MA427 Lecture 6
Total unimodularity

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Today's lecture

- Facility location
- Matching and assignment
- Totally unimodular matrices
- Network flow problems

MILP Formulations

$$
\begin{aligned}
z_I = \max c^\top x & \\
Ax \leq b & \\
x \geq 0 & \\
x_i \in \mathbb{Z}, & \qquad i \in I.
\end{aligned}
$$

$$X = \{x \in \mathbb{R}^n \,:\, Ax \leq b,\, x \geq 0,\, x_i \in \mathbb{Z} \text{ for all } i \in I\}$$

A formulation for the above (MILP) is a system

$$
\begin{aligned}
A'x \leq b' \\
x \geq 0
\end{aligned}
$$

such that

$$X = \{x \in \mathbb{R}^n \,:\, A'x \leq b',\, x \geq 0,\, x_i \in \mathbb{Z} \text{ for all } i \in I\}.$$

MILP Formulations

Two formulations for X:

$$\begin{array}{c} Ax \leq b \\ x \geq 0 \end{array} \quad \text{and} \quad \begin{array}{c} A'x \leq b \\ x \geq 0 \end{array}$$

The first formulation is better than the second if the polyhedron determined by the first system of constraints is contained in the one determined by the second system.

Note: if $Ax \leq b$, $x \geq 0$ is a better formulation than $A'x \leq b'$, $x \geq 0$, then

$$z_I \leq z_L \leq z'_L.$$

Better formulations give tighter lower-bounds.

Facility location

- n locations, m customer.
- d_i: demand of customer i, $i = 1, \ldots, m$.
- Costs:
  - c_{ij}: unit cost of servicing i from j;
  - f_j: operating/fixed cost in location j;

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Variables:

-
$$x_j = \begin{cases} 1 & \text{if facility is built at } j; \\ 0 & \text{otherwise.} \end{cases}$$

- y_{ij}: fraction of annual demand d_i provided from j to i;

Facility location: two possible formulations

Aggregated formulation

$$
\begin{array}{lrcll}
\min & \sum_{i=1}^m \sum_{j=1}^n c_{ij} d_i y_{ij} & + & \sum_{j=1}^n f_j x_j & \\
 & \sum_{j=1}^n y_{ij} & = & 1 & i = 1, \ldots, m \\
 & \sum_{i=1}^m y_{ij} & \leq & m x_j & j = 1, \ldots, n \\
 & y & \geq & 0 & \\
 & x & \in & \{0,1\}^n & .
\end{array}
$$

Facility location: two possible formulations

Aggregated formulation

$$
\begin{array}{lrcll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} d_i y_{ij} & + & \sum_{j=1}^{n} f_j x_j & \\
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 & y & \geq & 0 & \\
 & x & \in & \{0,1\}^n & .
\end{array}
$$

Disaggregated formulation

$$
\begin{array}{lrcll}
\min & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} d_i y_{ij} & + & \sum_{j=1}^{n} f_j x_j & \\
 & \sum_{j=1}^{n} y_{ij} & = & 1 & i = 1, \ldots, m \\
 & y_{ij} & \leq & x_j & i = 1, \ldots, m,\ j = 1, \ldots, n \\
 & y & \geq & 0 & \\
 & x & \in & \{0,1\}^n & .
\end{array}
$$

Facility location: comparing formulations

The disaggregated formulation is better than the aggregated one, because if (x, y) satisfies the disaggregated constraints, it also satisfies the aggregated ones.

$$\begin{array}{rcl} y_{1j} & \leq & x_j \\ y_{2j} & \leq & x_j \\ & \vdots & \\ y_{mj} & \leq & x_j \\ \hline \sum_i y_{ij} & \leq & mx_j \end{array}$$

Strictly better: consider $n = 2$, $m = 4$

Ideal formulations

Given a set $X \subseteq \mathbb{R}^n$, the convex hull of X, denoted by $\mathrm{conv}(X)$, it the minimal convex set containing X.

