MA427 – Mathematical Optimisation
Lecture 1

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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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General information

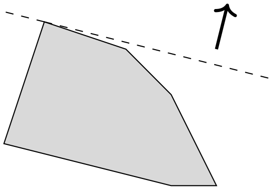
- Instructor: Giacomo Zambelli, `g.zambelli@lse.ac.uk`
- Office hours: Mondays 14:30-16:30
- Assessment: 90% written exam, 3 hours in ST. 10% course work (2 exercise sets)
- Formative work: 1 exercise set will be collected and feedback will be given.

Course overview

- **Linear programming (4 lectures)**
- **Linear mixed-integer programming (3 lectures)**
  Totally unimodular matrices, ideal formulations
  branch-and-bound, cutting planes.
- **Convex optimization (3 lectures)**
  Lagrangian duality, Karush-Kuhn-Tucker conditions, gradient descent.

Linear Programming (LP)

$$
\begin{array}{rrcl}
\max & 2x_1 + 8x_2 & & \\
\text{s.t.} & 2x_1 + x_2 & \leq & 10 \\
 & x_1 + 2x_2 & \leq & 10 \\
 & x_1 + x_2 & \leq & 6 \\
 & x_1 + 3x_2 & \leq & 12 \\
 & 3x_1 - x_2 & \geq & 0 \\
 & x_1 + 4x_2 & \geq & 4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$



Linear programming: MA423 and MA427

MA423 lectures 1-3

- Basic concepts, standard forms of LP
- Duality and optimality conditions
- Simplex Method (dictionary form)

MA427 lectures 1-4

- *Fourier-Motzkin elimination, Farkas' lemma, and the duality theorem*
- Geometry of linear programs
- Simplex Method: tableau form, two phase and dual simplex

Reminder: systems of linear equations

$$Ax = b$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

Gaussian elimination

- Efficient method to solve a system of linear equations.
- Elementary operations:
  - Multiply a row by a nonzero real number.
  - Add a multiple of one row to another one.
  - Swap two rows.

Proposition

The system $Ax = b$ is feasible, if and only if $\mathrm{rk}(A) = \mathrm{rk}(A|b)$.

Fourier-Motzkin elimination

Joseph Fourier (1768-1830)

Theodore Motzkin (1908-1970)

Linear feasibility problem: system of linear inequalities

$$Ax \leq b$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

- Goal: try to mimic Gaussian elimination for systems of linear inequalities.
- Elementary operations that preserve feasibility:
  - Multiply a row by a positive real number.
  - Add a positive multiple of one row to another one.
  - Swap two rows.

Example

Are the following two systems feasible?

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ -4x_1 + 2x_2 &\leq -30 \\ x_1 - x_2 &\leq 8 \\ -x_2 &\leq 4 \end{aligned}$$

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Fourier-Motzkin elimination

$$Ax \le b$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x = (x_1, x_2, \ldots, x_{n-1}, x_n)$.

- Reduce to a system of $n-1$ variables, by eliminating x_n.
- Let $I = \{1, \ldots, m\}$ and

$$I^+ = \{i \in I \,:\, a_{in} > 0\}, \quad I^- = \{i \in I \,:\, a_{in} < 0\}, \quad I^0 = \{i \in I \,:\, a_{in} = 0\}.$$

- Divide the ith row by $|a_{in}|$ for each $i \in I^+ \cup I^-$ to get $a'_{in} = \pm 1$ or 0 everywhere.

$$\begin{array}{llll} \sum_{j=1}^{n-1} a'_{ij} x_j & +x_n & \le b'_i, & i \in I^+ \\ \sum_{j=1}^{n-1} a'_{ij} x_j & -x_n & \le b'_i, & i \in I^- \\ \sum_{j=1}^{n-1} a_{ij} x_j & & \le b_i, & i \in I^0 \end{array}$$

Fourier-Motzkin elimination

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Eliminate x_n by:

- For each pair $i \in I^+$ and $k \in I^-$, sum the two inequalities indexed by i and k.
- Remove all inequalities in $I^+ \cup I^-$.

New system:

$$\begin{array}{rcll} \sum_{j=1}^{n-1}(a'_{ij} + a'_{kj}) x_j & \leq & b'_i + b'_k, & i \in I^+,\ k \in I^-, \\ \sum_{j=1}^{n-1} a_{ij} x_j & \leq & b_i, & i \in I^0. \end{array}$$

Fourier-Motzkin elimination

Original system:

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Theorem

A vector $(\bar{x}_1, \ldots, \bar{x}_{n-1})$ satisfies the system new system if and only if there exists $\bar{x}_n$ such that $(\bar{x}_1, \ldots, \bar{x}_{n-1}, \bar{x}_n)$ satisfies the original system.

