MA427 Lecture 3
Basic solutions and the Simplex Method

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Today's lecture

- Standard equality form: basic solutions and their existence
- Carathéodory's theorem

Simplex Method

- the tableau form
- connection to duality
- pivot steps
- cycling: Bland's rule

Systems in standard equality form

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

$$\boxed{\text{Assumption: } \mathrm{rk}(A) = m.}$$

Definition

A set $B \subseteq \{1, \ldots, n\}$ is said a *basis* of A if

- $|B| = m$;
- the vectors A_j, $j \in B$, are linearly independent.

Example

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & -2 & -1 \\ 1 & 2 & 1 & 3 & -1 & -5 \end{bmatrix} x = \begin{bmatrix} 11 \\ 6 \\ 13 \end{bmatrix}.$$

$$x \geq 0$$

$B = \{1, 2, 6\}$ is a basis.

$$A_B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 1 & 2 & -5 \end{bmatrix}.$$

$B = \{2, 5, 6\}$ is not a basis.

$$A_B = \begin{bmatrix} 2 & 0 & -6 \\ 1 & -2 & -1 \\ 2 & -1 & -5 \end{bmatrix}.$$

Systems in standard equality form

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Proposition

A point $x^* \in \mathbb{R}^n$ is a basic feasible solution of $Ax = b$, $x \geq 0$ if and only if it is feasible and $\exists$ a basis B such that $x^*_j = 0$ for every $j \notin B$.

Basic feasible solutions

Proposition

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Proof.

- x^* is basic feasible to $Ax = b$, $x \geq 0$ if and only if there n linearly independent inequalities.

Basic feasible solutions

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Proof.

- x^* is basic feasible to $Ax = b$, $x \geq 0$ if and only if there n linearly independent inequalities.
- $Ax = b$ already gives m linearly independent ones, therefore we need $n - m$ binding constraints $x_j \geq 0$, $j \in N$, $|N| = n - m$.

$$R = \left(\begin{array}{c|c} A_B & A_N \\ \hline \mathbf{0} & I \end{array} \right),$$

- $\det(R) = \det(A_B)$. Consequently, x^* is a basis if and only if $\det(A_B) \neq 0$, that is, the columns of A_B are lin. independent.

Systems in standard equality form

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & -2 & -1 \\ 1 & 2 & 1 & 3 & -1 & -5 \end{bmatrix} x = \begin{bmatrix} 11 \\ 6 \\ 13 \end{bmatrix}.$$

$$x \geq 0$$

$\bar{x} = (7, 8, 0, 0, 0, 2)$ basic (feasible) solution.

- It is feasible.
- $B = \{1, 2, 6\}$ is a basis
- $\bar{x}_3, \bar{x}_4, \bar{x}_5 = 0$

To see that the previous point is basic: the inequalities binding at $\bar{x}$.

$$\left[\begin{array}{cc|ccr|r} 1 & 2 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & -2 & -1 \\ 1 & 2 & 1 & 3 & -1 & -5 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] x = \left[\begin{array}{c} 11 \\ 6 \\ 13 \\ 0 \\ 0 \end{array}\right].$$

The above matrix has the same determinant as

$$A_B = \left[\begin{array}{ccr} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 1 & 2 & -5 \end{array}\right].$$

Systems in standard equality form

We will need this more general notion.

Definition

A point $x^* \in \mathbb{R}^n$ is a *basic solution* of $Ax = b$, $x \geq 0$, if $Ax^* = b$ and there exists a basis B of A such that $x_j^* = 0$ for every $j \notin B$.

That is, we consider also basic solutions that are not feasible. This will be needed when discussing the dual simplex method.

Systems in standard equality form

$$\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

Lemma

Given a basis B, the point

$$\begin{aligned} \bar{x}_B &= A_B^{-1} b; \\ \bar{x}_N &= 0 \end{aligned}$$

is the only one such that $A\bar{x} = b$ and $\bar{x}_j = 0 \ \forall j \notin B$.

The above is the basic solution relative to B. If $\bar{x}$ is feasible, then B is feasible basis.

