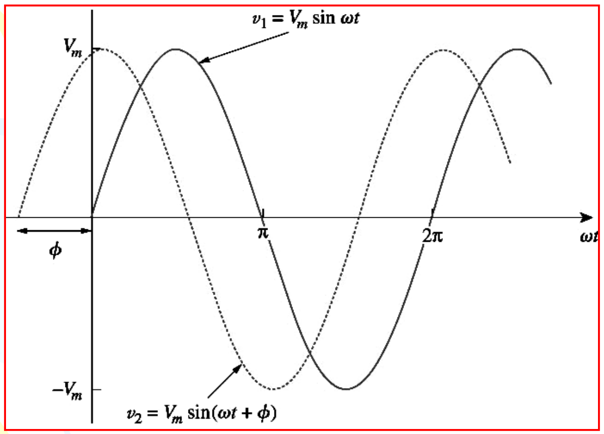
Chapter 9
Sinusoidal Steady-State Analysis

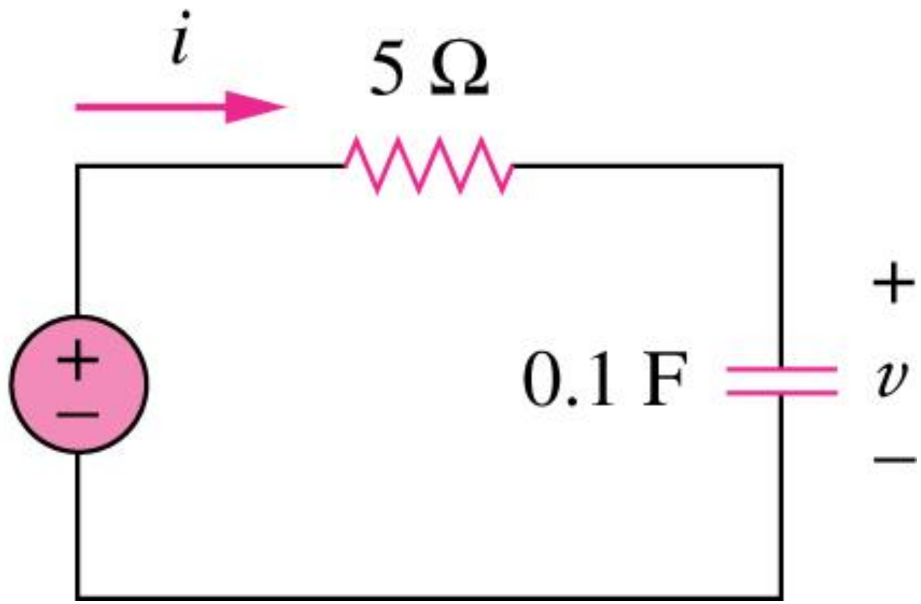
Sinusoids and Phasor
Chapter 9

9.1 Motivation (1)

How to determine *v(t)* and *i(t)?*



$v_s(t) = 10V$

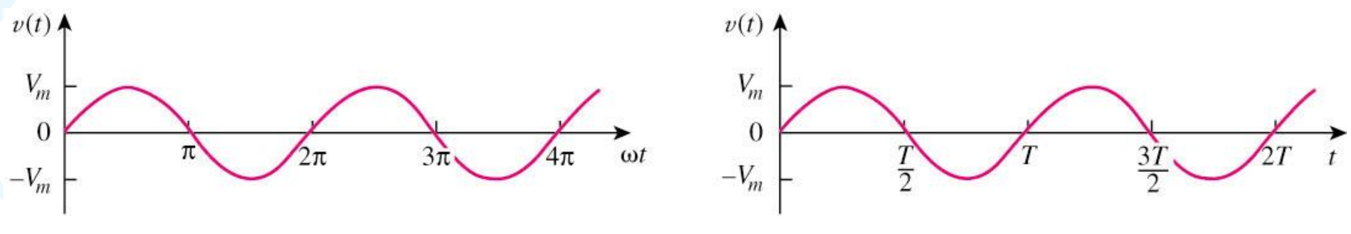


How can we apply what we have learned before to determine *i(t)* and *v(t)*?

