leq M \int_{t_0}^{t} ds \leq b$

By induction, $|y_k(t) - y_0| \leq \int_{t_0}^{t} |f(s, y_{k-1}(s))|\,ds \leq M \int_{t_0}^{t} ds \leq b$, $\forall k$, $|t - t_0| < \alpha$.

$k=0$; $|y_1(t) - y_0(t)| \leq M(t - t_0)$ ($\leq M\alpha$)

$k=1$; $|y_2(t) - y_1(t)| \leq L \int_{t_0}^{t} |y_1(s) - y_0(s)|\,ds \leq L \int_{t_0}^{t} M(s - t_0)\,ds = LM \frac{(t-t_0)^2}{2!}$

Recursively we get $|y_{k+1}(t) - y_k(t)| \leq L^k M \frac{(t-t_0)^{k+1}}{(k+1)!}$

By Weierstrass-M test $y_0(t) + \sum_{k=0}^{\infty} (y_{k+1}(t) - y_k(t))$ converges uniformly on R

($\sum_{k=0}^{\infty} \frac{L^k M (t-t_0)^k}{L\,k!} \to \frac{M}{L} e^{L(t-t_0)}$ by Taylor theorem)

Because $|y'_{k+1}(t) - y'_k(t)| = |f(t, y_k(t)) - f(t, y_{k-1}(t))| \leq L\,|y_k(t) - y_{k-1}(t)|$

$\leq L^k M \frac{(t-t_0)^{k+1}}{(k+1)!}$

, we get (by M test) $\sum_{k=0}^{\infty} (y'_{k+1}(t) - y'_k(t)) + y'_0(t)$ converges uniformly on R.

$\therefore$ According to term-by-term differentiability ($y \triangleq y_0(t) + \sum_{k=0}^{\infty} (y_{k+1} - y_k)$)

$\begin{cases} y' = y'_0(t) + \sum_{k=0}^{\infty} (y'_{k+1} - y'_k) = \lim_{k\to\infty} f(t, y_k(t)) = f(t,y) \\ y(t_0) = y_0(t_0) + \sum_{k=0}^{\infty} (y_{k+1}(t_0) - y_k(t_0)) = y_0. \end{cases}$

By validation, y is a solution. ($y(t) = y_0(t) + \sum_{k=0}^{\infty} [y_{k+1}(t) - y_k(t)]$

$t \in (t_0, t_0 + \alpha)$)

Another way. y is a solution to (I) (take limit). + term-by-term continuity $\Rightarrow$ differentiable y.

$t < t_0$ same way.

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(e.g. $\begin{cases} y' = 1+y^2 \\ y(0)=0 \end{cases}$. $y_0(t)=0$; $y_1(t)=t$; $y_2(t)=t+\frac{t^3}{3}$

$\alpha = \max_{b>0} \frac{b}{1+b^2} = \frac{1}{2}$ $[-\frac{1}{2}, \frac{1}{2}]$

$\underbrace{y_3 = t + \frac{t^3}{3} + \frac{2t^5}{15}} + \frac{1}{63}t^7$ ...... $\{y_k(t)\}_{k=0}^{\infty}$

several terms of expasion of tant (which is y)

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n+2)!} t^{2n+1}$$

(B_n — Bernoulli number, Several expressions $\begin{cases} \frac{x}{e^x-1} = \sum_{n=0} \frac{B_n}{n!} x^n \\ B_n = \frac{n!}{2\pi i} \oint_C \frac{z}{e^z-1} \frac{dz}{z^{n+1}} \end{cases}$)

10) Exact Equations & unsolvable ODEs

1° Bring linear & seperable equations together, we get the form $\frac{d}{dt}\Phi(t,y) = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial y} \cdot \frac{dy}{dt} = 0$.

2° Consider more general form: $M(t,y) + N(t,y)\frac{dy}{dt} = 0$

(1) $N = \frac{\partial \Phi}{\partial y}$, $M = \frac{\partial \Phi}{\partial t}$ (If holds) $\Rightarrow \frac{\partial N}{\partial t} = \frac{\partial^2 \Phi}{\partial y \partial t} = \frac{\partial^2 \Phi}{\partial t \partial y} = \frac{\partial M}{\partial t}$ ($N_t = M_y$) $\to$ exact equation $\to$ necessary condition

Remark: $\boxed{\text{"iff" condition (Thm) Given } M(t,y) \,\&\, N(t,y) \in C^1(R) \text{ when } R \text{ is a rectangle on the } t\text{-}y \text{ plane. Then, } (\exists \Phi = \Phi(t,y) \text{ s.t. } \Phi_t = M \,\&\, \Phi_y = N) \text{ exactly when } (N_t = M_y)}$

gives integrals

(Proof. "$\Rightarrow$" Done because of $C^1(R)$

"$\Leftarrow$" Set $\Phi(t,y) = \int M(t,y)dt + \int [N(t,y) - \int M_y(t,y)dt] dy$

$\Phi_t = M + \int (N_t - M_y) dy = M$; $\Phi_y = \int M_y dt + N(t,y) - \int M_y dt$)

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(e.g $\frac{y^2}{2} + 2ye^t + (y+e^t)\frac{dy}{dt} = 0$

$N_t = e^t \neq y + 2e^t = M_y$)

(2) Add integrating factor $\mu(t,y)$. $\mu M + \mu N y' = 0$

"$\Leftrightarrow$" condition gives $\boxed{(\mu N)_t = (\mu M)_y}$

Special: (i) $\mu = \mu(t)$, $\mu_t N = \mu(M_y - N_t) \Rightarrow \boxed{\frac{\mu_t}{\mu} = \frac{M_y - N_t}{N}}$

(e.g. (cont'd) $\frac{\mu_t}{\mu} = \frac{M_y - N_t}{N} = 1 \Rightarrow \mu = e^t$

$\therefore \frac{e^t y^2}{2} + 2ye^{2t} + (ye^t + e^{2t})\frac{dy}{dt} = 0$

$\Phi_t = \frac{e^t y^2}{2} + 2ye^{2t} \Rightarrow \Phi = \frac{y^2}{2}e^t + ye^{2t} + \boxed{g(y)}$

$\Phi_y = ye^t + e^{2t} + g'(y) = ye^t + e^{2t} \Rightarrow g'(y) = 0$ Thus, $g(y) \equiv const$ (e.g. 0)

$\Rightarrow \frac{y^2}{2}e^t + e^{2t}y = const \Rightarrow y = \frac{-e^{2t} \pm \sqrt{e^{4t} + 2Ce^t}}{e^t}$)

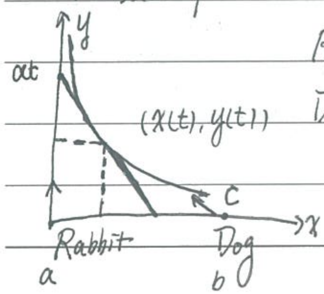
(ii) $\mu = \mu(y)$ similar to case (i)

Remark: $\mu = \mu(t)$ & $\mu = \mu(y)$ may NOT appear.

3° Riccati Equation & unsolvable equations

$\boxed{y' = t^2 + y^2}$ — No solution with elementary methods (Liouville)

11) Pursuit problem



Rabbit: speed a, direction y^+

Dog: speed b, direction Rabbit

$\boxed{\frac{dy}{dx} = \frac{at - y(t)}{0 - x(t)}} \Rightarrow xy' = y(t) - at$

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For dog: $bt = \int_x^c \sqrt{1+y'^2}\,d\tau$. $\therefore \frac{dt}{dx} = -\frac{1}{b}\sqrt{1+y'(x)^2}$

$\therefore xy'' = -y' + y' - a\frac{dt}{dx} = \frac{a}{b}\sqrt{1+y'(x)^2}$. Set $z = y'(x)$, $\frac{a}{b} = r$

$\Rightarrow \frac{z'}{\sqrt{1+z^2}} = \frac{r}{x} \Rightarrow \ln\left(\sqrt{z^2+1}+z\right) = r\ln x + const$

$\therefore y' = z = \frac{(\frac{x}{c})^r - (\frac{x}{c})^{-r}}{2}$. $\Rightarrow y = \frac{c}{2}\left(\frac{1}{r+1}\left(\frac{x}{c}\right)^{r+1} - \frac{1}{1-r}\left(\frac{x}{c}\right)^{1-r}\right)$

$+ \frac{rc}{1-r^2}$. (when $x=0$, $y = \frac{rc}{1-r^2}$, $t_{final} = \frac{y}{a} = \frac{bc}{b^2-a^2}$) $(r<1)$

(Back to e.g.2 in "Introductions")

A bolt is shot straightly upward with initial velocity $v_0 = 49$ m/sec at the ground level from a crossbow. Compute max height & t_{up}, t_{down}. (Consider air resistance $f_r = kv^p$, $1 \le p \le 2$)

"Newton" $\begin{cases} \text{low speed: } p=1 \\ \text{high speed: } p=2 \end{cases}$

(sol.)

$m\frac{dv}{dt} = -mg - kv$

$\Rightarrow \frac{dv}{dt} = -g - \frac{k}{m}v$ ($\rho = \frac{k}{m}$ drag coefficient) $= -\rho v - g$

$v(t) = v = e^{-\rho t}\left[\int_0^t -e^{\rho\tau}g\,d\tau + v_0\right] = \left(v_0+\frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}$

$h(t) = h(t) - h(0) = \int_0^t v(s)\,ds = \int_0^t\left[\left(v_0+\frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}\right]dt = -\frac{1}{\rho}\left(v_0+\frac{g}{\rho}\right)e^{-\rho t} - \frac{gt}{\rho} + \frac{1}{\rho}\left(v_0+\frac{g}{\rho}\right)$

max height: $h = h(t_{up}) = \frac{v_0}{\rho} - \frac{g}{\rho^2}\ln\left(\frac{v_0\rho}{g}+1\right)$ ($g = 9.8\,m/s^2$, $\rho = 0.04\,kg^{-1}$; $t_{up} \approx 4.56s$, $h_{max} \approx 107.8m$)

t_{up}: $0 = v(t_{up}) \Rightarrow t_{up} = \frac{1}{\rho}\ln\left(\frac{v_0\rho}{g}+1\right)$

t_{down}: $t_{down} = t_{total} - t_{up}$ ($h(t_{total}) = 0 \xRightarrow{\text{Newton's Method}} t_{total} = 9.41(s)$)

$= 4.85s$

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• Second-order Differential Equation ($y'' = f(t,y,y')$)

1) 2^{nd} order linear DE:

Form: $y'' + p(t)y' + q(t)y = r(t)$

IDEA: $L[y] = y'' + p(t)y' + q(t)y = 0$ ($y \in Ker(L)$)

☆ Proposition: Ker(L) is a $\boxed{\text{vector space of dimension 2}}$

$$\boxed{(IVP)\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = \alpha \\ y'(t_0) = \beta \end{cases}}$$

(Thm) Existence & Uniqueness

If $p(t), q(t)$ are continuous on (a,b), (IVP) has unique solution on (a,b).

(Riccati Equation in 2^{nd}-linear form (some 2^{nd}-linear equations do NOT have elementary solutions)

$\left(\frac{y'}{y}\right)' = t^2 + \left(\frac{y'}{y}\right)^2 \Rightarrow y'' + ty^2 = 0$.)

(Proof of proposition) $y_1, y_2 \in Ker L \Rightarrow c_1y_1 + c_2y_2 \in Ker L$

$\Rightarrow Ker L$ is a vector space.

Consider y_1, y_2 ($\boxed{\text{Fundamental Solutions}}$) such that

$\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 = 0 \\ y_1(t_0) = 0 \\ y_1'(t_0) = 1 \end{cases}$ $\begin{cases} y_2'' + p(t)y_2' + q(t)y_2 = 0 \\ y_2(t_0) = 1 \\ y_2'(t_0) = 0 \end{cases}$

"$\alpha y_1 + \beta y_2$" + uniqueness $\Rightarrow y_1, y_2$ form a basis ($\dim(Ker L) = 2$)

linear independent $c_1y_1 + c_2y_2 = 0 \Rightarrow c_1 = c_2 = 0$

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2) The Wronskian of y_1, y_2:

$$W[y_1,y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

($\underbrace{y'' + p(t)y' + q(t)y}_{L[y]} = 0$ (*))

☆ property (i) $\boxed{W'(t) + p(t)W(t) = 0}$

Thus, $W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)ds}$

☆ property (ii) In particular, $W(t) \equiv 0$ or $W(t) \neq 0$ ($\forall t$)

(proof.) $W' = y_1y_2'' - y_2y_1'' = y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1)$

$= -p(y_1y_2' - y_2y_1') = -pW. \Rightarrow W' + pW = 0$

$\therefore W = W(t_0)e^{-\int_{t_0}^{t} p(s)ds} \begin{cases} \equiv 0, \text{ if } W(t_0) = 0. \\ \neq 0, \text{ if } W(t_0) \neq 0. \end{cases}$

☆ property (iii) ($0 \equiv W[y_1,y_2] \Leftrightarrow y_1, y_2$ are linearly dependent.)

(proof.) "$\Leftarrow$" $W[y_1,y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} y_1 & \lambda y_1 \\ y_1' & \lambda y_1' \end{vmatrix} = 0$

"$\Rightarrow$" $W[y_1,y_2] = 0 = y_1y_2' - y_2y_1' \Rightarrow y_1y_2' = y_2y_1'$

case(I): $y_1y_2 = y_1(t)y_2(t)$ is never 0

$\Rightarrow \frac{y_2'}{y_2} = \frac{y_1'}{y_1} \Rightarrow \ln|y_2| = \ln|y_1| + c$

$\Rightarrow \left|\frac{y_2}{y_1}\right| = e^c \Rightarrow y_2 = Ay_1$, $A \in \mathbb{R}$ (continuity + never 0)

case(II): $\exists t_* \in \mathbb{R}$ s.t. $y_1(t_*)y_2(t_*) = 0$. w.l.o.g. let $y_1(t_*) = 0$

$\Rightarrow \begin{cases} y_1'(t_*) = 0 \Rightarrow \text{"uniqueness"} \Rightarrow y_1 \equiv 0 \text{ (linearly dependent)} \\ y_1'(t_*) \neq 0 \Rightarrow y_2(t_*) = 0, \text{ let } \lambda = \frac{y_2'(t_*)}{y_1'(t_*)}. \ \lambda y_1 - y_2 \text{ is a} \end{cases}$

solution with $y_3 = \lambda y_1 - y_2$, $y_3(t_*) = y_3'(t_*) = 0$.

$\xRightarrow{\text{"uniqueness"}}$ $y_3 \equiv 0$ (linearly dependent)

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Back to the basic solutions $W[y_1,y_2](t_0) = \det(I_2) = 1$.

(linearly independent based on prop. 3)

3) Solve second order linear DE:

(e.g. $y'' - 3y' + 2 = 0 \Rightarrow \left(\frac{d}{dt} - 2\right)\underbrace{\left[\left(\frac{d}{dt} - 1\right)y\right]}_{z} = 0$

$\Rightarrow \left(\frac{d}{dt} - 2\right)z = 0 \Rightarrow z = C_1e^{+2t} \Rightarrow \left(\frac{d}{dt} - 1\right)y = z = C_1e^{t}$

$\Rightarrow y' - y = C_1e^{2t} \Rightarrow (ye^{-t})' = C_1e^{t} \Longrightarrow y = C_1e^{2t} + Ce^{t}$.)

2 solutions

(i) General 2nd linear DE with constant coefficients.

$$ay'' + by' + cy = 0$$

☆ Characteristic Equation $ar^2 + br + c = 0$. $r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

cases $\begin{cases} (1)\ b^2 - 4ac > 0, \text{ 2 real roots } r_1, r_2 \\ \quad y = c_1\boxed{e^{r_1t}} + c_2\boxed{e^{r_2t}} \\ (2)\ b^2 - 4ac = 0, \text{ root } r_1 \text{ (2 equal roots } r_1 = r_2 = -\frac{b}{2a}) \\ \quad y = c_1\boxed{e^{r_1t}} + c_2\boxed{te^{r_1t}} \\ (3)\ b^2 - 4ac < 0, \ r_{1,2} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a} \triangleq u \pm iv \end{cases}$

$$y = \underbrace{(C_1 + C_2)}_{C_1}\boxed{\underbrace{e^{ut}\cos(vt)}_{R(t)}} + \underbrace{i(C_1 - C_2)}_{C_2}\boxed{\underbrace{e^{ut}\sin(vt)}_{I(t)}}$$

$0 = L[R(t) + iI(t)] = \underbrace{(aR'' + bR' + cR)}_{0} + i(aI'' + bI' + cI)$

(e.g. for repeated roots case: $y'' + 4y' + 4y = 0$

$\Rightarrow r = -2$. $y_1 = e^{-2t}$. (Variation of parameters)

let $y = c(t)e^{-2t}$, $y'' + 4y' + 4y = [c''(t) - 4c'(t) + 4c(t) + 4c'(t) - 8c(t) + 4c(t)]e^{-2t}$

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$c''(t)e^{-2t}=0$, which gives $c''(t)=0 \Rightarrow c(t)=c_1t+c_2$ with $c_1, c_2 \in \mathbb{R}$. Another basic solution: te^{r_1t})

(ii) 2^{nd} linear equations with variable as coefficients

(e.g. $(1-t^2)y''+2ty'-2y=0$, $-1<t<1$

1^{st} solution – easy one $y_1=t$ is a solution.

2^{nd} solution – (Variation of Parameters) $y_2=v(t)\cdot t$.

$\Rightarrow (1-t^2)(v''t+2v')+2t(v't+v)-2vt=(t-t^3)v''+2v'=0$

(first order case) $\Rightarrow e^{\int\frac{2}{t(1-t^2)}dt}v'=\text{const}$

$\int\frac{2}{t(1-t^2)}dt=\ln\frac{t^2}{1-t^2}+C$, thus $\frac{t^2}{1-t^2}v'=\text{const} \Rightarrow v'=c(\frac{1}{t^2}-1)$

$\therefore v=c(\frac{-1}{t}-t)=\tilde{c}(t+\frac{1}{t})$. $y_2=t^2+1$.)

(iii) Non-homogeneous cases for 2^{nd} linear DE

$$\boxed{y''+p(t)y'+q(t)=g(t)}$$

Suppose y_1, y_2 are 2 linear independent sols to

$y''+p(t)y'+q(t)=0$. Use (variation of parameters)

$y=u_1(t)y_1+u_2(t)y_2$ for (*) $\rightarrow$ 1 sol $y_{particular}$

$y'=(u_1'y_1+u_2'y_2)+(u_1y_1'+u_2y_2')$ $\quad y_{all}=y_{particular}+\ker[y]$

Let $=0$ (find such u_1, u_2)

$y''=u_1'y_1'+u_2'y_2'+u_1y_1''+u_2y_2''$

$g=y''+p(t)y'+q(t)y=u_1'y_1'+u_2'y_2'$

& $u_1'y_1+u_2'y_2=0 \Rightarrow \begin{bmatrix}y_1' & y_2'\\ y_1 & y_2\end{bmatrix}\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}g\\ 0\end{bmatrix}$

$\therefore \boxed{\begin{cases}u_1'=\dfrac{-y_2g}{W[y_1,y_2]}\\ u_2'=\dfrac{y_1g}{W[y_1,y_2]}\end{cases}}$ (Cramer's Rule)

(e.g. $y''+y'+y=t^2$ $\begin{cases}\text{① Variation of parameters}\\ \text{② Ad hoc}\end{cases}$

☆ method ① $r^2+r+1=0 \Rightarrow r=\frac{-1\pm i\sqrt{3}}{2}$. Thus

$y_1(t)=e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$, $y_2(t)=e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$W[y_1,y_2](t)=y_1y_2'-y_2y_1'=W[y_1,y_2](0)e^{-\int_0^t 1ds}=\frac{\sqrt{3}}{2}e^{-t}$

Then, $u_1'=-\frac{2}{\sqrt{3}}t^2e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$, $u_2'=\frac{2}{\sqrt{3}}t^2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$.

Derive u_1 & u_2 by integration.

some techniques for guess

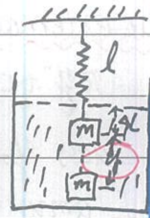
☆ method ②: Ad Hoc. $y_p=a_0+a_1t+a_2t^2$. Plug into the primitive equation $\Rightarrow a_2=1, a_1=-2, a_0=0$

$y_p=t^2-2t$ $\therefore y=y_p+c_1y_1+c_2y_2=t^2-2t+c_1e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t+c_2e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$, $c_1, c_2\in\mathbb{R}$.)

4) Applications of 2^{nd} linear DE

(i) Mechanical Vibrations

spring-mass-dashpot system



① weight mg; ② restoring force

$mg=k\Delta l$, $k(\Delta l+y)$

③ $\boxed{\text{Damping force}}$: $-cy'$ ④ external force F.

propotional to velocity

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Newton's second law. $my'' = mg - k(\Delta l + y) - cy' + F$

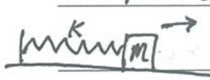
$\Rightarrow \boxed{my'' + cy' + ky = F \quad \text{(equation of motion)}}$

only when y is small

other-wise restoring force is never linear.

Case I: Free vibrations $(c=0, F=0)$; $y'' = -\frac{k}{m}y = -\omega_0^2 y$

$r = \pm\omega_0 i$, $y = a\cos\omega_0 t + b\sin\omega_0 t$ periodic with $2\pi/\omega_0 = T$



$= \underbrace{\sqrt{a^2+b^2}}_{A}\cos(\omega_0 t + \varphi)$ (with $\varphi = -\arctan\left(\frac{b}{a}\right)$)

Case II: Damped Free vibrations $(c>0, F=0)$

$my'' + cy' + ky = 0$. two roots $r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

determined by the medium

(1) $c^2 - 4km > 0 \Rightarrow r_1 \neq r_2$ both negative

$y = ae^{r_1 t} + be^{r_2 t}$ ($\to 0$ as $t \to \infty$, energy loss)

(2) $c^2 = 4km$, $r_1 = r_2 = -\frac{c}{2m}$ both real

$y = e^{-\frac{c}{2m}t}(a + bt)$ ($\to 0$ as $t \to \infty$)

(3) $c^2 - 4km < 0$, $r_{1,2} = \frac{-c}{2m} \pm i\frac{\sqrt{4km - c^2}}{2m}$

$y = e^{-\frac{c}{2m}t}\left(a\cos\frac{\sqrt{4km-c^2}}{2m}t + b\sin\frac{\sqrt{4km-c^2}}{2m}t\right)$

$\triangleq \underbrace{\sqrt{a^2+b^2}\, e^{-\frac{c}{2m}t}}_{A}\cos(\mu t + \theta)$ ($\to 0$ as $t \to \infty$)

Case III: Periodic external force (when $\omega \to \omega_0$, resonance phenomenon)

Damped Forced Vibrations

$my'' + cy' + ky = F_0\cos(\omega t)$ e.g.3 in "introduction"

Guess $y_p = a\cos\omega t + b\sin\omega t$

$\Rightarrow \cos\omega t[-am\omega^2 + bc\omega + ak] + \sin\omega t[-bm\omega^2 + ac\omega + bk]$

$= F_0\cos\omega t$. thus $\begin{cases} -am\omega^2 + bc\omega + ak = F_0 \\ -bm\omega^2 + ac\omega + bk = 0 \end{cases}$

Cramer's Rule: $a = \frac{F_0(k - m\omega^2)}{(k-m\omega^2)^2 + c^2\omega^2}$ & $b = \frac{c\omega F_0}{(k-m\omega^2)^2 + c^2\omega^2}$

$y_p = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\left[(k-m\omega^2)\cos\omega t + c\omega\sin\omega t\right]$

$= \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}\cos(\omega t + \delta)$ $\left(\delta = -\arctan\left(\frac{c\omega}{k-m\omega^2}\right)\right)$

Let $\omega \to \omega_0 = \sqrt{\frac{k}{m}}$, still periodic $y_p = \frac{F_0}{c\omega_0}\cos(\omega_0 t + \delta)$

But what if $c=0$ cases? (forced free)

Case IV: Forced Free Vibration $(c=0,\ F = F_0\cos\omega t)$

consider $my'' + ky = F_0\cos\omega t$ (special case of (iii))

① when $\omega \neq \omega_0$, $y_p = \frac{F_0/m}{|\omega_0^2 - \omega^2|}\cos\omega t$ $(\omega \to \omega_0, A\uparrow)$

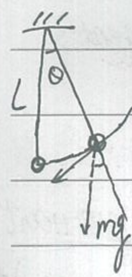
② when $\omega = \omega_0$, $y_p = \frac{F_0}{2m\omega_0}t\sin\omega_0 t$. $(t \to \infty, y_p \to \infty)$ "resonance"

(Derive. $c = 0$ & $\omega < \omega_0$

$y_p = -\frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t - \frac{F_0/m}{\omega_0^2 - \omega}\cos\omega_0 t$

let $\omega \to \omega_0$ + L'Hôpital Rule)

(ii) Principle of "Taipei 101" design (Damper)



$\begin{cases} E = \frac{1}{2}mv^2 + P = \frac{1}{2}m(l\theta')^2 + mg(l - l\cos\theta) \\ -mg\sin\theta = ml\theta'' \end{cases}$

$l\theta'' + g\sin\theta = 0$

approximation (θ small, $\sin\theta \approx \theta$)

$l\theta'' + g\theta = 0$, $\omega = \sqrt{\frac{g}{l}}$ period $T = 2\pi\sqrt{\frac{L}{g}}$

5) Series Solutions

(e.g.1 $y'' - 2ty' - 2y = 0$, guess a power series $y = \sum_{n=0}^{\infty} a_n x^n$

$$y''-2ty'-2y = \sum_{n=3}^{\infty}[n(n-1)a_n-2(n-2)a_{n-2}-2a_{n-2}]\,t^{n-2}+2a_2-2a_0$$
$$=\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2}-2(n+2)a_n]\,t^n=0$$

Thus, $\left[a_{n+2}=\frac{2}{n+2}a_n\right]$ (d.o.f = 2, a_0 & a_1) determined by initial conditions

2 Fundamental sols: $\begin{cases} y'(0)=0 \\ y(0)=1 \end{cases} \left(\Leftrightarrow \begin{cases} a_0=0 \\ a_1=1 \end{cases}\right)$ & $\begin{cases} y'(0)=1 \\ y(0)=0 \end{cases} \left(\Leftrightarrow \begin{cases} a_0=1 \\ a_1=0 \end{cases}\right)$

$\begin{cases} a_{2k}=\frac{1}{k!} \\ a_{2k-1}=0 \end{cases}$ or $\begin{cases} a_{2k}=0 \\ a_{2k-1}=\frac{2^k}{(2k-1)!!} \end{cases} \Rightarrow y_2 = t+\frac{2}{3}t^3+\frac{4}{15}t^5+\cdots$

$\Downarrow$

$y_1 = 1+t^2+\frac{t^4}{2!}+\cdots = e^{t^2}$ (By variation of parameters, $y_2 = ve^{t^2} = ce^{t^2}\int e^{-t^2}dt$)

(e.g.2 $(1+t^2)y''+3ty'+y=0$, solutions (sometimes) $\boxed{\text{may NOT converge!}}$)

(Thm) Consider the eqn $P(t)y''+Q(t)y'+R(t)y=0$ $(*)$, [if $\frac{Q(t)}{P(t)}$ & $\frac{R(t)}{P(t)}$ are analytic] (i.e., Taylor seires converges) on $|t-t_0|<\rho$, then every solution of $(*)$ has a [convergent Taylor series expansion on $|t-t_0|<\rho$]

Back to e.g.2, $y=\sum_{n=0}^{\infty}a_nt^n$, $y'=\sum_{n=0}^{\infty}na_nt^{n-1}$, $y''=\sum_{n=0}^{\infty}n(n-1)a_nt^{n-2}$

$\Rightarrow (1+t^2)y''+3ty'+y = \sum_{n=0}^{\infty}\left([n(n-1)+3n+1]a_n+(n+1)(n+2)a_{n+2}\right)t^n = 0,\ \forall t$. Then $a_{n+2}=-\frac{n+1}{n+2}a_n$.

