MA427 Lecture 4
The two phase method and the Dual Simplex

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Today's lecture

- Finite termination of Simplex: Bland's rule
- Finding an initial basis
- Dual basic solutions
- The Dual Simplex Method

Termination of the Simplex

$$\begin{array}{llcl} \max & z & & \\ & z & -\sum_{j\in N} \bar{c}_j x_j & = & \bar{z} \\ & & x_{B[i]} + \sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\ & & x & \geq & 0. \end{array}$$

New solution

$$\begin{array}{rcl} x_k(\bar{t}) & = & \bar{t}; \\ x_{B[i]}(\bar{t}) & = & \bar{b}_i - \bar{t}\bar{a}_{ik}, \quad i = 1, \ldots, m; \\ x_j(\bar{t}) & = & 0, \quad j \in N \setminus \{k\}. \end{array}$$

where

$$\bar{t} = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \ : \ i = 1, \ldots, m, \ \bar{a}_{ik} > 0 \right\}.$$

Entering variable: some x_k such that $\bar{c}_k > 0$.
Exiting variable: some $x_{B[h]}$ such that $\bar{a}_{hk} > 0$ and $\bar{t} = \frac{\bar{b}_h}{\bar{a}_{hk}}$

Termination

$$\begin{aligned} x_k(\bar{t}) &= \bar{t}; \\ x_{B[i]}(\bar{t}) &= \bar{b}_i - \bar{t}\bar{a}_{ik}, \quad i = 1, \ldots, m; \\ x_j(\bar{t}) &= 0, \quad j \in N \setminus \{k\}. \end{aligned}$$

$$\bar{t} := \min_{i \in \{1,\ldots,m\} \,:\, \bar{a}_{ik} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \right\}$$

- Objective value goes from $\bar{z}$ to $\bar{z} + \bar{c}_k \bar{t}$.
- If $\bar{t} > 0$, the objective value increases strictly.
- If $\bar{t} = 0$, the basis changes, but the corresponding basic feasible solution remains the same.

Cycling example

Bad *tie-breaking rule:* choose the entering variable with highest reduced cost, and the exiting variable with highest column coefficient.

1	-2.3	-2.15	13.55	0.4	0	0	0
0	0.4	0.2	-1.4	-0.2	1	0	0
0	-7.8	-1.4	7.8	0.4	0	1	0

1	0	-1	5.5	-0.75	5.75	0	0
0	1	0.5	-3.5	-0.5	2.5	0	0
0	0	2.5	-19.5	-3.5	19.5	1	0

1	0	0	-2.3	-2.15	13.55	0.4	0
0	1	0	0.4	0.2	-1.4	-0.2	0
0	0	1	-7.8	-1.4	7.8	0.4	0

This is the same tableau as in the beginning, only shift by two position. Repeating other two times (i.e, after 4 other pivots), we return the original tableau.

Degeneracy

Definition
A basis B is said to be degenerate if $\bar{b}_i = 0$ for some $i \in \{1, \ldots, m\}$ (where $\bar{b} = A_B^{-1} b$).

- If all bases are non-degenerate, then the Simplex Method terminates regardlessly of how we choose the variables that enters or leaves.
- If there are degenerate bases, we could cycle.

To prevent cycling, we need to be careful in how we choose the entering and exiting variables.

An anti-cycling rule

Bland's rule:

- Among all variables with positive reduced cost, choose as entering variable the variable x_k such that the index k is the smallest possible.
- Let $\bar{t} = \min\{\frac{\bar{b}_i}{\bar{a}_{ik}} \ : \ \bar{a}_{ik} > 0\}$. Choose as exiting variable the variable $x_{B[h]}$ such that $\bar{a}_{hk} > 0$, $\frac{\bar{b}_h}{\bar{a}_{hk}} = \bar{t}$, and such that $B[h]$ is smallest possible.