Systems in standard equality form

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & -2 & -1 \\ 1 & 2 & 1 & 3 & -1 & -5 \end{bmatrix} x = \begin{bmatrix} 11 \\ 6 \\ 13 \end{bmatrix}.$$

$$x \geq 0$$

$B = \{1, 2, 6\}$ is a basis.

$$A_B^{-1} = \begin{bmatrix} -3 & -2 & 4 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

$\bar{x} = (7, 8, 0, 0, 0, 2)$ basic *(feasible)* solution relative to B.

Systems in standard equality form

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & -6 \\ 0 & 1 & 1 & 3 & -2 & -1 \\ 1 & 2 & 1 & 3 & -1 & -5 \end{bmatrix} x = \begin{bmatrix} 11 \\ 6 \\ 13 \end{bmatrix}.$$

$$x \geq 0$$

$B = \{1, 3, 6\}$ is a basis.

$$A_B^{-1} = \begin{bmatrix} -2 & -3 & 3 \\ -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}.$$

$\bar{x} = (-1, 0, 4, 0, 0, -2)$ basic *(infeasible)* solution relative to B.

Systems in standard equality form

Theorem
If the LP problem

$$\begin{aligned} \max c^\top x & \\ Ax = & \; b \\ x \geq & \; 0 \end{aligned}$$

has a finite optimum, there exists an optimal solution x^* which is a basic feasible solution.

Two proofs:

- Simple direct proof now.
- Consequence of the Simplex Method.

Systems in standard equality form

Theorem

If the LP problem

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has a finite optimum, there exists an optimal solution x^* which is a basic feasible solution.

Proof.

- Select an optimal solution x^* with the *highest number of zero components*.
- For a contradiction, assume that x^* is not basic. Let $S = \{j : x_j^* > 0\}$.

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Theorem

If the LP problem

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has a finite optimum, there exists an optimal solution x^* which is a basic feasible solution.

Proof.

- Select an optimal solution x^* with the *highest number of zero components*.
- For a contradiction, assume that x^* is not basic. Let $S = \{j : x_j^* > 0\}$.
- Thus, $\{A_j \ : \ j \in S\}$ is not linearly independent. Thus, $\exists z_j, j \in S \ : \ \sum_j A_j z_j = 0$. Let

$$d_j = \begin{cases} z_j & \text{if } j \in S \\ 0 & \text{if } j \notin S \end{cases}$$

Systems in standard equality form

- $Ad = 0, d \neq 0$, *and* $d_j = 0$ *if* $j \notin S$.

Systems in standard equality form

- *$Ad = 0, d \neq 0$, and $d_j = 0$ if $j \notin S$.*
- For every $t \in \mathbb{R}$, $A(x^* - td) = b$ and $x_j^* - td_j = 0$ for $j \notin S$.

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- For every $t \in \mathbb{R}$, $A(x^* - td) = b$ and $x_j^* - td_j = 0$ for $j \notin S$.
- For a small $\varepsilon > 0$, $x^* - \varepsilon d, x^* + \varepsilon d$ are both feasible. Thus, $c^\top d = 0$ and therefore the are both optimal.

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- Assume $\exists i : d_i > 0$. Select $\bar{t} \geq 0$ as the largest value such that $x^* - \bar{t}d$ is feasible.

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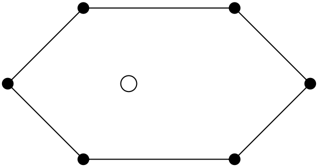
- For every $t \in \mathbb{R}$, $A(x^* - td) = b$ and $x^*_j - td_j = 0$ for $j \notin S$.
- For a small $\varepsilon > 0$, $x^* - \varepsilon d, x^* + \varepsilon d$ are both feasible. Thus, $c^\top d = 0$ and therefore the are both optimal.
- Assume $\exists i \, : \, d_i > 0$. Select $\bar{t} \geq 0$ as the largest value such that $x^* - \bar{t}d$ is feasible.
- $x' = x^* - \bar{t}d$ is also optimal and has more zero components than x^*, a contradiction.