2 solns: $y_1=\sum_{n=0}^{\infty}\underbrace{\frac{(2n-1)!!}{(2n)!!}}_{\frac{2n!}{4^n(n!)^2}}(-1)^nt^{2n}$; $y_2=\sum_{n=0}^{\infty}\underbrace{\frac{(2n)!!}{(2n+1)!!}}_{\frac{4^n(n!)^2}{(2n+1)!}}(-1)^nt^{2n+1}$

Radius of convergence (ratio test / root test + stirling formula) — 1 in that case

Overcome Singularity

(e.g.3 Euler's Equation $t^2y''+\alpha ty'+\beta y=0$

Singularity $t=0$ with $y''+\frac{\alpha}{t}y'+\frac{\beta}{t^2}y=0$. Let $\boxed{x=\ln t}$ $\Rightarrow$

$\frac{d^2y}{dx^2}+(\alpha-1)\frac{dy}{dx}+\beta y=0$ with $r_{1,2}=\frac{1-\alpha\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

case I: $(\alpha-1)^2>4\beta$, t^{r_1}, t^{r_2} (2 solns)

case II: $(\alpha-1)^2=4\beta$, t^r, $t^r\ln t$ (2 solns)

case III: $(\alpha-1)^2<4\beta$, $t^{\frac{1-\alpha}{2}}\cos\left(\frac{\sqrt{(\alpha-1)^2-4\beta}}{2}\ln t\right)$, $t^{\frac{1-\alpha}{2}}\sin\left(\frac{\sqrt{(\alpha-1)^2-4\beta}}{2}\ln t\right)$ (2 solns)

$\Downarrow$ generalized — leading terms same as Euler's

$y''+p(t)y'+q(t)=0$, with $q(t)=\frac{q_0}{t^2}+\frac{q_1}{t}+q_2+q_3t+\cdots$ & $p(t)=\frac{p_0}{t}+p_1+p_2t+\cdots$. Try $\left[y=\sum_{n=0}^{\infty}a_nt^{n+r},\ a_0\neq0\right]$

$\xRightarrow{\text{validation}}$ $t^2y''+t^2p(t)y'+t^2q(t)y=0$, i.e., $t^r\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^n+\sum_{m=0}^{\infty}p_mt^m\sum_{n=0}^{\infty}(n+r)a_nt^n+\sum_{m=0}^{\infty}q_mt^m\sum_{n=0}^{\infty}a_nt^n\right)=0$

constant term $r^2+(p_0-1)r+q_0=0$ (determined by q_0, p_0)

☆ Bessel's equation of order n (→ in Engineering) $\boxed{t^2y''+ty'+(t^2-n^2)y=0}$

Guess $y=\sum_{m=0}^{\infty}a_mt^{m+r}$, $a_0\neq0$ using the technique above

$\Rightarrow \sum_{m=2}^{\infty}[(m+r)(m+r-1)a_m+(m+r)a_m-n^2a_m+a_{m-2}]t^{m+r}+[r(r-1)a_0+ra_0-n^2a_0]t^r+[r(r+1)a_1+(1+r)a_1-n^2a_1]t^{r+1}$

$\Rightarrow r^2=n^2$ & $[(r+1)^2-n^2]a_1=0$. Then $\begin{cases} r=n \\ a_1=0 \end{cases}$ &

$a_m=(-1)\frac{1}{(m+r)^2-n^2}a_{m-2}=(-1)\frac{1}{m(m+2n)}a_{m-2}$. Therefore, $a_{2k-1}=0$

$a_{2k} = (-1)\frac{1}{2k(2k+2n)} a_{2k-2} = (-1)^k \frac{1}{2^{2k}\cdot k!(n+k)!/n!} a_0$

Let $a_0 = \frac{1}{2^n n!}$, solution $\boxed{y = \left(\frac{t}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!}\left(\frac{t}{2}\right)^{2m}}$

Special case $n=\frac{1}{2}$ ($\frac{1}{2}$-order)

$$\begin{cases} y_1 = \left(\frac{t}{2}\right)^{\frac{1}{2}}/t \sum_{m=0}^{\infty} \frac{(-1)^m}{(\frac{1}{2})!(2m+1)!} t^{2m+1} = \frac{1}{\sqrt{2\pi t}} \sin t \quad (\Gamma(\tfrac{1}{2}) = \sqrt{\pi}) \\ y_2 = \cdots = \sqrt{\frac{t}{2\pi}} \cos t \end{cases}$$

Another type of solution: $\boxed{J_n(t) = \frac{1}{\pi}\int_0^{\pi} \cos(t\sin\theta - n\theta)\, d\theta}$

validation $\begin{cases} J_{n-1}(t) + J_{n+1}(t) = \frac{2n}{t} J_n(t) \\ J_{n-1}(t) - J_{n+1}(t) = 2J_n'(t) \end{cases}$

6) Laplace Transform

(i) (Defn) Let f be a piecewise continuous function on $t>0$ which has the property $|f(t)| \le Ce^{at}$ for large t where c, a are 2 constants. Define $\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} e^{-st} f(t)\, dt$, $s>0$, $s>a$, large

& we call $\mathcal{L}\{f\}$ the Laplace Transform of f. well-defined

(ii) Properties of Laplace Transform

☆ Prop I: $\mathcal{L}$ is linear.

☆ Prop II: $\mathcal{L}\{f'(t)\}(s) = \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\, dt = \lim_{T\to\infty} \left(\left[e^{-st} f(t)\right]_0^T + s \int_0^T e^{-st} f(t)\, dt \right) = s\mathcal{L}\{f(t)\}(s) - f(0)$.

Induction $\Rightarrow$ $\mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0)$ ☆

(Note: $f^{(0)} = f$, $f^{(1)} = f'$, ...)

☆ Prop III: $\mathcal{L}\{1\}(s) = \int_0^{\infty} e^{-st}\, dt = \frac{1}{s}$.

☆ Prop IV: $\mathcal{L}\{e^{ct}\}(s) = \int_0^{\infty} e^{-st} e^{ct}\, dt = \frac{1}{s-c}$, $\forall c<s$.

☆ Prop V: $\mathcal{L}\{e^{ct} f(t)\}(s) = \int_0^{\infty} e^{ct} e^{-st} f(t)\, dt = \mathcal{L}\{f(t)\}(s-c)$. $\forall c<s$

(e.g.) $\begin{cases} y'' - 3y' + 2y = e^{3t} \\ y(0)=1,\ y'(0)=0, \end{cases}$ $Y(s) = \mathcal{L}\{y(t)\}(s)$

linearity & prop II: $s^2 Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s-3}$.

$\Rightarrow Y(s) = \frac{s-3}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s-3)} = \frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{2(s-1)} - \frac{1}{s-2} + \frac{1}{2(s-3)} = \frac{5/2}{s-1} - \frac{2}{s-2} + \frac{1/2}{s-3} \Rightarrow y(t) = \frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}$, $|y(t)| \le (3+2)e^{3t} = 5e^{3t}$.

(iii) Uniqueness of (Inverse) Laplace Transform

(Thm) Suppose f, g are piecewise continuous functions. If $\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s)$, for large s. Then, $f(t) = g(t)$, $\forall t$.

(e.g.) Vibration with discontinuous external force

$$\begin{cases} y'' - 3y' + 2y = \begin{cases} \frac{1}{c}, & 1 \le t \le 1+c \\ 0, & \text{otherwise} \end{cases} \\ y'(0) = y(0) = 0 \end{cases}$$

with $c \to 0$ → Dirac δ-function

*Note: Heaviside Function $H_c(s) = \begin{cases} 0, & s<c \\ 1, & s \ge c \end{cases}$

☆ Prop VI: $\mathcal{L}\{H_c(t) f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f\}(s)$

(explanation $LHS = \int_c^{\infty} e^{-st} f(t-c)\, dt = e^{-cs}\int_0^{\infty} e^{-s(t-c)} f(t-c)\, d(t-c) = RHS$

(sol. for e.g.) $y'' - 3y' + 2y = [H_1(t) - H_{1+c}(t)]\frac{1}{c}$

$\therefore (s^2 - 3s + 2)Y(s) = \frac{1}{cs}(e^{-s} - e^{-(1+c)s})$.

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$$Y(s) = \left[\frac{e^{-s}}{c} - \frac{e^{-(1+c)s}}{c}\right]\cdot\left[\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\right] \overset{prop.6}{=} \frac{1}{c}\mathcal{L}\{H_1(t)[\tfrac12 - e^{t-1} + \tfrac12 e^{2t-2}]\}(s) - \frac{1}{c}\mathcal{L}\{H_{1+c}(t)[\tfrac12 - \tfrac12 e^{t-1-c} + \tfrac12 e^{2t-2-2c}]\}(s).$$

$$\therefore y(t) = \begin{cases} 0, & 0<t<1 \\ \frac1c[\frac12 - e^{t-1} + \frac12 e^{2t-2}], & 1\le t<1+c \\ \frac1c[-e^{t-1} + \frac12 e^{2t-2} + e^{t-1-c} - \frac12 e^{2t-2-2c}], & t\geq 1+c. \end{cases}$$

Take $c\to 0$. $y(t) = \begin{cases} 0, & 0<t<1 \\ e^{2t-2} - e^{t-1}, & t\geq 1 \end{cases}$

7) The Dirac Delta Function (functional) → Schwarz: theory of distributions

(i) (Defn) the Dirac-δ function, denoted as δ_ξ, where ξ stands for the singularity, is defined by $\delta_\xi[\phi] = \phi(\xi)$, where ϕ is a smooth function symbolically $\delta_\xi[\phi] = \int_{\mathbb{R}} \delta_\xi(t)\phi(t)dt = \phi(\xi)$

(ii) ☆ Approximate to Identity.

Let η to be a function $\eta(t) = \begin{cases} c e^{-\frac{1}{1-t^2}}, & |t|<1 \\ 0, & \text{otherwise} \end{cases}$ C^∞ function (smooth) NOT C^ω function

(c is chosen s.t. $\int_{-\infty}^{\infty} \eta(t)dt = 1$)

(Defn) $\eta_\varepsilon(t) = \frac{1}{\varepsilon}\eta(\frac{t}{\varepsilon})$. (with property $\int_{-\infty}^{\infty}\frac{1}{\varepsilon}\eta(\frac{t}{\varepsilon})dt = 1$)

(Lemma) ($\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\eta_\varepsilon(t)\phi(t)dt = \phi(0)$ $\forall\phi$ which is C^∞.)

(proof.) $\left|\int_{-\infty}^{\infty}\eta_\varepsilon(t)\phi(t)dt - \phi(0)\right| = \left|\int_{-\infty}^{\infty}\eta_\varepsilon(t)\phi(t)dt - \int_{-\infty}^{\infty}\eta_\varepsilon(t)\phi(0)dt\right|$

$\leq \int_{-\infty}^{\infty}|\eta_\varepsilon(t)||\phi(t)-\phi(0)|dt = \int_{-\varepsilon}^{\varepsilon}|\eta_\varepsilon(t)||\phi(t)-\phi(0)|dt \leq \max_{-\varepsilon\le t\le\varepsilon}|\phi(t)-\phi(0)|$

$\int_{-\varepsilon}^{\varepsilon}\eta_\varepsilon(t)dt = \max_{-\varepsilon\le t\le\varepsilon}|\phi(t)-\phi(0)| \to 0$ because ϕ is C^∞ function as $\varepsilon\to 0$.

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Back to (e.g.) $\begin{cases} y''-3y'+2y = \delta_1 \\ y'(0)=y(0)=0 \end{cases}$, using Laplace transform

$Y(s)(s^2-3s+2) = \mathcal{L}(\delta_1) = \int_{-\infty}^{\infty} e^{-st}\delta_1(t)dt = e^{-s}$ $(\phi(1))$

$\Rightarrow Y(s) = \frac{e^{-s}}{(s-1)(s-2)} = \frac{e^{-s}}{s-2} - \frac{e^{-s}}{s-1} = \mathcal{L}\{e^{2(t-1)}H_1(t)\} - \mathcal{L}\{e^{t-1}H_1(t)\}$

$\therefore y(t) = (e^{2t-2} - e^{t-1})H_1(t)$, $t\in\mathbb{R}$

(iii) Physical meaning: $\begin{cases} ay''+by'+cy=0 \\ y(0)=y_0,\ y'(0)=y_0' \end{cases}$ at $t_0>0$, give the partical an extral impulsive force $I_0\delta(t-t_0)$.

Then, $Y(s)(as^2+bs+c) = (as+b)y_0 + ay_0' + \mathcal{L}(I_0\delta(t-t_0))$

$\therefore Y(s) = \left(\frac{as+b}{as^2+bs+c}\right)y_0 + \left(\frac{a}{as^2+bs+c}\right)y_0' + \frac{I_0 e^{-st_0}}{as^2+bs+c}$

2 basic solutions as defined ($y_1\begin{cases} y_0=1 \\ y_0'=0 \end{cases}$, & $y_2\begin{cases} y_0=0 \\ y_0'=1 \end{cases}$.)

$\mathcal{L}\{\frac{y_2}{a}\} = \frac{1}{as^2+bs+c}$, $\mathcal{L}\{y_1\} = \frac{as+b}{as^2+bs+c}$. Thus, $y(t) = y_0y_1(t) + y_0'y_2(t) + \mathcal{L}^{-1}\left(\frac{I_0e^{-st_0}}{as^2+bs+c}\right) = y_0y_1(t) + y_0'y_2(t) + \left[\frac{I_0}{a}H_{t_0}(t)y_2(t-t_0)\right] \to y_3$

$\begin{cases} y_3(t_0) = \frac{I_0}{a}y_2(0) = 0 \leftarrow \text{position change} \\ y_3'(t_0) = \frac{I_0}{a}y_2'(0) = \frac{I_0}{a} \leftarrow \text{velocity change (momentum theorem)} \end{cases}$

impulse → I_0, mass → a

8) The convolution integral.

(Lemma) $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$, where

$f*g(t) = \int_0^t f(t-u)g(u)du = \int_0^t f(u)g(t-u)du$

$f*g = g*f$.

(proof.) $\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}(f*g)(t)\,dt = \int_0^\infty e^{-st}\int_0^t f(t-u)g(u)\,du\,dt$

$= \int_0^\infty\int_u^\infty e^{-st} f(t-u)g(u)\,dt\,du \overset{\tau=t-u}{=\!=\!=} \int_0^\infty\int_0^\infty e^{-s\tau}\cdot f(\tau)\cdot e^{-su} g(u)\,d\tau\,du$

$= \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}.$

- System of Differential Equations.

(e.g.1) $\begin{cases} y'' - 3y' + 2y = e^t \\ y(0)=1,\ y'(0)=2 \end{cases}$. Let $Y = \begin{bmatrix} y \\ y' \end{bmatrix}$, then we get

$Y' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} Y + \begin{bmatrix} 0 \\ e^t \end{bmatrix} \Rightarrow \left[\begin{cases} Y'(t) = A\,Y(t) + \vec{g}(t) \\ Y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{cases}\right]$ first-order system

Not necessarily linear

(e.g.2-extension) $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$

Let $\begin{cases} x_1 = y \\ x_2 = y' \\ \vdots \\ x_n = y^{(n-1)} \end{cases} \Longrightarrow \begin{cases} x_1' = y' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = -a_0(t)x_1 - a_1(t)x_2 - \cdots - a_{n-1}(t)x_n + g(t) \end{cases}$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & \\ \vdots & I_{n-1} \\ 0 & \\ -a_0(t) & -a_1(t) \cdots\cdots -a_{n-1}(t) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}}_{\vec{g}},$$ which is ...

(1) General Linear system of 1^{st}-order

(i) Form: $\left[\vec{X}'(t) = A(t)\vec{X}(t) + \vec{g}(t)\right]$ with $\vec{X}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{g}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$ & $A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$.

(ii) (Thm) Existence & Uniqueness Thm

Assume $a_{ij}(t)$, $g_k(t)$, $i,j,k = 1,\dots,n$ are continuous on an interval I, with $t_0 \in I$. Then ☆ $\boxed{(IVP) \begin{cases} \vec{X}'(t) = A(t)\vec{X}(t) + \vec{g}(t) \\ \vec{X}(t_0) = \vec{X}_0 \end{cases}}$

has a unique solution on the entire I.

Recall: in 1-dim (1^{st} order equation case), if $f(t,y)$ is Lipschitz-continuous, we could find α to be a. (i.e., the entire interval of interest.)

$\Downarrow$ $|f(t_1,y_1) - f(t_2,y_2)| \le C(|t_1-t_2| + |y_1-y_2|)$.

Apply it to system $\begin{cases} \vec{X}'(t) = f(t,\vec{X}(t)) \\ \vec{X}(t_0) = \vec{X}_0 \end{cases}$. $\|f(t_1,\vec{X}_1) - f(t_2,\vec{X}_2)\| \le C(|t_1-t_2| + \|\vec{X}_1 - \vec{X}_2\|)$

(iii) Homogeneous case:

$(\dagger) \begin{cases} \vec{X}'(t) = A(t)\vec{X}(t) \\ \vec{X}(t_0) = \vec{X}_0 = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \end{cases}$. ☆ Proposition: [The set of all solns for $(\dagger)$, V, is a vector space of dimension n]

(proof.) $\vec{X}_1(t), \vec{X}_2(t) \in V$, we get $\alpha\vec{X}_1(t) + \beta\vec{X}_2(t) \in V$.

Let $\vec{X}_k(t)$ be soln of $(\dagger)$ with $\vec{X}_k(t_0) = \vec{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$

$\vec{X}(t) = \sum_{k=1}^n c_k\vec{X}_k(t)$ (uniqueness).

Suppose $\sum_{k=1}^n c_k\vec{X}_k(t_0) = 0 \Rightarrow c_i = 0,\ 1 \le i \le n \Rightarrow \sum_{k=1}^n c_k\vec{X}_k(t) = 0$

Thus ☆(Thm) Let $\vec{X}_1(t), \dots, \vec{X}_m(t)$ be solns of $(\dagger)$.

Then [$\vec{X}_1(t), \dots, \vec{X}_m(t)$ are linearly independent exactly when $\vec{X}_1(t_0), \dots, \vec{X}_m(t_0)$ are linearly independent.] $m \ge 2$.

(2) The Wronskian of $\vec{X}_1,\dots,\vec{X}_n$: (same in Chapter 2) (solns of (†))
is defined as $W[\vec{X}_1,\dots,\vec{X}_n](t) = \left|\vec{X}_1(t) \cdots \vec{X}_n(t)\right|$.
(determinant of solns in a matrix)

Property: $W[\vec X_1,\dots,\vec X_n](t)$ is $\boxed{\text{either zero or never zero}}$

(proof.) $W'[\vec X_1,\dots,\vec X_n](t) \overset{\text{Abel's thm}}{=\!=} tr[A](t)\,W[\vec X_1, \vec X_2,\dots,\vec X_n](t)$ (cofactor formula)

Then $W[\vec X_1,\dots,\vec X_n](t) = W[\vec X_1,\dots,\vec X_n](t_0)\, e^{+\int_{t_0}^{t} tr(A(\tau))\,d\tau}$

(e.g.) $X' = \begin{bmatrix}1&2\\4&3\end{bmatrix} X$, $X(0)=\begin{bmatrix}1\\1\end{bmatrix}$. Diagonalize the matrix (A), we can see that $0=\det(A-\lambda I) \Rightarrow \lambda_1=-1;\ \lambda_2=5$ (eigenvalues), eigenvectors $\begin{bmatrix}1\\-1\end{bmatrix}; \begin{bmatrix}1\\2\end{bmatrix}$ $\Rightarrow \begin{bmatrix}1&2\\4&3\end{bmatrix} = \begin{bmatrix}1&1\\-1&2\end{bmatrix}\begin{bmatrix}-1&0\\0&5\end{bmatrix}\begin{bmatrix}1&1\\-1&2\end{bmatrix}^{-1}$. (i) Method I: $e^{\lambda t}\cdot\vec v$, with eigenvector $\vec v$ is always a solution. (Reason: $X' = \lambda e^{\lambda t}\cdot\vec v = e^{\lambda t}\cdot A\vec v = AX$). Then solutions are $e^{-t}\begin{bmatrix}1\\-1\end{bmatrix}; e^{5t}\begin{bmatrix}1\\2\end{bmatrix}$. $X(0)= c_1e^{-0}\begin{bmatrix}1\\-1\end{bmatrix}+c_2e^{0}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$.
Then $c_1=\frac13$, $c_2=\frac23$. $X(t)=\begin{bmatrix}\frac13e^{-t}+\frac23e^{5t}\\ -\frac13e^{-t}+\frac43e^{5t}\end{bmatrix}$.
(ii) Method II: $e^{At}=Qe^{\Lambda t}Q^{-1} = \begin{bmatrix}1&1\\-1&2\end{bmatrix}\begin{bmatrix}e^{-t}&0\\0&e^{5t}\end{bmatrix}\begin{bmatrix}1&1\\-1&2\end{bmatrix}^{-1}$. Then, solution $X(t) = X(0)e^{At} = \begin{bmatrix}1&1\\-1&2\end{bmatrix}\begin{bmatrix}e^{-t}&0\\0&e^{5t}\end{bmatrix}\begin{bmatrix}2&-1\\1&1\end{bmatrix}\frac13\cdot\begin{bmatrix}1\\1\end{bmatrix}$.)

(3) General linear system of 1st-order with constant coefficients.
(3×3 is taken as an instance): $X'=AX$ with $A\in\mathbb{R}^{3\times3}$.
(all elements in A are in $\mathbb{R}$)

3 cases for $\det(A-\lambda I_3)=0$. (Knowledge of polynomials)

- case I: λ_1,λ_2 & λ_3 are real & distinct
- case II: $\lambda_1,\lambda_2,\lambda_3$ are real but NOT distinct
- case III: $\lambda_1\in\mathbb{R}$, $\lambda_2=\bar\lambda_3\in\mathbb{C}\setminus\mathbb{R}$

($\lambda_1,\lambda_2,\lambda_3$ are roots of that polynomial.)

(i) Case I: 3 distinct real eigenvalues $\Longrightarrow$ 3 linearly independent eigenvectors
suppose $\vec v_1,\vec v_2,\vec v_3$ (similar to e.g. above)

(ii) Case III: 1 real λ & 2 complex conjugate eigenvalues $\alpha\pm i\beta$. $\alpha,\lambda,\beta\in\mathbb{R}$.
Let $\vec v$ be an ei-vector of λ & $\underbrace{\vec w_1+i\vec w_2}_{\triangleq\vec w}$ be ...... of $\alpha+i\beta$.

Observations I: $\vec w\in\mathbb{C}^3$ & $\vec w\notin\mathbb{R}^3$. (Reason: $\underset{\in\mathbb{R}}{A\vec w} \ne \alpha\vec w+\underset{\notin\mathbb{R}}{i\beta\vec w}$)
Observation II: $\vec w_1$ & $\vec w_2$ must be (linearly independent)
(For otherwise, $A\vec w = A(1+ci)\vec w_1 = (\alpha+\beta i)(1+ci)\vec w_1$ contradictory!)

3 solutions $e^{\lambda t}\vec v$, $e^{(\alpha+\beta i)t}(\vec w_1+i\vec w_2)$, $(e^{(\alpha-\beta i)t}(\vec w_1-\vec w_2 i))$ → Apply conjugate on both sides

simplify: $\underbrace{e^{\alpha t}(\cos\beta t\,\vec w_1-\sin\beta t\,\vec w_2)}_{R(t)} + i\underbrace{e^{\alpha t}(\sin\beta t\,\vec w_1+\cos\beta t\,\vec w_2)}_{I(t)}$

Observation III: Both $R(t)$ & $I(t)$ are solutions.
(Reason: $[R(t)+iI(t)]' = A[R(t)+iI(t)]$, compare real & im parts)

Observation IV: $R(t)$ & $I(t)$ are linearly independent.
(Reason: $0 = c_1R(t)+c_2I(t) = e^{\alpha t}[\vec w_1\cdot(c_1\cos\beta t+c_2\sin\beta t)+\vec w_2(c_2\cos\beta t-c_1\sin\beta t)]$
$\xrightarrow[\vec w_1,\vec w_2]{\text{indep. of}} \underbrace{\begin{bmatrix}\cos\beta t & \sin\beta t\\ -\sin\beta t & \cos\beta t\end{bmatrix}}_{\text{Rotation matrix}}\begin{bmatrix}c_1\\c_2\end{bmatrix}=0 \Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix}=\vec 0$)

(e.g. $X'=\begin{bmatrix}1&0&0\\0&1&-1\\0&1&1\end{bmatrix}X$. $\lambda=1$, $\alpha=1$, $\beta=1$, $\vec v=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec w_1=\begin{bmatrix}0\\0\\1\end{bmatrix}$, $\vec w_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$ (solve for $\vec w$)
Then 3 solns, $\begin{bmatrix}e^t\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\-e^t\sin t\\e^t\cos t\end{bmatrix}$, $\begin{bmatrix}0\\e^t\cos t\\e^t\sin t\end{bmatrix}$)

(iii) Case II. 2 or 3 repeated real eigen-values.

(e.g.) $A=\begin{bmatrix}-1&1&2\\-1&1&1\\-2&1&3\end{bmatrix}$, $|A-\lambda I_3|=0 \Rightarrow \lambda_{1,2,3}=1,1,1 \Longrightarrow$ eigenvector $\vec{v}=\begin{bmatrix}1\\0\\1\end{bmatrix}$. ($e^{At}\cdot\vec{c}$) is a solution, $\forall \vec{c}\in R^3$

(Generalized Eigenvector) $\left\{\begin{array}{l}\boxed{(A-\lambda I_3)\cdot\vec{w}=\vec{v}} \Rightarrow \vec{w}=\begin{bmatrix}\omega\\1\\\omega\end{bmatrix}\overset{\omega=0}{=}\begin{bmatrix}0\\1\\0\end{bmatrix}\\ \boxed{(A-\lambda I_3)\cdot\vec{u}=\vec{w}} \Rightarrow \vec{u}=\begin{bmatrix}0\\-2\\1\end{bmatrix}\end{array}\right.$

3 solutions $\left\{\begin{array}{l}e^{At}\cdot\vec{v}=e^{(A-I)t}\cdot e^{It}\cdot\vec{v}=e^t\cdot I\cdot e^0\cdot\vec{v}=e^t\cdot\vec{v}\\ e^{At}\cdot\vec{w}=e^t\cdot I\cdot e^{(A-I)t}\cdot\vec{w}=e^t(t\vec{v}+\vec{w})\\ e^{At}\cdot\vec{u}=e^t\cdot I\cdot e^{(A-I)t}\cdot\vec{u}=e^t(\vec{u}+\vec{w}t+\frac{t^2}{2}\vec{v})\end{array}\right.$

General solution: $e^t[(c_1+c_2t+c_3\frac{t^2}{2})\vec{v}+(c_2+c_3t)\vec{w}+c_3\vec{u}]=e^t\begin{bmatrix}c_1+c_2t+c_3\frac{t^2}{2}\\c_2+c_3(t-2)\\c_1+c_2t+c_3(\frac{t^2}{2}+1)\end{bmatrix}$.

$\{\vec{v},\vec{w},\vec{u}\}_\beta$, then $[A]_\beta=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$ (in general $\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}$) → 1 block, 1 eigen-vector

$[\vec{v}\,\vec{w}\,\vec{u}]^{-1}A[\vec{v}\,\vec{w}\,\vec{u}]=[I]_\gamma^\beta[A]_\gamma[I]_\beta^\gamma=[A]_\beta$ Jordan Normal Form

$\begin{bmatrix}\lambda&0&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}$ → 2 blocks, 2 eigen-vectors

(4) Fundamental Matrix Solutions & e^{At}

(*) $\vec{X}'=A\vec{X}$

(i) (Defn) $\underline{X}(t)$ is called a fundamental matrix solution for (*) if the columns of $\underline{X}(t)$ form a [linearly independent set of n solutions of (*)]

(Back to e.g. above, $\underline{X}(t)=\begin{bmatrix}e^t&te^t&\frac{t^2}{2}e^t\\0&e^t&(t-2)e^t\\e^t&te^t&(\frac{t^2}{2}+1)e^t\end{bmatrix}$.)

(ii) (Thm) Suppose $\underline{X}(t)$ is a fundamental matrix solution for (*), we then get $[\,e^{At}=\underline{X}(t)\underline{X}^{-1}(0)\,]$ → any fundamental matrix

(proof.) consider $\underline{X}(t)\underline{X}^{-1}(0)$, every column is a solution of (*).

when it comes to e^{At}, $C(e^{At})$ (column space) (vectors in it) are all solutions of (*)

Then, compare initial value of the 2 matrices. $\underline{X}(0)\cdot\underline{X}^{-1}(0)=I_n=e^{A0}$.

$\Rightarrow e^{At}=\underline{X}(t)\underline{X}^{-1}(0)$ by uniqueness of the solutions.

(iii) Examples:

(e.g. 1, back to above Case III e.g., complex roots)

Compute e^{At} as $\begin{bmatrix}e^t&0&0\\0&-e^t\sin t&e^t\cos t\\0&e^t\cos t&e^t\sin t\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}^{-1}=e^t\begin{bmatrix}1&0&0\\0&\cos t&-\sin t\\0&\sin t&\cos t\end{bmatrix}$ → rotation by t

(e.g. 2, back to above Case II e.g., equal roots)

Compute e^{At} as $\begin{bmatrix}e^t&te^t&\frac{t^2}{2}e^t\\0&e^t&(t-2)e^t\\e^t&te^t&(\frac{t^2}{2}+1)e^t\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&1&-2\\1&0&1\end{bmatrix}^{-1}=e^t\begin{bmatrix}1-2t+t^2&t&2t+\frac{t^2}{2}\\t&1&t\\-2t-\frac{t^2}{2}&t&2t+1+\frac{t^2}{2}\end{bmatrix}$

(iv) Computation Rules:

$e^{A+B}=e^Ae^B$, if $AB=BA$.

(e.g. Jordan Form $\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}=N$, then $e^N=e^{\lambda I+J}=e^\lambda\cdot e^J=e^\lambda\cdot(\sum_{i=0}^{n}\frac{J^i}{i!})$)

(5) Non-homogeneous Linear Systems.

$\vec{X}'=A\vec{X}+\vec{f}(t)$; (+)

Firstly, solve $\vec{X}'=A\vec{X}$ (*). Let $\underline{X}(t)=[\vec{x}_1(t)\cdots\vec{x}_n(t)]$ be a fundamental solution (matrix).

Then, apply Variation of Parameters, $\vec{c}(t)=\begin{bmatrix}c_1(t)\\\vdots\\c_n(t)\end{bmatrix}\in R^n$, let $\vec{X}(t)=\underline{X}(t)\vec{c}(t)$. $\vec{X}'=\underline{X}'(t)\cdot\vec{c}(t)+\underline{X}(t)\vec{c}'(t)=A\underline{X}(t)\vec{c}(t)+\underline{X}(t)\vec{c}'(t)$, then let $\underline{X}(t)\vec{c}'(t)=\vec{f}(t)$, which means $\vec{c}'(t)=\underline{X}^{-1}(t)\vec{f}(t)$.

$\vec{c}(t)=\vec{c}(t_0)+\int_{t_0}^{t}\underline{X}^{-1}(s)\vec{f}(s)ds$.

$[\,\vec{X}(t)=\underline{X}(t)\vec{c}(t_0)+\underline{X}(t)\int_{t_0}^{t}\underline{X}^{-1}(s)\vec{f}(s)ds\,]$ (compared with $\vec{X}(t_0)$ to find $\vec{c}(t_0)=\underline{X}^{-1}(t_0)$)

Because [$\underline{X}(t)\underline{X}^{-1}(s) = e^{A(t-s)}$ & $c(t_0)$ (previous page)], $\vec{X}(t) = e^{A(t-t_0)}\vec{X}(t_0) + \int_{t_0}^{t} e^{A(t-s)}\vec{f}(s)ds$

(e.g. Judge whether a given matrix is a fundamental matrix solns

1) Necessary condition I: non-singular for $t\in R$

2) $\underline{X}(t)\underline{X}^{-1}(0) = e^{At}$. Then $A = I + \frac{d}{dt}e^{At} - e^{At}$ or $\left[\frac{d}{dt}e^{At}\right]_{t=0}$) $\underline{X}(0)\underline{X}^{-1}(0)$

needs validation (Necessary condition II)

$(\underline{X}'(t) = A\underline{X}(t))$

• Qualitative Theory of Differential Equations

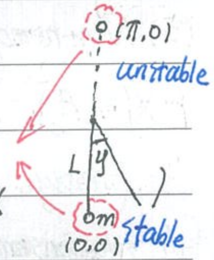
$\left[\vec{X}' = f(t,\vec{X}) \quad (*)\right]$

1) (Defn) Suppose $f(t,\vec{\xi}) = 0$, then $\vec{X}(t) \equiv \vec{\xi}$ ($\forall t\in R$) is a solution to (*), which we call an equilibrium point (or, a steady state)

special case ⇒ [($\hat{*}$) $\vec{X}' = f(\vec{X})$] ⇒ which is called autonomous.

