MA423 – Fundamentals of Operations Research
Lecture 1

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To contact me

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- Office hours Michaelmas Term : Tuesday 15:30–17:30

A few details

- Lectures: 2 hours with exercises given
- Exercises: should be attempted before next week's seminars
- Seminars: You should be assigned a seminar by Rebecca Batey `r.batey@lse.ac.uk`. If you want to change seminar group please email Rebecca.
- Formative work: 3 exercise sets will be collected and feedback will be given. This is not compulsory but highly recommended.
- Exam: 3 hour examination in ST. I will provide a mock exam that will be solved at the revision session week 1 of ST.

Content of the course

MA423 Part 1: Linear programming and integer programming

- Optimisation problems/LP formulations/Standard forms.
- Duality.
- The simplex method.
- Integer programming: branch and bound, formulations.

MA423 Part 2: Markov/Queueing

- Markov Chains.
- Queueing theory.

MA423 Part 3: Other OR methods.

- Inventory Models.
- Dynamic Programming.
- Game Theory.

Lecture 1

- Introduction
  - Optimization problems
  - Examples
- Linear programming
  - Terminology
  - Possible outcomes: fundamental theorem
  - LP in standard forms
  - Proving optimality: dual values

Mathematical Programming $\equiv$ Mathematical Optimization

$$
\begin{aligned}
\max \ (\text{or } \min) \quad & \underbrace{f_0(x)}_{\text{objective function}} \\
\text{s.t.} \quad & \left.\begin{aligned} f_1(x) &\le b_1 \\ &\vdots \\ f_m(x) &\le b_m \end{aligned}\right\} \text{constraints} \\
& x \in \mathcal{D}
\end{aligned}
$$

- x: n-dimensional vector of decision variables
- $\mathcal{D} \subseteq \mathbb{R}^n$: Domain of the problem
- $f_i : \mathcal{D} \to \mathbb{R}$.

Example

Factory produces Orange Juice (OJ) and Orange Concentrate (OC).

	OJ	OC
Profit (£/liter)	3	2
Electricity (unit/Liter)	1	1
Oranges (unit/liter)	1	2
Water (unit/liter)	1	-1

	Available
Electricity	6
Oranges	10
Water	4

Example: nurse scheduling

Hospital must choose how many nurses to staff.

- On day i of the week ($i = 1, \ldots, 7$), the hospital needs d_i nurses.
- Every nurse rests two consecutive days every week.

What is the minimum number of nurses needed?

Example: Markowitz Portfolio Optimization

Optimally allocate budget B to n assets $i = 1, \ldots, n$.

- w_i: proportion of budget B allocated to stock i: $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$, $i = 1, \ldots, n$.

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- Minimize risk while guaranteeing an expected return $\geq r_{\mathrm{min}}$.

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- Minimize risk while guaranteeing an expected return $\geq r_{\min}$.

$$
\begin{gathered}
\min \ \mathrm{Var}(r) \\
\text{s.t.} \\
\bar{r} \geq r_{\min} \\
\sum_{i=1}^{n} w_i = 1 \\
w \geq 0
\end{gathered}
$$

Example: Markowitz Portfolio Optimization

- p: random vector of returns.
- $\bar{p}$: vector of expected returns, $\bar{p}_i = \mathbb{E}[p_i]$.
- Σ: covariance matrix of p $(\Sigma_{ij} = \mathbb{E}[(p_i - \bar{p}_i)(p_j - \bar{p}_j)])$

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It follows ...

- $r = p^\top w$;
- $\bar{r} = \bar{p}^\top w$;
- $\mathrm{Var}(r) = w^\top \Sigma w$.

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$$
\begin{gathered}
\min \; w^\top \Sigma w \\
\text{s.t.} \\
\bar{p}^\top w \geq r_{\min} \\
\sum_{i=1}^n w_i = 1 \\
w \geq 0
\end{gathered}
$$

Example: cutting stock problem

A paper mill produces large rolls of paper of width W which are then cut into smaller rolls to meet orders.

- Roll widths: w_i for $i = 1, \ldots, m$
- Demand for width i: b_i, $i = 1, \ldots, m$

What is the minimum number of big rolls needed to meet demand?

Linear Programming problems

linear objective function

$$\max \boxed{c^\top x}$$

$$\boxed{\begin{array}{rcll} a_i^\top x & = & b_i & i = 1, \ldots, k \\ a_i^\top x & \leq & b_i & i = k+1, \ldots, r \\ a_i^\top x & \geq & b_i & i = r+1, \ldots, m \end{array}}$$

linear constraints

If some of the variables are required to be integer, then the problem is a Mixed-integer Linear Programming problem.

Terminology

objective function

$$
\begin{array}{rl}
\max \ \text{(or min)} & \boxed{f_0(x)} \\
\text{s.t.} & f_1(x) \le b_1 \\
& \vdots \\
& f_m(x) \le b_m \\
& x \in \mathcal{D}
\end{array}
$$

constraints

- **Feasible solution:** any point $\bar{x}$ satisfying the constraints.
- **Feasible region:** the set of all feasible solutions.

Possible outcomes

- Optimal solution (maximization): A feasible solution x^* such that, for every feasible solution x,

$$f_0(x^*) \geq f_0(x).$$

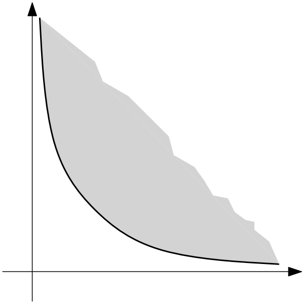
  If an optimal solution exists, we say that the problem has a finite optimum.
- Infeasible problem: A problem that has no feasible solution.
- Unbounded problem: A problem such that, for every number α, there exists a feasible solution x such that

$$f_0(x) > \alpha.$$

Possible outcomes

There are optimization problems for which none of these three outcomes occurs.

$$\begin{aligned} \min x_1 \\ x_1 x_2 \geq 1 \\ x_1, x_2 \geq 0. \end{aligned}$$



There are feasible points with objective value arbitrarily close to 0, but no point of value zero.

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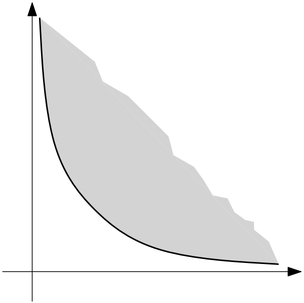


There are feasible points with objective value arbitrarily close to 0, but no point of value zero.

For LP problems, one of the three outcomes always occurs.

Fundamental theorem

Theorem (Fundamental Theorem of Linear Programming)

For any linear programming problem, exactly one of the following holds.

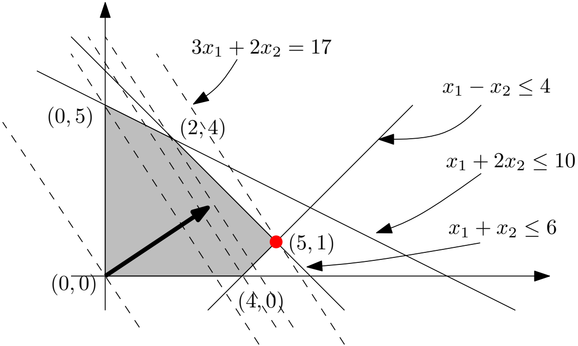
1. *The problem has a finite optimum;*
2. *The Problem is infeasible;*
3. *The problem is unbounded.*

Example: a problem with an optimum

The model from the OJ production problem:

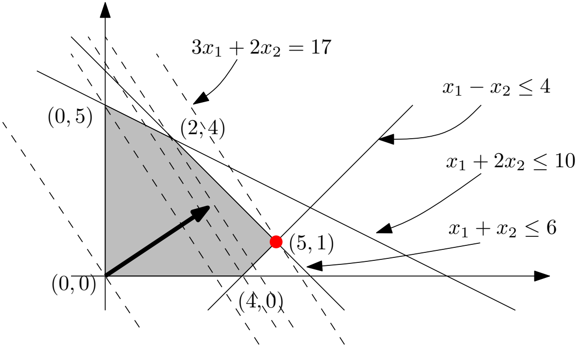
$$\begin{array}{lrcrcr}
\max & 3x_1 & + & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 4 \\
 & & & x_1, x_2 & \geq & 0
\end{array}$$

Example: a problem with an optimum



The optimum is $(5, 1)$ with value 17. The optimum is a vertex of the feasible region.

Example: a problem with an optimum



The optimum is $(5, 1)$ with value 17. The optimum is a vertex of the feasible region.

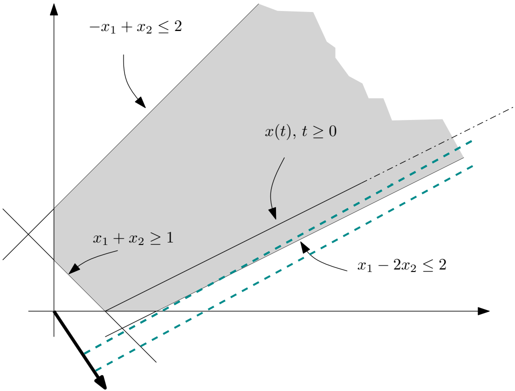
- When the LP has an optimum, it has one which is a vertex of the feasible region.
- How to prove that a point is an optimum? (LP duality).

Example: an infeasible problem

$$
\begin{array}{lrcrcrcr}
\max & x_1 & + & x_2 & & & & \\
s.t. & x_1 & - & 2x_2 & + & 2x_3 & \leq & 2 \\
 & -2x_1 & + & 6x_2 & - & 2x_3 & \leq & -6 \\
 & & & x_1, x_2, x_3 & & & \geq & 0
\end{array}
$$

Example: an unbounded problem

$$
\begin{array}{rrcrcc}
\max & 2x_1 & - & 3x_2 & & \\
s.t. & x_1 & + & x_2 & \geq & 1 \\
 & -x_1 & + & x_2 & \leq & 2 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$



Standard forms

An LP problem is in standard form if it is of the form

$$\begin{aligned} z^* = \quad & \max c^\top x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

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An LP problem is in standard equality form if it is of the form

$$\begin{aligned} z^* = \quad & \max c^\top x \\ & Ax = \quad b \\ & x \geq \quad 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

Converting to Standard forms

An LP in general form:

$$\begin{array}{llcll} \max(\text{resp. min}) & c^\top x & & & \\ & a_i^\top x & = & b_i & i = 1, \ldots, k \\ & a_i^\top x & \leq & b_i & i = k+1, \ldots, r \\ & a_i^\top x & \geq & b_i & i = r+1, \ldots, m \end{array} \tag{1}$$

where $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $c, a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and x is a vector of variables in $\mathbb{R}^n$.

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where $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $c, a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and x is a vector of variables in $\mathbb{R}^n$.

How do we convert such an LP in standard form?

Converting to Standard forms

- **Objective**: $\min c^T x$ becomes $\max -c^T x$

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- **Objective**: $\min c^T x$ becomes $\max -c^T x$
- **"$\geq$" Constraints**: $a_i^T x \geq b_i$ become $-a_i^T x \leq -b_i$
- **"$=$" Constraints**: $a_i^T x = b_i$ become $a_i^T x \leq b_i$ and $-a_i^T x \leq -b_i$

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- **Variables**: $x_i \geq 0$ are called nonnegative variables, $x_i \leq 0$ are called nonpositive variables otherwise x_i is called a free variable:

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  - Nonpositive variables: We introduce a new nonnegative variable, $x_i' \geq 0$ and we set $x_i' = -x_i$.

Converting to Standard forms

- **Objective**: $\min c^T x$ becomes $\max -c^T x$
- **"$\geq$" Constraints**: $a_i^T x \geq b_i$ become $-a_i^T x \leq -b_i$
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- **Variables**: $x_i \geq 0$ are called nonnegative variables, $x_i \leq 0$ are called nonpositive variables otherwise x_i is called a free variable:
  - Nonpositive variables: We introduce a new nonnegative variable, $x_i' \geq 0$ and we set $x_i' = -x_i$.
  - Free variables: We introduce two nonnegative variables, x_i^+ and x_i^-, and we set

$$x_i = x_i^+ - x_i^-, \quad x_i^+, x_i^- \geq 0$$

Converting to Standard equality form

- Convert to standard form
- For each "$\leq$" constraint $a_i^T x \leq b_i$, introduce nonnegative variable $s_i \geq 0$ and replace previous constraint with:

$$a_i^T x + s_i = b_i$$

Proving optimality

$$\begin{array}{rrcrcr} \max & 3x_1 & + & 2x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 10 \\ & x_1 & - & x_2 & \leq & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Optimal solution $(5, 1)$, with value 17.

Proving optimality

Let us combine the inequalities

$$
\begin{array}{rrcrcrr}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \frac{5}{2} \\
 & x_1 & + & 2x_2 & \leq & 10 & 0 \\
 & x_1 & - & x_2 & \leq & 4 & \frac{1}{2} \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Resulting inequality:

$$3x_1 + 2x_2 \leq 17.$$

Thus no feasible solution has value greater than 17.