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$$
\begin{array}{ccccccccccc}
z & & - & 3x_2 & - & 7x_3 & & & & = & 26 \\
& x_1 & + & \frac{3}{2}x_2 & + & \frac{2}{3}x_3 & & & & = & 18 \\
& & & 0.4x_2 & - & 0.2x_3 & & & + \; x_5 & = & 3.6 \\
& & & \frac{1}{3}x_2 & - & \frac{2}{3}x_3 & + & x_4 & & = & 3
\end{array}
$$

QUIZ: Which are the entering and exiting variables according to Bland's rule?

(A) Enter: x_2, exit: x_4. *(B)* Enter: x_2, exit: x_5.

(C) Enter: x_3, exit: x_1.

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- Let $\bar{t} = \min\{\frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0\}$. Choose as exiting variable the variable $x_{B[h]}$ such that $\bar{a}_{hk} > 0$, $\frac{\bar{b}_h}{\bar{a}_{hk}} = \bar{t}$, and such that $B[h]$ is smallest possible.

Theorem

The Simplex Method with Bland's pivot rule terminates for every possible instance of an LP problem and every possible choice of starting feasible basis.

Starting Simplex: how to find a feasible basis?

$$\begin{pmatrix} 3 & 1 & 4 & 2 & 1 \\ -2 & 1 & -2 & -1 & -2 \\ 1 & 1 & 2 & -1 & -1 \end{pmatrix} x = \begin{pmatrix} 10 \\ -5 \\ 4 \end{pmatrix}$$

$$x \geq 0$$

Starting Simplex: how to find a feasible basis?

$$\begin{pmatrix} 3 & 1 & 4 & 2 & 1 \\ -2 & 1 & -2 & -1 & -2 \\ 1 & 1 & 2 & -1 & -1 \end{pmatrix} x = \begin{pmatrix} 10 \\ -5 \\ 4 \end{pmatrix}$$

$$x \geq 0$$

First step: make the r.h.s. positive.

$$\begin{array}{rcrcrcrcrcr} 3x_1 & + & x_2 & + & 4x_3 & + & 2x_4 & + & x_5 & = & 10 \\ 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & + & 2x_5 & = & 5 \\ x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & x_5 & = & 4 \end{array}$$

$$x \geq 0$$

Starting the Simplex: how to find a feasible basis?

Second step: construct the auxiliary problem.

$$\min x_6 + x_7 + x_8$$

$$\begin{array}{rcr} 3x_1+x_2+4x_3+2x_4+x_5+x_6 & = & 10 \\ 2x_1-x_2+2x_3+x_4+2x_5+x_7 & = & 5 \\ x_1+x_2+2x_3-x_4-x_5+x_8 & = & 4 \end{array}$$

$$x_1, \dots, x_5, x_6, x_7, x_8 \geq 0$$

Starting the Simplex: how to find a feasible basis?

Second step: construct the auxiliary problem.

$$w^* = \max -x_6 - x_7 - x_8$$

$$\begin{array}{rcl}
3x_1 + x_2 + 4x_3 + 2x_4 + x_5 + x_6 & = & 10 \\
2x_1 - x_2 + 2x_3 + x_4 + 2x_5 + x_7 & = & 5 \\
x_1 + x_2 + 2x_3 - x_4 - x_5 + x_8 & = & 4 \\
x_1, \ldots, x_5, x_6, x_7, x_8 \geq 0 & &
\end{array}$$

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2x_1-x_2+2x_3+x_4+2x_5+x_7 & = & 5 \\
x_1+x_2+2x_3-x_4-x_5+x_8 & = & 4 \\
\multicolumn{3}{c}{x_1,\dots,x_5,x_6,x_7,x_8 \geq 0}
\end{array}$$

NOTE:

- The auxiliary problem always has an optimum, with $w^* \leq 0$.
- If there exists a feasible solution for the initial problem, then $w^* = 0$. If there is no feasible solution, then $w^* < 0$.
- $\{6, 7, 8\}$ is a feasible basis for the auxiliary problem!

