There is no α-approximation alg. for the k-center problem for $\alpha < 2$ unless P = NP.

Dominating Set Problem (DS)

- Given $G = (V, E)$, integer k
- Decide if there exists a set $S \subseteq V$ of size k such that each vertex is either in S or adjacent to a vertex in S.

$$\boxed{DS \lesssim k\text{-center}}$$

$$X_{DS} = (G, k) \xrightarrow{\boxed{f}} X_{k\text{-center}} = (G^c, k)$$

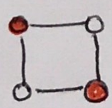
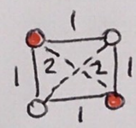
The D.S problem is on a graph G that might not be complete and also G has no cost on the edges.

We complete the graph G by adding all missing edges and call it G^c. We assign the following cost on the edges:

$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) \in G \quad \text{(old edge)} \\ 0 & \text{otherwise} \quad \text{(new edge)} \end{cases}$$

Claim: $\boxed{\text{Dominating set of size k on G}} \iff \boxed{\text{Optimal radius for k-center on } G^c \text{ is } r^* = 1}$

If there is a set of k nodes, where all other nodes are adjacent to them in G... ...then let these k nodes be the k centers; all other nodes in G^c will have distance 1 from them $\Rightarrow$ radius = 1.

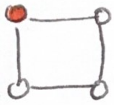
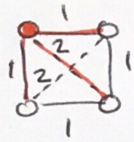
Yes instance: G =  $\Rightarrow$ G^c = 

k = 2 (DS) — k = 2 (k-center)

$\underbrace{\text{D.S. of size 2}}$ $\longrightarrow$ $\underbrace{\text{radius: } r^* = 1 \text{ for k center}}$

All nodes are either on D.S. or adjacent to it.

All nodes are either centers or distance 1 from the nearest center. $\longrightarrow$

No instance: $G =$  $\Longrightarrow$ $G^c =$ 

$\underbrace{k=1}_{\text{No D.S. of size 1}}$

$\underbrace{k=1}_{\text{The maximum distance from the center is 2.}}$ $\Rightarrow r^* = 2.$

Suppose $\exists$ α-approximation alg for the k-centre problem, call it APX, where $\alpha < 2$.

We must have:

$$r^* \leq \text{cost(APX)} \leq \alpha \underbrace{r^*}_{\text{OPT for k-center}}$$

r^* only takes values 1 OR 2.

If $r^* = 1 \Rightarrow 1 \leq \text{cost(APX)} \leq \alpha < 2 \Rightarrow \underbrace{\text{cost(APX)} = 1}_{\text{must be integer}}$

If $r^* = 2 \Rightarrow 2 \leq \text{cost(APX)} \leq 2\alpha \Longrightarrow \underbrace{\text{cost(APX)} = 2}_{\text{max distance on } G^c \text{ is 2.}}$

This implies that if cost(APX) = 1 we have $r^* = 1$ in G^c, which means there is a D.S. problem of size k.

& if cost(APX) = 2 we have $r^* = 2$ in G^c, which means there is no D.S. problem of size k.

$\therefore$ An α-approx. alg. with $\alpha < 2$ would solve the D.S problem which is NP-complete. Thus, it only exists if P = NP. //