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$\begin{array}{lrcrcrl}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\
 & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\
 & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

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\end{array}$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$\begin{array}{rcl}
y_1 + y_2 + y_3 & \geq & 3 \\
y_1 + 2y_2 - y_3 & \geq & 2
\end{array}$$

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$\begin{array}{rrcrcrl} \max & 3x_1 & + & 2x_2 & & & \\ s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\ & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\ & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$\begin{array}{rcl} y_1 + y_2 + y_3 & \geq & 3 \\ y_1 + 2y_2 - y_3 & \geq & 2 \end{array}$$

To obtain tightest upper-bound:

$$\min 6y_1 + 10y_2 + 4y_3$$

Dual problem

The dual of the original problem is

$$
\begin{array}{lcccccccc}
\min & 6y_1 & + & 10y_2 & + & 4y_3 & & \\
s.t. & y_1 & + & y_2 & + & y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & - & y_3 & \geq & 2 \\
 & & & & y_1, y_2, y_3 \geq 0 & & &
\end{array}
$$

The solution to the above gives the tightest possible upper-bound on the optimal value that we can infer by taking linear combinations of the constraints.

MA423 – Fundamentals of Operations Research
Lecture 2: LP Duality

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October 5, 2017

Last week's example

$$\begin{array}{lrcrcr} \max & 3x_1 & + & 2x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 10 \\ & x_1 & - & x_2 & \leq & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Optimal solution $(5, 1)$, with value 17.

Last week's example

$$\begin{array}{lrcrcrl}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \quad \frac{5}{2} \\
& x_1 & + & 2x_2 & \leq & 10 & \quad 0 \\
& x_1 & - & x_2 & \leq & 4 & \quad \frac{1}{2} \\
& & & x_1, x_2 & \geq & 0 &
\end{array}$$

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$$\begin{array}{lrcrcrl}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \quad \color{red}{\tfrac{5}{2}} \\
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Combining the inequalities:

$$3x_1 + 2x_2 \leq 17.$$

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Optimal solution $(5, 1)$, with value 17.
Combining the inequalities:

$$3x_1 + 2x_2 \leq 17.$$

No feasible solution has value greater than 17 $\Longrightarrow$ $(5, 1)$ is optimal.

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$\begin{array}{rrcrcrl}
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Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

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Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$\begin{array}{rcl} y_1 + y_2 + y_3 & \geq & 3 \\ y_1 + 2y_2 - y_3 & \geq & 2 \end{array}$$

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$\begin{array}{rrcrcrl} \max & 3x_1 & + & 2x_2 & & & \\ s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\ & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\ & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$\begin{array}{rcl} y_1 + y_2 + y_3 & \geq & 3 \\ y_1 + 2y_2 - y_3 & \geq & 2 \end{array}$$

To obtain tightest upper-bound:

$$\min 6y_1 + 10y_2 + 4y_3$$

Dual problem

The dual of the original problem is

$$\begin{array}{rrcrcrcl} \min & 6y_1 & + & 10y_2 & + & 4y_3 & & \\ s.t. & y_1 & + & y_2 & + & y_3 & \geq & 3 \\ & y_1 & + & 2y_2 & - & y_3 & \geq & 2 \\ & & & & y_1, y_2, y_3 \geq 0 & & & \end{array}$$

The solution to the above gives the tightest possible upper-bound on the optimal value that we can infer by taking linear combinations of the constraints.

Weak Duality Theorem

$$\begin{array}{rll} \max\ c^\top x & & \\ Ax \leq & b & (P) \\ x \geq & 0 & \end{array} \qquad \begin{array}{rll} \min\ b^\top y & & \\ A^\top y \geq & c. & (D) \\ y \geq 0 & & \end{array}$$

Theorem (Weak duality theorem)

Given any feasible solution x^* for the primal (P), and any feasible solution y^* for the dual (D), then

$$c^\top x^* \leq b^\top y^*.$$

A few consequences

A dual solution provides a certificate of optimality:

Corollary

Let x^* be a feasible sol. for the primal and y^* be a feasible solution for the dual. If $c^\top x^* = b^\top y^*$, then x^* is optimal for the primal, and y^* is optimal for the dual.

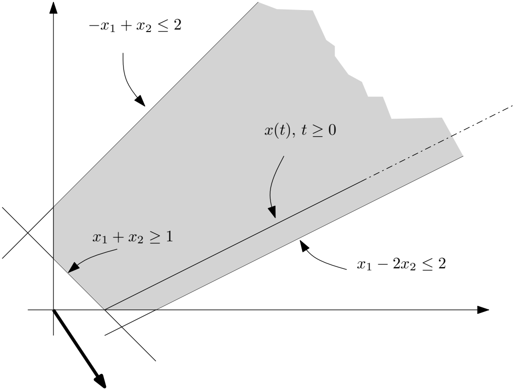
Corollary

(i) *Primal unbounded $\Longrightarrow$ dual infeasible.*

(ii) *Dual unbounded $\Longrightarrow$ primal infeasible.*

Example: unbounded primal

$$\begin{array}{lrcr} \max & 2x_1 - 3x_2 & & \\ s.t. & -x_1 - x_2 & \leq & -1 \\ & -x_1 + x_2 & \leq & 2 \\ & x_1 - 2x_2 & \leq & 2 \\ & x_1, x_2 & \geq & 0 \end{array}$$



$$\begin{array}{lrcr} \min & -y_1 + 2y_2 + 2y_3 & & \\ s.t. & -y_1 - y_2 + y_3 & \geq & 2 \\ & -y_1 + y_2 - 2y_3 & \geq & -3 \\ & y_1, y_2, y_3 \geq 0 & & \end{array}$$

Strong Duality Theorem

$$\begin{aligned} \max\ & c^\top x \\ Ax \leq & \ b \qquad (P) \\ x \geq & \ 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^\top y \\ A^\top y \geq & \ c. \qquad (D) \\ y \geq & \ 0 \end{aligned}$$

Theorem (Strong duality theorem)

If (P) has an optimum, then (D) has an optimum. Given an optimum x^* for (P) and an optimum y^* for (D), then

$$c^\top x^* = b^\top y^*.$$

Strong Duality Theorem (revisited)

$$\begin{aligned} \max\ & c^\top x \\ Ax &\le b \quad (P) \\ x &\ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^\top y \\ A^\top y &\ge c. \quad (D) \\ y &\ge 0 \end{aligned}$$

What is the dual of the dual?

Strong Duality Theorem (revisited)

$$\begin{aligned} \max\ & c^\top x \\ Ax &\leq b \quad (P) \\ x &\geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^\top y \\ A^\top y &\geq c. \quad (D) \\ y &\geq 0 \end{aligned}$$

What is the dual of the dual?

Theorem (Strong duality theorem)

If one among (P) and (D) has an optimum, then they both have an optimum. Given an optimum x^* for (P) and an optimum y^* for (D), then

$$c^\top x^* = b^\top y^*.$$

Possible outcomes

		Primal		
		Fin. opt.	Infeasible	Unbounded
Dual	Fin. opt.	*Possible*	*NO*	*NO*
	Infeasible	*NO*	*?*	*Possible*
	Unbounded	*NO*	*Possible*	*NO*

An example with both primal and dual infeasible

$$
\begin{array}{llcr}
\max & 2x_1 - x_2 & & \\
 & x_1 - x_2 & \leq & 1 \\
 & -x_1 + x_2 & \leq & -2 \\
 & x_1, x_2 \geq 0 & &
\end{array}
\quad , \quad
\begin{array}{llcr}
\min & y_1 - 2y_2 & & \\
 & y_1 - \ \ y_2 & \geq & 2 \\
 & -y_1 + \ \ y_2 & \geq & -1 \\
 & y_1, y_2 \geq 0 & &
\end{array}
$$

Duality: general form

$$\text{A variable } x_j \text{ is } \begin{cases} \text{nonnegative} & \text{if there is constraint } x_j \geq 0 \\ \text{nonpositive} & \text{if there is constraint } x_j \leq 0 \\ \text{free} & \text{otherwise} \end{cases}$$

Let us call the remaining constraints "resource constraints".

- If primal is a maximisation problem, dual is a minimisation
- If primal is a minimisation problem, dual is a maximisation
- Dual has one variable for every primal resource constraint. The coefficient of the ith dual variable is the right-hand-side of the ith primal constraint.
- Dual has one resource constraint for each primal variable x_j. The right-hand-side of the jth dual constraint is the coefficient of the jth primal variable.

Duality: general form

max	min
$\leq$ constraint	nonnegative variable
$\geq$ constraint	nonpositive variable
$=$ constraint	free variable
nonnegative variable	$\geq$ constraint
nonpositive variable	$\leq$ constraint
free variable	$=$ constraint

Duality: general form

max	min
$\leq$ constraint	nonnegative variable
$\geq$ constraint	nonpositive variable
$=$ constraint	free variable
nonnegative variable	$\geq$ constraint
nonpositive variable	$\leq$ constraint
free variable	$=$ constraint

$$
\begin{array}{lrcr}
\min & 3x_1 - 2x_2 - x_3 & & \\
 & -x_1 + x_2 + 2x_3 & = & 4 \\
 & 2x_1 \phantom{{}+x_2} - x_3 & \geq & -2 \\
 & -2x_1 - x_2 + x_3 & \leq & 1 \\
 & x_1 \leq 0,\ x_3 \geq 0 & &
\end{array}
$$

Duality: general form

max	min
$\leq$ constraint	nonnegative variable
$\geq$ constraint	nonpositive variable
$=$ constraint	free variable
nonnegative variable	$\geq$ constraint
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 & 2x_1 \quad\quad - x_3 & \geq & -2 \\
 & -2x_1 - x_2 + x_3 & \leq & 1 \\
 & x_1 \leq 0,\ x_3 \geq 0 & &
\end{array}
\qquad
\begin{array}{lrcr}
\max & 4y_1 - 2y_2 + y_3 & & \\
 & -y_1 + 2y_2 - 2y_3 & \geq & 3 \\
 & y_1 \quad\quad - y_3 & = & -2 \\
 & 2y_1 - y_2 + y_3 & \leq & -1 \\
 & y_2 \geq 0,\ y_3 \leq 0 & &
\end{array}
$$

An economics interpretation of the dual

It is often the case that also the variables, constraints, and objective function of the dual can be interpreted and give further information on the original problem. A typical case when such an interpretation of the dual is possible is in the case of resource allocation problems.

Economics interpretation of the dual - example

- A chip's manufacturer produces four types of memory chips in one of their factories.
- The main resources used in the chips' production are labor and silicon wafers.

The factory's problem for the next month is

$$\begin{array}{llcll}
\text{maximise} & 15x_1 + 24x_2 + 32x_3 + 40x_4 & & & \\
\textit{subject to} & x_1 + 2x_2 + 8x_3 + 7x_4 & \leq & 2000 & \textit{(Labour)} \\
 & 6x_1 + 8x_2 + 12x_3 + 15x_4 & \leq & 15000 & \textit{(Silicon wafers)} \\
 & x_1,\ x_2,\ x_3,\ x_4 \geq 0. & & &
\end{array}$$

- x_j: number of chips produced (thousands),
- Objective function coefficients £/unit of the four chips,
- 2000 hours of labour available, 15000 silicon wafers available.

Economic interpretation of duality

- Mango Inc., a giant consumer electronics corporation, urgently needs as many units as possible of a new type of memory chip, due to a stronger-than-expected demand for their new smart phone.

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- Mango Inc. would like the chip manufacturer to devote all resources of the factory to the production the new type of chip. Mango Inc. intends to determine prices to offer for each of the resources in order to convince the manufacturer to sell them, while minimising the total sum paid.

Economic interpretation of duality

- Mango Inc., a giant consumer electronics corporation, urgently needs as many units as possible of a new type of memory chip, due to a stronger-than-expected demand for their new smart phone.
- Mango Inc. would like the chip manufacturer to devote all resources of the factory to the production the new type of chip. Mango Inc. intends to determine prices to offer for each of the resources in order to convince the manufacturer to sell them, while minimising the total sum paid.
- Can we formulate this as a linear program?

Decision variables

- y_1: price that the buyer intends to pay per hour of labour (thousand £),
- y_2: price that Mango Inc. intends to pay for each silicon wafer.

Decision variables

- y_1: price that the buyer intends to pay per hour of labour (thousand £),
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Clearly these prices must be greater than or equal to zero, that is, $y_1, y_2 \geq 0$.

Determining the prices

- In order to persuade the manufacturer to sell, the buyer must offer prices such that the manufacturer is not tempted to retain its resources to produce chips of type 1.

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Determining the prices

- In order to persuade the manufacturer to sell, the buyer must offer prices such that the manufacturer is not tempted to retain its resources to produce chips of type 1.
- So the buyer must offer prices such that the total value of the resources used in producing chips of type 1 is at least the amount of profit the manufacturer could attain.
- Producing 1000 chips of type one requires 1 hour of labour and 6 silicon wafers, with a profit of £15,000. Thus the prices need to satisfy

$$y_1 + 6y_2 \geq 15.$$

Determining the prices

- Similarly, for chips 2, 3, and 4 Mango Inc. must pitch his prices so that the manufacturer is at least as well off selling as he would be by not selling:

$$\begin{aligned} 2y_1 + 8y_2 &\geq 24 \\ 8y_1 + 12y_2 &\geq 32 \\ 7y_1 + 15y_2 &\geq 40 \end{aligned}$$

Determining the prices

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- On the other hand, the buyer wants to pay the minimum amount possible for the entire amount of resources, that is, it wants to minimise $2000y_1 + 15000y_2$.