2) Stability of Steady States

(e.g. $\begin{pmatrix}x_1'\\ x_2'\end{pmatrix} = \begin{pmatrix}x_2\\ -\frac{g}{l}\sin x_1\end{pmatrix}$. Equilibrium points $\begin{cases}x_2 = 0\\ x_1 = k\pi,\ k\in Z\end{cases}$



(i) (Defn) A solution of ($\hat{*}$) $\vec{X}(t) = \vec{\phi}(t)$ is said to be stable if $\forall \varepsilon>0, \exists \delta>0$ s.t. $\forall$ sol $\vec{X} = \vec{\psi}(t)$ with $\|\vec{\psi}(0) - \vec{\phi}(0)\| < \delta$ ($\delta = \delta(\varepsilon)$), then $\|\vec{\psi}(t) - \vec{\phi}(t)\| < \varepsilon, \forall t\in R^+$

(i.e., any other sol which starts close to $\vec{\phi}$ will stay close to $\vec{\phi}(t)$)

(ii) Stability Properties for Linear systems:

(†) $\vec{X}' = A\vec{X}$, $A\in R^{n\times n}$

★ proposition (reduction): Every solution of (*), say $\vec{\phi}(t)$ is stable if & only if the steady state $\vec{X}(t) = \vec{0}$ is stable.

(proof.) (⇒) trivial; (⇐) $\|\underbrace{\vec{\phi}(t) - \vec{\psi}(t)}_{\text{a solution } (\triangleq \vec{p}(t))}\| < \varepsilon$, when $\|\vec{p}(t)\|\big|_{t=0} = \|\vec{\phi}(0) - \vec{\psi}(0)\| < \delta$. (Because $\vec{0}$ is stable & by defn)

3 cases to decide the stability of the solution $\vec{0}$ for (†)

① (At least 1) of eigenvalues of A is positive or has a positive real part. "unstable"

② (All eigenvalues) of A are negative or have negative real parts. "stable"

③ Some eigenvalues of A are zero or pure imaginary numbers, w[hile] all eigenvalues of A are non-positive or have non-pos. real parts "inconclusive" (sometimes stable)

Case ①

proposition: All solns in case ① are unstable.

(proof.) Need to show: $\exists \varepsilon_0 > 0, \forall \delta > 0, \exists$ a sol $\vec{\phi}_\delta(t)$ s.t. $\|\vec{\phi}_\delta(t_k)\| \geq \varepsilon_0$ & $\|\vec{\phi}_\delta(0)\| < \delta$, for a sequence $\{t_k\}\to\infty$. Suppose $\lambda > 0$ is an eigenvalue of A with eigenvector $\vec{v}$. Then $e^{\lambda t}\cdot\vec{v}$ is a solution. & Let $\vec{\phi}_\delta(t) = C_\delta e^{\lambda t}\cdot\vec{v}$, with fixed δ. $\|\vec{\phi}_\delta(0)\| = |C_\delta|\cdot\|\vec{v}\| < \delta$. (make $C_\delta = \frac{\delta}{2\|\vec{v}\|}$)

But $\|\vec{\phi}_\delta(t)\| \to \infty$ when $t\to\infty$, which means 0 is unstable.

If $\alpha + \beta i$, with $\alpha > 0$ is an eigenvalue, with $\vec{w}$ as eigenvector correspondingly. Let $\vec{\phi}_\delta(t) = C_\delta e^{\alpha t}(\cos\beta t\,\vec{w}_1 - \sin\beta t\,\vec{w}_2)$, $\vec{w} = \vec{w}_1 + i\vec{w}_2$, we can get very similar result to above. ($C_\delta = \frac{\delta}{2\|\vec{w}\|}$, $t_k = \frac{2k\pi}{|\beta|}$, $\forall k \to \infty$)

(Note that $0<\inf_{t>0}\{\|\cos\beta t\,\vec{w}_1-\sin\beta t\,\vec{w}_2\|\}\triangleq\gamma$ (periodic), so $\|\vec{\phi}_g(t)\|\to\infty$ ($\geq e^{\alpha t}\cdot\text{const}$))

Case (ii) All solns are stable & go to zero in case (ii) — asymptotically stable ($\to 0$ as $t\to\infty$)

(proof.) In this case, all solns to (†) are linear combinations of $e^{\lambda_1 t}\vec{v}_1,\dots,e^{\lambda_k t}\vec{v}_k$, $e^{\lambda_{k+1}t}(\vec{v}_{k+1}+t\vec{w}_1^{k+1}+t^2\vec{w}_2^{k+1}+\dots+t^{j_1}\vec{w}_{j_1}^{k+1}),\dots,e^{\lambda_m t}(\vec{v}_m+t\vec{w}_1^{m-k}+\dots+t^{j_{m-k}}\vec{w}_{j_{m-k}})$ & $e^{\alpha_{m+1}t}(\cos\beta_{m+1}t\,\vec{u}_{m+1}-\sin\beta_{m+1}t\,\vec{z}_{m+1})t^{l_{m+1}},\dots,e^{\alpha_n t}(\cos\beta_n t\,\vec{u}_n-\sin\beta_n t\,\vec{z}_n)t^{l_n}$

with $\lambda_1,\dots,\lambda_k,\lambda_{k+1},\dots,\lambda_m$ & $\alpha_{m+1},\dots,\alpha_n$ are all <0. all with ① real distinguished e-values ② repeated ③ imaginary

Let $\alpha\triangleq\max\{\lambda_1,\dots,\lambda_k,\lambda_{k+1},\dots,\lambda_m,\alpha_{m+1},\dots,\alpha_n\}$, then any soln,

$\vec{\phi}(t)=\sum_{i=1}^n c_i\vec{\phi}_i(t)$, $\|\vec{\phi}(t)\|=\|\sum_{i=1}^n c_i\vec{\phi}_i(t)\|\leq e^{\alpha t}\|P(t,\sin t,\cos t)\|\to 0$.

(all solns (fundamental) $\xrightarrow{\text{modulus}} 0$, which means ......)

Case (iii) suppose that $\lambda_1,\dots,\lambda_k,\lambda_{k+1},\dots,\lambda_n$ are eigenvalues of A, with $\lambda_j=i\beta_j$, $j=1,\dots,k$ & $\lambda_{k+1},\dots,\lambda_n$ are either negative or have negative real parts. Suppose further that $\lambda_1,\dots,\lambda_k$ have multiplicities (# of repeated roots!) $m_1,\dots,m_k$. Then all solns to (†) are $\boxed{\text{stable}}$ if & only if A has m_j linear independent eigenvectors corresponding to λ_j, $j=1,\dots,k$. $P=(\lambda-i\beta_1)^{m_1}\cdots(\lambda-i\beta_j)^{m_j}$

(e.g.1 $\vec{X}'=\begin{bmatrix}0&1\\0&0\end{bmatrix}\vec{X}$, then $\lambda_{1,2}=0,0$; $\vec{v}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{w}=\begin{bmatrix}0\\1\end{bmatrix}$

2 independent solns: $e^{tA}\vec{v}=\vec{v}$, $e^{tA}\vec{w}=\vec{w}+t\vec{v}$. (unstable!) unstable

($c\begin{bmatrix}t\\1\end{bmatrix}\to 0$ (small c) but $c\begin{bmatrix}t\\1\end{bmatrix}\to\begin{bmatrix}\infty\\1\end{bmatrix}$) — because $\vec{X}=0$ is unstable)

(e.g.2 $\vec{X}'=\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}_{A}\vec{X}$, then $\lambda_{1,2}=\pm i$; corresponding $\vec{v}_1=\begin{bmatrix}i\\1\end{bmatrix}$; $\vec{v}_2=\begin{bmatrix}-i\\1\end{bmatrix}$

2 independent solns: $\begin{bmatrix}-\sin t\\\cos t\end{bmatrix}$, $\begin{bmatrix}\cos t\\\sin t\end{bmatrix}$. $\phi=c_1\begin{bmatrix}-\sin t\\\cos t\end{bmatrix}+c_2\begin{bmatrix}\cos t\\\sin t\end{bmatrix}$

$\|\phi(0)-0\|=\sqrt{c_1^2+c_2^2}<\delta$, then $\|\phi(t)-0\|=\sqrt{c_1^2+c_2^2}<\delta=\varepsilon$, $\forall t\in\mathbb{R}$ stable 0 & all stable)


solns on the circle

(iii) Asymptotical stability.

(Defn) A solution $\vec{X}=\vec{\phi}(t)$ is said to be asymptotically stable if $\exists\,\delta>0$ s.t. for all solns $\vec{\Psi}(t)$ s.t. $\|\vec{\phi}(0)-\vec{\Psi}(0)\|<\delta$, we have $\lim_{t\to\infty}\|\vec{\Psi}(t)-\vec{\phi}(t)\|=0$ (stronger than stability) → stay close to some extent → approaches as time goes by

(e.g.3 $\vec{X}'=\begin{bmatrix}2&-3&0\\0&-6&-2\\-6&0&-3\end{bmatrix}\vec{X}$, $\lambda_{1,2,3}=0,0,-7$, but only one eigenvector $\begin{bmatrix}3\\2\\-6\end{bmatrix}$ corresponding to 0

Then, 0 is unstable from above ⇒ all solns are unstable).

3) Stability for Non-linear systems

($\tilde{†}$) $\vec{X}'(t)=A\vec{X}(t)+\vec{g}(\vec{X}(t))$, with $A\in\mathbb{R}^{n\times n}$, $\vec{g}(\vec{X})=o(\|\vec{X}\|)$ near $\vec{0}$. (pertubation)

(i) (Thm) Case I: $\vec{X}=0$ is $\boxed{\text{stable}}$ for ($\tilde{†}$) if all eigenvalues of A are negative or own negative real parts. (asymptotically stable, more precisely)

Case II: $\vec{X}=0$ is $\boxed{\text{unstable}}$ for ($\tilde{†}$) if A has at least an eigenvalue which is positive or owns positive real parts.

Case III: $\vec{X}=0$ cannot be determined "by A" if all eigenvalues of A are non-positive or own non-positive real parts, within which at least one is 0 or pure imaginary number.

(e.g. for case III $\vec{X}'=\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}_{A}\vec{X}\pm\overbrace{\begin{bmatrix}-x_1(x_1^2+x_2^2)\\-x_2(x_1^2+x_2^2)\end{bmatrix}}^{\vec{g}(\vec{X})}$, $\vec{X}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$

eigenvalues of A, $\lambda_{1,2}=\pm i$. $\frac{d}{dt}\|\vec{X}\|_2^2=2\vec{X}^T\vec{X}'=2\vec{X}^TA\vec{X}\pm2\vec{X}^T\vec{g}(\vec{X})=\mp2\|\vec{X}\|_2^4$

"+": $\frac{d}{dt}\|\vec{X}\|_2^2=-2\|\vec{X}\|_2^4$, $\|\vec{X}\|\searrow 0$ ⇒ stable $\vec{0}$ → asymptotically stable, in real

"−": $\frac{d}{dt}\|\vec{X}\|_2^2=+2\|\vec{X}\|_2^4$, $\|\vec{X}\|\nearrow$ ⇒ unstable $\vec{0}$).

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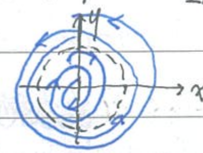
$\Rightarrow \|\vec{X}(t)\|_2^2 = \dfrac{\|\vec{X}(0)\|}{\pm 2t\|\vec{X}(0)\|_2^2+1}$ $\begin{cases} "+" \Rightarrow \|\vec{X}(t)\|_2^2 \to 0 & (\vec{X}(t)\to 0, \text{asympt. stable}) \\ "-" \Rightarrow \|\vec{X}(t)\|_2^2 \to \infty & (t \text{ from } 0 \text{ to } \frac{1}{2\|\vec{X}(0)\|_2^2}) \text{ unstable} \end{cases}$



(proof. of case I)

Preliminaries: ① $\vec{X}(t) = e^{At}\vec{X}(0) + \int_0^t e^{A(t-s)}\vec{g}(\vec{X}(s))\,ds$

(variation of parameters)

② Under assumptions in case I, $\exists$ constant α, k $(\alpha, k>0)$ s.t. $\|e^{At}\vec{c}\| \le k e^{-\alpha t}\|\vec{c}\|$, $\forall \vec{c}$ & $t>0$. (e.g. $\alpha = \frac{1}{2}\min_{i=1}^{n}\{-Re(\lambda_i)\}$)

③ $\forall \varepsilon>0, \exists \delta>0$ s.t. $\|\vec{g}(\vec{X})\| < \varepsilon\|\vec{X}\|$, $\forall \|\vec{X}\|<\delta$. (little-oh notation)

strategy: $\exists \eta>0$ s.t. $\forall \|\vec{X}(0)\|<\eta$, $\vec{X}(t)\to 0$ as $t\to\infty$. (choose small enough η s.t. $\|\vec{X}\|<\delta$, if $\|\vec{X}(0)\|<\eta$.)

$\|\vec{X}(t)\| \le \|e^{At}\vec{X}(0)\| + \|\int_0^t e^{A(t-s)}\vec{g}(\vec{X}(s))ds\| \overset{②}{\le} ke^{-\alpha t}\|\vec{X}(0)\| + \int_0^t \|e^{A(t-s)}\vec{g}(\vec{X}(s))\|ds$

$\overset{③ \text{ choose } \varepsilon}{\le} ke^{-\alpha t}\|\vec{X}(0)\| + ke^{-\alpha t}\varepsilon\int_0^t e^{\alpha s}\|\vec{X}(s)\|ds \Longrightarrow \left(\underbrace{e^{\alpha t}\|\vec{X}(t)\|}_{Z(t)} \le k\|\vec{X}(0)\| + \varepsilon k \underbrace{\int_0^t e^{\alpha s}\|\vec{X}(s)\|ds}_{Z(s)}\right) \Longrightarrow \left(Z(t) \le k\|\vec{X}(0)\| + \varepsilon k \underbrace{\int_0^t Z(s)ds}_{u(t)}\right) \Rightarrow u'(t) \le k\|\vec{X}(0)\| + \varepsilon k u(t)$

1st order $\Rightarrow [e^{-\varepsilon k t}u(t)]' \le k\|\vec{X}(0)\| e^{-\varepsilon k t}$. $\Rightarrow \left(u(t) \le e^{\varepsilon k t}\int_0^t k\|\vec{X}(0)\|e^{-\varepsilon k \tau}d\tau = \frac{\|\vec{X}(0)\|}{\varepsilon}(e^{\varepsilon k t}-1)\right)$

$\therefore Z(t) \le k\|\vec{X}(0)\| + \varepsilon k u(t) \le k\|\vec{X}(0)\|e^{\varepsilon k t} \Leftrightarrow \|\vec{X}(t)\| \le k\|\vec{X}(0)\|e^{(\varepsilon k-\alpha)t}$ $\varepsilon = \frac{\alpha}{2k}$ $e^{-\frac{\alpha}{2}t}$

Then choose $\eta = \frac{\delta}{2k}$ (determined by ε) (let $k\eta \cdot e^{-\frac{\alpha}{2}t} \le k\eta < \delta$) $\Rightarrow \|\vec{X}(t)\| < \delta$, $\forall t\in R$ → stability

& when $t\to\infty$ $\|\vec{X}(t)\| \le$ thus $\to 0$ → asymptotical stability

Remark:

in General Autonomous Cases:

$\vec{X}' = \vec{f}(\vec{X})$, with $\vec{f}(\vec{X}_0) = 0$. (i.e. $\vec{X}(t) = \vec{X}_0$ $\forall t\in R$ is a soln.)

Let $\vec{Y} = \vec{X} - \vec{X}_0$ & by Taylor's expansion, $\underbrace{\vec{Y}' = D_{\vec{f}(\vec{X}_0)}\vec{Y}}_{\text{linearization}} + O(\|\vec{Y}\|)$

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In most cases, linearization (almost) determines the stability of a steady state.

(e.g. Find all equilibrium points of $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1-xy \\ x-y^3 \end{bmatrix}$ & determine their stabilities. All steady states $\begin{cases} 1-xy=0 \\ x-y^3=0 \end{cases} \Rightarrow (1,1)$ & $(-1,-1)$ are all

At $(1,1)$, $D_{\vec{f}(1,1)} = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix}_{(1,1)} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$, $|A-\lambda I| = (1+\lambda)(3+\lambda)+1 = \lambda^2+4\lambda+4$

$\Rightarrow \lambda_{1,2} = -2$, which means $(1,1)$ is stable. (asymptotically)

At $(-1,-1)$, $D_{\vec{f}(-1,-1)} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. $|A-\lambda I| = (\lambda-1)(3+\lambda)-1 = \lambda^2+2\lambda-4 = 0$, $\lambda_{1,2} = -1\pm\sqrt{5}$

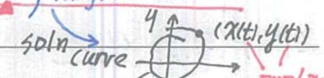
Then, $(-1,-1)$ is unstable. NOT know global behavior

"local" is to guarantee that sol → (1,1) as $t\to\infty$, we need initial value → (1,1). (very close)

)

4) Phase Plane: (autonomous & $n=2$)

(i) consider $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ $\Rightarrow \left(\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x,y)}{f(x,y)} \triangleq h(x,y)\right)$

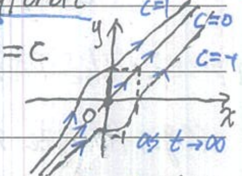


implicitly

trajectory / orbit

(e.g. $\begin{cases} x' = y^2 \\ y' = x^2 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2} \Rightarrow y^2dy = x^2dx \Rightarrow y^3 - x^3 = C$

trajectory



(ii) $\begin{cases} x' = y \\ y' = -x \end{cases}$ (1) $\begin{cases} x' = (1+x^2+y^2)y \\ y' = -(1+x^2+y^2)x \end{cases}$ (2) & $\begin{cases} x' = y(1-x^2-y^2) \\ y' = -x(1-x^2-y^2) \end{cases}$ (3)

(1) Linear. $y^2+x^2=c^2$. (or using eigenvalues)

$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

→ $x^2+y^2=c^2$

(2) circle in the phase plane, too. All solns are periodic. period $\frac{2\pi}{1+c^2}$, 2π respectively

IVP ⟺ $x' = (1+c^2)y$, with $c^2 = x_0^2+y_0^2$, $y' = -(1+c^2)x$

then linear case $\lambda_{1,2} = (1+c^2)\cdot(\pm i)$ ⇒ period $\frac{2\pi}{1+c^2}$. when c small → 2π

c ↑ - faster

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(3) $x^2+y^2\neq 1 \Rightarrow x^2+y^2=c^2$; $(0,0) \Rightarrow x^2+y^2=1$. period $\frac{2\pi}{|1-c^4|}$ (moving slowly on the circle close to unit) Directions for $c>1$ & $c<1$ are opposite.

↳ every pt on unit circle is a soln & does NOT move.

(1)(2)(3) hold the same phase plane (i.e., graph) but different behaviors

↳ shortages of phase planes.

5) Existence & Uniqueness Property (autonomous systems)

(*) $\vec{X}' = \vec{f}(\vec{X}(t)) = \begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_n(x_1,\cdots,x_n) \end{bmatrix}$

(i) Suppose $f_1,\cdots,f_n$ are C^1, then $\forall$ point $\vec{X}_0 \in \mathbb{R}^n$, (*) has a unique orbit $\vec{X}(t)$ going through $\vec{X}_0$.

(NO different 2 orbits can intersect each other!)

(ii) Let $\vec{X}(t)=\vec{\phi}(t)$ be a soln of (*), if for some t_0, $\exists$ a T s.t. $\vec{\phi}(t_0+T)=\vec{\phi}(t_0)$. Then $\vec{\phi}(t)$ is periodic with period T & orbit of $\vec{\phi}(t)$ is closed.

(Every close orbit is periodic (in its corresponding function).)

(proof.) Use Picard Iteration $\vec{X}(t)=\vec{X}(t_0)+\int_{t_0}^{t}\vec{f}(\vec{X}(s))\,ds$ with $\|\cdot\|_\infty$

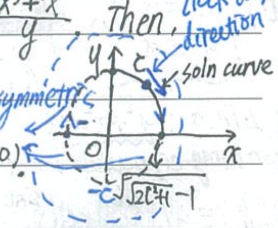
(e.g.) Analyze solns of $z''+z+z^3=0$

set $x=z$, $y=z'$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x-x^3 \end{bmatrix} \Rightarrow \frac{dy}{dx} = -\frac{x^3+x}{y}$. Then,

$x^4+2x^2+2y^2=2C^2$ (constant c^2)

All solns are periodic (all obits closed) excluding (0,0).

(0,0) is an equilibrium point. (stable)



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6) Phase Portraits of 2×2 linear systems

(1) $\vec{X}'=A\vec{X}$, with $\vec{X}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. $A\in\mathbb{R}^{2\times 2}$.

(i) A has 2 real eigenvalues λ_1,λ_2: with $\lambda_1 \geq \lambda_2$ w.l.o.g

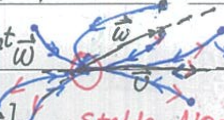
sub-cases:
- $\lambda_2<\lambda_1<0$ approaching ei-vector corsp. to λ_1 first, then to origin.
- $0<\lambda_1<\lambda_2$ go far away from ei-vector corspd. to λ_1 first, then to ∞.
- $\lambda_2<0<\lambda_1$ $t: -\infty \to +\infty$; From approaching e-vec crspd to λ_2 to ... λ_1
- $\underbrace{\lambda_2=\lambda_1}_{\lambda}<0$ { 2 indep. ei-vectors (diff) $\Leftrightarrow A=\lambda I$ → direction NOT change; 1 same ei-vector $\Leftrightarrow A\neq\lambda I$ → eigenvector dominates direction as $t\to\infty$ }
- $\underbrace{\lambda_2=\lambda_1}_{\lambda}>0$ { $A=\lambda I$; $A\neq\lambda I$ }

(ii) A has a pair of complex conjugate eigenvalues $\alpha \pm i\beta$

sub-cases:
- $\alpha=0$ period, with stable centre
- $\alpha>0$ spiral-out, with unstable focus
- $\alpha<0$ spiral-in, with stable focus

[Explanations] (i) If $\lambda_2<\lambda_1<0$, 2 eigenvectos $\vec{v},\vec{w}$; then $e^{\lambda_1 t}\vec{v}$; $e^{\lambda_2 t}\vec{w}$ are 2 solns.

$\vec{X}=c_1e^{\lambda_1 t}\vec{v}+c_2e^{\lambda_2 t}\vec{w}$
$= e^{\lambda_1 t}[c_1\vec{v}+e^{(\lambda_2-\lambda_1)t}c_2\vec{w}]$

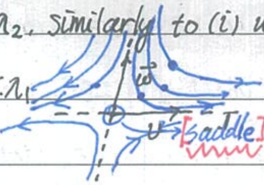


orbit, $\forall$ point (approaching $\vec{v}$ first then 0)

Stable Node

(ii) If $0<\lambda_1<\lambda_2$, similarly to (i) with reversed arrows (go to ∞) Unstable Node

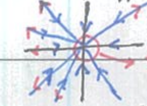
(iii) If $\lambda_2<0<\lambda_1$,



$\vec{X}(t)=c_1e^{\lambda_1 t}\vec{v}+c_2e^{\lambda_2 t}\vec{w}$

(From $-\infty \to +\infty$, approaching $\vec{w}$ → approaching $\vec{v}$)

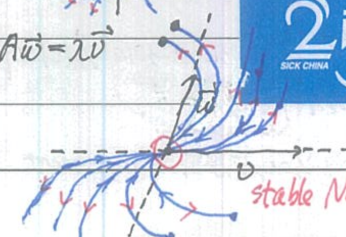
(iv) ① $\lambda=\lambda_1=\lambda_2<0$ & $A=\lambda I$, $c_1e^{\lambda t}\vec{v}$ soln. $+c_2e^{\lambda t}\vec{w}$



stable Node (unstable f(v)

② Jordan form for $A \Rightarrow$ generalized e-vector $\vec{w}$, $A\vec{w}=\lambda\vec{v}$

$\vec{X}=c_1e^{\lambda t}\vec{v}+c_2e^{\lambda t}(\vec{w}+t\vec{v})$
$= e^{\lambda t}[c_2\vec{w}+(c_1+c_2t)\vec{v}]$



stable Node (unstable for

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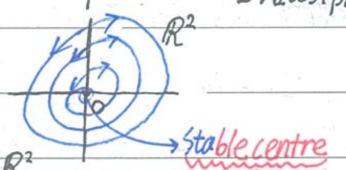
When eigenvalues $\alpha \pm \beta i$ (mark $\vec{u} + i\vec{v}$ corresponding to $\alpha + i\beta$):

Then, all solns are $\vec{X}(t) = e^{\alpha t}\begin{bmatrix} c_1(\cos\beta t\, u_1 - \sin\beta t\, v_1) + c_2(\cos\beta t\, v_1 + \sin\beta t\, u_1) \\ c_1(\cos\beta t\, u_2 - \sin\beta t\, v_2) + c_2(\cos\beta t\, v_2 + \sin\beta t\, u_2)\end{bmatrix}$

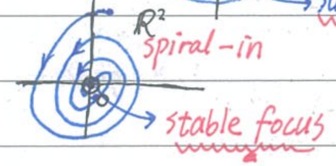
$= e^{\alpha t}\begin{bmatrix} c_1u_1+c_2v_1 & c_2u_1 - c_1v_1 \\ c_1u_2+c_2u_2 & c_2u_2 - c_1v_2\end{bmatrix}\begin{bmatrix}\cos\beta t\\ \sin\beta t\end{bmatrix} = e^{\alpha t}\begin{bmatrix}A_1\cos(\beta t+\varphi_1)\\ A_2\cos(\beta t+\varphi_2)\end{bmatrix}$

(vi) when $\alpha = 0$, periodic

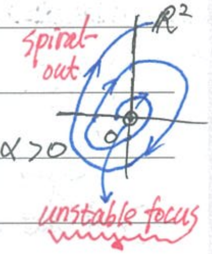
(Direction: determined by A)



(vii) when $\alpha < 0$,



(viii) when $\alpha > 0$



7) Phase Portraits for non-linear systems (2x2)

(*) $\vec{X}' = \vec{f}(\vec{X})$, with a steady state $\vec{X}_0$

(Thm) Let λ_1, λ_2 be 2 eigenvalues of the Jacobian matrix $D\vec{f}(\vec{X}_0)$. Then the local behaviors of solns to (*) are: (the soln starts close to $\vec{X}_0$).

(i) $\lambda_2 < \lambda_1 < 0$: stable Node; (ii) $\lambda_1 = \lambda_2 < 0$: stable Node / stable focus

(iii) $\lambda_2 < 0 < \lambda_1$: unstable saddle; (iv) $0 < \lambda_2 = \lambda_1$: unstable Node / unstable focus

(v) $0 < \lambda_1 < \lambda_2$: unstable node; (vi) $\lambda_{1,2} = \alpha \pm \beta i$, $\alpha = 0$: stable/unstable focus or center

(vii) $\lambda_{1,2} = \alpha \pm \beta i$, $\alpha > 0$: unstable focus; (viii) ... $\alpha < 0$, stable focus

8) Predator-Pray Model:

(i) (property) Suppose a soln $\vec{X}(t)$ of (*) converges, i.e. $\vec{X}(t) \to \vec{\xi}$ as $t \to \infty$. Then $\vec{f}(\vec{\xi}) = \vec{0}$.

(proof.) Consider 1st component $x_1(t)$ of $\vec{X}(t)$. Then $x_1(t_n+h) - x_1(t_n) \xrightarrow{n\to\infty} 0$, $\{t_n\} \to \infty$ as $n \to \infty$

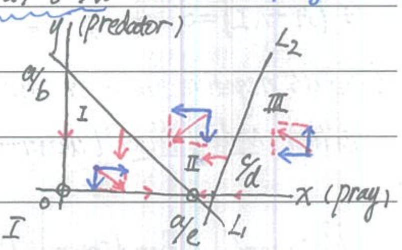
However, by MVT, $h x_1'(\tilde{t}_n) \to 0$, as $n \to \infty$ ($t_n \le \tilde{t}_n \le t_n + h$). Then $\lim_{n\to\infty} x_1'(\tilde{t}_n) \overset{0}{=} \lim_{n\to\infty} f_1(\vec{X}(\tilde{t}_n)) = f_1(\vec{\xi})$. (Holds for $f_2, \dots, f_n$ ($\vec{\xi}$) $= 0$) ($\vec{X}(\tilde{t}_n) \xrightarrow{n\to\infty} \vec{X}(\lim_{n\to\infty} \tilde{t}_n) = \vec{\xi}$)

(ii) [e.g.] consider $\begin{cases} x' = ax - bxy - ex^2 \\ y' = -cy + dxy - fy^2 \end{cases}$ with $a, b, c, d, e, f \in \mathbb{R}^+$ $\left(\frac{a}{e} < \frac{c}{d}\right)$, and all solns with $x(0), y(0) > 0$ converge to $(\frac{a}{e}, 0)$ as $t \to \infty$ (predator-pray model)

2 equilibrium points $(0,0)$, $(\frac{a}{e}, 0)$

(other 2 are with $y < 0$, NOT include)

"Never outside the region — by analysis"



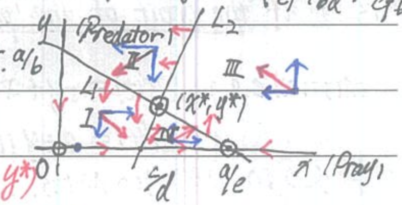
① a soln starts in III must enter II

② a soln (starts) in II has 2 possibilities { enter I ; stay in II forever (property) └ $y\downarrow, x\downarrow \Rightarrow y = 0 \Rightarrow x\downarrow \to a/e$ (stop) (x(t), y(t)) converges (in Region II) }

③ a soln in I stay in I forever └→ $y\downarrow, x\uparrow \Rightarrow y = 0 \Rightarrow x\uparrow \to a/e$ (stop) (property) (x(t), y(t)) converges

Consider "If $\left(\frac{a}{e} > \frac{c}{d}\right)$"
└→ 3 equilibrium points $(0,0)$; $(a/e, 0)$ & $(x^*, y^*) \triangleq \left(\frac{af+bc}{ef+bd}, \frac{ad-ce}{ef+bd}\right)$

No solns can go across into/out 1st quadrant.



