MA427 – Exercise set 2

Academic year 2018-19

Exercise 2.1

(a) Show that, if a linear programming problem $\max\{c^T x : x \in P\}$ (where P denotes the feasible region) has two distinct optimal solutions x', x'', then all solutions in the line segment joining x' and x'' are optimal.

(b) Show that in any LP, the number of distinct optimal solutions can be $0, 1$, or ∞.

Exercise 2.2 Let $x^0, x^1, \ldots, x^q \in \mathbb{R}^n$. Show that the following statements are equivalent:

(i) $x^0, x^1, \ldots, x^q$ are affinely independent;

(ii) $x^1 - x^0, \ldots, x^q - x^0$ are linearly independent;

(iii) The vectors $\binom{x^0}{1}, \binom{x^1}{1}, \ldots, \binom{x^q}{1}$ are linearly independent.

Exercise 2.3 Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polytope, and $z \in P$ a feasible solution. Assume that for some $r \in \mathbb{R}^n$, P contains the infinite ray starting from z in the direction r, that is

$$\{z + \lambda r : \lambda \geq 0\}.$$

Show that for every $x \in P$, P also contains the infinite ray

$$\{x + \lambda r : \lambda \geq 0\}.$$

Exercise 2.4 Let $S \subseteq \{0,1\}^n$, that is, S is a set of n dimensional vectors with all coordinates 0 or 1. Let $P = \text{conv}(S)$. Show that S is the set of vertices of P.

Exercise 2.5 The *n-dimensional hypercube* is defined as

$$\{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, i = 1, 2, \ldots, n\}.$$

What is the number of facets and the number of vertices? What is the total number of proper faces?

Exercise 2.6 Consider the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ for $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Assume that for any $i = 1, \ldots, m$, there exists a point $x \in P$ such that the ith inequality is strict: $a_i^\top x < b_i$. Show that there exists a point $\bar{x} \in P$ such that all inequalities are strict for $\bar{x}$.