Exercícios

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Disciplina: Introdução à Física de Partículas

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Mostre que $(\sigma \cdot \mathrm{p})^2 = \mathrm{p}^2$

$$(\sigma \cdot \mathrm{p})^2 = (\sigma \cdot \mathrm{p})(\sigma \cdot \mathrm{p}) = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)$$
$$= \sigma_1 p_1 \sigma_1 p_1 + \sigma_1 p_1 \sigma_2 p_2 + \sigma_1 p_1 \sigma_3 p_3 +$$
$$+ \sigma_2 p_2 \sigma_1 p_1 + \sigma_2 p_2 \sigma_2 p_2 + \sigma_2 p_2 \sigma_3 p_3 +$$
$$+ \sigma_3 p_3 \sigma_1 p_1 + \sigma_3 p_3 \sigma_2 p_2 + \sigma_3 p_3 \sigma_3 p_3$$

Usando a relação da anticomutação: algebra de Clifford

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathrm{I} \Rightarrow \sigma_1\sigma_2 = -\sigma_2\sigma_1 \quad \sigma_k^2 = \mathrm{I}. \quad i, j, k = 1, 2, 3$$

$$\sigma_1 p_1 \sigma_2 p_2 + \sigma_2 p_2 \sigma_1 p_1 = p_1 p_2 (\sigma_1\sigma_2 + \sigma_2\sigma_1) = p_1 p_2 (2\delta_{12}) = 0$$
$$\sigma_1 p_1 \sigma_3 p_3 + \sigma_3 p_3 \sigma_1 p_1 = p_3 p_1 (\sigma_3\sigma_1 + \sigma_1\sigma_3) = p_3 p_1 (2\delta_{31}) = 0$$
$$\sigma_2 p_2 \sigma_3 p_3 + \sigma_3 p_3 \sigma_2 p_2 = p_3 p_2 (\sigma_3\sigma_2 + \sigma_2\sigma_3) = p_3 p_2 (2\delta_{32}) = 0$$

Por tanto,

$$(\sigma \cdot \mathrm{p})^2 = \sigma_1 p_1 \sigma_1 p_1 + \sigma_2 p_2 \sigma_2 p_2 + \sigma_3 p_3 \sigma_3 p_3 = \sigma_1^2 p_1^2 + \sigma_2^2 p_2^2 + \sigma_3^2 p_3^2 = p_1^2 + p_2^2 + p_3^2$$

$$\boxed{(\sigma \cdot \mathrm{p})^2 = \mathrm{p}^2}$$

Duvidas?

Lembrar

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \{\sigma_a, \sigma_b\} = 2\delta_{ab}\mathrm{I}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad [\sigma^2, \sigma_j] = 0$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}$$

Mostre que $\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = (\gamma^\mu p_\mu + m\mathrm{I}) = \not{p} + m$

A soma pode ser realizada usando a base da helicidade ou os espinores u1 e u2:

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) \equiv u_1(p)\bar{u}_1(p) + u_2(p)\bar{u}_2(p).$$

Na representação de Dirac-Pauli, os espinores u1 e u2 podem ser escritos como

$$u_s(p) = \sqrt{E+m}\begin{pmatrix} \phi_s \\ \frac{\sigma\cdot\mathrm{p}}{E+m}\phi_s \end{pmatrix} \quad \text{with} \quad \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{u}_s = u_s^\dagger\gamma^0 = \sqrt{E+m}\begin{pmatrix} \phi_s^T & \phi_s^T\frac{(\sigma\cdot\mathrm{p})^\dagger}{E+m} \end{pmatrix}\begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} \phi_s^T & -\phi_s^T\frac{(\sigma\cdot\mathrm{p})}{E+m} \end{pmatrix}$$

porque as matrizes de Pauli são hermitianas $(\sigma\cdot\mathrm{p})^\dagger = \sigma\cdot\mathrm{p}$