The pricing problem

$$\begin{array}{llcl}
\textit{minimise} & 2000y_1 + 15000y_2 & & \\
\textit{subject to} & y_1 + 6y_2 & \geq & 15 \\
 & 2y_1 + 8y_2 & \geq & 24 \\
 & 8y_1 + 12y_2 & \geq & 32 \\
 & 7y_1 + 15y_2 & \geq & 40 \\
 & y_1,\ y_2 & \geq & 0
\end{array}$$

The pricing problem

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\begin{array}{lrcl}
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 & y_1,\ y_2 & \geq & 0
\end{array}
$$

- The manufacturer's and the Mango Inc.'s problems are dual to each other.

The pricing problem

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- The manufacturer's and the Mango Inc.'s problems are dual to each other.
- The dual values represent prices of resources (dual values are often also called "shadow prices").

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 & y_1,\ y_2 & \geq & 0
\end{array}
$$

- The manufacturer's and the Mango Inc.'s problems are dual to each other.
- The dual values represent prices of resources (dual values are often also called "shadow prices").
- The total amount that the buyer will have to pay is the same as the amount of profit that the manufacturer would achieve by carrying on its usual production (strong duality).

Complementary slackness

Consider an LP problem and its dual.

$$\begin{array}{rcl} \max \sum_{j=1}^{n} c_j x_j & & \\ \sum_{j=1}^{n} a_{ij} x_j \leq & b_i & i = 1, ..., m \\ x_j \geq & 0 & j = 1, ..., n \end{array} \qquad \begin{array}{rcl} \min \sum_{i=1}^{m} b_i y_i & & \\ \sum_{i=1}^{m} a_{ij} y_i \geq & c_j & j = 1, ..., n \\ y_i \geq & 0 & i = 1, ..., m \end{array}$$

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$$\begin{array}{rcll} \max \sum_{j=1}^{n} c_j x_j & & & \\ \sum_{j=1}^{n} a_{ij} x_j \leq & b_i & i = 1, \ldots, m \\ x_j \geq & 0 & j = 1, \ldots, n \end{array} \qquad \begin{array}{rcll} \min \sum_{i=1}^{m} b_i y_i & & & \\ \sum_{i=1}^{m} a_{ij} y_i \geq & c_j & j = 1, \ldots, n \\ y_i \geq & 0 & i = 1, \ldots, m \end{array}$$

Theorem (Complementary slackness theorem)

Given feasible solutions x^* and y^*, they are optimal if and only if

$$\begin{array}{lll} \forall\, j \in \{1, \ldots, n\}, & x_j^* = 0 \;\; \textbf{or} & \sum_{i=1}^{m} a_{ij} y_i^* - c_j = 0, \\ \forall\, i \in \{1, \ldots, m\}, & y_i^* = 0 \;\; \textbf{or} & \sum_{j=1}^{n} a_{ij} x_i^* - b_i = 0. \end{array} \qquad (SC)$$

Complementary slackness

$$\begin{array}{rrcr} \max & 3x_1+2x_2 & & \\ s.t. & x_1+\ x_2 & \leq & 6 \\ & x_1+2x_2 & \leq & 10 \\ & x_1-\ x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rrcr} \min & 6y_1+10y_2+4y_3 & & \\ & y_1+\ y_2+\ y_3 & \geq & 3 \\ & y_1+2y_2-\ y_3 & \geq & 2 \\ & y_1, y_2, y_3 & \geq & 0 \end{array}$$

Optimal primal solution $x^* = (5, 1)$.
Optimal dual solution $y^* = (\frac{5}{2}, 0, \frac{1}{2})$.

Complementary slackness: general form

Theorem (Complementary slackness theorem)

Given a LP problem and feasible primal/dual solutions x^* and y^*, they are optimal if and only if

- *For every primal variable x_j, either $x_j^* = 0$ or the corresponding dual constraint is binding at y^*.*
- *For every dual variable y_i, either $y_i^* = 0$ or the corresponding primal constraint is binding at x^*.*

Complementary slackness: example

$$\begin{array}{rrcr} \min & 3x_1 - 2x_2 - x_3 & & \\ & -x_1 + x_2 + 2x_3 & = & 4 \\ & 2x_1 - x_3 & \geq & -2 \\ & -2x_1 - x_2 + x_3 & \leq & 1 \\ & x_1 \leq 0,\ x_3 \geq 0 & & \end{array} \qquad \begin{array}{rrcr} \max & 4y_1 - 2y_2 + y_3 & & \\ & -y_1 + 2y_2 - 2y_3 & \geq & 3 \\ & y_1 - y_3 & = & -2 \\ & 2y_1 - y_2 + y_3 & \leq & -1 \\ & y_2 \geq 0,\ y_3 \leq 0 & & \end{array}$$

Is any of these points optimal?

$x^* = (0, \frac{2}{3}, \frac{5}{3})$

$x^* = (0, 4, 0)$

$x^* = (-1, 3, 0)$.

MA423 – Fundamentals of Operations Research
Lecture 3

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London School of Economics and Political Science

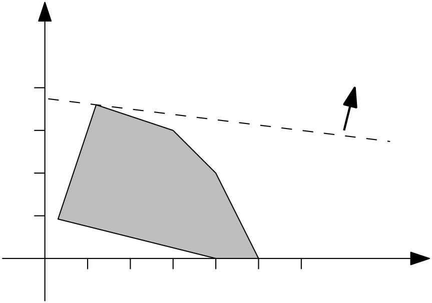
October 16, 2018

This Lecture

- Basic solutions and extreme points
- Effective constraints and basic variables
- Degenerate examples and multiple optimal solutions
- The simplex method
- Issues that arise in the simplex method

Basic solutions and extreme points

In the example, the optimal solution is a “corner point” of the polygon defining the feasible region.



This fact is true in general, even for problems with more than two variables.
Firstly, we need to define what we mean by “corner points”, which in the language of Linear Programming are called extreme points.

Basic solutions and extreme points

In two dimensions a corner point is one at the intersection of two non-parallel inequalities. What about in higher dimensions? Recall:

Independent constraints: Given a system of n linear inequalities in n variables, we say that the n inequalities are *independent* if there exists only one solution that satisfies all n of them at equality.

In 2 dimensions, two equations are not independent if they define parallel lines:

- If the two parallel lines are distinct, there are NO SOLUTIONS.
- If the two parallel lines are identical, there are INFINITELY MANY SOLUTIONS.

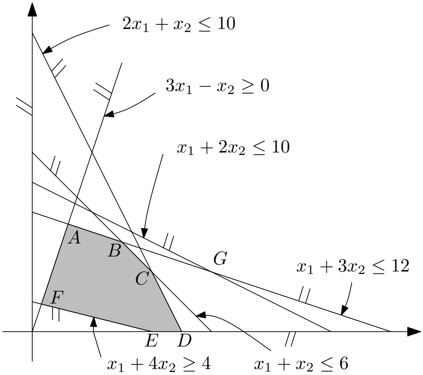
Basic solutions and extreme points

In two dimensions a corner point is one at the intersection of two non-parallel inequalities. What about in higher dimensions? Recall:

Independent constraints: Given a system of n linear inequalities in n variables, we say that the n inequalities are *independent* if there exists only one solution that satisfies all n of them at equality.

Basic point: Given a system of linear constraints in n variables, a basic point for the system is a point satisfying at equality n independent constraints from the system.
Defining constraints: Given a basic point $\bar{x}$ for the system, a set of defining constraints for $\bar{x}$ (*active constraints for* $\bar{x}$) is any choice of n independent constraints satisfied at equality by $\bar{x}$.

Example.



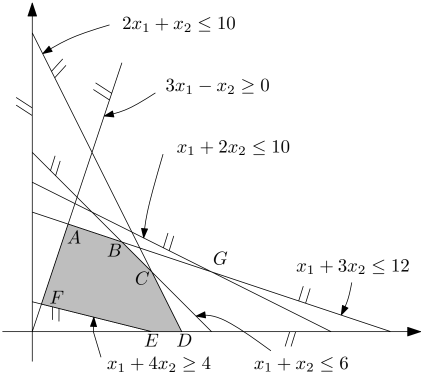
Point B

Basic: it is the solution of

$$\begin{aligned} x_1 + x_2 &= 6 \\ x_1 + 3x_2 &= 12 \end{aligned}$$

that is, $x_1 = 3$, $x_2 = 3$.

Example.



Point G

Basic: it is the solution of

$$\begin{aligned} x_1 + 2x_2 &= 10 \\ x_1 + 3x_2 &= 12 \end{aligned}$$

that is, $x_1 = 6$, $x_2 = 2$.

The point is NOT feasible.

Extreme points

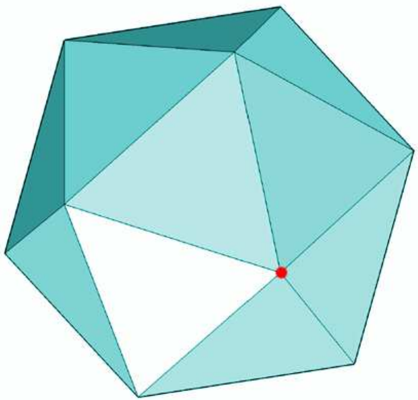
Extreme point: A point that is both basic and feasible for a given system of linear constrains.

Example: Is the point $x_1 = 4$, $x_2 = 2$ extreme?

$$
\begin{array}{rrcllll}
\text{maximise} & 2x_1 + 8x_2 & & & & & \\
\text{subject to} & 2x_1 + x_2 & \leq & 10 & 2 \cdot 4 + 1 \cdot 2 & = 10 & \surd \\
& x_1 + 2x_2 & \leq & 10 & 1 \cdot 4 + 2 \cdot 2 & = 8 & \\
& x_1 + x_2 & \leq & 6 & 1 \cdot 4 + 1 \cdot 2 & = 6 & \surd \\
& x_1 + 3x_2 & \leq & 12 & 1 \cdot 4 + 3 \cdot 2 & = 10 & \\
& 3x_1 - x_2 & \geq & 0 & 3 \cdot 4 - 1 \cdot 2 & = 10 & \\
& x_1 + 4x_2 & \geq & 4 & 1 \cdot 4 + 4 \cdot 2 & = 12 & \\
& x_1, x_2 & \geq & 0 & 4, 2 & &
\end{array}
$$

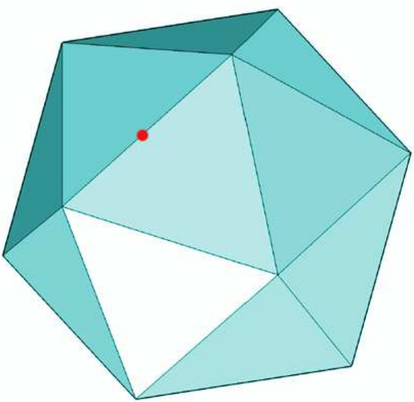
The point is feasible and satisfies two independent constraints: EXTREME.

Example



BASIC: there are three independent constraints satisfied at equality.

Example

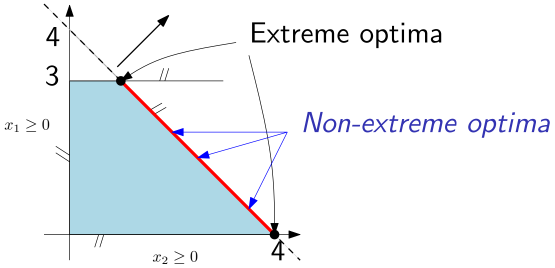


NON-BASIC: there are only two constraints satisfied at equality.

Why do we care about extreme solutions?

Extreme optimal solutions. Whenever an LP admits an optimal solution, there exists some optimal solution that is an extreme point of the feasible region.

Note! Some non-extreme points may also have optimal value, but there is ALWAYS at least an extreme point on the optimal contour.



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Note! Some non-extreme points may also have optimal value, but there is ALWAYS at least an extreme point on the optimal contour.

One only needs to search the optimum among the extreme points, and pick the one with best objective value!

Effective Constraints and basic variables

Given an extreme point x^* and a set S of n defining constraints:

> The resource constraints which define the extreme point are the effective constraints at that point with respect to S.

> If a nonnegativity constraint in S contributes to the definition of an extreme point, say constraint $x_j \geq 0$, x_j is a non-basic variable with respect to S. The other variables are basic variables at that point with respect to S.

- Non-basic variables always have value 0.
- Basic variables may take values $\neq 0$.
- An extreme point is defined by n independent constraints $\Longrightarrow$ *(# non-basic variables) + (# effective constraints)* $= n$.

> number of effective constraints = number of basic variables.

