

$T$  is a spanning tree:  
Cut Optimality Conditions: (C.O.C.)

MST

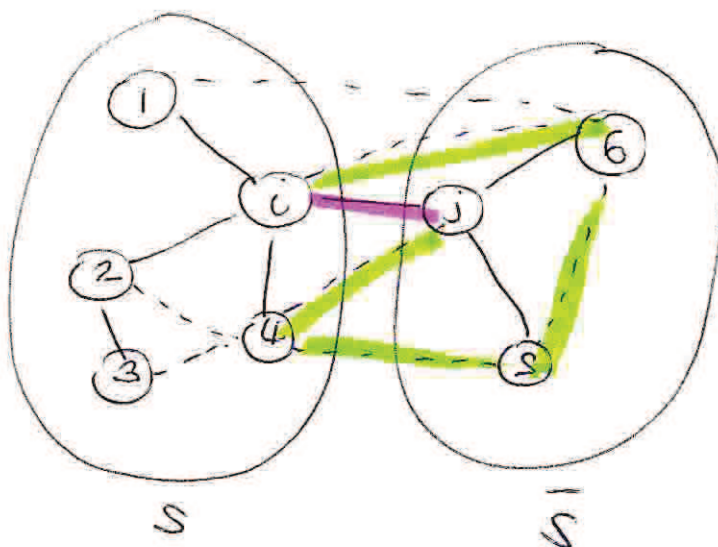
For every  $(i,j) \in T$ , we have  $c_{ij} \leq c_{kl}$  for every edge  $(k,l)$  contained in the cut formed by deleting edge  $(i,j)$  from  $T$ .

~~MST~~ |  $MST \iff C.O.C.$  |

$MST \Rightarrow C.O.C.$ : Suppose C.O.C. do not hold. Then  $\exists (i,j) \in T$  such  $c_{ij} > c_{kl}$  for some non-tree edge  $(k,l)$  contained in the cut of  $(i,j)$ . Replacing  $(k,l)$  with  ~~$(i,j)$~~  in  $T^*$  forms a spanning tree with lesser cost, contradicting MST of  $T^*$ .  $\square$

$C.O.C \Rightarrow MST$ : Let  $T^*$  satisfy C.O.C.  
 Let  $T^0$  be a MST,  $T^0 \neq T^*$  }  $T^*, T^0$  spanning trees  
 $\Rightarrow \exists (i,j) \in T^*$  that is not in  $T^0$ .

- Removing  $(i,j)$  from  $T^*$ , creates cut  $[S, \bar{S}]$  in  $T^*$
- Adding  $(i,j)$  in  $T^0$ , creates cycle  $C \subseteq T^0$



$T^*$  —

$T^0$  ---

Cycle C:

$(i,j), (i,6), (6,s), (s,4), (4,j)$

$C \cap [S, \bar{S}]$  contains:  $(i,j)$  and an odd # of edges of  $T^0$ .

- These odd # of edges cannot be in  $T^*$ .  
(because they ~~are~~ belong to the cut  $[s, \bar{S}]$ , &  $T^*$  only has one edge  $(i, j)$  in the cut).
- Pick one of these odd # of edges and call it  $(u, e)$ .
- Remove  $(u, e)$  from  $T^0$  and add  $(i, j)$  to  $T^0$ .  
 [Adding  $(i, j)$  to  $T^0$  creates a cycle, removing  $(u, e)$  which is part of the cycle restores the tree-ness of  $T^0$ .]  
 Thus,  $T^0$  is acyclic, connected  $\Rightarrow$  spanning tree.
- $C_{ij} \leq C_{ue} \Rightarrow$ 
 { Cost of  $T^0$  not increased  $\Rightarrow T^0$  MST.  
 Has one more edge in common with  $T^*$
- Continue:  $T^0_{(MST)} \rightarrow \rightarrow \rightarrow T^*$ .

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MST  $\Leftrightarrow$  P.O.C See notes.