Carathéodory's theorem

Theorem (Carathéodory)

If a point $z \in \mathbb{R}^n$ is a convex combination of points in some set $X \subseteq \mathbb{R}^n$, then it is a convex combination of at most $\dim(X) + 1$ *affinely independent points in X.*



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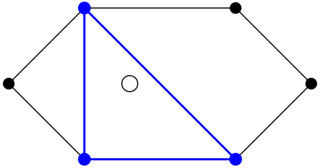

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Proof.

- $X = \{v^1, \ldots, v^k\}$. Define $A \in \mathbb{R}^{n+1 \times k}$: $A_i = \binom{v^i}{1}$.
- z is a convex combination if the following system is feasible.

$$\begin{aligned} A\lambda &= z \\ \lambda &\geq 0 \end{aligned}$$

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- Select a basic feasible solution. The columns $\{A_i : \lambda_i > 0\}$ are linearly independent, and there are at most $n + 1$ of them.

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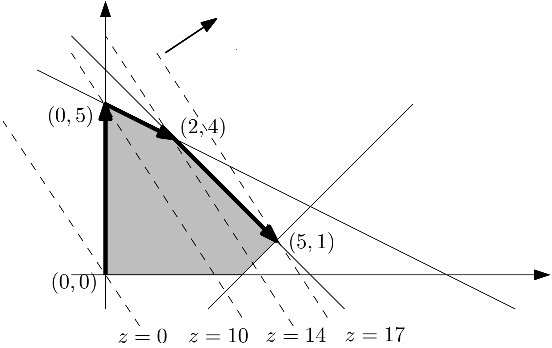
- $X = \{v^1, \dots, v^k\}$. Define $A \in \mathbb{R}^{n+1 \times k}$: $A_i = \binom{v^i}{1}$.
- z is a convex combination if the following system is feasible.

$$\begin{aligned} A\lambda = & \; z \\ \lambda \geq & \; 0 \end{aligned}$$

- Select a basic feasible solution. The columns $\{A_i : \lambda_i > 0\}$ are linearly independent, and there are at most $n + 1$ of them.
- This shows that $\{v^i : \lambda_i > 0\}$ are affinely independent.

The Simplex Method



The Simplex Method: example

$$
\begin{array}{rrcrcr}
\max & 3x_1 & + & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 4 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

In standard equality form

$$
\begin{array}{rrcrcrcrcrcr}
\max & 3x_1 & + & 2x_2 & & & & & & & & \\
s.t. & x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\
 & x_1 & + & 2x_2 & & & + & x_4 & & & = & 10 \\
 & x_1 & - & x_2 & & & & & + & x_5 & = & 4 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

Feasible basis: $\{3, 4, 5\}$ Basic solution: $(0, 0, 6, 10, 4)^\top$

The Simplex Method: example

$$
\begin{array}{rrcrcrcrcrcrcr}
\max z = & 3x_1 & + & 2x_2 & & & & & & & & \\
& x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\
& x_1 & + & 2x_2 & & & + & x_4 & & & = & 10 \\
& x_1 & - & x_2 & & & & & + & x_5 & = & 4 \\
& & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

Basic solution: $(0, 0, 6, 10, 4)^\top$. Basis: $\{3, 4, 5\}$.
If we increase x_2 by $t \geq 0$ and leave $x_1 = 0$, the objective value increases by $2t$.
The remaining components must become

$$
\begin{aligned}
x_3(t) &= 6 - t \\
x_4(t) &= 10 - 2t \\
x_5(t) &= 4 + t
\end{aligned}
$$

The maximum t we can choose is $t = 5$.

The Simplex Method: example

$$
\begin{array}{lrcrcrcrcrcl}
\max & 3x_1 & + & 2x_2 & & & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\
 & x_1 & + & 2x_2 & & & + & x_4 & & & = & 10 \\
 & x_1 & - & x_2 & & & & & + & x_5 & = & 4 \\
 & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

- Current solution $(0, 5, 1, 0, 9)$. Basis: $\{2, 3, 5\}$.
- We express everything in terms of the nonbasic variables.