Starting the Simplex: how to find a feasible basis?

Second step: construct the auxiliary problem.

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$$\begin{array}{rcr} 3x_1+x_2+4x_3+2x_4+x_5+x_6 & = & 10 \\ 2x_1-x_2+2x_3+x_4+2x_5+x_7 & = & 5 \\ x_1+x_2+2x_3-x_4-x_5+x_8 & = & 4 \end{array}$$

$$x_1, \dots, x_5, x_6, x_7, x_8 \geq 0$$

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2x_1-x_2+2x_3+\ x_4+2x_5 \quad +x_7 & = & 5 \\
x_1+x_2+2x_3-\ x_4-\ x_5 \qquad +x_8 & = & 4 \\
x_1, \ldots, x_5, x_6, x_7, x_8 \geq 0 & &
\end{array}$$

We write the objective function as $\max\ w$, where

$$w + x_6 + x_7 + x_8 = 0.$$

To write the problem in tableaux form w.r.t. the basis $B = \{6, 7, 8\}$ we need to eliminate the basic variables x_6, x_7 x_8 from the objective function.

Starting the Simplex: how to find a feasible basis?

$$\max w$$

$$\begin{array}{rcr}
w-6x_1-x_2-8x_3-2x_4-2x_5 & = & -19 \\
3x_1+x_2+4x_3+2x_4+x_5+x_6 & = & 10 \\
2x_1-x_2+2x_3+x_4+2x_5+x_7 & = & 5 \\
x_1+x_2+2x_3-x_4-x_5+x_8 & = & 4 \\
x_1,\ldots,x_8 \geq 0 & &
\end{array}$$

Solving, we get the following optimal tableau ...

Starting the Simplex: how to find a feasible basis?

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\begin{array}{rcr}
\multicolumn{3}{c}{\max w} \\
w - 6x_1 - x_2 - 8x_3 - 2x_4 - 2x_5 & = & -19 \\
3x_1 + x_2 + 4x_3 + 2x_4 + x_5 + x_6 & = & 10 \\
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x_1 + x_2 + 2x_3 - x_4 - x_5 + x_8 & = & 4 \\
\multicolumn{3}{c}{x_1, \ldots, x_8 \geq 0}
\end{array}
$$

Solving, we get the following optimal tableau ...

$$
\begin{array}{cccccccc|c}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\
1 & 0 & 0 & 6 & 3 & 2 & -1 & -3 & 3 \\
0 & 0 & 1 & -\frac{9}{2} & -2 & -\frac{3}{2} & 1 & \frac{5}{2} & 0
\end{array}
$$

Starting the Simplex: how to find a feasible basis?

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& x_1, \ldots, x_8 \geq 0 &
\end{array}
$$

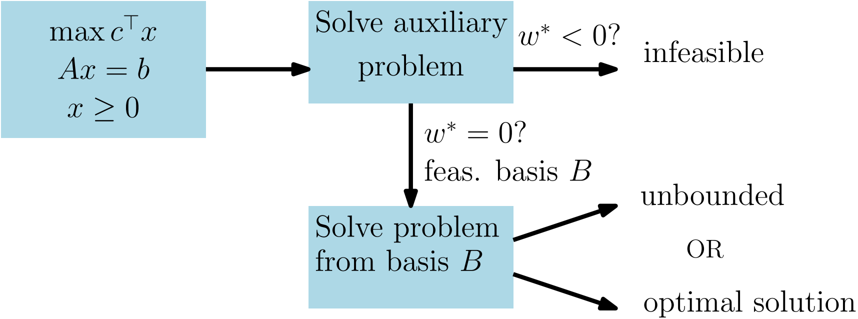
Solving, we get the following optimal tableau ...

