MA427 Lecture 2
The geometry of LP

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Today's lecture

- Proving the duality theorem and characterising unbounded problems
- Linear, affine, conic, and convex combinations
- Faces of polyhedra
- Vertices of polyhedra

LP duality

Theorem (Strong Duality Theorem)

Given a matrix $A \in \mathbb{R}^{m \times n}$ *and vectors* $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, *let*

$$P := \{x \,:\, Ax \le b\}, \quad D := \{u \,:\, u^\top A = c^\top,\ u \ge 0\}.$$

If P *and* D *are both nonempty, then*

$$\max\{c^\top x \,:\, Ax \le b\} = \min\{u^\top b \,:\, u^\top A = c^\top,\ u \ge 0\},$$

and there exist $x^* \in P$ *and* $y^* \in D$ *such that* $c^\top x^* = {u^*}^\top b$.

Proof of the Strong Duality Theorem

- Direction max $\leq$ min

$$c^\top x = (u^\top A)x = u^\top (Ax) \leq u^\top b.$$

- Direction min $\leq$ max: via Fourier-Motzkin elimination.

Proof of the Strong Duality Theorem

- Consider the feasibility problem

$$\begin{aligned} z - c^\top x &\leq 0 \\ Ax &\leq b \end{aligned}$$

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- There exists a solution $(z, \bar{x})$ to the original system with $z = \beta$; we get $\max\{c^\top x : Ax \leq b\} = \beta$.

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$$z \leq \beta$$

- There exists a solution $(z, \bar{x})$ to the original system with $z = \beta$; we get $\max\{c^\top x : Ax \leq b\} = \beta$.
- We can express $z \leq \beta$ as a nonnegative combination (u_0, u^*) of the original system. It follows that $u^{*\top} A = c^\top$ and $u^{*\top} b = \beta$.

Unbounded objectives

Theorem

Given a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, let

$$P := \{x \,:\, Ax \leq b\}, \quad D := \{u \,:\, u^\top A = c^\top,\; u \geq 0\}.$$

Assume that $P \neq \emptyset$. Then the primal program $\max\{c^\top x \,:\, x \in P\}$ is unbounded if and only if $D = \emptyset$, which is equivalent to the existence of a vector $\bar{y}$ with $A\bar{y} \leq 0$, $c^\top \bar{y} > 0$.

Proof.

- *Farkas' lemma:* $D = \emptyset \Leftrightarrow \exists \bar{y} : \; A\bar{y} \leq 0, c^\top \bar{y} > 0.$

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Proof.

- *Farkas' lemma: $D = \emptyset \Leftrightarrow \exists \bar{y} :\ A\bar{y} \leq 0, c^\top \bar{y} > 0$.*
- *$D \neq \emptyset$:* Strong duality gives an upper bound on $\max\{c^\top x \,:\, x \in P\}$.

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Proof.

- *Farkas' lemma: $D = \emptyset \Leftrightarrow \exists \bar{y} :\ A\bar{y} \leq 0, c^\top \bar{y} > 0$.*
- *$D \neq \emptyset$:* Strong duality gives an upper bound on $\max\{c^\top x \,:\, x \in P\}$.
- *$D = \emptyset$:* For any $\bar{x} \in P$, $\lambda > 0$,

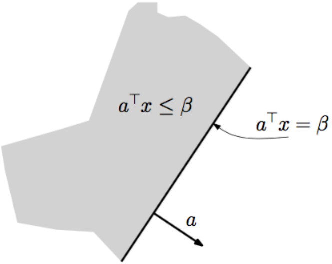
$$\bar{x} + \lambda \bar{y} \in P, \quad \lim_{\lambda \to \infty} c^\top(\bar{x} + \lambda \bar{y}) = \infty.$$

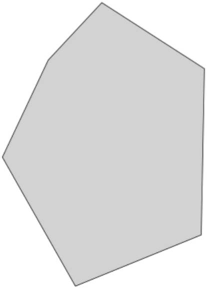
The geometry of Linear Programming

Hyperplanes, half-spaces and polyhedra

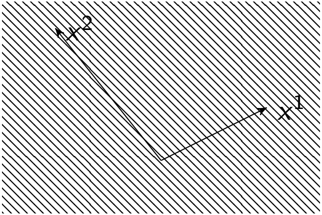
Given $a \in \mathbb{R}^n$, $a \neq 0$, and $\beta \in \mathbb{R}$:

- *hyperplane*: $\{x \in \mathbb{R}^n \mid a^\top x = \beta\}$.
- *half-space*: $\{x \in \mathbb{R}^n \mid a^\top x \leq \beta\}$.
- *polyhedron*: intersection of half-spaces $=$ feasible region of LP.