The Simplex Method: example

$$
\begin{array}{rccccccccccccc}
\max z = 10+ & 2x_1 & & & & & - & x_4 & & & & \\
 & \frac{1}{2}x_1 & & & + & x_3 & - & \frac{1}{2}x_4 & & & = & 1 \\
 & \frac{1}{2}x_1 & + & x_2 & & & + & \frac{1}{2}x_4 & & & = & 5 \\
 & \frac{3}{2}x_1 & & & & & + & \frac{1}{2}x_4 & + & x_5 & = & 9 \\
 & & & & & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0
\end{array}
$$

Basic solution: $(0, 5, 1, 0, 9)^\top$. Feasible basis: $\{2, 3, 5\}$.
If we increase x_1 by $t \geq 0$ and leave $x_4 = 0$, the objective value increases by $2t$.
The remaining components must become

$$
\begin{aligned}
x_2(t) &= 5 - \frac{1}{2}t \\
x_3(t) &= 1 - \frac{1}{2}t \\
x_5(t) &= 9 - \frac{3}{2}t
\end{aligned}
$$

The Simplex Method: example

$$
\begin{array}{rcrcrcrcrcrcl}
\max z = 14 & & & & & - & 4x_3 & + & x_4 & & & & \\
 & & x_1 & & & + & 2x_3 & - & x_4 & & & = & 2 \\
 & & & + & x_2 & - & x_3 & + & x_4 & & & = & 4 \\
 & & & & & - & 3x_3 & + & 2x_4 & + & x_5 & = & 6 \\
 & & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}
$$

Basic solution: $(2, 4, 0, 0, 6)^\top$. Feasible basis: $\{1, 2, 5\}$.

The Simplex Method: example

$$
\begin{array}{lcccccccccccc}
\max z = 14 & & & & - & 4x_3 & + & x_4 & & & & & \\
 & x_1 & & & + & 2x_3 & - & x_4 & & & = & 2 \\
 & & + & x_2 & - & x_3 & + & x_4 & & & = & 4 \\
 & & & & - & 3x_3 & + & 2x_4 & + & x_5 & = & 6 \\
 & & & & & & & \multicolumn{3}{c}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}
$$

Basic solution: $(2, 4, 0, 0, 6)^\top$. Feasible basis: $\{1, 2, 5\}$.

...

Continue until

- either we cannot increase the objective value any further
- we find an unbounded direction.

The Simplex Method

We assume that the initial LP is in standard equality form.

$$\begin{aligned} \max \quad & c^\top x \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

$\boxed{\text{Assumption: } \mathrm{rk}(A) = m.}$

- At every iteration we maintain a basis B, defining a basic feasible solution.
- The LP is transformed, via row reductions, to one where
  - The objective function is expressed in terms of the nonbasic variables only,
  - each basic variable is written in terms of the nonbasic ones.
- Increasing the value of a nonbasic variable with positive coefficient in the objective function gives a solution with higher value.

Tableau form

$$\begin{array}{llcl} \max & z & & \\ & z & -\bar{c}_N x_N & = & \bar{z} \\ & & x_B + \bar{A}_N x_N & = & \bar{b} \\ & & x & \geq & 0 \end{array}$$

where

$$\begin{aligned} \bar{A}_N &= A_B^{-1} A_N; \\ \bar{b} &= A_B^{-1} b; \\ \bar{c} &= c - A^\top {A_B^{-1}}^\top c_B; & \textit{(reduced costs)} \\ \bar{z} &= c_B^\top A_B^{-1} b. \end{aligned}$$

