

T is a spanning tree.

Cut Optimality Conditions: (C.O.C.)

MST

For every $(i,j) \in T$, we have $c_{ij} \leq c_{ke}$ for every edge (k,e) contained in the cut formed by deleting edge (i,j) from T .

~~Exercise~~

$\boxed{\text{MST} \Leftrightarrow \text{C.O.C.}}$

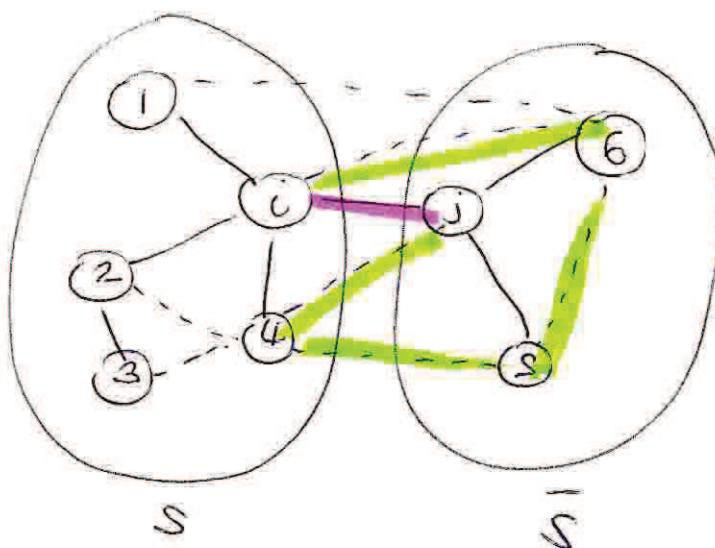
$\text{MST} \Rightarrow \text{C.O.C.}$: Suppose C.O.C. don't hold. Then $\exists (i,j) \in T$ such $c_{ij} > c_{ke}$ for some non-tree edge (k,e) contained in the cut of (i,j) . Replacing (k,e) with ~~(i,j)~~ (i,j) in T^* forms a spanning tree with lesser cost, contradicting MST of T^* . \square

$\text{C.O.C.} \Rightarrow \text{MST}$:

Let T^* satisfy C.O.C.

Let T^o be a MST, $T^o \neq T^*$ } $\begin{cases} T^*, T^o \text{ spanning trees.} \\ \Rightarrow \exists (i,j) \in T^* \text{ that is not in } T^o \end{cases}$

- Removing (i,j) from T^* , creates cut $[S, \bar{S}]$ in T^* .
- Adding (i,j) in T^o , creates cycle $C \subseteq T^o$.



T^* —

T^o - - -

Cycle C :

$(i,j), (i,6), (6,s), (s,4), (4,j)$

$C \cap [S, \bar{S}]$ contains: (i,j) and an odd # of edges of T^o .

- These odd # of edges cannot be in T^* :
 (because they belong to the cut $[s, \bar{s}]$, & T^* only has one edge (i,j) in the cut).
- Pick one of these odd # of edges and call it (k,e) .
- Remove (k,e) from T^0 and add (i,j) to \bar{T}^0 .
 [Adding (i,j) to T^0 creates a cycle, removing (k,e) which is part of the cycle restores the tree-ness of T^0 .]
 Thus, T^0 is acyclic, connected \Rightarrow spanning tree.
- $c_{ij} \leq c_{ke}$ \Rightarrow { Cost of T^0 not increased $\Rightarrow T^0$ MST.
 { Has one more edge in common with T^* .
- Continue: $T^0 \rightarrow \rightarrow \rightarrow T^*$.

MST \Leftrightarrow P.O.C See notes.