MAT1002

Calculus Further

Reference book

Thomas' Calculus (13th Edition in SI Units), George B. Thomas Jr., Maurice D. Weir, Joel R. Hass. Publisher: Pearson.

Calculus: Early Transcendentals (8th Edition), James Stewart. Publisher: Brooks Cole

Chapter 10

"Intuitive" calculus (without real ~~anyl~~ analysis) ①

§ 10.1 Sequences

• Convergence & Divergence of sequences

1) Def. ① $\{a_n\}_{n=1}^{\infty}$ "convergence"/converges to a finite number "L" as "n" goes to infinity. "a_n" can be as close to "L" as we wish. If so, we say limit of $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$ is "L". Namely, "$\lim\limits_{n\to\infty} a_n = L$" ("$\xi-\delta$" Language definition in the textbook) "initial index"

connecting with calculation analysis

② If this "L" cannot be found, $\{a_n\}$ diverges.

2) Remarks. ① Finitely changing many numerical "a_n" does not affect the convergence/divergence of the sequence $\{a_n\}$, and also does not affect "$\lim\limits_{n\to\infty} a_n$" if it exists.

② Let "p" be a fixed number, consider $\{a_{n+p}\}_{n=1}^{\infty}$.

"$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+p}$" if $\lim\limits_{n\to\infty} a_n$ exists as a finite number.

3) Rules for calculating "limits": omit it.

Q: ~~If~~ Does $\lim\limits_{n\to\infty} \sin n$ exist?

Suppose it exists. (Let it be "L")

$$\lim_{n\to\infty} \sin(n+1) = \lim_{n\to\infty}(\sin n \cos 1 + \cos n \sin 1)$$

$$\overset{\text{Remark "2"}}{=} \lim_{n\to\infty} \sin n$$

So $(1-\cos 1)\lim\limits_{n\to\infty} \sin n = \sin 1 \lim\limits_{n\to\infty} \cos n$.

Namely, $\lim\limits_{n\to\infty} \cos n = \dfrac{1-\cos 1}{\sin 1} \cdot \lim\limits_{n\to\infty} \sin n = \dfrac{1-\cos 1}{\sin 1} \cdot L$

But, on the other hand $\sin 2n = 2\sin n \cos n$

$$L = \lim_{n\to\infty} \sin 2n = 2 \lim_{n\to\infty} \sin n \cdot \lim_{n\to\infty} \cos n = 2 \cdot L \cdot \frac{1-\cos 1}{\sin 1} L \quad (*)$$

1° If $L = 0$, then $\lim\limits_{n\to\infty} \cos n = 0$. Because $\sin^2 n + \cos^2 n = 1$, contradictory.

2° If $L \neq 0$, $\cos 2n = 2\cos^2 n - 1$

$\lim\limits_{n\to\infty} \cos n = \lim\limits_{n\to\infty} \cos 2n = \lim\limits_{n\to\infty} (2\cos^2 n - 1)$ contradictory with $\lim\limits_{n\to\infty} \cos n = \frac{1}{2}$ (from $(*)$)

$\Rightarrow \lim\limits_{n\to\infty} \sin n$ "DNE" $\rightarrow$ does not exists

• Theorem: Suppose $\exists f(x)$ satisfied

$$\begin{cases} f(n) = a_n & \forall n \geq M \quad (M \text{ is a fixed number}) \\ \lim\limits_{x\to\infty} f(x) = L \end{cases}$$

Then $\lim\limits_{n\to\infty} a_n = L$.

e.g. for "theorem"

② Prof. $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$

- Squeeze Theorem. the same as what is in "Calculus I"

  eg. Prof. $\lim_{n\to\infty}\frac{x^n}{n!}=0$

$$-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!} \xrightarrow[\text{and if } n>M]{\text{Choose } M>|x|} \frac{|x|^{n-M}}{\frac{n!}{M!}}\cdot\frac{|x|^M}{M!} < \frac{|x|^{n-M}}{(M+1)^{n-M}}\cdot\frac{|x|^M}{M!}$$

Take the limits. $\lim_{n\to\infty}\frac{x^n}{n!} \le \lim_{n\to\infty}\frac{|x|^n}{n!} < \frac{|x|^M}{M!}\lim_{x\to\infty}\left(\frac{|x|}{M+1}\right)^{n-M}=0$

similarly, we can prove the left side.

Q: What if we want to know the existence of $\lim_{n\to\infty} a_n$, but does not need to compute it? Namely, confirm the convergence / divergence.

Theory for "monotonic function"

Terminology
① If $a_{n+1}\geqslant a_n$, $\forall n\geqslant 1$, then $\{a_n\}_{n=1}^{\infty}$ is non-decreasing.
② If $\{a_n\}$ is either nondecreasing or nonincreasing, it is a monotone.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a monotonic sequence
- If it is bounded, $\{a_n\}_{n=1}^{\infty}$ converges. $\lim_{n\to\infty} a_n = L \to$ a finite number
- If it is ~~un~~bounded, $\{a_n\}_{n=1}^{\infty}$ diverges. $\lim_{n\to\infty} a_n = \infty / -\infty$

• §. 10.2 (infinite) series

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{i=1}^{\infty} a_i$$

Q: How to add up infinite many numbers?

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty}\sum_{i=1}^{n} a_i = \lim_{n\to\infty} \underline{S_n} \text{ partial sums}$$

consider if $\lim_{n\to\infty} S_n$ exists as a finite number.

If so, then we can say $\sum_{i=1}^{\infty} a_i$ converges.

If not, then we say the infinite ~~numbe~~ series diverges.

Central Q: Given the $\sum_{i=1}^{\infty} a_i$, know if it is convergent or divergent.

Remarks: 1) Changing / adding / deleting finite terms cannot affect $\sum_{i=1}^{\infty} a_i$'s convergence/divergence

2) Reindexing $\sum_{i=1}^{\infty} a_i = \sum_{j=M+1}^{\infty} a_{j-M}$ (the same)

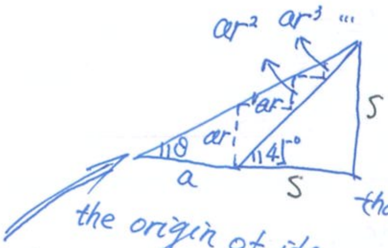
§ 10.2

Two important types of series.

1) Geometric series

given a common ratio "r"

$\underbrace{a + ar + ar^2 + \cdots + ar^n + \cdots}$ is called a geometric series



tanθ = r

the case that 0<r<1

the origin of its name

$r = \tan\theta = \frac{S}{a+S} \Rightarrow \begin{cases} S = \frac{ar}{1-r} \\ a+S = \frac{a}{1-r} \end{cases}$

$$\begin{cases} 0<|r|<1 & \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \\ r=1 & \lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty / -\infty \\ |r|>1 & \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \pm\infty \end{cases}$$

} depends on "a"'s sign.

Conclusion

$$\sum_{n=0}^{\infty} ar^n < \begin{matrix} \text{convergent}, = \frac{a}{1-r} \text{ when } 0<|r|<1 \\ \text{divergent, wher } |r| \geq 1 \end{matrix}$$

2) Telescoping series

- Divergence test.

If $\sum^{\infty} a_n$ converges. then general term a_n must converge to 0.
($\lim_{n\to\infty} a_n = 0$)

(i.e.) If $\lim_{n\to\infty} a_n$ does not exist / exists but not 0,
then $\sum^{\infty} a_n$ diverges.

Prof. " $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_n$ "

$S_{n+1} = S_n + a_{n+1}$

So $a_n = S_n - S_{n-1}$

Since $\sum^{\infty} a_n$ converges

S_n converges to a finite number "L"

$\Rightarrow a_n \to 0$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0$

$\square$

- Basic Rules.

Suppose $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ both converge

then (i) $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converge $= \left(\sum_{n=1}^{\infty} a_n\right) \pm \left(\sum_{n=1}^{\infty} b_n\right)$

(ii) $\sum_{n=1}^{\infty} a_n b_n$ $\underbrace{\text{coverges / diverges}}_{\text{cannot be determined}}$

Remarks.

(i) $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} b_n$ diverges

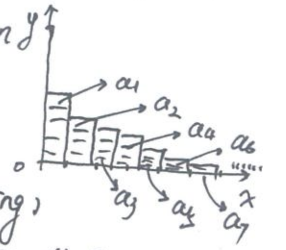
$\Rightarrow \sum_{n=1}^{\infty} (a_n \pm b_n)$ diverges.

(ii) If $c \neq 0$, $\sum_{i=1}^{\infty} c a_i$ has the same convergence/divergence as $\sum_{i=1}^{\infty} a_i$

④ • §. 10.3 Integral test

Always assume $\sum a_n$ is non-negative,
(i.e. $\forall n.\ a_n \geq 0$)

Geometric interpretation:



If $\sum a_n$ converges.

$S_n \uparrow$ (non-decreasing)

Case I $S_n \uparrow$, converges # S
Case II $S_n \uparrow$, ∞

$\Longleftarrow$ Want to use monotonic sequences test.

Sum of all areas of rectangulars is a finite number.

• Integral (improper) test:

Let $\{a_n\}$ be a nonnegative sequence, such that
$\exists$ a continue: decreasing function $f(x)$ — three fundamental conditions
satisfying $f(n) = a_n$ if $n > N$ (finite number)

Then $\sum_{n=1}^{\infty} a_n$ & $\int_N^{\infty} f(x)\,dx$ converge and diverge at the same time

$a_N + a_{N+1} + \cdots \geq \int_N^{\infty} f(x)dx \geq a_{N+1} + a_{N+2} + \cdots$
By this inequation

• P — test:

$p>0$, $\sum_{n=1}^{\infty} \frac{1}{n^p} \longleftrightarrow \int_1^{\infty} \frac{dx}{x^p}$ $\begin{cases} \text{converges}, & p>1 \\ \text{diverges}, & p \leq 1 \end{cases}$

harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \Rightarrow \infty$ special case (when $p=1$)

Approximate:

$R_n = \text{error} = S - S_n = a_{n+1} + a_{n+2} + \cdots$

Recall $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$ provided "f" is decreasing on $[1, +\infty)$

·§. 10.4 Comparison Tests

more general than limit C.T.

THEOREM 10—The Comparison Test Let Σa_n, Σc_n, and Σd_n be series with nonnegative terms. Suppose that for some integer N

$$d_n \le a_n \le c_n \qquad \text{for all} \qquad n > N.$$

(a) If Σc_n converges, then Σa_n also converges.

(b) If Σd_n diverges, then Σa_n also diverges.

Prof. Theorem 11. Case I.

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$

then for every $\varepsilon > 0$, there exists an N

such that $0 < \left|\frac{a_n}{b_n} - c\right| < \varepsilon \Leftarrow n > N$

assume $\varepsilon = \frac{c}{2}$, $N = N_0$

so when $n > N_0$ $\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}$

so $\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}$

$\frac{c}{2}\sum\limits_{n=N_0}^{\infty} b_n =$ then $\sum\limits_{n=N_0}^{\infty} \frac{c b_n}{2} < \sum\limits_{n=N_0}^{\infty} a_n < \sum\limits_{n=N_0}^{\infty} \frac{3c}{2} b_n = \frac{3c}{2}\sum\limits_{n=N_0}^{\infty} b_n$

$\Rightarrow \Sigma a_n$, Σb_n both converge & diverge.

Case II.

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0 \,/\, \infty$

Similarly, let $\varepsilon = 1$. $n > N_1 \Rightarrow \left|\frac{a_n}{b_n}\right| < 1$

or

$n > N_1 \Rightarrow \left|\frac{a_n}{b_n}\right| > 1$

then by comparison test

★**THEOREM 11—Limit Comparison Test** Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer).

Similar to limit test of functions

1. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then Σa_n and Σb_n both converge or both diverge.

2. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and Σb_n converges, then Σa_n converges.

3. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n diverges, then Σa_n diverges.

Only non-negtive sequences.

Consider their power

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$ means they grow at the ~~s~~ same rate.

·§. 10.5 Absolute covergence & Ration, root test

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM 12—The Absolute Convergence Test If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Prof. $\underbrace{\sum_{n=1}^{\infty}-|a_n|}_{\text{convergent}} \leq \sum_{n=1}^{\infty} a_n \leq \underbrace{\sum_{n=1}^{\infty}|a_n|}_{\text{convergent.}}$ ($a_n = \underbrace{|a_n| + a_n}_{\text{non-negative}} - |a_n| \leq 2|a_n|$, comparison

eg. If divergence, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. Because $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ does not converge

the theorem does not work.

Sol. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)$

$= \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}-\frac{2\times1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right)$

$= \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$

Because $\ln\frac{n+1}{n} \geq \frac{1}{n+1} \geq \ln\frac{n+2}{n+1}$

$\ln 2 = \lim_{n\to\infty}\ln\frac{2n}{n} \leq \lim_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right) \leq \lim_{n\to\infty}\left(\frac{2+}{1+}\right.$ $= \ln$

So, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2$

THEOREM 13—The Ratio Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$. eg. $\rightarrow$ { convergent $\sum^{\infty}\frac{1}{n^2}$; divergent $\sum^{\infty}\frac{1}{n}$ or $\sum^{\infty}$

prof. By definition, let $\varepsilon = \frac{1-\rho}{2}$ (if $\rho<1$), $\exists N_0$

(a) $n > N_0$ $\quad 0 < \left|\frac{a_{n+1}}{a_n}\right| < \underbrace{\left|\frac{1+\rho}{2}\right|}_{= r_0} < 1$

$|a_{n+1}| < r_0|a_n| < \cdots < r_0^{n-N_0}|a_{N_0}|$

$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N_0-1}|a_n| + \sum_{n=N_0}^{\infty}|a_n| \leq \underbrace{\sum_{n=1}^{N_0-1}|a_n|}_{\text{conv.}} + \underbrace{|a_{N_0}|\cdot\sum_{n=N_0}^{\infty} r_0^{n-N_0}}_{\text{conv.}}$ convergent

(b) Let $\varepsilon = \frac{\rho-1}{2}$ (if $\rho>1$) $\exists N_1$. $n>N_1$ then $\frac{3\rho-1}{2} > \left|\frac{a_{n+1}}{a_n}\right| > \underbrace{\frac{\rho+1}{2}}_{r_1} > 1$

So. $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N_1-1}|a_n| + \sum_{n=N_1}^{\infty}|a_n|$

$\geq \sum_{n=1}^{N_1-1}|a_n| + |a_{N_1}|\ldots$

$= \underbrace{\sum_{n=1}^{N_1-1}|a_n|}_{\text{conv}} + \underbrace{|a_{N_1}|\sum_{n=N_1}^{\infty} r_1^{n-N_1}}_{\text{div.}}$ div

THEOREM 14—The Root Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$. eg. $\rightarrow$ { convergent $\sum^{\infty}\frac{1}{n^2}$; divergent $\sum^{\infty}1$ or $\sum^{\infty}n$

★ Can be proved in the same way as above (omit)

§10.6 Alternating series

THEOREM 15—The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Prof. Assume $N=1$, say that $n=2m$

$$S_{2m} = u_1 - (u_2 - u_3) - \cdots - u_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) < u_1$$

because $S_{2m+2} \geq S_{2m}$. $\{S_{2m}\}$ non-decreasing and bounded

It has limit. $\lim_{m\to\infty} S_{2m} = L$

$$\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} S_{2m+2} + u_{2m+2} = \lim_{m\to\infty} S_{2m+2} + \lim_{m\to\infty} u_{2m+2} = L.$$

(This theorem doesnot work necessarily.) eg. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} + \frac{1}{n^2}$

THEOREM 16—The Alternating Series Estimation Theorem If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

$$|L - S_n| = \left|\sum_{i=n+1}^{\infty}(-1)^{i+1}u_i\right| = |u_{n+1} - \underbrace{(u_{n+2} - u_{n+3}) - \cdots}_{\text{the sum of negative things / non-positive things.}}| \leq |u_{n+1}|$$

(Because $L - S_n = (-1)^{n+1}\left(\sum_{i=n+1}^{\infty} \underbrace{u_i \cdot (-1)^{i-n-1}}_{\text{positive}}\right)$, it has the same sign as $(-1)^{n+1}$, namely, the first term)

DEFINITION A convergent series that is not absolutely convergent is **conditionally convergent**.

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THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

★ Rearrangement will change the outcome of conditionally convergent series.

Prof. Because $\sum a_n$ converges absolutely, $\forall \varepsilon > 0$ (ε is a real number) $\exists N_1 > 0$

$\sum_{n=N_1}^{\infty} |a_i| < \frac{\varepsilon}{2}$, find $N_2 \geq N_1$, let $\sum_{n=1}^{\infty} a_n = L$, $S_m = \sum_{n=1}^{m} a_n$

$$|S_{N_2} - L| = |\sum_{n=N_2}^{\infty} a_n - \sum_{n=1}^{\infty} L| \leq |\sum_{n=N_1}^{\infty} a_n| \leq \sum_{n=N_1}^{\infty} |a_n| < \frac{\varepsilon}{2}.$$

According to the conditions. there is an $N_3 \geq N_2$ s.t. $n \geq N_3$

$\sum_{i=1}^{n} b_i$ contains $a_1, a_2, \ldots a_{N_2}$

So $|\sum_{i=1}^{n} b_i - L| = |\sum_{i=1}^{n} b_i - S_{N_2} + S_{N_2} - L| \leq |\sum_{i=1}^{n} b_i - S_{N_2}| + |S_{N_2} - L| \leq \sum_{n=N_1}^{\infty} |a_i| + |S_{N_2} - L| < \varepsilon$

So $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} a_i = L$. For the same reason $|\sum_{i=1}^{n} |b_i| - \Sigma| \leq \sum_{n=N_1}^{\infty} |a_i| + \sum_{n=N_1}^{\infty} |a_i|$, $\sum_{i=1}^{\infty} |a_i| = \Sigma$

1. **The *n*th-Term Test:** If it is not true that $a_n \to 0$, then the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Properties of convergent sequences:

- property (I) Uniqueness of limits:

  If $\{x_n\}$ converges, it has only one limit

prof. Assume we have $x_n \to a$, $x_n \to b$ $(b > a)$. given $\varepsilon = \frac{b-a}{2}$ then

$$\exists N_1, N_2 \in \mathbb{N}^* \begin{cases} |x_n - a| < \frac{b-a}{2}, & n > N_1 \text{ ①} \\ |x_n - b| < \frac{b-a}{2}, & n > N_2 \text{ ②} \end{cases}$$

get $N = \max\{N_1, N_2\}$

when $n > N$, then $x_n < \frac{b+a}{2}$ by ① but $x_n > \frac{a+b}{2}$ by ②. contradictory.

- property (II) Boundary Nature.

  If $\{x_n\}$ converges, it has bounds.

prof. Assume $\lim_{n\to\infty} x_n = a$, get $\varepsilon = 1$

then $\exists N \in \mathbb{N}^*$ $|x_n - a| < 1$, $n > N$

So $|x_n| - |a| < 1$, $|x_n| < |a| + 1$. $n > N$

So $M = \max\{|x_1|, |x_2|, \ldots |x_{N-1}|, |x_N|, |a| + 1\}$ is its bound.

