MA427 Lecture 7
More on total unimodularity and the cutting plane method

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Today's lecture

- More on totally unimodular matrices
- The cutting planes method

Totally unimodular matrices

Definition

A matrix A is said totally unimodular if, for every square submatrix B of A, $\det(B) \in \{0, +1, -1\}$.

Theorem

Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. Given a vector $b \in \mathbb{Z}^m$, all vertices of the polyhedron $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ are integer. Similarly, all vertices of the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ are integer.

Theorem

The incidence matrix of a bipartite graph is totally unimodular.

TU matrices

Theorem

Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. The following hold

i) *Submatrices of A are T.U.*

ii) $A^\top$ *is T.U.*

iii) *If matrix A' is obtained from A by multiplying one row or column by -1, then A' is T.U.*

iv) *The matrix $(A| - A)$, obtained by juxtaposing the matrices A and $-A$, is T.U.*

v) *The matrix $(A|e)$ is T.U., where e is a unit vector (one entry 1, all others 0).*

vi) *The matrix $(A|I)$, obtained by juxtaposing the matrix A and the identity matrix I, is T.U.*

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ & & & & & \\ & & & & & \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & \\ 0 & 1 & 0 & & 1 & 0 & \\ 1 & 0 & 0 & & 0 & 1 & \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

TU matrices

Corollary

Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. Given vector $b, d \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ all vertices of the polyhedron

$$\{x \in \mathbb{R}^n \,:\, b \leq Ax \leq d,\, \ell \leq x \leq u\}$$

are integer.

Network problems

Theorem

Let A be a matrix with all entries in $\{0, 1, -1\}$, such that in every column of A there is exactly one entry of value 1, one entry of value -1, and all other entries with value 0. Then A is totally unimodular.

Example

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Network problems

Directed graph $G = (V, E)$, source $s \in V$, sink $t \in V$, edge capacities $u : E \to \mathbb{R}$.

Maximum flow problem: find a vector $x : E \to \mathbb{R}_+$ such that

- the total incoming amount equals the total outgoing amount at every node $v \in V \setminus \{s, t\}$;
- the flow on every edge is between 0 and the upper bound: $0 \leq x \leq u$.

Maximize the total amount of flow leaving s.

Network problems

Directed graph $G = (V, E)$, costs $c : E \to \mathbb{R}$, lower and upper capacity bounds $\ell, u : E \to \mathbb{R}$.

Feasible circulation: vector $x : E \to \mathbb{R}$ such that

- the total incoming amount equals the total outgoing amount at every node $v \in V$.
- it is between the upper and lower bounds: $\ell \leq x \leq u$.

Find a minimum cost feasible circulation.

Ideal formulations

Given a set $X \subseteq \mathbb{R}^n$, the convex hull of X, denoted by $\mathrm{conv}(X)$, is the minimal convex set containing X.

Theorem (Fundamental theorem of Integer Programming)

Given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, let $X = \{x \in \mathbb{R}^n : Ax \leq b,\, x \geq 0,\, x_i \in \mathbb{Z} \text{ for } i \in I\}$. Then $\mathrm{conv}(X)$ is a polyhedron.

$\Longrightarrow$ there exists $\tilde{A} \in \mathbb{Q}^{\tilde{m} \times n}$ and $\tilde{b} \in \mathbb{Q}^{\tilde{m}}$ such that

$$\mathrm{conv}(X) = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b},\, x \geq 0\}.$$

$\tilde{A}x \leq \tilde{b}$, $x \geq 0$ is the ideal formulation for X.

Perfect matchings: ideal formulation [Edmonds, 1965]

For every graph G, the ideal formulation for the maximum weight perfect matching problem is

$$\begin{array}{rcll} \min \sum_{e \in E} c_e x_e & & & \\ \sum_{u\,:\,uv \in E} x_{uv} & = & 1 & v \in V, \\ \sum_{e \in E[U]} x_e & \leq & \frac{|U|-1}{2} & U \subseteq V,\ |U| \text{ odd}, \\ x_e & \geq & 0 & e \in E. \end{array}$$

where $E[U] := \{uv \in E \,:\, u, v \in U\}$.

