MA421 Advanced Algorithms — Lent 2019

Exercises 3

1. Consider the following reduction from Circuit Satisfiability Problem to Formula Satisfiability Problem.

   > We can use induction to express any boolean circuit as a boolean formula. We simply look at the gate that produces the circuit output and inductively express each of the gate's inputs as formulas. We then obtain the formula for the circuit by writing an expression that applies the gate's function to its inputs' formulas.

   It turns out that this straightforward reduction is not computable in polynomial time (and that is why we dismissed it in the lecture). Describe a circuit of size n that, when converted to a formula by this method, yields a formula whose size is exponential in n. Note that the size of a circuit is defined as the total number of gates and wires in the circuit.

2. Suppose that someone gives you a polynomial-time algorithm to decide formula satisfiability. Describe how to use this algorithm to find satisfying assignments in polynomial time.

3. The Hamiltonian Path problem is the problem of deciding whether a given undirected graph contains a path that visits each vertex exactly once. Show that the Hamiltonian Path problem is $\mathcal{NP}$-complete. (Hint: You can use that Hamiltonian Cycle problem is $\mathcal{NP}$-complete.)

4. The Longest Simple Cycle problem is the problem of determining a simple cycle (with no repeated vertices) of maximum length in a graph. Formulate a related decision problem and show that the decision problem is $\mathcal{NP}$-complete.

5. Show that the problem of determining the satisfiability of boolean formulas in disjunctive normal form (as defined on Page 1083 of the textbook, an OR of ANDs) is polynomial-time solvable.

6. Let 2-CNF-SAT be the set of satisfiable boolean formulas in CNF with exactly 2 literals per clause. Show that 2-CNF-SAT is in $\mathcal{P}$. Make your algorithm as efficient as possible. (Hint: Observe that $x \vee y$ is equivalent to $\neg x \rightarrow y$. Reduce 2-CNF-SAT to an efficiently solvable problem on a directed graph.)

7. The subgraph-isomorphism problem takes two undirected graphs G_1 and G_2, and it asks whether G_1 is isomorphic to a subgraph of G_2. Show that the subgraph-isomorphism problem is $\mathcal{NP}$-complete.

8. The set-partition problem takes as input a set S of positive integers. The question is whether the numbers can be partitioned into two sets A and $\bar{A} = S \setminus A$ such $\sum_{x \in A} x = \sum_{x \in \bar{A}} x$. Show that the set-partition problem is $\mathcal{NP}$-complete. (Hint: You can consider a reduction from the SUBSET-SUM problem).

9. In the half 3-CNF satisfiability problem, we are given a 3-CNF formula ϕ with n variables and m clauses, where m is even. We wish to determine whether there exists a truth

assignment to the variables of ϕ such that exactly half the clauses evaluate to 0 and exactly half the clauses evaluate to 1. Prove that the half 3-CNF satisfiability problem is $\mathcal{NP}$-complete.

10. The decision problem HITTING-SET is the following problem:

    Input: A collection $C = \{S_1, \ldots, S_n\}$ of subsets of a finite set S, and an integer $K \geq 1$.

    Question: Is there a *hitting set* for C of size K, that is, a subset $S' \subseteq S$ with $|S'| = K$ so that $S_i \cap S' \neq \emptyset$ for all i?

    Show that HITTING-SET is $\mathcal{NP}$-complete by showing VERTEX-COVER $\leq_{\mathrm{P}}$ HITTING-SET.

Solutions to these exercises are to be handed in at the lecture on **5 February 2019**.