Example

$$
\begin{array}{rrcl}
\text{maximise} & 3x_1 + 5x_2 + 2x_3 & & \\
\text{subject to} & 2x_1 + 2x_2 + x_3 & \leq & 4 \\
& 3x_1 - x_2 + 2x_3 & \leq & 5 \\
& x_1 - x_2 - x_3 & \geq & 1 \\
& x_1 \qquad\qquad & \geq & 0 \\
& x_2 \quad\; & \geq & 0 \\
& x_3 & \geq & 0
\end{array}
$$

At extreme point $x^* = (1.5,\ 0.5,\ 0)$:

- Defining constraints: $S = \{1, 3, 6\}$
- Effective constraints: *constraint 1 and constraint 3*.
- Non-basic variables: variable x_3.
- Basic variables: $x_1,\ x_2$.
- Number of effective constraints = number of basic variables

Proving Optimality: Combining uniform constraints

- How did we prove optimality in lecture 1?
- We used positive multipliers (dual variables) for the resource constraints.
- We used the nonnegativity of the primal variables.
- We obtained a new inequality, called a derived inequality, satisfied by all points satisfying the original constraints, that gave a bound to the objective function.
- It turns out that for basic points we only need to consider linear combinations of the defining constraints at that point.

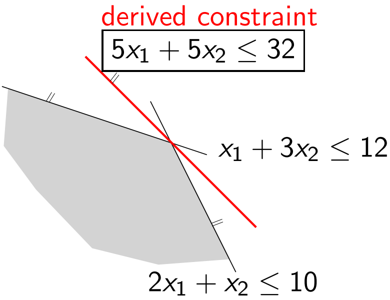
Proving Optimality: Combining uniform constraints

Example:

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + 3x_2 &\leq 12 \end{aligned}$$

If we choose weights of 2 for the first constraint and 1 for the second

$$\begin{array}{rrcl} 2 \cdot (& 2x_1 + x_2 & \leq & 10) \\ 1 \cdot (& x_1 + 3x_2 & \leq & 12) \\ \hline & 5x_1 + 5x_2 & \leq & 32 \end{array}$$

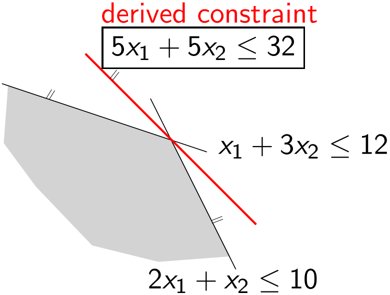


- The derived inequality is satisfied by all points satisfying the two original inequalities.
- The derived inequality is satisfied at equality by the intersection point of the two original inequalities.

Proving Optimality: Combining uniform constraints

Example: Consider the LP

$$\max 5x_1 + 5x_2$$
$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + 3x_2 &\leq 12 \\ x_1 + x_2 &\leq 8 \end{aligned}$$



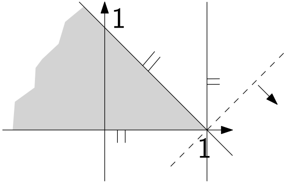
Point $(3.6, 2.8)$ has defining constraints 1,2.

- The derived inequality can be derived from the defining constraints 1 and 2.
- Constraint 3 can be ignored.

A degenerate example

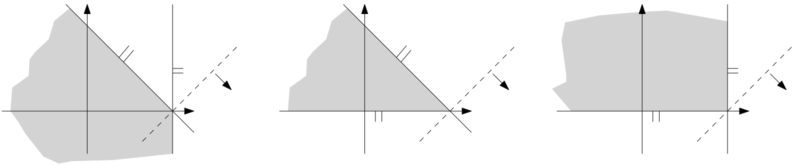
Consider the LP

$$\begin{array}{rrcl} \max & x_1 - x_2 & & \\ & x_1 + x_2 & \leq & 1 \\ & x_1 & \leq & 1 \\ & -x_2 & \leq & 0 \end{array}$$



- The optimal point is $x^* = (1, 0)$.
- This is called a degenerate solution because x^* satisfies 3 constraints at equality but only 2 of them are linearly independent.
- Sets of defining constraints for x^* are: 1,2 or 1,3 or 2,3.
- For every set of defining constraints we have a potential optimality proof.

A degenerate example: optimality proof



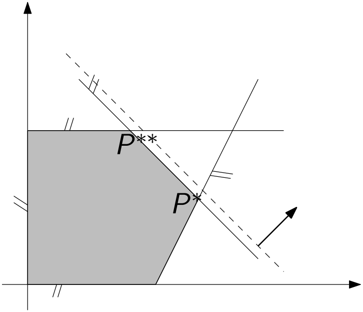
- Constraints 1,2
- $y_1 = -1,\ y_2 = 2$
- Does not prove optimality of x^*

- Constraints 1,3
- $y_1 = 1,\ y_3 = 2$
- *Proves optimality* of x^*

- Constraints 2,3
- $y_2 = 1,\ y_3 = 1$
- *Proves optimality* of x^*

Multiple optima

$$
\begin{aligned}
\text{maximise} \quad & 3x_1 + 3x_2 \\
\text{subject to} \quad x_1 + x_2 &\leq 5 \\
2x_1 - x_2 &\leq 5 \\
x_2 &\leq 3 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

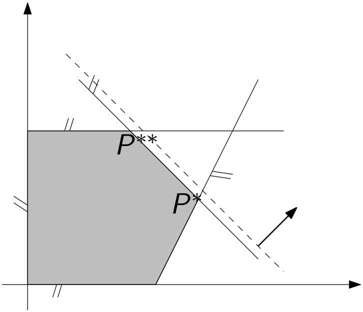


- Optimal extreme solutions: P^*, P^{**},
- All points in the line segment between P^* and P^{**} are also optimal, but not extreme.

If we have one optimal solution can we tell whether or not another optimal point exists?

Multiple optima

$$\begin{array}{rrcl} \text{maximise} & 3x_1 + 3x_2 & & \\ \text{subject to} & x_1 + x_2 & \leq & 5 \\ & 2x_1 - x_2 & \leq & 5 \\ & x_2 & \leq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$



P^* is defined by constraints 1 and 2. Solving the equations gives $P^* = (\frac{10}{3}, \frac{5}{3})$. $Value = 15$.

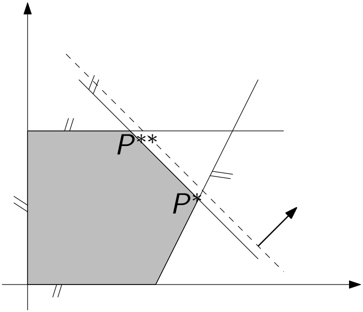
To prove $3x_1 + 3x_2 \leq 15$ using constraints 1 and 2 we need to solve

$$\begin{array}{rcl} y_1 + 2y_2 & = & 3 \\ y_1 - y_2 & = & 3 \end{array}$$

which gives $y_1 = 3,\ y_2 = 0$.

Multiple optima

$$
\begin{array}{lrcl}
\text{maximise} & 3x_1 + 3x_2 & & \\
\text{subject to} & x_1 + x_2 & \leq & 5 \\
 & 2x_1 - x_2 & \leq & 5 \\
 & x_2 & \leq & 3 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$



P^{**} is defined by constraints 1 and 3. Solving the equations gives $P^{**} = (2,3)$. $\mathit{Value} = 15$.

To prove $3x_1 + 3x_2 \leq 15$ using constraints 1 and 3 we need to solve

$$
\begin{array}{rcl}
y_1 & = & 3 \\
y_1 + y_3 & = & 3
\end{array}
$$

which gives $y_1 = 3$, $y_3 = 0$.

Necessary condition for multiple optimal solutions

- At either points, the dual value of one of the defining constraints is 0.
- The only constraint with a positive dual value is constraint 1, which defines the line where both P^* and P^{**} lie.

In general, if there are at least two distinct extreme optimal solutions x^*, x^{**}, the only dual variables that can take non-zero value are the ones corresponding to inequalities that contribute in the definition of both x^*, x^{**}.

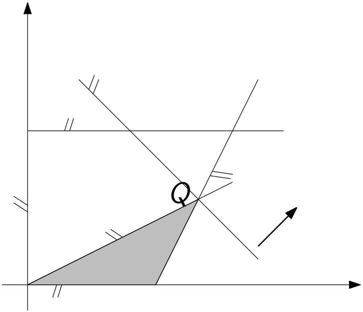
Necessary condition for multiple optimal solutions Given an LP problem, let x^* be an optimal extreme solution. If the problem has multiple optimal solutions then some constraint defining x^* must have dual value 0.

If all dual values associated to the constraints that define an optimal extreme solution are $\neq 0$, then the solution is the UNIQUE optimum.

Necessary, but not sufficient ...

There are cases were there is a unique optimal solution, even though some constraint defining the optimum has zero dual value. Example:

$$\begin{array}{lrcl} \text{maximise} & 3x_1 + 3x_2 & & \\ \text{subject to} & x_1 + x_2 & \leq & 5 \\ & 2x_1 - x_2 & \leq & 5 \\ & x_2 & \leq & 3 \\ & -x_1 + 2x_2 & \leq & 0 \\ & x_1, x_2 & \geq & 0 \end{array}$$



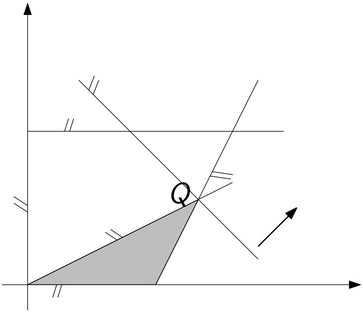
Unique optimal solution is the point $Q = (\frac{10}{3}, \frac{5}{3})$ defined by constraints 1 and 2. Dual values: $y_1 = 3,\ y_2 = 0$.

Constraint 2 defines Q, $y_2 = 0$, but Q is the **unique optimum**.

Necessary, but not sufficient ...

There are cases were there is a unique optimal solution, even though some constraint defining the optimum has zero dual value. Example:

$$\begin{array}{rrcl} \text{maximise} & 3x_1 + 3x_2 & & \\ \text{subject to} & x_1 + x_2 & \leq & 5 \\ & 2x_1 - x_2 & \leq & 5 \\ & x_2 & \leq & 3 \\ & -x_1 + 2x_2 & \leq & 0 \\ & x_1, x_2 & \geq & 0 \end{array}$$



But optimal point $Q = (\frac{10}{3}, \frac{5}{3})$ is also defined by constraints 1,4 and constraints 2,4. If we chose constraints 2 and 4 (that define the feasible region) we would get:
Dual values: $y_2 = 3,\ y_4 = 3$.

This proves that point Q is the **unique optimum**.

The simplex method

- Linear programming emerged after the end of world war II as a practical, powerful tool in a wide array of applications.
- This was made possible by the convergence of two events:
  1. Invention of computers,
  2. The development of the first effective method for solving LP problems: *the Simplex Method* (devised by George Dantzig in the 1940's).
- The method is still today at the core of all commercial Linear Programming solvers:
  - Cplex (IBM),
  - Gurobi,
  - Xpress (FICO),
  - MOSEK
  - SAS/OR
  - ...

The Simplex Method: example

$$\begin{array}{rrcrcr} \max & 3x_1 & + & 2x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 10 \\ & x_1 & - & x_2 & \leq & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Transform the problem, in standard equality form:

$$\begin{array}{rrcrcrcrcrcr} \max & 3x_1 & + & 2x_2 & & & & & & & & \\ s.t. & x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\ & x_1 & + & 2x_2 & & & + & x_4 & & & = & 10 \\ & x_1 & - & x_2 & & & & & + & x_5 & = & 4 \\ & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

Basic solution: $(0, 0, 6, 10, 4)$
Basic variables x_3, x_4, x_5.

The Simplex Method: example

$$\begin{array}{rrrrrrrrrrrrl}
\max & 3x_1 & + & 2x_2 & & & & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\
 & x_1 & + & 2x_2 & & & + & x_4 & & & = & 10 \\
 & x_1 & - & x_2 & & & & & + & x_5 & = & 4 \\
 & & & & & & & x_1, x_2, x_3, x_4, x_5 & & & \geq & 0
\end{array}$$

Basic solution: $(0, 0, 6, 10, 4)$. Value $= 0$. If we increase x_2 by $t \geq 0$ and leave $x_1 = 0$, the objective value increases by $2t$. The remaining components must become

$$\begin{aligned}
x_3(t) &= 6 - t \\
x_4(t) &= 10 - 2t \\
x_5(t) &= 4 + t
\end{aligned}$$

What is the maximum t we can choose? $t = 5$

New solution $(0, 5, 1, 0, 9)$. Value $= 2 \cdot 5 = 10$.

The Simplex Method: example

$$\begin{array}{rrcrcrcrcrcrcr}
\max & 3x_1 & + & 2x_2 & & & & & & & & \\
& x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\
& x_1 & + & 2x_2 & & & + & x_4 & & & = & 10 \\
& x_1 & - & x_2 & & & & & + & x_5 & = & 4 \\
& & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}$$

- Current solution $(0, 5, 1, 0, 9)$. Can we find a better solution?
- Properties we used to improve our solution before:
  1. There was a nonbasic variable with positive objective coefficient (i.e. x_2).
  2. No basic variable appeared in the objective function.
     $\Longrightarrow$ Increasing the nonbasic variable with positive objective coefficient and leaving the remaining nonbasic variables at 0 we increased the objective value.
- Can we get back to that situation? We need to write objective function in terms of the nonbasic variables.