9.2 Sinusoids (1)

- A sinusoid is a signal that has the form of the sine or cosine function.
- A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$



(a) (b)

where

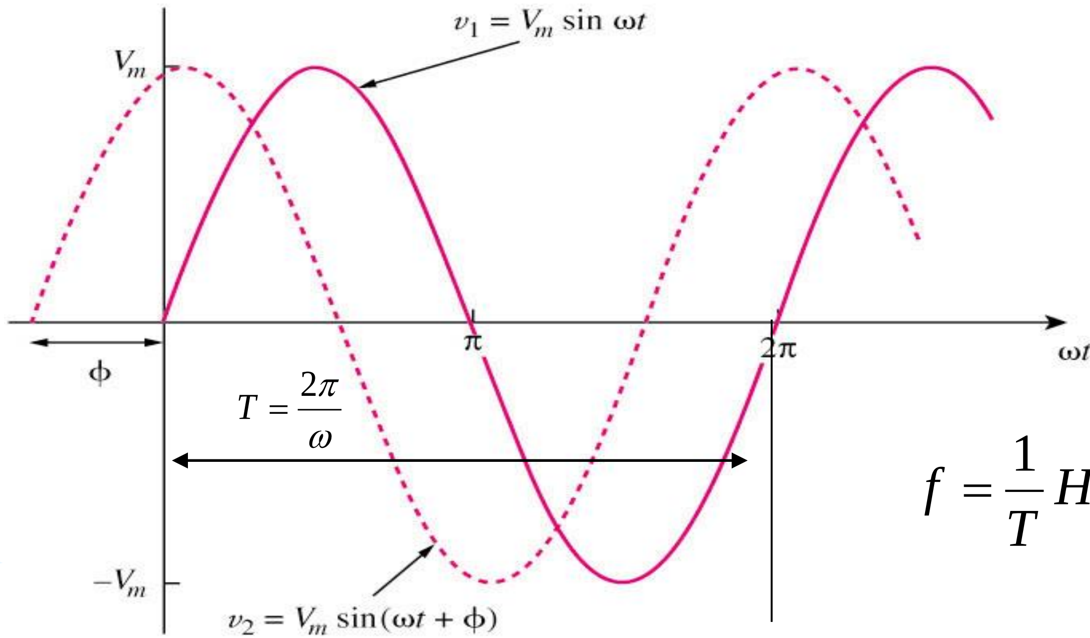
Vm = the **amplitude** of the sinusoid

ω = the angular frequency in radians/s

Φ = the phase

9.2 Sinusoids (2)

A <u>periodic function</u> is one that satisfies *v(t) = v(t + nT)*, for all t and for all integers n.



$$f = \frac{1}{T} Hz \quad \omega = 2\pi f$$

- Only two sinusoidal values with the <u>same frequency</u> can be compared by their amplitude and phase difference.
- If phase difference is zero, they are in phase; if phase difference is not zero, they are out of phase.

9.2 Sinusoids (3)

Example 1

Given a sinusoid, $5\sin(4\pi t - 60^o)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Solution:

Amplitude = 5, phase = -60^o, angular frequency = 4π rad/s, Period = 0.5 s, frequency = 2 Hz.

9.2 Sinusoids (4)

Example 2

Find the phase angle between $i_1 = -4\sin(377t + 25^o)$ and $i_2 = 5\cos(377t - 40^o)$, does i_1 lead or lag i_2?

Solution:

Since $\sin(\omega t+90^o) = \cos \omega t$

$$i_2 = 5\sin(377t - 40^o + 90^o) = 5\sin(377t + 50^o)$$

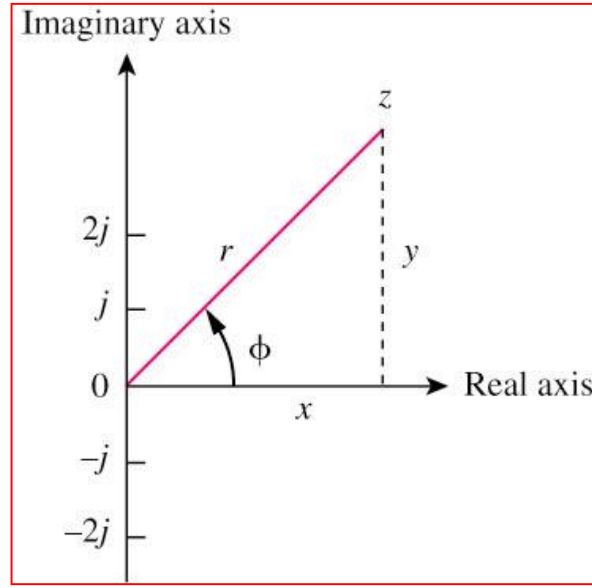
$$i_1 = -4\sin(377t + 25^o) = 4\sin(377t + 180^o + 25^o) = 4\sin(377t + 205^o)$$

therefore, i_1 leads i_2 155^o.