Theorem (Fundamental theorem of Integer Programming)

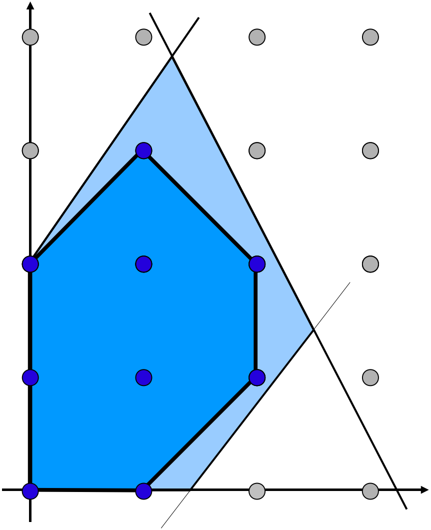
Given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, let $X = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0, x_i \in \mathbb{Z} \text{ for } i \in I\}$. Then $\mathrm{conv}(X)$ is a polyhedron.

$\Longrightarrow$ there exists $\tilde{A} \in \mathbb{Q}^{\tilde{m} \times n}$ and $\tilde{b} \in \mathbb{Q}^{\tilde{m}}$ such that

$$\mathrm{conv}(X) = \{x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b}, x \geq 0\}.$$

$\tilde{A}x \leq \tilde{b}$, $x \geq 0$ is the ideal formulation for X.

Ideal formulations



Ideal formulations

Theorem

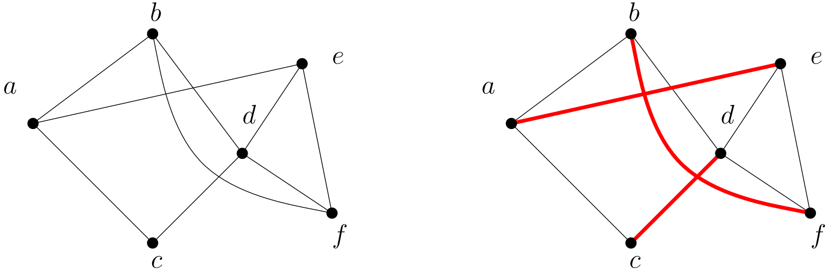
For any $c \in \mathbb{R}^n$, let

$$z_I = \max_{x \in X} c^\top x \qquad \text{and} \qquad \tilde{z} = \max_{x \in \mathrm{conv}(X)} c^\top x \quad .$$

Then $z_I = \tilde{z}$. Furthermore, all vertices of $\mathrm{conv}(X)$ are elements of X.

Matchings and assignment problem

Graph $G = (V, E)$: finite set V of elements, called *nodes*, and a set E of unordered pairs of nodes, called *edges*.



Matching in G: set of edges $M \subseteq E$ such that the elements of M are pairwise disjoint.

Maximum cardinality matching problem: Find a matching of G with the largest possible number of elements

Perfect matching: if every node of G belongs to exactly one edge of M.

Perfect matchings: a formulation

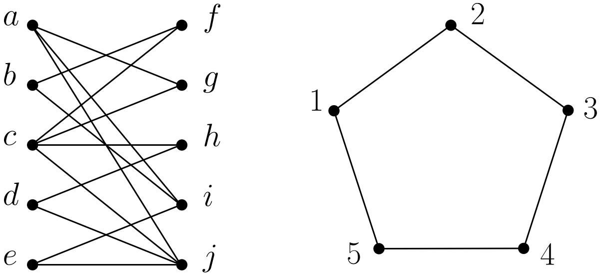
Perfect matching problem: Given weights c_e on the edges, find a perfect matching of minimum total weight.

$$
\begin{aligned}
\min \sum_{e \in E} c_e x_e & \\
\sum_{u\,:\,uv \in E} x_{uv} = 1, & \quad v \in V \\
x_{uv} \geq 0, & \quad uv \in E \\
x_{uv} \in \mathbb{Z}, & \quad uv \in E.
\end{aligned}
$$

Not an ideal formulation for arbitrary graphs!

