MA423 – Fundamentals of Operations Research
Lecture 2: LP Duality

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Last week's example

$$\begin{array}{lrcrcr} \max & 3x_1 & + & 2x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 10 \\ & x_1 & - & x_2 & \leq & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Optimal solution $(5, 1)$, with value 17.

Last week's example

$$
\begin{array}{rrcrcrl}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \quad \color{red}{\tfrac{5}{2}} \\
 & x_1 & + & 2x_2 & \leq & 10 & \quad \color{red}{0} \\
 & x_1 & - & x_2 & \leq & 4 & \quad \color{red}{\tfrac{1}{2}} \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

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Combining the inequalities:

$$3x_1 + 2x_2 \leq 17.$$

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\end{array}$$

Optimal solution $(5, 1)$, with value 17.
Combining the inequalities:

$$3x_1 + 2x_2 \leq 17.$$

No feasible solution has value greater than 17 $\implies$ $(5, 1)$ is optimal.

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$\begin{array}{lrcrcrl}
\max & 3x_1 & + & 2x_2 & & & \\
s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\
 & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\
 & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}$$

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 & x_1 & - & x_2 & \leq & 4 & \qquad y_3 \\
 & & & x_1, x_2 & \geq & 0 &
\end{array}
$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

Proving optimality: LP duality

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\end{array}$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$\begin{array}{rcl}
y_1 + y_2 + y_3 & \geq & 3 \\
y_1 + 2y_2 - y_3 & \geq & 2
\end{array}$$

Proving optimality: LP duality

Nonnegative multipliers y_1, y_2, y_3 for three constraints

$$\begin{array}{rrcrcrl} \max & 3x_1 & + & 2x_2 & & & \\ s.t. & x_1 & + & x_2 & \leq & 6 & \quad y_1 \\ & x_1 & + & 2x_2 & \leq & 10 & \quad y_2 \\ & x_1 & - & x_2 & \leq & 4 & \quad y_3 \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Resulting inequality:

$$(y_1 + y_2 + y_3)x_1 + (y_1 + 2y_2 - y_3)x_2 \leq 6y_1 + 10y_2 + 4y_3.$$

To obtain upper-bounds, we need

$$\begin{array}{rcl} y_1 + y_2 + y_3 & \geq & 3 \\ y_1 + 2y_2 - y_3 & \geq & 2 \end{array}$$

To obtain tightest upper-bound:

$$\min 6y_1 + 10y_2 + 4y_3$$

Dual problem

The dual of the original problem is

$$\begin{array}{lccccccc}
\min & 6y_1 & + & 10y_2 & + & 4y_3 & & \\
s.t. & y_1 & + & y_2 & + & y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & - & y_3 & \geq & 2 \\
 & & & & y_1, y_2, y_3 \geq 0 & & &
\end{array}$$

The solution to the above gives the tightest possible upper-bound on the optimal value that we can infer by taking linear combinations of the constraints.

Weak Duality Theorem

$$\begin{aligned} \max\ & c^\top x \\ Ax \le\ & b \qquad (P) \\ x \ge\ & 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^\top y \\ A^\top y \ge\ & c. \qquad (D) \\ y \ge\ & 0 \end{aligned}$$

Theorem (Weak duality theorem)

Given any feasible solution x^* for the primal (P), and any feasible solution y^* for the dual (D), then

$$c^\top x^* \le b^\top y^*.$$

A few consequences

A dual solution provides a certificate of optimality:

Corollary

Let x^* be a feasible sol. for the primal and y^* be a feasible solution for the dual. If $c^\top x^* = b^\top y^*$, then x^* is optimal for the primal, and y^* is optimal for the dual.

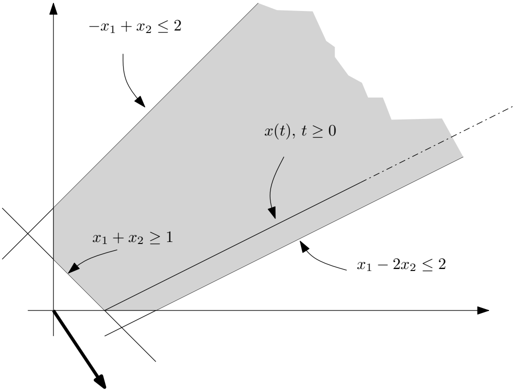
Corollary

(i) *Primal unbounded $\Longrightarrow$ dual infeasible.*

(ii) *Dual unbounded $\Longrightarrow$ primal infeasible.*

Example: unbounded primal

$$
\begin{array}{rrcr}
\max & 2x_1 - 3x_2 & & \\
s.t. & -x_1 - \ x_2 & \leq & -1 \\
 & -x_1 + \ x_2 & \leq & 2 \\
 & x_1 - 2x_2 & \leq & 2 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$



$$
\begin{array}{rrcr}
\min & -y_1 + 2y_2 + 2y_3 & & \\
s.t. & -y_1 - \ y_2 + \ y_3 & \geq & 2 \\
 & -y_1 + \ y_2 - 2y_3 & \geq & -3 \\
 & y_1, y_2, y_3 \geq 0 & &
\end{array}
$$

