**Theorem 6.1** Semidefinite programs are equivalent with vector ~~gr~~ programs

**Proof**: Let $X \in \mathbb{R}^{n\times n}$ be a feasible solution for SDP. Then $X = V^T V$ for some $n\times n$ matrix $V$. Let $v_i$ be the $i$-th column of $V$. Then $v_1, \cdots, v_n$ is feasible for vector program and the value stays the same.

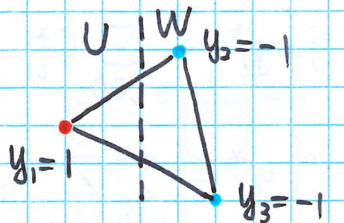
If $v_1, \cdots, v_n$ is feasible for VP, then define $V = [v_1 | v_2 | \cdots | v_n]$ and $X = V^T V$. Then $X = (x_{ij})$ is feasible for SDP and the value stays the same.

## 6.2 Finding large cuts.

Assume w.l.o.g. that the graph is complete since we define $w_{ij} = 0$ for any missing edge.

(QP) $\max \frac{1}{2} \sum_{(i,j)} (1 - y_i y_j) w_{ij}$.

s.t. $y_i \in \{-1, 1\}$, $i = 1, \cdots, n$

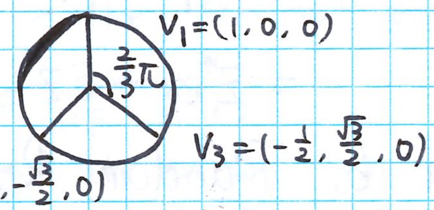



$i = U$ if $y_i = 1$

$i = W$ if $y_i = -1$.

(VP) $\max \frac{1}{2} \sum_{(i,j)} (1 - v_i v_j) w_{ij}$

s.t. $v_i \cdot v_i = 1$, $v_i \in \mathbb{R}^n$, $i = 1, \cdots, n$.




It is easy to see that (VP) really is a relaxation of (QP).

An optimal solution to this (QP) is $(y_1, y_2, y_3) = (1, -1, -1)$ and optimal value is 2.

The corresponding solution to (VP) is $V_1 = (1, 0, 0)$, $V_2 = V_3 = (-1, 0, 0)$.

An optimal solution to this (VP) is shown in the figure, and optimal value is $\frac{1}{2} \sum_{i,j} (1 - \cos \frac{2\pi}{3}) w_{ij} = \frac{1}{2} \cdot 3 \cdot (1 + \frac{1}{2}) \cdot 1 = \frac{9}{4} > 2$.

**Algo.** Step 1: Solve the VP-relaxation $\to v_1^*, \cdots v_n^*, Z_{VP}^*$

Step 2: (Randomized rounding) Take a vector $r \in \mathbb{R}^n$ of length 1 uniformly at random. Add vertex $i$ to $U$ if $v_i^* \cdot r \geq 0$ and add it to $W$ otherwise.

**Theorem 6.2** The algo. above is a 0.878-approximation algo.

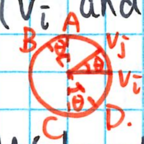
**Proof**: Let $\phi_{ij}$ be the angle between $v_i^*$ and $v_j^*$. Then, $v_i^* \cdot v_j^* = \cos \phi_{ij}$ and we can express the optimal value of the VP in terms of $\phi_{ij}$

$$Z_{VP}^* = \frac{1}{2} \sum_{(i,j)} (1 - v_i^* \cdot v_j^*) w_{ij} = \frac{1}{2} \sum_{i,j} (1 - \cos \phi_{ij}) w_{ij}.$$

Pr (edge $(i,j)$ is in the cut)

$= \Pr(v_i^*$ and $v_j^*$ are separated by the hyperplane perpendicular to $r)$

$= \frac{\phi_{ij}}{\pi}$




let $r'$ be the projection of $r$. if $r'$ in AB, $i \in W$, $j \in U$. if $r'$ in CD, $i \in U$, $j \in W$. $\Pr = 2\theta/2\pi = \theta/\pi$.

Let $W$ be the weight of the cut. Then $E(W) = \sum_{i,j} \frac{\phi_{ij}}{\pi} w_{ij}$.

Let $\alpha = \min_{0 \le \phi \le \pi} \left(\frac{\phi}{\pi} \cdot \frac{2}{1 - \cos\phi}\right) > 0.878$.