0	0	0	0	0	1	1	1	0
0	1	0	2	0	1	-1	-1	1
1	0	0	6	3	2	-1	-3	3
0	0	1	$-\frac{9}{2}$	-2	$-\frac{3}{2}$	1	$\frac{5}{2}$	0

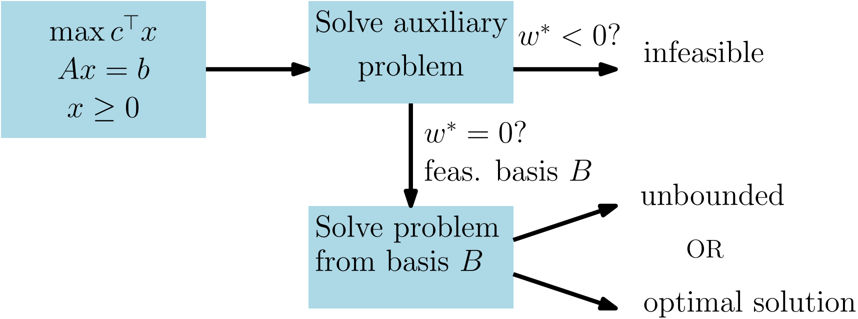
$w^* = 0 \implies x_6^*, x_7^*, x_8^* = 0.$

This gives a basic feasible solution to start the original LP: $(3, 1, 0, 0, 0)$, determined by the basis $\{1, 2, 3\}$.

The Two-Phase method



The Two-Phase method



Theorem (Fundamental Theorem of LP)

For every LP problem, one of the following holds: the problem has an optimum, the problem is infeasible, or the problem is unbounded.

Dual solutions for problems in standard form

We assume that the LP is in standard equality form.

$$\begin{aligned} \max \quad & c^\top x \\ Ax = & \; b \\ x \geq & \; 0 \end{aligned}$$

$$\boxed{\text{Assumption: } \mathrm{rk}(A) = m.}$$

The dual problem is

$$\begin{aligned} \min \; b^\top y & \\ A^\top y \geq & \; c. \end{aligned} \qquad (D)$$

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If the primal has an optimal solution, the dual also has an optimal solution. Furthermore, there exists a dual optimal solution which is basic.

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How does a basic dual solution look like?

Dual solutions for problems in standard form

$$\begin{aligned} \min\ & b^\top y \\ & A^\top y \geq \ c. \end{aligned} \qquad (D)$$

- A vector $\bar{y} \in \mathbb{R}^m$ is a basic solution for (D) if it satisfies m linearly independent constraints of the system at equality.

Dual solutions for problems in standard form

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- A vector $\bar{y} \in \mathbb{R}^m$ is a basic solution for (D) if it satisfies m linearly independent constraints of the system at equality.
- I.e., there exists a set $B \subseteq \{1, \ldots, n\}$ with m elements such that

  $$A_i^\top \bar{y} = c_i, \forall i \in B$$

  and such that all the vectors A_i, $i \in B$, are linearly independent.

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- This means that $\bar{y}$ is basic for (D) if and only if there exists a basis B such that

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  and such that all the vectors A_i, $i \in B$, are linearly independent. $\Longrightarrow$ B is a basis for $Ax = b$!
- This means that $\bar{y}$ is basic for (D) if and only if there exists a basis B such that

  $$A_B^\top \bar{y} = c_B.$$

$$\Longrightarrow \qquad \bar{y} = (A_B^\top)^{-1} c_B.$$

Simplex and duality

$$\begin{aligned} \max c^\top x & \\ Ax = & \quad b \\ x \geq & \quad 0 \end{aligned} \quad (P) \qquad \begin{aligned} \min\ b^\top y & \\ A^\top y \geq & \quad c. \end{aligned} \quad (D)$$

Let B be a basis. Primal solution associated to B:

$$\bar{x} = \begin{pmatrix} \bar{x}_B \\ \bar{x}_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}.$$

Dual solution associated to B

$$\bar{y} = (A_B^{-1})^\top c_B.$$

Note:

$$c^\top \bar{x} = c_B^\top \bar{x}_B = c_B^\top A_B^{-1} b = b^\top \bar{y}.$$

$\Longrightarrow$ If $\bar{x}$ and $\bar{y}$ are feasible, $\bar{x}$ and $\bar{y}$ are optimal.