MAT 1002 Calculus II

⑨

§. 10.7 Power series

• Definition. A power series about "$x=a$" is the series of the form:

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{i=0}^{\infty} c_i (x-a)^i$$

"c_i" — coefficient ; "a" — centre of the series

Central Question: 1) Given $f(x)$, expressing $f(x)$ to a power series (represent) about $x=a$

Talor expansion of "f" about "$x=a$" $\rightarrow$ $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ using partial sum (polynomial) to estimate functions

(especially. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$

2) Given $\sum_{n=0}^{\infty} c_n (x-a)^n$, want to know for what "x" does this series converge.

※ Theorem. (i) If $c \neq a$ s.t. (such that) $\sum_{n=0}^{\infty} c_n (x-a)^n \Big|_{x=c}$ converges, then for abitrary x s.t. $|x-a| < |c-a|$, the series converges absolutely.

(ii) If $\exists d$ s.t. $\sum_{n=0}^{\infty} c_n (x-a)^n \Big|_{x=d}$ diverges, then for x s.t $|x-a| > |d-a|$, the series diverges

Prof. (i) $\sum |c_n| \cdot |x-a|^n = \sum |c_n| \left|\frac{x-a}{c-a}\right|^n \cdot |c-a|^n = \sum |c_n (c-a)^n| \cdot \left|\frac{x-a}{c-a}\right|^n$

$\because$ $\sum c_n (c-a)^n$ converges, $\lim_{n\to\infty} c_n (c-a)^n = 0$ $\{c_n (c-a)^n\}$ is bounded.

(i.e.) $\exists M > 0$ s.t. $|c_n (c-a)^n| \leq M$, $\forall n \geq 0$

$\Rightarrow \sum |c_n| |x-a|^n = \sum |c_n (c-a)^n| \left|\frac{x-a}{c-a}\right|^n \leq M \cdot \sum \left|\frac{x-a}{c-a}\right|^n$

$\Rightarrow$ converges absolutely Geometric Series & comparison test

(ii) Assume $|x_0 - a| > |d-a|$ and $\sum c_n (x_0 - a)^n$ converges, then by (i), $\sum c_n (d-a)^n$ converges, contradictory

(Arguement by contradict)

Corollary (推論):

3 possibilities of $\sum c_n (x-a)^n$ for its convergence

1) $\exists\, 0 < R < \infty$ s.t. $\sum c_n (x-a)^n$ converges absolutely for $x \in (a-R, a+R)$ $\Rightarrow$ may or may not converge at the point $a-R$ & $a+R$

2) $\sum c_n (x-a)^n$ converges absolutely, $\forall x$ $R = \infty$

3) $\sum c_n (x-a)^n$ diverges, $\forall x$ except for $x=a$ $R=0$

R — Radius for convergence of $\sum c_n (x-a)^n$

⑩ Q: How to find R (all "xs" such that $\sum c_n(x-a)^n$ converges)

"root test" or "ratio test"   always a interval

- Power series operations.

Given $\sum_{n=0}^{\infty} a_n(x-a)^n$ & $\sum_{n=0}^{\infty} b_n(x-a)^n$ both converge absolutely for $|x-a|<R\neq 0$

(i) $\sum_{n=0}^{\infty} a_n(x-a)^n \sum_{n=0}^{\infty} b_n(x-a)^n = \sum_{n=0}^{\infty} c_n(x-a)^n$

$(c_n = a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0)$

Then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely in $|x-a|<R\neq 0$

(ii) $\dfrac{\sum_{n=0}^{\infty} a_n(x-a)^n}{\sum_{n=0}^{\infty} b_n(x-a)^n} = \sum_{n=0}^{\infty} d_n(x-a)^n$ provided $b_0 \neq 0$

$\sum_{n=0}^{\infty} d_n(x-a)^n$ converges absolutely in $|x-a|<R\neq 0$

eg. $\dfrac{1}{(1-x)^2} = \left(\dfrac{1}{1-x}\right)^2 = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}(n+1)x^n$ way I

$\left(\dfrac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty}(n+1)x^n$ way II

- Differentiation of power ~~theorem~~ series $\sum_{n=0}^{\infty} c_n(x-a)^n = f(x)$ $|x-a|<R$

"f" is differentiable $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$, $|x-a|<R$.

$f'''\cdots$ (n-order derivative can be done)

- Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $x\in(a-R, a+R)$, $R>0$

Then $\forall n \geq 0$ $f(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + O((x-a)^{n+1})$ as $x\to a$

$(O(x-a) = f'(a)(x-a) + o(x-a))$

Proof. $f(x) - \sum_{i=0}^{n} c_i(x-a)^i = c_{n+1}(x-a)^{n+1} + \cdots$ (as $\sum c_n(x-a)^n$ converges to $f(x)$)

$= (x-a)^{n+1}\left(c_{n+1} + c_{n+2}(x-a) + \cdots\right)$ converges for $x\in(a-R, a+R)$

$\therefore$ It is continuous at $x=a$

so $c_{n+1} + c_{n+2}(x-a) + \cdots = O(1)$, $(x-a)^{n+1}O(1) = O((x-a)^{n+1})$

$(\forall \varepsilon>0\ \exists\delta,\ |x-a|<\delta,\ |c_{n+1} + c_{n+2}(x-a) + \cdots - L| < \varepsilon)$

MAT 1002 Calculus II

(11)

- Integration of $\sum_{n=0}^{\infty} c_n(x-a)^n$, $|x-a|<R\neq 0$ (converges)

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n(x-a)^n dx = \int \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1})$$

when $|x-a|<R$

eg.

① $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \Rightarrow \underbrace{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}_{x\in(-1,1]}$

$|x|<1$

② $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \Rightarrow \arctan x = \int (1-x^2+x^4-x^6+\cdots)dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1}$

$|x|<1$

$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots$ $\quad |x|<1 \Rightarrow x\in[-1,1]$

- §. 10.8 Taylor & Maclauren series

Central Q: Given $f(x)$, want to $\overset{\text{write}}{f(x)} = \sum c_n(x-a)^n$

How to determine c_n? By "derivative"

1) " $f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

$= \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n}$ " (general term $\frac{f^{(n)}(a)}{n!}(x-a)^n$, "Taylor formula/series" $(0!=1)$ (Defined by $T(x)$)

If $f(x)$ has a power series expansion near "$x=a$", then that expansion has to be given by "Taylor formula"

2) Maclauren series: a special case when "a" $=0$ in Taylor formula

eg. $\begin{cases} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!} & \text{converges for any "}x\text{"} \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i \cdot x^{2i+1}}{(2i+1)!} \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!} \end{cases}$

whether these Taylor series converge? For what x it's convergent?

★Caution: No, in general

$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x\neq 0 \\ 0 & x=0 \end{cases}$ $\quad f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$

definition

By Taylor series. $f(x)\neq 0$ "not all converges to $f(x)$"

⑫ Just need to prove partial sum $\underbrace{\lim_{n\to\infty}\sum_{i=0}^{n}}\frac{f^{(i)}(a)(x-a)^i}{i!} = f(x)$

By definition $\lim_{n\to\infty} P_n(x) = f(x)$

polynominal

• §. 10.9 Convergence of Taylor series to $f(x)$

• Taylor's Theorem: Suppose $f, f', \cdots f^{(n)}$ exist and ~~close~~ continuous on $[a,b]$. $f^{(n)}(x)$ differentiable and $f^{(n+1)}(x)$ exists on (a,b)

Then $\forall$ fixed $x_0 \in [a,b]$, $\forall x \in [a,b]$, we have

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Remainder}}$$

for some c ~~between~~ x_0 and x

Taylor's formula "$f(x) = P_n(x) + R_n(x)$"

Remainder $\Leftrightarrow R_n(x)$

(special case: when $n=0$. MVT)

Cauchy's Remainder

Proof. for any fixed "x" $(x \neq x_0)$. Let $M = \frac{f(x) - P_n(x)}{(x-x_0)^{n+1}}$

without loss of generality $a \le x_0 < x \le b$.

(不失一般性)

Define $g(t) = f(t) - P_n(t) - M(t-x_0)^{n+1}$

so $\Rightarrow g(x) = 0$, $g(x_0) = 0 = g'(x_0) = g''(x_0) = \cdots = g^{(n)}(x_0)$

Rolle's Theorem

$g'(c_1) = 0$ $\quad c_1 \in (x_0, x)$

$g''(c_2) = 0$ $\quad \cdots\cdots$ $\quad c_2 \in (x_0, c_1)$

$g'''(c_3) = 0$ $\quad \cdots\cdots$

Let $c = c_{n+1}$ satisfying $g^{(n+1)}(c) = g^{(n+1)}(c_{n+1}) = 0$

Because $g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (n+1)!\,M$ then $M = \frac{f^{(n+1)}(c)}{(n+1)!}$ Q.E.D

Just need to show "Remainder $\to 0$" $\lim_{n\to\infty} R_n = 0$

$$\lim_{n\to\infty}|R_n| = \lim_{n\to\infty}\frac{|f^{(n+1)}(c)|\,|x-x_0|^{n+1}}{(n+1)!} \le \lim_{n\to\infty}\frac{\overbrace{\max_{a\le x\le b}|f^{(n+1)}(x)|}^{\le M}\,|x-x_0|^{n+1}}{(n+1)!} = \begin{cases} 0, & \text{if } M \text{ independent of } n \\ \text{cannot determine} & \text{if } M \text{ dependent of } n. \end{cases}$$

$\boxed{\frac{|x-x_0|^{n+1}}{(n+1)!} \xrightarrow{n\to\infty} 0}$ squeeze theorem

e.g. for $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$0 \le \lim_{n\to\infty}\frac{f^{(n+1)}(c)\cdot x^{n+1}}{(n+1)!} \le \lim_{n\to\infty}\frac{e^{|x|}\,x^{n+1}}{(n+1)!} = 0 \quad \forall x \in \mathbb{R}$$

$0 \le c \le x$

for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ Remainder: $\frac{f^{(2n+2)}(c)}{(2n+2)!}x^{2n+2}$

$0 \le c \le x$

$$0 \le \lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{x^{2n+2}}{(2n+2)!} = 0 \quad \forall x \in \mathbb{R}$$

eg. $\tan x = \dfrac{\sin x}{\cos x} = \dfrac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$ By long-division

$= x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \cdots \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

·§· 10.10 Binominal series application.

$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^{n}, \quad r \in \mathbb{R}$

Let $f(x) = (1+x)^r$. Talor series about $x=0$

$1 + rx + \dfrac{r(r-1)}{2!}x^2 + \dfrac{r(r-1)(r-2)}{3!}x^3 + \cdots + \dfrac{r(r-1)\cdots(r-k+1)}{k!}x^k + \cdots + \cdots$

$= \sum_{n=0}^{\infty} \binom{r}{n} x^n$

Interval convergence of this series: ratio test $\Rightarrow$ for $x \in (-1, 1)$

eg. $\lim_{x \to 0} \dfrac{1 - \cos(\sqrt{\tan x - \sin x})}{\sqrt[3]{1+x^3} - \sqrt[3]{1-x^3}}$ (using Talor expansion)

Another application

$\int \dfrac{\sin x}{x} dx = \int \dfrac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots)}{x} dx = \int 1 dx - \int \dfrac{x^2}{3!} dx + \int \dfrac{x^4 dx}{5!} - \cdots$

Euler's formula ($\forall \theta \in \mathbb{R}$)

$e^{i\theta} = 1 + i\theta + \dfrac{(i\theta)^2}{2!} + \cdots + \dfrac{(i\theta)^n}{n!} + \cdots$

$= (1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \cdots) + i(\theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!}) = \cos\theta + i\sin\theta$

Chapter 11. Parametric equations & Polar coordinates.

·§· 11.1 How to represent curves analytically?

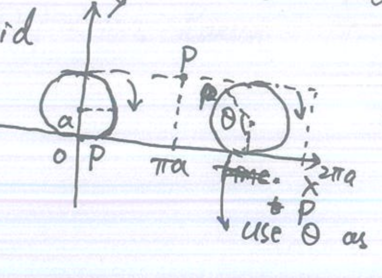
Way I: $f(x,y) = 0$ (graph of Cartesian)

Way II: Think of curves on x-y-planes as trajectory of a particle's moving.

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ t - parametric, "time"

Nature parametrication of the graph $y = f(x)$, $x \in I$: $\begin{cases} x = t \\ y = f(t) \end{cases}, \ t \in I$

Cycloid



palametric equation: $\begin{cases} x = \theta a - a\sin\theta \\ y = a - a\cos\theta \end{cases}$, $\theta \in (0, 2\pi)$ $\theta > 0$

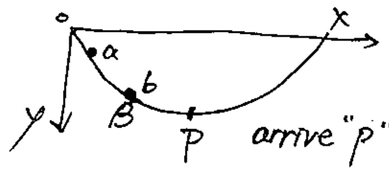
In Reality θ can be anything larger than 0

(14) History of cycloid:

1) Galileo: area below cycloid

2) Huygens: "same time property"

3) Bernoulli: "shortes time property"



arrive "p" at the same time.

from "B" to "p" this way ~~takes~~ takes least time.

Helen of Geometrics

§10.2 Calculus of Parametric Curves

① Slope of parametric curves $\left.\frac{dy}{dx}\right|_{t=t_0}$ (at some special points)

② High-order derivative of parametric curves

③ Area below curves.

Q: $A = \int_0^{2\pi a} y\,dx$ (take cycloid as an example)

$$= \int_0^{2\pi a} a(1-\cos\theta)\,d\,a(\theta-\sin\theta) = \int_0^{2\pi} a^2(1-\cos\theta)^2\,d\theta$$

$$= \int_0^{2\pi} a^2 d\theta - 2a^2\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{0} + a^2\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= 3\pi a^2$$

④ Arc-length of parametric curves

Q: $L = \int_0^{2\pi}\sqrt{(dx)^2+(dy)^2} = \int_0^{2\pi}\sqrt{(1-\cos\theta)^2+\sin^2\theta}\,d\theta$

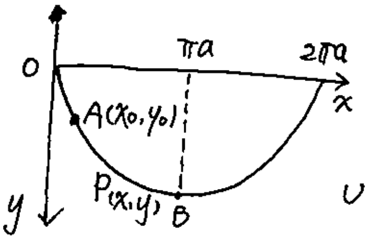
$$= \int_0^{2\pi}\sqrt{2-2\cos\theta}\,d\theta$$

$$= \int_0^{2\pi} 4\sin\frac{\theta}{2}\,d\frac{\theta}{2} = -4\cos\frac{\theta}{2}\Big|_0^{2\pi} = 8$$

- Combine with physics

*①

1) Proof of same time property in cycloid:



$T_A^B = t$ independent of A

$\theta = \theta(t)$. let s be the distance

$$v = \frac{ds}{dt} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dt} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot\frac{d\theta}{dt}$$

$$= a\sqrt{2-2\cos\theta}\cdot\frac{d\theta}{dt}$$

$$= 2a\sin\frac{\theta}{2}\cdot\frac{d\theta}{dt}, \quad 0\le\theta\le 2\pi$$

$$\int_0^{T_A^B}\left(\frac{1}{v(t)}\frac{ds}{dt}\right)dt = \int_0^{T_A^B} 1\,dt = T_A^B$$

$$\Rightarrow \int_0^{T_A^B}\frac{1}{v}\frac{ds}{dt}\cdot dt = T_A^B \Rightarrow \int_0^{T_A^B}\frac{1}{v}\,2a\sin\frac{\theta}{2}\frac{d\theta}{dt}\cdot dt = T_A^B$$

$$\Rightarrow T_A^B = \int_{\theta_0}^{\pi}\frac{2a\sin\frac{\theta}{2}}{v}\,d\theta$$

$$mg(y-y_0) = \text{work} = \tfrac{1}{2}mv^2$$

$$\Rightarrow t = T_A^B = \int_{\theta_0}^{\pi}\frac{2a\sin\frac{\theta}{2}}{\sqrt{2g(y-y_0)}}\,d\theta = \int_{\theta_0}^{\pi}\left(\sqrt{\frac{2a}{g}}\right)\frac{\sin\frac{\theta}{2}\cdot d\theta}{\sqrt{\cos\theta_0-\cos\theta}} = \int_{\theta_0}^{\pi}\sqrt{\frac{2a}{g}}\cdot\frac{d\left(-2\cos\frac{\theta}{2}\right)}{\sqrt{2\cos^2\frac{\theta_0}{2}-2\cos^2\frac{\theta}{2}}}$$

$$= \int_{\theta_0}^{\pi}-2\sqrt{\frac{a}{g}}\cdot\frac{d\cos\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_0}{2}-\cos^2\frac{\theta}{2}}} = -2\sqrt{\frac{a}{g}}\left(\underbrace{\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta_0}{2}}}_{\arcsin}\right)\Bigg|_{\theta_0}^{\pi} = 2\sqrt{\frac{a}{g}}\arcsin\frac{\cos\frac{\theta_0}{2}}{\cos\frac{\theta_0}{2}} = \pi\sqrt{\frac{a}{g}}$$

(t has no relationship with θ_0 or (x_0, y_0).)

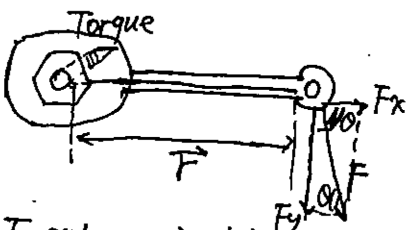
Huygen's clock.

All are cycloids.

the period stays the same.

because of same time principle.

2)



$\underline{|\text{Torque}| \propto |\vec{r}|\cdot|\vec{F}\sin\theta|} \Longrightarrow |\text{Torque}| = |\vec{r}\times\vec{F}|$

Definition Reason $\quad \underset{(\text{bolt})}{\text{Torque}} \overset{\text{def}}{=} \vec{r}\times\vec{F}$

* ②

☆ Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\frac{d\vec{r}}{ds}\cdot\frac{ds}{dt}\right) = \frac{d}{dt}\left(\vec{T}\cdot\frac{ds}{dt}\right)$$

$$= \frac{d}{dt}\vec{T}\,\frac{ds}{dt} + \frac{d^2s}{dt^2}\cdot\vec{T} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\,\frac{d}{dt}\left(\frac{d\vec{T}}{ds}\cdot ds\right)$$

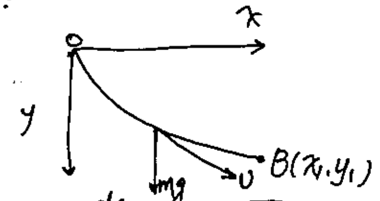
$$= \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa \vec{N} \triangleq a_\tau \vec{T} + a_n \vec{N}$$

$a_\tau = \frac{d^2s}{dt^2}$, $a_n = \left(\frac{ds}{dt}\right)^2 \kappa$ are tangential & nomal scalar components of acceleration respectively.