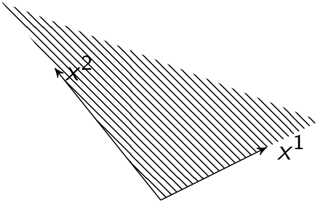




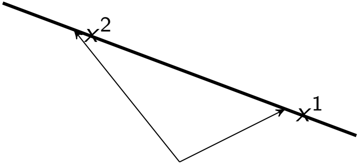
Linear, affine, convex, and conic combinations



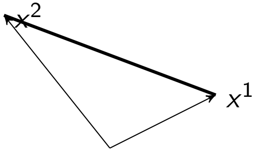
Linear combination



Conic combination



Affine combination

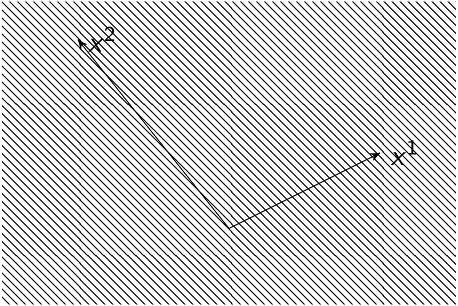


Convex combination

Linear combinations and linear spaces

- $x \in \mathbb{R}^n$ is a *linear combination* of $x^1, \dots, x^q \in \mathbb{R}^n$ if $\exists \lambda_1, \dots, \lambda_q$:
$$x = \sum_{j=1}^{q} \lambda_j x^j.$$
- $x^1, \dots, x^q \in \mathbb{R}^n$ are *linearly independent*, if $\sum_{j=1}^{q} \lambda_j x^j = 0$ implies that $\lambda_i = 0$, $j = 1, \dots, q$.
- *Linear space*: set closed under taking linear combinations = intersection of hyperplanes through the origin
$$\mathcal{L} = \{x \in \mathbb{R}^n : Ax = 0\}$$
- *Basis*: maximal set of linearly independent vectors in $\mathcal{L}$.
- *Dimension* of $\mathcal{L}$: cardinality of any basis, equals $\dim(\mathcal{L}) = n - \mathrm{rk}(A)$.

Linear combinations and linear spaces



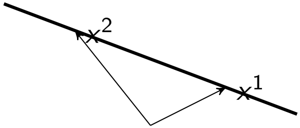
Affine combinations and affine spaces

- $x \in \mathbb{R}^n$ is a *affine combination* of $x^1, \ldots, x^q \in \mathbb{R}^n$ if $\exists \lambda_1, \ldots, \lambda_q$, $\sum_{j=1}^q \lambda_j = 1$,

$$x = \sum_{j=1}^{q} \lambda_j x^j.$$

- $x^0, x^1, \ldots, x^q \in \mathbb{R}^n$ are *affinely independent*, if $\sum_{j=0}^q \lambda_j x^j = 0$, $\sum_{j=0}^q \lambda_j = 0$ implies that $\lambda_i = 0$, $j = 0, \ldots, q$.
- Equivalently, none of the vectors can be written as an affine combination of the others.
- *Affine space*: set closed under taking affine combinations:

$$\mathcal{A} = \{x \in \mathbb{R}^n : Ax = b\}$$

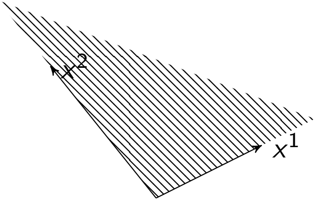


Dimension

- *Basis of affine space*: maximal set of affinely independent vectors in $\mathcal{A}$.
- *Dimension* of set $S \subseteq \mathbb{R}^n$: maximum number of affinely independent vectors in S minus one.
- $dim(\emptyset) = -1$.
- If $\mathcal{A} = \{x \in \mathbb{R}^n : Ax = b\} \neq \emptyset$, then $dim(\mathcal{A}) = n - \mathrm{rk}(A)$.