Strong Duality Theorem

$$\begin{aligned} \max\ & c^\top x \\ Ax \le\ & b \qquad (P) \\ x \ge\ & 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^\top y \\ A^\top y \ge\ & c. \qquad (D) \\ y \ge\ & 0 \end{aligned}$$

Theorem (Strong duality theorem)

If (P) has an optimum, then (D) has an optimum. Given an optimum x^* for (P) and an optimum y^* for (D), then

$$c^\top x^* = b^\top y^*.$$

Strong Duality Theorem (revisited)

$$\begin{aligned} \max\ & c^\top x \\ Ax \leq\ & b \qquad (P) \\ x \geq\ & 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^\top y \\ A^\top y \geq\ & c. \qquad (D) \\ y \geq\ & 0 \end{aligned}$$

What is the dual of the dual?

Strong Duality Theorem (revisited)

$$\begin{array}{rlr} \max\ c^\top x & & \\ Ax \leq & b & (P) \\ x \geq & 0 & \end{array} \qquad\qquad \begin{array}{rlr} \min\ b^\top y & & \\ A^\top y \geq & c. & (D) \\ y \geq 0 & & \end{array}$$

What is the dual of the dual?

Theorem (Strong duality theorem)

If one among (P) and (D) has an optimum, then they both have an optimum. Given an optimum x^* for (P) and an optimum y^* for (D), then

$$c^\top x^* = b^\top y^*.$$

Possible outcomes

		Primal		
		Fin. opt.	Infeasible	Unbounded
Dual	Fin. opt.	*Possible*	*NO*	*NO*
	Infeasible	*NO*	*?*	*Possible*
	Unbounded	*NO*	*Possible*	*NO*

An example with both primal and dual infeasible

$$
\begin{array}{llcr}
\max & 2x_1 - x_2 & & \\
 & x_1 - x_2 & \leq & 1 \\
 & -x_1 + x_2 & \leq & -2 \\
 & x_1, x_2 \geq 0 & &
\end{array}
\quad , \quad
\begin{array}{llcr}
\min & y_1 - 2y_2 & & \\
 & y_1 - \ y_2 & \geq & 2 \\
 & -y_1 + \ y_2 & \geq & -1 \\
 & y_1, y_2 \geq 0 & &
\end{array}
$$

Duality: general form

$$\text{A variable } x_j \text{ is } \begin{cases} \text{nonnegative} & \text{if there is constraint } x_j \geq 0 \\ \text{nonpositive} & \text{if there is constraint } x_j \leq 0 \\ \text{free} & \text{otherwise} \end{cases}$$

Let us call the remaining constraints "resource constraints".

- If primal is a maximisation problem, dual is a minimisation
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- Dual has one variable for every primal resource constraint. The coefficient of the ith dual variable is the right-hand-side of the ith primal constraint.
- Dual has one resource constraint for each primal variable x_j. The right-hand-side of the jth dual constraint is the coefficient of the jth primal variable.

Duality: general form

max	min
$\leq$ constraint	nonnegative variable
$\geq$ constraint	nonpositive variable
$=$ constraint	free variable
nonnegative variable	$\geq$ constraint
nonpositive variable	$\leq$ constraint
free variable	$=$ constraint

Duality: general form

max	min
$\leq$ constraint	nonnegative variable
$\geq$ constraint	nonpositive variable
$=$ constraint	free variable
nonnegative variable	$\geq$ constraint
nonpositive variable	$\leq$ constraint
free variable	$=$ constraint

$$
\begin{array}{lrcl}
\min & 3x_1 - 2x_2 - x_3 & & \\
 & -x_1 + x_2 + 2x_3 & = & 4 \\
 & 2x_1 \phantom{{}+x_2} - x_3 & \geq & -2 \\
 & -2x_1 - x_2 + x_3 & \leq & 1 \\
 & x_1 \leq 0,\ x_3 \geq 0 & &
\end{array}
$$