Proof of shortest time Rule:

$y = y(x)$

$y(0) = 0$, $y(x_1) = y_1$



$$mgh = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gh} \quad , v = \frac{ds}{dt} = \sqrt{1+y'^2}\,\frac{dx}{dt}$$

$$\int_0^T 1\,dt = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx \Longleftrightarrow T = R(y) = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

☆ Another calculation for κ

$$|\vec{a}\times\vec{v}| = |\underbrace{a_\tau \vec{T}\times\vec{v}}_{\vec{0}} + a_n \vec{N}\times\vec{v}| = |\vec{v}|^3\kappa$$

Therefore, $\kappa = \left|\frac{\vec{a}\times\vec{v}}{|\vec{v}|^3}\right|$

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{1}$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{2}$$

Assume: $\left.\frac{dx}{dt}\right|_{t=t_0} = f'(t_0) \neq 0$, let (t_1, t_2) around t_0 be a monotonic interval about x

(At $t=t_0$) $y \sim t \sim x$ then $t = f^{-1}(x)$ exists

Chain Rule: $\frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}$. So $\left.\frac{dy}{dx}\right|_{t=t_0} = \left.\frac{dy}{dt}\right|_{t=t_0} \cdot \frac{1}{\left.\frac{dx}{dt}\right|_{t=t_0}} = \frac{dy/dt\,|_{t=t_0}}{dx/dt\,|_{t=t_0}}$

Second-Order derivative: (at $t=t_0$)

$$\left.\frac{d^2y}{dx^2}\right|_{t=t_0} = \left.\frac{d(\frac{dy}{dx})}{dx}\right|_{t=t_0} = \frac{\left.\frac{d(\frac{dy}{dx})}{dt}\right|_{t=t_0}}{dx/dt|_{t=t_0}} = \frac{dy'/dt\,|_{t=t_0}}{dx/dt\,|_{t=t_0}}$$

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Using MVT $\Rightarrow$ the definition

$\uparrow$ continuously infinitesimal

$ds = \sqrt{(dx)^2 + (dy)^2}$

$\Uparrow$

By $S(t) = \int_a^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2}\, d\tau$

$\Rightarrow \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis ($y \geq 0$):**

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

2. **Revolution about the y-axis ($x \geq 0$):**

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

(16) • §. 11.3 Polar Coordinates

positive & negative angles ; no uniqueness

try "$-\theta$", "$\pi-\theta$" & "$\pi+\theta$"

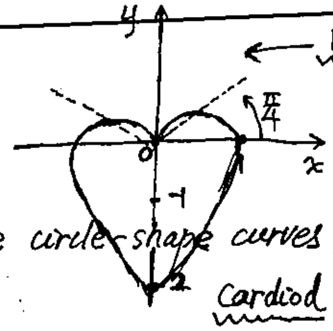
Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.27a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.27b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.27c).

$(r,\theta)=(r,\ \theta+2n\pi),\ n\in\mathbb{Z}$

$=(-r,\ \theta+(2n+1)\pi)$

for convenience, allow $r<0$



$r=1-\sin\theta$

$f(r,\theta)=0 \Rightarrow f(r,\pi-\theta)$ symmetry about y-axis

Polar coordinates are used to describe circle-shape curves.

• §. 11.4 Graphing polar equation

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $dx/d\theta \neq 0$ at (r, θ).

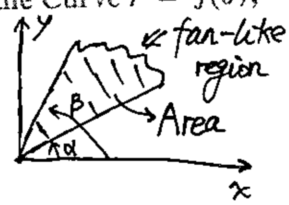
$f(r,\theta)=0$, sketch all points on x-y plane, whose polar coordinates satisfy this equation

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{(r,\theta)} = \left.\frac{d\,r\sin\theta/d\theta}{d\,r\cos\theta/d\theta}\right|_{(r,\theta)} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\underset{r=f(\theta)}{=} \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

• §. 11.5 Areas & Length in polar coordinates

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$



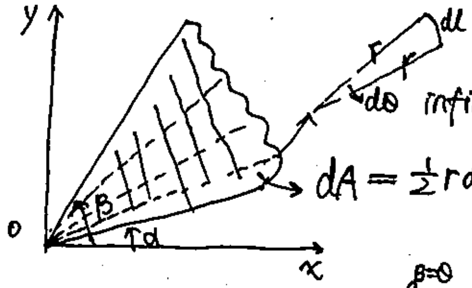
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta$$

This is the integral of the **area differential** (Figure 11.32)

$$dA = \frac{1}{2} r^2\, d\theta = \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 \, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$



$dA = \frac{1}{2} r\,dl = \frac{1}{2} r^2 d\theta$ ← area differential

$$A = \int_{\alpha=\theta}^{\beta=\theta} 1\, dA = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

Corollary, $A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\, d\theta \rightarrow$ region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad (3)$$

Recall $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \cdot d\theta = \int_\alpha^\beta \sqrt{\left(\frac{d\, r\cos\theta}{d\theta}\right)^2 + \left(\frac{d\, r\sin\theta}{d\theta}\right)^2}\, d\theta$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{r^2(\sin^2\theta + \cos^2\theta) + \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \Longrightarrow \text{Length of a polar curve.}$$

(18) .§. 12.4 MAT1002 Calculus II

DEFINITION The **cross product u × v** ("**u cross v**") is the vector

area of parallelogram $|\vec{u}\times\vec{v}|$

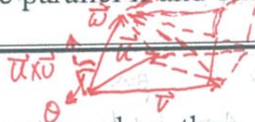
$$\mathbf{u}\times\mathbf{v} = (|\mathbf{u}||\mathbf{v}|\sin\theta)\,\mathbf{n}.$$

$\vec{n}$ — normal unit vector (right-hand rule)

(A vector is a directed line segment.)

Parallel Vectors

Nonzero vectors **u** and **v** are parallel if ~~and only~~ if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$.



volumn of cylinder

Properties of the Cross Product

If **u**, **v**, and **w** are any vectors and *r*, *s* are scalars, then

7. Triple scalar product $(\vec{u}\times\vec{v})\cdot\vec{w} = V$

Geometric mean $|(\vec{u}\times\vec{u})\cdot\vec{w}| = |\vec{u}\times\vec{v}||\vec{w}||\cos\theta|$

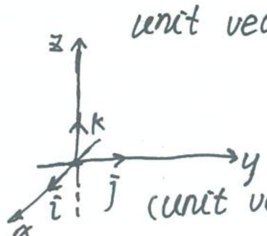
1. $(r\mathbf{u})\times(s\mathbf{v}) = (rs)(\mathbf{u}\times\mathbf{v})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
3. $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$
4. $(\mathbf{v}+\mathbf{w})\times\mathbf{u} = \mathbf{v}\times\mathbf{u} + \mathbf{w}\times\mathbf{u}$
5. $\mathbf{0}\times\mathbf{u} = \mathbf{0}$ ($\vec{0}$)
6. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$

$(\vec{u}\times\vec{v})\times\vec{w} \neq$ must parallel to $\vec{v},\vec{w}$ plane.

$\overrightarrow{AB} = (\underbrace{x_1-x_2,\ y_1-y_2,\ z_1-z_2}_{\text{components}})$; 1D vectors $\vec{v} = \langle v_1 \rangle$, $v_1 \in (-\infty, \infty)$ (Scalar) 标量 ("two directions": positive / negative)

unit vectors: length of which $= 1$

(for any vector $\vec{v}$, $\frac{\vec{v}}{|\vec{v}|}$ is unit vector of it) expressing "direction of $\vec{v}$"



(unit vectors – standard/special)

parallelogram (平行四边形)

$\vec{u} \parallel k\cdot\vec{u}$ (scalar multiplication) (scalar product)

$\vec{u}\cdot\vec{v}$ dot product → how to get θ, origin, $\in [0,\pi]$

law of cosines $\cos\theta = \frac{u_1v_1+u_2v_2+u_3v_3}{\sqrt{u_1^2+u_2^2+u_3^2}\sqrt{v_1^2+v_2^2+v_3^2}}$ $\hat{=}$ → $\vec{u}\cdot\vec{v}$

($\vec{v}$ onto $\vec{u}$)

$\text{project}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}\vec{u}|}\cdot\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}\cdot\frac{\vec{u}}{|\vec{u}|}$ → scalar component

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then (cross product)

Motivation: $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$; want: $\vec{w}\perp\vec{u},\vec{v}$ ⇓ $\begin{cases}\vec{w}\cdot\vec{u} = 0\\ \vec{w}\cdot\vec{v} = 0\end{cases}$

$$\vec{w} = \mathbf{u}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix} = (u_2v_3-u_3v_2)\vec{i} + (u_3v_1-u_1v_3)\vec{j} + (u_1v_2-u_2v_1)\vec{k}$$

← determinant

($\vec{w} = (a,b,c)$) (take $c = (u_1v_2-v_1u_2)$)

Calculating the Triple Scalar Product as a Determinant

Another way to calculate: $\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix} + a_2\begin{vmatrix}b_3 & b_1\\ c_3 & c_1\end{vmatrix} + a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}$

$$V = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}$$

→ volumn of cylinder

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit L** as t approaches t_0 and write

truncated neighborhood (去心邻域)

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

the length

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous over its interval domain.

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f, g, and h have derivatives at t. The derivative is the vector function

direction of $\frac{d\vec{r}}{dt}$ = tangent line direction

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

If exists $\Rightarrow$ differentiable of $\vec{r}$ at t

DEFINITIONS If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

• curve traced by $\vec{r}(t)$ is smooth at $t=t_0$ if $\frac{d\vec{r}(t)}{dt}\Big|_{t=t_0}$ exists & does not $= \vec{0}$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

•§• why do we need $\vec{r}'(t) \neq \vec{0}$?

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

eg. $\begin{cases} x = t^3 \\ y = t^2 \end{cases} \xrightarrow{get} y = x^{\frac{2}{3}}$

$\vec{r}'(t)|_{t=0} = \vec{0}$

$\Longrightarrow$ not differentiable at $t=t_0$ ($f(x,y)$, $y = \ldots$ / $x = x(t_0)$) — cusp

#13.1

• Think of curves in space is trajectory / path of particles.

make geometric sense

$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ parametric equations

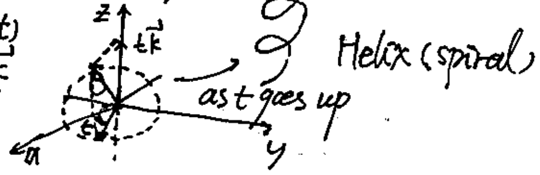
(for calculation reason)

$\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ (standard position.

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

vector-value function

eg.• The curve of $\vec{r}(t) = \underbrace{\cos t\,\vec{i} + \sin t\,\vec{j}}_{\vec{r}_1(t)} + \underbrace{t\vec{k}}_{\vec{r}_2(t)}$



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Differentiation Rules for Vector Functions

Let **u** and **v** be differentiable vector functions of t, **C** a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$ product Rule
   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

5, 6 – by distributive law in dot & cross product

If $|\vec{r}(t)| = \text{const } (r)$ $\forall t\in I$ (differentiable)

$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2 = r^2 \Rightarrow \frac{d}{dt}\vec{r}^2(t) = 2\vec{r}(t)\vec{r}'(t) = 0$

$\Rightarrow \vec{r}(t)\cdot\vec{r}'(t) = 0$, $\vec{r}(t)\perp\vec{r}'(t)$

Arclength variable.

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2 + \left(\frac{dz}{dz}\right)^2}\,dz$$
$$= \int_a^t |\vec{v}(z)|\,dz$$

Remark. sometimes defined as from t_0 ($a \to t_0$ – substitution)

$s(t)$ – directed/signed arclength

DEFINITION The **indefinite integral** of **r** with respect to t is the set of all antiderivatives of **r**, denoted by $\int \mathbf{r}(t)\,dt$. If **R** is any antiderivative of **r**, then

$$\int \mathbf{r}(t)\,dt = \underbrace{\mathbf{R}(t)}_{\text{vector function}} + \overbrace{\mathbf{C}}^{\text{a vector } \vec{C}}.$$

DEFINITION If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is **r**, and the **definite integral** of **r** from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \underbrace{\left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}}_{\text{constant vector}}.$$

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. = \int_a^b |\vec{v}(t)|\,dt \qquad (1)$$

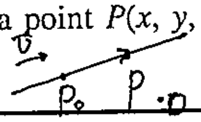
§12.5 Lines & planes in space

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.



$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}\ (t\vec{v})$ (parametric equations)

(coordinary polar)

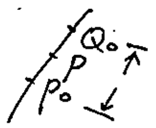
Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$

Q①: $P_0(x_0, y_0, z_0)$; $Q_0(x_1, y_1, z_1)$

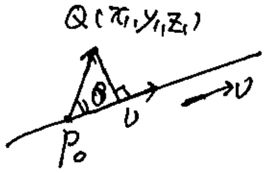
equation of segement in line from P_0 to Q_0



Let $\vec{v} = \overrightarrow{P_0Q_0} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

So $r(t) = \vec{r_0} + t\vec{v}$, $0 \le t \le 1$

Q②: Distance from Q to L



$$d = |\overrightarrow{P_0Q}| \cdot |\sin\theta|$$

$$= \frac{|\overrightarrow{P_0Q}|\,|\vec{v}|\,|\sin\theta|}{|\vec{v}|}$$

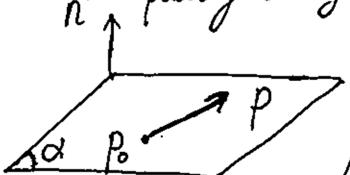
$$= \frac{|\overrightarrow{P_0Q} \times \vec{v}|}{|\vec{v}|}$$

$\Downarrow$ we get the distance formula.

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{5}$$

Q③. $P_0(x_0, y_0, z_0)$ & vector $\vec{n} = (A, B, C)$ (normal vector) of the plane passing through P_0.



$\vec{n} \cdot \overrightarrow{P_0P} = 0$ (vector function)

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ (component equation)

Namely $Ax + By + Cz = D$

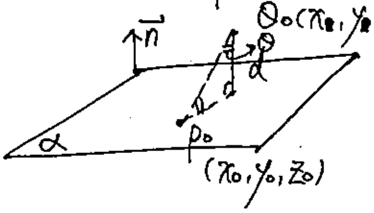
(next page)

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:	$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$
Component equation:	$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
Component equation simplified:	$Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

Distance from Q_0 to the Plane:



$$d = |\overrightarrow{P_0Q_0}||\cos\theta| = \frac{|\overrightarrow{P_0Q_0}||\vec{n}||\cos\theta|}{|\vec{n}|} = \left|\frac{\overrightarrow{P_0Q_0}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\overrightarrow{P_0Q_0}\cdot\frac{\vec{n}}{|\vec{n}|}\right|$$

The Distance from a Point to a Plane

If P is a point on a plane with normal $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$. That is, the distance from S to the plane is

$$d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right| \tag{6}$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

In xyz-space, we have $f(x,y,z)=0$ as a surface.

- ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
- elliptical paraboloids/cones: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ (when $z=d$ ($\frac{d}{c}>0$), getting a ellipsoid, ($c>0$ upward opening, $c<0$ downward opening)
- hyperboloids: $\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}$
  / hyperbolic paraboloid — intersection with $z=d$, $\frac{d}{c}>0$

Cylinders $f(x,y)=0$ in xyz plane ($f(x,z)=0$; $f(y,z)=0$ are the same.)

A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder (Figure 12.43). In solid geometry, where *cylinder* means *circular*

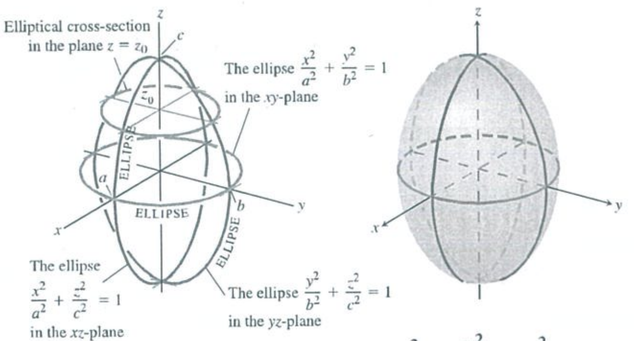
Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equation in x, y, and z. In this section we study quadric surfaces given by the equation

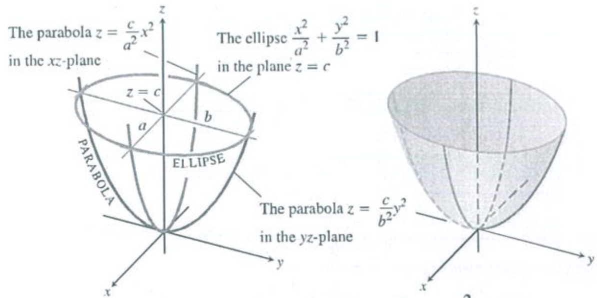
$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

where A, B, C, D, and E are constants. The basic quadric surfaces are **ellipsoids, paraboloids, elliptical cones,** and **hyperboloids**. Spheres are special cases of ellipsoids. We pres-

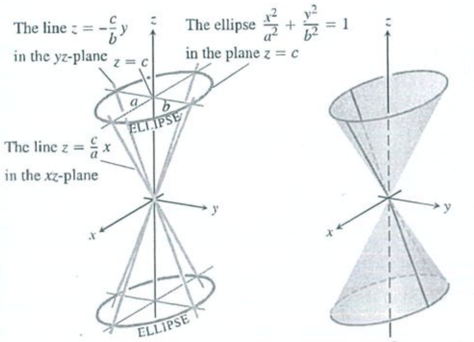
TABLE 12.1 Graphs of Quadric Surfaces



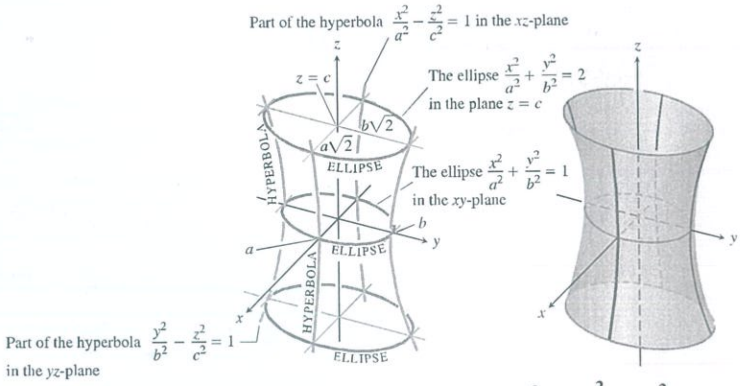
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



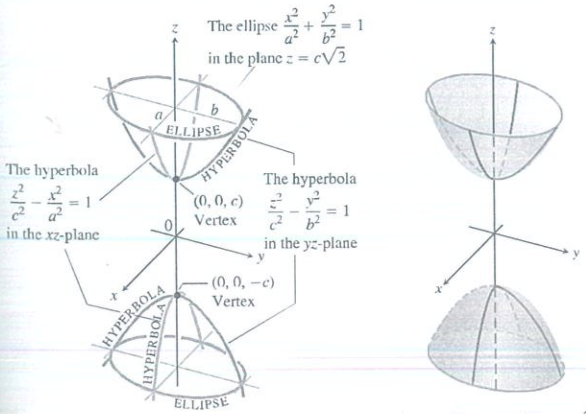
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



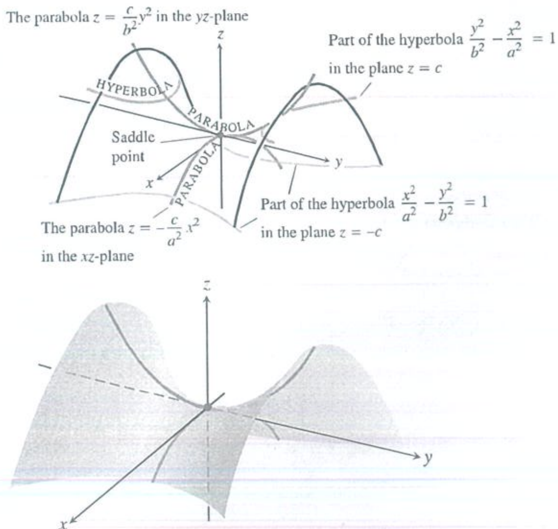
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Reparametrization of curve by using arclength variable $-s$:

Suppose the curve traced out by $\vec{r}(t)$, $a \le t \le b$ is smooth.
$\Rightarrow |\vec{v}(t)| \ne 0$

$\Rightarrow \frac{ds}{dt} = |\vec{v}(t)| > 0$, $\forall t \in (a,b)$ s – strictly increases,

so s has a inverse function $s^{-1}(t) = t$, $0 \le s \le L$ – arclength of the curve

$\Rightarrow \vec{r} = \vec{r}(t) = \underbrace{\vec{r}(s^{-1}(t))}_{\text{a function of } s}$ or $\vec{r} = L(t(s))$ ← reparametrization by using arclength parameter.