Conic combinations and cones

- $x \in \mathbb{R}^n$ is a *conic combination* of $x^1, \ldots, x^q \in \mathbb{R}^n$ if $\exists \lambda_1, \ldots, \lambda_q \geq 0$:
$$x = \sum_{j=1}^{q} \lambda_j x^j.$$
- *Cone*: set closed under taking conic combinations.
- *cone*(S): cone generated by set $S \subseteq \mathbb{R}^n$.



- *Ray*: $\text{cone}(r) = \{\lambda r : \lambda \geq 0\}$.

The geometric interpretation of Farkas' lemma

Theorem (Farkas' lemma)

Exactly one of the following two systems has a feasible solution:

- $Ax = b$, $x \geq 0$
- $u^\top A \leq 0,\ u^\top b > 0$

The geometric interpretation of Farkas' lemma

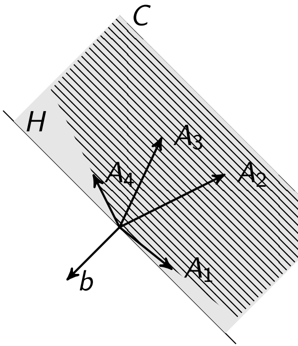
Theorem (Farkas' lemma)

Exactly one of the following two systems has a feasible solution:

- $Ax = b,\ x \geq 0$
- $u^\top A \leq 0,\ u^\top b > 0$

- Columns of A: $A^1, A^2, \ldots, A^n$

$$\begin{aligned} C &= cone(\{A_1, A_2, \ldots, A_n\}) \\ &= \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, x \geq 0, y = Ax\}. \end{aligned}$$



- Primal system feasible if and only if $b \in C$
- Dual system is feasible: $H = \{y \in R^m : u^\top y \leq 0\}$

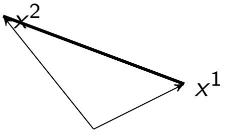
$$b \notin H \supseteq C$$

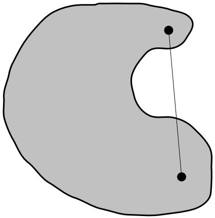
Convex combinations and convex sets

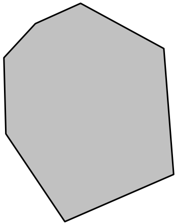
- $x \in \mathbb{R}^n$ is a *convex combination* of $x^1, \ldots, x^q \in \mathbb{R}^n$ if $\exists \lambda_1, \ldots, \lambda_q \geq 0$, $\sum_{j=1}^q \lambda_j = 1$:

$$x = \sum_{j=1}^{q} \lambda_j x^j.$$

- *Convex set*: closed under taking convex combinations.







Convex combinations and convex sets

- Half-spaces are convex: $H = \{x \in \mathbb{R}^n : a^\top x \leq \beta\}$.

$$x^1, x^2 \in H,\ x = \lambda_1 x^1 + \lambda_2 x^2,\ 0 \leq \lambda_1, \lambda_2,\ \lambda_1 + \lambda_2 = 1$$
$$a^\top x = a^\top(\lambda_1 x^1 + (1 - \lambda_1)x^2) \leq \lambda_1 \beta + (1 - \lambda_1)\beta = \beta$$

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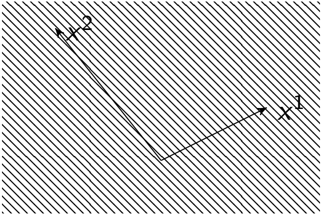
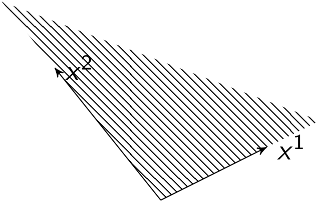
- Intersections of convex sets are convex.
- Consequently, *every polyhedron is convex*.