Tableau with respect to B

1	$\mathbf{0}$	$-\bar{c}_N^\top$	$\bar{z}$
$\mathbf{0}$	I	$\bar{A}_N$	$\bar{b}$

Example of problem in tableau form

Basis: $B = \{1, 2, 5\}$

$$
\begin{array}{rcrcrcrcrcrcl}
\max z & & & & & & & & & & & & \\
z & & & & & + & 4x_3 & - & x_4 & & & = & 14 \\
& x_1 & & & & + & 2x_3 & - & x_4 & & & = & 2 \\
& & + & x_2 & & - & x_3 & + & x_4 & & & = & 4 \\
& & & & & - & 3x_3 & + & 2x_4 & + & x_5 & = & 6 \\
& & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}
$$

$$
\begin{array}{|c|cccccc|c|}
\hline
1 & 0 & 0 & 4 & -1 & 0 & 14 \\
\hline
0 & 1 & 0 & 2 & -1 & 0 & 2 \\
0 & 0 & 1 & -1 & 1 & 0 & 4 \\
0 & 0 & 0 & -3 & 2 & 1 & 6 \\
\hline
\end{array}
$$

The Simplex Method

If $B = \{B[1], \ldots, B[m]\}$, we can write the problem in tableau form

$$\begin{array}{llllcl}
\max & z & & & & \\
 & z & & -\sum_{j \in N} \bar{c}_j x_j & = & \bar{z} \\
 & & x_{B[i]} & +\sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
 & & & x & \geq & 0.
\end{array}$$

- $\bar{c}_j$, $j = 1, \ldots, n$ are said the reduced costs.
- $x_{B[h]}$ is said the basic variable in row h.

The Simplex Method

If $B = \{B[1], \dots, B[m]\}$, we can write the problem in tableau form

$$\begin{array}{lllcl}
\max & z & & & \\
& z & \quad - \sum_{j \in N} \bar{c}_j x_j & = & \bar{z} \\
& & x_{B[i]} \quad + \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \dots, m \\
& & x & \geq & 0.
\end{array}$$

- $\bar{c}_j$, $j = 1, \dots, n$ are said the reduced costs.
- $x_{B[h]}$ is said the basic variable in row h.

Let $\bar{x}$ be the basic feasible solution relative to B.

- *Case 1.* $\bar{c}_j \leq 0$ *for all* $j \in N$: $\bar{x}$ is optimal

The Simplex Method

If $B = \{B[1], \ldots, B[m]\}$, we can write the problem in tableau form

$$
\begin{array}{llllll}
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 & & x_{B[i]} & +\sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
 & & & x & \geq & 0.
\end{array}
$$

- $\bar{c}_j$, $j = 1, \ldots, n$ are said the reduced costs.
- $x_{B[h]}$ is said the basic variable in row h.

Let $\bar{x}$ be the basic feasible solution relative to B.

- *Case 1.* $\bar{c}_j \leq 0$ *for all* $j \in N$*:* $\bar{x}$ is optimal

We can read off a dual optimal solution certifying the optimality of $\bar{x}$*.*

Dual optimal solution

$$
\begin{array}{lllll}
\max & z & & & \\
& z & & -\bar{c}_N x_N & = & \bar{z} \\
& & x_B & +\bar{A}_N x_N & = & \bar{b} \\
& & & x & \geq & 0
\end{array}
$$

$$
\begin{aligned}
\bar{A}_N &= A_B^{-1} A_N; \\
\bar{b} &= A_B^{-1} b; \\
\bar{c} &= c - A^\top {A_B^{-1}}^\top c_B \leq 0; \qquad \textit{(reduced costs)} \\
\bar{z} &= c_B^\top A_B^{-1} b.
\end{aligned}
$$

$$
\bar{x}_j = \begin{cases} \bar{b}_i & \text{if } j \in B \\ 0 & \text{if } j \in N \end{cases} \qquad\qquad \bar{y} = {A_B^{-1}}^\top c_B
$$

Dual optimal solution

$$
\begin{array}{lllll}
\max & z & & & \\
 & z & -\bar{c}_N x_N & = & \bar{z} \\
 & & x_B \quad +\bar{A}_N x_N & = & \bar{b} \\
 & & x & \geq & 0
\end{array}
$$

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\bar{A}_N &= A_B^{-1} A_N; \\
\bar{b} &= A_B^{-1} b; \\
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\bar{z} &= c_B^\top A_B^{-1} b.
\end{aligned}
$$

$$
\bar{x}_j = \begin{cases} \bar{b}_i & \text{if } j \in B \\ 0 & \text{if } j \in N \end{cases} \qquad\qquad \bar{y} = {A_B^{-1}}^\top c_B
$$

$\bar{x}$ is a feasible primal, $\bar{y}$ a feasible dual solution, and they satisfy complementary slackness.