Fourier-Motzkin elimination

Original system:

$$\begin{array}{llll} \sum_{j=1}^{n-1} a'_{ij} x_j & +x_n & \leq b'_i, & i \in I^+ \\ \sum_{j=1}^{n-1} a'_{ij} x_j & -x_n & \leq b'_i, & i \in I^- \\ \sum_{j=1}^{n-1} a_{ij} x_j & & \leq b_i, & i \in I^0 \end{array}$$

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Proof: Let

$$\ell = \max_{k \in I^-} \left\{ \sum_{j=1}^{n-1} a'_{kj} \bar{x}_j - b'_k \right\}, \quad u = \min_{i \in I^+} \left\{ b'_i - \sum_{j=1}^{n-1} a'_{ij} \bar{x}_j \right\}$$

Selecting any $\bar{x}_n$ such that $\ell \leq \bar{x}_n \leq u$ gives a feasible solution to the original system.

Fourier-Motzkin elimination

Given a system of linear inequalities $Ax \leq b$*, let* $A^n := A$*,* $b^n := b$*;*
For $i = n, \ldots, 1$*, eliminate variable* x_i *from* $A^i x \leq b^i$ *to obtain system* $A^{i-1}x \leq b^{i-1}$.

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For $i = n, \ldots, 1$, eliminate variable x_i from $A^i x \leq b^i$ to obtain system $A^{i-1}x \leq b^{i-1}$.

- Final system $A^0 x \leq b^0$ has inequalities $0 \leq b_i^0$. It is feasible if and only if all b_i^0's are nonnegative.
- Given a solution to $A^i x \leq b^i$, we can obtain a solution to $A^{i+1}x \leq b^{i+1}$ for $i = 0, 1, 2, \ldots, n-1$.

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- Final system $A^0 x \leq b^0$ has inequalities $0 \leq b^0_i$. It is feasible if and only if all b^0_i's are nonnegative.
- Given a solution to $A^i x \leq b^i$, we can obtain a solution to $A^{i+1}x \leq b^{i+1}$ for $i = 0, 1, 2, \ldots, n-1$.
- Number of iterations can be large: $|I^+| + |I^-|$ inequalities are replaced by $|I^+| \cdot |I^-|$ new ones: number of inequalities may grow exponentially.

Fourier-Motzkin elimination

$$
\begin{array}{rrrrrrcr}
- & x_1 & & & & & \leq & -1 \\
& & - & x_2 & & & \leq & -1 \\
& & & & - & x_3 & \leq & -1 \\
- & x_1 & - & x_2 & & & \leq & -3 \\
- & x_1 & & & - & x_3 & \leq & -3 \\
& & - & x_2 & - & x_3 & \leq & -3 \\
& x_1 & + & x_2 & + & x_3 & \leq & 6
\end{array}
$$

Quiz

We have a system of linear inequalities with 20 inequalities having $+x_n$, and 2 inequalities not containing x_n. (That is, $|I^+| = 20$, $|I^-| = 0$, $|I^0| = 2$.) How many inequalities do we get after eliminating x_n?

(A) 2

(B) 22

(C) 40

Characterising feasibility: Farkas' lemma

Gyula Farkas (1847-1930)

Expressing inequalities in Fourier-Motzkin

Proposition

Every inequality added during Fourier-Motzkin elimination is a nonnegative combination of the inequalities in the original system $Ax \leq b$. That is, for every inequality $c^\top x \leq d$ in the system $A^i x \leq b^i$, we can find a nonnegative vector $u \in \mathbb{R}^m$, $u \geq 0$, such that $c^\top = u^\top A$, and $d = u^\top b$.

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Theorem (Farkas' lemma)

Exactly one of the following two systems has a feasible solution:

- $Ax \leq b$
- $u^\top A = 0, \; u^\top b < 0, \; u \geq 0$

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Proof:

1. *Both systems cannot be simultaneously feasible:*

$$0 = (u^\top A)x = u^\top (Ax) \leq u^\top b < 0$$

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1. *Both systems cannot be simultaneously feasible:*

$$0 = (u^\top A)x = u^\top (Ax) \leq u^\top b < 0$$

2. *If $Ax \leq b$ is infeasible, then $u^\top A = 0,\ u^\top b < 0,\ u \geq 0$ is feasible:*
   FM-elimination gives infeasible $A^0 x \leq b^0$, including an inequality $0 \leq b_i^0$, for $b_i^0 < 0$.

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   FM-elimination gives infeasible $A^0 x \leq b^0$, including an inequality $0 \leq b_i^0$, for $b_i^0 < 0$.
   $\exists u \geq 0$ $u^\top A = 0$, $u^\top b = b_i^0 < 0$.

Farkas' lemma

Theorem (Farkas' lemma, standard equality form)

Exactly one of the following two systems has a feasible solution:

- $Ax = b, x \geq 0$
- $u^\top A \leq 0,\ u^\top b > 0$

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Exactly one of the following two systems has a feasible solution:

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Proof: Rewrite $Ax = b, x \geq 0$ as

$$\begin{pmatrix} A \\ -A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

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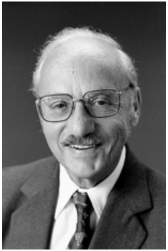
Previous form gives $(v, v', w) \geq 0$ such that
$v^\top A - v'^\top A - Iw = 0$, $v^\top b - v'^\top b - w0 < 0$. Set $u = v' - v$.

