MA427 – Exercise set 8

Academic year 2018-19

Exercise 8.1 Show the following.

(a) If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $g : \mathbb{R} \to \mathbb{R}$ is convex and non-decreasing[1], then $g \circ f$ is convex[2].

(b) For $x \in \mathbb{R}^n$, we denote $x_{[i]}$ the ith largest component of x, that is,

$$x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}.$$

are the components sorted in non-increasing order. Then for any $1 \leq r \leq n$, the function

$$f(x) = \sum_{i=1}^{r} x_{[i]},$$

i.e., the sum of the r largest components, is convex.

(c) The Euclidean norm $f(x_1, x_2, \ldots, x_n) = \|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$ on $\mathbf{dom}\, f = \mathbb{R}^n$ is convex.

(d) Let $C \subseteq \mathbb{R}^n$ be an arbitrary set. Show that the distance to the farthest point in C, that is,

$$f(x) = \sup_{y \in C} \|x - y\|$$

is convex.

Exercise 8.2 Prove that the point $x^* = (1, 1)$ is an optimal solution to the following convex optimization problem

$$\begin{aligned} \min \quad & 2e^{2-x_1} + e^{x_2} \\ & \frac{x_1^2}{x_2} - 1 \leq 0 \end{aligned}$$

Exercise 8.3 Determine the optimal solution to the following convex optimization problem

$$\begin{aligned} \min \quad & \tfrac{1}{2}(x_1 - 4)^2 + \tfrac{1}{2}(x_2 - 4)^2, \\ & x_1 + x_2 \leq 4, \\ & x_1, x_2 \geq 0 \end{aligned}$$

Exercise 8.4 (Equality constrained convex minimization)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex differentiable function, and let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that a feasible solution x^* for the convex optimization problem

$$\begin{aligned} \min \quad & f(x) \\ & Ax = b \end{aligned}$$

is optimal if and only if there exists $\lambda \in \mathbb{R}^m$ such that $\nabla f(x^*) = A^\top \lambda$.

[1] g is non-decreasing if $g(x) \leq g(y)$ for all $x, y \in \mathbf{dom}\, g$ such that $x \leq y$.

[2] $g \circ f : \mathbb{R}^n \to \mathbb{R}$ is the *composition* of g and f, where $(g \circ f)(x) = g(f(x))$