The Simplex Method

If $B = \{B[1], \ldots, B[m]\}$, we can write the problem in tableau form

$$
\begin{array}{llllll}
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& z & & -\sum_{j\in N} \bar{c}_j x_j & = & \bar{z} \\
& & x_{B[i]} & +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
& & & x & \geq & 0.
\end{array}
$$

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- $x_{B[h]}$ is said the basic variable in row h.

Let $\bar{x}$ be the basic feasible solution relative to B.

- *Case 1.* $\bar{c}_j \leq 0$ *for all* $j \in N$: $\bar{x}$ is optimal

The Simplex Method

If $B = \{B[1], \ldots, B[m]\}$, we can write the problem in tableau form

$$
\begin{array}{llcrcl}
\max & z & & & & \\
& z & & -\sum_{j \in N} \bar{c}_j x_j & = & \bar{z} \\
& & x_{B[i]} & +\sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
& & & x & \geq & 0.
\end{array}
$$

- $\bar{c}_j$, $j = 1, \ldots, n$ are said the reduced costs.
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- *Case 2. There exists $k \in N$ such that $\bar{c}_k > 0$:* if we increase x_k by $t \geq 0$ leaving $x_j = 0$ for all $j \in N \setminus \{k\}$, then the objective value increases by $\bar{c}_k t$.

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  if we increase x_k by $t \geq 0$ leaving $x_j = 0$ for all $j \in N \setminus \{k\}$,
  then the objective value increases by $\bar{c}_k t$.
  What is the maximum we can increase t?

Minimum ratio rule

The largest $\bar{t}$ such that the new solution $x(\bar{t})$ is feasible is

$$\bar{t} = \min_{i \in \{1,\ldots,m\} \,:\, \bar{a}_{ik} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \right\}.$$

One iteration

$$
\begin{array}{rcrcrcrcrcrcr}
z & & & & & & 4x_3 & - & x_4 & & & = & 14 \\
& x_1 & & & & + & 2x_3 & - & x_4 & & & = & 2 \\
& & + & x_2 & & - & x_3 & + & x_4 & & & = & 4 \\
& & & & & - & 3x_3 & + & 2x_4 & + & x_5 & = & 6 \\
& & & & & & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0
\end{array}
$$

Basis: $\{1, 2, 5\}$.
Entering variable: x_4.
Exiting variable: x_5 (since $\min\{\cdot, \frac{4}{1}, \frac{6}{2}\} = 3$).

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Pivot.

$$
\begin{array}{rrrrrrrrrrrr}
z & & & & + & \frac{5}{2}x_3 & & & + & \frac{1}{2}x_5 & = & 17 \\
 & x_1 & & & + & \frac{1}{2}x_3 & & & + & \frac{1}{2}x_5 & = & 5 \\
 & & + & x_2 & + & \frac{1}{2}x_3 & & & - & \frac{1}{2}x_5 & = & 1 \\
 & & & & - & \frac{3}{2}x_3 & + & x_4 & + & \frac{1}{2}x_5 & = & 3 \\
 & & & & & & & & x_1, x_2, x_3, x_4, x_5 & & \geq & 0
\end{array}
$$

Example of an unbounded problem

$$\begin{array}{ccccccccccc} z & & & + & 1.5x_3 & - & 0.25x_4 & = & 3 \\ & x_1 & & + & 0.5x_3 & - & 0.25x_4 & = & 2 \\ & & x_2 & - & 0.5x_3 & - & 0.25x_4 & = & 1 \end{array}$$

Feasible basis: $\{1, 2\}$. Basic solution: $(2, 1, 0, 0)^\top$.

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If we increase x_4 by $t \geq 0$ leaving $x_3 = 0$, the objective value increases by $0.25t$.