① Every soln starts in III { goes to II ; stays in III forever $\xrightarrow{t\to\infty}$ converges to (x^*, y^*) }

② Every soln in II { goes to I ; stays in II forever $\xrightarrow{t\to\infty}$ converges to (x^*, y^*) }

③ Every soln in I { goes to IV ; stays in I forever $\xrightarrow{t\to\infty}$ converges (x^*, y^*) }

④ Every soln in IV { goes to III ; stay in IV forever $\xrightarrow{t\to\infty}$ converges to (x^*, y^*) or converges to $(a/e, 0)$ when $y = 0$ } Not possible X

In summary, solns either converge to (x^*, y^*) or go around from III → II → I → IV & continue.

otherwise, 2 orbits intersect at 1 point between $(0, a/e)$

Date: 2022 Apr 29

Local analysis of (x^*, y^*). $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x(a-by-ex) \\ y(-c+dx-fy) \end{bmatrix}$

$$\text{Jacobian} \left(\frac{\partial(x',y')}{\partial(x,y)}\right) = \begin{vmatrix} a-by-2ex & -bx \\ dy & -c+dx-2fy \end{vmatrix}_{(x^*,y^*)} = bdx^*y^* + ef\,x^*y^* = (ef+bd)x^*y^* = \frac{(af+bc)(ad-ce)}{ef+bd}$$

Linearize equation: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -ex^* & -bx^* \\ dy^* & -fx^* \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + O\left(\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|\right)$

$\Rightarrow |A-\lambda I| = 0 \Leftrightarrow \lambda_{1,2} = \dfrac{-(ex^*+fy^*) \pm \sqrt{(ex^*-fy^*)^2 - 4bdx^*y^*}}{2}$ real part < 0

($\lambda_{1,2}$ are negative / have negative real parts) $\Longrightarrow$ (x^*, y^*) is stable (asymptotically)

either Node / Focus (determined by $(ex^*-fy^*)^2 - 4bdx^*y^* \wedge 0$)

9) Poincaré – Bendixon Theorem

(i) (Thm) Suppose a solution of a system $\begin{cases} x'(t) = f(x,y) \\ y'(t) = g(x,y) \end{cases}$ (*) remains in a bounded closed region of the phase plane, which contains No equilibrium points of the system (*). Then $((x(t), y(t))$ must spiral into a simple closed curve, which is itself the orbit of the periodic solution of (*).

(ii) (Illustration e.g.) $\begin{cases} x'(t) = -y + x(1-x^2-y^2) \\ y'(t) = x + y(1-x^2-y^2) \end{cases}$ (only equilibrium $(0,0)$)

"See in MAT 3300 Notes, which is just in this Note book"

Using Poincaré-Bendixon thm $\Rightarrow$ consider $\Omega := \{(x,y) \mid \frac{1}{4} < x^2+y^2 < 4\}$.

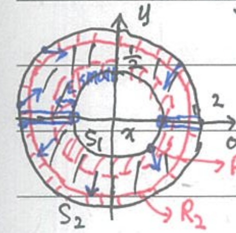
Any point starting in Ω stays in Ω forever by analysis of r^2.

(e.g.2) $x''(t) + (x^2(t) + 2x'^2(t) - 1)x'(t) + x(t) = 0$

Let $y = x'$, we get $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x - y(x^2+2y^2-1) \end{bmatrix}$

Firstly, pick Ω. Because $(\frac{1}{2}r^2)' = xx'+yy' = xy + [-xy - y^2(x^2+2y^2-1)]$
$= y^2(1-x^2-2y^2)$, let $\Omega := \{(x,y) \mid \frac{1}{4} < x^2+y^2 < 4\}$.

Date: 2022 May 06



On S_1: $1-2y^2-x^2 \geq 1-2(x^2+y^2) \geq \frac{1}{2} > 0$, $r^2 \uparrow$ ($\Rightarrow (r^2)' \geq 0$ on the red region) R_1

On S_2: $1-2y^2-x^2 \leq 1-x^2-y^2 = -1 < 0$, $r^2 \downarrow$ ($\Rightarrow (r^2)' \leq 0$ on the red region) R_2

$\therefore$ There is a periodic solutions in Ω.

(iii) [Limit cycles]: (Defn) A closed trajectory (needs to be a soln) $\mathcal{C}$ is called a limit cycle of (*) $\begin{cases} x' = f(x,y) \\ y' = f(x,y) \end{cases}$ if its orbits spiral into or away from it

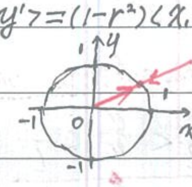
(e.g.3) Find all limit cycles of the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - x^3 - xy^2 \\ y - y^3 - x^2y \end{bmatrix}$

Let $\begin{bmatrix} x' \\ y' \end{bmatrix} = \vec{0}$, we get $\begin{cases} x = 0 \text{ or } x^2+y^2 = 1 \\ y = 0 \text{ or } x^2+y^2 = 1 \end{cases}$

√Case I: $(0,0)$; Case II: $(0, \pm 1)$; Case III: $(\pm 1, 0)$; √Case IV: $x^2+y^2 = 1$ (Equilibrium points)

$(\frac{1}{2}r^2)' = xx'+yy' = r^2(1-r^2) \begin{cases} >0, & r<1 \uparrow \\ 0, & r=1 \text{ —} \\ <0, & r>1 \downarrow \end{cases}$. $\langle x', y' \rangle = (1-r^2)\langle x, y \rangle$

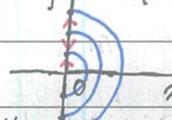
$x^2+y^2 = 1$ is NOT a limit cycle, it is a set of equilibrium points.



(e.g.4) $\begin{cases} x' = xy + x\cos(x^2+y^2) \\ y' = -x^2 + y\cos(x^2+y^2) \end{cases}$. Let $x' = y' = 0$, then $\begin{cases} x(y+\cos r^2) = 0 \\ -x^2 + y\cos r^2 = 0 \end{cases}$

$\Rightarrow$ All equilibrium points are $(0,0)$; $(0, \pm\sqrt{(n+\frac{1}{2})\pi})$, $n \in \mathbb{N}$ & $(\frac{1}{2}r^2)' = xx'+yy' = r^2\cos r^2$. $\begin{cases} >0, & r \in (\sqrt{(n-\frac{1}{2})\pi}, \sqrt{(n+\frac{1}{2})\pi}) \\ =0, & r = \sqrt{(n+\frac{1}{2})\pi} \\ <0, & r \in (\sqrt{(n+\frac{1}{2})\pi}, \sqrt{(n+\frac{3}{2})\pi}) \end{cases}$

↳ No orbit can cross y-axis

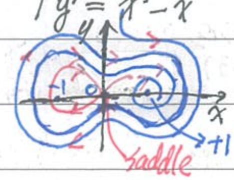


(iv) Some comments on Poincaré – Bendixon thm:

① In practice, we show in fact, all solns start in Ω will stay in Ω for all t large by [examining the slope field $\langle f(x,y), g(x,y) \rangle$] on the boundary of Ω, making sure it's point into Ω

② Let $\mathcal{C}$ be a closed orbit of (*), then the region enclosed by $\mathcal{C}$ must contain [at least one steady state.]

(e.g $\begin{cases} x' = -y - \varepsilon\left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)(x^3 - x) \\ y' = x^3 - x - \varepsilon\left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right) y \end{cases}$ when $\varepsilon = 0$, $\begin{cases} x' = -y \\ y' = x^3 - x \end{cases}$)



10) Predator-Prey Model.

Volterra $\begin{cases} x' = x(a - by) \\ y' = y(-c + dx) \end{cases}$ & $\begin{cases} x' = x(a - by - ex) \\ y' = y(-c + dx - fy) \end{cases}$ with $\begin{cases} a/e > c/d \\ a/e < c/d \end{cases}$ 2 cases

(see in MAT 3300, previous Notes of this notebook)

11) Competition - Exclusion

From Logistic-equation $\begin{cases} \dot{x} = x(a - bx - cy) \\ \dot{y} = y(m - nx - ly) \end{cases}$, $a, b, c, m, n, l \in \mathbb{R}^+$

(Lotka-Votera competition system)

(i) $\frac{b}{n} < \frac{a}{m} < \frac{c}{l}$ — strong competition

(ii) $\frac{b}{n} > \frac{a}{m} > \frac{c}{l}$ — weak competition

Equilibrium points: $(0,0)$; $(0, \frac{m}{l})$; $(\frac{a}{b}, 0)$; (x^*, y^*) → s.t. $\begin{bmatrix} b & c \\ n & l \end{bmatrix}\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} a \\ m \end{bmatrix}$ i.e. $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \frac{1}{bl - cn}\begin{bmatrix} al - cm \\ bm - an \end{bmatrix}$

eliminates — coexistence steady state

Prepositions.

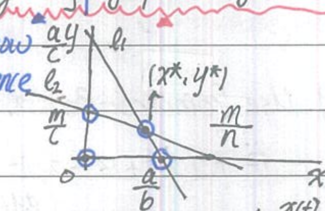
① (x^*, y^*) is unstable if $\frac{b}{n} < \frac{c}{l}$ (strong) & stable if $\frac{b}{n} > \frac{c}{l}$ (weak)

(proof.) Linearize the system $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \doteq \begin{bmatrix} a - 2bx^* - cy^* & -cx^* \\ -ny^* & m - nx^* - 2ly^* \end{bmatrix}\begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix} = \begin{bmatrix} -bx^* & -cx^* \\ -ny^* & -ly^* \end{bmatrix}\begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$. Eigenvalues $\lambda_{1,2} = \frac{1}{2}\left[-(bx^* + ly^*) \pm \sqrt{(bx^* + ly^*)^2 - 4(bl - nc)x^*y^*}\right]$. Case I — if $bl < cn$, 2 eigenvalues one > 0, one < 0, point

(x^*, y^*) is a saddle, thus unstable. Case II: if $bl > cn$, 2 eigenvalues must both have negative real parts, thus, stable.

Remark: In reality, (x^*, y^*) in weak case is globally asymptotically stable for $(x > 0, y > 0)$. use a decreasing non-negative quantity to show convergence

(proof.) Lyapunov Functional Method



Define $\varphi(x, y) = n(x - x^* - x^* \ln\frac{x}{x^*}) + c(y - y^* - y^* \ln\frac{y}{y^*})$, parametrize as $E(t) = \varphi[x(t), y(t)] = n(x(t) - x^* - x^* \ln\frac{x(t)}{x^*}) + c \cdot (y(t) - y^* - y^* \ln\frac{y(t)}{y^*})$. Claim: $E'(t) \leq 0$, $\forall t \in \mathbb{R}^+$ & "=" only holds for $\begin{cases} x = x^* \\ y = y^* \end{cases}$

$\nabla\varphi = \langle n(1 - \frac{x^*}{x}), c(1 - \frac{y^*}{y}) \rangle = \vec{0}$, if $x = x^*$ & $y = y^*$. (always ≥ 0)

$$\frac{dE(t)}{dt} = n\left(x'(t) - \frac{x^* x'(t)}{x(t)}\right) + c\left(y'(t) - \frac{y^* y'(t)}{y(t)}\right) = n\frac{x'(t)}{x(t)}(x(t) - x^*) + c\frac{y'(t)}{y(t)}(y(t) - y^*)$$

$$= n(x - x^*)(a - bx - cy) + c(y - y^*)(m - nx - ly) = n(x - x^*)[-b(x - x^*) - c(y - y^*)]$$

$$+ c(y - y^*)[-n(x - x^*) - l(y - y^*)] = -[bn(x - x^*)^2 + 2cn(x - x^*)(y - y^*) + cl(y - y^*)^2] \leq 0$$

because $bn(x - x^*)^2 + cl(y - y^*)^2 \geq 2\sqrt{bcln}\,|(x - x^*)(y - y^*)| > 2cn|x - x^*||y - y^*|$.

"=" hold iff $x = x^*$ & $y = y^*$.

12) Epidemic Model - SIR Model

N: total populations; $S(t)$: suspectible; $I(t)$: infective; $R(t)$: removed (recover/death)

Assumptions. ① $N = S + I + R$ (neglect unrelated death)

② Rate of change of $S(t)$ is proportional to the product of S & I.

③ Rate of change for $R(t)$ is proportional to the size of I.

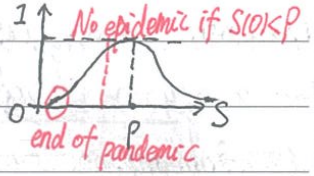
$\begin{cases} \dot{S} = -rSI \\ \dot{I} = rSI - \gamma I \\ \dot{R} = \gamma I \end{cases}$ → Just consider the first 2.

Date: 2022 May 13

$\frac{dI}{dS} = -1 + \frac{\gamma}{r}\cdot S^{-1} \triangleq -1 + \frac{\rho}{S} \Rightarrow I(S) = -S + \rho \ln(S) + C$

set $t=0 \Rightarrow I(S) = \rho \ln \frac{S}{N} - (S-N)$. (with $\begin{cases} S(0)=N \\ I(0)=R(0)=0 \end{cases}$)

(Equilibrium points: $I=0$, $S \in \mathbb{R}^+$)



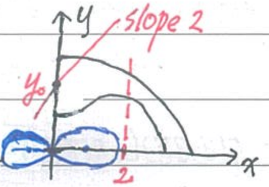
(e.g. Use Poincaré–Bendixson to show that

$\ddot{z} + (\dot{z}^4 + z^2 - 2)\dot{z} = 0$ has non-trivial periodic solutions.

Let $x = z$, $y = \dot{z}$, $\begin{cases} \dot{x} = y \\ \dot{y} = -x(x^2+y^4-2) \end{cases}$; $dy/dx = (\frac{2-x^2-y^4}{y})\cdot x$.

claim that $y(x)$ is symmetric w.r.t. x & y-axis, because $d(y^2) = (2-x^2-y^4)\,dx^2 \Rightarrow dY = (2-X-Y^2)\,dX \Rightarrow \frac{dY}{dX} = 2-X-Y^2$ / directly considering $y(x)$, $y(-x)$ & $-y(x)$, all are or are NOT orbits.

Starting with any point $(0, y_0)$ on y-axis, there is an orbit that is closed because $\frac{dY}{dX} \leq 2$ in the 1st quadrant, the curve must pass 2 because the curve is symmetric. After passing $x=2$, the curve goes down to cross x-axis.)



Addition I: proof of Abel's thm in Chapter III.

Let $W(t) = W[\vec{x}_1, \vec{x}_2, \dots \vec{x}_n](t) = \begin{vmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{vmatrix} \triangleq \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$

$\frac{d}{dt}W(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial W}{\partial x_{ij}} \frac{dx_{ij}}{dt}$; because $\forall\, 1 \leq j \leq n$ & fixed j, $W(t) = \sum_{i=1}^{n} C_{ij} x_{ij}$ with cofactors C_{ij}. Then $\frac{\partial W}{\partial x_{ij}} \frac{dx_{ij}}{dt} = C_{ij} \cdot \dot{x}_{ij}$. Let $C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$

$\therefore \frac{d}{dt}W(t) = tr(C^T \dot{X}) = tr(C^T A X) = tr(X C^T A) = tr(W I A) = tr(A) W$

$tr(AB) = tr(BA)$

Date: 2022 May 23

(A brief note) Introduction to Machine Learning (Stanford CME 107)

- Overview

1) Generic Task: build a "model" from some data ⇒ map raw data to features; → choose form; → choose parameter

general concept

test/validate the model (on unseen data)

2) Taxonomy: supervised — predict & give some others (prediction model)
- regression — predict scalar/vector value
- classification — predict value from finite set (e.g. TRUE, FLASE)

unsupervised ↳ create a model for data

point estimate → a single value prediction

probability estimate → a distribution of values

3) Performance Metrics
- mean square / RMS prediction error (for regression) ← root mean square
- error rate (for classification)
- log likelihood (for probabilistic model)

4) Training & Validation …

5) Learning a model (Empirical Risk Minimization)
- choose model structure / form, containg a number of parameters
- choose a loss function, rating how badly it perfoms on a single example
- choose the parameter value by minimizing avg loss over the training data

- Predictors

1) (Defn) A predictor is a function $g: \mathbb{R}^d \to \mathbb{R}^m$

(i) Use u to represent raw data (input), e.g. vector, word, image, …; $\vec{x} = \phi(u)$ is the corresponding feature vector

Homework I, Chapter I in the textbook

WANG Yuzhe*

February 20, 2022

1 First-order Linear Differential Equations

Selected Questions from 1.2

1.1 q9 & q10

The hardest part is to integrate $\sqrt{1+t^2}e^{-t}$. An alternative choice is to use it implicitly. (Writing as $\exp(-\int_{t_0}^{t}\sqrt{1+\tau^2}e^{-\tau}d\tau)$.)

1.2 q19

One of the thoughts:
General solutions to y is $y = e^{-\int^t a(t)dt}\int^t f(t)e^{\int^t a(t)}dt$. Because $\lim_{x\to\infty} f(x) = 0$, which means $\forall\epsilon > 0, \exists M > 0$, s.t. $|f(x)| < c\epsilon, \forall x > M$.
Therefore, $|y(x)| = e^{-\int^t a(t)dt}\left|\int^t f(t)e^{\int^t a(t)dt}dt\right| \leq e^{-\int^t a(t)dt}\int^t \left|f(t)e^{\int^t a(t)dt}\right|dt < c\epsilon e^{-\int^t a(t)dt}\int^t e^{\int^t a(t)dt}dt = c\epsilon\dfrac{\int^t e^{\int^t a(t)dt}dt}{\int^t a(t)e^{\int^t a(t)dt}dt} \leq c\epsilon\dfrac{\int^t e^{\int^t a(t)dt}dt}{c\int^t e^{\int^t a(t)dt}dt} = \epsilon, \forall x > M$.
By definition, $y(x)\to 0$ when $x\to\infty$.

2 Seperable Equations

Selected Questions from 1.4

2.1 q13

Methodology: $\dfrac{dy}{dt} = f(\dfrac{y}{t}) \xrightarrow{v=\frac{y}{t}} t\dfrac{dv}{dt} + v = f(v)$.

2.2 q21

Methodology: $$\frac{dy}{dt} = \frac{at+by+m}{ct+dy+n} \longrightarrow \begin{cases} \xrightarrow[Y=y+\frac{an-cm}{ad-bc}]{T=t+\frac{dm-bn}{ad-bc}} \dfrac{dY}{dT} = \dfrac{aT+bY}{cT+dY} & ad\neq bc \\ \xrightarrow[abcd\neq 0,\text{ otherwise seperable}]{Y=at+by+m} \dfrac{dY}{dt} = f(Y) & ad = bc \end{cases}$$

3 Population Models

Selected Questions from 1.5

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3.1 q2 (a)

Solve the system of equation about a, b. (For convenience, mark $p(t_1) = p_1, p(t_2) = p_2, \triangle t = t_1 - t_0 = t_2 - t_1$).

The result: $\begin{cases} a = \dfrac{1}{\triangle t} \ln \dfrac{p_2(p_1 - p_0)}{p_0(p_2 - p_1)} \\ b = a \dfrac{p_1^2 - p_2 p_0}{p_1(p_2 p_1 + p_2 p_0 - 2p_2 p_0)} = \dfrac{p_1^2 - p_2 p_0}{p_1 \triangle t(p_2 p_1 + p_2 p_0 - 2p_2 p_0)} \ln \dfrac{p_2(p_1 - p_0)}{p_0(p_2 - p_1)} \end{cases}$

4 Exact Equations

Selected Questions from 1.9

4.1 q20 Bernoulli DE

Case I: $y \equiv 0$, LHS $= 0 =$ RHS.

Case II: $y \neq 0$, the original DE is equivalent to $[a(t)y^{1-n} - b(t)] + y^{-n}\dfrac{dy}{dt} = 0$. Thus, $\dfrac{\mu_t}{\mu} = \dfrac{M_y - N_t}{N} = (1-n)a(t)$. Thus, $\mu = \mu(t) = \exp\left[(1-n)\int a(t)dt\right]$.

From this we can easily get $y = y(t) = \left[(1-n)e^{(n-1)\int a(t)dt} \int \exp((1-n)\int a(t)dt) b(t) dt\right]^{\frac{1}{1-n}}$

5 The Existence-Uniqueness Theorem

Selected Questions from 1.10

5.1 q5,q7~q9

Strategy: Choose appropriate a, b (i.e., rectangular) to make sure $\alpha = \min(a, \frac{b}{M})$ such that t is in the range. (e.g., for q5, α must $\geq \frac{1}{3}$)

Pay attention to example 6 (Suppose that $|f(t, y)| \leq K$ in the strip $t_0 \leq t < \infty$, $-\infty < y < \infty$. The solution $y(t)$ of the initial-value problern $y' = f(t, y), y(t_0) = y_0$ exists for all $t \geq t_0$.)

5.2 q17~q19

Note: conditions (f, $\frac{\partial f}{\partial y}$ continuous); range (a rectangular); initial value.

may happen. In Problems 20–22 determine the behavior of all solutions of the given differential equation as $t \to 0$, and in Problem 23 determine the behavior of all solutions as $t \to \pi/2$.

20. $\frac{dy}{dt} + \frac{1}{t}y = \frac{1}{t^2}$ **21.** $\frac{dy}{dt} + \frac{1}{\sqrt{t}}y = e^{\sqrt{t}/2}$

$$y(t) = c(t)\, e^{-\int \frac{1}{t}\,dt}$$
$$= c(t)\, e^{-\ln|t|}$$
$$= c(t)\frac{1}{|t|}, \quad t \neq 0.$$

$\underline{t>0}$: $c'(t)\frac{1}{t} = \frac{1}{t^2}$

$\int c'(t)dt = \int \frac{1}{t}\,dt \qquad c(t) = \ln|t| + \tilde{C}_1$

$y(t) = (\ln t + \tilde{C}_1)\frac{1}{t}.$

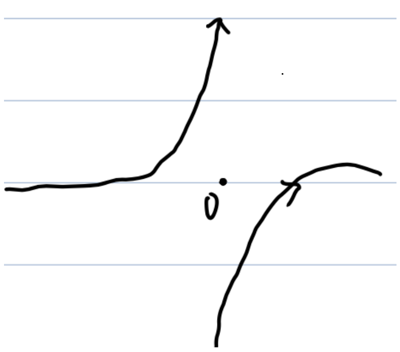
$\underline{t<0}$: $-c'(t)\frac{1}{t} = \frac{1}{t^2}$

$-\int c'(t) = \int \frac{1}{t}\,dt \qquad c(t) = -\ln|t| + \tilde{C}_2$

Then, y(t) = 1/t * [ln|t| + const], $\forall t \in \mathbb{R}$

$t \to 0^+ \quad y(t) = (\underbrace{\ln t}_{\to -\infty} + \tilde{C}_1)\underbrace{\frac{1}{t}}_{\to +\infty} \to -\infty.$

$t \to 0^- \quad y(t) = (\underbrace{\ln t}_{\to -\infty} + \tilde{C}_2)\underbrace{\frac{1}{t}}_{\to -\infty} \to +\infty.$



Homework II, Chapter II in the textbook

WANG Yuzhe*

March 20, 2022

1 Algebraic Properties of Solutions

Selected Questions from 2.1

1.1 q18

Firstly get 2 consecutive zeros of y_2, marked as $\alpha < \beta$, respectively. Thus consider $f(t) = \dfrac{y_1(t)}{y_2(t)}$ on (α, β), because $f'(t) = -\dfrac{W[y_1, y_2](t)}{y_2^2(t)}$, and y_1, y_2 form a fundamental set of solutions ($W[y_1, y_2](t) \neq 0$), we get $f'(t) \neq 0$ on the interval (α, β).

Thus, $f(t)$ is monotonic on (α, β) ($f'(t)$ exists on $\mathbb{R} \Rightarrow f(t)$ is continuous on $\mathbb{R}$). $f(t)$ has at most one zero in (α, β), meaning that $y_1(t)$ has at most one zero in (α, β)

Assume for contradicts, $y_1(t)$ has no zero on (α, β), which gives that $f(t) \neq 0, t \in (\alpha, \beta)$. Considering $f^{-1}(t) = \dfrac{1}{f(t)}$, we have $[f^{-1}(t)]' = \dfrac{W[y_1, y_2](t)}{y_1^2(t)} \neq 0$. However, $f^{-1}(\alpha) = f^{-1}(\beta) = 0$, by Rolle's Theorem, $\exists\ t_0 \in (\alpha, \beta)$ s.t. $[f^{-1}(t_0)]' = 0$, contradictory!

Thus, $y_1(t)$ has and only has one zero between consecutive zeros of $y_2(t)$.

2 The Method of Variation of Parameters

Selected Questions from 2.4

2.1 q10

Wronskian is constant $\Rightarrow p(t) = 0$, plug in $y_1(t) \Rightarrow q(t) = -\dfrac{2}{(1+t)^2} \xrightarrow{\text{Variation of Parameters}} y_2 = -\dfrac{1}{3(t+1)} \xrightarrow{\text{Non-homogeneous case}} u_1', u_2' \Rightarrow y_{\text{particular}}$.

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3 The Method of Judicious Guessing / Ad Hoc Method

Techniques from 2.5 – guess solutions of constant coefficients non-homogeneous equation (i.e., $L[y](t) = a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = g(t)$).

3.1 $g(t)$ as a polynomial $P(t)$

Idea: guess a polynomial $\psi(t)$ such that $a\psi''(t) + b\psi'(t) + c\psi(t)$ is a polynomial with degree n.

Concrete Operations: $\psi(t) = \begin{cases} A_0 + A_1 t + \cdots + A_n t^n & c \neq 0 \\ t(A_0 + A_1 t + \cdots + A_n t^n) & c = 0, b \neq 0 \\ t^2(A_0 + A_1 t + \cdots + A_n t^n) & c = b = 0 \end{cases}$.

3.2 $g(t)$ as $P(t)e^{\alpha t}$

Idea: reduce to case I by variation of parameters, $y(t) = e^{\alpha t}v(t)$. (Note: different cases in case I is corresponding to different homogeneous solutions in this case.)

Concrete Operations:

$$\psi(t) = \begin{cases} (A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t} & e^{\alpha t} \text{ is NOT a homogeneous solution} \\ t(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t} & e^{\alpha t} \text{ is a homogeneous solution, but } te^{\alpha t} \text{ is NOT} \\ t^2(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t} & e^{\alpha t}, te^{\alpha t} \text{ are homogeneous solutions} \end{cases} \cdot$$

3.3 $g(t)$ as a complex function

Idea: If $y(t) = y_1(t) + iy_2(t)$ is a solution of the equation with $g(t) = g_1(t) + ig_2(t)$, then $L[y_1](t) = g_1(t)$ and $L[y_2](t) = g_2(t)$.

Special case: $g(t) = P(t)e^{i\omega t} = P(t)\cos\omega t + iP(t)\sin\omega t$, with solutions as $\mathrm{Re}(\psi(t))$ and $\mathrm{Im}(\psi(t))$, respectively.

Remark: the technique of addition – for $L[y] = a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = \sum_{j=1}^{n} p_j(t)e^{\alpha_j t}$. Take apart as $L[y] = p_j(t)e^{\alpha_j t}$ to get solution $\psi_j(t)$, then $\psi(t) = \sum_{j=0}^{n} \psi_j(t)$ is a particular solution.

4 Series Solutions

Selected Questions from 2.8

4.1 q5

Guess in the form of series $\sum_{n=0}^{\infty} a_n(t-1)^n \longrightarrow \sum_{n=0}^{\infty}(n+1)(t-1)^{2n} = \frac{d}{d(t-1)^2}\sum_{n=0}^{\infty}(t-1)^{2n+2} = \frac{d}{d(t-1)^2}\frac{(t-1)^2}{1-(t-1)^2} = \frac{1}{t^2(2-t)^2}$.

4.2 q12(b)

Guess in the form of $\left(\sum_{n=0}^{\infty} a_n t^n\right)e^t \longrightarrow a_0 = a_1 = 0, a_2 = \frac{1}{2}, a_3 = -\frac{1}{3}, a_4 = \frac{1}{8} \longrightarrow \left(\frac{1}{2}t^2 - \frac{1}{3}t^3 + \frac{1}{8}t^4 + O(t^5)\right)\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + O(t^5)\right) \longrightarrow$ first 5 coefficients of $t^0, \cdots, t^4$.

5 The Laplace Transform

Selected Questions from 2.9, 2.10

5.1 2.9 q14

$f(t)$ is of exponential order $\iff \exists\ M, c$ such that $|f(t)| \le Me^{ct}$, $0 \le t < \infty$.

$0 \le |\mathcal{L}\{f(t)\}(s)| = \left|\int_0^\infty f(t)e^{-st}dt\right| \le \int_0^\infty \left|f(t)e^{-st}\right| dt \le \int_0^\infty |M|\, e^{-(s-c)t}dt = \frac{|M|}{c-s}, \forall s > c$.

Then, Squeeze Theorem shows that $0 \le \lim_{s\to\infty} |\mathcal{L}\{f(t)\}(s)| \le \lim_{s\to\infty}\frac{|M|}{c-s} = 0 \Rightarrow \lim_{s\to\infty}\mathcal{L}\{f(t)\}(s) = 0$.

5.2 2.10 q6

Two properties:

Property I:

$$\mathcal{L}\{-tf(t)\}(s) = \frac{d}{ds}\mathcal{L}\{f(t)\}(s). \tag{1}$$

Property II: Suppose $\lim_{t\to 0}\frac{f(t)}{t}$ exists.

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty \mathcal{L}\{f(t)\}(u)du. \tag{2}$$

5.3 2.10 q16

Ans. $\mathcal{L}\left\{\frac{\sin t - t\cos t}{2}\right\}(s) = \frac{1}{(s^2+1)^2}$.