The Simplex Method: some definitions

Let us introduce a new variable z equal to the objective function.

$$
\begin{array}{rrcrcrcrcrcrcr}
\max z & & & & & & & & & & & & & \\
z = & 3x_1 & + & 2x_2 & & & & & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & & & & = & 6 \\
 & x_1 & + & 2x_2 & & & + & x_4 & & & & = & 10 \\
 & x_1 & - & x_2 & & & & & + & x_5 & & = & 4 \\
 & & & & & & & x_1, x_2, x_3, x_4, x_5 & & & \geq & 0 &
\end{array}
$$

- The above form of writing an LP is called a dictionary: z and the basic variables are expressed in terms of the nonbasic variables.
- The basic solution that comes from setting the nonbasic variables to 0 in the dictionary is called the dictionary solution: $(0, 0, 6, 10, 4)$
- The set of basic variables is called a basis B: $B = \{3, 4, 5\}$, $N = \{1, 2\}$.
- For nondegenerate problems:
  dictionary $\Leftrightarrow$ basis $\Leftrightarrow$ basic solution

The Simplex Method: example

Dictionary 1

$$\begin{array}{rrcrcrcrcrcrcr}
\max z & & & & & & & & & & & & & \\
z = & 3x_1 & + & 2x_2 & & & & & & & & & & \\
 & x_1 & + & x_2 & + & x_3 & & & & & & & = & 6 \\
 & x_1 & + & 2x_2 & & & + & x_4 & & & & & = & 10 \\
 & x_1 & - & x_2 & & & & & + & x_5 & & & = & 4 \\
 & & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0 &
\end{array}$$

New basis $B = \{2, 3, 5\}$, $N = \{1, 4\}$: we want to write z in terms of the non-basic variables x_1 and x_4.
Note from the second constraint: $x_2 = 5 - \frac{1}{2}x_1 - \frac{1}{2}x_4$
Substituting for x_2 in the objective function we get
$z = 3x_1 + 2(5 - \frac{1}{2}x_1 - \frac{1}{2}x_4)$, that is

$$z = 10 + 2x_1 - x_4$$

The Simplex Method: example

Current solution $(0, 5, 1, 0, 9)$. The objective function is

$$\max \quad z = 10 + 2x_1 - x_4$$

If we increase x_1 by $t \geq 0$ and leave $x_4 = 0$, the objective value increases by $2t$.
How does this change affect the basic variables x_2, x_3, x_5?
We need to express each of the basic variables x_2, x_3, x_5 in terms of x_1 and x_4.
As we did for the objective function, this can be done by substituting $5 - \frac{1}{2}x_1 - \frac{1}{2}x_4$ for x_2 in all constraints.
For example, constraint $x_1 + x_2 + x_3 = 6$ becomes

$$x_1 + (5 - \frac{1}{2}x_1 - \frac{1}{2}x_4) + x_3 = 6$$

that is

$$\frac{1}{2}x_1 + x_3 - \frac{1}{2}x_4 = 1$$

We do this for all constraints.

The Simplex Method: example

Dictionary 2 , $B = \{2, 3, 5\}$, dictionary solution $(0, 5, 1, 0, 9)$:

$$\begin{array}{rrrrrrrrrrrr}
\max z = 10+ & 2x_1 & & & & & - & x_4 & & & & \\
 & \frac{1}{2}x_1 & & & + & x_3 & - & \frac{1}{2}x_4 & & & = & 1 \\
 & \frac{1}{2}x_1 & + & x_2 & & & + & \frac{1}{2}x_4 & & & = & 5 \\
 & \frac{3}{2}x_1 & & & & & + & \frac{1}{2}x_4 & + & x_5 & = & 9 \\
 & & & & & & \multicolumn{4}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}$$

If we increase x_1 by $t \geq 0$ and leave $x_4 = 0$, the objective value increases by $2t$.
The remaining components must become

$$\begin{array}{rcl}
x_2(t) & = & 5 - \frac{1}{2}t \\
x_3(t) & = & 1 - \frac{1}{2}t \\
x_5(t) & = & 9 - \frac{3}{2}t
\end{array}$$

The largest t for which the variables remain ≥ 0 is $t = 2$ (x_3).
New solution: $(2, 4, 0, 0, 6)$. Value $= 10 + 2 \cdot 2 = 14.$
x_1 entering basis; x_3 leaving basis.

The Simplex Method: example

$$
\begin{array}{lcccccccccccc}
\max z = 10+ & 2x_1 & & & & & - & x_4 & & & & \\
& \frac{1}{2}x_1 & & & + & x_3 & - & \frac{1}{2}x_4 & & & = & 1 \\
& \frac{1}{2}x_1 & + & x_2 & & & + & \frac{1}{2}x_4 & & & = & 5 \\
& \frac{3}{2}x_1 & & & & & + & \frac{1}{2}x_4 & + & x_5 & = & 9 \\
& & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

Basic solution: $(2, 4, 0, 0, 6)$, $B = \{1, 2, 5\}$.
As before, we want write z and the basic variables x_1, x_2, x_5 in terms of the nonbasic variables x_3, x_4.
We substitute $x_1 = 2 - 2x_3 + x_4$ everywhere, obtaining

$$
\begin{array}{lcccccccccccc}
\max z = 14 & & & & - & 4x_3 & + & x_4 & & & & \\
& x_1 & & & + & 2x_3 & - & x_4 & & & = & 2 \\
& & + & x_2 & - & x_3 & + & x_4 & & & = & 4 \\
& & & & - & 3x_3 & + & 2x_4 & + & x_5 & = & 6 \\
& & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

The Simplex Method: example

Dictionary 3:

$$
\begin{array}{rcrcrcrcrcrcl}
\max z = 14 & & & & & - & 4x_3 & + & x_4 & & & & \\
 & x_1 & & & & + & 2x_3 & - & x_4 & & & = & 2 \\
 & & + & x_2 & & - & x_3 & + & x_4 & & & = & 4 \\
 & & & & & - & 3x_3 & + & 2x_4 & + & x_5 & = & 6 \\
 & & & & & & & & x_1, x_2, x_3, x_4, x_5 & & & \geq & 0
\end{array}
$$

Basic solution: $(2, 4, 0, 0, 6)$. Objective value $= 14$.
If we increase x_4 by $t \geq 0$ and leave $x_3 = 0$, the objective value increases by t.
The remaining components must become

$$
\begin{aligned}
x_1(t) &= 2 + t \\
x_2(t) &= 4 - t \\
x_5(t) &= 6 - 2t
\end{aligned}
$$

The largest t for which the variables remain ≥ 0 is $t = 3$.
New solution: $(5, 1, 0, 3, 0)$, $B = \{1, 2, 4\}$.

The Simplex Method: example

Dictionary 4: Again, we write z and the basic variables x_1, x_2, x_4 in terms of the nonbasic variables x_3, x_5.

$$
\begin{array}{rcccccccccccl}
\max z = 17 & & & & & - & \frac{5}{2}x_3 & & & - & \frac{1}{2}x_5 & & \\
 & x_1 & & & & + & \frac{1}{2}x_3 & & & + & \frac{1}{2}x_5 & = & 5 \\
 & & + & x_2 & & + & \frac{1}{2}x_3 & & & - & \frac{1}{2}x_5 & = & 1 \\
 & & & & & - & \frac{3}{2}x_3 & + & x_4 & + & \frac{1}{2}x_5 & = & 3 \\
 & & & & & & & & & & x_1, x_2, x_3, x_4, x_5 & \geq & 0
\end{array}
$$

Basic solution: $(5, 1, 0, 3, 0)$. Objective value $= 17$.

Can there be a better solution?

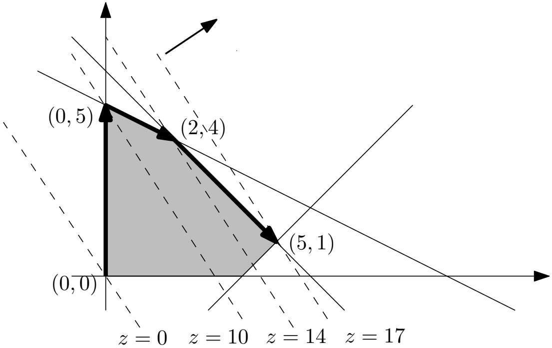
NO! It is clear that no solution can achieve value better than 17 in the objective function, because all coefficients of all variables in the expression

$$z = 17 - \frac{5}{2}x_3 - \frac{1}{2}x_5$$

are ≤ 0.

It follows that $(5, 1, 0, 3, 0)$ is the optimal solution!

The Simplex Method: example



Visited solutions:

$(0,0,6,10,4)$,
$(0,5,1,0,9)$,
$(2,4,0,0,6)$,
$(5,1,0,3,0)$.

The Simplex Method

- The simplex method requires the problem to be in standard equality form:

$$\begin{aligned} \max cx & \\ Ax = & \quad b \\ x \geq & \quad 0 \end{aligned}$$

- Since all resource constraints are of type "$=$", they are effective in every extreme solution. Any extreme solution is thus completely defined by the choice of basic variables. If there are m resource constraints, there must be m basic variables.
- The Simplex Method needs an initial extreme solution to start from, say $\bar{x}$, and the corresponding set of basic variables that define it.

The Simplex Method

$$
\begin{aligned}
\max z & \\
z = & \quad cx \\
Ax = & \quad b \\
x \geq & \quad 0
\end{aligned}
$$

- Denote the basic variables by $x_{B[1]}, x_{B[2]} \ldots, x_{B[m]}$, and let us denote by N the set of indices of the n *nonbasic variables*.
- We write the variables z, $x_{B[1]}, x_{B[2]} \ldots, x_{B[m]}$ in terms of the nonbasic variables, so that the problem is in dictionary form:

$$
\begin{array}{llllll}
\max & z & & & & \\
& z = & \bar{z} & + \sum_{j \in N} \bar{c}_j x_j & & \\
& & x_{B[i]} & + \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
& & & x & \geq & 0.
\end{array}
$$

The Simplex Method

$$
\begin{array}{llllll}
\max & z & & & & \\
& z = & \bar{z} & + \sum_{j \in N} \bar{c}_j x_j & & \\
& & x_{B[i]} & + \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
& & & x & \geq & 0.
\end{array}
$$

The extreme solution $\bar{x}$ is defined by

$$
\begin{aligned}
\bar{x}_{B[i]} &= \bar{b}_i; \quad i = 1, \ldots, m \\
\bar{x}_j &= 0; \quad j \in N
\end{aligned}
$$

Objective value $= \bar{z}$.

- When a problem is written in dictionary form with basis B and nonbasic variables N, we call the coefficients $\bar{c}_j$ of the nonbasic variables reduced cost. The reduced cost of the basic variables is 0.
- Intuition: reduced cost is the marginal amount the objective would increase when a nonbasic variable is increased (currently set at 0).

The Simplex Method: there are two cases

Case 1: There exists a nonbasic variable x_k such that $\bar{c}_k > 0$.

$$
\begin{array}{llllll}
\max & z & & & & \\
 & z = & \bar{z} & + \sum_{j \in N} \bar{c}_j x_j & & \\
 & & x_{B[i]} & + \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
 & & & x & \geq & 0.
\end{array}
$$

- If we increase the value of x_k from 0 to $t \geq 0$, while leaving at 0 all other nonbasic variables, then the objective function value increase by $\bar{c}_k \cdot t$.
- In order to do so, we need to adjust the values of the basic variables to maintain feasibility:

$$
x_{B[i]}(t) = \bar{b}_i - t\bar{a}_{ik}, \qquad i = 1, \ldots, m;
$$

- What is the largest value of t that we can choose?

The Simplex Method - Case 1

We need

$$x_{B[i]}(t) \quad = \quad \bar{b}_i - t\bar{a}_{ik} \geq 0, \qquad i = 1, \ldots, m;$$

We would like to increase the value of t as much as possible. The only thing preventing t from increasing indefinitely is the nonnegativity of the variables. 2 cases.

a) For some value $t = t^*$, some basic variable x_ℓ, becomes 0. Variable x_k becomes basic, x_ℓ becomes nonbasic, and we repeat.

b) The values of the variables remain nonnegative for any positive value of $t \Longrightarrow$ the problem is unbounded, STOP.

The Simplex Method - Case 2

Case 2: $\bar{c}_j \leq 0$ for all indices $j \in N$.

$$
\begin{array}{llllll}
\max & z & & & & \\
& z = & \bar{z} & + \sum_{j \in N} \bar{c}_j x_j & & \\
& & x_{B[i]} & + \sum_{j \in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \quad i = 1, \ldots, m \\
& & & x & \geq & 0.
\end{array}
$$

- This means that all coefficient in the objective function (reduced costs) of the LP are ≤ 0.
- No solution can have value greater than $\bar{z}$.
- The current solution $\bar{x}$ has value $\bar{z} \implies$ **it is optimal**.

The Simplex Method (maximisation)

Start from an extreme solution, with basic variables $x_{B[1]}, \ldots, x_{B[m]}$.

