MA423 – Fundamentals of Operations Research
Lecture 1

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- Office hours Michaelmas Term : Tuesday 15:30–17:30

A few details

- Lectures: 2 hours with exercises given
- Exercises: should be attempted before next week's seminars
- Seminars: You should be assigned a seminar by Rebecca Batey `r.batey@lse.ac.uk`. If you want to change seminar group please email Rebecca.
- Formative work: 3 exercise sets will be collected and feedback will be given. This is not compulsory but highly recommended.
- Exam: 3 hour examination in ST. I will provide a mock exam that will be solved at the revision session week 1 of ST.

Content of the course

MA423 Part 1: Linear programming and integer programming

- Optimisation problems/LP formulations/Standard forms.
- Duality.
- The simplex method.
- Integer programming: branch and bound, formulations.

MA423 Part 2: Markov/Queueing

- Markov Chains.
- Queueing theory.

MA423 Part 3: Other OR methods.

- Inventory Models.
- Dynamic Programming.
- Game Theory.

Lecture 1

- Introduction
  - Optimization problems
  - Examples
- Linear programming
  - Terminology
  - Possible outcomes: fundamental theorem
  - LP in standard forms
  - Proving optimality: dual values

Mathematical Programming $\equiv$ Mathematical Optimization

objective function

$$
\begin{aligned}
\max \ (\text{or } \min) \quad & f_0(x) \\
\text{s.t.} \quad & f_1(x) \leq b_1 \\
& \vdots \\
& f_m(x) \leq b_m \\
& x \in \mathcal{D}
\end{aligned}
$$

constraints

- x: n-dimensional vector of decision variables
- $\mathcal{D} \subseteq \mathbb{R}^n$: Domain of the problem
- $f_i : \mathcal{D} \to \mathbb{R}$.

Example

Factory produces Orange Juice (OJ) and Orange Concentrate (OC).

	OJ	OC
Profit (£/liter)	3	2
Electricity (unit/Liter)	1	1
Oranges (unit/liter)	1	2
Water (unit/liter)	1	-1

	Available
Electricity	6
Oranges	10
Water	4

Example: nurse scheduling

Hospital must choose how many nurses to staff.

- On day i of the week ($i = 1, \ldots, 7$), the hospital needs d_i nurses.
- Every nurse rests two consecutive days every week.

What is the minimum number of nurses needed?

Example: Markowitz Portfolio Optimization

Optimally allocate budget B to n assets $i = 1, \dots, n$.

- w_i: proportion of budget B allocated to stock i: $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$, $i = 1, \dots, n$.

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- Minimize risk while guaranteeing an expected return $\geq r_{\min}$.

$$
\begin{gathered}
\min \ \mathrm{Var}(r) \\
\text{s.t.} \\
\bar{r} \geq r_{\min} \\
\sum_{i=1}^{n} w_i = 1 \\
w \geq 0
\end{gathered}
$$

Example: Markowitz Portfolio Optimization

- p: random vector of returns.
- $\bar{p}$: vector of expected returns, $\bar{p}_i = \mathbb{E}[p_i]$.
- Σ: covariance matrix of p $(\Sigma_{ij} = \mathbb{E}[(p_i - \bar{p}_i)(p_j - \bar{p}_j)])$

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It follows ...

- $r = p^\top w$;
- $\bar{r} = \bar{p}^\top w$;
- $\mathrm{Var}(r) = w^\top \Sigma w$.

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\begin{gathered}
\min \;\; w^\top \Sigma w \\
\text{s.t.} \\
\bar{p}^\top w \geq r_{\min} \\
\sum_{i=1}^n w_i = 1 \\
w \geq 0
\end{gathered}
$$

Example: cutting stock problem

A paper mill produces large rolls of paper of width W which are then cut into smaller rolls to meet orders.

- Roll widths: w_i for $i = 1, \ldots, m$
- Demand for width i: b_i, $i = 1, \ldots, m$

What is the minimum number of big rolls needed to meet demand?

Linear Programming problems

linear objective function

$$\max \boxed{c^\top x}$$

$$\boxed{\begin{array}{lcll} a_i^\top x & = & b_i & i = 1, \ldots, k \\ a_i^\top x & \leq & b_i & i = k+1, \ldots, r \\ a_i^\top x & \geq & b_i & i = r+1, \ldots, m \end{array}}$$

linear constraints

If some of the variables are required to be integer, then the problem is a Mixed-integer Linear Programming problem.

Terminology

$$\begin{array}{rl} \max\ (\text{or } \min) & \boxed{f_0(x)} \text{ objective function} \\ \text{s.t.} & \boxed{\begin{array}{c} f_1(x) \le b_1 \\ \vdots \\ f_m(x) \le b_m \end{array}} \text{ constraints} \\ & x \in \mathcal{D} \end{array}$$

- Feasible solution: any point $\bar{x}$ satisfying the constraints.
- Feasible region: the set of all feasible solutions.