Thoughts: $\displaystyle\int_s^\infty \frac{1}{(\mathbf{u}^2+1)^2}d\mathbf{u} = -\frac{s}{2(s^2+1)} + \frac{1}{2}\left(\frac{\pi}{2} - \arctan s\right) \xrightarrow[\text{2.10 exercise 7(a)}]{\text{Property II}} -\frac{s}{2(s^2+1)} + \frac{1}{2}\left(\frac{\pi}{2} - \arctan s\right) = \mathcal{L}\left\{\frac{\sin t}{2t} - \frac{\cos t}{2}\right\}(s) = \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s)$.

Another way: 2.14 q12, the Convolution Integral and theoerem 9, $f(t) = g * g(t) = \displaystyle\int_0^t \sin(u-t)\sin(u)du = \frac{1}{2}(\sin t - t\cos t)$.

Homework III, Chapter III in the textbook

WANG Yuzhe*

April 9, 2022

1 Dimention of A Vector Space

Selected Questions from 3.3

1.1 q7

Two steps: (1) $\dim(\mathbf{V}) = 3 - 1 = 2$ (proof by considering IVP); (2) Use chracteristic equations to find 2 solutions $e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - e^{-t}; e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$ independent, thus a basis.

1.2 q14

Hint: (1) vector space: definition + triangle inequality; (2) $\mathbf{V}$ contains all polynomials, argue by contradicts.

2 Applications of Linear Algebra to Differential Equations

Selected Questions from 3.4

2.1 q4

Method I: Directly use generalized eigenvalues and Jordan normal form to find 3 independent solutions;

Method II: Let $x_1 = x_2 = 0, x_3 \neq 0$, $x_1 = 0, x_2, x_3 \neq 0$ and $x_1, x_2, x_3 \neq 0$, respectively.

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2.2 q14

$\mathbf{x}^1(t_0), \cdots, \mathbf{x}^n(t_0)$ linearly independent in $\mathbb{R}^n$ for some t_0 implies $\mathbf{x}^1(t), \cdots, \mathbf{x}^n(t)$ are linearly independent functions (argue by contradicts).

$\mathbf{x}^1(t_0), \cdots, \mathbf{x}^n(t_0)$ linearly dependent in $\mathbb{R}^n$ for some t_0 does NOT imply $\mathbf{x}^1(t), \cdots, \mathbf{x}^n(t)$ are linearly dependent functions. (a counter-exapmle, $\mathbf{x}^m(t) = (t, \cdots, t, 0, \cdots, 0)$, with the first m variables to be t. Then they are independent at $t \neq 0$, but at $t = 0$ they are dependent.)

3 Linear Transformations

Selected Questions from 3.7

3.1 Some definitions

(*Lemma.*) Let A be an $n \times n$ matrix and $\mathcal{T}$ be the linear transformation defined by the equation $\mathcal{T}(x) = Ax$. Then, $\mathcal{T}$ has an inverse if and only if, the matrix A has an inverse. Moreover, if A^{-1} exists, then $\mathcal{T}^{-1}(x) = A^{-1}x$.

3.2 q21

(a) F.T.C $\Longrightarrow (Kf)(t)$ is differentiable, thus continuous.

(b), (c) By association rule of matrix, we get $(KD)f = [K(Df)]$.

4 Equal Roots

Selected Questions from 3.10

4.1 q15

$$\alpha e^{At} = \sum_{i=0}^{\infty} \alpha \frac{A^n}{n!} t^n = A \sum_{i=0}^{\infty} \frac{(\alpha t)^n}{n!} = A e^{\alpha t}. \text{ Then, } e^{At} = \frac{e^{\alpha t}}{\alpha} A.$$

4.2 q17 (b)

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^n}{n!} t^n \xlongequal{A^2 = -I} I \cos t + A \sin t.$$

5 Fundamental matrix solutions

Selected Questions from 3.11

5.1 q1, q3, q7

Compute e^{At} from A: find all (generalized) eigenvalues of $A \Longrightarrow$ find a fundamental matrix solution $\mathcal{X}(t) \Longrightarrow e^{At} = \mathcal{X}(t)\mathcal{X}^{-1}(0)$;

Compute A from e^{At}: $\dfrac{de^{At}}{dt} = Ae^{At} \Longrightarrow A = Ae^0 = \left.\dfrac{de^{At}}{dt}\right]_{t=0}$.

5.2 q9, q11

Determine whether the given matrix is a funda- mental matrix solution:

Necessary Condition (I): linear independent columns for all $t \in \mathbb{R}^n$;

Necessary Condition (II): use $e^{At} = \mathcal{X}(t)\mathcal{X}^{-1}(0)$ to find A ($A = \mathcal{X}'(0)\mathcal{X}^{-1}(0)$);

Validation: use A to show that $\mathcal{X}'(t) = A\mathcal{X}(t)$.

6 The nonhomogeneous equation; variation of parameters

Selected Questions from 3.12

6.1 q7

$$L[y] = f(t) = 0 \Longleftrightarrow ((y, y', \cdots, y^{(n-1)})^T)' = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} (y, y', \cdots, y^{(n-1)})^T;$$

Then, apply variation of parameters: $\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-s)}(0, 0, \cdots, f(s))^T ds$. Thus, $y(t) = \int_0^t v(t-s)f(s)ds$.

Homework IV, Chapter IV in the textbook

WANG Yuzhe*

May 17, 2022

1 Introduction and The Equillibrium Point

Selected Questions from 4.1

1.1 q10

Firstly, find the general solution $\begin{bmatrix} x \\ y \end{bmatrix}(t) = e^{-t}\left(x(0)\begin{bmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + y(0)\begin{bmatrix} -\frac{\sqrt{2}}{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{bmatrix}\right)$.

Suppose that we make an error of magnitude 10^{-4} in rneasuring $x(0)$ and $y(0)$, then for $x(t)$ and $y(t)$, the largest error is no larger than that of magnitude 10^{-3} (by Cauchy inequality or auxiliary angle formula).

1.2 q11 (b)

Key point: use the similar idea to Chapter II (or another method with Chapter III), let $\mathbf{x}_{nh}$ be a soln of $\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}\mathbf{x}$, then $\mathbf{x}_{nh} + e^{-t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a soln with $\mathbf{x}(0) \neq 0$. Then, show that equillibrium points are unstable for the linear system.

2 Stability of Linear Systems

Selected Questions from 4.2

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2.1 q11,q12,q13

Directly solve the equation and apply the definition. For instance, in q11, $x \equiv 0$ is unstable and $x \equiv 1$ is stable (moreover, asymptotically stable).

2.2 q15

Similar to 4.1 q11(b) above, consider difference of 2 solutions of nonhomogeneous case $\mathbf{x}_{nh} - \mathbf{y}_{nh}$, which is a solution of homogeneous case.

2.3 q21

Let $M = \dfrac{1}{\sqrt{n}}$ and $N = 1$, we can directly get this solution. (Standard way: applying method of Lagrangian multipliers, see details in MAT2007 HW6).

3 Qualitative Properties of Orbits

Selected Questions from 4.6

3.1 q1, q5

Technique: analysis the boundary (steady states) and then use that 2 orbits cannot intersect.

3.2 q12

Wrong conditions: (correction version) $0 < \dot{z}(0)^2 + z(0)^2 - z(0)^4 < \dfrac{1}{4}$ and $|z(0)| \leq \dfrac{\sqrt{2}}{2}$. Divide into cases and discuss the diagrams and possible cases.

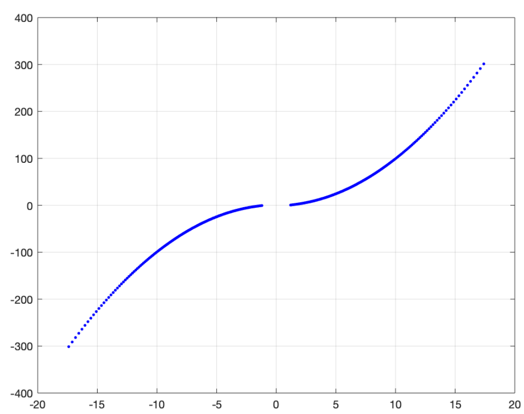


Figure 1: Case I: $0 > \dot{z}(0)^2 + z(0)^2 - z(0)^4$

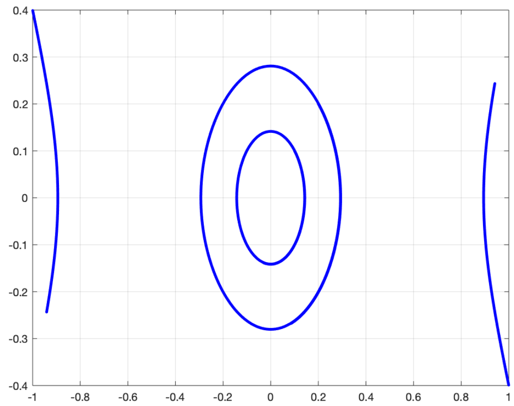


Figure 2: Case II: $0 < \dot{z}(0)^2 + z(0)^2 - z(0)^4 < \frac{1}{4}$

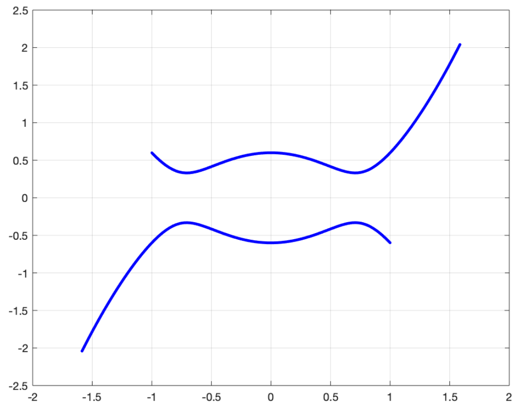


Figure 3: Case III: $\dot{z}(0)^2 + z(0)^2 - z(0)^4 \geq \frac{1}{4}$

Case 1. $\tilde{C} < 0$

let $y=0$, $x^2 - x^4 = \tilde{C} \Rightarrow (x^2 - \frac{1}{2})^2 = -\tilde{C} + \frac{1}{4}$

$x^2 - \frac{1}{2} = \pm\sqrt{\underbrace{\frac{1}{4} - \tilde{C}}_{> \frac{1}{4}}}$

$\frac{dy}{dx} = \frac{(2x^3 - x)}{y} > 0$

$\frac{dx}{dt} = y$

$x^2 = \frac{1}{2} \pm \sqrt{\underbrace{\frac{1}{4} - \tilde{C}}}$

$\frac{dy}{dt} = 2x^3 - x$.

two roots $\pm x_0$, $x_0 > \frac{\sqrt{2}}{2}$

$y^2 = (x^2 - \frac{1}{2})^2 + (\tilde{C} - \frac{1}{4})$.

let $x = 0$, no root for y

Case 2. $0 < \tilde{C} < \frac{1}{4}$.

$y^2 = (x^2 - \frac{1}{2})^2 + (\tilde{C} - \frac{1}{4})$

$\frac{dy}{dx} = \frac{2x^3 - x}{y}$.

- let $x = 0$ $y^2 = \tilde{C} > 0$ $y = \pm y_0$, $y_0 > 0$.
- let $y = 0$, $(x^2 - \frac{1}{2}) = \pm\sqrt{\underbrace{\frac{1}{4} - \tilde{C}}_{\frac{1}{4} > \ > 0}}$

$\frac{dx}{dt} = y$

$\frac{dy}{dt} = 2x^3 - x$

$x^2 = \frac{1}{2} \pm \sqrt{\underbrace{\frac{1}{4} - \tilde{C}}_{< \frac{1}{2}}}$

y^2

$(\frac{\sqrt{2}}{2}, \overset{>0}{y_1})$ plug into

$y^2 = (x^2 - \frac{1}{2})^2 + (\tilde{C} - \frac{1}{4})$

roots $x = \pm x_0, \pm x_1$. $0 < x_0 < \frac{\sqrt{2}}{2} < x_1$

$\Rightarrow \tilde{C} = \frac{1}{4} + y_1^2 > \frac{1}{4}$

Case 3 $\quad \tilde{C} > \frac{1}{4}$, $\quad y^2 = (x^2-\frac{1}{2})^2 + (\tilde{C}-\frac{1}{4})$.

let $y=0$, $\quad (x^2-\frac{1}{2})^2 + \underbrace{(\tilde{C}-\frac{1}{4})}_{>0} = 0$. $\quad$ no roots.

let $x=0$, $\quad \uparrow \quad y = \pm y_0$

$\frac{dy}{dt} = 2x^3 - x$

$x > \frac{\sqrt{2}}{2}$

$\frac{dy}{dt} > \alpha > 0$.

$y \uparrow$ at least in a linear rate.

3.3 q14

Addition (operator norm): suppose $A \in \mathbb{R}^{m\times n}$, define $\|A\|_{a,b}$ as maximium of $\|Ax\|_a$ subject to $\|x\|_b \le 1$. Then, we will have the property $\|Ax\| = \left\|A\frac{x}{\|x\|}\right\|\|x\| \le \|A\|\|x\|$.

Applying the Picard iterations $\mathbf{x}_{j+1}(t) = \mathbf{x}^0 + A\int_0^t \mathbf{x}_j(s)ds$, show by M-test that it converges to the solution $\mathbf{x}(t) = e^{At}\mathbf{x}^0$.

4 Long Time Behavior of Solutions; the Poincare-Bendixson Theorem

Selected Questions from 4.8

4.1 q9

A possible solution:

(e.g. Use Poincaré–Bendixson to show that

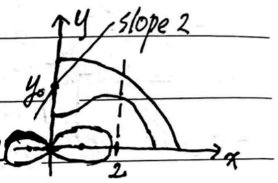
$\ddot{z} + (\dot{z}^4 + z^2 - 2)\dot{z} = 0$ has non-trivial periodic solutions.

Let $x = z$, $y = \dot{z}$, $\begin{cases} \dot{x} = y \\ \dot{y} = -x(x^2+y^4-2) \end{cases}$; $dy/dx = (\frac{2-x^2-y^4}{y}) \cdot x$,

claim that $y(x)$ is symmetric w.r.t. x & y-axis, because $d(y^2) =$

$(2-x^2-y^4)dx^2 \Rightarrow dY = (2-X-Y^2)dX \Rightarrow \frac{dY}{dX} = 2-X-Y^2$ / directly considering

$y(x)$, $y(-x)$ & $-y(x)$, all are or are NOT orbits.



Starting with any point $(0, y_0)$ on y-axis, there is an orbit that is closed because $\frac{dY}{dX} \le 2$ in the 1st quadrant, the curve must pass 2 because the curve is symmetric. After passing $x=2$, the curve goes down to cross x-axis.)

4.2 q11-13

A method to show no non-trivial solution: $f_x + g_y$ stays with the same sign across a region. Assume by contradicts, and use Green's thm $0 = \oint_C f dy - g dx = \iint_R (f_x + g_y) dA \neq 0$, contradictory.

5 The Princip1e of competitive Exclusion in Popu1ation Bio1ogy

Selected Questions from 4.11

5.1 q5

Lyapunov Functional Method: define

$$F(N_1, N_2) = \frac{a_2\beta}{K_2}\left(N_1 - N_1^0 - N_1^0 \ln\frac{N_1}{N_1^0}\right) + \frac{a_1\alpha}{K_1}\left(N_2 - N_2^0 - N_2^0 \ln\frac{N_2}{N_2^0}\right)$$

$E'(t) = \frac{d}{dt}F[x(t), y(t)] \leq 0$, equality only holds for $N_1 = N_1^0$ and $N_2 = N_2^0$.

5.2 q6

Show stability by linearization.

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Midterm Exam
April 17, 2022

Show your work! Answers without justifications will receive no credit. No books, notes or calculators are allowed.

1. (20 pts) Compute the first 3 Picard iterates for the following initial value problem of the Riccati equation (of which the solution cannot be expressed in terms of elementary functions and integrals)

$$y' = t^2 + y^2, \;\; y(0) = 0.$$

2. It has been observed that a mothball of radius 1 cm evaporates to leave a ball of radius 1/2 cm at the end of six months.

(i) (20 pts) Find the radius as a function of time.

(ii) (5 pts) After how many more months will it disappear altogether?

3. (i) (10 pts) Find the general solution of the equation $y'' + 4y' + 4y = 0$.

(ii) (20 pts) Find the solution of the initial value problem

$$t^2y'' - 3ty' + 4y = 0, \quad y(1) = 0, \quad y'(1) = 1.$$

4. (25 pts) Suppose that a nonhomogeneous second-order linear equation

$$y'' + p(t)y + q(t)y = g(t)$$

is known to have three solutions $y_1(t) = 3e^t + e^{t^2}$, $y_2(t) = 7e^t + e^{t^2}$, and $y_3(t) = 5e^t + e^{-t^3} + e^{t^2}$. Find the solution of the initial-value problem

$$y'' + p(t)y + q(t)y = g(t), \;\; y(0) = 1, \;\; y'(0) = 1.$$

5. Suppose that the Wronskian of any two solutions of the equation

$$y'' + p(t)y' + q(t)y = 0$$

where p, q are two continuous functions on $\mathbb{R}$, is a constant, and that the equation is known to have one solution e^t.

(i) (10 pts) Find $p(t)$ and $q(t)$.

(ii) (10 pts) Find the general solution of the equation.

(iii) (10 pts) Find the general solution of the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = t^2 + 1.$$

6. (30 pts) Determine all initial-value vectors $\mathbf{x_0}$ such that the solution of the initial-value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x_0}$ where $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$, is a periodic function of time.

MAT2002 Student ID #:________________ Name:_______________________________
Final Exam
May 17, 2022

Show your work! Answers without justifications will receive no credit. No book-s, notes, calculators, cell phone or internet are allowed.

1. (30 pts) Suppose that y_1 and y_2 are two linearly independent solutions of the equation

$$y'' + p(t)y' + q(t)y = 0$$

where p, q are two continuous functions on $\mathbb{R}$. Show that between two consecutive zeros of y_1 there is exactly one zero of y_2.

2. (30 pts) Find two linearly independent solutions of the equation $y'' - 2ty' + \lambda y = 0$, where $\lambda \neq 2n$, for any integer n, is a constant.

3. (30 pts) A particle of mass 1 kg is attached to a spring dashpot mechanism. The stiffness constant of the spring is 3 N/m and the drag force on the particle by the dashpot mechanism is 4 times its velocity. At time $t = 0$, the particle is stretched 1/4 m from its equilibrium position. At time $t = 3$ seconds, an impulsive force of very short duration is applied to the system and this force imparts an impulse of 2 N·s to the particle. Find the displacement of the particle from its equilibrium position.

4. (30 pts) Let $f(t) \geq 0$ be a continuous function and $k > 0$ be a constant. Suppose that $\psi(t)$ is continuous and satisfies

$$\psi(t) \leq k + \int_0^t f(u)\psi(u)du.$$

Show that

$$\psi(t) \leq k\, exp\Big(\int_0^t f(u)du\Big).$$

5. (30 pts) Does the following system

$$\begin{aligned} x' &= y \\ y' &= x - x^3 \end{aligned}$$

have infinitely many periodic solutions? Justify your answer.

6. Consider the system

$$\begin{aligned} x' &= x(1-2x-y) \\ y' &= y(1-x-2y). \end{aligned}$$

(i) (15 pts) Find all the equilibrium points and determine their local stability properties.

(ii) (15 pts) Is any of the equilibrium points in (i) globally stable in the first quadrant $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$? Justify your answer.

7. (30 pts) Show that all solutions of

$$\begin{aligned} x' &= -1 - y + x^2 \\ y' &= x + xy \end{aligned}$$

which starts inside the unit circle $x^2 + y^2 = 1$ must remain there for all time.

8. (30 pts) Show that the following system of differential equations has no nontrivial periodic solution which lies inside the circle $x^2 + y^2 = 4$:

$$\begin{aligned} x' &= x - xy^2 + y^3 \\ y' &= x^3 + 3y - yx^2 \end{aligned}$$

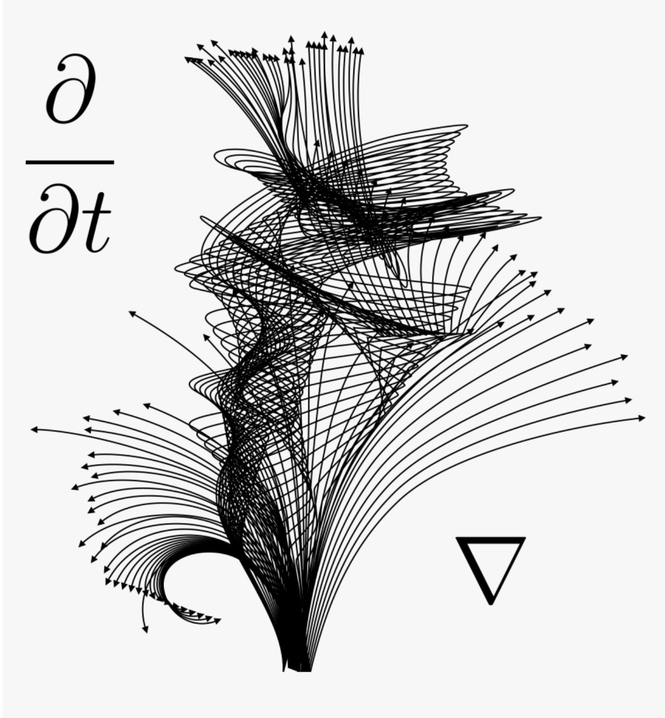
9. (30 pts) Prove the existence of a nontrivial periodic solution of the equation

$$z'' + [\,ln(z^2 + 4z'^{\,2})]\, z' + z = 0.$$

10. (30 pts) Find all the limit cycles of the system

$$\begin{aligned} x' &= y + x(x^2+y^2-1)(x^2+y^2-2) \\ y' &= -x + y(x^2+y^2-1)(x^2+y^2-2) \end{aligned}$$

and determine their stability properties.



Partial Differential Equations

MAT 4220 Notebook

Youthy WANG

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Date: 2022 Sep 5

MAT 4220 Partial Differential Equations

• Introduction

1) For PDE: $\begin{cases}\text{more than 1 } (\geq 1) \text{ variables} \\ \text{dependent variables (e.g. } u = u(x,y))\end{cases}$

2) Classification of PDEs

- Order: highest derivative that appears
- Scalar / System: # of dependent variables
- Linearity: F is linearly dependent of u & derivatives of u
  represented by $F = c(\vec{x}) + \mathcal{L}u$ with indep. var $\vec{x}$.

($\mathcal{L}$ – linear operator, e.g. $\mathcal{L} = \partial_x,\ x\partial_x + y\partial_y$, etc.)

3) Principle of $\boxed{\text{super-position}}$:

linear PDEs $<$ homogeneous: $c(\vec{x}) \equiv 0$ (H)
inhomogeneous: non-constant (zero) $c(\vec{x})$ (IH)

(i) If $u_1, \dots, u_n$ are solns of (H)
then $[c_1u_1 + \dots + c_nu_n]$ with coefficients $c_1, \dots, c_n$ is also a soln of (H).

(ii) If u_0 is a soln to $\mathcal{L}u = g(u)$ (IH), then $[u_0 + \sum_{i=1}^n c_i u_i]$ is also a soln of (IH), given (i) general linear function

4) Subject for PDEs

$\left.\begin{array}{l}\text{Solve basic eqns} \\ \text{existence \& uniqueness} \\ \text{stability}\end{array}\right\}$ well-posedness of eqns

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• Sources of PDEs

1) 1^{st}-order linear equations:

(i) With constant coefficients: $\mathcal{L}u = au_x + bu_y$; $\mathcal{L}u = 0$

① Geometric Method: $\mathcal{L}u = 0 \Leftrightarrow \underbrace{\begin{bmatrix}a\\b\end{bmatrix}}_{\text{directional derivative}}\begin{bmatrix}\partial_x\\\partial_y\end{bmatrix} u = 0$

$\therefore \vec{v} = \begin{bmatrix}a\\b\end{bmatrix} \Rightarrow \vec{v}^T \nabla u = 0,$

that means along $\vec{v}$, u does NOT change $\therefore$ u stay the same along lines $-bx + ay = c$, u does NOT change, then $u(x,y) = u(c) = f(ay - bx)$

② Coordinate Method: $\boxed{\text{Rotate}}$ the coordinates $\begin{cases} X = ax + by \\ Y = -bx + ay\end{cases}$

Then $a\partial_x + b\partial_y = a(\frac{\partial X}{\partial x}\partial_X + \frac{\partial Y}{\partial x}\partial_Y) + b(\frac{\partial X}{\partial y}\partial_X + \frac{\partial Y}{\partial y}\partial_Y)$
$= a(a\partial_X - b\partial_Y) + b(b\partial_X + a\partial_Y) = (a^2+b^2)\partial_X$

$\therefore (a^2+b^2)\tilde{u}_X = (a\partial_x + b\partial_y)u = \mathcal{L}u = 0 \Rightarrow u(x,y) = \tilde{u}(X,Y) = f(Y)$
$= f(ay - bx)$.

(ii) Variable coefficient eqns: $a(x,y)\partial_x + b(x,y)\partial_y = \mathcal{L} = 0$

Similar to above, $\begin{bmatrix}a(x,y)\\b(x,y)\end{bmatrix}^T \nabla u = 0 \Rightarrow \begin{cases} x'(s) = a(x,y) = a(s) \\ y'(s) = b(x,y) = b(s)\end{cases}$ Characteristic line / curves

(e.g. 1) $\begin{cases} u_x + y u_y = 0 \\ u(0,y) = y^2 \end{cases} \Rightarrow$ Let $\underbrace{\frac{\partial u}{\partial x}}_{u_x} \underbrace{x'(s)}_{1} + \underbrace{\frac{\partial u}{\partial y}}_{u_y} \underbrace{y'(s)}_{y} = 0 \approx \frac{du}{ds}$

Along curve $\begin{cases} x' = 1 \\ y' = y \end{cases}$ u does NOT change
$\Rightarrow y = Ae^{x}$, then $u = f(A) = f(ye^{-x})$

$\therefore u(0,y) = y^2 = f(y)$, which means $u(x,y) = y^2 e^{-2x}$.

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IDEA: parametrize characteristic curves $\begin{cases} x = x(s) \\ y = y(s) \end{cases}$. $\phi(x,y) = \phi[x(s), y(s)]$

Especially, when $x(s) = s$, we can get $\phi = \phi(x, y(x))$.

☆ Initial data cannot be given arbitrarily. (to get a general unique soln)

Back to the last e.g. if $u(x,0) = x$ is given, then

$f(0) = f(0)e^{-x} \neq x$ ← On a characteristic line, it must be a const.

The IDEA can be used also in general $a(x,y)u_x + b(x,y)u_y = f(x,y)$

(iii) Inhomogeneous Cases: $a(x,y)\partial_x + b(x,y)\partial_y + f(x,y) = \mathcal{L}$

(e.g.2) $u_x + 4u_y = e^x$. let $\Phi(x) = u[x, y(x)]$, then $\Phi'(x) = u_x + \underset{4}{\frac{dy}{dx}} u_y$

$= e^x \Rightarrow \Phi(x) = e^x + C = e^x - 1 + \Phi(0)$ represent $y = 4x + C$, $y(0) = C$

Meantime, $\Phi(0) = u[0, y(0)] = u[0, y - 4x] = f(y - 4x)$ different (diverse) fs

☆ Note: by geometric/coordinate methods, the [PDE reduces to ODE case] (Solvable ODE ⇒ solvable 1st-order PDE (linear))

2) 2nd-order linear equations $a_{11}\partial_x^2 + 2a_{12}\partial_{xy} + a_{22}\partial_y^2 + f(u, u_x, u_y) = \mathcal{L}$

(i) (Laplace Equations; Heat Equations & Wave Equations forms the whole general cases)

☆ (Thm) Define $\Delta = a_{12}^2 - a_{11}a_{22}$ (w.l.o.g., let $a_{11} > 0$) Then we have (by changing coordinates under some rules)

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elliptic eqns → (i) $\Delta < 0$, the eqn becomes $u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} + \tilde{f} = 0$ ← Laplace eqns

parabolic eqns → (ii) $\Delta = 0$, the eqn becomes $u_{\tilde{x}\tilde{x}} + \tilde{f} = 0$ ← Heat eqns

hyperbolic eqns → (iii) $\Delta > 0$, the eqn becomes $u_{\tilde{x}\tilde{x}} - u_{\tilde{y}\tilde{y}} + \tilde{f} = 0$ ← Wave eqns

(proof.) $a_{11} > 0$, we only consider case (i), $\Delta < 0$

$$a_{11}\partial_x^2 + 2a_{12}\partial_{xy} + a_{22}\partial_y^2 = a_{11}\left(\partial_x + \frac{a_{12}}{a_{11}}\partial_y\right)^2 + \frac{a_{22}a_{11} - a_{12}^2}{a_{11}}\partial_y^2$$

Let $\begin{cases} x = \sqrt{a_{11}}\,\tilde{x} \\ y = \frac{a_{12}}{\sqrt{a_{11}}}\tilde{x} + \sqrt{\frac{-\Delta}{a_{11}}}\,\tilde{y} \end{cases} \longrightarrow \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 = a_{11}\partial_x^2 + 2a_{12}\partial_{xy} + a_{22}\partial_y^2$

Similarly for case (ii) & (iii)

Remark: $\vec{x} \in R_n$ with $\sum_{i,j} a_{ij}\partial_{x_i}\partial_{x_j} u = f(u, Du, \vec{x})$

Let $\vec{x} = B\vec{y}$, with $B = (B_{ij})_{n\times n}$. Then $\partial_{x_i} = \sum_l B_{il}\partial_{y_l}$.

Then $\partial_{x_i}\partial_{x_j} = \sum_{i,j,m,l} B_{il}B_{jm}\partial_{y_l}\partial_{y_m} \Rightarrow a_{ij}\partial_{x_j}\partial_{x_i} = \sum_{i,j,m,l}(a_{ij}B_{il}B_{jm})$.

$\partial_{y_l}\partial_{y_m} = \sum_{m,l}\left(\sum_{i,j} a_{ij}B_{il}B_{jm}\right)\partial_{y_l}\partial_{y_m}$. Moreover, with $A \in S_{n\times n}$

[$(B^TAB)_{n\times n} \to l,m$ term] Some B makes B^TAB = diagonal

eigen-values > 0, elliptic, (or < 0)

1 ei-value different $\begin{cases} 0 \to \text{parabolic} \\ \text{opp sign} \to \text{hyperbolic} \end{cases}$

(ii) Wave Equations: $\partial_t^2 u = c^2\partial_x^2 u$ (come from waves)

Solve the eqn: ① $(\partial_t + c\partial_x)\underbrace{(\partial_t - c\partial_x)u}_{v} = 0 \Rightarrow \begin{cases} (\partial_t - c\partial_x)u = v \\ (\partial_t + c\partial_x)v = 0 \end{cases}$

⇒ Solve for 1st order to get v, then get u ({Geometric / Coordinate Methods)

② Characteristic Coordinates. $\begin{cases} \xi = ct + x \\ \eta = -ct + x \end{cases}$

$\partial_t = c\partial_\xi - c\partial_\eta$ & $\partial_x = \partial_\xi + \partial_\eta$. Then, $4c^2\partial_{\xi\eta}u = 0$.