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = (E+m)\sum_{s=1}^{2}\begin{pmatrix} \phi_s\phi_s^T & -\frac{\sigma\cdot \mathrm{p}}{E+m}\phi_s\phi_s^T \\ \frac{\sigma\cdot \mathrm{p}}{E+m}\phi_s\phi_s^T & -\frac{(\sigma\cdot \mathrm{p})^2}{(E+m)^2}\phi_s\phi_s^T \end{pmatrix}$$

onde nós usamos:

$$\sum_{s=1}^{2}\phi_s\phi_s^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I} \quad \text{e} \quad (\sigma\cdot \mathrm{p})^2 = \mathrm{p}^2 = E^2 - m^2 = (E+m)(E-m),$$

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \begin{pmatrix} (E+m)\mathrm{I} & -\sigma\cdot \mathrm{p} \\ \sigma\cdot \mathrm{p} & (-E+m)\mathrm{I} \end{pmatrix} = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} E - \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}\cdot \mathrm{p} + m\begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix},$$

$$\boxed{\sum_{s=1}^{2} u_s\bar{u}_s = \left(\gamma^\mu p_\mu + m\mathrm{I}\right) = \not{p} + m} \quad ,$$

com $\not{p} \equiv \gamma^\mu p_\mu = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3$

Lembrar que as matrizes gamma são

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{pmatrix} \quad k = 1, 2, 3$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu \equiv \left(1, \sigma^i\right), \quad \bar{\sigma}^\mu \equiv \left(1, -\sigma^i\right)$$

Duvidas?

Mostre que $\sum_{r=1}^{2} v_r \bar{v}_r = \left(\gamma^\mu p_\mu - m\mathrm{I}\right) = \not{p} - m$

Na representação de Dirac-Pauli,

$$v_r(p) = \sqrt{E+m} \begin{pmatrix} \frac{\sigma \cdot \mathrm{p}}{E+m}\chi_r \\ \chi_r \end{pmatrix} \quad \text{com} \quad \chi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{e} \quad \chi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{v}_r = v_r^\dagger \gamma^0 = \sqrt{E+m} \begin{pmatrix} \chi_r^T \frac{(\sigma \cdot \mathrm{p})^\dagger}{E+m} & \chi_r^T \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} = \sqrt{E+m} \begin{pmatrix} \chi_r^T \frac{(\sigma \cdot \mathrm{p})}{E+m} & -\chi_r^T \end{pmatrix}$$

porque as matrizes de Pauli são hermitianas $(\sigma \cdot \mathrm{p})^\dagger = \sigma \cdot \mathrm{p}$

$$\sum_{r=1}^{2} v_r(p)\bar{v}_r(p) = (E+m)\sum_{r=1}^{2}\begin{pmatrix} \frac{(\sigma\cdot\mathrm{p})^2}{(E+m)^2}\chi_r\chi_r^T & -\frac{\sigma\cdot\mathrm{p}}{E+m}\chi_r\chi_r^T \\ \frac{\sigma\cdot\mathrm{p}}{E+m}\chi_r\chi_r^T & -\chi_r\chi_r^T \end{pmatrix}$$

onde nós usamos:

$$\sum_{s=1}^{2}\chi_r\chi_r^T = \mathrm{I} \quad \text{e} \quad (\sigma\cdot\mathrm{p})^2 = \mathrm{p}^2 = E^2 - m^2 = (E+m)(E-m),$$

$$\sum_{r=1}^{2} v_r(p)\bar{v}_r(p) = \begin{pmatrix} (E-m)\mathrm{I} & -\sigma\cdot\mathrm{p} \\ \sigma\cdot\mathrm{p} & -(E+m)\mathrm{I} \end{pmatrix} = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} E - \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}\cdot\mathrm{p} - m\begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix},$$

$$\boxed{\sum_{r=1}^{2} v_r\bar{v}_r = \left(\gamma^\mu p_\mu - m\mathrm{I}\right) = \not{p} - m} \quad ,$$

com $\not{p} \equiv \gamma^\mu p_\mu = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3$