9.3 Phasor (1)

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.

- It can be represented in one of the following three forms:



a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential $z = re^{j\phi}$

where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\dfrac{y}{x}$

9.3 Phasor (2)

Example 9.3

- Evaluate the following complex numbers:

Evaluate these complex numbers:

Example 9.3

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

Solution:

(a) Using polar to rectangular transformation,

$$40\angle 50^\circ = 40(\cos 50^\circ + j\sin 50^\circ) = 25.71 + j30.64$$

$$20\angle -30^\circ = 20[\cos(-30^\circ) + j\sin(-30^\circ)] = 17.32 - j10$$

Adding them up gives

$$40\angle 50^\circ + 20\angle -30^\circ = 43.03 + j20.64 = 47.72\angle 25.63^\circ$$

Taking the square root of this,

$$(40\angle 50^\circ + 20\angle -30^\circ)^{1/2} = 6.91\angle 12.81^\circ$$

9.3 Phasor (3)

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\dfrac{1}{z} = \dfrac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugate $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

9.3 Phasor (4)

- Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \quad \longleftrightarrow \quad V = V_m \angle \phi$$

(time domain) (phasor domain)

- <u>Amplitude</u> and <u>phase difference</u> are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the <u>cosine function</u> in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

9.3 Phasor (5)

Example 4

Transform the following sinusoids to phasors:

$i = 6\cos(50t - 40^\circ)$ A

$v = -4\sin(30t + 50^\circ)$ V

Solution:

a. $I = 6\angle -40^\circ$ A

b. Since $-\sin(A) = \cos(A+90^\circ)$;

$v(t) = 4\cos(30t+50^\circ+90^\circ) = 4\cos(30t+140^\circ)$ V

Transform to phasor => $V = 4\angle 140^\circ$ V

9.3 Phasor (6)

Example 5:

Transform the sinusoids corresponding to phasors:

a. $\mathbf{V} = -10\angle 30^\circ$ V

b. $\mathbf{I} = j(5 - j12)$ A

Solution:

a) *v(t) = 10cos(ωt + 210°)* V

b) Since $\mathbf{I} = 12 + j5 = \sqrt{12^2 + 5^2} \angle \tan^{-1}(\frac{5}{12}) = 13\angle 22.62^\circ$

i(t) = 13cos(ωt + 22.62°) A

9.3 Phasor (7)

The differences between v(t) and V:

- v(t) is instantaneous or time-domain representation
  V is the frequency or phasor-domain representation.
- v(t) is time dependent, V is not.
- v(t) is always real with no complex term, V is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

9.3 Phasor (8)

Relationship between differential, integral operation in phasor listed as follow:

$$v(t) \longleftrightarrow \mathbf{V} = V\angle\phi$$

$$\frac{dv}{dt} \longleftrightarrow j\omega\mathbf{V}$$

$$\int v\,dt \longleftrightarrow \frac{\mathbf{V}}{j\omega}$$

9.3 Phasor (9)

Example 6

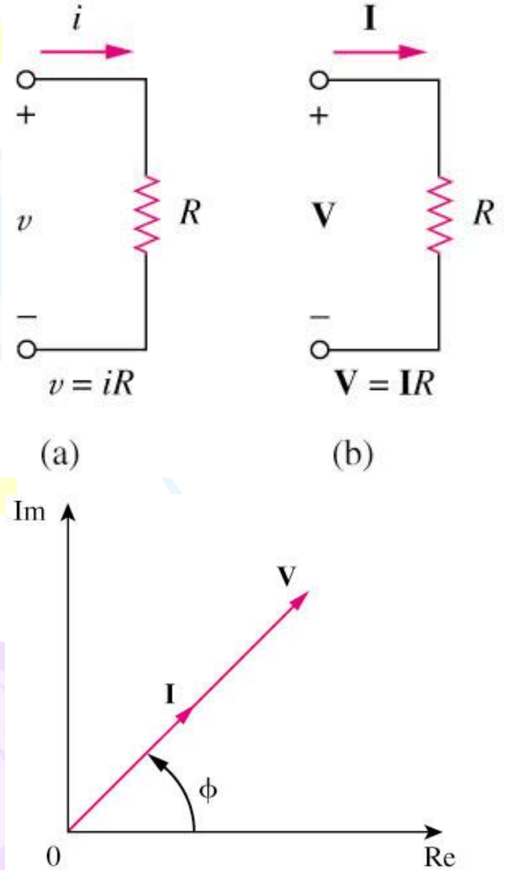
Use phasor approach, determine the current *i(t)* in a circuit described by the integro-differential equation.