$\therefore u = f(\xi) + g(\eta) = f(x + ct) + g(x - ct)$ (general soln)

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Initial Value Problem (I.V.P) $\begin{cases} u_{tt}=c^2u_{xx}, & x\in\mathbb{R}\ \&\ t>0 \\ u(x,0)=\phi(x)\ \&\ u_t(x,0)=\psi(x) \end{cases}$

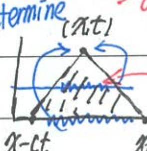
2 initial data for 2nd functions!

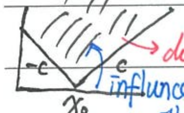
Then $\begin{cases} f(x)+g(x)=\phi(x) \\ cf'(x)-cg'(x)=\psi(x) \end{cases}$ thus (take derivatives of above) $\Rightarrow \begin{cases} f'(x)=\frac{1}{2c}\psi(x)+\frac{1}{2}\phi'(x) \\ g'(x)=\frac{1}{2}\phi'(x)-\frac{1}{2c}\psi(x) \end{cases}$

Integrations & the back to above $\Rightarrow f(x), g(x)$.

☆ $u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds+\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]$ → d'Alembert Formula

Causality & Energy:





$\Rightarrow$ The signal at x_0 cannot move faster than light speed c.

☆ Energy Conservation: $\frac{1}{\rho}E=\frac{1}{2}\int_{\mathbb{R}}(u_t^2+c^2u_x^2)\,dx$

velocity (kinetic energy $\int_{\mathbb{R}}\frac{1}{2}\rho u_t^2\,dx$) force
potential energy $\frac{1}{2}T\int u_x^2\,dx$

Suppose u is compactly supported (i.e., u is supported on $[-R,R]\subseteq\mathbb{R}$), & a soln to wave eqn.

changing variables $\int_\Omega fg'=fg|_{\partial\Omega}-\int_\Omega f'g$

$\frac{1}{\rho}\frac{d}{dt}E=\int_{\mathbb{R}}u_tu_{tt}\,dx+c^2\int_{\mathbb{R}}u_xu_{tx}\,dx=\int_{\mathbb{R}}u_tu_{tt}\,dx-c^2\int_{\mathbb{R}}u_{xx}u_t\,dx=\int_{\mathbb{R}}u_t(u_{tt}-c^2u_{xx})\,dx=0$. Then E is const.

(iii) Diffusion Equations. $\partial_t u=k\partial_x^2 u$ ($k>0$, also heat eqns)

☆ properties. ① Maximum Principle: (weak) (thm) Suppose $u=u(x,t)$ is a soln to diffusion eqn on the domain $\Omega_T=[0,l]\times[0,T]$

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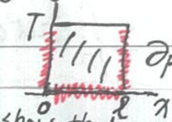
thus, maximum of the soln of Ω_T will be give at the parabolic boundary $\partial_p\Omega_T=\{x=0\}\cup\{x=l\}\cup\{t=0\}$



(proof.) Let $M=\max_{\{x=0,l\}\cup\{t=0\}}u(x,t)$, then we need to show that $u(x,t)\le M$ on R (rectangle). Let $v(x,t)=u(x,t)+\varepsilon x^2$ with any fixed $\varepsilon>0$. Clearly on $t=0$, $x=0$ & $v(x,t)\le M+\varepsilon l^2$. $v_t-kv_{xx}=u_t-ku_{xx}-2k\varepsilon=-2k\varepsilon<0$ (given $k>0$ as usual). Suppose then $v(x,t)$ attaches "max" at interior point (x_0,t_0), then $\nabla v(x_0,t_0)=0$ & $H_v(x_0,t_0)$ must be neg-semidef. Thus, $v_{xx}(x_0,t_0)\le 0$, contradictory with v_t-kv_{xx}.
Suppose maximum on the top most, $v_t(x_0,t_0)=\lim_{\delta\to0}\frac{v(x_0,t_0)-v(x_0,t_0-\delta)}{\delta}\ge0$, $v_{xx}(x_0,t_0)\le0$, still contradictory. Thus $u(x,t)+\varepsilon x^2\le M+\varepsilon l^2$ on the rectan
$\Rightarrow u(x,t)\le M+\varepsilon(l^2-x^2)\Rightarrow u(x,t)\le M$ on the rectangle R.

☆ Some comments: ① Similarly we can get $\min_R u=\min_{\partial R\setminus\{t=T\}}u$.

② Generalize: $\partial_t u=k\Delta u$, with $\Delta=\partial_{x_1}^2+\partial_{x_2}^2+\dots+\partial_{x_n}^2$
parabolic domain $\Omega_T=\bar{\Omega}\times[0,T]$, $\Omega\subseteq\mathbb{R}^n$, we have the similar results

③ More generalized: $\partial_t u=k\Delta u+cu$.

(thm) Assume $\Omega\subseteq\mathbb{R}^n$ is bounded & have smooth boundary ($\partial\Omega$) suppose $\partial_t u=k\Delta u$, on $\Omega_T=\Omega\times[0,T]$, with soln $u=u(\vec{x},t)$

Then, $\boxed{\max_{\Omega_T}u=\max_{\partial_p\Omega_T}u}$, where $\partial_p\Omega_T=\partial\Omega\times[0,t]+\Omega\times\{0\}$ "bottom" surface

(proof.) Similarly, $v=u+\varepsilon\|x\|_2^2$. $v_t-k\Delta v=u_t-k\Delta u-2nk\varepsilon<0$, $\forall\varepsilon>0$. If inside ..., neg-semi def. $H_v\Rightarrow tr(H_v)\le0\Rightarrow\Delta v\le0$, contradictory! (with $v_t=0$)

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Energy of the Heat / Diffusion Eqn: $E = \frac{1}{2}\int_\Omega u^2 d\vec{x}$

$\frac{d}{dt}E = \frac{1}{2}\int_\Omega \frac{d}{dt}u^2 d\vec{x} = \int_\Omega u \cdot k\Delta u\, d\vec{x} \overset{\text{divergence thm}}{=} k\int_{\partial\Omega} u n\cdot\nabla u\, ds - k\int_\Omega \nabla u^T \nabla u = k\int_{\partial\Omega} u\frac{\partial u}{\partial n}ds \boxed{- k\int_\Omega \|\nabla u\|_2^2\, d\vec{x}} \le 0$

$\int_\Omega \vec{F}\nabla g + \int_\Omega (\nabla\cdot\vec{F})g = \int_{\partial\Omega} g\vec{F}\cdot\vec{n}\, ds$

suppose $u=0$ on $\partial\Omega$ / $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$
(Dirichlet) (Neumann)

$\frac{dE}{dt} \le 0$ — energy dissipated

② Uniqueness: $\partial_t u = k\Delta u$, with $u|_{\partial n} = \varphi(\vec{x})$ (Dirichlet-BVP) has unique soln
& $u(0,\vec{x}) = \phi(\vec{x})$

(proof.) Suppose u_1, u_2 are 2 solns, let $v = u_1 - u_2$, "Note that $u|_{\partial n}$ or $\frac{\partial u}{\partial n} = 0$"
$\partial_t v = k v$. $v|_{\partial n} = 0$ & $v(0,\vec{x}) = 0$. Thus, $E(v) = 0$ because "energy dissipated!"
$0 \le E(v(t)) \le E(v(0)) = 0$, thus $v \equiv 0$, $u_1 = u_2$

Remark: Maximum & Minimum principle can also be used for proofs.

③ Stability: Define $\|\cdot\|$ as a metric, with $f, g \in L^2$, (which means $\|f\|_{L^2} = (\int_\Omega f^2 dx)^{\frac{1}{2}} < \infty$, same for g)
L_2-norm

☆ Suppose u_1 & u_2 satisfies the equation (D-BVP). then
$[\|u_1 - u_2\|_{L^2} \le C\|\phi_1 - \phi_2\|_{L^2}]$ (stability s.t. initial data) [when $\varphi_1(\vec{x}) = \varphi_2(\vec{x})$]

(proof.) Using $\frac{d}{dt}E \le 0$ for $v = u_1 - u_2$ we get $(\int_\Omega (u_1-u_2)^2 dx)^{\frac{1}{2}}$
$\le [\int_\Omega (v(0))^2 dx)^{\frac{1}{2}} = [\int_\Omega (\phi_1 - \phi_2)^2 dx]^{\frac{1}{2}}$, which means $\|u_1 - u_2\|_{L^2} \le \|\phi_1 - \phi_2\|_{L^2}$

(Note: with maximum principle, $[\|u_1 - u_2\|_{L^\infty} \le C(\|\phi_1 - \phi_2\|_{L^\infty} + \|\varphi_1 - \varphi_2\|_{L^\infty})]$
[when $\varphi_1(\vec{x}) \ne \varphi_2(\vec{x})$]

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Weak / Strong maximum principle.

$\max_{\Omega_T} u = \max_{\partial_p\Omega_T} u$; $\max_{\Omega_T} u = \max_{\partial_p\Omega_T} u$ & $[u < \max_{\Omega_T} u$, with $\vec{x} \in \Omega_T \setminus \partial_p\Omega_T]$
otherwise $u = \text{const}$

Δ Diffusion Equation on $\mathbb{R}$: $\partial_t u = k\,\partial_{xx} u$; $x \in \mathbb{R}, t > 0$

⇒ Invariance: if $u = u(x,t)$ is a soln, $u(x+a,t)$, $\partial_x u$, $\partial_t u$, $\partial_{xx} u$, $\int_{\mathbb{R}} u(x-y,t)dy$ are all solns; when u_1, u_2 are 2 solns, linear combination $c_1u_1 + c_2u_2$ is a soln. (Under dilation / scaling, $u(\frac{x}{\sqrt{\lambda}}, \frac{t}{\lambda})$ is a solution.)

Try: $u(x,t) = f(\frac{x}{\sqrt{t}})$ (note that scaling invariance holds)
$0 = \partial_t u - k\partial_{xx} u = f'(\frac{x}{\sqrt{t}}) \cdot -\frac{x}{2t^{\frac{3}{2}}} - k f''(\frac{x}{\sqrt{t}}) \cdot \frac{1}{t} \overset{x/\sqrt{t} = p}{=\!=} [f'(p)\cdot\frac{p}{2} + kf''(p)]$
$\cdot -\frac{1}{t}$. Thus, $pf'(p) + 2kf''(p) = 0 \Rightarrow f'(p) = ce^{-\frac{p^2}{4k}}$, $f(p) = \int_0^p ce^{-\frac{s^2}{4k}}ds$
$+ f(0) = \underbrace{[\int_0^{\frac{x}{\sqrt{4kt}}} ce^{-y^2}dy + f(0)]}_{\triangleq Q(x,t)}$ (No explicit form!)

Note derivatives also give solns: $\partial_x Q = \tilde{C}\frac{1}{\sqrt{t}}e^{-\frac{x^2}{4kt}} \triangleq \phi(x,t)$
s.t. $\int_{-\infty}^{\infty} \phi(x,t)dx = 1 \Rightarrow \tilde{C} = \frac{1}{\int_{\mathbb{R}} \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4kt}}dx} = \frac{1}{\sqrt{4k\pi}}$.

(Defn) $\boxed{\Phi(x,t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{x^2}{4kt}}}$, $x \in \mathbb{R}, t > 0$

Fundamental Soln to 1-d heat / diffusion Eqn.

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(Thm) Consider $\begin{cases} \partial_t u = k\partial_{xx} u & x\in\mathbb{R},\ t>0 \\ u(x,0)=\varphi(x). \end{cases}$ ← continuous on $\mathbb{R}$ & bounded $|\varphi(x)|\le C,\ \forall x\in\mathbb{R}$ (IVP for Diffusion eqn)

Invoke F. soln, we get $u(x,t) = \Phi * \varphi = \int_{-\infty}^{\infty}\Phi(x-y,t)\varphi(y)\,dy$
convolution

(proof.) $\partial_t u = \int_{-\infty}^{\infty}\Phi_t(x-y,t)\varphi(y)\,dy = k\int_{-\infty}^{\infty}\Phi_{xx}(x-y,t)\varphi(y)\,dy = k\partial_{xx}u.$

$\lim_{t\to0^+}[u(x,t)-u(x,0)] = \lim_{t\to0^+}\left[\int_{-\infty}^{\infty}\Phi(y,t)\varphi(x-y)\,dy - \int_{-\infty}^{\infty}\Phi(y,t)\varphi(x)\,dy\right]$

(since $\int_{-\infty}^{\infty}\Phi(x,t)dx=1$) $= \lim_{t\to0^+}\int_{-\infty}^{\infty}\Phi(y,t)[\varphi(x-y)-\varphi(x)]\,dy$

$= \lim_{t\to0^+}\int_{-\delta}^{\delta}\Phi(y,t)[\varphi(x-y)-\varphi(x)]\,dy + \lim_{t\to0^+}\left[\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right]\Phi(y,t)[\varphi(x-y)-\varphi(x)]\,dy < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ (since $\le 2C\left(\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}\right)e^{-\frac{x^2}{4kt}}\frac{1}{\sqrt{4k\pi t}}\,dx$) $\to 0$ as $t\to0^+$

continuity of φ

$\left(\frac{4C}{\sqrt{\pi}}\int_{\frac{\delta}{2\sqrt{kt}}}^{\infty}e^{-s^2}ds \to 0 \text{ as } t\to0^+, \text{ for } \frac{\delta}{2\sqrt{kt}}\to\infty\right)$

☆ Note that ☆ $\Phi\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$ — piecewise cts + bdd; singular pts: $\lim_{t\to0^+}\Phi*\varphi = \frac{1}{2}(\varphi(x^+)+\varphi(x^-))$

Comments: (i) sometimes: $\begin{cases}\partial_t\Phi = k\partial_x^2\Phi \\ \Phi(x,0)=\delta\end{cases}$ ← dirac-δ functional; $\int_{-\infty}^{\infty}\delta\,dx=1$, $\delta=\begin{cases}\infty, & x=0\\ 0, & x\neq0\end{cases}$

(ii) F.S. on $\mathbb{R}^n$: ☆ $\Phi(x,t) = \frac{1}{(4k\pi t)^{\frac{n}{2}}}e^{-\frac{\|x\|_2^2}{4kt}}$

(iii) Semig Group: $e^{kt\Delta}\varphi$ (same as $\Phi*\varphi$) operator form

(iv) If $\Phi\ge0$ & $\Phi\not\equiv0$, $u(x,t)=\int_{-\infty}^{\infty}\Phi(x-y)\varphi(y)\,dy>0$

(e.g.1. with singularities $\varphi(x)=\begin{cases}1, & x<0\\ 3, & x>0.\end{cases}$

then with F. Soln $\Rightarrow u(x,t) = \frac{1}{\sqrt{4k\pi t}}\left[\int_{-\infty}^{0}+\int_{0}^{\infty}\right]e^{-\frac{(x-y)^2}{4kt}}\cdot\varphi(y)\,dy$

$= 1+\frac{2}{\sqrt{\pi}}\int_{-\frac{x}{\sqrt{4kt}}}^{\infty}e^{-z^2}dz,\quad t>0.$

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Check: $x>0, t\to0^+$, $x<0, t\to0^+$ singularity: $x=0$, $u(0,t)=2$)

(e.g.2, if $\varphi(x)\notin L^\infty(\mathbb{R})$, e.g. $\varphi=e^{x^3}$. then $\Phi*\varphi$ makes No sense!)

(e.g.3. $\begin{cases}\partial_t u - k\partial_{xx}u + Vu_x = 0\\ u(x,0)=\varphi(x).\end{cases}$ $y = x-vt \Rightarrow v(y,t)=u(x+vt,t)$ → $v_t = kv_{yy}$

techs! $\begin{cases}\partial_t u - k\partial_{xx}u + cu = 0\\ u(x,0)=\varphi(x)\end{cases}$ $v = e^{ct}\cdot u$

(iv) Comparison between Wave & Diffusion Eqns

① Maximum / Minimum Principles (D)

② Singularity (recall d'Alembert) (W) → $\frac{1}{2}(\varphi^+ + \varphi^-)$ transported along

③ Well-posedness ($t>0$ D, W ; $t<0$ W). ← bdd φ

④ Long time behaviors of soln: $t\to\infty$

- Not decay to zero (energy conservation) (W)
- Decay to zero (energy dissipated) (D)

Integrable φ — Not generally true (e.g. $f(x+at)$)

3) Reflections & Sources

(i) Diffusion Eqns on the half-line:

$\begin{cases}\partial_t u = \partial_x^2 u\\ u(x,0)=\varphi(x)\\ u(0,t)=0\end{cases}$ → zero Dirichlet boundary conditions

reflect to another half!

let $\tilde\varphi(x) = \begin{cases}\varphi(x), & x>0\\ 0, & x=0\\ -\varphi(-x), & x<0\end{cases}$ $\Rightarrow \tilde u(x,t) = \begin{cases}u(x,t), & x>0\\ 0, & x=0\\ -u(-x,t), & x<0\end{cases}$ ← aim!

Odd Extension

solve for $\tilde u$ on $\mathbb{R}$; can prove that u is odd

$$\left[\tilde{u}(x,t) = \frac{1}{\sqrt{4\pi k t}}\int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right)\varphi(y)\,dy\right]$$ care about $t>0$

singularity at $t=0$ NOT matter

☆ with Neumann Boundary contition

$$\begin{cases} \partial_t u = \partial_x^2 u & t, x>0 \\ u(x,0) = \varphi(x) \\ \partial_x u(0,t) = 0 \end{cases}$$ ← zero Neumann

even extension!

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x>0 \\ \varphi(-x), & x<0 \end{cases}$$

$$\& \ \tilde{u}(x,t) = \begin{cases} u(x,t), & x>0 \\ u(-x,t), & x<0 \end{cases}$$ ↙ solve for $\tilde{u}$ on $\mathbb{R}$

Even Extension

$$\left[\tilde{u}(x,t) = \frac{1}{\sqrt{4\pi k t}}\int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}\right)\varphi(y)\,dy\right]$$

☆ Inhomogeneous Case. (e.g. $u(0,t)=1$)

change to $v = u-1$ ← homogeneous

(ii) Wave Eqns on the half-line: transform $\tilde{\psi}$ & $\tilde{\phi}$ first, then solve

$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u & x>0 \\ u(x,0) = \phi(x); \ u_t(x,0) = \psi(x) \\ u(0,t) = 0 \end{cases}$$ zero Dirichlet

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & x>0 \\ 0, & x=0 \\ -u(-x,t), & x<0 \end{cases}$$ Odd Extension

similarly → $\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(y)\,dy$

(transform $\tilde{\phi}$, $\tilde{\psi}$ like $\tilde{u}$)

3 cases: $x+ct<0$; $x+ct>0,\ x-ct<0$; both >0

☆ with Neumann B.C. (even extension)

$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u, & x>0 \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x) \\ u_x(0,t) = 0 \end{cases}$$ zero Neumann

Even Extension

☆ Two boundaries! $0<x<\ell$ periodic extension!

(I) $$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u, & 0<x<\ell \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x) \\ u(0,t) = u(\ell,t) = 0 \end{cases}$$ Dirichlet

"odd" + "periodic" extension to $\tilde{u}$

$$\tilde{\phi} = \begin{cases} \phi(x), & 0<x<\ell \\ 0, & x=0 \\ -\phi(-x), & 0>x>-\ell \end{cases} \ \& \ \tilde{\phi}(x+2\ell) = \tilde{\phi}(x)$$ Odd + Periodic Extension } extend to $\mathbb{R}$!

(similar for $\tilde{\psi}$ extension)

(II) $$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u, & 0<x<\ell \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x) \\ u(0,t) = 0 \ \& \ u_x(\ell,t) = 0 \end{cases}$$

Dirichlet, Neumann

odd + symmetric (w.r.t. $x=\ell$) Extension

(iii) Diffusion with Source

$$\begin{cases} \partial_t u - k\partial_x^2 u = f(x,t) & x\in\mathbb{R}, t>0 \\ u(x,0) = \varphi(x) \end{cases}$$ ← source / inhomogeneous term

(IDEA: integrating factor $e^{\cdot}$ in 1st-order ODE)

$$\frac{d}{dt}\left[e^{-kt\Delta}\cdot u\right] = e^{-kt\Delta} f(x,t) \quad (\text{for } \partial_t u - k\Delta u = f(x,t))$$

$$\rightarrow u(x,t) = e^{kt\Delta}\varphi(x) + \int_0^t e^{k(t-s)\Delta} f(s)\,ds$$

Note that $e^{kt\Delta}\varphi = \phi * \varphi = \int_{\mathbb{R}} \phi(x-y,t)\varphi(y)\,dy$ (operator)

$$\& \ u(x,t) = \int_{\mathbb{R}} \phi(x-y,t)\cdot\varphi(y)\,dy + \int_0^t\int_{\mathbb{R}} \phi(x-y,t-s) f(s,y)\,dy\,ds$$

(Thm) [Duhamel's Principle]: Suppose $u=u(t,x;s)$ satisfies

$$\begin{cases}\partial_t u = k\partial_x^2 u \\ u(s,x;s)=f(x,s)\end{cases}$$

(s is a parameter). Then, $\tilde{u}(t,x)=\int_0^t u(t,x;s)\,ds$

solves inhomogeneous eqn $\begin{cases}\partial_t u = k\partial_x^2 u + f(x,t) \\ u(0,x)=0 \leftarrow \text{homo}\end{cases}$

(Proof.) $\partial_t \tilde{u}(t,x) = u(t,x;t) + \int_0^t u_t(t,x;s)\,ds$

$$= f(x,t) + \int_0^t u_t(t,x;s)\,ds$$

$\partial_x^2\tilde{u} = \int_0^t u_{xx}(t,x;s)$. Thus $\partial_t\tilde{u} - k\partial_x^2\tilde{u} = f(x,t)$ since $u_t = k u_{xx}$.

& $\tilde{u}(0,x) = \int_0^0 u(t,x;s) = 0$, obviously.

According to the thm, let $w(x,t) = u(t+s,x;s)$

$\Rightarrow w$ s.t. $\begin{cases}\partial_t w = k\partial_x^2 w \\ w(x,0)=f(x,s)\end{cases} \Rightarrow w = \int_{\mathbb{R}} \phi(x-y,t) f(y,s)\,dy$

& $u(t,x;s) = \int_{\mathbb{R}} \phi(x-y,t-s) f(y,s)\,dy \Rightarrow u_2 = \int_0^t\int_{\mathbb{R}} \phi(x-y,t-s) f(y,s)\,dy\,ds$ ← 2nd term of above $u(x,t)$, satisfying $\begin{cases}\partial_t u_2 - k\partial_x^2 u_2 = f(x,t) \\ u_2(0,x)=0.\end{cases}$

$\therefore u = u_1 + u_2$ (← homo-case; ← inhomo-case (with 0 initial))

☆ Source on half-line:

(I) $\begin{cases}k\partial_x^2 u = \partial_t u - f(x,t) \\ u(0,x)=\varphi(x) \\ u(t,0)=h(t) \text{ (Dirichlet B.C.)}\end{cases}$ $t,x>0$; $v(x,t)=u(x,t)-h(t)$ $\Rightarrow \begin{cases}\partial_t v - k\partial_x^2 v = f(x,t)+h'(t) \\ v(0,x)=\varphi(x)-h(0) \\ v(t,0)=0 \text{ zero-Dirich}\end{cases}$

(II) $\begin{cases}\partial_t u = k\partial_x^2 u - f(x,t) \\ u(0,x)=\varphi(x) \quad t,x>0 \\ u_x(t,0)=h(t) \text{ (Neumann B.C.)}\end{cases}$; $v(x,t)=u(x,t)-xh(t)$ $\Rightarrow \begin{cases}\partial_t v - k\partial_x^2 v = f(x,t)+xh'(t) \\ v(0,x)=\varphi(x)-xh(0) \\ v_x(t,0)=0 \text{ zero-Neum}\end{cases}$

(iv) Wave with Source

$$\begin{cases}\partial_t^2 u - c^2\partial_x^2 u = f(x,t) \leftarrow \text{source} \\ u(0,x)=\phi(x),\ u_t(0,x)=\psi(x).\end{cases}$$

(Thm) [Duhamel's Principle] suppose $u=u(t,x;s)$ solves

$\begin{cases}[\partial_t^2 - c^2\partial_x^2]u = 0 \\ u(s,x;s)=0 \ \&\ u_t(s,x;s)=f(x,s).\end{cases}$ then $\tilde{u}(t,s) = \int_0^t u(t,x;s)\,ds$

solves inhomogeneous eqn $\begin{cases}\partial_t^2 u = c^2\partial_x^2 u + f(x,t) \\ u(0,x)=0 \ \&\ u_t(0,x)=0 \leftarrow \text{homo}\end{cases}$

(Proof.) $\partial_t^2\tilde{u} = \partial_t\tilde{u}(t,x;s) + \int_0^t \partial_t^2\tilde{u}(t,x;s) = f(x,t) + \int_0^t \partial_t^2\tilde{u}(t,x;s)$

& $\partial_x^2\tilde{u} = \int_0^t \partial_x^2\tilde{u}(t,x;s)$. $\tilde{u}(0,x)=0$ & $\tilde{u}_t(0,x) = u(0,t;0) = 0$

Similarly, let $w(x,t)=u(t+s,x;s)$, we get

$$u(x,t) = \frac{1}{2}\left(\phi(x-ct)+\phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

☆ Source on half-line

$\begin{cases}[\partial_t^2 - c^2\partial_x^2]u = f(x,t) \\ u(0,x)=\phi(x),\ u_t(0,x)=\psi(x) \\ u(t,0)=h(t) \text{ (Dirichlet B.C.)}\end{cases}$ ← $v(x,t)=u(x,t)-h(t)$

Uniqueness & Stability: let u_k solve $\begin{cases}[\partial_t^2 - c^2\partial_x^2]u_k = f_k(x,t) \\ u_k(0,x)=\phi_k,\ \partial_t u_k(0,x)=\psi_k\end{cases}$ $k=1,2$

$$u_1-u_2 = \frac{1}{2}[(\phi_1-\phi_2)(x+ct) + (\phi_1-\phi_2)(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}(\psi_1-\psi_2)(y)\,dy + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}[f_1-f_2](y,s)\,dy\,ds$$

$$\Rightarrow \|u_1-u_2\|_\infty \le \left[\|\phi_1-\phi_2\|_\infty + t\|\psi_1-\psi_2\|_\infty + \frac{t^2}{2}\|f_1-f_2\|_\infty\right]$$

→ bold by initial data for finite time

- Boundary Value Problems

1) Seperation of Variables

(e.g. i) $\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u, \quad 0<x<\ell,\ t\in\mathbb{R} \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0)=\phi(x),\ u_t(x,0)=\psi(x) \end{cases}$ Method: $u(t,x) = T(t)\,X(x)$

$\Rightarrow \begin{cases} X''(x) = -\lambda X(x) \\ T''(t) = -\lambda c^2 T(t) \\ X(0)=X(\ell)=0 \end{cases}$ ($\exists$ unknown λ, X, T)

[Defn] Say λ is an [eigenvalue] of $\begin{cases} X''(x) = -\lambda X(x) \\ X(0)=X(\ell)=0 \end{cases}$ (S) ← eigenvalue problem

if $\exists$ a function $X \not\equiv 0$ s.t. (λ, X) is a soln to (S)

In this case, X is called an [eigen-function] & (λ, X) is an [eigen-pair]

Back to e.g. i, $\lambda \geq 0$, for otherwise, $\lambda = -\beta^2$, $X(x) = Ae^{\beta x} + Be^{-\beta x}$, A, B NOT zero simutaneously. Plug in boundary conditions, $(\beta \neq 0)$

$\begin{bmatrix} 1 & 1 \\ e^{\beta\ell} & e^{-\beta\ell} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \vec{0} \Rightarrow$ contradictory!

(Another approach: $\int_0^\ell X''(x)\,X(x)\,dx = \int_0^\ell -\lambda X(x)\cdot X(x)\,dx$

$\Rightarrow \int_0^\ell \underbrace{|X'(x)|^2}_{\geq 0} dx = \lambda \int_0^\ell \underbrace{X^2(x)}_{\geq 0} dx \Rightarrow \lambda \geq 0$ since $X(x) \not\equiv 0$.)

$\therefore X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$, with b.cs, $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$ with $\sin(\sqrt{\lambda}\ell) = 0$ $(b\in\mathbb{R}\ \&\ b\neq 0)$

$\therefore X(x) = b\cdot\sin(\frac{n\pi}{\ell}\cdot x)$, with $\lambda_n = (\frac{n\pi}{\ell})^2$, $n\in\mathbb{Z}\setminus\{0\}$ all ei-values

For T: $T_n''(t) = -\lambda_n c^2 T_n(t) \Rightarrow T_n(t) = \tilde{A}\cos(\frac{cn\pi}{\ell}t) + \tilde{B}\sin(\frac{cn\pi}{\ell}t)$

Thus, $u_n(t,x) = [\hat{A}\cos(\frac{cn\pi}{\ell}\cdot t) + \tilde{B}\sin(\frac{cn\pi}{\ell}\cdot t)]\cdot b\sin(\frac{n\pi}{\ell}x)$

(w.l.o.g., $b=1$ is fine)

solves $\begin{cases} \partial_t^2 u = c^2\partial_x^2 u \\ u(0,t) = u(\ell,t) = 0 \end{cases}$, $n = 1, 2, \cdots$

$\left[\phi(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{\ell}x);\ \psi(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{\ell}x)\right]$ Fourier Series!

(odd extension to $x\in(-\ell,\ell)$.)

For Most cases, Fourier series converges.

Remark: for Neumann B.C., it becomes $X'(0) = X'(\ell) = 0$

still $\lambda \geq 0$, $\lambda_n = (\frac{n\pi}{\ell})^2$ & $X_n(x) = \cos(\frac{n\pi}{\ell}\cdot x)$, thus $n\in\mathbb{Z}$.