The cutting planes method

Cutting planes: motivation

- Start with the LP relaxation, and move towards the ideal formulation
- Repeatedly add *valid inequalities* to the current formulation, which cuts off the current fractional solution.

Valid inequalities

$$\begin{array}{c} \max c^\top x \\ x \in X \end{array} \quad (P_I)$$

$$X = \{x \in \mathbb{R}^n \,:\, Ax \leq b,\, x \geq 0,\, x_i \in \mathbb{Z} \text{ for all } i \in I\}$$

Definition

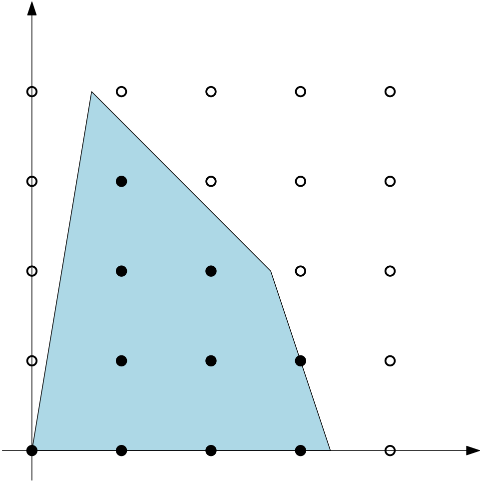
A linear inequality $\alpha^\top x \leq \beta$, ($\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$) is valid for X if, for all $x \in X$,

$$\alpha^\top x \leq \beta.$$

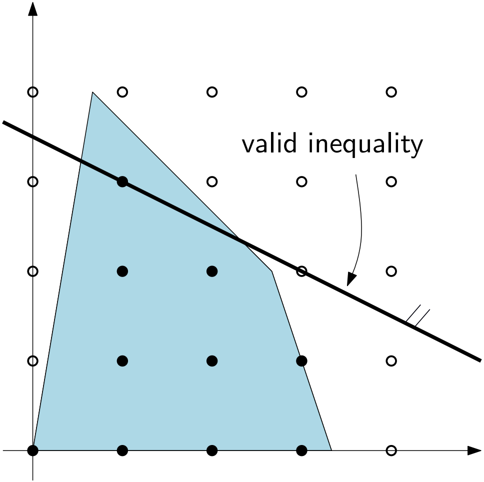
Note: If we append a valid inequality $\alpha^\top x \leq \beta$ for X to the initial formulation $Ax \leq b$ $x \geq 0$, we obtain a new (tighter) formulation:

$$\begin{array}{c} Ax \leq b \\ \alpha^\top x \leq \beta \\ x \geq 0 \end{array}$$

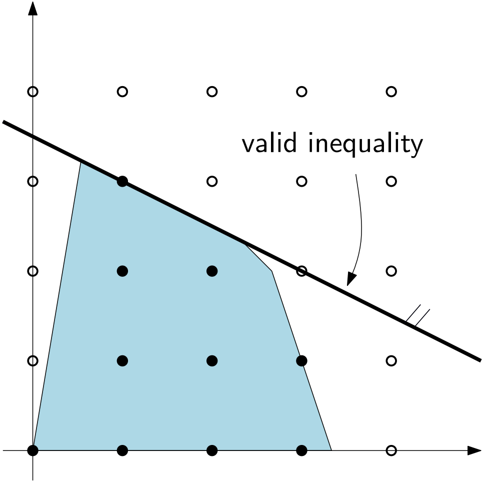
Valid inequalities



Valid inequalities



Valid inequalities



Example: matchings in nonbipartite graphs

Starting formulation

$$\begin{array}{rcl} \min \sum_{e \in E} c_e x_e & & \\ \sum_{u\,:\,uv \in E} x_{uv} & = & 1 \quad v \in V, \end{array}$$

For every odd set U,

$$\sum_{e \in E[U]} x_e \leq \frac{|U|-1}{2}$$

is valid.