1. Write the LP expressing the variables $z, x_{B[1]}, \ldots, x_{B[m]}$ in terms of the nonbasic variables.
2. If $\bar{c}_j \leq 0$ for all $j \in N$, then the current solution is optimal, STOP.
3. Otherwise, pick a nonbasic variable x_k such that $\bar{c}_k > 0$ to be increased by t.
   Compute the largest value t^* of t such that the new solution is feasible.
   - 3a. If $t^* < +\infty$, then some basic variable x_ℓ takes value 0. Compute the new extreme solution, and replace x_ℓ with x_k as basic variable. Return to 1.
   - 3b. If $t^* = +\infty$, then the problem is unbounded. STOP.

Simplex: issues that arise

- Choosing the entering basic variable
- Problems with degeneracy
- Multiple optimal solutions

MA423 Fundamentals of Operations Research

Lecture 4 : Integer Programming

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Lecture outline: first part

- Integer programming vs Linear programming
- Rounding integer programs
- The branch and bound method

Integer programming

- Example: our decision variables correspond to a number of expensive machines to buy.

- Optimal solution: 2.3

General integer program

Pure integer program

$$\max c^T x$$

subject to

$$Ax \le b$$

$$x \text{ is integer}$$

Mixed integer program

$$\max c_1{}^T x_1 + c_2{}^T x_2$$

subject to

$$A_1 x_1 + A_2 x_2 \le b$$

$$x_1 \text{ is integer}$$

$$x_2 \text{ is real}$$

Binary variables

- Only 0/1 values
- Model “yes”/ “no” decisions
- Most important class of integer variables.

The trade-off

	LP	IP
Modelling power	-	+
Efficient solvability	+	-

General integer program

Linear programming

VS

Integer programming

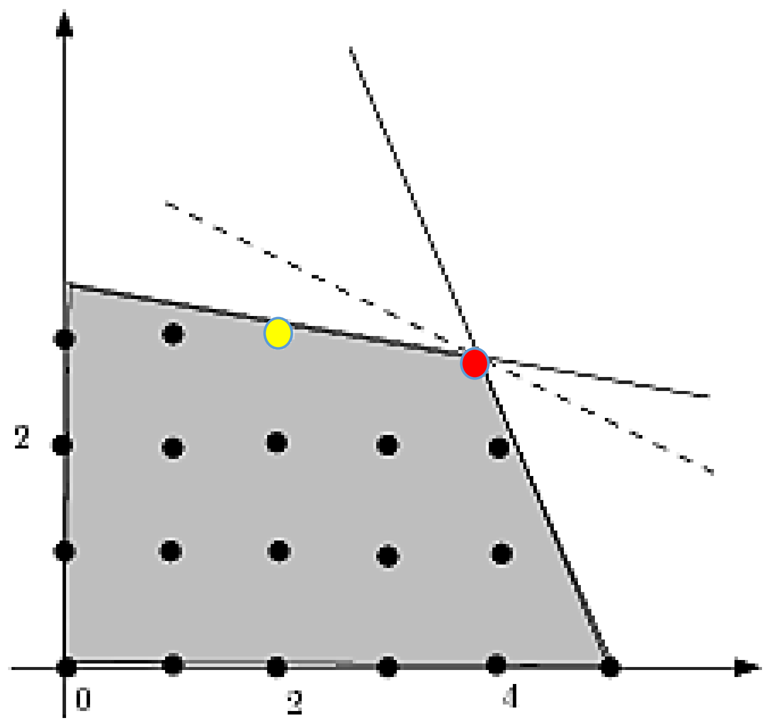
IP feasible regions

$$\max \; 2x_1 + 5x_2$$

$$s.t. \quad 12x_1 + 5x_2 \le 60$$

$$2x_1 + 10x_2 \le 35$$

$$x_1, x_2 \ge 0 \;\; and \;\; Integer$$



LP optimum

$x_1 = 3.864$, $x_2 = 2.727$

objective function

21.364

IP optimum

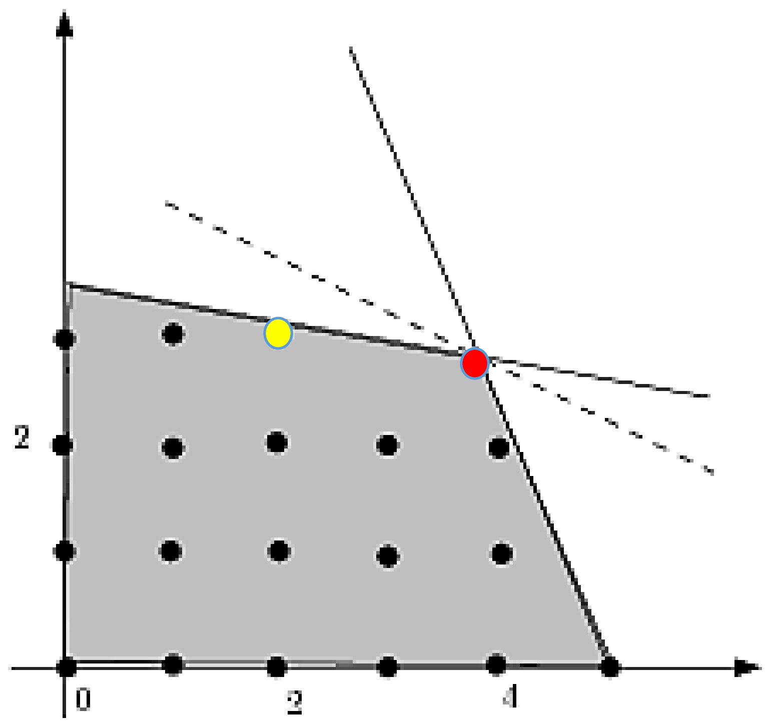
$x_1 = 2$, $x_2 = 3$

objective function

19

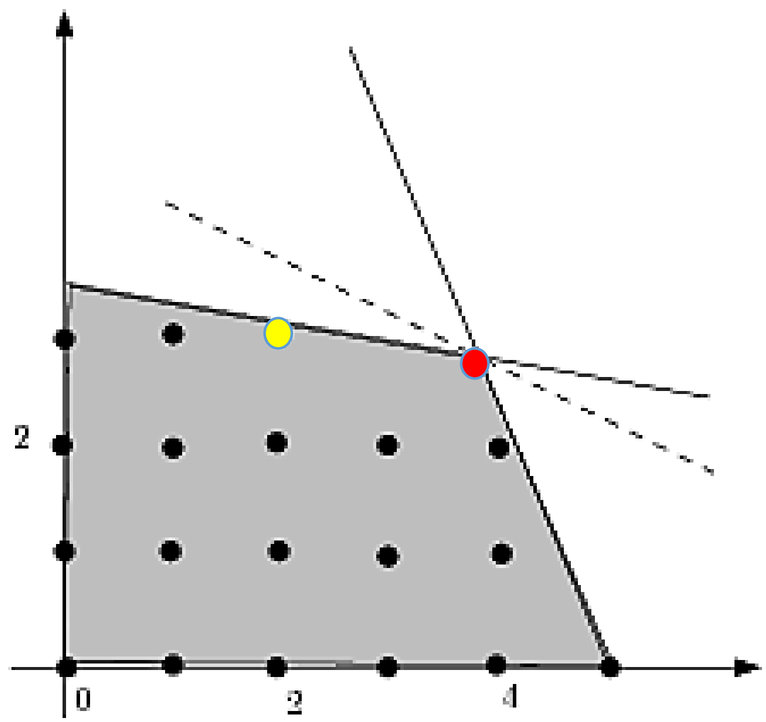
Integer Programming vs Linear Programming

- Local optimum $\neq$ global optimum
- No dual solutions
- Cannot easily convince someone about optimality: no easy optimality proof



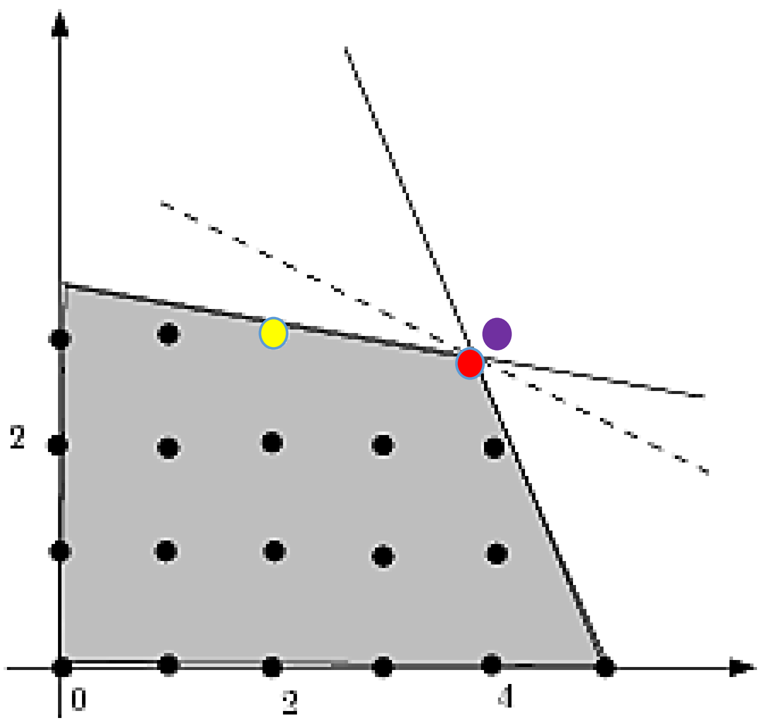
LP relaxation of IP

- Remove all integrality constraints from the IP.
- LP relaxations provide an upper (lower) bound for the objective for maximisation (minimisation) problems



Rounding

- Rounded solution can be infeasible
- Even if feasible, possibly not optimal.



Branch and bound

Ailsa Land

Alison Doig

LSE Operational Research Department, 1960

Branch and bound

- Divide-and-conquer method
- Solve the LP relaxation
- Based on the solution, decompose the problem into subproblems
- Iterate
- Subproblems represented by a tree graph.
- Branch = decompose
- Bound = compare subproblems

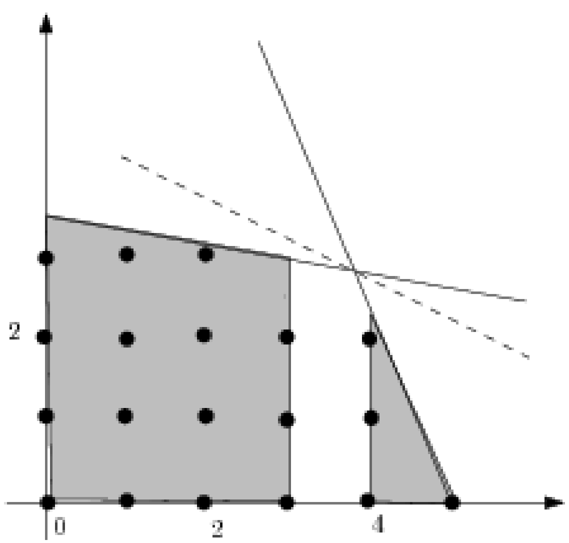
Example

$$\max\ z = 2x_1 + 5x_2$$

$$\begin{aligned} s.t. \quad & 12x_1 + 5x_2 \le 60 \\ & 2x_1 + 10x_2 \le 35 \\ & x_1, x_2 \ge 0 \ \ and \ \ Integer \end{aligned}$$

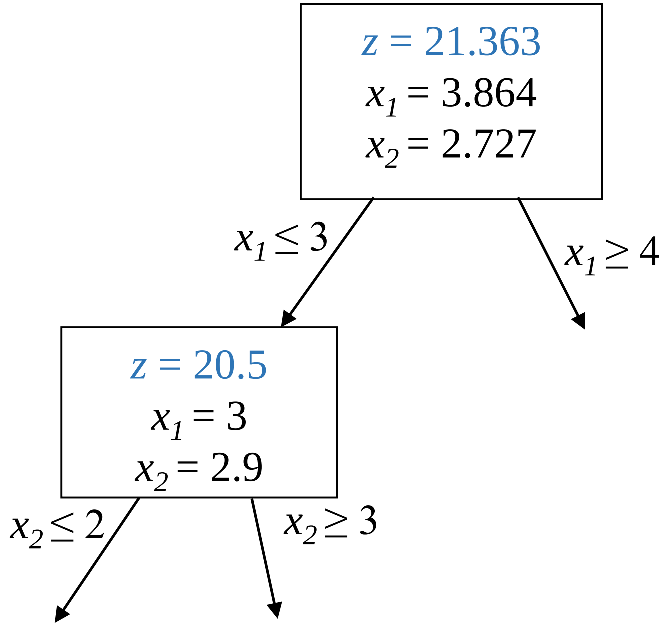
$z = 21.363$
$x_1 = 3.864$
$x_2 = 2.727$