Duality: general form

max	min
$\leq$ constraint	nonnegative variable
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 & -2x_1 - x_2 + x_3 & \leq & 1 \\
 & x_1 \leq 0,\ x_3 \geq 0 & &
\end{array}
\qquad
\begin{array}{lrcr}
\max & 4y_1 - 2y_2 + y_3 & & \\
 & -y_1 + 2y_2 - 2y_3 & \geq & 3 \\
 & y_1 \qquad - y_3 & = & -2 \\
 & 2y_1 - y_2 + y_3 & \leq & -1 \\
 & y_2 \geq 0,\ y_3 \leq 0 & &
\end{array}
$$

An economics interpretation of the dual

It is often the case that also the variables, constraints, and objective function of the dual can be interpreted and give further information on the original problem. A typical case when such an interpretation of the dual is possible is in the case of resource allocation problems.

Economics interpretation of the dual - example

- A chip's manufacturer produces four types of memory chips in one of their factories.
- The main resources used in the chips' production are labor and silicon wafers.

The factory's problem for the next month is

$$\begin{array}{llll} \text{maximise} & 15x_1 + 24x_2 + 32x_3 + 40x_4 & & & \\ \textit{subject to} & x_1 + 2x_2 + 8x_3 + 7x_4 & \leq & 2000 & \textit{(Labour)} \\ & 6x_1 + 8x_2 + 12x_3 + 15x_4 & \leq & 15000 & \textit{(Silicon wafers)} \\ & x_1, x_2, x_3, x_4 \geq 0. & & & \end{array}$$

- x_j: number of chips produced (thousands),
- Objective function coefficients £/unit of the four chips,
- 2000 hours of labour available, 15000 silicon wafers available.

Economic interpretation of duality

- Mango Inc., a giant consumer electronics corporation, urgently needs as many units as possible of a new type of memory chip, due to a stronger-than-expected demand for their new smart phone.

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- Mango Inc. would like the chip manufacturer to devote all resources of the factory to the production the new type of chip. Mango Inc. intends to determine prices to offer for each of the resources in order to convince the manufacturer to sell them, while minimising the total sum paid.

Economic interpretation of duality

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- Mango Inc. would like the chip manufacturer to devote all resources of the factory to the production the new type of chip. Mango Inc. intends to determine prices to offer for each of the resources in order to convince the manufacturer to sell them, while minimising the total sum paid.
- Can we formulate this as a linear program?

Decision variables

- y_1: price that the buyer intends to pay per hour of labour (thousand £),
- y_2: price that Mango Inc. intends to pay for each silicon wafer.

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Clearly these prices must be greater than or equal to zero, that is, $y_1, y_2 \geq 0$.

Determining the prices

- In order to persuade the manufacturer to sell, the buyer must offer prices such that the manufacturer is not tempted to retain its resources to produce chips of type 1.

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- So the buyer must offer prices such that the total value of the resources used in producing chips of type 1 is at least the amount of profit the manufacturer could attain.

Determining the prices

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- So the buyer must offer prices such that the total value of the resources used in producing chips of type 1 is at least the amount of profit the manufacturer could attain.
- Producing 1000 chips of type one requires 1 hour of labour and 6 silicon wafers, with a profit of £15,000. Thus the prices need to satisfy
$$y_1 + 6y_2 \geq 15.$$

Determining the prices

- Similarly, for chips 2, 3, and 4 Mango Inc. must pitch his prices so that the manufacturer is at least as well off selling as he would be by not selling:

$$
\begin{aligned}
2y_1 + 8y_2 &\geq 24 \\
8y_1 + 12y_2 &\geq 32 \\
7y_1 + 15y_2 &\geq 40
\end{aligned}
$$

Determining the prices

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$$\begin{aligned} 2y_1 + 8y_2 &\geq 24 \\ 8y_1 + 12y_2 &\geq 32 \\ 7y_1 + 15y_2 &\geq 40 \end{aligned}$$

- On the other hand, the buyer wants to pay the minimum amount possible for the entire amount of resources, that is, it wants to minimise $2000y_1 + 15000y_2$.

The pricing problem

$$
\begin{array}{lrcl}
\textit{minimise} & 2000y_1 + 15000y_2 & & \\
\textit{subject to} & y_1 + 6y_2 & \geq & 15 \\
 & 2y_1 + 8y_2 & \geq & 24 \\
 & 8y_1 + 12y_2 & \geq & 32 \\
 & 7y_1 + 15y_2 & \geq & 40 \\
 & y_1,\ y_2 & \geq & 0
\end{array}
$$

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- The manufacturer's and the Mango Inc.'s problems are dual to each other.