Cutting planes

If we solve the LP relaxation

$$\begin{gathered} z_L = \max c^\top x \\ Ax \leq b \\ x \geq 0 \end{gathered}$$

and the optimal basic solution x^* does not satisfy the integrality conditions, then $x^* \notin \mathrm{conv}(X)$

Cutting planes

If we solve the LP relaxation

$$
\begin{aligned}
z_L = \max c^\top x \\
Ax \le b \\
x \ge 0
\end{aligned}
$$

and the optimal basic solution x^* does not satisfy the integrality conditions, then $x^* \notin \mathrm{conv}(X)$

$$\Longrightarrow$$

there exists a valid inequality $\alpha^\top x \le \beta$ cutting off x^*.
Append $\alpha^\top x \le \beta$ and solve again.

Cutting plane method

1. Solve the current relaxation, and let x^* be the optimal solution found;
2. If $x^* \in X$, then x^* is an optimal solution to the MILP, STOP.
3. Otherwise, find a valid inequality $\alpha^\top x \leq \beta$ for X cutting-off x^*;
4. Add the constraint $\alpha^\top x \leq \beta$ to the current linear relaxation and return to 1.

Cutting plane method

1. Solve the current relaxation, and let x^* be the optimal solution found;
2. If $x^* \in X$, then x^* is an optimal solution to the MILP, STOP.
3. Otherwise, find a valid inequality $\alpha^\top x \leq \beta$ for X cutting-off x^*;
4. Add the constraint $\alpha^\top x \leq \beta$ to the current linear relaxation and return to 1.

How do we find a valid inequality cutting off the current solution?

Gomory cuts

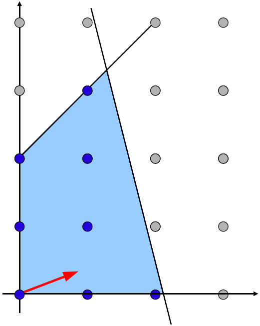
WARNING: Gomory cuts work only for pure integer programs. There exists a generalization, called Gomory mixed-integer cuts that work for general problems.

Problem in standard form (we can assume without loss of generality).

$$
\begin{gathered}
z_I = \max c^\top x \\
Ax = b \\
x \geq 0 \\
x \in \mathbb{Z}^n
\end{gathered}
$$

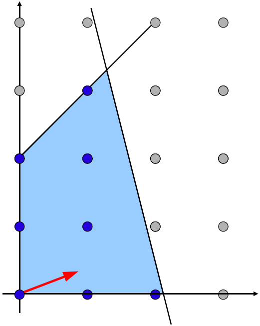
Gomory cuts

$$
\begin{aligned}
\max\ z &= 11x_1 + 4.2x_2 \\
-x_1 + x_2 &\leq 2 \\
8x_1 + 2x_2 &\leq 17 \\
x_1, x_2 &\geq 0 \text{ integer.}
\end{aligned}
$$



Gomory cuts

$$\begin{aligned} \max\ z &= 11x_1 + 4.2x_2 \\ -x_1 + x_2 &\leq 2 \\ 8x_1 + 2x_2 &\leq 17 \\ x_1, x_2 &\geq 0 \text{ integer.} \end{aligned}$$



Standard form

$$\begin{aligned} z - 11x_1 - 4.2x_2 &= 0 \\ -x_1 + x_2 + x_3 &= 2 \\ 8x_1 + 2x_2 + x_4 &= 17 \\ x_1, x_2, x_3, x_4 &\geq 0 \text{ integer.} \end{aligned}$$

Gomory cuts

Optimal tableau:

$$\begin{array}{llllrrcl}
z & & & & +1.16x_3 & +1.52x_4 & = & 28.16 \\
& & x_2 & & +0.8x_3 & +0.1x_4 & = & 3.3 \\
& x_1 & & & -0.2x_3 & +0.1x_4 & = & 1.3
\end{array}$$

Gomory cuts

Optimal tableau:

$$\begin{array}{ccccccc} z & & & +1.16x_3 & +1.52x_4 & = & 28.16 \\ & & x_2 & +0.8x_3 & +0.1x_4 & = & 3.3 \\ & x_1 & & -0.2x_3 & +0.1x_4 & = & 1.3 \end{array}$$

$$x_2 \leq 3$$

is valid for X. It cuts off the current optimum.

Gomory cuts

Optimal tableau:

$$\begin{array}{cccccccc} z & & & +1.16x_3 & +1.52x_4 & = & 28.16 \\ & & x_2 & +0.8x_3 & +0.1x_4 & = & 3.3 \\ & x_1 & & -0.2x_3 & +0.1x_4 & = & 1.3 \end{array}$$

$$x_2 \leq 3$$

is valid for X. It cuts off the current optimum.