The assignment problem

- A graph $G = (V, E)$ is said to be bipartite if its node set V can be partitioned into two disjoint sets V_1, V_2 such that every edge $uv \in E$, exactly one node among u and v is in V_1, and he other is in V_1.
- The pair V_1, V_2 is a bipartition of G.



- The graph on the left is bipartite, the graph on the right is not.

The assignment problem

Given a bipartite graph $G = (V, E)$ with bipartition V_1, V_2 such that $|V_1| = |V_2|$ and costs c_e on every edge $e \in E$, assign to every node u in V_1 exactly one node v in V_2 so that $uv \in E$, every element of V_2 is assigned to exactly one element of V_1, and the total cost of the pairs selected is minimized.

In other words, find a perfect matching of minimum total cost.

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$$\min \sum_{uv \in E} c_{uv} x_{uv}$$

$$\begin{array}{rcll} \sum_{v: uv \in E} x_{uv} & = & 1, & u \in V_1, \\ \sum_{u: uv \in E} x_{uv} & = & 1, & v \in V_2, \\ x_{uv} & \geq & 0, & uv \in E \\ x_{uv} & \in & \mathbb{Z}, & uv \in E \end{array}$$

Ideal formulation for bipartite graphs.

The assignment problem

$$\begin{array}{rcll} \min \sum_{uv \in E} c_{uv} x_{uv} & & & \\ \sum_{v: uv \in E} x_{uv} & = & 1, & u \in V_1, \\ \sum_{u: uv \in E} x_{uv} & = & 1, & v \in V_2, \\ x_{uv} & \geq & 0, & uv \in E \\ x_{uv} & \in & \mathbb{Z}, & uv \in E \end{array}$$

It is of the form

$$\begin{gathered} \min c^T x \\ A(G)\, x = \mathbf{1} \\ x \geq 0 \\ x \in \mathbb{Z}^{|E|}, \end{gathered}$$

How does the matrix $A(G)$ look?
Incidence matrix of G: $0, 1$ matrix $A(G)$ with $|V|$ *rows* and $|E|$ *columns*, where

$$a_{v,e} = \begin{cases} 1 & \textit{if } v \in e, \\ 0 & \textit{otherwise.} \end{cases} \qquad v \in V,\ e \in E.$$

Incidence matrix

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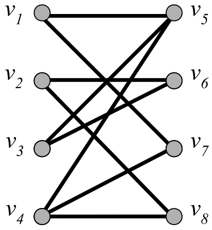

	v_1v_5	v_1v_7	v_2v_6	v_2v_8	v_3v_5	v_3v_6	v_4v_5	v_4v_7	v_4v_8
v_1	1	1	0	0	0	0	0	0	0
v_2	0	0	1	1	0	0	0	0	0
v_3	0	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1	1
v_5	1	0	0	0	1	0	1	0	0
v_6	0	0	1	0	0	1	0	0	0
v_7	0	1	0	0	0	0	0	1	0
v_8	0	0	0	1	0	0	0	0	1

Assignment problem

$$\begin{gathered}\min c^T x \\ A(G)\, x = \mathbf{1} \\ x \geq 0,\, x \in \mathbb{Z}^{|E|},\end{gathered}$$

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Definition

A matrix A is said totally unimodular if, for every square submatrix B of A, $\det(B) \in \{0, +1, -1\}$.

Assignment problem

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Theorem

Let $A \in \mathbb{R}^{m\times n}$ be totally unimodular matrix, and let $b \in \mathbb{Z}^m$. Then all basic solutions of

$$\begin{gathered}Ax = b\\ x \geq 0\end{gathered}$$

are integer.