The pricing problem

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- The dual values represent prices of resources (dual values are often also called "shadow prices").

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\end{array}
$$

- The manufacturer's and the Mango Inc.'s problems are dual to each other.
- The dual values represent prices of resources (dual values are often also called "shadow prices").
- The total amount that the buyer will have to pay is the same as the amount of profit that the manufacturer would achieve by carrying on its usual production (strong duality).

Complementary slackness

Consider an LP problem and its dual.

$$
\begin{array}{rcll}
\max \sum_{j=1}^{n} c_j x_j & & & \\
\sum_{j=1}^{n} a_{ij} x_j \leq & b_i & i = 1, ..., m \\
x_j \geq & 0 & j = 1, ..., n
\end{array}
\qquad
\begin{array}{rcll}
\min \sum_{i=1}^{m} b_i y_i & & & \\
\sum_{i=1}^{m} a_{ij} y_i \geq & c_j & j = 1, ..., n \\
y_i \geq & 0 & i = 1, ..., m
\end{array}
$$

Complementary slackness

Consider an LP problem and its dual.

$$\begin{array}{rcll} \max \sum_{j=1}^{n} c_j x_j & & & \\ \sum_{j=1}^{n} a_{ij} x_j \leq & b_i & i = 1, \ldots, m \\ x_j \geq & 0 & j = 1, \ldots, n \end{array} \qquad \begin{array}{rcll} \min \sum_{i=1}^{m} b_i y_i & & & \\ \sum_{i=1}^{m} a_{ij} y_i \geq & c_j & j = 1, \ldots, n \\ y_i \geq & 0 & i = 1, \ldots, m \end{array}$$

Theorem (Complementary slackness theorem)

Given feasible solutions x^* and y^*, they are optimal if and only if

$$\begin{array}{lll} \forall\, j \in \{1, \ldots, n\}, & x_j^* = 0 \quad \textbf{or} & \sum_{i=1}^{m} a_{ij} y_i^* - c_j = 0, \\ \forall\, i \in \{1, \ldots, m\}, & y_i^* = 0 \quad \textbf{or} & \sum_{j=1}^{n} a_{ij} x_i^* - b_i = 0. \end{array} \qquad (SC)$$

Complementary slackness

$$
\begin{array}{lrcr}
\max & 3x_1 + 2x_2 & & \\
s.t. & x_1 + x_2 & \leq & 6 \\
& x_1 + 2x_2 & \leq & 10 \\
& x_1 - x_2 & \leq & 4 \\
& x_1, x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcr}
\min & 6y_1 + 10y_2 + 4y_3 & & \\
& y_1 + y_2 + y_3 & \geq & 3 \\
& y_1 + 2y_2 - y_3 & \geq & 2 \\
& y_1, y_2, y_3 & \geq & 0
\end{array}
$$

Optimal primal solution $x^* = (5, 1)$.
Optimal dual solution $y^* = (\frac{5}{2}, 0, \frac{1}{2})$.

Complementary slackness: general form

Theorem (Complementary slackness theorem)

Given a LP problem and feasible primal/dual solutions x^* and y^*, they are optimal if and only if

- *For every primal variable x_j, either $x_j^* = 0$ or the corresponding dual constraint is binding at y^*.*
- *For every dual variable y_i, either $y_i^* = 0$ or the corresponding primal constraint is binding at x^*.*

Complementary slackness: example

$$\begin{array}{lrcr} \min & 3x_1 - 2x_2 - x_3 & & \\ & -x_1 + x_2 + 2x_3 & = & 4 \\ & 2x_1 - x_3 & \geq & -2 \\ & -2x_1 - x_2 + x_3 & \leq & 1 \\ & x_1 \leq 0,\ x_3 \geq 0 & & \end{array}$$

$$\begin{array}{lrcr} \max & 4y_1 - 2y_2 + y_3 & & \\ & -y_1 + 2y_2 - 2y_3 & \geq & 3 \\ & y_1 - y_3 & = & -2 \\ & 2y_1 - y_2 + y_3 & \leq & -1 \\ & y_2 \geq 0,\ y_3 \leq 0 & & \end{array}$$

Is any of these points optimal?

$x^* = (0, \frac{2}{3}, \frac{5}{3})$

$x^* = (0, 4, 0)$

$x^* = (-1, 3, 0)$.