Mixed (e.g. $u(0,t)=0$, $u_x(\ell,t)=0$) or Robin B.C. ($u_x + au =$

↪ similar approach: $X_n = \cos(\frac{(2n+1)\pi}{2\ell}\cdot x)$) or Periodic B.C. ($u(0,t)$ $= u(\ell,t)$ & $u_x(0,t) = u_x(\ell,t)$) → $\lambda \geq 0$

☆ Consider "Robin" B.C.: $\begin{cases} u_x(0,t) + a_0 u(0,t) = 0 \\ u_x(\ell,t) + a_\ell u(\ell,t) = 0 \end{cases}$ with $a_0, a_\ell \in\mathbb{R}$

$\begin{cases} X''(x) = -\lambda^2 X(x) \\ X'(0) + a_0 X(0) = 0 \\ X'(\ell) + a_\ell X(\ell) = 0 \end{cases}$ similarly $\int_0^\ell X''(x)X(x)\,dx = \int_0^\ell -\lambda X^2(x)\,dx$

$\Rightarrow -a_0 X^2(0) + a_\ell X^2(\ell) + \int_0^\ell |X'(x)|^2 dx = \lambda\int_0^\lambda X^2(x)$

The sign of LHS depends on a_0 & a_ℓ.

① pos. ei-values: $X(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

$\Rightarrow \begin{cases} A\sqrt{\lambda} + a_0 B = 0 \\ (A\sqrt{\lambda} + a_\ell B)\cos(\sqrt{\lambda}\ell) + (Aa_\ell - B\sqrt{\lambda})\sin(\sqrt{\lambda}\ell) = 0 \end{cases}$ Try to check

the sigularity of $\begin{bmatrix} \sqrt{\lambda} & a_0 \\ \sqrt{\lambda}\cos\sqrt{\lambda}\ell + a_\ell\sin\sqrt{\lambda}\ell & a_\ell\cos\sqrt{\lambda}\ell - \sqrt{\lambda}\sin\sqrt{\lambda}\ell \end{bmatrix}$

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∃ a non-trivial soln if $(a_\ell - a_0)\sqrt{\lambda}\cos\sqrt{\lambda}\ell - \sin\sqrt{\lambda}\ell \cdot (\lambda + a_0 a_\ell) = 0$

$\Leftrightarrow \tan\sqrt{\lambda}\ell = \frac{(a_\ell - a_0)\sqrt{\lambda}}{\lambda + a_0 a_\ell}$ & find such λ (or $\sqrt{\lambda}$)

In general — hard to solve: can use curve to find existence.

☆ $\lambda_n \overset{n\to\infty}{\sim} (\frac{\pi}{\ell}\cdot n)^2$ (long term behavior)

② 0-eivalues: $\begin{cases} X''(x) = 0 \\ X'(0) + a_0 X(0) = 0 \\ X'(\ell) + a_\ell X(\ell) = 0 \end{cases} \Rightarrow X(x) = b(1 - a_0 x)$, with

($a_\ell = \frac{a_0}{1 - \ell a_0}$, otherwise no soln)

③ neg.-ei-values: $X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x)$ "hyperbolic functions"

$\begin{cases} B\sqrt{-\lambda} + a_0 A = 0 \\ A(\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\ell) + a_\ell\cosh(\sqrt{-\lambda}\ell)) + B(\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\ell) + a_\ell\sinh(\sqrt{-\lambda}\ell)) = 0 \end{cases}$

similarly, ∃ a non-trivial soln if $\frac{(a_\ell - a_0)\sqrt{-\lambda}}{\lambda + a_0 a_\ell} = \tanh(\sqrt{-\lambda}\ell)$

has such $\sqrt{-\lambda} > 0$ soln. (No stability!) ↳ check diff. cases with

($a_\ell - a_0 > 0$ or < 0) & ($a_0 a_\ell > 0$ or < 0)

• Fourier Series

1) Fourier series & its orthogonal basis on Hilbert space (see in MAT 2007 Notes)

Note: $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t}\sin(\frac{n\pi}{\ell}x)$ solves (H-IBVP) Dirichlet

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t}\cos(\frac{n\pi}{\ell}x) + \frac{A_0}{2}$ solves (H-IBVP) Neumann

2) Fourier series for solving PDEs:

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(e.g.) $\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = x, & u_t(x,0) = 0 \end{cases}$. $u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(\frac{n\pi}{\ell}t) + B_n\sin(\frac{n\pi}{\ell}t)]\cdot\sin(\frac{n\pi}{\ell}x)$

with $u(x,0) = x$, $u_t(x,0) = 0$,

we get $A_n = \frac{2}{n\pi}[1 - (-1)^n]$ & $B_n = 0$.

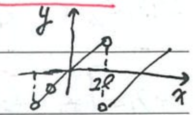
2) Fourier series with even, odd & periodic functions.

(i) period: f has period T if $f(x+T) = f(x)$, $\forall x \in \mathbb{R}$

Then, directly $\int_a^{a+T} f\,dx = \int_0^T f\,dx$.

(ii) periodic extension: $\tilde{\Phi}(x) = \tilde{\Phi}(x + 2\ell)$, $\forall x \in \mathbb{R}$ for $\phi(x)$, $-2\ell < x < 2\ell$

Note that $\tilde{\Phi}(2n\ell)$ may NOT exist & $\tilde{\Phi}$ may NOT be smooth. ← in most cases, NOT



Every function on $\mathbb{R}$ can be written as

$f(x) = \underbrace{\frac{1}{2}[f(x) + f(-x)]}_{\text{even}} + \underbrace{\frac{1}{2}[f(x) - f(-x)]}_{\text{odd}}$

(iii) Complex Form (Fourier series)

On $(-\ell, \ell)$, the fourier series have the form $\left[\sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{\ell}}\right]$

(Note: $C_n = \frac{A_{|n|}}{2} + \mathrm{sgn}(n)\frac{B_{|n|}}{2i}$, $n \neq 0$ & $C_0 = \frac{A_0}{2}$) → $|C_n|^2 = \frac{1}{4}(A_n^2 + B_n^2)$.

☆ property: $\int_{-\ell}^{\ell} e^{i\frac{n\pi x}{\ell}} \cdot e^{-i\frac{m\pi x}{\ell}}\,dx = \begin{cases} 0, & \forall m \neq n,\ m,n \in \mathbb{Z} \\ 2\ell, & m = n \end{cases}$ ← Othogonal basis (with $\frac{1}{\sqrt{2\ell}}$ coefficient)

$\therefore C_n = \frac{1}{2\ell}\int_{-\ell}^{\ell}\varphi(x)e^{-i\frac{n\pi x}{\ell}}\,dx$

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(3) Orthogonality & General Fourier series

consider $\begin{cases} X'' = -\lambda X \\ \text{B.C.s} \end{cases}$ (ei-value prob.)

B.C.s ← (Dirichlet, Neumann, Robin, Period, ... any)

(i) [Defn] L^2-inner product: Let f, g be L^2-measurable (real-valued continuous) functions, on (a,b). Define L^2-inner product as

$\langle f, g\rangle_{L^2} = \int_a^b f(x)g(x)\,dx$ (L^2-norm: $\|f\|_{L^2} := \langle f, f\rangle_{L^2}^{1/2}$)

linear trans.

(ii) What kind of B.C.s gives orthogonality of $\{X_n\}$?

Denote $A = -\frac{d^2}{dx^2}$ (operator). $\langle A X_n, X_m\rangle_{L^2} = \langle \lambda_n X_n, X_m\rangle_{L^2}$

$LHS = -X_n' X_m \Big]_a^b + \int_a^b X_n' X_m'\,dx$. Similarly $\langle A X_m, X_n\rangle_{L^2} = \langle \lambda_m X_m, X_n\rangle_{L^2}$

$LHS_{(of\ 2)} = -X_m' X_n\Big]_a^b + \int_a^b X_m' X_n'\,dx$. Thus $\Big[\underbrace{(X_m' X_n - X_n' X_m)\Big]_a^b}_{\text{Boundary Term}} = (\lambda_n - \lambda_m)\langle X_m, X_n\rangle_{L^2}\Big]$ (*)

Zero Dirichlet, Neumann, Periodic, Robin conditions ⇒

$LHS_{(of\ (*))} = 0$ ⇒ orthogonality of X_m & X_n.

(iii) [Defn] Symmetric Boundary condition: $\forall f, g$ satisfying a s.B.C. s.t. $f'g - fg'\Big]_a^b = 0$ (gives X_m, X_n orthogonal to each other)

(Thm) If the B.C.s are symmetric, any 2 eigenfunctions with different eigenvalues are orthogonal.

(Note: with repetitive ei-values, $\{X_1, \dots\}$ can work as a basis of eigenspace corresponding to some λ. By Gram-Schmidt ⇒ find orthogonal basis)

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(iv) Complex-valued Functions:

[Defn] Inner product, $\langle f, g\rangle_{L^2} = \int_a^b f\bar{g}\,dx$. (generalization)

Then, symmetric B.C.: $(f'\bar{g} - \bar{f}g')\Big]_a^b = 0$ → gives real number λ_n

(By taking conjugate, $\underbrace{(\overline{X_n} X_n' - \overline{X_n}' X_n)\Big]_a^b}_{0} = (\bar{\lambda}_n - \lambda_n)\int_a^b \underbrace{\|X_n\|_{L^2}^2\,dx}_{\neq 0}$.)

∴ We can choose real-valued eigen-functions!

(If $X(x) = Y(x) + iZ(x)$ is ..., then $Y(x), Z(x)$ are both ei-functs w.r.t. $\lambda \in \mathbb{R}$) real

Then, we can define "Fourier series" by

$f(x) \sim \sum_{n=1}^{\infty} A_n X_n$, where $A_n = \frac{\langle f, X_n\rangle_{L^2}}{\|X_n\|_{L^2}^2}$

(4) Convergence of Fourier Series:

(i) $\begin{cases} \text{Uniform convergence: } \lim_{n\to\infty} \max_{x\in X} |f_n(x) - f(x)| = 0 \\ L^2\text{-convergence: } \lim_{n\to\infty} \|f_n - f\|_{L^2} = 0 \quad (\text{i.e. } (\int_X |f_n - f|^2 dx)^{\frac{1}{2}} \to 0) \end{cases}$

(Note: uniform convergence ⇒ L^2-convergence;

L^2-convergence ⇏ pointwise-convergence)

⇏ pointwise-convergence $\boxed{a.e.}$ (see in MAT3006)

(ii) [Thm 1] (Uniform Convergence). For general "Fourier series" on (a,b) that is $f(x) \sim \sum_{n=1}^{\infty} A_n X_n$, then $\boxed{S_N(f) = \sum_{n=1}^{N} A_n X_n}$ ← partial sum converges uniformly to f, if $\begin{cases} f \in C^2[a,b] & \text{No need for } f'' \text{ for classical Fourier!} \\ f \text{ satisfies the given B.C.} \end{cases}$

[Thm 2] (L^2 Convergence) If $f \in L^2$, then $S_N(f) \xrightarrow{N\to\infty} f$ in L^2/mean-square sense.

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[Thm 3] (Pointwise Convergence) $S_N(f) \xrightarrow{N\to\infty} f$ pointwisely on (a,b), ← Classical Fourier! (cos, sin/full)
provided that f is $\boxed{\text{a continuous function}}$ on $[a,b]$ & simutaneously, let f be $\boxed{\text{piecewisely differentiable}}$ on $[a,b]$ (i.e., f' continuous piecewisely on $[a,b]$.) Then, $S_N(f) \to \frac{1}{2}[f(x^+)+f(x^-)]$, $\forall x \in (a,b)$ ← if f is piecewisely cts

Do extension f_{ext}, $S_N(f) \to \frac{1}{2}[f_{ext}(x^+)+f_{ext}(x^-)]$, $\forall x\in\mathbb{R}$.

(Comments) $\sum_{n=1}^{\infty} A_n X_n(x)$ (Fourier series) always satisfies the B.C.
(For) sine series: Dirichlet B.C. (0) ; Full seriers: Periodic
cosine series: Neumann B.C. (0)
All thms are only "tests" (i.e., sufficient conditions).

(iii) Further properties: in complex sense
(Least square approximation) $\|f-\sum_{m=1}^{n} A_m X_m\|_{L^2} \le \|f - \sum_{m=1}^{n} C_m X_m\|_{L^2}$
$\forall$ orthogonal basis $\{X_1,\dots,X_n\}$, $C_m\in\mathbb{C}$ & $A_m = \frac{\langle f, X_m\rangle_{L^2}}{\|X_m\|^2_{L^2}}$
(proof.) RHS $= \|f\|^2_{L^2} + \sum_{m=1}^{n}\left[|C_m|^2\|X_m\|^2_{L^2} - C_m\overline{\langle f, X_m\rangle}_{L^2} - \overline{C_m}\langle f, X_m\rangle_{L^2}\right]$
a polynomial w.r.t. C_m. The 2nd-term $= \sum_{m=1}^{n}\left(\|X_m\|^2_{L^2}|C_m - A_m|^2 - \frac{|\langle f, X_m\rangle_{L^2}|^2}{\|X_m\|^2_{L^2}}\right)$ when $C_m = A_m$, RHS for sure attains its L^2 minimum.

(Coro.) [Bessel's Inequality] $\|S_N(f)\|^2_{L^2} = \sum_{n=1}^{N} |A_n|^2\|X_n\|^2_{L^2} \le \|f\|^2_{L^2}$
[Parseval's Identity] when $S_N(f)\to f$ in L^2 sense, $\|S_N(f)\|^2_{L^2} \to \|f\|^2_{L^2}$, that is $\|f\|^2_{L^2} = \sum_{n=1}^{\infty} |A_n| \|X_n\|^2_{L^2}$

(iv) Proofs of Convergence Results:

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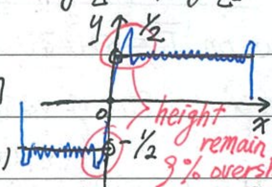
$A_n\cos nx + B_n \cos ny = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos n(y-x)\,dy$. Thus, $S_N(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \boxed{K_N(y-x)}\,dy$. Thus, since $K_N(y-x) = 1 + 2\sum_{n=1}^{N} \cos n(y-x)$
Dirichlet kernel $= 1 + \frac{1}{\sin\frac{y-x}{2}}\left[\sin\left(\frac{2N+1}{2}(y-x)\right) - \sin\left[\frac{1}{2}(y-x)\right]\right]$
(use $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$) $= \frac{\sin(2N+1)\theta}{\sin\theta}$, where $\theta = \frac{y-x}{2}$.

$S_N(f) - f = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(N+\frac{1}{2})y}{\sin(\frac{1}{2}y)}(f(x-y)-f(x))\,dy = -\langle \phi_x, X_N\rangle$, with $X_n = \sin(n+\frac{1}{2})y$
$N\to\infty$, $\to 0$ (aim)
& $\phi_x \triangleq \frac{f(x)-f(x-y)}{2\pi(\sin(\frac{1}{2}y))}$ which is a function of y, given x.
With Bessel's inequality: $\sum_{n=1}^{\infty} \frac{|\langle\phi_x, X_n\rangle|^2_{L^2}}{\|X_n\|^2_{L^2}} \le \|\phi_x\|^2_{L^2} < \infty$ → 0
Since $\lim_{y\to 0^+}\phi_x(y) \overset{\text{L'Hospital}}{=} \frac{2f'(x)}{1} < \infty$ ∴ $|\phi_x(y)|<\infty$ on $(-\pi,\pi)$ ⇒ $\|\phi_x\|^2_{L^2}<\infty$.
Note removable/jump discontinuity case, may NOT exist but still finite. (f cts, piecewisely-diff. & finite f'), ∀ fixed x ⇒ pointwise-convergence.

Suppose $f\in C^2[a,b]$ (i.e., $C^2[-\pi,\pi]$), then $\begin{cases} A_n = -\frac{1}{n} B_n' \\ B_n = \frac{1}{n} A_n'\end{cases}$
$|A_n\cos nx + B_n\cos ny| \le \frac{1}{n}|B_n'| + \frac{1}{n}|A_n'| \underset{C-S}{\le} \frac{1}{n^2} + \frac{1}{2}(|A_n'|+|B_n'|) \xrightarrow{n\to\infty} 0$
Since Bessel's inequality ⇒ $\sum_{n=1}^{\infty}(|A_n'|^2+|B_n'|^2) \le \|f'\|^2_{L^2}\pi < \infty$
(+ pointwise result) ⇒ uniformly convergence (to f)
Recall approximation by cts functions $\|f-g\| < \varepsilon/3$, for some $g\in C_0^\infty$
$\|S_N f - f\|_{L^2} \le \|S_N f - S_N g\|_{L^2} + \|g - S_N g\|_{L^2} + \|f-g\|_{L^2} < \varepsilon$ ⇒ L^2 convergence.

(v) Gibbs Phenomenon:
let $f(x) = \begin{cases} 1/2, & x\in(0,\pi] \\ 0, & x=0 \\ -1/2, & x\in[-\pi,0)\end{cases}$



Then $S_N(f) \not\to f(x)$, since $S_N(f) = \frac{1}{4\pi}\left[\int_{-Mx}^{Mx} - \int_{Mx-M\pi}^{Mx-M\pi}\right]\frac{\sin\theta}{M\sin\frac{\theta}{2M}}\,d\theta$
∴ $S_N(f) - f = \frac{1}{4\pi}\int_{Mx+M\pi}^{Mx}\frac{\sin\theta}{M\sin\frac{\theta}{2M}}\,d\theta$. chose $x_N = \frac{\pi}{M}$, $\not\to 0$, NOT uniformly → f

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(5) Inhomogeneous B.C.s:

• Harmonic Functions

(1) Laplacian Equation: $\nabla\cdot\nabla u=\Delta u=0$

[when a diffusion or wave is stationary (indep. of time), they reduce to Laplacian equations.]

(i) Origins: $\boxed{\text{Steady fluid flow}}$: if $\nabla\times\vec{v}=\vec{0}$, assume $\vec{v}=\vec{v}(x,y,z)$ indep. of time, the fluid is incompressible & no sources and sinks.

$\therefore \nabla\cdot\vec{v}=0$, hence, $\vec{v}=-\nabla\phi$ ($\Delta\phi=-\nabla\cdot\vec{v}=0$) → simply-connected! → velocity potential

☆ From Maxwell's equations $\Rightarrow \begin{cases}\nabla\times\vec{E}=\vec{0}\\ \nabla\cdot\vec{E}=4\pi\rho.\end{cases}$ inhomogeneous laplacian

$\therefore \nabla\times\vec{E}=\vec{0}\Rightarrow\vec{E}=-\nabla\phi$ → Poisson's eqn (D-BVP)

Thus, $\nabla\cdot\vec{E}=\boxed{-\Delta\phi=4\pi\rho}$.

Another application is Brownian Motion $\begin{cases}\Delta u=0,\text{ in } D\\ u=1 \text{ on } C_1 \ \& \ u=0 \text{ on } C_2\end{cases}$

with $u(x,y,z)$ — probability that starts from (x,y,z), stops hitting C_1

(ii) Properties:

☆ Maximum Principle (weak-version) $u\in C^2(D)\cap C(\bar{D})$

Let D be a connected bounded domain (open). Let u s.t. $\Delta u=0$ in D & is continuous on $\bar{D}=D\cup\partial D$. Then, $\max\limits_{D} u=\max\limits_{\partial D} u$ & $\min\limits_{\partial D} u=\min\limits_{D} u$. (Strong version — neither inside unless $u\equiv$ const)

(proof.) Let $v=u+\varepsilon e^{x_1}$ $(\varepsilon>0)$, $\Delta v=\varepsilon e^{x_1}>0$ Not negative semi-def.

Since $\bar{D}$ is bdd, $\max\limits_{x\in\bar{D}} w=\max\limits_{\partial D} w \Rightarrow \begin{cases}\max\limits_{D} v\le\max\limits_{\partial D} v+\varepsilon M\\ \max\limits_{\partial D} v\le\max\limits_{D} v+\varepsilon M\end{cases}$ $(\varepsilon\to 0)$ ✓

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☆ Invariance under all rigid motions (No preference direction)

Let $\vec{x}'=B\vec{x}$, $(B^TB=I)$ then $\Delta_{\vec{x}'}u=\Delta_{\vec{x}}u=0$. (Also translation invariant)

In 2-D. by polar coordinates $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ it is "rotation invariant" (can be seen as rotation matrix) for some special x', y'

$\partial_x^2+\partial_y^2=\Delta=\partial_{x'}^2+\partial_{y'}^2=$ ☆ $\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$

(proof.) $\partial_{x_i}u=\sum\limits_j\partial_{x_j'}u\,\frac{\partial x_j'}{\partial x_i}$ (b_{ji}). $\Rightarrow \nabla_x u=B^T\nabla_{x'}u$. (Suppose $B^TB=I$)

Thus $\nabla_x\cdot\nabla_x u$ (Δu) $=\sum\limits_i\sum\limits_j b_{ji}\,\partial_{x_i}\partial_{x_j'}u=\sum\limits_i\sum\limits_j b_{ji}\sum\limits_k b_{ki}\,\partial_{x_k'}\partial_{x_j'}u$

$=\sum\limits_{j,k}\left(\sum\limits_i b_{ji}b_{ki}\right)u_{x_j'x_k'}=\sum\limits_{j,k}(BB^T)_{jk}u_{x_j'x_k'}=\sum\limits_i u_{x_i'x_i'}=\Delta_{x'}u$. □

change sum to get matrix form

In 2D $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ $[\Delta_{2D}$ $]$

In 3D, Let $\rho=r\sin\theta\Rightarrow\begin{cases}x=\rho\cos\varphi\\ y=\rho\sin\varphi\end{cases}$ as an intermediant:

$\Rightarrow\left[\Delta_{3D}=\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right.$

$\left.=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left[\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right]$

(2) Series Soln. on rectangles & cubes

(i) $\begin{cases}u_{xx}+u_{yy}=0\\ \text{B.C.s (e.g. symmetric)}\end{cases}$ on $\Omega=(0,a)\times(0,b)$ Let $u=X(x)Y(y)$

$\Rightarrow \left|\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda\right|$ (eigen-value problems)

Suppose $u(0,y)=u(a,y)=0$ as B.C. along x-direction

$\Rightarrow u(x,t)=\sum\limits_{n=0}^{\infty}\left(A_n\cosh\left(\frac{n\pi}{a}\right)y+B_n\sinh\left(\frac{n\pi}{a}y\right)\right)\sin\left(\frac{n\pi}{a}\right)x$

$\{A_n\}_{n=0}^{\infty}$ & $\{B_n\}_{n=0}^{\infty}$ are determined by B.C. along y-direction (e.g. $\begin{cases}u(x,0)=\phi(x)\\ u(x,h)=\psi(x)\end{cases}$) Fourier

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(ii) On a 3-d cube $\begin{cases} u_{xx}+u_{yy}+u_{zz}=0 \\ \text{B.C.s (on } x,y,z \text{ directions)} \end{cases}$ on $(0,a)\times(0,b)\times(0,c)$

$u(x,y,z)=X(x)Y(y)Z(z) \Rightarrow \frac{X''(x)}{X} = -\left(\frac{Y''}{Y}+\frac{Z''}{Z}\right) = -\lambda$

(eigen-value problem on 3-D)

Firstly, sym B.C. along x-direction: $X_n(x)=\sin\left(\frac{n\pi}{a}x\right)$ (or $\cos\left(\frac{n\pi}{a}x\right)$ Neumann)

$\frac{Y''}{Y} = -\frac{Z''}{Z}+\lambda \triangleq -\mu$ (if sym B.C. along y-direction: $Y_m=\sin\left(\frac{m\pi}{b}y\right)$)

$+\frac{Z''}{Z} = -\lambda-\mu$ ($\left[\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}\right]$ for the n,m^{th} value)

$\Rightarrow \left[u=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[A_{mn}\cosh\frac{\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}}{c}z + B_{mn}\sinh\frac{\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}}{c}z\right]\cdot\sin\left(\frac{m\pi}{b}y\right)\right.$

$\left.\cdot\cos\left(\frac{n\pi}{a}x\right)\right]$ Along z direction, the basis $\{\cos(\frac{n\pi}{a}x), \sin(\frac{m\pi}{b}y)\}$ orthogonal under $L^2((0,\pi)^2)$. $\Rightarrow A_{mn}, B_{mn}$.

(3) Disk Domain; Poisson's Formula

(i) $\begin{cases}\Delta u=0 \\ u|_{\partial\Omega}=h(x,y)\end{cases}$ $\Omega:=\{x^2+y^2<a^2\}$.

with coordinate rotation (polar) $\begin{cases}\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r+u_{rr}=0 \\ u|_{\partial\Omega}=h(x,y)\triangleq g(\theta)\ (u(a,\theta))\end{cases}$

Seperation of variables: $-\frac{r^2R''+rR'}{R}=\frac{\Theta''}{\Theta}=-\lambda$

Note: B.C.s for θ: $u(r,\theta)=u(r,\theta+2\pi),\ \forall\theta\in\mathbb{R}$

(initial data) $u(a,\theta)=g(\theta)$

$\Theta_n=A_n\cos n\theta+B_n\sin n\theta$, Euler's eqn for $R_n(r)=C_nr^n+D_nr^{-n}$ ← Sigularity at o

& $R_0=C_0\ln r+D_0$. Singularity at $r=0$ ← drop

series soln: $u(r,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}[A_n\cos n\theta+B_n\sin n\theta]r^n$

Together with initial data, $\begin{cases}A_n=\frac{1}{a^n\pi}\int_0^{2\pi}g(\theta)\cos n\theta\,d\theta \\ B_n=\frac{1}{a^n\pi}\int_0^{2\pi}g(\theta)\sin n\theta\,d\theta\end{cases}$

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$\Rightarrow u(r,\theta)=\frac{1}{2a^0\pi}\int_0^{2\pi}g(\theta)d\theta+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\frac{1}{\pi}\int_0^{2\pi}g(\varphi)\cos(n(\varphi-\theta))d\varphi$

$\overset{\text{D.C.T}}{\underset{\text{(coro)}}{=}}\ \frac{1}{2\pi}\int_0^{2\pi}g(\theta)d\theta+\frac{1}{\pi}\int_0^{2\pi}g(\varphi)\boxed{\sum_{n=1}^{\infty}\cos(n(\varphi-\theta))\cdot\left(\frac{r}{a}\right)^n}d\varphi$

$\operatorname{Re}(z^n)$ if $z=\frac{r}{a}e^{i(\varphi-\theta)}$, $|z|<1$

$=\frac{1}{2\pi}\int_0^{2\pi}g(\theta)d\theta+\frac{1}{\pi}\int_0^{2\pi}g(\varphi)\operatorname{Re}\left(\frac{z}{1-z}\right)d\varphi=\frac{1}{\pi}\int_0^{2\pi}g(\varphi)\left[\frac{1}{2}+\operatorname{Re}\left(\frac{z}{1-z}\right)\right]d\varphi$

$\Rightarrow \boxed{u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}g(\varphi)P_a(\theta-\varphi,r)d\varphi}$ $\quad \frac{1}{2}+\frac{\operatorname{Re}(z)-|z|^2}{|1-z|^2}=\boxed{\frac{1}{2}\frac{a^2-r^2}{a^2-2ar\cos(\varphi-\theta)+r^2}}$

with Poisson's kernel $P_a(\theta,r)=\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}$ Poisson's kernel

☆ Higher-Dim Form: $u(\vec{x})=\frac{a^2-\|\vec{x}\|^2}{2\pi a}\oint_{\partial D_a}\frac{u(\vec{x}')}{\|\vec{x}-\vec{x}'\|^2}ds$ ← B.C. $\quad$ arc length $ds=a\,d\varphi$

(compared with Cauchy Integration formula)

(ii) Properties of the solution:

① $\Delta_{(\theta,r)}P_a(\theta,r)=0$ (since $\left(\frac{r}{a}\right)^n\cos(n\varphi-n\theta)$ is harmonic) ← term-by-term differentiation + M-test

② $\frac{1}{2\pi}\int_0^{2\pi}P_a(\theta,r)d\theta=1$. (use C.I.F)

[Thm] If $g(\theta)=u(x,y)$ on ∂D_a is continuous, Poisson's formula gives the only $C^2(D_a)\cap C(\overline{D_a})$ harmonic function s.t. $\lim_{\vec{x}\to\vec{x}_0}u(\vec{x})=u(\vec{x}_0)$ $\forall\vec{x}_0\in\partial D$

(proof.) $C_\delta=B_\delta(\vec{x}_0)\cap\partial D_a$, $D_a=C_\delta\cup C_\delta^c$ (Define: $C_\delta^c=\partial D_a\setminus C_\delta$)

$u(\vec{x})-u(\vec{x}_0)=\frac{a^2-\|\vec{x}\|^2}{2\pi a}\oint_{\partial D_a}\frac{u(\vec{x}')-u(\vec{x}_0)}{\|\vec{x}-\vec{x}'\|^2}ds=\underbrace{\frac{a^2-\|\vec{x}\|^2}{2\pi a}\int_{C_\delta}\frac{u(\vec{x}')-u(\vec{x}_0)}{\|\vec{x}-\vec{x}'\|^2}ds}_{<\varepsilon/2 \text{ continuity!}}$

$+\underbrace{\frac{a^2-\|\vec{x}\|^2}{2\pi a}\int_{C_\delta^c}\frac{u(\vec{x}')-u(\vec{x}_0)}{\|\vec{x}-\vec{x}'\|^2}ds}_{<\varepsilon/2 \text{ since } \vec{x}\to\vec{x}_0,\ a^2-\|\vec{x}\|^2 \text{ small enough.}}$. $<\varepsilon$

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(Mean-value property) If u is a harmonic function in D_a & $u \in C(\bar{D}_a)$, then $u(0) = \frac{1}{2\pi a}\oint_{\partial D_a} u\,ds = \fint_{\partial D_a} u\,ds$

~~(proof.)~~ $u(0) = \frac{a^2}{2\pi a}\oint_{\partial D_a} \frac{u(\vec{x}')}{\|\vec{x}'\|^2} ds = \frac{1}{2\pi a}\oint_{\partial D_a} u(\vec{x}')ds$

[Coro.] $u(x) = \frac{1}{2\pi a}\oint_{\partial D_a(x)} u\,ds$ (with assumption changing to $D_a(x)$)

(proof.) Let $\tilde{u}(x) = u(x-a)$

(2^{nd} form) $u(x) = \frac{1}{\pi a^2}\int_{D_a(x)} u\,dV$ (with above assumptions)

~~(proof.)~~ $\int_{D_a(x)} u\,dV = \int_0^a \int_{\partial D_r(x)} u\,ds\,dr = \pi a^2 u(x)$

(Strong maximum principle) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be harmonic in a bounded connected domain Ω. Then the maximum value only holds on the boundary unless it's a const.

