**There is no $\alpha$-approx. alg for $\alpha > 1$ for the T.S.P. unless P = NP.**



## Hamiltonian Cycle Problem (HC)

- Given $G = (V, E)$ $\longrightarrow$ $G$ not complete
- Decide whether it has a H.C.

$$\boxed{HC \leq D.V.T.S.P.}$$

$$X_{HC} = G \xrightarrow{\boxed{f}} X_{TSP} = G^c$$

The H.C. problem is on a graph $G$ that might not be complete and also $G$ has no cost on the edges.

We complete the graph $G$ by adding all missing edges and call it $G^c$. We assign the following cost on the edges:

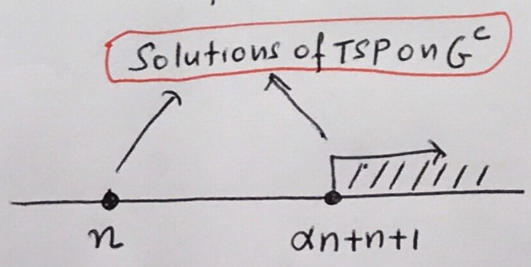
$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) \in G \quad \text{(old edge)} \\ \alpha n + 2 & \text{otherwise} \quad \text{(new edge)} \end{cases}$$

**Claim:** $\boxed{\text{Hamiltonian Cycle in } G} \Longleftrightarrow \boxed{\text{Tour in } G^c \text{ of length } n}$

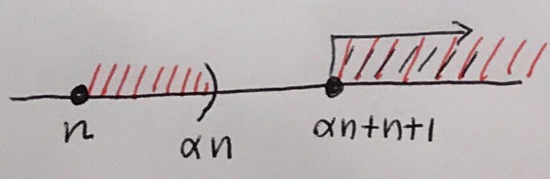
If there is an H.C in G... ... then this H.C is a tour in $G^c$ of length $n$.

For the T.S.P on $G^c$ there are two options:

- it has a tour of size $n$
- tour of size $\geq \underbrace{(\alpha n + 2)}_{\text{1 expensive edge}} + \underbrace{n - 1}_{\text{n-1 cheap edges (cost 1 each)}} = \alpha n + n + 1$




An $\alpha$-approx. alg for the TSP on $G^c$ would give a solution in $[n, \alpha n]$ if the optimal tour on $G^c$ is of size $n$, and would give a solution $\geq \alpha n + n + 1$ if the optimal tour on $G^c$ is of size $\geq \alpha n + n + 1$.




$\therefore$ $\boxed{\exists \text{ H.C. in } G} \Longleftrightarrow \boxed{\text{Opt. tour in } G^c \text{ is of length } n} \Longleftrightarrow \boxed{\alpha\text{-approx. finds a tour} \leq \alpha n}$

Thus an $\alpha$-approx. would solve the H.C problem, which is NP-complete.

Thus, there exists an $\alpha$-approx. with $\alpha > 1$ only if $P = NP$ //