Can be added to the relaxation. Slack variable x_5:

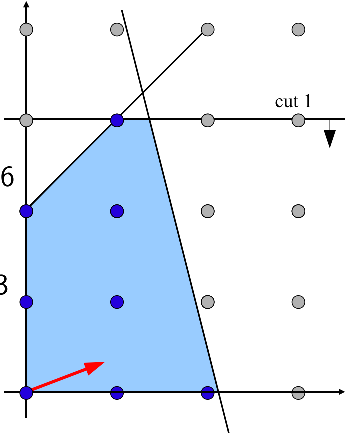
$$x_2 + x_5 = 3$$

$\{1,2,5\}$ is a basis. Write tableau w.r.t. $\{1,2,5\}$. Subtracting the first constraint we cancel x_2:

$$-0.8x_3 - 0.1x_4 + x_5 = -0.3$$

Gomory cuts

$$
\begin{array}{lllllll}
z & & +1.16x_3 & +1.52x_4 & & = & 28.16 \\
 & x_2 & +0.8x_3 & +0.1x_4 & & = & 3.3 \\
x_1 & & -0.2x_3 & +0.1x_4 & & = & 1.3 \\
 & & -0.8x_3 & -0.1x_4 & +x_5 & = & -0.3
\end{array}
$$



Basis is dual feasible! We can solve using the dual simplex method.

Gomory cuts

Optimal tableau:

$$\begin{array}{llcrcl}
\max & z & & & & \\
 & z & & -\sum_{j\in N}\bar{c}_j x_j & = & z_B \\
 & & x_{B[i]} & +\sum_{j\in N}\bar{a}_{ij} x_j & = & \bar{b}_i, \quad i=1,\dots,m \\
 & & & x & \geq & 0.
\end{array}$$

- If $\bar{b}_i \in \mathbb{Z}$ for $i = 1, \dots, m$, then x^* is the integer optimum!
- Otherwise, choose $h \in \{1, \dots, m\}$ such that $\bar{b}_h \notin \mathbb{Z}$.

Gomory cuts

Optimal tableau:

$$\begin{array}{lllcll}
\max & z & & & & \\
& z & -\sum_{j\in N} \bar{c}_j x_j & = & z_B & \\
& x_{B[i]} & +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, & i = 1,\ldots,m \\
& & x & \geq & 0. &
\end{array}$$

- If $\bar{b}_i \in \mathbb{Z}$ for $i = 1,\ldots,m$, then x^* is the integer optimum!
- Otherwise, choose $h \in \{1,\ldots,m\}$ such that $\bar{b}_h \notin \mathbb{Z}$.
  Any solution must also satisfy

$$x_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq \bar{b}_h$$

Gomory cuts

Optimal tableau:

$$\begin{array}{llllll}
\max & z & & & & \\
 & z & & -\sum_{j\in N} \bar{c}_j x_j & = & z_B \\
 & & x_{B[i]} & +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1,\dots,m \\
 & & & x & \geq & 0.
\end{array}$$

- If $\bar{b}_i \in \mathbb{Z}$ for $i = 1, \dots, m$, then x^* is the integer optimum!
- Otherwise, choose $h \in \{1, \dots, m\}$ such that $\bar{b}_h \notin \mathbb{Z}$.
  Any solution must also satisfy

  $$x_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq \bar{b}_h$$

  Any integer solution must satisfy

  $$x_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq \lfloor \bar{b}_h \rfloor$$

Gomory cuts

Optimal tableau:

$$\begin{array}{llcrcl} \max & z & & & & \\ & z & & -\sum_{j\in N}\bar{c}_j x_j & = & z_B \\ & & x_{B[i]} & +\sum_{j\in N}\bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1,\ldots,m \\ & & & x & \geq & 0. \end{array}$$

Gomory cut:

$$x_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq \lfloor \bar{b}_h \rfloor$$

Note: Gomory cut cuts off x^*: indeed, $x^*_{B[h]} = \bar{b}_h$, $x^*_j = 0$ for $j \in N$, hence

$$x^*_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x^*_j = \bar{b}_h > \lfloor \bar{b}_h \rfloor$$