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If we increase x_4 by $t \geq 0$ leaving $x_3 = 0$, the objective value increases by $0.25t$.

$$\begin{array}{rcl} x_1(t) & = & 2 + 0.25t \\ x_2(t) & = & 1 + 0.25t \\ x_3(t) & = & 0 \\ x_4(t) & = & t \end{array}$$

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$$\begin{aligned} x_1(t) &= 2 + 0.25t \\ x_2(t) &= 1 + 0.25t \\ x_3(t) &= 0 \\ x_4(t) &= t \end{aligned}$$

$x(t)$ is a family of feasible solutions,

$$\lim_{t \to +\infty} c^\top x(t) = +\infty.$$

The Simplex Method

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, a feasible basis $B = \{B[1], \ldots, B[m]\}$ for $Ax = b$, $x \geq 0$;

Output: An optimal solution for $\max\{c^\top x : Ax = b, x \geq 0\}$, or we determine that the problem is unbounded.

1. Compute the tableau relative to the current basis B;
2. If $\bar{c}_j \leq 0$ for all $j \in N$, then the basic feasible solution relative to B is *optimal, STOP*.
3. Otherwise, choose k such that $\bar{c}_k > 0$;
   - 3a. If $\bar{a}_{ik} \leq 0 \; \forall i \in \{1, \ldots, m\}$, then the problem is *unbounded, STOP*.
   - 3b. Otherwise, choose $h \in \{1, \ldots, m\}$ such that

$$\bar{a}_{hk} > 0 \quad \text{and} \quad \bar{b}_h / \bar{a}_{hk} = \min_{i \,:\, \bar{a}_{ik} > 0} \bar{b}_i / \bar{a}_{ik};$$

     Set $B[h] := k$, return to 1. (x_k enters the basis in row h, $x_{B[h]}$ leaves the basis)

Pivots: x_k enters, x_h leaves

z	x_1 $\cdots$	x_h	$\cdots$ x_m	x_{m+1} $\cdots$	x_k	$\cdots x_n$	
1	0 $\ldots$	0	$\ldots$ 0	$\ldots$	$-\bar{c}_k$	$\ldots$	$\bar{z}$
0	1	0			$\bar{a}_{1k}$		$\bar{b}_1$
$\vdots$	$\ddots$	$\vdots$			$\vdots$		$\vdots$
0		1			$\bar{a}_{hk}$		$\bar{b}_h$
$\vdots$		$\vdots$	$\ddots$		$\vdots$		$\vdots$
0		0	1		$\bar{a}_{mk}$		$\bar{b}_m$

1	0 $\ldots$	$\frac{\bar{c}_k}{\bar{a}_{hk}}$	$\ldots$ 0	$\ldots$	0	$\ldots$	$\bar{z} + \frac{\bar{c}_k \bar{b}_h}{\bar{a}_{hk}}$
0	1	$-\frac{\bar{a}_{1k}}{\bar{a}_{hk}}$			0		$\bar{b}_1 - \frac{\bar{a}_{1k} \bar{b}_h}{\bar{a}_{hk}}$
$\vdots$	$\ddots$	$\vdots$			$\vdots$		$\vdots$
0		$\frac{1}{\bar{a}_{hk}}$			1		$\frac{\bar{b}_h}{\bar{a}_{hk}}$
$\vdots$		$\vdots$	$\ddots$		$\vdots$		$\vdots$
0		$-\frac{\bar{a}_{mk}}{\bar{a}_{hk}}$	1		0		$\bar{b}_m - \frac{\bar{a}_{mk} \bar{b}_h}{\bar{a}_{hk}}$

Termination of the Simplex

$$\begin{array}{llrcl}
\max & z & & & \\
& z & -\sum_{j\in N}\bar{c}_j x_j & = & \bar{z} \\
& x_{B[i]} & +\sum_{j\in N}\bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1,\ldots,m \\
& & x & \geq & 0.
\end{array}$$

New solution

$$\begin{array}{rcl}
x_k(\bar{t}) & = & \bar{t}; \\
x_{B[i]}(\bar{t}) & = & \bar{b}_i - \bar{t}\bar{a}_{ik}, \quad i = 1,\ldots,m; \\
x_j(\bar{t}) & = & 0, \quad j \in N \setminus \{k\}.
\end{array}$$

where

$$\bar{t} = \min\left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \ : \ i = 1,\ldots,m, \ \bar{a}_{ik} > 0 \right\}.$$

Entering variable: some x_k such that $\bar{c}_k > 0$.