Possible outcomes

- Optimal solution (maximization): A feasible solution x^* such that, for every feasible solution x,

$$f_0(x^*) \geq f_0(x).$$

  If an optimal solution exists, we say that the problem has a finite optimum.

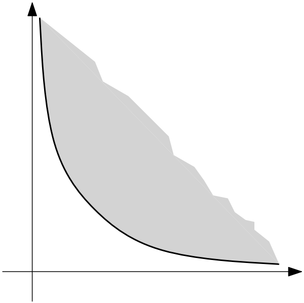
- Infeasible problem: A problem that has no feasible solution.
- Unbounded problem: A problem such that, for every number α, there exists a feasible solution x such that

$$f_0(x) > \alpha.$$

Possible outcomes

There are optimization problems for which none of these three outcomes occurs.

$$\begin{aligned} \min x_1 \\ x_1 x_2 \geq 1 \\ x_1, x_2 \geq 0. \end{aligned}$$



There are feasible points with objective value arbitrarily close to 0, but no point of value zero.

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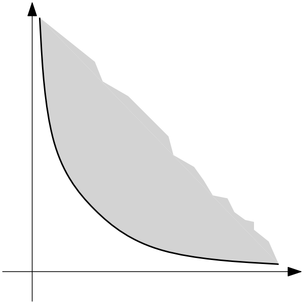


There are feasible points with objective value arbitrarily close to 0, but no point of value zero.

For LP problems, one of the three outcomes always occurs.

Fundamental theorem

Theorem (Fundamental Theorem of Linear Programming)

For any linear programming problem, exactly one of the following holds.

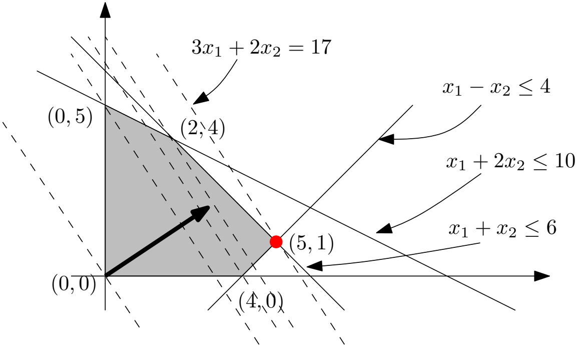
1. *The problem has a finite optimum;*
2. *The Problem is infeasible;*
3. *The problem is unbounded.*

Example: a problem with an optimum

The model from the OJ production problem:

$$
\begin{array}{lrcrcr}
\max & 3x_1 & + & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 4 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$

Example: a problem with an optimum



The optimum is $(5, 1)$ with value 17. The optimum is a vertex of the feasible region.

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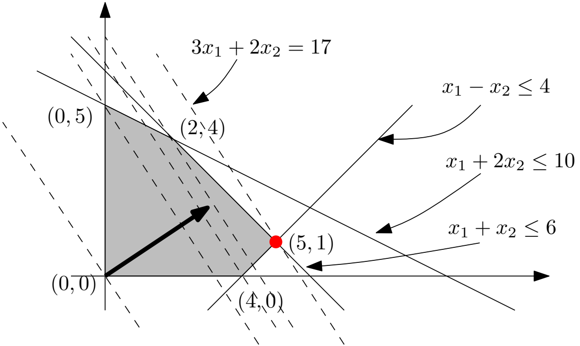


The optimum is $(5,1)$ with value 17. The optimum is a vertex of the feasible region.

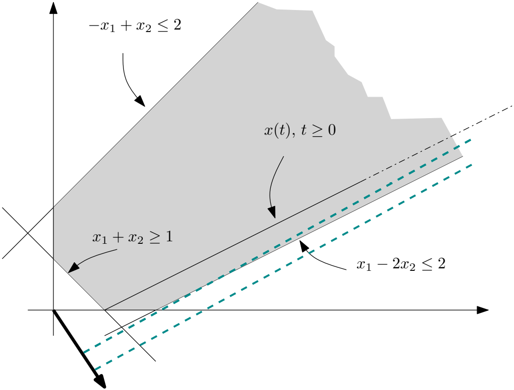
- When the LP has an optimum, it has one which is a vertex of the feasible region.
- How to prove that a point is an optimum? (LP duality).

