MA427 – Exercise set 10

Academic year 2018-19

Exercise 10.1 Assume that a differentiable function f is m-strongly convex for some $m > 0$, in particular,

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{m}{2}\|y - x\|^2$$

holds for any $x, y \in \mathbf{dom}\, f$.

(a) Show that the function $h(x) = f(x) - \frac{m}{2}\|x\|^2$ is also convex.

(b) Show that f is also *strictly convex*: for any $x, y \in \mathbf{dom}\, f$, $x \neq y$ and $0 < \lambda < 1$, we have

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda) f(y).$$

Hint: use part (a) to show part (b)

Exercise 10.2 Consider the function

$$f(x) = -\sum_{i=1}^{n} \log x_i$$

over the domain $\mathbf{dom}\, f = \{x \in \mathbb{R}^n \,:\, 0 < x_i \leq U,\, i = 1, 2, \ldots, n\}$ for some constant U. (Clearly, the minimum of f will be taken at $x_i = U$ for all $i = 1, 2, \ldots, n$.)

(a) Show that this function is not M-smooth for any finite $M > 0$ over the entire $\mathbf{dom}\, f$.

(b) Assume that we are given a point $s \in \mathbb{R}^n$, and consider the sublevel set

$$S = \mathbf{dom}\, f \cap \{x \in \mathbb{R}^n : f(x) \leq f(s)\}.$$

Show that f is M-smooth for some finite value of M on S.

Exercise 10.3 Let $\gamma > 1$. Consider the convex quadratic function

$$f(x) = \frac{1}{2}(x_1^2 + \gamma x_2^2)$$

on the domain $\mathbb{R}^2$. Clearly, the optimal solution is $x^* = 0$ with $p^* = f(x^*) = 0$.

(a) What is the condition number κ for this function?

(b) We use gradient descent with exact line search starting from the point $x^{(0)} = (\gamma, 1)$. Show that the kth iterate will be

$$x^{(k)} = \left(\gamma\left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k\right).$$

(c) Compute the value $f(x^{(k)})$. What is the ratio

$$\frac{f(x^{(k)}) - p^*}{f(x^{(0)}) - p^*}?$$

How does this relate to the bound we derived on the lecture, using the condition number κ?

Exercise 10.4 Consider the following optimisation problem

$$\begin{aligned} \min \quad & e^{x_1} + e^{x_2} + x_1 x_2 \\ & x_1 + x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Show that this is a convex optimisation problem, and that the objective function is strongly convex on the feasible region.

(b) What is the optimal solution to this problem?

(c) Apply the Frank-Wolfe method with exact line search, from the starting point $x^{(0)} = (2, 0)$. Perform two iterations of the algorithm.