Simplex and duality

$\bar{y}$ is feasible if $A^\top \bar{y} \geq c^\top$:

$$\begin{array}{rcl} A_B^\top \bar{y} & \geq & c_B \\ A_N^\top \bar{y} & \geq & c_N \end{array}$$

By definition of $\bar{y}$, $A_B^\top \bar{y} = c_B$. Hence $\bar{y}$ is feasible iff

$$A_N^\top (A_B^{-1})^\top c_B \geq c_N.$$

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$$A_N^\top (A_B^{-1})^\top c_B \geq c_N.$$

$$\Longrightarrow$$

$$c_N - A_N^\top (A_B^{-1})^\top c_B \leq 0.$$

Simplex and duality

$\bar{y}$ is feasible if $A^\top \bar{y} \geq c^\top$:

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By definition of $\bar{y}$, $A_B^\top \bar{y} = c_B$. Hence $\bar{y}$ is feasible iff

$$A_N^\top (A_B^{-1})^\top c_B \geq c_N.$$

$$\Longrightarrow$$

$$\bar{c}_N = c_N - A_N^\top (A_B^{-1})^\top c_B \leq 0.$$

The slacks of $\bar{y}$ are the reduced costs!

Simplex and duality

$\bar{y}$ is feasible if $A^\top \bar{y} \geq c^\top$:

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By definition of $\bar{y}$, $A_B^\top \bar{y} = c_B$. Hence $\bar{y}$ is feasible iff

$$A_N^\top (A_B^{-1})^\top c_B \geq c_N.$$

$$\Longrightarrow$$

$$\bar{c}_N = c_N - A_N^\top (A_B^{-1})^\top c_B \leq 0.$$

The slacks of $\bar{y}$ are the reduced costs!
$\bar{y}$ is feasible for the dual if and only if the reduced costs associated to B are non-positive.

Simplex and duality

Definition

- The basis B is *primal feasible* if the corresponding basic solution is feasible, i.e. if $A_B^{-1} b \geq 0$.
- The basis B is *dual feasible* if the corresponding dual solution is feasible, i.e. if $c_N - A_N^\top (A_B^{-1})^\top c_B \leq 0$.
- If B is both primal feasible and dual feasible, then we say that B is an *optimal basis*.

Simplex and duality: example

$$A = \begin{bmatrix} 1.5 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0.3 & 0.5 & 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix}, \qquad c = \begin{bmatrix} 130 \\ 100 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$B = \{1, 4, 5\}$.

$$A_B^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{bmatrix} \qquad A_B^{-1} b = \begin{bmatrix} 18 \\ 3 \\ 3.6 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{y} = (A_B^{-1})^\top c_B = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 130 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{260}{3} \\ 0 \\ 0 \end{bmatrix}$$

Basis is primal feasible but not dual feasible ($\bar{y}$ violates the second dual constraint).

Simplex and duality: example

$$A = \begin{bmatrix} 1.5 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0.3 & 0.5 & 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 27 \\ 21 \\ 9 \end{bmatrix}, \qquad c = \begin{bmatrix} 130 \\ 100 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$B = \{1, 2, 5\}$.

$$A_B^{-1} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 0 \\ 0.4 & -0.9 & 1 \end{bmatrix} \qquad A_B^{-1} b = \begin{bmatrix} 12 \\ 9 \\ 0.9 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{y} = (A_B^{-1})^\top c_B = \begin{bmatrix} 2 & -2 & 0.4 \\ -2 & 3 & -0.9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 130 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \\ 0 \end{bmatrix}.$$

Basis is primal feasible and dual feasible $\Longrightarrow$ optimal basis.