(Why is it possible?)

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\cdot\frac{dt}{ds} = \vec{r}'(t)\cdot s^{-1\prime}(t) = \vec{r}'(t)\frac{1}{s'(\underset{t}{s^{-1}(t)})} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{unit tangent vector } (\vec{T})}$$

§ 13.4 Curvature & Unit normal vectors for space curves

- curvature is meant to measure how severe the curve is bent.

$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$ ← unit tagent line/vector   idea: measurement of the rate of change of $\vec{T}$

Suppose the curve can be parametrized by $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

$$\vec{T}(t) = \frac{\frac{d\vec{r}(t)}{dt}}{\left|\frac{d\vec{r}(t)}{dt}\right|} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{r}(t)/dt}{ds/dt} = \frac{d\vec{r}(t)}{ds(t)}$$

$$\Rightarrow \kappa = \underbrace{\left|\frac{d\vec{T}(t)}{ds}\right|}_{\text{not user-friendly}} = \left|d\left(\frac{d\vec{r}/dt}{ds/dt}\right)\frac{1}{ds}\right| = \underbrace{\left|\frac{d\vec{T}/dt}{ds/dt}\right|}$$

process: $\frac{d\vec{T}(t)}{dt}$ – vector – not good
$\Downarrow$
$\left|\frac{d\vec{T}(t)}{dt}\right|$ – not good – depends on parameters used
$\Downarrow$
$\left|\frac{d\vec{T}(t)}{ds(t)}\right|$ – everything depends on $s(t)$

eliminate the effect

$\left|\frac{d\vec{T}(t)}{dt}\right|\frac{1}{|\vec{v}(t)|}$ ⇒ consider two ways to parametrize the unit circle.

- normal vectors of curves in space

$\vec{T}\cdot\vec{T} = 1 \Rightarrow \frac{d}{ds}\vec{T}\cdot\vec{T} = 0 \Leftrightarrow \left(\frac{d\vec{T}}{ds}\right)\cdot 2\vec{T} = 0$

$\Rightarrow \frac{d\vec{T}}{ds} \perp \vec{T}$, normal vector: $\vec{n} = \frac{d\vec{T}}{ds}$ principal normal vector

Unit vector $\vec{N} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{n}}{\kappa}$ $\left(\frac{d\vec{T}/ds}{\kappa}\right) = \frac{d\vec{T}/dt}{\kappa|\vec{v}(t)|} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$

(24) $\vec{N}$ for straight line: undefined

Chapter 14 Partial Derivatives

· §14.1 Functions of several variables

e.g.) Temperature: $T = T(x, y, z, t)$

- Euclidean space $\mathbb{R}^n \to$ set: consists of n-tuples $(x_1, x_2, x_3 \ldots x_n)$

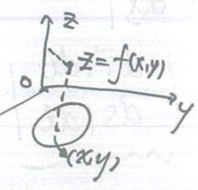
  $\mathbb{R}^1 = \mathbb{R}$ — set of all real numbers
  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ — all points on x-y plane
  ......

- Definition:
  functions of n-variables, Let D be a subset of $\mathbb{R}^n$ (domain of f)
  $f(x_1, x_2 \ldots x_n) = W$
  Range of f — a subset of $\mathbb{R}'$

Jargons:
① region in $\mathbb{R}^n$: subset of $\mathbb{R}^n$
② open ball centred at the point $\bar{P}$ $(\bar{x}_1, \bar{x}_2 \ldots \bar{x}_4)$ with $r > 0$: $\to$ Notation $O_r(\bar{p})$
set of all points $(x_1, x_2, \ldots x_4)$ such that $\sqrt{(x_1 - \bar{x}_1)^2 + \ldots + (x_4 - \bar{x}_4)^2} < r$
③ A point P in region R is call an interior point if $\exists$ an open ball $O_r(p) \subset R$
④ A point P in region R is call a boundary point if $\forall r > 0$, $O_r(p)$ contains points in R & points not in R
⑤ Region R is called to be open if every point in R is an interior point. (closed — if R contains all its boundary points)
⑥ R is called bounded — if R is contained in an open ball

$\mathbb{R}^2$ & $\emptyset$ are both open & closed.

- How to visualize graph of $z = f(x, y)$?
  Way I: $f(x, y) - z = 0$, in the x-y-z space



  Way II: level curve, $\forall c$ in Range of f
  consider curve $f(x, y) = c$ in x-y plane
  (level curves & level surfaces (for what $n=4$))

· §14.2 Limits & Continuity

- Definition: ~~f(x) has~~ $f(x_1, x_2, \ldots x_n)$ has limits "L" at $(x_1^\circ, x_2^\circ, \ldots x_n^\circ)$ (P)
  if for any abitrary ε, there exists $\delta > 0$ such that for any point in $O_\delta(P) \setminus P$ $(x_1', x_2', \ldots x_n')$
  $|f(x_1', x_2', \ldots x_n') - L| < \varepsilon$

  neighbor area without heart.

* $O(x^n; \delta)$ — meaning that for any y^n ($x^n, y^n \in \mathbb{R}^n$), $|y^n - x^n| < \delta$, it is an open ball

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

25

Remark:

1) No restriction on the way in which (x,y) approaches (a,b)

If $\exists$ 2 paths along which $f(x,y)$ has different limits, $f(x,y)$ has no limit at (a,b)

- Definition: $f(x,y)$ is continuous at $(x,y)=(a,b)$ if (a,b) is in Domain of f & $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Fact: All elementary functions are continuous in their domains respectively.

§ 14.3 Partial Derivatives

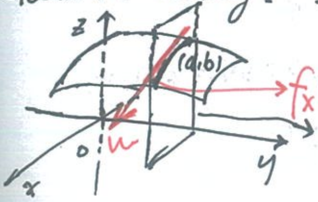
- $$\frac{\partial f}{\partial x}\Big|_{x=x_0} = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x} \quad \left(= \frac{df(x, y_0)}{dx}\Big/_{x=x_0} \right)$$

Notation: $f_x(a,b)$, $\frac{\partial f}{\partial x}\Big|_{\substack{x=a\\y=b}}$, $\frac{\partial f}{\partial x}(a,b)$, $D_x f(a,b)$

meanings of partial derivatives:

$\frac{\partial f}{\partial x}$: rate of change of f with respect to x at some points

Geometric meaning: $z = f(x,y)$



$\to f_x(x,y)$ at (a,b) (for $f_y(x,y)$ the same)

(an area parallel to xz-plane)

eg. $f(x,y) \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = 0$$

Caution:

Partial differentiabily does not imply continuity.

- Def. We say $f(x,y)$ is differentiable at (a,b) if there exists $f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b) * (x-a) + \frac{\partial f}{\partial y}(a,b) * (y-b) + o(\sqrt{(x-a)^2 + (y-b)^2})$ as $(x,y) \to (a,b)$

$$\left(\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + o(\sqrt{\Delta x^2 + \Delta y^2}) \right)$$

☆ $o(\Delta x) = o(\Delta y) = o(\sqrt{\Delta x^2 + \Delta y^2})$ when $\Delta x \to 0$, $\Delta y \to 0$

proof: $\Longrightarrow \lim_{\Delta x\to 0} \frac{o(1)\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} = \lim_{\substack{\Delta y \\ \Delta x\to 0}} o(1)\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} = 0$

(or $\lim_{\Delta x\to 0} \frac{o(1)\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$)

(26) differentiable $\Rightarrow$ continuity

$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f(a+\Delta x, b+\Delta y) = f(a,b)$ by the equation $\Rightarrow$ continuity

- Theorem: If f_x & f_y are continuous at (a,b), f is differentiabe at (a,b)

Proof: by MVT $\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y) + f(x_0, y_0+\Delta y) - f(x_0,y_0)$

$= f_x(x_0+\theta_1\Delta x, y_0+\Delta y)\Delta x + f_y(x_0, y_0+\theta_2\Delta y)\Delta y \quad \theta_1, \theta_2 \in (0,1)$

Because f_x, f_y continuous, $f_x(x_0+\theta_1\Delta x, y_0+\Delta y) = f_x(x_0, y_0) + O(1)$ (when $\sqrt{\Delta x^2+\Delta y^2}\to 0$, $O(1)\to 0$)

f_y still ...

$\Rightarrow \Delta f = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + O(\Delta x) + O(\Delta y)$

- High-Order partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2 f}{\partial x^2}(x,y) = f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial^2 f}{\partial y\,\partial x}(x,y) = f_{xy}(x,y)$$

Theorem: If f_{xy} & f_{yx} are both continous on (x_0,y_0) then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$

$\Phi(x) = f(x, y_0+\Delta y) - f(x, y_0)$

Proof: By MVT, $f_{xy}(x_0,y_0) = \lim_{\Delta y\to 0}\dfrac{f_x(x_0, y_0+\Delta y) - f_x(x_0,y_0)}{\Delta y}$

$\varphi(y) = f(x_0+\Delta x, y) - f(x_0, y)$

Consider $I = \dfrac{1}{\Delta x\Delta y}\left(f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) + f(x_0+\Delta x, y_0) - f(x_0,y_0)\right)$

$= \dfrac{1}{\Delta x\Delta y}\left(\Phi(x_0+\Delta x) - \Phi(x_0)\right) = \dfrac{\Phi'(x_0+\alpha_1\Delta x_1)}{\Delta y} = \dfrac{f_x(x_0+\alpha_1\Delta x, y_0+\Delta y) - f_x(x_0+\alpha_1\Delta x, y_0)}{\Delta y}$

$= f_{xy}(x_0+\alpha_1\Delta x, y_0+\alpha_2\Delta y)$, $0<\alpha_1,\alpha_2<1$, similarly $I = f_{yx}(x_0+\alpha_3\Delta x, y_0+\alpha_4\Delta y)$

$\overset{\text{continuity}}{\Rightarrow} f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$

- §·14.4 Chain Rule

- Suppose $z=f(x,y)$ is differentiable at (a,b), $x=x(u,v)$, $y=y(u,v)$ are partial differentiable at (u_0,v_0), $x_0 = x(u_0,v_0) = a$, $y_0 = y(u_0,v_0) = b$

$$\frac{\partial z}{\partial u}(u_0,v_0) = \frac{\partial z}{\partial x}(a,b)\frac{\partial x}{\partial u}(u_0,v_0) + \frac{\partial z}{\partial y}(a,b)\frac{\partial y}{\partial u}(u_0,v_0)$$

(For $\frac{\partial z}{\partial v}$, it is the same as above.)

$$\left[\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(\cdots)\frac{\partial x}{\partial u}(\cdots) + \frac{\partial z}{\partial y}(\cdots)\frac{\partial y}{\partial u}(\cdots),\ \text{if } x=g(t),\ y=f(t),\ \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right]$$

$O(1) \rightarrow$ a bounded function

(Rally of all these functions)

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial z}{\partial z}\right)}_{\text{gradient of } w} \cdot \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

($\frac{\partial f}{\partial l}$ → directional derivative

· §. 14.5 Directional derivatives & gradient vectors

☆ $\text{grad} f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$ (Remark) → or written as $\nabla f(x_0, y_0)$

for unit vector $\vec{v}$, $|\vec{v}| = 1$

$\frac{dx}{d\vec{v}}$: meaning that x changes along $\vec{v}$ direction. differential

$$\frac{d}{d\vec{v}} f(x_0, y_0) = \frac{\partial}{\partial x} f(x_0, y_0) \frac{dx}{d\vec{v}} + \frac{\partial}{\partial y} f(x_0, y_0) \frac{dy}{d\vec{v}}$$

directly $= \left\langle \frac{\partial}{\partial x} f(x_0, y_0), \frac{\partial}{\partial y} f(x_0, y_0) \right\rangle \cdot \vec{v}$

$$= \text{grad} f(x_0, y_0) \cdot \vec{v} = |\text{grad} f(x_0, y_0)| \cos\langle \text{grad} f(x_0, y_0), \vec{v} \rangle$$

$$\left(\frac{df}{d\vec{v}} = \lim_{\Delta\vec{v} \to \vec{0}} \frac{f(\vec{v} + \Delta\vec{v}) - f(\vec{v})}{|\Delta\vec{v}|} = \lim_{\Delta\vec{v} \to \vec{0}} \frac{f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|} \right)$$

directional derivative (方向導數) $\leq |\text{grad} f(x_0, y_0)|$ — change in the direction of unit vector $\vec{v}$

☆ In the direction of

- $-\text{grad} f(x_0, y_0)$ decreases
- $\text{grad} f(x_0, y_0)$ increases

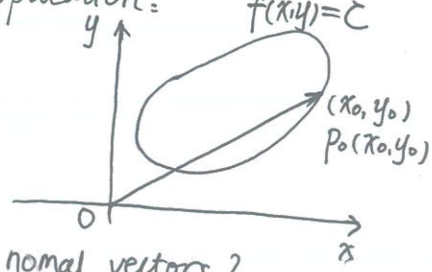
most rapidly at $P_0(x_0, y_0)$

$f(x_0, y_0)$

Properties:

- $\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g$ (linear)
- $\nabla(fg) = (\nabla f)g + (\nabla g)f$; $\nabla\left(\frac{f}{g}\right) = \frac{(\nabla f)g - (\nabla g)f}{g^2}$ $(g \neq 0)$ (product & quote)

• Application:



nomal vectors? (at P_0)

parametrize by $\vec{r}(t) = \langle x(t), y(t) \rangle$

$\Rightarrow f(\vec{r}(t)) = C$

$0 = \frac{df}{dt} \nabla f(\vec{r}(t)) \vec{r}'(t) \Big|_{t=0} = \nabla f(P_0) \cdot \vec{r}'(P_0)$

$\Rightarrow \nabla f(P_0) \perp \vec{r}'(P_0)$ tangent vector

nomal vector:

☆ $\nabla f(P_0)$ $(\text{grad} f(x_0, y_0))$

$$\left(\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \vec{r}(t) \right)$$

· §. 14.6 Tangent planes

• Tangent surface: (at t_0

$\vec{r}_0 = \vec{OP_0}$

Because $\vec{r}(t)$ on the plane, $f(\vec{r}(t)) = C$

$\Rightarrow \nabla f(P_0) \cdot r'(0) = 0$ (tangent vector) $\Rightarrow \nabla f(P_0)$ the nomal vector of ~~curve~~ to the plane

has no relationship with $\vec{r}$

$\nabla f(P_0) \perp$ all the tangent vector of arbitrary curves

(28)

The plane perpendicular to $\nabla f(p_0)$, passing p_0 is called the tangent plane (of p_0) of the original plane (曲面) curved. $[f_x(p_0)(x-x_0)+f_y(p_0)(y-y_0)+f_z(p_0)(z-z_0)=0]$ (equation)

vector form of normal line at p_0:

$\vec{r}=\vec{op_0}+t\nabla f(p_0)$

*Special case: $z=f(x,y)$, $p(x_0,y_0,f(x_0,y_0)) \Rightarrow$ Rewrite as $\underbrace{f(x,y)-z}_{g(x,y,z)}=0$

$\xrightarrow[\text{plane}]{\text{tangent}} z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

• Linearization of f at (x_0,y_0)

If f is differentiable at (x_0,y_0)

$f(x,y)\approx L(x,y)$, when $(x,y)\to(x_0,y_0)$

(By definition of differentiability) $\to f(x,y)=L(x,y)+\underbrace{O\sqrt{(x-x_0)^2+(y-y_0)^2}}_{\text{can be omit in approximation}}$

Accuracy: error $E(x,y)=f(x,y)-L(x,y)$ (Suppose f & all its partial derivatives (up to order 2) are continuous in a closed circle/rectangle centered at (x_0,y_0)

$\Longrightarrow$ then we have

$E(x,y)\le \frac{M}{2}(|x-x_0|+|y-y_0|)^2$ M s.t. $M=\max\{|f_{xx}|,|f_{xy}|,|f_{yy}|\}$ on the rectangle/circle)

$df=f_x(x_0,y_0)dx+f_y(x_0,y_0)dy \to$ "differential of f at (x_0,y_0)"

$df|_{(x_0,y_0)}\approx\Delta f$

• §. 14.7 Extreme values & saddle points

local / globa minimum/ maximum ......

• saddle point: (recall: critical point)

critical pt. { a point where $\nabla f(x_0,y_0)=\vec{0}$; a point where f is NOT differentiable

saddle pt: a critical pt. which satisfies $f(x_0,y_0)$ is NOT local extreme. $(x_0,y_0,f(x_0,y_0))$

• First derivative test: (at local minimum \ maximum, the horizontal tangent plane)

Suppose $f(x_0,y_0)$ is a local extreme value, then necessary condition: (and f is differentiable at (x_0,y_0); (x_0,y_0) is an interior point)

$f_x(x_0,y_0)=0=f_y(x_0,y_0)$ not sufficient

(proof. $g(x)=f(x,y_0)\Rightarrow g(x_0)$ is a local extreme, by 1st derivative test, $g'(x_0)=f_x(x_0,y_0)=0$)

Cautions: not every critical point yields a local extreme

eg. $z=f(x,y)=y^2-x^2$, $\nabla f(x,y)=\langle -2x,2y\rangle$. critical point $\langle 0,0\rangle$ (not a local extreme)

- Second derivative test :

Let (a,b) be a critical point of f & suppose all particial derivative of f (up to order 2) s.t. (***)

(These partials exists & continues on a neighborhood of (a,b))

Then (i) $f_{xx}(a,b) > 0$ & $\underbrace{f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)}_{\text{Hessian of } f \text{ at pt. } (a,b)\text{ / Discriminant}} > 0$ ($\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$) ← Hessian

$\Rightarrow f(a,b)$ is a local minimum.