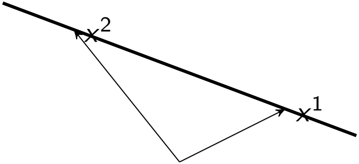
Linear, affine, convex, and conic combinations



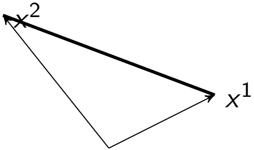
Linear combination



Conic combination



Affine combination



Convex combination

Valid inequalities

An inequality $c^\top x \leq \delta$ is *valid* for $P \subseteq \mathbb{R}^n$ if $c^\top x \leq \delta$ is satisfied by every point in P.

Theorem

Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty polyhedron. An inequality $c^\top x \leq \delta$ is valid for P if and only if there exists $u \geq 0$ such that $u^\top A = c^\top$ and $u^\top b \leq \delta$.

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Proof - "if" part: If $u^\top A = c^\top$, $u^\top b \leq \delta$, $u \geq 0$:

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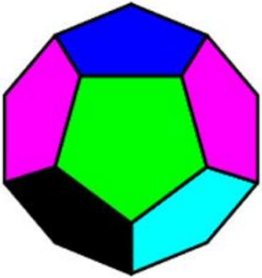
Proof - "if" part: If $u^\top A = c^\top$, $u^\top b \leq \delta$, $u \geq 0$:

$$c^\top x = (u^\top A)x = u^\top (Ax) \leq u^\top b \leq \delta.$$

Proof - "only if" part: If $c^\top x \leq \delta$ is valid, then $\max\{c^\top x : x \in P\} \leq \delta$. Apply duality.

Faces of polyhedra



Faces of polyhedra

For a polyhedron P and a valid inequality $c^\top x \leq \delta$, a *face* is

$$F := P \cap \{x \in \mathbb{R}^n : c^\top x = \delta\}$$

Faces of polyhedra

For a polyhedron P and a valid inequality $c^\top x \leq \delta$, a *face* is

$$F := P \cap \{x \in \mathbb{R}^n : c^\top x = \delta\}$$

- The inequality $c^\top x \leq \delta$ *defines* the face F.
- The hyperplane $\{x \in \mathbb{R}^n : c^\top x = \delta\}$ is the *supporting hyperplane* of F.
- Every face of a polyhedron is a polyhedron.
- $\emptyset$ and P are always faces. If $F \neq \emptyset, P$, then F is a *proper* face.
- *Facet*: inclusionwise maximal proper face.

Faces of polyhedra

Theorem

Let $P := \{x \in \mathbb{R}^n : a_i^\top x \le b_i,\ i \in M\}$, *assume* $P \ne \emptyset$. *For any* $I \subseteq M$, *the set*

$$F_I := \{x \in \mathbb{R}^n : a_i^\top x = b_i, i \in I,\ a_i^\top x \le b_i, i \in M \setminus I\}$$

is a face of P. *Conversely, if* F *is a nonempty face of* P, *then* $F = F_I$ *for some* $I \subseteq M$.

Faces of polyhedra

Theorem

Let $P := \{x \in \mathbb{R}^n : a_i^\top x \leq b_i,\ i \in M\}$*, assume* $P \neq \emptyset$*. For any* $I \subseteq M$*, the set*

$$F_I := \{x \in \mathbb{R}^n : a_i^\top x = b_i, i \in I,\ a_i^\top x \leq b_i, i \in M \setminus I\}$$

is a face of P*. Conversely, if* F *is a nonempty face of* P*, then* $F = F_I$ *for some* $I \subseteq M$*.*

Proof - F_I is a face: set $c = \sum_{i \in I} a_i$, $\delta = \sum_{i \in I} b_i$.

Faces of polyhedra

Theorem

Let $P := \{x \in \mathbb{R}^n : a_i^\top x \leq b_i,\ i \in M\}$, assume $P \neq \emptyset$. For any $I \subseteq M$, the set

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is a face of P. Conversely, if F is a nonempty face of P, then $F = F_I$ for some $I \subseteq M$.