Proof of the Strong Duality Theorem

Theorem (Strong Duality Theorem)

If the primal problem has an optimal solution x^*, then also the dual has an optimal solution y^*, and

$$c^\top x^* = b^\top y^*.$$

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If the primal problem has an optimal solution x^*, then also the dual has an optimal solution y^*, and

$$c^\top x^* = b^\top y^*.$$

Proof.

Apply the two phases method with Bland's rule. It terminates. The Primal Simplex maintains a primal feasible basis at every iteration. It terminates when reduced costs are all non-positive. This means that the basis is also dual feasible. The primal and dual solution determined have the same value. □

Dual Simplex Method

- **The (Primal) Simplex Method** maintains at each iteration a primal feasible basis until it finds an optimal basis (i.e., a basis that is also dual feasible).

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- **The Dual Simplex Method** maintains at each iteration a dual feasible basis until it finds an optimal basis (i.e., a basis that is also primal feasible).
- Main motivations for studying the Dual Simplex Method:
  - It is more efficient in practice;
  - It is widely used in Integer Programming, both in the Branch-and-Bound method and in the Cutting Planes method. (We will see this in a few lectures)

Dual Simplex Method

Let B be dual feasible.

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- if $\bar{b}_i \geq 0$ for $i = 1, \ldots, m$, then B is optimal.
- Suppose there exists h such that $\bar{b}_h < 0$.
  We select $x_{B[h]}$ to leave the basis. Who enters?

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If x_k enters, the new reduced costs $\tilde{c}_j$ are

$$\begin{aligned} \tilde{c}_{B[h]} &= -\frac{\bar{c}_k}{\bar{a}_{hk}}, \\ \tilde{c}_j &= \bar{c}_j - \frac{\bar{c}_k}{\bar{a}_{hk}} \bar{a}_{hj}, j \in N \setminus \{k\} \end{aligned}$$

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- This happens if and only if we select x_k such that $\bar{a}_{hk} < 0$ and

$$\frac{\bar{c}_k}{\bar{a}_{hk}} \leq \frac{\bar{c}_j}{\bar{a}_{hj}} \text{ for every } j \in N \setminus \{k\} \text{ such that } \bar{a}_{hj} < 0.$$

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- **Q.** What if $\bar{a}_{hj} \geq 0$ for all $j \in N$?

Dual Simplex Method

Input: *$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, a dual feasible basis $B = \{B[1], \ldots, B[m]\}$;*

Output: *Either an optimal solution $\bar{x}$ for (P), or we determine that (P) is infeasible.*

1. Compute the tableau with respect to the current basis B;
2. If $\bar{b} \geq 0$, then B is optimal, STOP.
3. Otherwise, choose an index h such that $\bar{b}_h < 0$;
   - 3a. If $\bar{a}_{hj} \geq 0 \ \forall j \in N$, then the problem is infeasible, STOP.
   - 3b. Otherwise, choose $k \in N$ such that

$$\bar{a}_{hk} < 0 \quad \text{and} \quad \frac{\bar{c}_k}{\bar{a}_{hk}} = \min\left\{ \frac{\bar{c}_j}{\bar{a}_{hj}} \ : j \in N, \, \bar{a}_{hj} < 0 \right\};$$

   Set $B[h] := k$, return to 1.