Gomory cuts

$$\begin{array}{llllll} \max & z & & & & \\ & z & & -\sum_{j\in N} \bar{c}_j x_j & = & z_B \\ & & x_{B[i]} & +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\ & & & x & \geq & 0. \end{array}$$

Append the Gomory cut to the tableau:

$$x_{B[h]} + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j \qquad \leq \lfloor \bar{b}_h \rfloor$$

Gomory cuts

$$
\begin{array}{llllll}
\max & z & & & & \\
& z & & -\sum_{j\in N} \bar{c}_j x_j & = & z_B \\
& & x_{B[i]} & +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1,\ldots,m \\
& & & x & \geq & 0.
\end{array}
$$

Append the Gomory cut to the tableau:

$$x_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x_j + x_{n+1} = \lfloor \bar{b}_h \rfloor$$

Slack variable x_{n+1} is an integer variable (why?).

Gomory cuts

$$\begin{array}{lllll}
\max & z & & & \\
& z & -\sum_{j\in N} \bar{c}_j x_j & = & z_B \\
& & x_{B[i]} \quad +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1,\ldots,m \\
& & x & \geq & 0.
\end{array}$$

Append the Gomory cut to the tableau:

$$x_{B[h]} + \sum_{j\in N} \lfloor \bar{a}_{hj} \rfloor x_j + x_{n+1} = \lfloor \bar{b}_h \rfloor$$

Slack variable x_{n+1} is an integer variable (why?).
Write the tableau w.r.t. to basis $B \cup \{n+1\}$: must cancel out variable $x_{B[h]}$. Subtract the equation

$$x_{B[h]} + \sum_{j\in N} \bar{a}_{hj} x_j = \bar{b}_h$$

We get the Gomory cut in fractional form:

$$\sum_{j\in N} (\lfloor \bar{a}_{hj} \rfloor - a_{hj}) x_j + x_{n+1} = \lfloor \bar{b}_h \rfloor - \bar{b}_h.$$

Gomory cuts

New tableau

$$
\begin{array}{llllll}
\max z & & & & & \\
-z & & +\sum_{j \in N} \bar{c}_j x_j & = & -z_B & \\
 & x_{B[i]} & +\sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, & i = 1, \ldots, m \\
 & & \sum_{j \in N} (\lfloor \bar{a}_{hj} \rfloor - a_{hj}) x_j \quad +x_{n+1} & = & \lfloor \bar{b}_h \rfloor - \bar{b}_h & \\
 & & \qquad x_1, \ldots, x_{n+1} \geq 0. & & &
\end{array}
$$

- Tableau is dual feasible: $\bar{c}_j \leq 0$ *for all* $j \in N$;

Gomory cuts

New tableau

$$
\begin{array}{lllcll}
\max z & & & & & \\
-z & & +\sum_{j\in N}\bar{c}_j x_j & = & -z_B & \\
 & x_{B[i]} & +\sum_{j\in N}\bar{a}_{ij} x_j & = & \bar{b}_i, & i = 1,\ldots,m \\
 & & \sum_{j\in N}(\lfloor \bar{a}_{hj} \rfloor - a_{hj}) x_j \quad +x_{n+1} & = & \lfloor \bar{b}_h \rfloor - \bar{b}_h & \\
 & & x_1,\ldots,x_{n+1} \geq 0. & & &
\end{array}
$$

- Tableau is dual feasible: $\bar{c}_j \leq 0$ *for all* $j \in N$;
- Tableau is not primal feasible: $\lfloor \bar{b}_h \rfloor - \bar{b}_h < 0$;

Gomory cuts

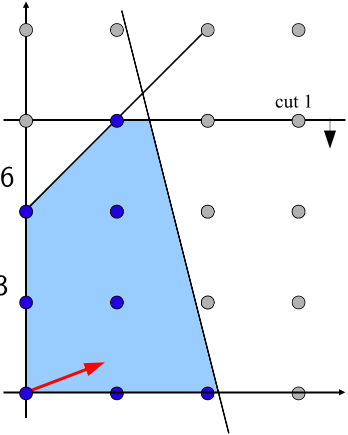
New tableau

$$
\begin{array}{llllll}
\max z & & & & & \\
-z & & +\sum_{j\in N} \bar{c}_j x_j & = & -z_B & \\
 & x_{B[i]} & +\sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, & i = 1, \ldots, m \\
 & & \sum_{j\in N}(\lfloor \bar{a}_{hj} \rfloor - a_{hj}) x_j \quad + x_{n+1} & = & \lfloor \bar{b}_h \rfloor - \bar{b}_h & \\
 & & & x_1, \ldots, x_{n+1} \geq 0. & &
\end{array}
$$