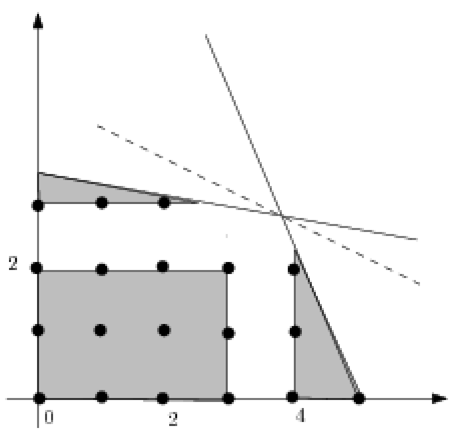
$x_1 \le 3$ $x_1 \ge 4$



Example

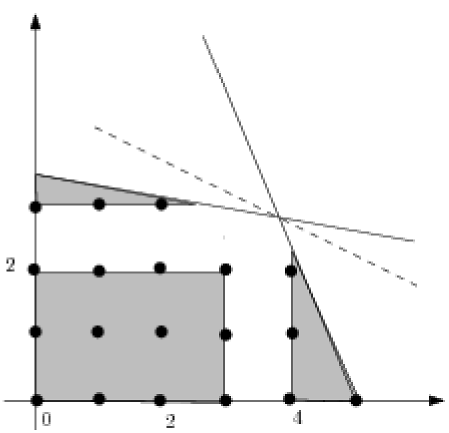
$$
\begin{aligned}
\max\ & z = 2x_1 + 5x_2 \\
s.t.\ & 12x_1 + 5x_2 \le 60 \\
& 2x_1 + 10x_2 \le 35 \\
& x_1, x_2 \ge 0 \ \ \textit{and Integer}
\end{aligned}
$$

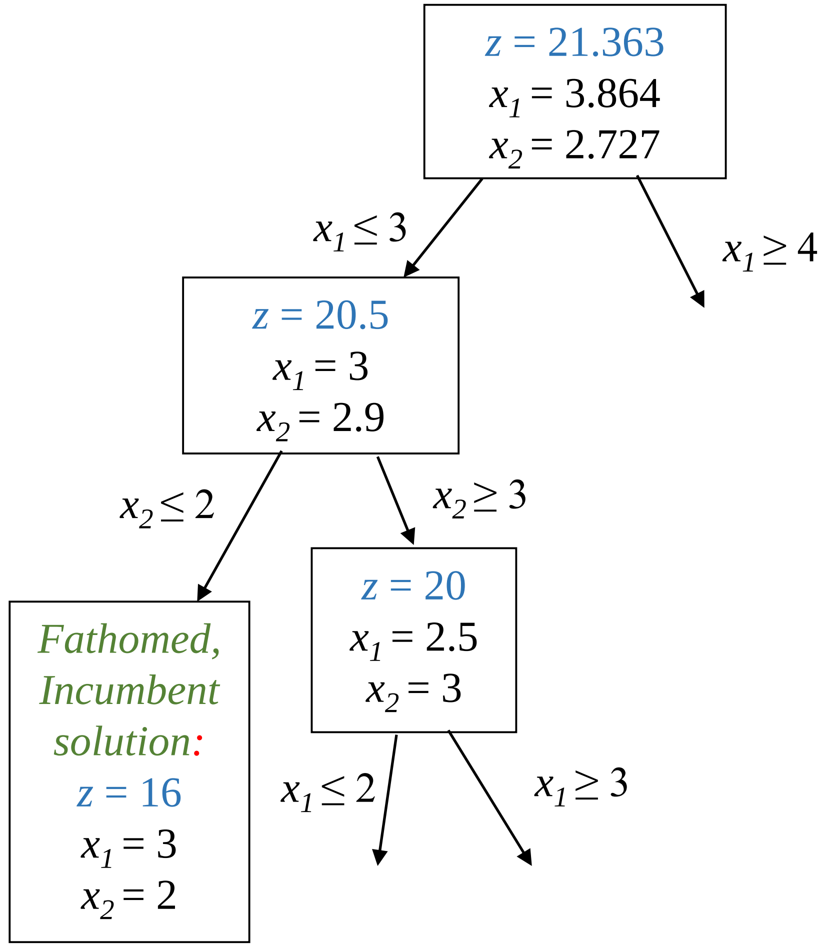




Example

$$\max \ z = 2x_1 + 5x_2$$
$$s.t. \qquad 12x_1 + 5x_2 \le 60$$
$$2x_1 + 10x_2 \le 35$$
$$x_1, x_2 \ge 0 \ \ and \ \ Integer$$





Example

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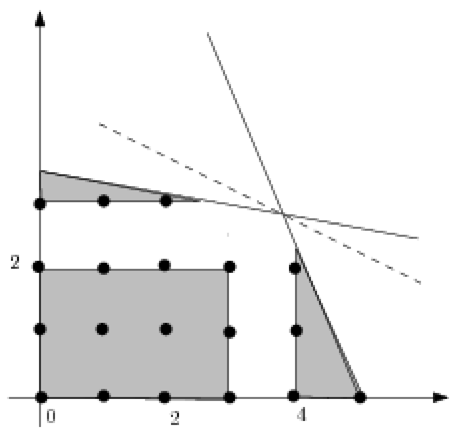

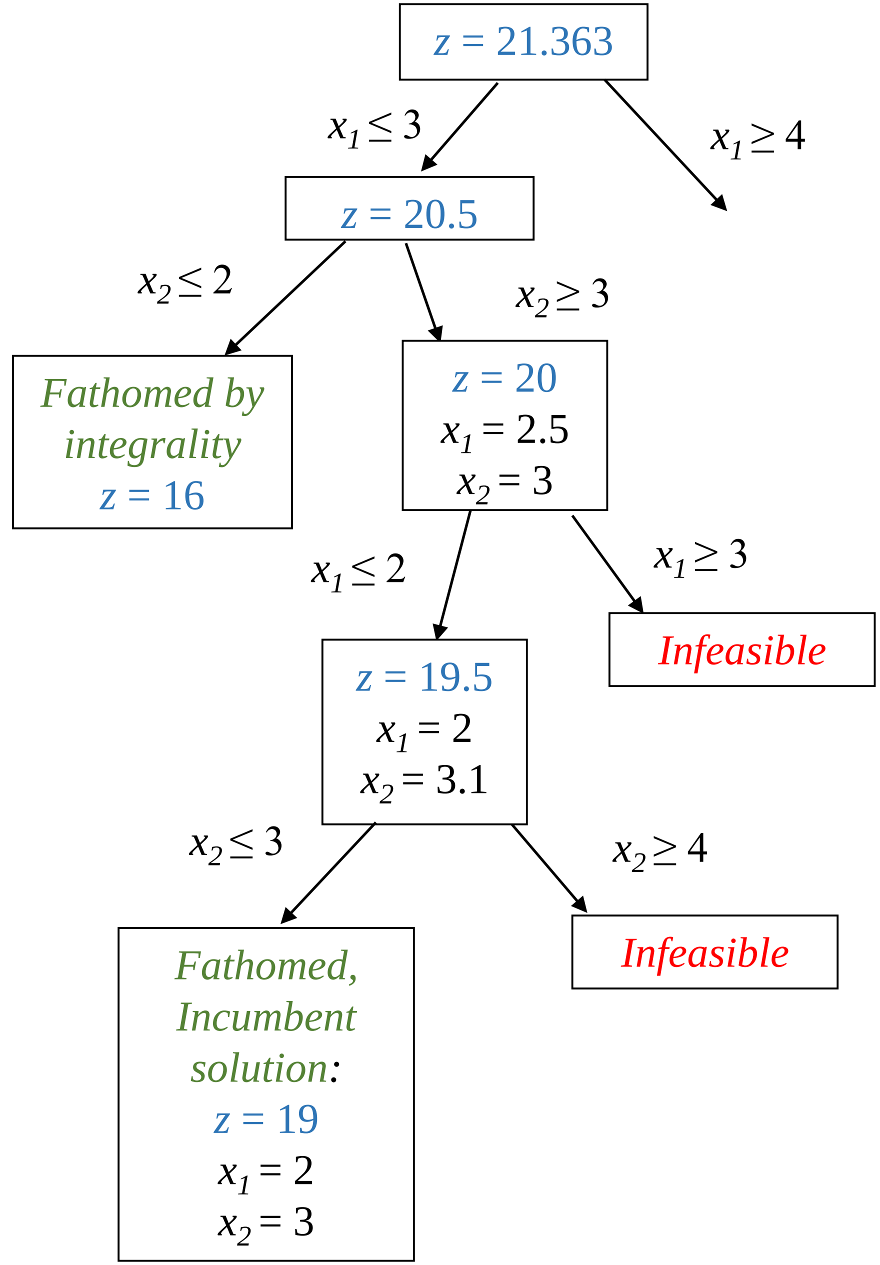

Example

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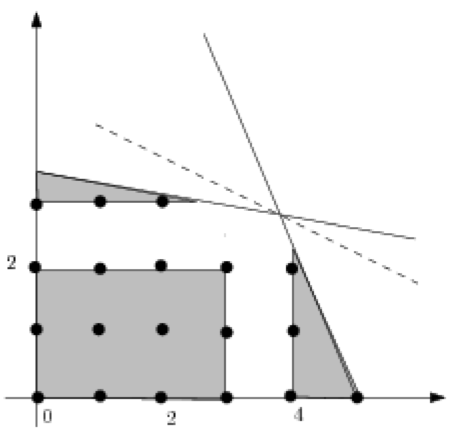


$z = 21.363$

$x_1 \le 3$

Fathomed,
Incumbent
solution:
$z = 19$
$x_1 = 2$
$x_2 = 3$

$x_1 \ge 4$

$z = 20$
$x_1 = 4$
$x_2 = 2.4$

$x_2 \le 2$

$z = 18.3$
$x_1 = 4.167$
$x_2 = 2$
Fathomed by bound.

$x_2 \ge 3$

Infeasible

Branch and Bound

Fathoming tests:

A branch or sub-problem can be fathomed (conquered) if one of the following hold:

1. Fathomed by integer solution:
   - LP relaxation of the sub-problem is integer.
   - We have solved the IP sub-problem optimally.
2. Fathomed by infeasibility:
   - LP relaxation of the sub-problem has no feasible solutions
   - IP sub-problem is also infeasible.
3. Fathomed by bound:
   - Objective value of LP relaxation of the sub-problem is less than the objective value of the incumbent solution.
   - If we branch further we will only get worse integer solutions.

In all of the above cases, we do not branch any further.

Branch and Bound

Initialize:

Apply the bounding step, fathoming step and optimality test to the whole problem. If not fathomed then classify as remaining sub-problem and perform:

Iteration steps:

1. Branching step: amongst the remaining unfathomed sub-problems select one. Branch on one of the variables that did not have integer value in the LP relaxation.
2. Bounding step: for each new sub-problem solve the LP relaxation to obtain an optimal solution and an objective value. This objective value is an upper bound on the objective value of the IP sub-problem.
3. Fathoming step: For each sub-problem apply the three fathoming tests and discard all the sub-problems that are fathomed by any of the three tests.

Optimality check: Stop when there are no remaining sub-problem. The current incumbent solution is optimal.

Branch and Bound

Remarks:

1. How to pick the next sub-problem:
    - Highest bound
    - The one created most recently.
2. Binary variables: branching x=0 and x=1
3. Objective coefficients not integer: no rounding of the bound
4. Number of sub-problems solved: up to 2^k for k binary variables

Mixed Integer Programming (MIP)

For MIP problems we can use branch and bound with the following alterations:

1. Branching step: You only branch on variables that are integer.
2. Bounding step: You do not round down the bound of the LP relaxation since the objective value of the MIP is most likely fractional.
3. Fathoming step: A solution is considered incumbent if it is integer in the integer variables.

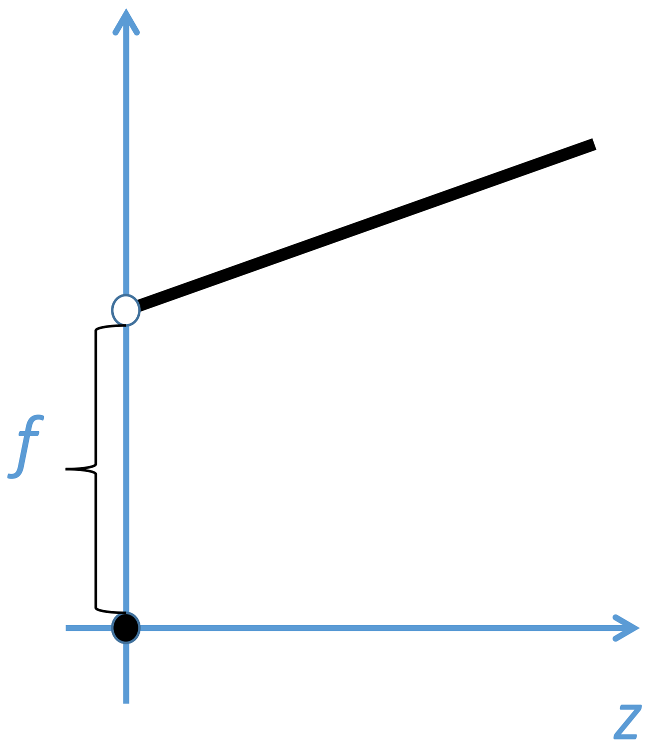
Lecture outline: second part
IP Formulations with binary variables

- Fixed costs: the big-M method
- Blending problems: the small-M method
- Facility location
- Expressing logical conditions
- Non-convex regions
- Variables with restricted value ranges

Binary variables

- Only 0/1 values
- Model “yes”/ “no” decisions
- Most important class of integer variables.