Exiting variable: some $x_{B[h]}$ such that $\bar{a}_{hk} > 0$ and $\bar{t} = \frac{\bar{b}_h}{\bar{a}_{hk}}$

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x_{B[i]}(\bar{t}) & = & \bar{b}_i - \bar{t}\bar{a}_{ik}, \quad i = 1, \ldots, m; \\
x_j(\bar{t}) & = & 0, \quad j \in N \setminus \{k\}.
\end{array}
$$

$$
\bar{t} := \min_{i \in \{1,\ldots,m\} \,:\, \bar{a}_{ik} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \right\}
$$

- If $\bar{t} > 0$, the objective value increases <u>strictly</u>.
- If $\bar{t} = 0$, the basis changes, but the corresponding basic feasible <u>solution remains the same</u>.

Cycling example

Bad *tie-breaking rule:* choose the entering variable with highest reduced cost, and the exiting variable with highest column coefficient.

1	-2.3	-2.15	13.55	0.4	0	0	0
0	0.4	0.2	-1.4	-0.2	1	0	0
0	-7.8	-1.4	7.8	0.4	0	1	0

1	0	-1	5.5	-0.75	5.75	0	0
0	1	0.5	-3.5	-0.5	2.5	0	0
0	0	2.5	-19.5	-3.5	19.5	1	0

1	0	0	-2.3	-2.15	13.55	0.4	0
0	1	0	0.4	0.2	-1.4	-0.2	0
0	0	1	-7.8	-1.4	7.8	0.4	0

This is the same tableau as in the beginning, only shift by two position. Repeating other two times (i.e, after 4 other pivots), we return the original tableau.

Degeneracy

Definition

A basis B is said to be **degenerate** if $\bar{b}_i = 0$ for some $i \in \{1, \ldots, m\}$ (where $\bar{b} = A_B^{-1} b$).

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- If all bases are non-degenerate, then the simplex method terminates regardlessly of how we choose the variables that enters or leaves.
- If there are degenerate bases, we could cycle.

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- If there are degenerate bases, we could cycle.

To prevent cycling, we need to be careful in how we choose the entering and exiting variables.

An anti-cycling rule

Bland's rule:

- Among all variables with positive reduced cost, choose as entering variable the variable x_k such that the index k is the smallest possible.
- Let $\bar{t} = \min\{\frac{\bar{b}_i}{\bar{a}_{ik}} : \bar{a}_{ik} > 0\}$. Choose as exiting variable the variable $x_{B[h]}$ such that $\bar{a}_{hk} > 0$, $\frac{\bar{b}_h}{\bar{a}_{hk}} = \bar{t}$, and such that $B[h]$ is smallest possible.

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$$
\begin{array}{cccccccccccc}
z & & - & 3x_2 & - & 7x_3 & & & & & = & 26 \\
& x_1 & + & \frac{3}{2}x_2 & + & \frac{2}{3}x_3 & & & & & = & 18 \\
& & & 0.4x_2 & - & 0.2x_3 & & & + & x_5 & = & 3.6 \\
& & & \frac{1}{3}x_2 & - & \frac{2}{3}x_3 & + & x_4 & & & = & 3
\end{array}
$$

QUIZ: Which are the entering and exiting variables according to Bland's rule?

(A) Enter: x_2, exit: x_4. *(B)* Enter: x_2, exit: x_5.
(C) Enter: x_3, exit: x_1.

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Theorem

The simplex method with Bland's pivot rule terminates for every possible instance of an LP problem and every possible choice of starting feasible basis.