Example: an infeasible problem

$$\begin{array}{lrcrcrcr}
\max & x_1 & + & x_2 & & & & \\
s.t. & x_1 & - & 2x_2 & + & 2x_3 & \leq & 2 \\
 & -2x_1 & + & 6x_2 & - & 2x_3 & \leq & -6 \\
 & & & x_1, x_2, x_3 & & & \geq & 0
\end{array}$$

Example: an unbounded problem

$$
\begin{array}{rrcrcl}
\max & 2x_1 & - & 3x_2 & & \\
s.t. & x_1 & + & x_2 & \geq & 1 \\
 & -x_1 & + & x_2 & \leq & 2 \\
 & x_1 & - & 2x_2 & \leq & 2 \\
 & & & x_1, x_2 & \geq & 0
\end{array}
$$



Standard forms

An LP problem is in standard form if it is of the form

$$\begin{aligned} z^* = \quad \max c^\top x & \\ Ax \leq & \quad b \\ x \geq & \quad 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

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An LP problem is in standard equality form if it is of the form

$$\begin{aligned} z^* = \quad & \max c^\top x \\ & Ax = \quad b \\ & x \geq \quad 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x is a vector of indeterminates in $\mathbb{R}^n$.

Converting to Standard forms

An LP in general form:

$$
\begin{array}{llcll}
\max(\text{resp. } \min) & c^\top x & & & \\
& a_i^\top x & = & b_i & i = 1, \ldots, k \\
& a_i^\top x & \leq & b_i & i = k+1, \ldots, r \\
& a_i^\top x & \geq & b_i & i = r+1, \ldots, m
\end{array}
\tag{1}
$$

where $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $c, a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and x is a vector of variables in $\mathbb{R}^n$.

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where $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $c, a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and x is a vector of variables in $\mathbb{R}^n$.

How do we convert such an LP in standard form?

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- **Objective**: $\min c^T x$ becomes $\max -c^T x$

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- **Variables**: $x_i \geq 0$ are called nonnegative variables, $x_i \leq 0$ are called nonpositive variables otherwise x_i is called a free variable:

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- **Variables**: $x_i \geq 0$ are called nonnegative variables, $x_i \leq 0$ are called nonpositive variables otherwise x_i is called a free variable:
  - Nonpositive variables: We introduce a new nonnegative variable, $x_i' \geq 0$ and we set $x_i' = -x_i$.

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- **Variables**: $x_i \geq 0$ are called nonnegative variables, $x_i \leq 0$ are called nonpositive variables otherwise x_i is called a free variable:
  - Nonpositive variables: We introduce a new nonnegative variable, $x_i' \geq 0$ and we set $x_i' = -x_i$.
  - Free variables: We introduce two nonnegative variables, x_i^+ and x_i^-, and we set

$$x_i = x_i^+ - x_i^-, \quad x_i^+, x_i^- \geq 0$$

Converting to Standard equality form

- Convert to standard form
- For each "$\leq$" constraint $a_i^T x \leq b_i$, introduce nonnegative variable $s_i \geq 0$ and replace previous constraint with:

$$a_i^T x + s_i = b_i$$

Proving optimality

$$\begin{array}{rrcrcr}
\max & 3x_1 & + & 2x_2 & & \\
s.t. & x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 4 \\
 & & & x_1, x_2 & \geq & 0
\end{array}$$

Optimal solution $(5, 1)$, with value 17.

Proving optimality

Let us combine the inequalities

$$
\begin{array}{rrcrcrc}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \color{red}{\frac{5}{2}} \\
 & x_1 & + & 2x_2 & \leq & 10 & \color{red}{0} \\
 & x_1 & - & x_2 & \leq & 4 & \color{red}{\frac{1}{2}} \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Resulting inequality:

$$\color{blue}{3x_1 + 2x_2 \leq 17}.$$

Thus no feasible solution has value greater than 17.

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$
\begin{array}{rrcrcrl}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\
 & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\
 & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

Proving optimality: LP duality

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To obtain upper-bounds, we need

$$
\begin{array}{rcl}
y_1 + y_2 + y_3 & \geq & 3 \\
y_1 + 2y_2 - y_3 & \geq & 2
\end{array}
$$

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s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\
 & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\
 & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$
\begin{array}{rcl}
y_1 + y_2 + y_3 & \geq & 3 \\
y_1 + 2y_2 - y_3 & \geq & 2
\end{array}
$$

To obtain tightest upper-bound:

$$\min 6y_1 + 10y_2 + 4y_3$$

Dual problem

The dual of the original problem is

$$\begin{array}{lcccccccc} \min & 6y_1 & + & 10y_2 & + & 4y_3 & & \\ s.t. & y_1 & + & y_2 & + & y_3 & \geq & 3 \\ & y_1 & + & 2y_2 & - & y_3 & \geq & 2 \\ \end{array}$$
$$y_1, y_2, y_3 \geq 0$$

The solution to the above gives the tightest possible upper-bound on the optimal value that we can infer by taking linear combinations of the constraints.