(proof.) Suppose $x_0 \in int(\Omega)$, $u(x_0) = M$. $\exists r_0$ s.t. $B(x_0, r_0) \subset \Omega$.
← centre $M \Rightarrow$ all M
apply MVP $\Rightarrow u = u(x_0) = \fint_{B_1(x_0,r_0)} u\,dx \le M$, "=" holds $\Rightarrow u \equiv M$ on $B(x_0, r_0)$

$\forall x \in \Omega$, connected $\Rightarrow$ path-connected $\exists \ell$ cts connecting x, x_0 &
← r or even $2r$ H.B.
ℓ is closed, bdd $\{B(x_i, r)\}$ is an open cover $\Rightarrow \exists$ finite subcover $\{B(x_i, r_i)\}_{i=0}^n$

$\Rightarrow u(x) = M$, since we can let $x_{k+1} \in B(x_k, r_k)$. if $r = \min_{x \in \ell, y \in \partial\Omega} \frac{\|x-y\|}{2}$, $\{B(x_i, r)\}$ is a subcover
← by defn. can enlarge
thus $x_{k+1} \in B(x_k, 2r)$. ▯

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• Green's Identity:

(1) Green's 1^{st} identity: $\iint_{\partial D} v\frac{\partial u}{\partial n}ds = \iiint_D \nabla v \cdot \nabla u\,dV + \iiint_D v\Delta u\,dV$

with $D \subset \mathbb{R}^3$ is bdd, closed, ∂D smooth
$u \in C^2(\Omega)$ & $v \in C^1(\Omega)$

(proof. apply divergence theorem on "$v \cdot \nabla u$".)

(e.g.1 $\begin{cases}\Delta u = f(\vec{x}) \\ \partial u/\partial n = h(x)\end{cases}$ Green's identity (1^{st}) $\iint_{\partial D} \frac{\partial u}{\partial n}ds = \iiint_D \Delta u\,dV$

$\Rightarrow$ exist only when $\iint_{\partial D} h\,ds = \iiint_D f\,dV$)

(e.g.2 MVP: "$\iint_{\partial B_1} \overset{\vec{n}\text{ (since }|\vec{z}|=1)}{\vec{z}} \cdot \nabla u(\vec{z}r)\,dS_1 = \iiint_{B_1} \Delta u(\vec{z}r)\,dS_1 = 0$"

define $f(r) = \frac{1}{Area(\partial B_r)}\int_{\partial B_r} u\,dS_r$, when $Area(\partial B_r) = r^{n-1}\alpha_n$ ← area of unit ball in $\mathbb{R}^n$

$= \frac{1}{\alpha_n}\int_{\partial B_1} u(\vec{z}r)dS_1 \Rightarrow f'(r) = \frac{1}{\alpha_n}\iint_{\partial B_1} \vec{z} \cdot \nabla u(\vec{z}r)\,dS_1 = 0$ constant f

$\Rightarrow f(r) = f(0)$ by taking $\lim_{r\to 0^+} f(r)$. $\lim_{r\to 0^+} f(r) = u(0)$ by definition

(use $\lim_{r\to 0^+}|f(r) - u(0)| = 0$.))

(e.g.3 Uniqueness of Poisson eqn $\begin{cases}\Delta u = f & \text{on } \Omega \\ u = \phi & \text{on } \partial\Omega\end{cases}$ (3-dim, & same for n-dim)

$u_1, u_2 \Rightarrow v = u_1 - u_2$ s.t. $\begin{cases}\Delta v = 0 & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega\end{cases}$ Green's 1^{st} identity.

$0 = \iiint_D \|\nabla v\|^2 dV \Rightarrow v = 0$ since ($v = 0$ on $\partial\Omega$).)

(ii) Dirichlet Principle: $E(w) = \frac{1}{2}\int_\Omega \|\nabla w\|^2 d\vec{x}$ (Dirichlet energy!)

On admissible set $\mathcal{A} = \{w: \Omega \mapsto \mathbb{R} \mid E(w) < \infty, w|_{\partial\Omega} = h\}$

(DP) The lowest energy $E(w)$ is attained by the unique harmonic function in $\mathcal{A}$

(proof) Let u be s.t. $\Delta u = 0$, $v = u - w$. then $v|_{\partial\Omega} = 0$. $w = u - v$

$E(w) = \frac{1}{2}\int_\Omega \|\nabla u - \nabla v\|^2 d\vec{x} = \frac{1}{2}\int_\Omega \|\nabla u\|^2 + \|\nabla v\|^2 - 2\nabla u \cdot \nabla v \, d\vec{x}$

$\overset{1^{st}GI}{=} E(u) + \int_\Omega \frac{1}{2}\|\nabla v\|^2 d\vec{x} \quad (\int_\Omega \nabla u \cdot \nabla v \, d\vec{x} = 0) \geq E(u).$

(r.m.k $\underbrace{\frac{d}{d\varepsilon} E(w+\varepsilon\varphi)}_{\text{Euler-Lagrangian eqn}} = -\int_\Omega \Delta u \cdot \varphi \, d\vec{x}$. $\forall \varphi$ s.t. $\varphi|_{\partial\Omega} = 0$ $\Rightarrow -\Delta u = 0$

$E(w) = \int_\Omega F(\nabla w, w, \vec{x}) d\vec{x}$.

(2) Green's 2^{nd} identity: $\left[\iint_{\partial\Omega} [u \frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}] dS = \iiint_\Omega (u\Delta v - v\Delta u) dV\right]$

with all conditions same as 1^{st} except $u, v \in C^2(\bar\Omega)$

(i) General soln for $\begin{cases}\Delta u = 0, \ \Omega \subset \mathbb{R}^3 \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = h\end{cases}$:

$u(x_0) = \frac{1}{4\pi}\int_{\partial\Omega} [-u(\vec{x}) \frac{\partial}{\partial n}\frac{1}{\|\vec{x}-\vec{x}_0\|} + \frac{1}{\|\vec{x}-\vec{x}_0\|}\overbrace{\frac{\partial u}{\partial n}}^{h}] dS$

(proof.) $v = -\frac{1}{4\pi}\frac{1}{\|\vec{x}-\vec{x}_0\|}$ new domain $\Omega \setminus B_\varepsilon(x_0)$. $\Delta v = (\partial_r^2 + \frac{2}{r}\partial_r)(\frac{1}{4\pi r})$

$= 0$. v is harmonic on $D \setminus B_\varepsilon(x_0) \Rightarrow \int_{\partial\Omega} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) ds = 0$.

$\Rightarrow \int_{\partial\Omega} - \int_{\partial B_\varepsilon} (\cdots) ds = 0$ with $u(x_0) = \int_{\partial B_\varepsilon} (u\underbrace{\frac{\partial v}{\partial n}}_{\text{polar coordinate } \frac{\partial}{\partial n} = \partial_r} - v\frac{\partial u}{\partial n}) ds$ since RHS converges to $\frac{1}{\text{Area}(\partial B_\varepsilon)}\int_{\partial B_\varepsilon} u \, ds$ $(+\varepsilon \cdot C)$ & by MVP. ☐)

(r.m.k. always called (representation formula of Laplace equation)

In 2-d case. $u(x_0) = \frac{1}{2\pi}\int_{\partial\Omega} [-u(\vec{x})\frac{\partial}{\partial n}\ln(|x-x_0|) - \ln(|x-x_0|)\frac{\partial u}{\partial n}] ds$

Dirichlet ← cannot → Newmann
get simutaneously

In n-d case, "$v = (-1)^n \frac{1}{\alpha_n}\frac{1}{\|\vec{x}-\vec{x}_0\|^{n-2}}$"

(3) Green's function:

(i) [Defn] Let $\Omega \subset \mathbb{R}^3$, then a function $G(x, x_0)$ is Green's functn of $-\Delta$ at $x_0 \in \Omega$ if $\begin{cases} G(x,x_0) \text{ is smooth \& harmonic on } \Omega\setminus\{x_0\} \\ u(x) \triangleq G(x,x_0) + \frac{1}{4\pi\|x-x_0\|} \text{ is smooth \& harmonic on } \Omega \\ G(x,x_0)|_{\partial\Omega} = 0 \end{cases}$

(r.m.k $\begin{cases} v = \frac{-1}{4\pi\|x-x_0\|} \text{ satisfies the first 2.} \\ G \text{ is unique since } G = H - \frac{1}{4\pi\|x-x_0\|} \ (\text{unique } H: \Delta H = 0, \ H|_{\partial\Omega} = -\frac{1}{4\pi\|x-x_0\|}) \end{cases}$

[Thm] $\Delta u = 0$ on Ω, then $u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial G}{\partial n}(x,x_0) ds$

(proof.) $v = -\frac{1}{4\pi\|x-x_0\|} = G(x,x_0) - H(x)$

From the representation formula: $u(x_0) = \iint_{\partial\Omega} [u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}] ds$

$= \iint_{\partial\Omega} [u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}] ds - \underbrace{\iint_{\partial\Omega} [u\frac{\partial H}{\partial n} - H\frac{\partial u}{\partial n}] ds}_{0 \text{ green's identity}} = \iint_{\partial\Omega} u\frac{\partial G}{\partial n} ds$. ☐

[Thm] (Symmetry) $G(x,y) = G(y,x) \quad \forall x, y \in \Omega$

(can be seen as a $\Omega\times\Omega$ dom functn)

(proof.) Let $a, b \in \Omega$. Define $w_a(x) = G(x,a)$, $v_b(x) = G(x,b)$

On $V_\varepsilon = \Omega \setminus (\overline{B_\varepsilon(a)} \cup \overline{B_\varepsilon(b)})$ closed, apply Green's 2^{nd} identity.

$\Rightarrow \underbrace{\iint_{\partial V_\varepsilon} [w_a\frac{\partial v_b}{\partial n} - v_b\frac{\partial w_a}{\partial n}] ds}_{0 \text{ Green's 2nd}\sim} = -\underbrace{\iint_{\partial B_\varepsilon(a)} [w_a\frac{\partial v_b}{\partial n} - v_b\frac{\partial w_a}{\partial n}] ds}_{\varepsilon\to 0^+: \ -w_a(b)} - \underbrace{\iint_{\partial B_\varepsilon(b)} [w_a\frac{\partial v_b}{\partial n} - v_b\frac{\partial w_a}{\partial n}] ds}_{\varepsilon\to 0^+: \ v_b(a)} + \iint_{\partial\Omega} \cdots ds \to$ vanished $(G|_{\partial\Omega} = 0)$ ☐

(ii) Poisson's Formula: $\begin{cases}\Delta u = f \text{ on } \Omega \\ u|_{\partial\Omega} = h\end{cases}$

$\left[u(x_0) = \iint_{\partial\Omega} h(y)\frac{\partial G(x_0,y)}{\partial n} dS_y + \iiint_\Omega f(y) G(x_0,y) dS_y\right]$

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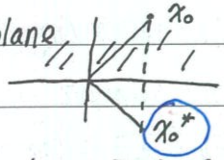
(iii) Green's Function on half plane $\mathbb{R}_+^n$ & sphere

On half plane,



rmv singularity: $\tilde{v}(x) = -\frac{1}{\|x-x_0^*\|}\cdot\frac{1}{4\pi}$

Then $H|_{\partial\Omega} = V(x) = \tilde{v}(x)$

Note: $\tilde{v}(x)$ harmonic & C^2 in $\mathbb{R}_+^3$ → Green's Functn on $\mathbb{R}_+^3$

Then $G(x,x_0) := V(x) - \tilde{v}(x)$ satisfies all conditions.

[Thm] Suppose $\Delta u = 0$ on $\mathbb{R}_+^3$, $\forall x_0 \in \mathbb{R}_+^3$, $u|_{\partial\mathbb{R}_+^3} = h$

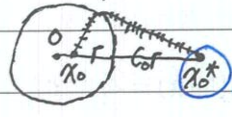
$$\left[\ u(\vec{x}_0) = \frac{x_{0,3}}{2\pi}\iint_{\partial\mathbb{R}_+^3}\frac{h(y_1,y_2)}{|\vec{x}_0-\vec{y}|^3}\,dy_1dy_2\ \right]$$

(proof.) Since $\frac{\partial G}{\partial n}(x_0,y) = -\frac{\partial G}{\partial y_3}(x_0,y) = \frac{1}{4\pi}\frac{\partial}{\partial y_3}\left[\frac{1}{|y-x_0|} - \frac{1}{|y-x_0^*|}\right]$

$\frac{\partial}{\partial y_3}\frac{1}{|y-x_0|} = \frac{-y_3+x_{0,3}}{|y-x_0|^3}$, $\frac{\partial}{\partial y_3}\frac{1}{|y-x_0^*|} = \frac{-y_3-x_{0,3}}{|y-x_0|^3}$ & $ds = dy_1dy_2$

since $\partial\mathbb{R}_+^3 = x$-$y$ plane.

On the sphere: cannot find such x_0^* as above ⇒ find $|x-x_0^*| = C_0|x-x_0|$ is possible



reflection pt

$G(x,x_0) = V(x) - C_0\tilde{v}(x)$

[Defn] $\vec{x}^*$ is called a reflection point of $\vec{x}$ w.r.t. $\partial B_a(0)$ (boundary of the sphere):
(i) It's colinear with the center O & $\vec{x}$
(ii) $\|O-\vec{x}\|\cdot\|O-\vec{x}^*\| = a^2$

Check: $\|\vec{x}-\vec{y}\|\cdot\boxed{\frac{a}{\|O-\vec{x}\|}} = \|\vec{x}^*-\vec{y}\|$, $\forall \vec{y}\in\partial B_a(0)$

C_0 what we want for $G|_{\partial\Omega}=0$

Thus, for a ball with center at origin

$$G(\vec{x},\vec{x}_0) = -\frac{1}{4\pi\|\vec{x}-\vec{x}_0\|} + C_0\frac{1}{4\pi\|\vec{x}-\vec{x}_0^*\|}$$

$$\Rightarrow u(\vec{x}_0) = \frac{a^2-\|\vec{x}_0\|^2}{4\pi a}\iint_{|\vec{x}|=a}\frac{h(\vec{x})}{\|\vec{x}-\vec{x}_0\|^3}\,ds$$

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• Distributions & Transformations

(1) Distributions: recall the heat kernel $\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4\pi t}}$, $x\in\mathbb{R}$

mollifier
Smooth the singularity

s.t. (i) $\int_{\mathbb{R}}\Phi\, dx = 1$ & (ii) $\boxed{\lim_{t\to0^+}\int_{\mathbb{R}}\Phi(x-y,t)\varphi(y)dy} = \boxed{\varphi(x)}$, if $\varphi\in C_0(\mathbb{R})$

$t\to0^+$: assign a function φ to a value $\varphi(x)$

If $t\to0^+$, Φ itself becomes dirac-delta function: $\delta_0(x) = \begin{cases}\infty, & x=0\\ 0, & \text{otherwise}\end{cases}$

& $\int_{\mathbb{R}}\delta_0(x)\,dx = 1$ — NOT common functions.

(i) [Defn] Distribution / Functional / General Function

$f: \mathcal{D}\mapsto\mathbb{R}$ is called a functional if ① f is $\boxed{\text{continuous}}$ in the sense: $f[\phi_n]\mapsto f[\phi_0]$, as $\phi_n\mapsto\phi_0$ $\boxed{\text{uniformly}}$ ② f is linear. (with $\mathcal{D}\subseteq C_0^\infty(\mathbb{R})$)

function space

(e.g.1 Dirac-delta functn: $\delta_x[\phi] = \phi(x)$, $\forall\phi\in\mathcal{D}$

e.g.2 $\mathcal{D} = C^1$, $f[g] \triangleq \int_\Omega f\cdot g\,dx$) check for continuity & linearity.

(r.m.k Take $\rho(\phi_1,\phi_2) = \max_\Omega|\phi_1(x)-\phi_2(x)|$ in function space $\mathcal{D}$ (metric space), we get the continuity of above sense.)

(ii) Weak convergence: [Defn] For sequence of distributions $\{f_n\}_{n=1}^\infty$, it converges $\boxed{\text{weakly}}$ to f if $f_n[\phi]\xrightarrow{n\to\infty}f[\phi]$, $\forall\phi\in\mathcal{D}$.

(e.g.1 $\int_\Omega\Phi(x-y,t)\varphi(y)dy \xrightarrow[\text{weakly}]{t\to0^+} \delta_0[\varphi]$

e.g.2 $S_N(f)\xrightarrow[\text{weakly}]{N\to\infty}\delta_x[f]$, where $S_N(f) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\overset{\text{Dirichlet kernel}}{K_N}(x-y)dy$)

(iii) Derivatives of distributions: given by $\partial_x f[\phi] = f[-\partial_x\phi] = -f[\partial_x\phi]$

(IDEA: $\int_\Omega\partial_x f[\phi]dx = f[\phi]\big|_{\partial\Omega} - \int_\Omega f[\partial_x\phi]dx$ (integrate by part))

(e.g.1 Heaviside functn: $H_c(x) = \begin{cases}0, & x<c\\1, & x\geq c\end{cases}$ Riesz-representation thm ⇒ !!!

$$\left[\int_{\mathbb{R}}H_c\phi(y)dy\right]' = \int_{\mathbb{R}}-H_c\cdot\phi'(y)dy = -\int_c^\infty\phi'(y)dy = \delta_c[\phi]$$

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e.g.2 Consider the wave eqn: we say that $u(x,t)$ is a distribution soln to $\begin{cases} u_{xx}\cdot c^2 = u_{tt} \\ u(x,0)=\phi(x) \\ u_t(x,0)=0 \end{cases}$ if $u[\partial_t^2\phi - c^2\partial_x^2\phi] = 0 \ \forall \phi\in C_0^\infty$

$(\Leftrightarrow \partial_t^2 u[\phi] = c^2\partial_x^2 u[\phi])$

(check: $\phi(x) = (1-|x|)^+$.)

(r.m.k For a function u, we can construct the corresponding functional by let $u[\phi] = \underbrace{\langle u, \phi\rangle}_{\text{inner product}} = \int_\Omega u\phi \, d\mu = \int_{\mu(x)} \phi \, d\varphi$ ← Riesz representation thm holds) positive measure

Thus $\langle \partial_x u, \phi\rangle = \langle u, -\partial_x \phi\rangle$) adjoint operator

(e.g.3 (Green's Functn) $\begin{cases} \Delta G(x,x_0) = \delta_{x-x_0} \\ G(x,x_0)|_{\partial\Omega} = 0 \end{cases}$

$u(x_0) = \int_D u(x)\delta_{x-x_0}\,dx = \int_D \Delta G u \, dx$)

(2) Fourier Transform:

(i) From discrete to continuous: with complex notation $f = \sum_{n=-\infty}^{\infty} c_n \cdot e^{\frac{in\pi x}{l}}$ (Fourier series on $(-l,l)$, coefficients $c_n = \frac{1}{2l}\int_{-l}^{l} f(y) e^{-\frac{in\pi y}{l}} dy$

$\Rightarrow f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} [\int_{-l}^{l} f(y) e^{-iky} dy] e^{ikx}\cdot\frac{\pi}{l}$, with $k = \frac{n\pi}{l}$

$l\mapsto\infty$, tends to integration $[f(x) = \frac{1}{2\pi}\int_{\mathbb{R}}[\int_{\mathbb{R}} f(y)e^{-iky}dy]e^{ikx}dk]$

(ii) (F. invT) $f(x) = \int_{\mathbb{R}} F(k) e^{ikx} \frac{dk}{2\pi}$ frequency, where $F(k) \triangleq \int_{\mathbb{R}} f(y) e^{-iky}$

(Fourier-Tran) $F(k) = \int_{\mathbb{R}} f(x) e^{-ikx} dx$

(r.m.k F.T. is well-defined on L^1, $|F| \le \|f\|_{L^1} < \infty$)

generally, $f \ne \tilde{F}$ (F.T. & inverse F.T. are NOT inverse operator)

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(iii) High-dim F.T.: $\vec{x}\in\mathbb{R}^n, \vec{k}\in\mathbb{R}^n$ $F(\vec{k}) = \int_{\mathbb{R}^n} f(\vec{x}) e^{-i\vec{k}\cdot\vec{y}} d\vec{y} \triangleq \tilde{f}(\vec{k})$

& $f(\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^n} F(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} d\vec{x} \triangleq \tilde{F}(\vec{x})$

(iv) Derivatives of F.T.: $F[\partial x_i f] = -i k_i F[f]$ ($\tilde{f}_{x_j} = -i\cdot k_j \tilde{f}$) reduce PDE ⇒ ODE

Translation: $f(x-a) \xrightarrow{F.T.} e^{-iak}\tilde{f}(k)$

$f(a\vec{x}) \xrightarrow{F.T.} \frac{1}{|a|^n}\tilde{f}(\frac{\vec{k}}{a})$ if $a \ne 0$

(v) Convolution: $\widetilde{f*g} = \tilde{f}\cdot\tilde{g}$, where $f*g = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$

(vi) Plancherel's thm: $[2\pi\int_{\mathbb{R}^n}|f|^2 d\vec{x} = \int_{\mathbb{R}^n}|F(f)|^2 d\vec{k}]$

(proof.) $\int_{\mathbb{R}^n}\tilde{f}\cdot\bar{\tilde{f}}\,d\vec{k} = \int_{\mathbb{R}^n}[\int_{\mathbb{R}^n} f(\vec{x})e^{-i\vec{k}\vec{x}}d\vec{x}][\int_{\mathbb{R}^n}\bar{f}(\vec{y})e^{-i\vec{k}\cdot\vec{y}}d\vec{y}]\,d\vec{k}$

$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\vec{x})\cdot\bar{f}(\vec{y}) e^{i\vec{k}\cdot(\vec{y}-\vec{x})} d\vec{y}\,d\vec{x}\,d\vec{k} = \iint f(\vec{x})\bar{f}(\vec{y})\,d\vec{x}\,d\vec{y}$

$\cdot \underbrace{\int_{\mathbb{R}^n} e^{ik\cdot(\vec{y}-\vec{x})}d\vec{k}}_{\tilde{1}(\vec{y}-\vec{x})} = \iint 2\pi f(\vec{x})\bar{f}(\vec{y})\,\delta(\vec{x}-\vec{y})\,d\vec{x}\,d\vec{y} = 2\pi\int_{\mathbb{R}^n}|f|^2 d\vec{x}$.

(e.g.1 Heat eqn: $\begin{cases}\Delta_{\vec{x}}\Phi = \partial_t\Phi, \ \vec{x}\in\mathbb{R}^n \\ \Phi(\vec{0}) = \delta \end{cases}$ (Initial V.P.)

① 1-d: Do Fourier transform ⇒ $\begin{cases}\partial_t\tilde{\Phi} + k^2\tilde{\Phi} = 0 \\ \tilde{\Phi}(\vec{0}) = 1\end{cases}$

ODE ⇒ $\tilde{\Phi} = e^{-k^2 t}$, then $\Phi = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$ (check table)

② high-dim (n): $\Phi = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{\|x\|^2}{4t}}$.

e.g.2 Wave eqn: $\begin{cases}\partial_t^2 S = c^2\partial_x^2 S \\ S(x,0) = 0 \ \& \ S_t(x,0) = \delta\end{cases} \xRightarrow{F.T.} \begin{cases}\partial_t^2\tilde{S} + c^2k^2\tilde{S} = 0 \\ \tilde{S}(k,0) = 0, \ \tilde{S}_t(k,0) = 1\end{cases}$

$\Rightarrow \tilde{S} = \frac{1}{kc}\sin(kct) \Rightarrow S = \frac{1}{2c}H(c^2t^2 - x^2)$ (check table)

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• Generalized Eigenvalue Problems

(I) Introduction:

(i) Let $\Omega \subset \mathbb{R}^n$, $\begin{cases} -\Delta u = \lambda u \\ u|_{\partial\Omega} = 0 \ (\text{on } \partial\Omega) \end{cases}$ is called general ei-value problem. ($u \not\equiv 0$). λ must be real positive, since $\langle -\Delta u, u\rangle = \lambda \langle u, u\rangle$ gives that $-\int_\Omega \Delta u \cdot \bar{u}\, d\vec{x} = -\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot \bar{u}\, d\vec{x} + \int_\Omega \nabla u \cdot \nabla \bar{u}\, d\vec{x} = \int_\Omega |\nabla u|^2 d\vec{x} = \lambda \int_\Omega |u|^2 d\vec{x}$

$\therefore \lambda = \int_\Omega |\nabla u|^2 d\vec{x} \big/ \int_\Omega |u|^2 d\vec{x}$, real positive.

(ii) (Thm) [Minimum Principle] Let $E(w) = \frac{\|\nabla w\|_2^2}{\|w\|_2^2}$ ← functional, the smallest ei-value is given by $\lambda_1 = \min_w \{E(w) \mid w|_{\partial\Omega} = 0,\ w \not\equiv 0\}$

(proof.) $\forall\, w, v$ smooth, $w|_{\partial\Omega} = 0 = v|_{\partial\Omega}$, $w \not\equiv 0$.

Let $f(\varepsilon) = E(w+\varepsilon v)$, we get $f'(\varepsilon) = 2\frac{\int_\Omega \nabla(w+\varepsilon v)^T \nabla v\, d\vec{x} \cdot \int_\Omega (w+\varepsilon v)^2 d\vec{x} - \int_\Omega v(w+\varepsilon v) d\vec{x} \int_\Omega |\nabla(w+\varepsilon v)|^2 d\vec{x}}{\left(\int_\Omega (w+\varepsilon v)^2 d\vec{x}\right)^2}$. Let w be a minimizer of E. then 0 should be the minimizer of $w \Rightarrow f'(0) = 0$, which gives $\int_\Omega \nabla w \nabla v = E(w) \int_\Omega w \cdot v\, d\vec{x}$, thus, $LHS = \underbrace{\int_{\partial\Omega} \frac{\partial w}{\partial n} \frac{v}{d\vec{x}}}_{0} - \int_\Omega \Delta w \cdot \frac{v}{d\vec{x}} = E(w) \int_\Omega w \cdot v\, d\vec{x}$ $\Rightarrow \forall\, v|_{\partial\Omega} = 0$, $-\int_\Omega (\Delta w + E(w) w) v\, d\vec{x} = 0$. $\Rightarrow -\Delta w = E(w) \cdot w = \lambda_1 \cdot w$, thus λ_1 is an ei-value.

Consider any λ (ei-value) s.t. $-\Delta u = \lambda u$, $u|_{\partial\Omega} = 0$ $\Rightarrow \langle -\Delta u, u\rangle = \lambda \langle u, u\rangle$

$\Rightarrow \lambda = \frac{\int_\Omega |\nabla u|^2 d\vec{x}}{\int_\Omega u^2 d\vec{x}} = E(u) \geq \lambda_1$. ($\lambda_1$ is the smallest)

(Thm') [Minimum Principle / n^{th} ei-value]

Suppose $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ are the first $(n-1)$ smallest ei-values (no need to be different), associated with ei-functions $v_1, \cdots, v_{n-1}$

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then λ_n ← the n^{th} smallest $= \min_w \{E(w) \mid w|_{\partial\Omega} = 0,\ w \not\equiv 0\ \&\ w \in S_{n-1}^\perp\}$ → orthogonal complement of the ei-space, spanned by $v_1, \cdots, v_{n-1}$

(proof.) similarly, but $v \in S_{n-1}^\perp$ this time, thus

$f'(\varepsilon) = 0 \Rightarrow \int_\Omega (-\Delta w - \lambda_n w) v\, d\vec{x} = 0$, $\forall\, v \in S_{n-1}^\perp$, since $\int_\Omega (-\Delta w - \lambda_n w) v_k\, d\vec{x} \overset{2^{nd}\text{ identity}}{=} \int_\Omega (\lambda_k - \lambda_n) w v_k\, d\vec{x} = (\lambda_k - \lambda_n)\langle w, v_k\rangle = 0$. $\Rightarrow -\Delta w = \lambda_n w$.

(λ_n is an ei-value). $\lambda_n \geq \lambda_{n-1}$, since $S_k^\perp \leq S_{k-1}^\perp$, & also same as above, λ_n is the minimal on $S_{n-1}^\perp$.

(iii) (Lemma) [Orthogonality] Any 2 eigen-pairs (λ_k, v_k), $k = 1, 2$, with $\lambda_1 \neq \lambda_2$, we have $\langle v_1, v_2\rangle_\Omega = 0$.

(proof.) Do Green's identity on $0 = \int_\Omega (-\Delta v_1 - \lambda_1 v_1) v_2\, d\vec{x} = (\lambda_2 - \lambda_1)\langle v_1, v_2\rangle$.

(Application) Given n functions $\{f_1, \cdots, f_n\}$, $f_k|_{\partial\Omega} = 0$, $k = 1, \cdots, n$.

$J_n = \text{span}\{f_1, \cdots, f_n\}$. We can approximate the k^{th} ei-value as

$(\lambda_k \sim \min_w \{E(w) \mid w \in J_n,\ w \in S_{k-1}^\perp\})$ → in finite dimension

$= \max_w \{E(w) \mid w \in J_n \cap S_k\}$

Rayleigh-Ritz Approximation ← the approx. accuracy depends on choice of $f_1, \cdots, f_n$.

(iv) Test function space.

[Defn] Let $\{w_k, k = 1, 2, \cdots, n\}$ be a set of n arbitrary functions, satisfying B.C.. Let $J_n = \text{span}\{w_k;\ k = 1, 2, \cdots, n\}$. we define $\mathcal{F}_n$ to be the collection of all such n-dimensional linear space.

Denote $\lambda_n^* = \max_{w \in J_n} E(w)$, as the approximation of the n^{th} smallest ei-value.

(Thm) (Min-max Principle) The n^{th} ei-value is given by

$$\left[\ \lambda_n = \min_{J_n \in \mathcal{F}_n} \lambda_n^*(J_n) = \min_{J_n \in \mathcal{F}_n} \max_{w \in J_n} E(w)\ \right]$$

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