Mostrar que o operador de carga elétrica é $\hat{C} = i\gamma^2$

É usualmente considerado a equação de Dirac com acoplamento eletromagnético

$$\gamma^\mu \left(\partial_\mu - ieA_\mu\right) \psi + im\psi = 0 \tag{1}$$

Tomando primeiro o conjugado complexo de e então pré-multiplicando por $i\gamma^2$ vai dar

$$-i\gamma^2 \left(\gamma^\mu\right)^* \left(\partial_\mu + ieA_\mu\right) \psi^* - im(-i\gamma^2)\psi^* = 0$$

com $\left(\gamma^0\right)^* = \gamma^0, \left(\gamma^1\right)^* = \gamma^1, \left(\gamma^2\right)^* = -\gamma^2$ and $\left(\gamma^3\right)^* = \gamma^3$ e $\gamma^2\gamma^\mu = -\gamma^\mu\gamma^2$ for $\mu \neq 2$.

$$\gamma^\mu \left(\partial_\mu + ieA_\mu\right) i\gamma^2\psi^* + imi\gamma^2\psi^* = 0.$$

Agora se nós definimos ψ' como $\boxed{\psi' = i\gamma^2\psi^* = \hat{C}\psi^*}$, podemos escrever a Eq. (1) como sigue,

$$\gamma^\mu \left(\partial_\mu + ieA_\mu\right) \psi' + im\psi' = 0, \text{ só } ieA_\mu \text{ aparece com o sinal oposto.}$$

Vamos a definir a conjugação de carga $\mathscr{C}\psi(x)\mathscr{C}^{-1} \quad = \quad C\bar{\psi}^T(x)$

Mostrar que o operador de paridade é dado por $P = \gamma^0$

A forma do operador de paridade pode ser deduzida considerando uma função de onda $\psi(x, y, z, t)$ que satisfaz a equação de Dirac de partícula livre,

$$i\gamma^1 \frac{\partial \psi}{\partial x} + i\gamma^2 \frac{\partial \psi}{\partial y} + i\gamma^3 \frac{\partial \psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial \psi}{\partial t} \tag{2}$$

Agora se nós definimos $\psi'(x', y', z', t') = \hat{P}\psi(x, y, z, t) \Rightarrow \psi = \hat{P}\psi'$

$$i\gamma^1 \frac{\partial \psi'}{\partial x'} + i\gamma^2 \frac{\partial \psi'}{\partial y'} + i\gamma^3 \frac{\partial \psi'}{\partial z'} - m\psi' = -i\gamma^0 \frac{\partial \psi'}{\partial t'}.$$

Por tanto, desde Eq. 2,

$$i\gamma^1 \hat{P} \frac{\partial \psi'}{\partial x} + i\gamma^2 \hat{P} \frac{\partial \psi'}{\partial y} + i\gamma^3 \hat{P} \frac{\partial \psi'}{\partial z} - m\hat{P}\psi' = -i\gamma^0 \hat{P} \frac{\partial \psi'}{\partial t}$$

Pré-multiplicando por γ^0, que introduz sinais de menos para todas as coordenadas de espaço:

$$-i\gamma^0\gamma^1\hat{P}\frac{\partial\psi'}{\partial x'}-i\gamma^0\gamma^2\hat{P}\frac{\partial\psi'}{\partial y'}-i\gamma^0\gamma^3\hat{P}\frac{\partial\psi'}{\partial z'}-m\gamma^0\hat{P}\psi'=-i\gamma^0\gamma^0\hat{P}\frac{\partial\psi'}{\partial t'}$$

$$i\gamma^1\gamma^0\hat{P}\frac{\partial\psi'}{\partial x'}+i\gamma^2\gamma^0\hat{P}\frac{\partial\psi'}{\partial y'}+i\gamma^3\gamma^0\hat{P}\frac{\partial\psi'}{\partial z'}-m\gamma^0\hat{P}\psi'=-i\gamma^0\gamma^0\hat{P}\frac{\partial\psi'}{\partial t'}.$$

Por conseguinte, este Eq. pode ser reduzido à Eq. 2, mas deveríamos ter $\gamma^0\hat{P}\propto I$. Assim, obtemos $\hat{P}=\gamma^0$,

$$\boxed{\psi\to\hat{P}\psi=\gamma^0\psi}$$

Vamos a definir a $\mathscr{P}\psi(\vec{x},t)\mathscr{P}^{-1} \quad = \lambda_P P\psi(-\vec{x},t)$