MA427 − Exercise set 5
Academic year 2018-19

Exercise 5.1 Consider two formulations for the same mixed integer set S, say

$$S = \{x \in \mathbb{R}^n \ : \ Ax \le b, \ x \ge 0, \ x_i \in \mathbb{Z} \text{ for } i \in I\} = \{x \in \mathbb{R}^n \ : \ A'x \le b', \ x \ge 0, \ x_i \in \mathbb{Z} \text{ for } i \in I\}.$$

Show that the first formulation is better than the second if, for every inequality $\alpha^\top x \le \beta$ of the system $A'x \le b'$, there exists a vector $u \ge 0$ of multipliers for the inequalities of $Ax \le b$ such that

$$\alpha \le A^\top u, \ \beta \ge b^\top u.$$

Exercise 5.2 Show that the following three formulations describe the same binary set. That is, $S_1 = S_2 = S_3$.

$$
\begin{aligned}
S_1 &= \{x \in \{0,1\}^4 \ : \ 97x_1 + 32x_2 + 25x_3 + 20x_4 \le 139\}, \\
S_2 &= \{x \in \{0,1\}^4 \ : \ 2x_1 + x_2 + x_3 + x_4 \le 3\}, \\
S_3 &= \left\{
\begin{array}{rll}
x \in \{0,1\}^4 \ : \ x_1 + x_2 + x_3 & \le & 2 \\
x_1 + x_2 \quad + x_4 & \le & 2 \\
x_1 \quad + x_3 + x_4 & \le & 2
\end{array}
\right\}.
\end{aligned}
$$

Show that among these formulations, both the 2nd and 3rd one are better than the first one, whereas the 2nd and 3rd are incomparable.

Exercise 5.3 Solve the following pure integer programming problem by branch-and-bound. Solve the first LP relaxation using the primal simplex method (after introducing slack variables x_3, x_4, x_5, the basis $B = \{3,4,5\}$) is primal feasible). The subsequent LP relaxations obtained after each branching should be solved with the dual simplex. Draw the feasible region and show each branching graphically.

$$
\begin{array}{rrcrcl}
\max & 2x_1 & + & 5x_2 & & \\
& 4x_1 & + & x_2 & \le & 28 \\
& x_1 & + & 4x_2 & \le & 27 \\
& x_1 & - & x_2 & \le & 1 \\
& \multicolumn{4}{r}{x_1, x_2 \ge 0} & \\
& \multicolumn{4}{r}{x_1, x_2 \in \mathbb{Z}} &
\end{array}
$$

Exercise 5.4 Show that the following mixed-integer programming formulation is ideal

$$
\begin{aligned}
\max \ & c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \\
& x_1 + 2x_2 + 3x_3 + 4x_4 \le 6 \\
& x \ge 0 \\
& x_1, x_2, x_3 \in \mathbb{Z}^3 \quad (x_4 \text{ continuous})
\end{aligned}
$$

Exercise 5.5 Solve the following MILP by branch-and-bound.

$$
\begin{array}{rlllll}
\min & 8y & +3x_1 & +4x_2 & +2x_3 & \\
& y & +x_1 & & & \geq & 0.2 \\
& y & & +x_2 & & \geq & 0.8 \\
& y & & & +x_3 & \geq & 1.5 \\
& y & & & & \geq & 0 \\
& \multicolumn{5}{c}{y, x_1, x_2, x_3 \geq 0} \\
& \multicolumn{5}{c}{x_1, x_2, x_3 \in \mathbb{Z}}
\end{array}
$$

(Observe that solving the LP relaxations of the subproblems can be easily done by hand, because it is easy to enumerate the basic solutions (there are few of them), and so you can just choose the best one.)