- Tableau is dual feasible: $\bar{c}_j \leq 0$ *for all* $j \in N$;
- Tableau is not primal feasible: $\lfloor \bar{b}_h \rfloor - \bar{b}_h < 0$;
- Can re-solve using the dual simplex method instead of starting from scratch.

Gomory cuts

$$\begin{array}{lllll} z & & +1.16x_3 + 1.52x_4 & = & 28.16 \\ & x_2 & +0.8x_3 \ +0.1x_4 & = & 3.3 \\ x_1 & & -0.2x_3 \ +0.1x_4 & = & 1.3 \\ & & -0.8x_3 \ -0.1x_4 + x_5 & = & -0.3 \end{array}$$



Gomory cuts

New optimal tableau

$$
\begin{array}{ccccccccc}
z & & & & +1.375x_4 & +1.45x_5 & = & 27.725 \\
& & x_2 & & & +x_5 & = & 3 \\
& x_1 & & & +0.125x_4 & -0.25x_5 & = & 1.375 \\
& & & x_3 & +0.125x_4 & -1.25x_5 & = & 0.375
\end{array}
$$

Gomory cuts

New optimal tableau

$$
\begin{array}{ccccccccc}
z & & & & +1.375x_4 & +1.45x_5 & = & 27.725 \\
& & x_2 & & & +x_5 & = & 3 \\
& x_1 & & & +0.125x_4 & -0.25x_5 & = & 1.375 \\
& & & x_3 & +0.125x_4 & -1.25x_5 & = & 0.375
\end{array}
$$

From the tableau equation $x_3 + 0.125x_4 - 1.25x_5 = 0.375$ we generate the Gomory cut

$$x_3 - 2x_5 \leq 0.$$

Gomory cuts

New optimal tableau

$$\begin{array}{cccccccc}
z & & & & +1.375x_4 & +1.45x_5 & = & 27.725 \\
& & x_2 & & & +x_5 & = & 3 \\
& x_1 & & & +0.125x_4 & -0.25x_5 & = & 1.375 \\
& & & x_3 & +0.125x_4 & -1.25x_5 & = & 0.375
\end{array}$$

From the tableau equation $x_3 + 0.125x_4 - 1.25x_5 = 0.375$ we generate the Gomory cut

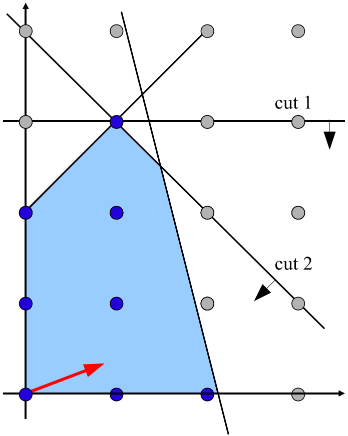
$$x_3 - 2x_5 \leq 0.$$

In fractional form:

$$-0.125x_4 - 0.75x_5 + x_6 = -0.375$$

Gomory cuts

$$\begin{array}{cccccccc}
z & & & +1.375x_4 & +1.45x_5 & & = & 27.725 \\
& & x_2 & & +x_5 & & = & 3 \\
& x_1 & & +0.125x_4 & -0.25x_5 & & = & 1.375 \\
& & x_3 & +0.125x_4 & -1.25x_5 & & = & 0.375 \\
& & & -0.125x_4 & -0.75x_5 & +x_6 & = & -0.375
\end{array}$$