Proof - F_I is a face: set $c = \sum_{i \in I} a_i$, $\delta = \sum_{i \in I} b_i$.

Proof - every nonempty face defined as $\{x \in P : c^\top x = \delta\}$ can be written as F_I: Apply duality:

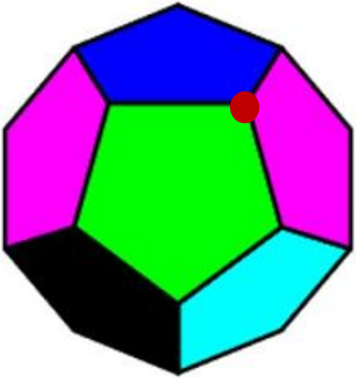
$$\max\{c^\top x : x \in P\} = \min\{u^\top b : u^\top A = c^\top, u \geq 0\}.$$

Dual optimal $\bar{u}$: set

$$I = \{i \in M : \bar{u}_i > 0\}.$$

Apply complementary slackness.

The three ways of defining vertices



Definition I - a face of dimension 0

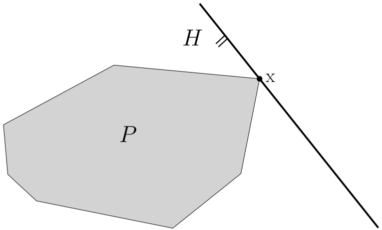
Definition

A face of dimension 0 is called a *vertex*.

- That is, for a polyhedron $P \subseteq \mathbb{R}^n$, $x^* \in P$ is a *vertex* of P if for some valid inequality $c^\top x \leq \delta$,

$$\{x^*\} = P \cap \{x \in \mathbb{R}^n : c^\top x = \delta\}.$$

- Equivalently, $c^\top x < c^\top x^*$ for every $x \in P \setminus \{x^*\}$.



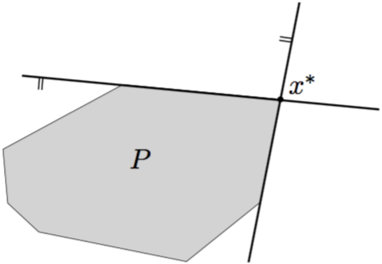
Definition II - basic feasible solution

Definition

A point $x^* \in \mathbb{R}^n$ is a *basic feasible solution* of the system

$$a_i^\top x \quad \leq \quad b_i, \quad i \in M$$

if there are n linearly independent constraints that are binding at x^*.



Quiz: basic solutions

$$
\begin{array}{rrrcl}
x_1 & +x_2 & & = & 1 \\
 & x_2 & +x_3 & \leq & 1 \\
x_1 & & +x_3 & \geq & 1 \\
x_1 & +2x_2 & +x_3 & \geq & 2 \\
x_1 & +x_2 & +x_3 & \geq & 1
\end{array}
$$

Which of these points are basic feasible solutions?

(A) $(1/2, 1/2, 1/2)$

(B) $(1, 0, 1)$

(C) $(0, 1, 0)$

Definition III - extreme points

Definition (Extreme points)

A point $x^* \in P$ is an *extreme point* of the polyhedron P, if x^* cannot be written as a proper convex combination of some points in P.

Equivalence of the three definitions

Theorem

Let $P = \{x \mid Ax \leq b\}$ *be a polyhedron in* $\mathbb{R}^n$*, where* $A \in \mathbb{R}^{m \times n}$*,* $b \in \mathbb{R}^m$*. Given a point* $x^* \in P$*, the following are equivalent.*

(i) x^* *is a vertex of* P.

(ii) x^* *is a basic feasible solution of the system* $Ax \leq b$.

(iii) x^* *is an extreme point of* P.

Equivalence of the three definitions

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(i) *x^* is a vertex of P.*

(ii) *x^* is a basic feasible solution of the system $Ax \leq b$.*

(iii) *x^* is an extreme point of P.*

Proof: (i) $\Rightarrow$ (iii): $x^* = P \cap \{x : c^\top x = \delta\}$, and assume $x^* = \lambda x' + (1 - \lambda)x''$, for $x', x'' \in P$, $0 < \lambda < 1$.

$$\delta = c^\top x^* = \lambda c^\top x' + (1 - \lambda)c^\top x'' \leq \delta$$

This implies $c^\top x' = c^\top x'' = \delta$, therefore $x' = x'' = x^*$.

