MA421 Advanced Algorithms Lent 2019

Exercises 5

1. Give an example on which First Fit algorithm for bin packing does at least as bad as $5/3 \cdot \mathrm{OPT}$, where OPT is the cost of an optimal solution.

2. Consider a more restricted algorithm than First Fit for bin packing, called Next Fit. Next Fit tries to pack the next item only in the most recently started bin. If it does not fit, it is packed in a new bin. Show that this algorithm also achieves an approximation factor of 2. Show that it is not possible to guarantee an approximation factor better than 2 for this algorithm.

3. We call a bin packing algorithm *monotonic* if the number of bins it uses for packing a subset of the items is at most the number of bins it uses for packing all items. Show that, whereas Next Fit is monotonic, First Fit is not.

4. In the knapsack problem, we have n items with sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$, and a threshold (total knapsack size) B, where $s_1, \ldots, s_n, v_1, \ldots, v_n$, and B are positive integers. We want to find a set $I \subseteq \{1, \ldots, n\}$, of total size $s(I) = \sum_{i \in I} s_i \leq B$, and for which the total value $v(I) = \sum_{i \in I} v_i$ is maximised. Suppose that the items are labeled in the order of non-increasing value-to-size ratio: that is,

$$\frac{v_1}{s_1} \geq \frac{v_2}{s_2} \geq \cdots \geq \frac{v_n}{s_n}.$$

   (a) Consider the following greedy approximation algorithm for the knapsack problem: place items in the knapsack in the given order (of labels) until the next item no longer fits. In other words, return the set $\{1, \ldots, k\}$ such that $\sum_{i=1}^{k} s_i \leq B$ but $\sum_{i=1}^{k+1} s_i > B$. Show that this greedy algorithm can return an arbitrarily bad solution: that is, the ratio of the total value obtained by an optimal solution to the total value of the greedy solution can be arbitrarily high.

   (b) Let $i_{\max}$ be an index for which $v_{i_{\max}}$ has the largest value. Modify the greedy algorithm in (a) so that it returns the set $\{1, \ldots, k\}$ as above or $\{i_{\max}\}$, whichever has greater value.

   Suppose that, given an instance of the knapsack problem, integer k is such that $\sum_{i=1}^{k} s_i \leq B$ but $\sum_{i=1}^{k+1} s_i > B$. Let OPT be the total value obtained by an optimal solution for that instance. Then, show that $\sum_{i=1}^{k+1} v_i \geq \mathrm{OPT}$.

   (c) Given (b), prove that the modified greedy algorithm is a $\frac{1}{2}$-approximation algorithm for the knapsack problem.

Solutions to these exercises are to be handed in at the lecture on **19 February 2019**.