(ii) $f_{xx}(a,b) < 0$ & $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) > 0$

$\Rightarrow f(a,b)$ is a local maximum.

(iii) $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) < 0 \Longrightarrow (a, b, f(a,b))$ is a saddle point.

(iv) $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b) = 0$ NO CONCLUSION

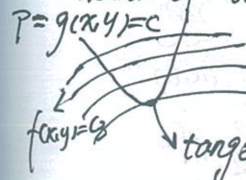
- Suppose f is continuous on closed & bounded region D,

f must has global maximum & global minimum over D

§14.8 Lagrange Mulitplier

Given $f(x,y)$, find extremes of f on a curve $\underset{(P)}{g(x,y)=c}$ (a restriction) $f|_P$ constrain

Idea : Think of temperature function as $f(x,y)$



If P is the location where $f|_P$ has local extreme value, then the two curves are tangent to each other at P.

$\Updownarrow$

normal vectors at P of these two all parallel

$\nabla g(x,y) \parallel \nabla f(x,y) \Longleftrightarrow \nabla g(x,y) = \lambda \nabla f(x,y)$ for some $\lambda \in \mathbb{R}$

Mathematical proof.

parametrize $P(g(x,y))$ as $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a < t < b$ $\quad \vec{r}(t_0) = (x_0, y_0)$ (P_0)

$\because f(p) = f(\vec{r}(t_0))$ is a local extreme value of f restrictly on P ($f|_P$)

$[f|_P = f(\vec{r}(t))]$ → as a function of t has local extreme at $t = t_0$

$$0 = \frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0} = \frac{d}{dt} f(\vec{r}(t))\Big]_{t_0} = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\Big]_{t=t_0} = \nabla f(P_0) \cdot \vec{v}(t_0)$$

$\nabla f(x_0, y_0)$ → (decrease / increases) most rapidly; reverse

$\Rightarrow (\nabla f)_{P_0} \perp \vec{v}(t_0)$, However $\vec{v}(t_0) \perp (\nabla g)_{P_0}$

$\Rightarrow (\nabla f)_{P_0} \parallel (\nabla g)_{P_0}$

$$\Rightarrow \text{Solve for } P_0(x_0, y_0) \begin{cases} g(x_0, y_0) = c \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \Rightarrow \begin{cases} f_x(x_0, y_0) = \lambda g_x(x_0, y_0) \\ f_y(x_0, y_0) = \lambda g_y(x_0, y_0) \end{cases} \end{cases}$$

(30)

λ — Lagrange multiplier

$$\varphi = f(x,y) - \lambda(g(x,y) - c) \Rightarrow \begin{cases} \varphi_x = 0 \\ \varphi_y = 0 \\ g^{\dots} = c \end{cases}$$ (Lagrange multiplier) method (general term)

• §.14.9 Taylor's theorem of two variables

Assume $f(x,y)$ & it's all partials up to order $n+1$ exists & are continuous in a closed rectangle R centered at point (a,b) $(a+h, b+k)$ in the rectangle

$$f(a+h, b+k) = f(a,b) + \underbrace{f_x(a,b)h + f_y(a,b)k}_{} + \underbrace{\frac{f_{xx}(a,b)}{2!}h^2 + \frac{2f_{xy}(a,b)}{2!}hk + \frac{f_{yy}(a,b)}{2!}k^2 + \cdots}_{} + \left(\frac{h\partial}{\partial x} + \frac{k\partial}{\partial y}\right)^{n+1}\frac{1}{(n+1)!}f(a+ch, b+ck)$$ (as below)

(Set $F(t) = f(a+th, b+tk)$, $-1 \le t \le 1$, $F, F', \cdots F^{(n+1)}$ exists & continues, single variable of Taylor's …

$$f(a+h, b+k) = F(1) = F(0) + F'(0)(1-0) + \cdots + \frac{F^{n}(0)}{n!}(1-0)^n + \frac{F^{n+1}(c)}{(n+1)!}(1-0)^{n+1} = f(a,b) + f_x(a+0, b+0)h\left(\text{Namely } h\cdot\frac{\partial f}{\partial x}\right)$$

(Simple proof)

$$+ f_y(a+0, b+0)k \left(k\frac{\partial f}{\partial x}\right) + \frac{1}{2!}\left[f_{xx}(a,b)h^2 + 2f_{xy}(a,b)kh + f_{yy}(a,b)k^2\right] + \cdots$$

☆ Taylor theorem:

Rewrite as

$$f(a+h, b+k) = f(a,b) + \sum_{m=1}^{n}\frac{\left(\frac{h\partial}{\partial x} + \frac{k\partial}{\partial y}\right)^m}{m!}$$ differential operator (微分算子)

$$f(a,b) + \frac{\left(\frac{h\partial}{\partial x} + \frac{k\partial}{\partial y}\right)^{n+1}}{(n+1)!}f(a+ch, b+ck)$$

where $c \in (0,1)$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b) \qquad \left(h^2\frac{\partial}{\partial x\partial x} + 2hk\frac{\partial}{\partial x\partial y} + k^2\frac{\partial}{\partial y\partial y}\right)f(a,b)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f^{(\dots)}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)g^{(\dots)}$$

Chapter 15 Multiple Integrals

• §.15.1 Double & iterated integrals

Motivations
(1) $z = f(x,y)$, defined on region $R \subset \mathbb{R}^2$, volume below the graph
(2) $\rho = f(x,y)$ — mass density function, how to implement?

$$f(x,y) = \lim_{\Delta A \to 0}\frac{\Delta m}{\Delta A} = \frac{dm}{dA}$$

want to find m (total mass) $= \int f(x,y)\,dA$

• §.15.2

special case: R is a rectangle on x-y plane $[a,b]\times[c,d]$

do split  $\to (x_k, y_k) \in R_k$ $\displaystyle\sum_{k=1}^{n}\lim_{\Delta A_k \to 0} f(x_k, y_k)\Delta A_k$

Do Riemann's sum, define $\displaystyle V = \lim_{\|P\| \to 0}\sum_{k=1}^{n} f(x_k, y_k)\underbrace{\Delta x_k \Delta y_k}_{\Delta A_k}$

P = partition of $R = \{R_1, R_2, \cdots R_n\}$

$\|P\| = \max\{D_1, D_2, \cdots D_n\}$ ($A \to$ area) ($D \to$ largest diameter) (Diameter, the longest line segment joining two dots.)

- Def. For general $f(x,y)$ defined on (rectangle) R.

  Double Integral on R is defined to be $\underbrace{\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k}_{\text{Riemann sum}}$

  (the limit is required to be independent of choice $(x_k,y_k)\in R_k$)

  If the limit exists, we say f is integrable on R

  (*) $\underbrace{\iint_R f(x,y)\,dA}$ (Notation) or $\underbrace{\iint_R f(x,y)\,dx\,dy} = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k$

- (*)Note:

  Not all functions are integrable.

  $f(x,y)=\begin{cases}0, & \text{irrational pt.}\\ 1, & \text{rational pt.}\end{cases}$ (Dericlet function of two variables)

  "$D(x)=\lim_{k\to\infty}\lim_{j\to\infty}(\cos(k!\pi x))^{2j}$" either 0 or 1

  Bounded & continuous functions on bounded regions (eg. rectangles) are integrable.

  (or piecewise continous / $\underbrace{\text{countable}}$ points or finite ~)
  pieces

  (Consider "Darboux sum")

- Non-rectangle Area R:

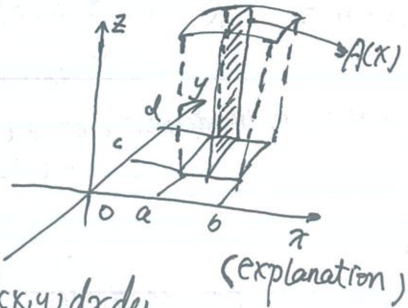
  If R is bounded, pick a large rectangle $\tilde{R}\supset R$.

  Extend f s.t. $\tilde{f}(x,y)=\begin{cases}f(x,y), & (x,y)\in R\\ 0, & (x,y)\in \tilde{R}\setminus R\end{cases}$

  $\therefore \iint_{\tilde{R}} \tilde{f}(x,y)\,dx\,dy = \iint_R f(x,y)\,dx\,dy$

- Compute $\iint_R f(x,y)\,dA$:

  (1) when R is a rectangle $[a,b]\times[c,d]$

  (explanation)

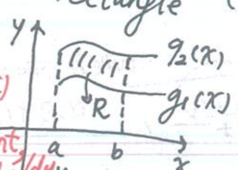
  $\iint_R f(x,y)\,dA = V \overset{\text{Calculus I}}{=} \int_a^b A(x)\,dx$

  $= \underbrace{\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy}$ → iterated integrals

  Fubini's theorem, special case

  (2) when R is not a rectangle (Fubini's theorem)

  special case I:

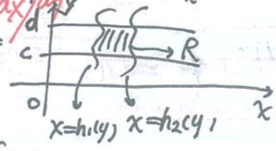
  

  do f on R

  $\iint_R f(x,y)\,dA = \int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx$

  If one of $g_1(x)$ / $g_2(x)$ or $h_1(y)$ / $h_2(y)$ is a constant, then we can exchange dx/dy

  special case II:

  $\iint_R f(x,y)\,dA = \int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$

  ($\iint_R dA = \iint_R 1\,dA$)

(32) General properties:

1) $\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

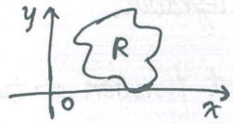
2) for constant c, $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$

3) Interiors of R_1 & R_2 do not intersect (only boundary pt. or not)

$$\iint_{R_1\cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

• §. 15.3

Q: area of an arbitrary region:

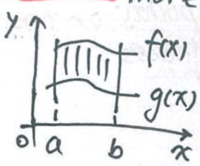


Take the density $f(x,y)\equiv 1$,

$$A = \iint_R (f(x,y)\equiv 1)\,dA = \iint_R 1\,dA$$

more general

$$A = \iint_R dA = \int_a^b \int_{g(x)}^{f(x)} dy\,dx$$



$$= \int_a^b f(x)\,dx$$ a special case with linear bounds

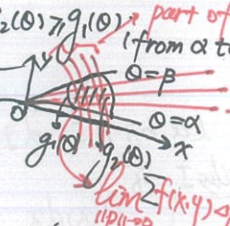
• §. average of f on the Region R: $\dfrac{\iint_R f\,dA}{A} = \bar{f}(x,y)$

15.4 Double integrals for polar co-ordinates

Q: $\iint_R f(x,y)\,dA$ $\quad R:\ 0\le \beta-\alpha\le 2\pi,\ g_2(\theta)\geq g_1(\theta)$ → part of circles (from α to β)

this region

Do Integration in the direction of $\theta\in[\alpha,\beta]$ (θ)



} partitions of Riemann's sum

$\lim\limits_{\|P\|\to\infty}\sum f(x,y)\Delta A$

$$\iint_R f(x,y)\,dA \Longleftarrow \Delta A = \tfrac{1}{2}\left((r+\Delta r)^2 - r^2\right)\Delta\theta = r\Delta r\Delta\theta + \tfrac{1}{2}\Delta r^2\Delta\theta$$

$dA = r\,dr\,d\theta$ $\quad$ (can be seen as a rectangular) $\quad \tfrac{1}{2}\Delta r^2\Delta\theta = O(\Delta r\Delta\theta)\to 0$

$$= \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

(as Calculus I shows)

$$\overset{\text{take apart}}{=} \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

eg. $$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy} = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy}$$

When should we use (r,θ)? R is circular

$$\overset{x=r\cos\theta}{\underset{y=r\sin\theta}{=}} \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta} = \sqrt{\tfrac{1}{2}\int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}\,d r^2} = \sqrt{\pi}$$

§. 15.5 Triple integrals

motivation: given mass density $f(x,y,z)$, want to know total mass

$$m=\iiint_D f(x,y,z)\,dV$$

(Cut D into tiny pieces)

$D_1, D_2 \cdots D_k$

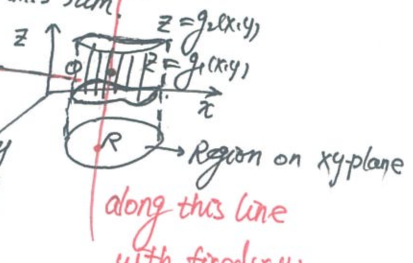
$\lim_{\|P\|\to 0}\sum f(x,y,z)\,\Delta V$ $\quad \Delta V = \Delta x\Delta y\Delta z$ as Riemann's sum of ⋯ exists & independent of the choice for (x_i, y_i, z_i) $\Rightarrow f(x,y,z)$ is integrable on D

$\left(m=\sum_{i=1}^{k}\Delta m_i \approx \sum_{i=1}^{k} f(x_i,y_i,z_i)\,\Delta V_i\right)$

"P" represents the largest diameter (Longest line segments)

Define: $m=\iiint_D f(x,y,z)\,dV$ to be the limit of Riemann's sum

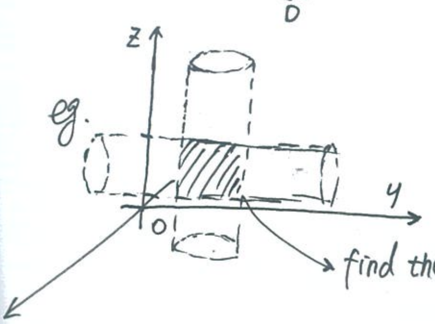
When D is an object as right:
(special case)



$$\iiint_D f(x,y,z)\,dV=\iint_R\left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right)dx\,dy$$

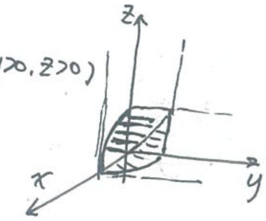
function of x & y
(fixed x & y)

eg.



find the volume of the intersection

$C_1: x^2+y^2\le 1,\ z\in\mathbb{R}$
$C_2: x^2+z^2\le 1,\ y\in\mathbb{R}$
⇓
consider first part $(x>0, y>0, z>0)$



$\left(m=\iiint_D f(x,y,z)\,dV \Longrightarrow V=\iiint_D dV\right)$

same as for area

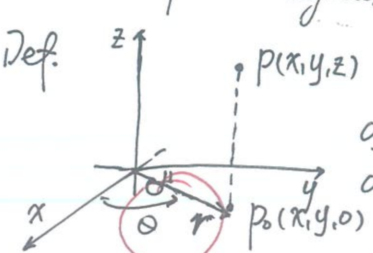
$$\begin{cases} R: x^2+y^2\le 1, & x,y\ge 0\\ g_2: z=\sqrt{1-x^2} \\ g_1: 0\end{cases}$$

$$\Rightarrow \frac{V}{8}=\iint_R\left(\int_0^{\sqrt{1-x^2}} dz\right)dA=\iint_R\sqrt{1-x^2}\,dA=\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy\,dx$$

$$=\frac{2}{3},\quad V=\frac{16}{3}$$

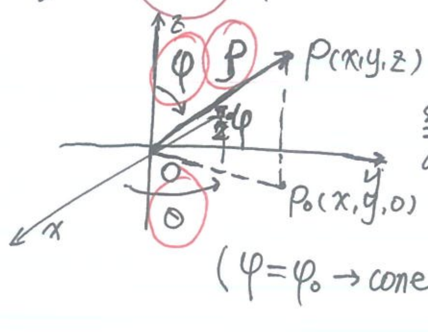
§. 15.7 Triple integrals in cylindrical coordinates & spherical coordinates

Def.



$P(x,y,z) \longrightarrow P(r\cos\theta, r\sin\theta, z)$
⇓
$\xrightarrow{\text{cylindrical coordinates}} P(r,\theta,z)$ of P

(r,θ): polar coordinates of P_0



$P(x,y,z) \longrightarrow P(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$

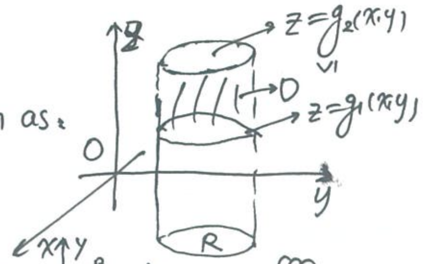
$\xrightarrow{\text{spherical coordinates}} P(\rho,\theta,\varphi)$ of P or (ρ,φ,θ)

"change of the meaning"!