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(i) *x^* is a vertex of P.*

(ii) *x^* is a basic feasible solution of the system $Ax \leq b$.*

(iii) *x^* is an extreme point of P.*

Proof: (iii) $\Rightarrow$ (ii):

- Let x^* be an extreme point, satisfying the inequalities $A'x \leq b'$ at equality. *Assume* $\mathrm{rk}(A') < n$.
- There exists $y \neq 0$: $A'y = 0$.
- For some $\varepsilon > 0$, both $x - \varepsilon y$ and $x + \varepsilon y$ are in P.

Equivalence of the three definitions

Theorem

Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Given a point $x^* \in P$, the following are equivalent.

(i) *x^* is a vertex of P.*

(ii) *x^* is a basic feasible solution of the system $Ax \leq b$.*

(iii) *x^* is an extreme point of P.*

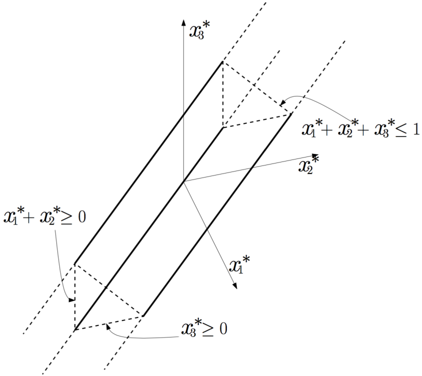
Proof: (ii) $\Rightarrow$ (i):

- x^* basic feasible, satisfying the inequalities $A'x \leq b'$ at equality with $\mathrm{rk}(A') = n$.
- Let $c = \sum_i a_i'$, $\delta = \sum_i b_i'$.
- Then, $P \cap \{x \,:\, c^\top x = \delta\} = \{x^*\}$.

Existence of vertices

Not all polyhedra have vertices.

$$
\begin{aligned}
x_1 + x_2 + x_3 &\leq 1,\\
x_1 + x_2 &\geq 0,\\
x_3 &\geq 0,
\end{aligned}
$$



Existence of vertices

Theorem

Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Assume that the problem is feasible, that is, $P \neq \emptyset$. Then, the following three properties are equivalent:

(i) *There exists a basic feasible solution in P.*

(ii) *The matrix A has $\mathrm{rk}(A) = n$.*

(iii) *The only solution of $Az = 0$ is $z = 0$.*

Existence of vertices

Theorem

Let $P = \{x \mid Ax \leq b\}$ be a polyhedron in $\mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Assume that the problem is feasible, that is, $P \neq \emptyset$. Then, the following three properties are equivalent:

(i) *There exists a basic feasible solution in P.*

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(iii) *The only solution of $Az = 0$ is $z = 0$.*

Theorem

If a linear programming problem that has a basic feasible solutions admits an optimum, then there exists an optimum which is a basic feasible solution.

An important consequence

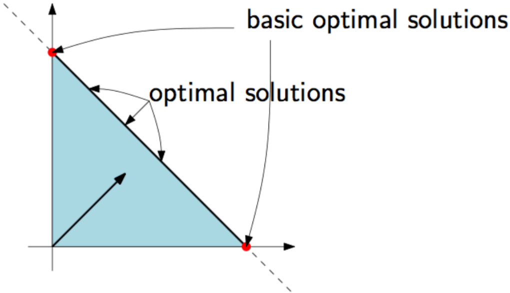
Theorem

If a linear programming problem that has a basic feasible solutions admits an optimum, then there exists an optimum which is a basic feasible solution.

- Note that the number of basic solutions to a linear system is finite (how many?)
- Therefore, to solve an LP problem, we only need to consider a finite number of possibilities.
- How to do this efficiently?

A common misunderstanding

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



The previous theorem says that at at least one optimal solution is basic, but there may be also non-basic optimal solutions.