Thus, for any $\phi_{ij}$, we have $\frac{\phi_{ij}}{\pi} \geq \alpha \frac{1 - \cos\phi_{ij}}{2}$

$$E(W) = \sum_{i,j} \frac{\phi_{ij}}{\pi} w_{ij} \geq \frac{\alpha}{2} \sum_{i,j} (1 - \cos \phi_{ij}) w_{ij} = \alpha Z_{VP}^* \geq \alpha OPT > 0.878 OPT.$$

## 6.3

Max 2-SAT.

QP variables: $y_i \in \{-1, 1\}$, $i = 0, 1, \cdots, n$

$y_i = y_0 \iff x_i = \text{true} \iff \text{val}(x_i) = 1$

$y_i \neq y_0 \iff x_i = \text{False} \iff \text{val}(x_i) = 0$

$\Rightarrow \text{val}(x_i) = \frac{1 + y_0 y_i}{2}$ and $\text{val}(\neg x_i) = \frac{1 - y_0 y_i}{2}$

$$\begin{aligned} \text{val}(x_i \vee x_j) &= 1 - \text{val}(\neg x_i)\, \text{val}(\neg x_j) \\ &= 1 - \left(\frac{1 - y_0 y_i}{2}\right)\left(\frac{1 - y_0 y_j}{2}\right) \\ &= (1 + y_0 y_i)/4 + (1 + y_0 y_j)/4 + (1 - y_i y_j)/4 \end{aligned}$$

(QP) $\max \sum_{i,j} a_{ij}(1 + y_i y_j) + b_{ij}(1 - y_i y_j)$

s.t. $y_i^2 = 1$, $i = 0, 1, \cdots, n$.

number $a_{ij}, b_{ij}$ depend on the instance.

(VP) $\max \sum_{i,j} a_{ij}(1 + v_i \cdot v_j) + b_{ij}(1 - v_i \cdot v_j)$

s.t. $v_i \cdot v_i = 1$, $v_i \in \mathbb{R}^{n+1}$, $i = 0, 1, \cdots, n$.

**Algo.** Step 1: Solve VP-relaxation

Step 2: Take $r \in \mathbb{R}^{n+1}$ of length 1 at random

Let $y_i = \begin{cases} 1 & \text{if } v_i^* \cdot r \geq 0 \\ -1 & \text{if } v_i^* \cdot r < 0 \end{cases}$

**Theorem 6.3** The algo. above is a 0.878-approximation algo.

**Proof** Let $\phi_{ij}$ be the angle between $v_i^*$ and $v_j^*$

$$Z_{VP}^* = \sum_{i,j} a_{ij}(1 + v_i^* \cdot v_j^*) + b_{ij}(1 - v_i^* \cdot v_j^*) = \sum_{i,j} a_{ij}(1 + \cos\phi_{ij}) + b_{ij}(1 - \cos\phi_{ij})$$

Let $W$ be the number of clauses by the algo. As we have seen for the max cut problem, $\Pr(y_i \neq y_j) = \phi_{ij}/\pi$. Hence

$$E(W) = 2\sum_{i,j} a_{ij} \Pr(y_i = y_j) + b_{ij} \Pr(y_i \neq y_j)$$

$$= 2\sum_{i,j} a_{ij}\left(1 - \frac{\phi_{ij}}{\pi}\right) + b_{ij}\frac{\phi_{ij}}{\pi}$$

Let $\alpha = 0.878$, $\boxed{\frac{\phi}{\pi} \geq \frac{\alpha}{2}(1-\cos\phi)}$, $0 \leq \phi \leq \pi$

Substitute $\phi = \pi - \theta$, $\forall\, 0 \leq \theta \leq \pi$

$$\frac{\pi-\theta}{\pi} \geq \frac{\alpha}{2}(1-\cos(\pi-\theta)) \iff \boxed{1-\frac{\theta}{\pi} \geq \frac{\alpha}{2}(1+\cos\theta)}$$

Therefore,

$$E(W) = 2\sum_{i,j} \boxed{a_{ij}\left(1-\frac{\phi_{ij}}{\pi}\right)} + \boxed{b_{ij}\frac{\phi_{ij}}{\pi}} \geq \alpha \sum_{i,j} \boxed{a_{ij}(1+\cos\phi_{ij})} + \boxed{b_{ij}(1-\cos\phi_{ij})}$$

$$= \alpha Z^*_{vp} \geq \alpha\, OPT.$$