Example

$$\begin{array}{lrrrcl}
\max & -3x_1 & -4x_2 & -5x_3 & & \\
 & 2x_1 & +2x_2 & +x_3 & \geq & 6 \\
 & x_1 & +2x_2 & -3x_3 & \geq & 5 \\
\end{array}$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

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Add "surplus variables" x_4 and x_5 and write in tableau form w.r.t. the basis $\{4, 5\}$, which is dual feasible because the reduced costs are nonpositive.

$$\begin{array}{lrrrrrcr} \multicolumn{8}{l}{\max z} \\ z & +3x_1 & +4x_2 & +5x_3 & & & = & 0 \\ & -2x_1 & -2x_2 & -x_3 & +x_4 & & = & -6 \\ & -x_1 & -2x_2 & +3x_3 & & +x_5 & = & -5 \\ & \multicolumn{3}{c}{x_1, x_2, x_3, x_4, x_5 \geq 0} & & & & \end{array}$$

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 & \multicolumn{5}{c}{x_1, x_2, x_3, x_4, x_5 \geq 0} & & &
\end{array}
$$

x_4 leaves the basis and x_1 enters because $\min\{3/2, 4/2, 5/1\} = 3/2$.

Example

$$
\begin{array}{cccccccc}
z & & +x_2 & +\frac{7}{2}x_3 & \frac{3}{2}x_4 & & = & -9 \\
 & x_1 & +x_2 & +\frac{1}{2}x_3 & -\frac{1}{2}x_4 & & = & 3 \\
 & & -x_2 & +\frac{7}{2}x_3 & -\frac{1}{2}x_4 & +x_5 & = & -2
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x_5 leaves the basis and x_2 enters because $\{\frac{1}{1}, \cdot, \frac{3/2}{1/2}\} = 1$

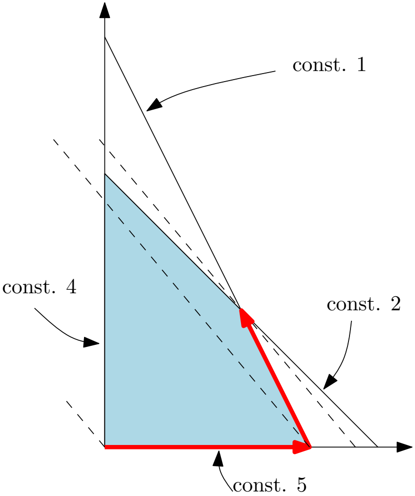
$$
\begin{array}{cccccccc}
z & & & +7x_3 & +x_4 & +x_5 & = & -11 \\
 & x_1 & & -2x_3 & -x_4 & +x_5 & = & 1 \\
 & & x_2 & +\frac{5}{2}x_3 & +\frac{1}{2}x_4 & +x_5 & = & 2
\end{array}
$$

In the dual space:

$$\begin{array}{lrrcl}
\min & -6y_1 & -5y_2 & & \\
 & -2y_1 & -y_2 & \geq & -3 \\
 & -2y_1 & -2y_2 & \geq & -4 \\
 & -y_1 & +3y_2 & \geq & -5 \\
 & y_1 & & \geq & 0 \\
 & & y_2 & \geq & 0
\end{array}$$

Active dual constraints:

- $\{4,5\}$ - Dual sol. $(0,0)$
- $\{1,5\}$ - Dual sol. $(\frac{3}{2},0)$
- $\{1,2\}$ - Dual sol. $(1,1)$



Termination of the Dual Simplex

Let B be a dual feasible basis,

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- Suppose $x_{B[h]}$ exits the basis and x_k enters. What is the new value in the objective function?

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- At every iteration the objective value does not increase (and in fact it decreases if $\bar{c}_k < 0$).
- Indeed, the dual problem is a minimization problem, so we are finding "better" dual solutions.

Bland's rule

- As in the Primal Simplex, the objective value might not improve from one iteration to the next (i.e. if $\bar{c}_k = 0$).

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- As in the Primal Simplex, the objective value might not improve from one iteration to the next (i.e. if $\bar{c}_k = 0$).
- The following "dual" variant of Bland's rule ensures that the Dual Simplex Method terminates.

> - Among all variables eligible to exit the basis (i.e. $x_{B[i]}$ s.t. $\bar{b}_i < 0$), choose the one with smallest subscript.
> - Among all variables eligible to enter the basis, choose the one with smallest subscript.