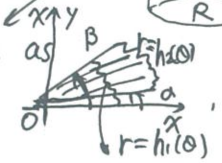
$0\le\varphi\le\pi$
$0\le\theta\le 2\pi$

$(\varphi=\varphi_0 \to \text{cone})$

Suppose D is given as:



Assume R is given as



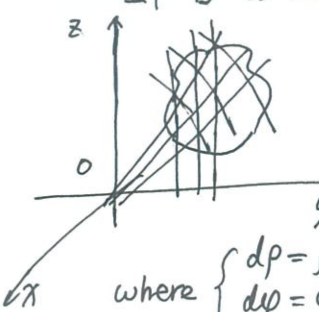
$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right) dA$$

$$\Rightarrow \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right) r\,dr\,d\theta$$

(integration on cylinderical coordinates)

Change to $\int_{g_1(r\cos\theta,\, r\sin\theta)}^{g_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz$ $\Rightarrow$ check whether appropriate

If D is not regular (strange).



use spheres $\rho=\rho_1, \rho_2, \cdots \rho_k$

& cones $\varphi = \varphi_1, \varphi_2, \cdots \varphi_m$

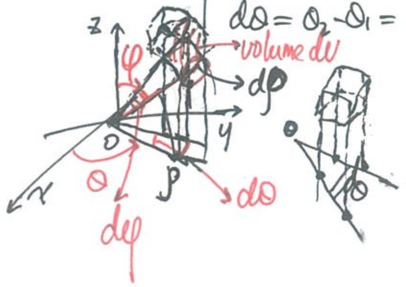
& half-planes $\theta=\theta_1, \theta_2, \cdots \theta_n$

where $\begin{cases} d\rho = \rho_2-\rho_1 = \rho_3-\rho_2 = \cdots = \rho_k-\rho_{k-1} \\ d\varphi = \varphi_2-\varphi_1 = \varphi_3-\varphi_2 = \cdots = \varphi_m-\varphi_{m-1} \\ d\theta = \theta_2-\theta_1 = \cdots \ \text{(infinitesimal)} \end{cases}$

$$\iiint_D f(x,y,z)\,\underline{dV}$$
volume of pieces

$$= \int_{\theta_1}^{\theta_2}\int_{\varphi=h_1(\theta)}^{\varphi=h_2(\theta)}\int_{\rho=g_1(\varphi,\theta)}^{\rho=g_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

(first, fixed φ & θ, change ρ, and then, fixed θ, change φ) ($\theta_1, \theta_2 \to$ shadow of D on x-y plane, range)



$$dv = (d\rho)\cdot(\rho\,d\varphi)\cdot(\rho\sin\varphi\,d\theta) = \underline{\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$$

approximate rectangular box (form like $(r\,d\theta)\cdot(dr)$)

§15.8 Substitution method for multiple integrals

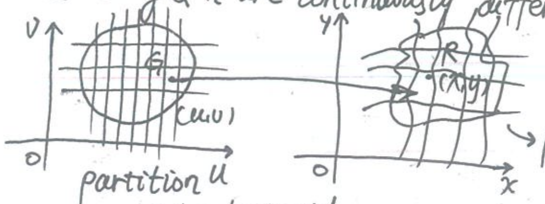
Given $\iint_R f(x,y)\,dx\,dy$, $\iiint_D f(x,y,z)\,dx\,dy\,dz$, want to do substitution

Suppose $\exists$ a mapping M: region G on uv plane $\to$ region R on xy plane

s.t. M is one-to-one and onto

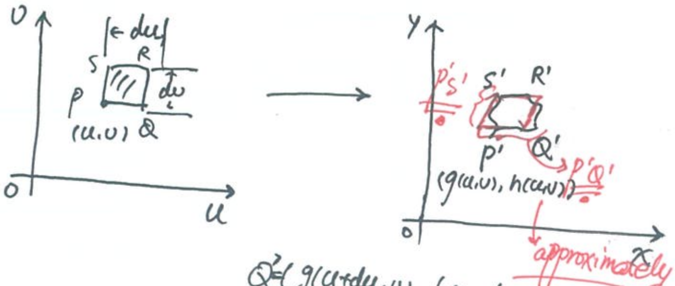
write $M(u,v) = (x,y) = (g(u,v), h(u,v))$

Assume g & h are continuously differentiable on G



partition G by horizontal & vertical lines

$\to$ partition according to the partition of G



$Q' = (g(u+du, v),\ h(u+du, v))$

$S' = (g(u, v+dv), h(u, v+dv))$

$dx\,dy = dA = S_{P'Q'R'S'}$

$\approx$ "= area of parallogram"

$= |\vec{P'Q'} \times \vec{P'S'}|$

$\approx \left| \langle \frac{\partial g}{\partial u} du, \frac{\partial h}{\partial u} du, 0 \rangle \times \langle \frac{\partial g}{\partial v} dv, \frac{\partial h}{\partial v} dv, 0 \rangle \right|$ at (u,v)

$= \left| \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\cdot\frac{\partial h}{\partial u} \right) du\,dv \right| \underset{at\,(u,v)}{=}$

$\left| \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} \right| du\,dv$

$\left(\left\| \begin{matrix} g_u & h_u \\ g_v & h_v \end{matrix} \right\| du\,dv \right)$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v)) \cdot \left\| \begin{matrix} g_u & h_u \\ g_v & h_v \end{matrix} \right\| du\,dv$$

Jacobian of M (not including "| |")

Another notation: $J(u,v) = \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \right| = \frac{\partial(x,y)}{\partial(u,v)}$

(not including | |)

absolute value

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v)) \cdot |J(u,v)|\,du\,dv = \iint_G f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

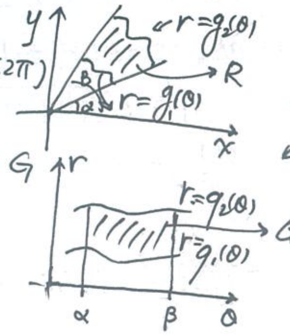
eg. Use the substitution method to ~~proof~~ polar coordinates substitution

$\iint_R f(x,y)\,dx\,dy$

$= \iint_G f(r\cos\theta, r\sin\theta) \left\| \begin{matrix} -r\sin\theta & r\cos\theta \\ \cos\theta & +\sin\theta \end{matrix} \right\| dr\,d\theta$ $(0 \le \beta - \alpha \le 2\pi)$

$= \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

(consider the boundary for substitution)



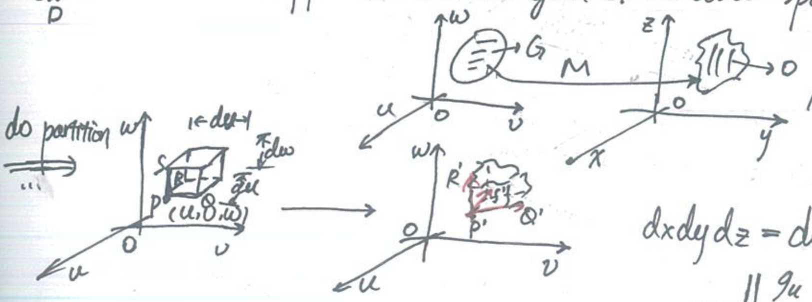
mapping

$x = r\cos\theta$

$y = r\sin\theta$

$M(\theta, r) = (r\cos\theta, r\sin\theta)$

$\iiint_D f(x,y,z)\,dx\,dy\,dz$ suppose a solid region G in uvw-space and mapping $M: G \to D$



$M \begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$ $M(u,v,w) = (x,y,z)$ one-to-one & onto interior of D

$dx\,dy\,dz = dV = |\vec{P'Q'} \cdot (\vec{P'S'} \times \vec{P'R'})|$

$= \left\| \begin{matrix} g_u & h_u & k_u \\ g_v & h_v & k_v \\ g_w & h_w & k_w \end{matrix} \right\| du\,dv\,dw$

$J(u,v,w)$ Jacobian of M

$= \left| \frac{\partial(g,h,k)}{\partial(u,v,w)} \right| du\,dv\,dw$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_G f(g(u,v,w), h(u,v,w), k(u,v,w)) \left| \frac{\partial(g,h,k)}{\partial(u,v,w)} \right| du\,dv\,dw$$

eg. volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(36)

$$V=\iiint_D dx\,dy\,dz=\iiint_G \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw=\iiint_G abc\,du\,dv\,dw \xlongequal[\text{sphere}]{\text{coordinates}} \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=abc\cdot\frac{4\pi}{3}$$

Chapter 16 Integrals on curves & surfaces

§. 16.1 Line integrals of scalar functions

Motivation: Give a wire whose density function is $\delta(x,y,z)$, want total mass m of the wire.

def of $\delta(x,y,z)$: $\frac{\Delta m}{\Delta \text{arc length}\,(\Delta s)}$ = average density.

Δm ≥ mass of the small piece ((x,y,z))

$\lim_{\Delta s\to 0}\frac{\Delta m}{\Delta s}=\delta(x,y,z)\approx\frac{\Delta m}{\Delta s}$, $\Delta s\to 0$

$\Delta m=\delta(x,y,z)\Delta s$, $\Delta s\to 0$

$\Longrightarrow \lim_{\|P\|\to 0}\sum_{i=1}^{n}\Delta m_i=\lim_{\|P\|\to 0}\sum_{i=1}^{n}\delta(x_i,y_i,z_i)\Delta s_i$

$$\boxed{m=\iiint_D f(x,y,z)\,ds} \xrightarrow{\text{compute}} \int_a^b f(\vec r(t))\,|\vec r\,'(t)|\,dt$$

$m \overset{\text{def}}{=} \int_C \delta(x,y,z)\,ds$ (C → curve (given))

$|\vec r\,'(t)|\,dt$ can be expressed as $=|d\vec r|$

step 1: parametrize curve C by $\vec r(t)=(x(t),y(t),z(t))$, $t\in[a,b]$

step 2: recall $ds=|\vec r\,'(t)|\,dt$

$=\sqrt{(dx)^2+(dy)^2+(dz)^2}$

Special case: $\delta(x,y,z)=1$, $m=\int_C 1\,ds=L$ (total arc length)

$=\int_a^b|\vec r\,'(t)|\,dt$

§. 16.2 vector fields & their line integrals

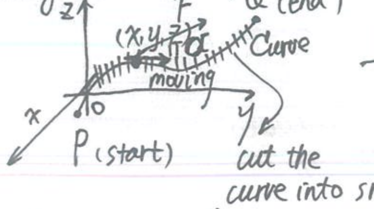
Motivation:

1) vector fields. (eg. velocity field, gravitational field)

2) tangent vector fields: $\vec T(x,y,z)$ tangent unit vector at every point.

3) gradient vector fields: $\nabla f=(f_x,f_y,f_z)$ $\nabla f(x,y,z)$ at (x,y,z) (every point - gradient)

Integrate vector fields on the curve C.



Force: $\vec F(x,y,z)$, want to calculate the work:

$$W=\int_C \vec F\cdot\vec T\,ds = \int_a^b \vec F(x(t),y(t),z(t))\cdot\vec T(x(t),y(t),z(t))\cdot|\vec r\,'(t)|\,dt=\int_a^b\vec F(\vec r(t))\cdot\vec r\,'(t)\,dt$$

parametrize

(make sure $\vec r(t)$ matches the orientation of T)

then, $W=\lim_{\|P\|\to 0}\sum_{i=1}^{n}|\vec F_i|\cos\theta_i\,\Delta l_i$ (Riemann's sum) (Δl_i: arc-length)

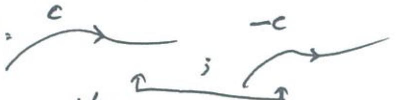
* $\int_C\vec F\cdot\vec T\,ds$ depends on the choice of $\vec T$ (unit tangent vector)

without knowing for the choice, can only determine the abs of $\int_C\vec F\cdot\vec T\,ds$.

Def.

A choice of direction of $\vec{T}$ on C is called orientation.

If a curve is given an orientation, say C is oriented.

Notation. 

the same curve with opposite orientations

$$\Longrightarrow \int_C \vec{F}\cdot\vec{T}\,ds + \int_{-C} \vec{F}\cdot\vec{T}\,ds = 0$$

(Others)

$\vec{F} = \langle f(x), g(y), h(z)\rangle$, $\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \left(f(x)\frac{dx}{dt} + g(y)\frac{dy}{dt} + h(z)\frac{dz}{dt}\right)dt = \int_C (f(x)dx + g(y)dy + h(z)dz)$$

$$= \int_C \vec{F}\cdot d\vec{r}$$

(consider as functions of x, y, z respectively for derivative)

($\int_C \vec{v}\cdot\vec{T}\,ds$ – flow of fluid along C, measuring intensity of motion)

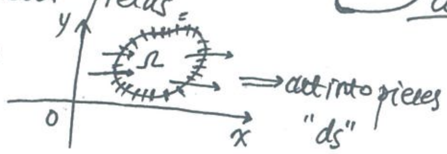
流量

when C is closed / loop: circulation $\oint_C \vec{v}\,d\vec{r}$

• Another type of line integrals of vector fields.

consider a fluid flowing on x-y-plane

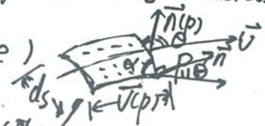
Suppose, $\delta(x,y)$ be the density function of fluid.



Motivation. the net rate at which fluid leaves Ω across its boundary

(consider 1 unit of time)

$|\vec{v}(p)|\,ds\cos\theta$



$(\cos\theta = \sin(\alpha))$

$\vec{n}(p)$ – unit outer normal vector of C at p (of the curve)

$= \delta(p)\vec{v}(p)\cdot\vec{n}(p)\,ds$ $\left(\frac{mass}{area}\right)$

$$\Rightarrow \int_{\Omega_2} \delta\vec{v}\cdot\vec{n}\,ds + \int_{\Omega_1} \delta\vec{v}\cdot\vec{n}\,ds \;\text{the rate}\; \overset{\Delta}{=} \underset{\text{flux 通量}}{=} \int_{\Omega} \delta\cdot\vec{v}\cdot\vec{n}\,ds$$

outer vector   inner vector

outer flux of $\vec{v}\cdot\delta$ across Ω

Compute general flux integral of f:

$$\oint_C \vec{F}\cdot\vec{n}\,ds \xrightarrow[\substack{a\le t\le b\\ \text{with counter-clockwise orientation (make sure)}}]{C=\vec{r}(t)=\langle x(t),y(t),0\rangle} \int_a^b \frac{\vec{F}(\vec{r}(t))\cdot\left\langle\frac{dy}{dt}, -\frac{dx}{dt}\right\rangle}{\left|\left\langle\frac{dy}{dt}, -\frac{dx}{dt}\right\rangle\right|}|\vec{r}'(t)|\,dt = \int_a^b \vec{F}(\vec{r}(t))\cdot\left\langle\frac{dy}{dt}, -\frac{dx}{dt}\right\rangle dt$$

closed curve

write $\vec{F}(\vec{r}(t))$ as $\langle f(y), g(x)\rangle$ (same as the above one)

$$\Rightarrow \int_a^b \left(f(y)\frac{dy}{dt} - g(x)\frac{dx}{dt}\right)dt = \oint_C (f(y)dy - g(x)dx)$$

• §. 16.3 Path independence, conservative fields

Goal: easy formula to compute $\int_C \vec{F}\cdot d\vec{r}$ for special $\vec{F}$

Motivation: suppose $\vec{F}(x,y,z) = \nabla \underbrace{f(x,y,z)}_{\text{scalar function}}$

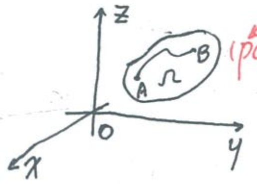
$$\int_C \vec{F}\,d\vec{r} = \int_C \nabla f(x,y,z)\frac{d\vec{r}}{dt}\,dt = \int_a^b \nabla f(\vec{r}(t))\frac{d\vec{r}}{dt}\,dt$$

$$\overset{\text{chain Rule}}{=} \int_a^b \frac{d f(\vec{r}(t))}{dt}\,dt \overset{\text{Fundamental Theorem}}{=} f(\vec{r}(t))\Big]_a^b = f(\vec{r}(b)) - f(\vec{r}(a))$$

(38) Because $\vec{r}(b)$ & $\vec{r}(a)$ are not relevant to path,
we call it path independence ($\int_C \nabla f(x,y,z) d\vec{r}$ – path independent)
If C is a loop, then $\oint_C \nabla f dr = 0$, $(\vec{r}(a)=\vec{r}(b))$

- Jargons: If $\vec{F} = \nabla f$, f is called potential function of $\vec{F}$.
  If $\int_C \vec{F}\cdot d\vec{r}$ is path independent, the say $\vec{F}$ is conservative.

Theorem: (i) Any gradint vector field is conservative. (conservative $\supseteq$ gradint vector fields)
(ii) The converse is true in the region Ω if Ω is connected.



meaning that (paths inside the Ω)

We say Ω is connected if $\forall$ pt. A & B, $\exists$ a path C connecting A & B s.t. C is completely included in Ω.

- Question: Given $\vec{F}(x,y,z)$, how to determine whether it is conservative?

necessary conditions: let $\vec{F} = \langle M, N, P \rangle$
then, $M_y = f_{xy} = N_x$, $M_z = f_{xz} = P_x$, $N_z = f_{yz} = P_y$

sufficient conditions: (counter example) $\vec{F}(x,y,z) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$ ⇑ x-y plane without origin

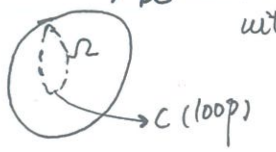
$M_y = N_x = \frac{y^2-x^2}{(x^2+y^2)^2}$, $M_z = N_z = P_x = P_y = 0$

However, $\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\langle -\sin t, \cos t \rangle dt = \int_0^{2\pi} \frac{y\sin t + x\cos t}{x^2+y^2} dt = \int_0^{2\pi} 1\, dt \neq 0$

meaning that $\vec{F}(x,y,z)$ is not conservative.

$M_y = N_x$, $M_z = P_x$, $N_z = P_y$ is sufficient provided region Ω is simply connected.

★ Ω is said to ~~by~~ be simply connected if any loop in Ω can be shrunk into a point (can go any way (not necessarily straight)) without ever leaving Ω. (a loop and its content in ...)



(gradint vector fields $\supseteq \nabla\times\vec{F} = \vec{0}$)

Define curl of $\vec{F} = \underset{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}{\nabla} \times \vec{F}$ (cross product) $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

$= (P_y - N_z)\vec{i} + (P_x - M_z)\vec{j} + (-M_y + N_x)\vec{k}$ ⟹ necessary condition ⟺ $\nabla\times\vec{F} = \vec{0}$

Recall $\int_C \vec{F} d\vec{r} = \int_C (M dx + N dy + P dz) \underset{\substack{\Uparrow \\ \vec{F} \text{ is conservative}}}{=} \int_C f_x dx + f_y dy + f_z dz = \int_C d f(x,y,z)$

Def. $M dx + N dy + P dz$ – differential form & if $\exists f(x,y,z)$ s.t. $df(x,y,z) = M dx + N dy + P dz$, then the differential form is exact.

Range: conservative > gradint vector fields $\nabla\times\vec{F} = 0$ ... <

eg. How to find f given $\nabla f = \vec{F}$

$\int_A^B (Mdx + Ndy + Pdz)$, $(\vec{F} = \nabla f = \langle M, N, P \rangle)$

get the answer for $C(y,z)$ (last two functions solving)

$$\begin{cases} f(x,y,z) = \int \underbrace{f_x(x,y,z)}_{M} dx + C(y,z) \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial(\int M dx)}{\partial y} + \dfrac{\partial C(y,z)}{\partial y} = N \\ \dfrac{\partial f}{\partial z} = \dfrac{\partial(\int M dx)}{\partial z} + \dfrac{\partial C(y,z)}{\partial z} = P \end{cases}$$

§16.4 Green's Theorem.

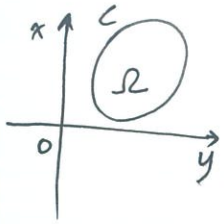
Recall: (the method of computing line integral)

$$\begin{cases} \int_C \vec{F} d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\ \int_C \nabla f \cdot d\vec{r} = f(B) - f(A). \text{ (when gradient vector fields)} \end{cases}$$

when $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$

- Green's Theorem:

Let C be a simple, closed curve on xy-plane, oriented counter-clockwise.
(not self-crossed)

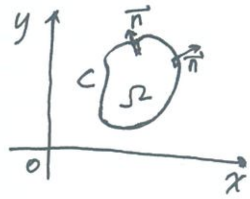


Suppose $M(x,y)$ & $N(x,y)$'s first order partials are continuous on $\Omega \cup C$, then

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C Mdx + Ndy = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

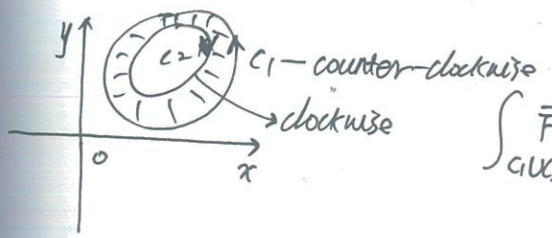
Extended: how to do integrals on annulus?