Assignment problem

$$\min c^T x$$
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$$x \geq 0,\, x \in \mathbb{Z}^{|E|},$$

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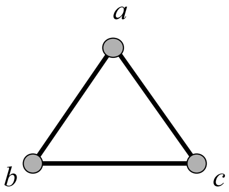
Theorem

The incidence matrix of a bipartite graph is totally unimodular.

General graphs

If a graph G is not bipartite, then $A(G)$ is not totally unimodular.

Example:



$$\begin{array}{c|ccc|} & ab & ac & bc \\ \hline a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 1 \\ \hline \end{array}$$

The incidence matrix in this case has determinant -2, hence it is not totally unimodular.

TU matrices

Theorem

Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. Given a vector $b \in \mathbb{Z}^m$, all vertices of the polyhedron $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ are integer. Similarly, all vertices of the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ are integer.

Fractional feasibility implies integer feasibility

Corollary

If A is a TU matrix, and the system $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is feasible, then it must have an integer feasible solution.

Example: Let $G = (V, E)$ be a bipartite graph such that every node has exactly k incident edges for some integer $k \geq 1$. (Called a *k-uniform bipartite graph*.)

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Show that G has a perfect matching.

TU matrices

Theorem

Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. The following hold

i) *Submatrices of A are T.U.*

ii) *$A^\top$ is T.U.*

iii) *If matrix A' is obtained from A by multiplying one row or column by -1, then A' is T.U.*

iv) *The matrix $(A|-A)$, obtained by juxtaposing the matrices A and $-A$, is T.U.*

v) *The matrix $(A|e)$ is T.U., where e is a unit vector (one entry 1, all others 0).*

vi) *The matrix $(A|I)$, obtained by juxtaposing the matrix A and the identity matrix I, is T.U.*

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ & & & & & \\ & & & & & \end{bmatrix}$$

TU matrices

For example, the following matrix is TU

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ & & & & \\ & & & & \end{bmatrix}$$

TU matrices

Corollary

Let $A \in \mathbb{R}^{m \times n}$ be a totally unimodular matrix. Given vector $b, d \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ all vertices of the polyhedron

$$\{x \in \mathbb{R}^n \,:\, b \leq Ax \leq d,\, \ell \leq x \leq u\}$$

are integer.

Network problems

Theorem

Let A be a matrix with all entries in $\{0, 1, -1\}$, such that in every column of A there is exactly one entry of value 1, one entry of value -1, and all other entries with value 0. Then A is totally unimodular.

Example

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Network problems

Directed graph $G = (V, E)$, source $s \in V$, sink $t \in V$, edge capacities $u : E \to \mathbb{R}$.

Maximum flow problem: find a vector $x : E \to \mathbb{R}$ such that

- the total incoming amount equals the total outgoing amount at every node $v \in V \setminus \{s, t\}$
- the flow on every edge is between 0 and the upper bound: $0 \leq x \leq u$.
- Maximize the total amount of flow leaving s.

Network problems

Directed graph $G = (V, E)$, costs $c : E \to \mathbb{R}$, lower and upper capacity bounds $\ell, u : E \to \mathbb{R}$.

Feasible circulation: vector $x : E \to \mathbb{R}$ such that

- the total incoming amount equals the total outgoing amount at every node $v \in V$.
- it is between the upper and lower bounds: $\ell \leq x \leq u$.

Find a minimum cost feasible circulation.

Perfect matchings: ideal formulation [Edmonds, 1965]

For every graph G, the ideal formulation for the maximum weight perfect matching problem is

$$\begin{array}{rcll} \min \sum_{e \in E} c_e x_e & & & \\ \sum_{u \,:\, uv \in E} x_{uv} & = & 1 & v \in V, \\ \sum_{e \in E[U]} x_e & \leq & \frac{|U|-1}{2} & U \subseteq V,\ |U| \text{ odd}, \\ x_e & \geq & 0 & e \in E. \end{array}$$

where $E[U] := \{uv \in E \,:\, u, v \in U\}$.