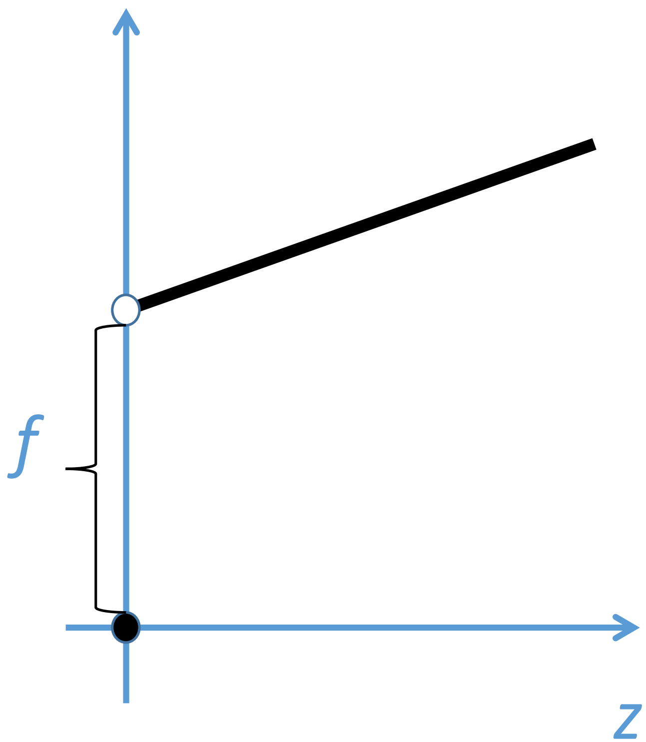
Fixed costs



- Unit cost of production: c
- Fixed cost: f
- $z \geq 0$ is amount produced
- The cost f has to be paid if we decide to produce any positive amount.

$$\text{Production cost} \;=\begin{cases} 0 & if\ z = 0 \\ f + cz & if\ z > 0 \end{cases}$$

Big-*M* method



- Binary (0-1) variable: δ
- Need to formulate

  "If $z > 0$ then $\delta = 1$"
- *M:* upper bound on *z.*

$$z \leq M\delta$$

Production cost:

$$f\delta + cz$$

Blending problem

- Ingredients: *A*, *B* and *C.*
- If we include *A* then we must also include *B.*
- x_A, x_B, x_C proportions (between 0 and 1)
- If x_B is included then we must have $x_B \geq 0.01$

$$x_A + x_B + x_C = 1,$$
$$x_A, x_B, x_C \geq 0$$

- δ: binary variable
- If $x_A > 0$ then $\delta = 1$

- Big M-constraint: $\boxed{x_A \leq \delta}$

"Small-*m* method"

- If we include *A* then we must also include *B.*
- We need:

$$(\delta = 1) \rightarrow (x_B > 0)$$

- *We know that m = 0.01 is a lower bound on* x_B
- *Instead:* $(\delta = 1) \rightarrow (x_B \geq 0.01)$

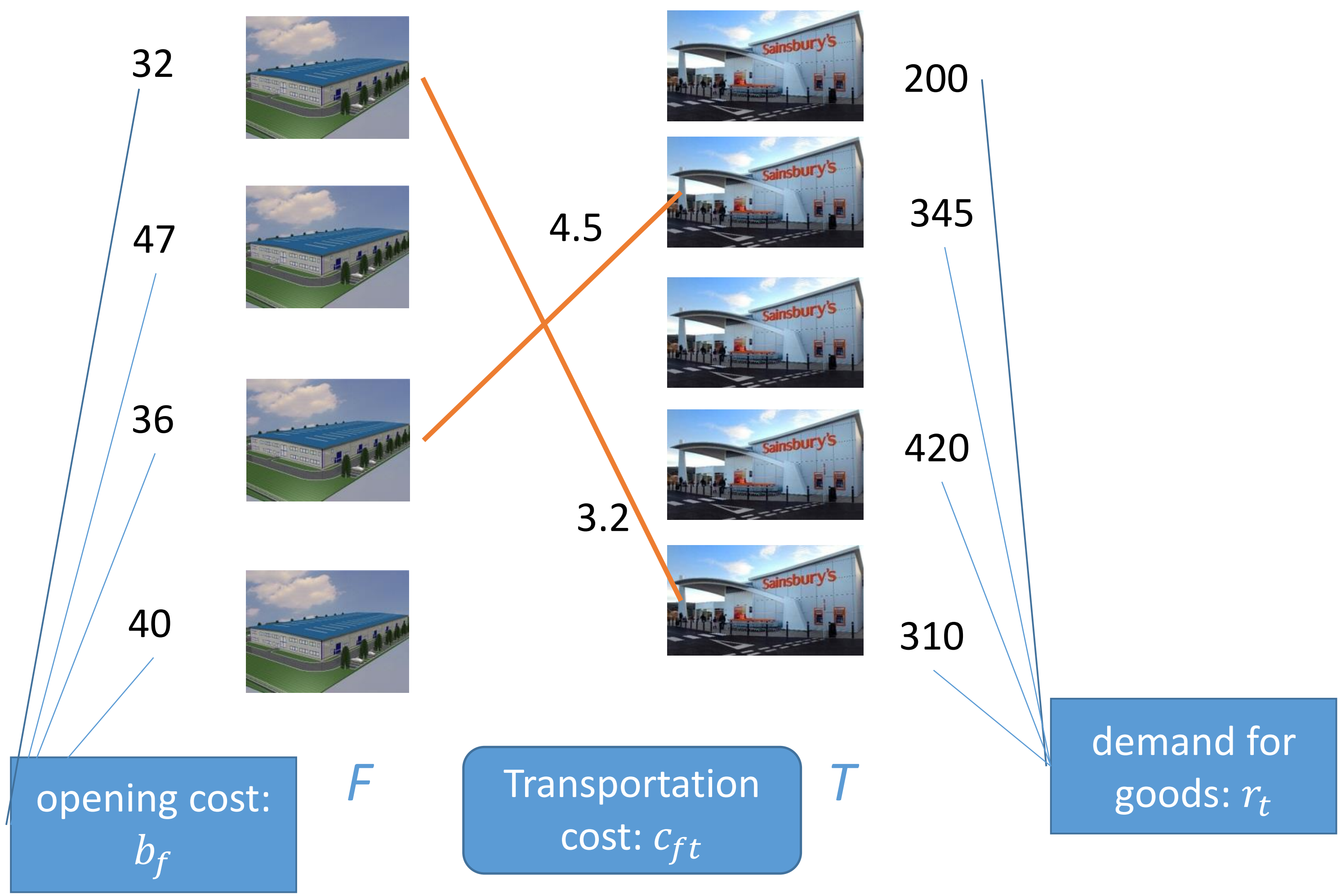
$$\boxed{x_B \geq 0.01\delta}$$

Formulation

$$x_A \leq \delta$$
$$x_B \geq 0.01\delta$$
$$x_A + x_B + x_C = 1$$
$$x_A, x_B, x_C \geq 0$$
$$\delta \epsilon \{0,1\}$$

Facility location

Need to supply a set T of stores from a set F of possible facilities.



Facility location

- Variables:
  - x_{ft}: amount of goods supplied by facility f to store t (continuous).
  - δ_f: variable indicating if facility f is open (binary).
- Big M-constraint:

if $x_{ft} > 0$ then $\delta_f = 1$

$$x_{ft} \leq r_t \delta_f \qquad f \epsilon F, t \epsilon T$$

Facility location: Formulation

$$min \sum_{f \epsilon F} b_f \delta_f + \sum_{f \epsilon F} \sum_{t \epsilon T} c_{ft} x_{ft}$$

$$\sum_{f \epsilon F} x_{ft} = r_t \qquad t \epsilon T$$

$$x_{ft} \leq r_t \delta_f \qquad f \epsilon F, t \epsilon T$$

$$x_{ft} \geq 0$$

$$\delta_f \epsilon \{0,1\}$$

Facility location: Additional constraint

- Every store has to be supplied from only one facility.
- β_{ft}: binary variable, 1 if t is supplied from f.

Supply from open facilities only:

$$\beta_{ft} \leq \delta_f \qquad f\epsilon F, t\epsilon T$$

$$x_{ft} = \beta_{ft} r_t \qquad f\epsilon F, t\epsilon T$$

$$\sum_{f\epsilon F} \beta_{ft} = 1 \qquad t\epsilon T$$

Logical conditions

Events X_1 and X_2

Indicator variables x_1 and x_2:

equal to 1 if corresponding event occurs and 0 otherwise.

Indicator variables x_1 and x_2, are to be related via logical conditions:

- 'X1 *or* X2': $x_1 + x_2 \geq 1$
- '*either* X1 *or* X2': $x_1 + x_2 = 1$
- 'X1 *and* X2': $x_1 = 1, x_2 = 1$
- 'X1 $\leftrightarrow$ X2': $x_1 = x_2$
- 'X1 $\rightarrow$ X2': $x_1 \leq x_2$

Logical conditions

n expressions $X_1, X_2, \ldots, X_n$

Indicator variables $x_1, x_2, \ldots, x_n$

At most *k* events can happen: $\sum_{i=1}^{n} x_i \leq k$

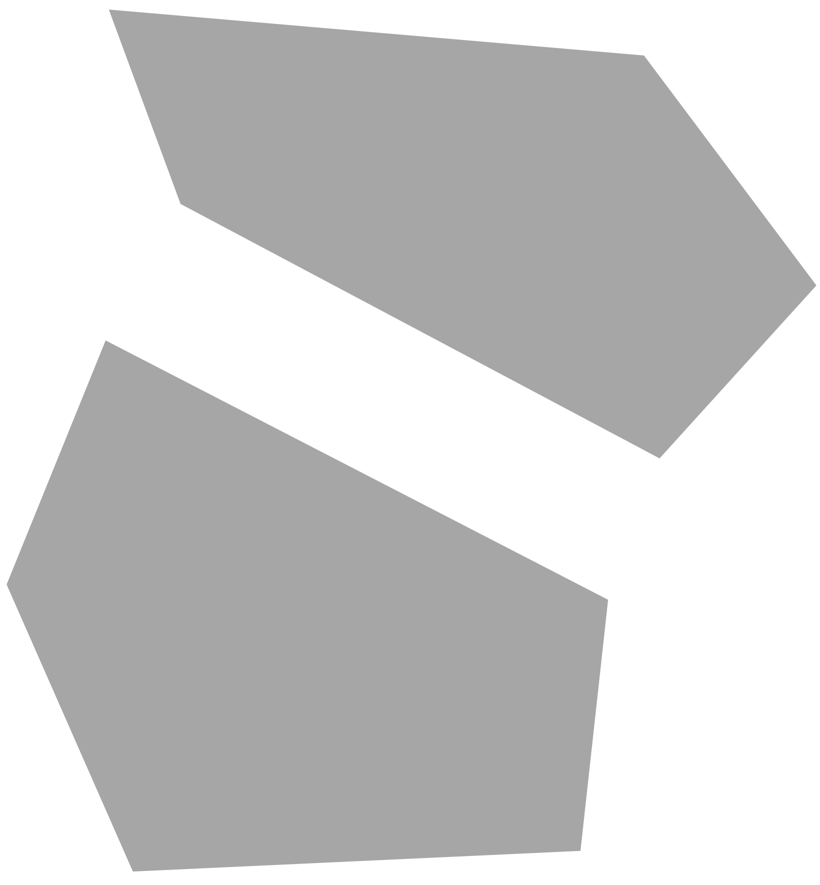
Exactly *k* events can happen : $\sum_{i=1}^{n} x_i = k$.

At least *k* events can happen: $\sum_{i=1}^{n} x_i \geq k$.

Logical conditions: Example

- Clothing company
- 4 possible shirts: A, B, C, D
- 2 possible ties: P, Q
- Condition*: if we decide to manufacture ties, then at least 3 different types of shirts have to be also manufactured.*

Non-convex feasible region



- Even if all variables are continuous, binary variables can be useful to model constraints that are non-linear.

The Beer Decision problem

- Brewery wants to produce ale and lager beer.
- Management vision: build a strong brand, with a dominant product.
- Either much more ale, or much more lager.

The Beer Decision problem

x_1: amount of lager

x_2: amount of ale

- At most 10,000 barrels can be produced.
- EITHER $x_1 \geq x_2 + 6000$

  OR $x_2 \geq x_1 + 4000$

The Beer Decision Problem

$$\delta = \begin{cases} 1 & if\ more\ lager\ is\ brewed \\ 0 & if\ more\ ale\ is\ brewed \end{cases}$$

$$x_1 - x_2 \geq 6000\delta - 10000(1-\delta)$$
$$x_1 - x_2 \leq -4000(1-\delta) + 10000\,\delta$$

Variables with restricted range

- Suppose we have a variable x with $l \leq x \leq u$ and x is an integer divisible by 3.

- Define a new variable y which is an integer and add a constraint $x = 3y$.

Variables with restricted range

- Now suppose x has to take one of the values 3,5,12,17,19.

- Define binary variables $\delta_3, \delta_5, \delta_{12}, \delta_{17}, \delta_{19}$ with

$$\delta_3 + \delta_5 + \delta_{12} + \delta_{17} + \delta_{19} = 1$$

and

$$x = 3\delta_3 + 5\delta_5 + 12\delta_{12} + 17\delta_{17} + 19\delta_{19}.$$