- Green's Theorem for flux ($\oint_C \vec{F} \cdot \vec{n}\, ds$) — "corollary"



$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C (Mdy - Ndx) = \iint_\Omega \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy$$

Recall $\langle -N, M \rangle \langle dx, dy \rangle$ (the form of Green's theorem)

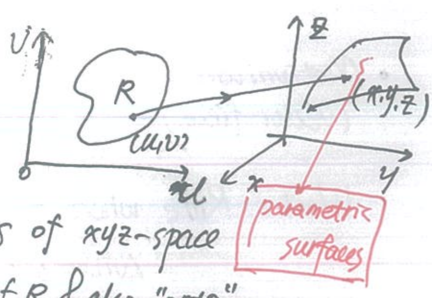


$$\int_{C_1 \cup C_2} \vec{F} \cdot d\vec{r} = \iint_\Omega (N_x - M_y) dx dy$$

§16.5 Surfaces & Areas

Recall $\begin{cases} z = f(x,y) \\ f(x,y,z) = 0 \end{cases}$ (represent surfaces)

- Parametrize Curves & Surfaces: Let R be a region on uv-plane



Suppose $\exists$ a mapping from R to subset S of xyz-space such that M is "one-to-one" in the interior of R & also "onto"

$(x,y,z) = (f(u,v), g(u,v), h(u,v))$, f, g, h are continous on R

(40) • vector formular for parametric surfaces:

$\vec{r}(u,v) = < f(u,v), g(u,v), h(u,v) >$, $(u,v) \in R$ (parametric domain)

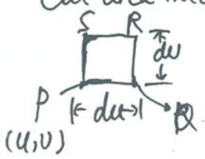
$\begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$

eg1. $z = f(x,y) \Rightarrow \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$ $(u,v) \in R$

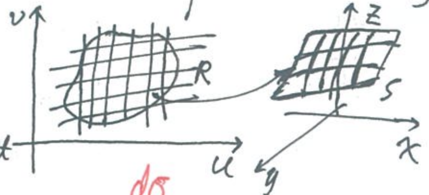
eg2. $x^2+y^2+z^2=a^2 \Rightarrow$ sphere coordinate ($\rho = a$)

• Area for general surfaces:

Cut area into pieces (partition)



→ can be seen as tangent plane.



$d\sigma$

$\approx$ area of parallelogram $|\vec{P'Q'} \times \vec{P'S'}| = |\vec{r}_u \times \vec{r}_v|\, du\, dv$

$\vec{P'Q'} = \vec{r}(u+du, v) - \vec{r}(u,v)$, $\vec{P'S'} = \vec{r}(u, v+dv) - \vec{r}(u,v)$ Area differential

$\approx \vec{r}_u(u,v)\, du$ $\qquad \approx \vec{r}_v(u,v)\, dv$

Area of parametric surfaces: $\iint_R |\vec{r}_u \times \vec{r}_v|\, du\, dv = A$ (→ normal vector)

* Special case: $z = f(x,y)$, $(x,y) \in R$

$\Rightarrow \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$ $A = \iint_R |<1,0,f_u> \times <0,1,f_v>|\, du\, dv = \iint_R \sqrt{f_u^2+f_v^2+1}\, du\, dv$

$= \iint_R \sqrt{\underbrace{f_x^2+f_y^2}_{|\nabla f|^2}+1}\, dx\, dy$

• §. 16.6 Surface Integrals

• Motivation (scalar). Let $\delta(x,y,z)$ be the mass density function on surface S

$\delta(x,y,z) = \frac{\Delta m}{\Delta \sigma}$ (Δm → mass ..., $\Delta\sigma$ → area of tiny pieces) "areal density"

defined when $\Delta\sigma \to 0$

cut into tiny pieces — do Riemann's sum $\lim_{\|P\| \to 0} \sum_{i=1}^{n} \delta_i \Delta\sigma_i$

$m = \iint_S \delta(x,y,z)\, d\sigma \underset{\text{parametrize}}{=} \iint_R \delta(f(u,v), g(u,v), h(u,v)) \cdot |\vec{r}_u \times \vec{r}_v|\, du\, dv$ $\quad = \delta(\vec{r}(u,v))$

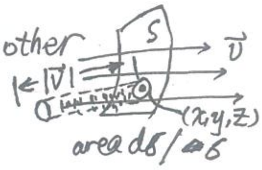
↑ how to compute

Special Case (when $z = f(x,y)$)

$m = \iint_R \delta(x, y, f(x,y)) \cdot \sqrt{1 + |\nabla f|^2}\, dx\, dy$ (R — QR surface)

• Motivation (vector field): Let $\vec{v}(x,y,z)$ be velocity vector field of a fluid flowing xyz-space, $\delta(x,y,z)$ be mass density of the fluid. ($\frac{\text{mass}}{\text{volume}}$)

Want: Rate where fluid mass crosses S from one side to the other



$\Delta m = \delta(x,y,z) \cdot (\vec{v}(x,y,z) \cdot \vec{n}(x,y,z))\, \Delta\sigma$

"unit normal vector"

$R=\sum \frac{\Delta m}{\Delta t} = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \delta \cdot \vec{v}(x_i,y_i,z_i)\cdot \vec{n}(x_i,y_i,z_i)\Delta \sigma$ (Riemann's sum)

$\Rightarrow R = \iint_S \delta\cdot\vec{v}(x,y,z)\vec{n}(x,y,z)\,d\sigma = \iint_S \delta\vec{v}\cdot\vec{n}\,d\sigma \xlongequal{\text{parametrize}} \iint_R \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$

(flux integral of $\delta\vec{v}$) ($\iint_S \vec{F}\cdot\vec{n}\,d\sigma$) for computing "depends on choice of $\vec{n}$"

($\vec{n}$ is chosen s.t. it matches the intended direction.)

"make sure $\vec{r}_u\times\vec{r}_v$ consistent with orientation of S"

"$\vec{r}_u\times\vec{r}_v$ is a normal vector of S."

Def. a choice for $\vec{n}$ on the surface S determines an "orientation" of S.
unit normal vector

special: a surface without orientation  (Möbius Surface) just one side

Special Case: "S: $z=f(x,y)$" $(x,y)\in R$

$\iint_S \vec{F}\times\vec{n}\,d\sigma \xlongequal{\text{paremetrize}} \iint_R \vec{F}(x,y,f(x,y))\cdot$ "$\pm$" $\langle f_x, f_y, 1\rangle\,dx\,dy$ choose sign every time

• §. 16.7 Stoke's Theorem.

Goal: generalize Green's Theorem to 3D case.

☆ Let S be an oriented surface (3-D space) with a boundary — a closed curve C with orientation of C & S s.t. the "right-hand-rule" is satisfied. (circular the right hand of orientation of C, thumb points to ... of S.) orientation of S (direction of $\vec{n}$)

Let $\vec{F}=\langle M,N,P\rangle$ (all functions about x,y & z)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma \quad \text{(3D case)}$$

• Special case: $\Omega = S$, $\vec{F}=\langle M,N,0\rangle$, $\oint_C \vec{F}\cdot d\vec{r} = \iint_\Omega (\nabla\times\vec{F})\vec{n}\,d\sigma$ ($\vec{n}$ & orientation of circulation satisfy the right-hand rule.)

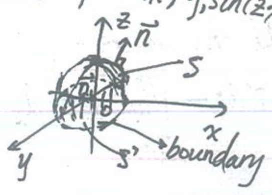
$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\vec{k}$, $\vec{n}=\vec{k}$

$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \underbrace{d\sigma = dx\,dy}_{\text{(2D space)}}$ — Green's Theorem

(When working on real problems, pay attention to the choice of S given C.)

eg. S — upper unit hemisphere oriented by outer normal vectors $\vec{F}=\langle x^2, y^2, \sin(z^2xy)\rangle$

$\underset{\downarrow\ \text{surface}}{\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma} \xlongequal[\text{Stoke's Theorem}]{\text{a trick}} \underset{\downarrow\ \text{unit disk}}{\iint_{S'} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma}$



$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & \sin(z^2xy) \end{vmatrix} = z^2x\cos(z^2xy)\vec{i} - z^2y\cos(z^2xy)\vec{j}$, $\vec{n}=\vec{k}$, $\Rightarrow \iint_{S'} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{S'} 0\,dx\,dy = 0$

(think S': $f(x,y)=z=0$, over $R=S'=x^2+y^2=1$)

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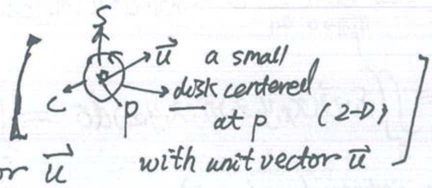
Meaning of curl ($\nabla \times \vec{F}$):



Fixed point $P=(x_0, y_0, z_0)$ & a unit vector $\vec{u}$

Take $S=$ disk, $C=$ circle $\dfrac{\oint_C \vec{F}\cdot d\vec{r}}{\text{Area of } S} = \dfrac{\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma}{\text{Area of } S}$ (Area arbitrary) $=\iint_S 1\, d\sigma$

shrink $S \to$ point P circulation

$\Rightarrow \lim\limits_{\text{area of } S\to 0} \dfrac{\oint \vec{F}\cdot d\vec{r}}{\text{area of } S} = \lim\limits_{\text{area of } S\to 0} \dfrac{\iint_S (\nabla\times\vec{F})\cdot\vec{n}\, d\sigma}{\text{area of } S} = \left(\nabla\times\vec{F}(x_0,y_0,z_0)\right)\cdot\vec{u}$

circulation density at (x_0, y_0, z_0)

circulation density at $P(x_0, y_0, z_0)$ attaches maxmium when $\vec{u}$ in direction of $\nabla\times\vec{F}(P)$

1) $\nabla\times\vec{F}(P)$ gives the direction of axis around which the fluid rotates the fastest.
(just $\vec{n}$'s direction)

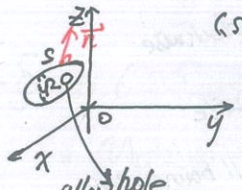
2) $|\nabla\times\vec{F}(P)|$ gives the maximium intensity of the rotation at P

• §. 16.8 Divergence Theorem

~~Addition: Green~~ "$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_\Omega \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right) dx\,dy$"

Goal: generalize Green's Theorem (flux form) to 3-D vector fields.

※ Let Ω be a bounded region in xyz-space, with boundary (surface) S.
(smooth or piece-wide smooth). $\vec{F}(x,y,z)=\langle M, N, P\rangle$ functions of x, y & z



$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_\Omega \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right) dv$$

unit outer normal vector field of S

allow hole.

inner boundary & outer normal

Def. the divergence of $\vec{F}$ is define to be $\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}$

$\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle\cdot\langle M, N, P\rangle = \nabla\cdot\vec{F} \longrightarrow$ notation of divergence of $\vec{F}$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_\Omega \nabla\cdot\vec{F}\, dx\,dy\,dz$$

eg. what if S: upper hemisphere of $x^2+y^2+z^2=1$ (not closed)

solution: add the lower disk "$x^2+y^2\leq 1$ with $z=0$"

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_{\Omega\cup D} \nabla\cdot\vec{F}\, dx\,dy\,dz - \iint_D \vec{F}\cdot\vec{n}\, d\sigma$$

$\to \begin{cases}\vec{n}=-\vec{k} \\ z=0\end{cases}$ & it is easy to compute

- Physical meaning of <u>divergence</u> (散度) of $\vec{F}$.

Fix P at $<x_0, y_0, z_0>$; S_a: a surface of a ball Ω_a, $\Omega_a \to P$

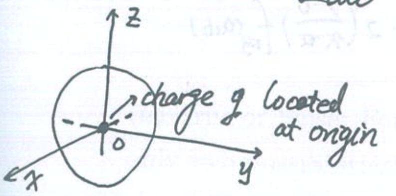
$$\lim_{\substack{V_{\Omega_a}\to 0\\(\text{or } a\to 0)}} \frac{\iint_{S_a} \vec{F}\cdot\vec{n}\,d\sigma}{\text{Volume of } \Omega_a} = \lim_{\substack{V_{\Omega_a}\to 0\\(\text{or } a\to 0)}} \frac{\iiint_{\Omega_a} \nabla\cdot\vec{F}\,dv}{\iiint_{\Omega_a} 1\,dv \ (\text{or Volume of } \Omega_a)} = \lim_{\substack{V_{\Omega_a}\to 0\\(\text{or } a\to 0)}} \operatorname{avg}(\nabla\cdot\vec{F})$$

$$= \left(\nabla\cdot\vec{F}\right)\Big|_{\text{at } P} \text{ or } \nabla\cdot\vec{F}(P)$$

→ flux density at $P(x_0, y_0, z_0)$

Sign of $\nabla\cdot\vec{F}(P)$:
- >0, P is source or fluid in expanding.
- $=0$, P is neither the above nor below. or fluid is <u>incompressible</u>.
- <0, P is sink or fluid in contracting.

- Addition. Gauss' Law



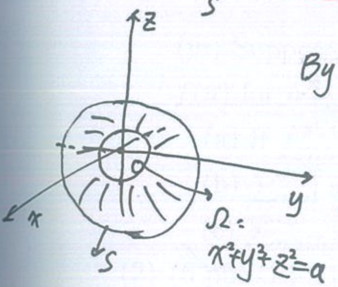
Electric vector field E generated by q

$$E(x,y,z) = < x\cdot\frac{q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}},\ y\cdot\frac{q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}},\ z\cdot\frac{q}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}} >$$

$$= \frac{q\cdot <x,y,z>}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{q\cdot\vec{r}}{4\pi\varepsilon_0 r^{\frac{3}{2}}}$$

Let S be a closed surface with q inside S.

$$\underline{\iint_S \vec{E}\cdot\vec{n}\,d\sigma = \frac{Q}{\varepsilon_0}} \quad \text{Gauss Law}$$



By divergence theorem,

$$\iint_{S\cup S_\Omega} \vec{E}\cdot\vec{n}\,d\sigma = \iiint_{\Omega} \underbrace{\nabla\cdot\vec{E}}_{0}\,dx\,dy\,dz = 0$$

$$\| $$

$$\iint_S \vec{E}\cdot\vec{n}\,d\sigma + \iint_{S_\Omega} \vec{E}\cdot\vec{n}\,d\sigma$$

$$-\iint_{S_\Omega} \vec{E}\cdot\underbrace{\vec{n}}_{\text{inner normal}}\,d\sigma = \iint_{S_\Omega} \frac{q\ <x,y,z>}{4\pi\varepsilon_0 (x^2+y^2+z^2)^{\frac{3}{2}}}\cdot\frac{<x,y,z>}{\sqrt{x^2+y^2+z^2}}\,d\sigma = \iint_{S_\Omega} \frac{q}{4\pi\varepsilon_0 a^2}\,d\sigma = \frac{q}{4\pi\varepsilon_0 a^2}\cdot \overset{=4\pi a^2}{S_\Omega} = \frac{q}{\varepsilon_0}$$

$\square$

- Laplace Equation. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \Leftrightarrow f$ is harmonic.

(properties) $\Rightarrow \iint_S \nabla f\cdot\vec{n}\,d\sigma = 0$; $\iint_S f\nabla f\cdot\vec{n}\,d\sigma = \iiint_D |\nabla f|^2\,dx\,dy\,dz$

- Green's first identity. $\iint_S f\nabla g\cdot\vec{n}\,d\sigma \xlongequal{\forall \text{ smooth } f \,\&\, g} \iiint_D (f\nabla^2 g + \nabla f\nabla g)\,dx\,dy\,dz$

(proved by div. theorem)

- Fact: $\nabla\cdot(\nabla\times\vec{F}) = 0$. Proof $\nabla\cdot(\nabla\times\vec{F}) = <\partial_x, \partial_y, \partial_z>\cdot<\frac{\partial N}{\partial z} - \frac{\partial P}{\partial y}, \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}, \frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}>$

$$= \cdots = 0$$

44 Additions.

- Proof of Second derivative test:

Taylor's theorem: $f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\left(\frac{\partial}{\partial x}\cdot(x-a) + \frac{\partial}{\partial y}\cdot(y-b)\right)^2 f(h,k)$

h between a & x, k between b & y (for (x,y) close to (a,b))

Because (a,b) is a critical point, $f_x(a,b) = f_y(a,b) = 0$,

$\Rightarrow f(x,y) = f(a,b) + \frac{1}{2}\left[f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\right]_{\text{at pt. }(h,k)}$

Consider $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{\text{at }(a,b)} > 0$ & $f_{xx}(a,b) > 0$:

Assume $x-a \neq 0$

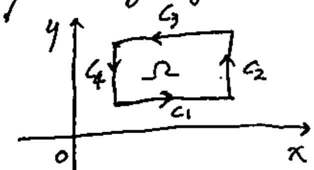
$\left((x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right)_{\text{at }(a,b)} = (x-a)^2\left(\left(\frac{y-b}{x-a}\right)^2 f_{yy}(a,b) + 2\left(\frac{y-b}{x-a}\right)f_{xy}(a,b) + f_{xx}(a,b)\right)$

because $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$ & $f_{xx} > 0$

$\Delta = 4f_{xy}^2(a,b) - 4f_{xx}(a,b)f_{yy}(a,b) < 0$.

Then, $f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2 > 0$. Continuity...

- Proof of Green's Theorem in rectangular regions / circularly regions



$\oint_{\partial\Omega} M(x,y)dx + N(x,y)dy = \sum_{i=1}^{4}\int_{C_i} Mdx + Ndy$

$(M=P, N=Q)$

$-\iint_D \frac{\partial P}{\partial y}dxdy = -\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}dy = -\int_a^b \left(P(x,\varphi_2(x)) - P(x,\varphi_1(x))\right)dx$

(for C_i: $a \le x \le b$, $\varphi_1(x) \le y \le \varphi_2(x)$)

$\oint_L Pdx = \int_a^b P(x,\varphi_1(x))dx + \int_b^a P(x,\varphi_2(x))dx \quad \Rightarrow \oint_L Pdx = \iint_D -\frac{\partial P}{\partial y}dxdy$

Similarly, $\oint_L Qdy = \iint_D \frac{\partial Q}{\partial x}dxdy$

MAT 1002-T13 Fall 2020 - Quiz 1, 20 Minutes

Show your work unless otherwise instructed.

(1) (9 Points) **Short questions** (no need to show your work for this problem)

(i) True or False: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

True *False*

(ii) True or False: For a nonnegative series $\sum_{n=1}^{\infty} a_n$ (i.e., every a_n is nonnegative), there are only two possibilities: either the series converges or diverges to ∞.

True *False*

(iii) Suppose that $f(x)$ is continuous, positive and strictly decreasing on $[1, \infty)$. Let $a_n = f(n)$ for $n \geq 1$. Which of the following statements is UNTRUE?