Gomory cuts

New optimal tableau

$$
\begin{array}{cccccccccc}
z & & & & +17/15x_4 & & +29/15x_6 & = & 27 \\
 & & x_2 & & -1/6x_4 & & +4/3x_6 & = & 2.5 \\
 & x_1 & & & +1/6x_4 & & -1/3x_6 & = & 1.5 \\
 & & & x_3 & & & +x_6 & = & 0 \\
 & & & & 1/6x_4 & +x_5 & -4/3x_6 & = & 0.5
\end{array}
$$

Gomory cuts

New optimal tableau

$$
\begin{array}{cccccccccc}
z & & & & +17/15x_4 & & +29/15x_6 & = & 27 \\
& & x_2 & & -1/6x_4 & & +4/3x_6 & = & 2.5 \\
& x_1 & & & +1/6x_4 & & -1/3x_6 & = & 1.5 \\
& & & x_3 & & & +x_6 & = & 0 \\
& & & & 1/6x_4 & +x_5 & -4/3x_6 & = & 0.5
\end{array}
$$

From the tableau equation $1/6x_4 + x_5 - 4/3x_6 = 0.5$ we generate the Gomory cut

$$x_5 - 2x_6 \leq 0$$

Gomory cuts

New optimal tableau

$$
\begin{array}{cccccccccc}
z & & & & +17/15x_4 & & +29/15x_6 & = & 27 \\
& & x_2 & & -1/6x_4 & & +4/3x_6 & = & 2.5 \\
& x_1 & & & +1/6x_4 & & -1/3x_6 & = & 1.5 \\
& & & x_3 & & & +x_6 & = & 0 \\
& & & & 1/6x_4 & +x_5 & -4/3x_6 & = & 0.5
\end{array}
$$

From the tableau equation $1/6x_4 + x_5 - 4/3x_6 = 0.5$ we generate the Gomory cut

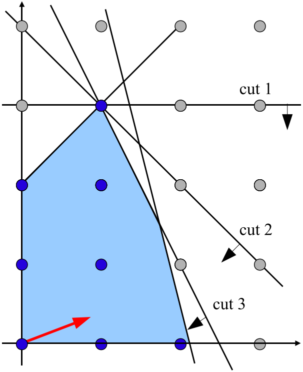
$$x_5 - 2x_6 \leq 0$$

In fractional form:

$$-1/6x_4 - 2/3x_6 + x_7 = -0.5$$

Gomory cuts

$$
\begin{array}{cccccccccc}
z & & & +17/15x_4 & & +29/15x_6 & & = & 27 \\
& x_2 & & -1/6x_4 & & +4/3x_6 & & = & 2.5 \\
x_1 & & & +1/6x_4 & & -1/3x_6 & & = & 1.5 \\
& & x_3 & & & +x_6 & & = & 0 \\
& & & 1/6x_4 & +x_5 & -4/3x_6 & & = & 0.5 \\
& & & -1/6x_4 & & -2/3x_6 & +x_7 & = & -0.5
\end{array}
$$



Gomory cuts

New optimal tableau

$$
\begin{array}{ccccccccc}
z & & & +13/15x_3 & & & & +76/15x_7 & = & 23.6 \\
& & x_2 & +2/3x_3 & & & & +1/3x_7 & = & 3 \\
& x_1 & & -1/3x_3 & & & & +1/3x_7 & = & 1 \\
& & & 4/3x_3 & +x_4 & & & -10/3x_7 & = & 3 \\
& & & -2/3x_3 & & +x_5 & & -1/3x_7 & = & 0 \\
& & & -1/3x_4 & & & x_6 & -2/3x_7 & = & 0
\end{array}
$$

Optimal integer solution to the original problem $(1, 3)$.

Branch-and-Bound or cutting planes?

Branch-and-Bound and cutting planes!

Branch-and-cut:
Apply branch and bound, but at each node decide whether or not to tighten the formulation by adding cuts, in order to obtain a better bound at the node.

Branch-and-Bound and cutting planes!

Branch-and-cut:
Apply branch and bound, but at each node decide whether or not to tighten the formulation by adding cuts, in order to obtain a better bound at the node.

- State-of-the-art solvers implement many different types of cutting planes, including Gomory cuts and Gomory mixed-integer cuts (a variant that works also for general mixed-integer programming problems).
- Sometimes we can exploit special structure in the problem at hand to generate strong cutting planes.
- An example: cover inequalities.
- State-of-the-art solvers employ many different types of cuts.