Characterising optimality: Duality theory

John von Neumann (1903-1957)



George Dantzig (1914-2005)

LP duality

Theorem (Strong Duality Theorem)

Given a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, let

$$P := \{x \,:\, Ax \leq b\}, \quad D := \{u \,:\, u^\top A = c,\ u \geq 0\}.$$

If P and D are both nonempty, then

$$\max\{c^\top x \,:\, Ax \leq b\} = \min\{u^\top b \,:\, u^\top A = c,\ u \geq 0\},$$

and there exist $x^* \in P$ and $y^* \in D$ such that $c^\top x^* = {u^*}^\top b$.

Proof of the Strong Duality Theorem

- Direction max $\leq$ min

$$c^\top x = (u^\top A)x = u^\top (Ax) \leq u^\top b.$$

- Direction min $\leq$ max: via Fourier-Motzkin elimination.

Proof of the Strong Duality Theorem

- Consider the feasibility problem

$$\begin{aligned} z - c^\top x &\leq 0 \\ Ax &\leq b \end{aligned}$$

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- The resulting system can be reduced to a single inequality

$$z \leq \beta$$

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- There exists a solution $(z, \bar{x})$ to the original system with $z = \beta$; we get $\max\{c^\top x : Ax \leq b\} = \beta$.

Proof of the Strong Duality Theorem

- Consider the feasibility problem

$$\begin{aligned} z - c^\top x &\leq 0 \\ Ax &\leq b \end{aligned}$$

- Apply Fourier-Motzkin elimination to all x_i variables, but keep z.
- The resulting system can be reduced to a single inequality

$$z \leq \beta$$

- There exists a solution $(z, \bar{x})$ to the original system with $z = \beta$; we get $\max\{c^\top x : Ax \leq b\} = \beta$.
- We can express $z \leq \beta$ as a nonnegative combination (u_0, u^*) of the original system. It follows that ${u^*}^\top A = c$ and ${u^*}^\top b = \beta$.

Complementary slackness

Theorem

Given a matrix $A \in \mathbb{R}^{m \times n}$ *and vectors* $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, *let*

$$P := \{x \,:\, Ax \le b\}, \quad D := \{u \,:\, u^\top A = c,\ u \ge 0\}.$$

Given $x^* \in P$ *and* $u^* \in D$, x^* *and* u^* *are optimal solutions for the primal and dual problem* $\max\{c^\top x \,:\, x \in P\}$ *and* $\min\{u^\top b \,:\, u \in D\}$, *respectively, if and only if the following complementary slackness conditions hold*

$$u_i^*({a_i}^\top x^* - b_i) = 0 \ \textit{for}\ i = 1, \ldots, m.$$

Proof.

$$c^\top x^* = ({u^*}^\top A)x = {u^*}^\top (Ax^*) \le {u^*}^\top b.$$

Unbounded objectives

Theorem

Given a matrix $A \in \mathbb{R}^{m \times n}$ *and vectors* $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, *let*

$$P := \{x : Ax \leq b\}, \quad D := \{u : u^\top A = c,\ u \geq 0\}.$$

Assume that $P \neq \emptyset$. *Then the primal program* $\max\{c^\top x : x \in P\}$ *is unbounded if and only if* $D = \emptyset$, *which is equivalent to the existence of a vector* $\bar{y}$ *with* $A\bar{y} \leq 0$, $c^\top \bar{y} > 0$.

Proof.

- *Farkas' lemma:* $D = \emptyset \Leftrightarrow \exists \bar{y} : A\bar{y} \leq 0, c^\top \bar{y} > 0$.

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Assume that $P \neq \emptyset$. Then the primal program $\max\{c^\top x \,:\, x \in P\}$ is unbounded if and only if $D = \emptyset$, which is equivalent to the existence of a vector $\bar{y}$ with $A\bar{y} \leq 0$, $c^\top \bar{y} > 0$.

Proof.

- *Farkas' lemma: $D = \emptyset \Leftrightarrow \exists \bar{y} :\ A\bar{y} \leq 0, c^\top \bar{y} > 0$.*
- *$D \neq \emptyset$:* Strong duality gives an upper bound on $\max\{c^\top x \,:\, x \in P\}$.

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Assume that $P \neq \emptyset$. Then the primal program $\max\{c^\top x \,:\, x \in P\}$ is unbounded if and only if $D = \emptyset$, which is equivalent to the existence of a vector $\bar{y}$ with $A\bar{y} \leq 0$, $c^\top \bar{y} > 0$.

Proof.

- *Farkas' lemma: $D = \emptyset \Leftrightarrow \exists \bar{y} :\ A\bar{y} \leq 0, c^\top \bar{y} > 0$.*
- *$D \neq \emptyset$:* Strong duality gives an upper bound on $\max\{c^\top x \,:\, x \in P\}$.
- *$D = \emptyset$:* For any $\bar{x} \in P$, $\lambda > 0$,

$$\bar{x} + \lambda \bar{y} \in P, \quad \lim_{\lambda \to \infty} c^\top (\bar{x} + \lambda \bar{y}) = \infty.$$