MA427 – Exercise set 3

Academic year 2018-19

Exercise 3.1 Let $H_1, \ldots, H_k$ be closed half-spaces in $\mathbb{R}^d$. Show that, if $H_1 \cap H_2 \cap \ldots \cap H_k = \emptyset$, then there exists a collection of at most $d+1$ half-spaces among $H_1, \ldots, H_k$ whose intersection is already empty.

This is a special case of Helly's theorem:
Helly's theorem. *Let $C_1, \ldots, C_k$ be closed convex sets in $\mathbb{R}^d$. If $C_1 \cap C_2 \cap \ldots \cap C_k = \emptyset$, then there exists a collection of at most $d+1$ sets among $C_1, \ldots, C_k$ whose intersection is already empty.*

Exercise 3.2 Consider the LP problem $\max\{c^\top x \mid Ax = b,\ x \geq 0\}$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & -6 \\ 0 & -1 & -1 & -3 & 2 & 1 \\ 1 & 2 & 1 & 3 & -1 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} 11 \\ -6 \\ 13 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ -2 \\ 3 \\ 4 \\ -10 \\ 5 \end{pmatrix}$$

1. Show that $B = \{2, 4, 5\}$ is a feasible basis, and write the problem in tableau form with respect to the basis B.
2. Apply the Simplex Method with Bland's rule to solve the LP problem starting from the feasible basis $B = \{2, 4, 5\}$.

Exercise 3.3 Consider the following LP problem.

$$\begin{array}{rrcrcl} \max & 2x_1 & + & x_2 & & \\ & x_1 & - & 2x_2 & \leq & 2 \\ & 2x_1 & - & x_2 & \leq & 5 \\ & -4x_1 & + & x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

Transform the above problem in standard equality form by adding the slack variables x_4, x_5, x_6, relative to the first, second and third constraint, respectively. Note that $\{4, 5, 6\}$ is a feasible basis, and apply the Simplex Method starting from such basis.
From the final tableau, determine a family of feasible solutions that can take arbitrarily large objective value.

Exercise 3.4 Let A be an $m \times n$ matrix. Given vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, consider the LP problem in standard equality form

$$\begin{array}{ll} \max & c^\top x \\ & Ax = b \qquad (P). \\ & x \geq 0. \end{array}$$

Let x^* be an optimal basic feasible solution for (P), and B be the basis that determines it.

1. Show that, if $\bar{c}_j < 0$ for every $j \in N$, then x^* is the unique optimal solution.
2. Suppose that $x^*_j > 0$ for every $j \in B$. Show that the dual solution is unique. What is the optimal dual solution for the problem?

Exercise 3.5 Consider an LP problem in standard equality form

$$\begin{aligned} \max \quad & c^\top x \\ & Ax = b \qquad (P) \\ & x \geq 0 \end{aligned}$$

and let B be a feasible basis for the problem. Let $\bar{x}$ be the feasible solution determined by B, and let $\bar{c}_j$, $j = 1, \dots, n$, be the reduced costs of the variables x_j with respect to the basis B.

1. We have seen that, if $\bar{c}_j \leq 0$ for all $j \in \{1, \dots, n\}$, then the solution $\bar{x}$ is optimal. Is the converse statement true? (That is, is it true that if $\bar{x}$ is optimal, then the reduced costs associated with B must all be non-positive?)
2. Suppose that B is the current basis at an iteration of the simplex method. Let x_k be the variable selected to enter the basis (hence $\bar{c}_k > 0$). Let x_ℓ the variable that leaves the basis. Hence at the new iteration the basis becomes $\tilde{B} := B \setminus \{\ell\} \cup \{k\}$. What is the reduced cost of x_ℓ with respect to the new basis $\tilde{B}$?
3. Use the previous point to conclude that the variable x_ℓ that leaves the basis cannot re-enter at the next iteration.