(a) If $\int_{2020}^{\infty} f(x)\,dx$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

(b) $\sum_{n=1}^{2020} a_n \geq \int_{1}^{2021} f(x)\,dx$.

(c) $\sum_{n=2020}^{\infty} a_n \leq \int_{2020}^{\infty} f(x)\,dx$.

(2) (6 points) Find the limit:

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right).$$

(3) (6 pints) Express the number $0.\bar{d} = 0.ddd\cdots$, where d is an integer between 1 and 9, as the ratio of two integers.

(4) (12 pints) Determine whether each of the following series converges or diverges.

(i) $\sum_{n=1}^{\infty} \frac{n!}{1000^n}$.

(ii) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.1}}$.

(5) (8 pints) Let $x_0 = 1$, $x_{n+1} = x_n - \frac{\tan x_n - 1}{\sec^2 x_n}$ for $n \geq 0$. Prove that the sequence $\{x_n\}$ converges to $\pi/4$.

MAT 1002-T15 Fall 2020 - Quiz 2, 15 Minutes

Show your work unless otherwise instructed.

(1) (6 Points) **Short questions** (no need to show your work for this problem)

(i) Absolute convergence of a series implies the convergence of the series.

True *False*

(ii) True or False: Consider an alternating series $\Sigma_{n=1}^{\infty}(-1)^{n+1}b_n$, where b_n decreases to 0 as n increases to infinity. Let S_n be the n-th partial sum of the series, then

$$|\Sigma_{n=1}^{\infty}(-1)^{n+1}b_n - S_n| \leq b_n.$$

True *False*

(2) (30 pints) Determine if the following series converge absolutely or conditionally or diverge.

(i) $\Sigma_{n=1}^{\infty} \frac{\sin n}{n^2}$.

(ii) $\Sigma_{n=1}^{\infty} \tan(\frac{1}{n^2})$.

(iii) $\Sigma_{n=1}^{\infty}(1-\frac{1}{n})^n$.

(iv) $\Sigma_{n=1}^{\infty}\frac{n3^n(n+1)!}{2^n n!}$.

(v) $\Sigma_{n=1}^{\infty}(-1)^n\frac{\ln n}{n-\ln n}$.

MAT 1002-T15 Fall 2020 - Quiz 3, 20 Minutes

Show your work unless otherwise instructed.

(1) (12 Points) **Short questions** (no need to show your work for this problem)

(i) Suppose power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = 1231$. Which of the following statement is correct?

(a) The radius R of convergence of the power series satisfies $R \geq 1231$, and the power series must converge also at $x = -1231$;

(b) the power series must converge absolutely at $x = 1231$;

(c) the power series must converge absolutely at $x = -1230$.

(ii) True or False: It is possible to expand $|\sin x|$ as a power series $\sum_{n=0}^{\infty} c_n x^n$ for x near 0.

By differentiability

True *False.*

(iii) True or False: Let $f(x)$ be infinitely differentiable on the interval $(-1, 1)$, and assume that the Taylor series of f about $x = 0$ converges for all $x \in (-1, 1)$. Then $f(x) =$ the Taylor series for all $x \in (-1, 1)$.

True *False.*

(iv) State Taylor's Theorem.

(2) (10 pints) Find the interval of convergence of the following power series. For what values of x does the series converge absolutely? conditionally?

$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n x^n}{3n}$.

(3) (10 pints) Find

$$\lim_{x \to 0} \frac{1 - \cos \sqrt{\tan x - \sin x}}{\sqrt[3]{1 + x^3} - \sqrt[3]{1 - x^3}}.$$

MAT 1002-T15 Fall 2020 - Quiz 4, 15 Minutes

Show your work unless otherwise instructed.

(1) (9 Points) **Short questions** (no need to show your work for this problem)

(i) State 3 ways to represent curves on the xy-plane analytically:

(ii) State 2 interesting physical properties of cycloids:

(iii) Let the point P have polar coordinates $(r, \theta) = (-1, \pi/4)$. Write down the Cartesian cycloid.
coordinates of P:

(2) (12 pints) Consider the cycloid which is parametrized by $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, where $0 \le \theta \le 2\pi$.

(i) Find the arclength of the cycloid;

(ii) Find the area of the region below the cycloid and above the interval $[0, 2\pi]$ on the x-axis.

(3) (6 pints) Assume the heart you sent to your girlfriend/boyfriend on Valentine's Day is $r = 1 - \sin\theta$. Find the area of your heart.

MAT 1002-T15 Fall 2020 - Quiz 5, 20 Minutes

Show your work unless otherwise instructed.

(1) (6 Points) Use a pair of equations to describe the circle center at point $(1, 2, 3)$ with radius 4, which lies on a plane perpendicular to the z-axis.

(2) (15 Points) Let $\vec{u} =< 1, 1, 1, >$ and $\vec{v} =< 1, 2, 3, >$.

(i) Compute $proj_{\vec{v}}\vec{u}$;

(ii) Compute $\vec{u} \times \vec{v}$;

(iii) Compute the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$

(3) (12 pints) Consider point $P(2, 2, 3)$ and plane $2x + y + 2z = 1231$.

(i) Find the parametric equations of the line L passing through P and normal (i.e., perpendicular) to the plane;

(ii) Find the distance from the origin to the line L.

MAT 1002-T15 Fall 2020 - Quiz 6, 20 Minutes

Show your work unless otherwise instructed.

(1) (6 Points) Sketch the graph of the equation of $y^2 - x^2 = z$.

(2) (10 pints) Compute the arclength of the helix parametrized by $\vec{r} = < \cos t,\ \sin t,\ 2t >$, $0 \le t \le 2\pi$; find the unit tangent vector of the helix at $t = 0$.

(3) (15 Points) Let $\vec{r} = (\cos t)\vec{i} - 2(\sin t)\vec{j}$, $t \geq 0$, be the position vector function of a particle moving on the xy-plane.

(i) Compute the velocity vector and acceleration vector of the particle;

(ii) Is the velocity vector orthogonal to the acceleration vector?

(iii) Sketch the trajectory of the particle and indicate direction of motion.

MAT 1002-T15 Fall 2020 - Quiz 7, 20 Minutes

Show your work unless otherwise instructed.

(1) (6 points) **Short questions** (no need to show your work for this problem)

(i) True or False: Partial differentiability of $f(x, y)$ at $(0, 0)$ (i.e., the first order partial derivatives of f exist at $(0, 0)$) implies continuity of f at $(0, 0)$.

True *False*

(ii) State the definition of differentiability of $f(x, y)$ at point (a, b):

(2) (12 Points) Compute the curvature and the principal unit normal vector of the helix, which is parametrized by $\vec{r(t)} = < \sin t, \cos t, 2t >$ at $t = 0$.

(3) (12 Points) Let

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Find $f_x(0,0)$ and $f_y(0,0)$; is f differentiable at $(0,0)$?

(4) (6 pints) Let $\overset{\omega}{z} = f(x,y)$ be differentiable at point $(12,31)$, and $f_x(12,31) = 20$; and $f_y(12,31) = 21$. Let $x = g(u,v)$ and $y = h(u,v)$ satisfy $g(0,0) = 12$, $g_u(0,0) = 1$, $g_v(0,0) = 2$; $h(0,0) = 31$, $h_u(0,0) = 3$, $h_v(0,0) = 4$. Compute

$$\frac{\partial w}{\partial u}, \ \frac{\partial w}{\partial v} \text{ at } (u,v) = (0,0).$$

MAT 1002-T15 Fall 2020 - Quiz 8, 20 Minutes

Show your work unless otherwise instructed.

(1) (4 points) **Short questions** (no need to show your work for this problem)
State two geometric or physical motivations of $\int\int_R f(x,y)dA$ where R is a bounded region on the xy-plane, and f is nonnegative on R.

(2) (6 Points) Let $f(x,y)$ and all its partial derivatives up to order 2 be continuous in a neighborhood of the origin $(0,0)$, such that

$$f(0,0) = 2021,\ f_x(0,0) = 4,\ f_y(0,0) = 5,\ f_{xx}(0,0) = 12,\ f_{yy} = 31,\ f_{xy}(0,0) = 10.$$

Approximate $f(x,y)$ by the second order Taylor polynomial for (x,y) near $(0,0)$

(3) (6 Points) Find the maximum value of $3x - y + 6$ on the circle $x^2 + y^2 = 4$.

(4) (6 pints) Consider the iterated integral

$$\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx dy.$$

(i) Sketch the region R of the integration

(ii) compute the above iterated integral.

MAT 1002-T15 Fall 2020 - Quiz 9, 20 Minutes

Show your work unless otherwise instructed.

(1) (6 points) Compute

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy.$$

(2) (6 Points) Find the volume of the part of the intersection in the first octant of the cylinder $x^2 + y^2 \leq 1$ and the cylinder $x^2 + z^2 \leq 1$.

(3) (6 pints) Compute

$$\int_0^1 \int_0^1 \int_{x^2}^1 xze^{zy^2}\, dy dx dz.$$

MAT 1002-T15 Fall 2020 - Quiz 10, 20 Minutes

Show your work unless otherwise instructed.

(1) (3 Points) **Short questions** (no need to show your work for this problem)
State a physical motivation of line integrals for scalar functions:

(2) (18 pints)
(i)Compute

$$\int\int_R (\sqrt{\frac{y}{x}} + \sqrt{xy})dxdy,$$

where R is the region in the fisrt quadrant of the xy-plane bounded by the curves $xy = 1$, $xy = 9$, $y = x$ and $y = 4x$.

(ii) Find the volume of the solid bounded below by the xy-plane, and above by the surface whose equation in spherical coordinates is given by $\rho = 1 + \cos\phi$.

(iii) Compute $\int_C |y|ds$, where circle on the xy-plane centered at the origin.

MAT 1002-T15 Fall 2020 - Quiz 11, 20 Minutes

Show your work unless otherwise instructed.

(1) **Short questions** (<u>no need</u> to show your work for this problem)

(i) (3 points) State two physical motivations of $\int_C \overrightarrow{F}(x,y,z)\cdot d\overrightarrow{r}$.

(ii) (6 points) Let C be a closed smooth curve on the xy-plane with unit outer normal vector field $\overrightarrow{n}(x,y)$, and $V(x,y)$ be the velocity vector field of a fluid flowing on the xy-plane, whose mass density function is $\delta(x,y)$. What does $\int_C \overrightarrow{F}(x,y,z)\cdot\overrightarrow{n}(x,y)ds$ represent physically? If this integral is positive, what does it say about the fluid inside the curve C? What if the integral is negative?

(iii) (3 points) Let $\overrightarrow{F} =< M(x,y,z), N(x,y,z), P(x,y,z) >$ be a vector field which is smooth everywhere in the xyz-space, except on the z-axis. Suppose $\nabla\times\overrightarrow{F}\equiv\overrightarrow{0}$ except on the z-axis. Is it necessarily true that F is conservative?

True *False*

(2) (6 Points) Compute $\int_C xydx + ydy - yzdz$, where C is parametrized by $\vec{r}(t) =< t, t^2, t >$, $0 \leq t \leq 2$.

(3) (12 Points) Let $\vec{F}(x, y, z) =< y \sin z, x \sin z, xy \cos z >$.

(i) Is $\vec{F}$ conservative?

(ii) If so, find a potential function f for $\vec{F}$.

(iii) Compute $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$ where C is the oriented curve parametrized by $\vec{r}(t) =< \arctan t, \sin(\sin(t\pi)), t^2 >$, $0 \leq t \leq 1$.

IAT 1001 Midterm Exam, 10:00am -Noon, March 27, 2021

our **Name and Student ID:**

our **Lecture** Class(e.g, L1) and **your tutorial class** (e.g, T01):

uction: (i) This is a closed-book and closed-notes exam; no calculators, no dictionaries o cell phones; (ii) Show your work unless otherwise instructed–a correct answer without ng your work when required shall be given no credits; (iii) Write down ALL your work our answers(including the answers for short questions) in the Answer Book.

(27 points) **Short Questions** (for these questions, NO need to show your work, just write down your answers in the Answer Book; NO partial credits for each question)

(i). Which of the following statements is false?

(a) Let f be a continuous function defined for all real numbers. If $a_1 = a$ and $a_{n+1} = f(a_n)$ (for all $n \geq 1$) define a convergent sequence $\{a_n\}$, then f has a fixed point (i.e. $f(x_0) = x_0$ for some x_0).

(b) If the sequence $\{a_1, a_3, a_5, \ldots\}$ converges to L_1 and $\{a_2, a_4, a_6, \ldots\}$ converges to L_2 with $L_1 \neq L_2$, then $\{a_n\}_{n\geq 1}$ cannot converge.

(c) If $\sum |a_n|$ diverges, then $\sum a_n$ cannot converge.

(ii). Which of the following statements is false?

(a) Given an alternating series, if it does not satisfy the conditions in the al-nating series test, then it must diverge.

(b) Rearranging the terms (which means changing the order of the terms $\displaystyle\sum_{n=1}^{\infty}(-1)^n(1/n^{\pi})$ will never change the value of the series. the

(c) If $\sum a_n$ is convergent but $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ must be di gent.

(iii). Which of the following is the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-1}{6(5+si}$?

(a) $\{x | \ x$ is not an integer multiple of $2\pi\}$

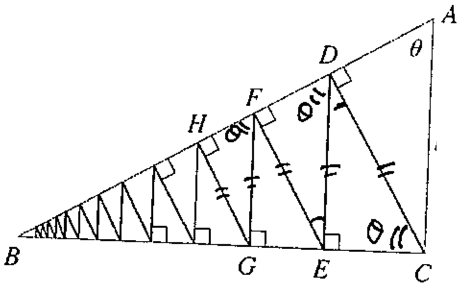
(b) $\{x | \ x$ is not an integer multiple of $\pi\}$

(c) $-\infty < x < \infty$

(d) diverges for all x.

(iv). Which of the following is the Maclaurin series for $\cos 2x$?

finite? If so, find it.



3. (24 points) For each of the following series, determine whether it is convergent a lutely, convergent conditionally, or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \ldots \cdot (3n-1)}$.

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, where $0 < p < 1$.

4. (8 points) Determine the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{2^n x^n}{4^n + 1}$.

5. (8 points) Find

$$\lim_{x \to 0} \frac{\tan x - \sin x}{(1 + x^3)^{\pi} - 1}. =$$

6. (10 points) Let $f(x)$ be continuously differentiable on the finite interval $[a, b]$ (i.e., f' exists and is continuous on $[a, b]$), and suppose f'' exists on the open interval (a, b). Prove the following special case of Taylor's Theorem: there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2!}(b - a)^2.$$

7. (24 points) Consider the cycloid parametrized by

$$x = t - \sin t, \quad y = 1 - \cos t, \qquad t \in [0, 2\pi].$$

   (a) Find its arclength.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^{2n} 2^{2n} x^{2n}}{n!}$ ✗ .

(b) $\sum_{n=0}^{\infty} \frac{(-1)^{2n} 2^{2n} x^{2n}}{(2n)!}$ ✗ .

(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} x^{2n}}{n!}$ ✗ .

(d) $\sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} x^{2n}}{(2n)!}$. ✓

(v). Which of the following is the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 8$?

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n n (x-8)^n}{8^{n+2}}$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-8)^n}{8^{n+1}}$.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(x-8)^n}{8^{n+2}}$ ✓ .

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n n (x-8)^n}{8^{n+1}}$.

(vi). $\frac{2}{3} - \frac{2^3}{3^3 . 3!} + \frac{2^5}{3^5 . 5!} - \frac{2^7}{3^7 . 7!} + \ldots =$

(a) $\sin \frac{3}{2}$.

(b) $\cos \frac{2}{3}$.

(c) $\ln \frac{2}{3}$.

(d) $\sin \frac{2}{3}$. ✓

(vii). Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be nonzero vectors in space. Which of the following statements is false?

(a) $\vec{u} \times \vec{v} = \vec{0}$ is equivalent to $\vec{u} = k\vec{v}$ for some scalar k. ✓

(b) The vector $\vec{u} \times (\vec{v} \times \vec{w})$ is parallel to the plane spanned by $\vec{v}$ and $\vec{w}$. ✓

(c) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$ if $\vec{u}$ and $\vec{v}$ are parallel. ✗

(viii). Which of the following statements is false?

(a) If we want to expand a function $f(x)$ as a power series about $x = 0$ in the interval $(-1, 1)$, then necessarily f has to be infinitely differentiable on the interval (i.e., derivative of all orders of f exist on the interval). ✓

(b) If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$, then $f(x) = \sum_{n=0}^{2021} c_n x^n + O(x^{2022})$ as $x \to 0$. ✓

(c) If $f(0) = 0 = f^{(n)}(0)$ for all positive integers n, then f has to be identically equal to 0, at least in a small open interval containing 0.

(ix). Write down the parametric equations of the line tangent to the curve $\vec{r}(t) = \cos(e^t)\vec{i} + (3 - t^2)\vec{j} + t\vec{k}$ at $t = 0$:

____________________________________.

2. (5 points) In the triangle ACB below, the angle at corner C is a right angle, and line segments CD, EF, GH, etc., are parallel, while AC, DE, FG, etc., are parallel. The drawing $CDEFGHI$... continues indefinitely. Is the length

$$|CD| + |DE| + |EF| + |FG| + |GH| + \ldots$$

(b) Find the area of the surface generated by revolving the cycloid about the x-axis.

(c) Find the curvature of the cycloid at $t = \pi$.

(d) Let the y-axis **point downward** (and the x-axis be horizontal, pointing to the right). Consider a particle sliding frictionlessly on the cycloid under the influence of gravity, from the origin $O = (0,0)$ to the bottom $B = (\pi, 2)$ of the cycloid, with 0 initial speed. Find the time T it takes for the particle to reach B from O. Length is measured in meters, and time in seconds.

8. (15 points) Consider the polar curve $r = \sin(2\theta)$, $\theta \in [0, \pi]$.

   (a) Sketch the curve on the xy-plane.

   (b) Compute the slope of the curve at $\theta = \pi/4$.

   (c) Find the area of the region bounded by the polar curve for $0 \leq \theta \leq \pi/2$.

9. (16 points) Let $P = (2, 4, 5)$, $Q = (1, 5, 7)$ and $R = (-1, 6, 8)$.

   (a) Find the area of the triangle ΔPQR.

   (b) Find an equation of the plane containing the triangle mentioned above.

   (c) Find parametric equations of the line which is perpendicular to the plane mentioned above, and passes through the origin $(0, 0, 0)$.

   (d) Find the distance from the origin to the plane mentioned above.

10. (16 points) Let $\vec{r}(t)$, $-\infty < t < \infty$, be the position vector function of a particle moving, with positive speed, along a smooth curve C in space. Suppose $\vec{r}(t)$ is twice differentiable (which means the second order derivative exists) for all t.

   (a) Prove that the speed of the particle is constant if and only if the velocity vector $\vec{v}(t)$ is always orthogonal to the acceleration vector $\vec{a}(t)$.

   (b) Prove that if the curvature of curve C is 0 everywhere, then the curve has to be a straight line. (space curve)