Max-flow Min-cut theorem:

MAX FLOW

① Weak max-flow min-cut theorem:

Theorem: The value V of any feasible flow x is less than or equal to the capacity of any s-t cut $[S, \bar{S}]$ in the network G:

$$V \leq c(S, \bar{S}) \quad \text{for } \forall (V, x), \ \forall \underset{\text{s-t cut}}{[S, \bar{S}]}.$$

$\Rightarrow$ $\boxed{\text{max flow } V \leq \text{capacity of the minimum capacity s-t cut.}}$

Lemma: For any feasible flow x of value V,

$$V = \text{net flow of s-t cut } [S, \bar{S}]$$

$$= \sum_{(i,j) \in (S, \bar{S})} x_{ij} - \sum_{(i,j) \in (\bar{S}, S)} x_{ij}$$

for any s-t cut $[S, \bar{S}]$.

Remember: $(S, \bar{S})$: forward arcs of cut.
$(\bar{S}, S)$: backward arcs of cut.

Proof of Lemma: Consider the flow conservation constraints for the max flow problem:

$$\sum_{j:(i,j) \in A} x_{ij} - \sum_{k:(k,i) \in A} x_{ki} = \begin{cases} V & i = s \\ 0 & i \in N \setminus \{s, t\} \\ -V & i = t \end{cases}$$

For $\forall$ s-t cut $[S, \bar{S}]$, sum the above for all $i \in S$:

$\rightarrow$

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$$\sum_{i \in S} \left[\sum_{j:(i,j)\in A} x_{ij} - \sum_{k:(k,i)\in A} x_{ki} \right] = V \quad (*)$$

x_{pq}: • when both $p,q \in S$, variable x_{pq} will appear in the term of p as $+x_{pq}$, since it is outflow, and in the term of q as $-x_{pq}$, since it is inflow. These will cancel.

- when both $p,q \in \bar{S}$ it will not appear.
- when $p \in S$, $q \in \bar{S}$ (i.e. $(p,q) \in (S,\bar{S})$ forward arc) it will appear as $+x_{pq}$ once.
- when $p \in \bar{S}$, $q \in S$ (i.e. $(p,q) \in (\bar{S},S)$ backward arc) it will appear as $-x_{pq}$ once.

Therefore $(*)$ becomes:

$$\underbrace{\sum_{(i,j)\in(S,\bar{S})} x_{ij}}_{\text{forward arcs}} - \underbrace{\sum_{(i,j)\in(\bar{S},S)} x_{ij}}_{\text{backward arcs}} = V$$

$\therefore$ V = net flow through s-t cut $[S,\bar{S}]$. □

Proof of theorem: From lemma:

$$V = \sum_{(i,j)\in(S,\bar{S})} x_{ij} - \sum_{(i,j)\in(\bar{S},S)} x_{ij} \le \sum_{(i,j)\in(S,\bar{S})} u_{ij} - \sum_{(i,j)\in(\bar{S},S)} 0 = C(S,\bar{S}).$$

□