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

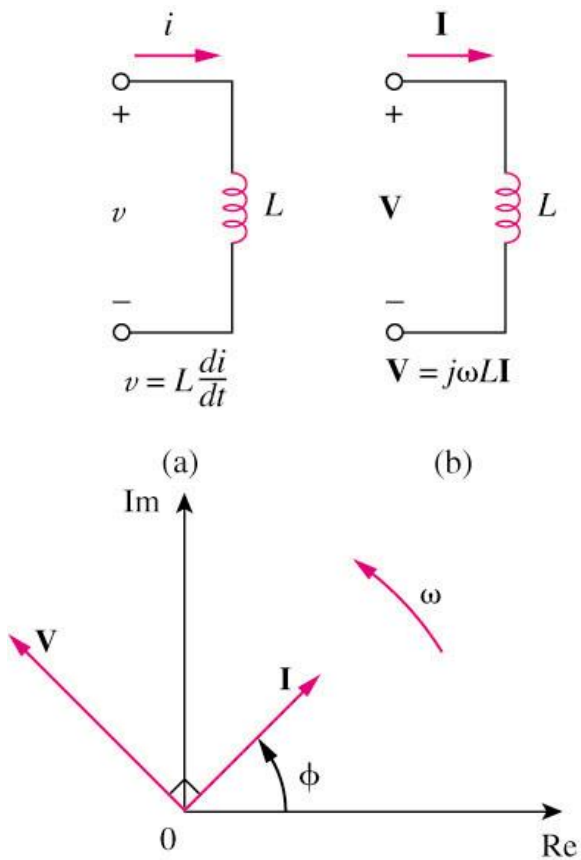
Answer: $i(t) = 4.642\cos(2t + 143.2^o)\ A$

9.4 Phasor Relationships for Circuit Elements (1)

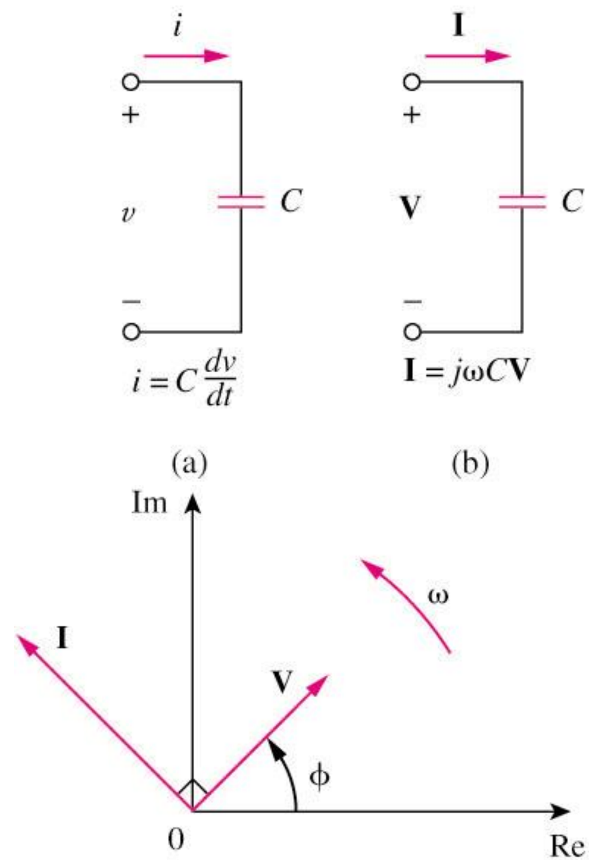
Resistor:



Inductor:



Capacitor:



9.4 Phasor Relationships for Circuit Elements (2)

Summary of voltage-current relationship		
Element	Time domain	Frequency domain
R	$v = Ri$	$V = RI$
L	$v = L\dfrac{di}{dt}$	$V = j\omega LI$
C	$i = C\dfrac{dv}{dt}$	$V = \dfrac{I}{j\omega C}$

9.4 Phasor Relationships for Circuit Elements (3)

Example 7

If voltage *v(t)* = $6\cos(100t - 30^o)$ is applied to a 50 μF capacitor, calculate the current, *i(t)*, through the capacitor.

Answer: i(t) = **$30 \cos(100t + 60^o)$ mA**

9.5 Impedance and Admittance (1)

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms Ω.

$$Z = \frac{V}{I} = R + jX$$

where R = Re, Z is the resistance and X = Im, Z is the reactance. Positive X is for L and negative X is for C.

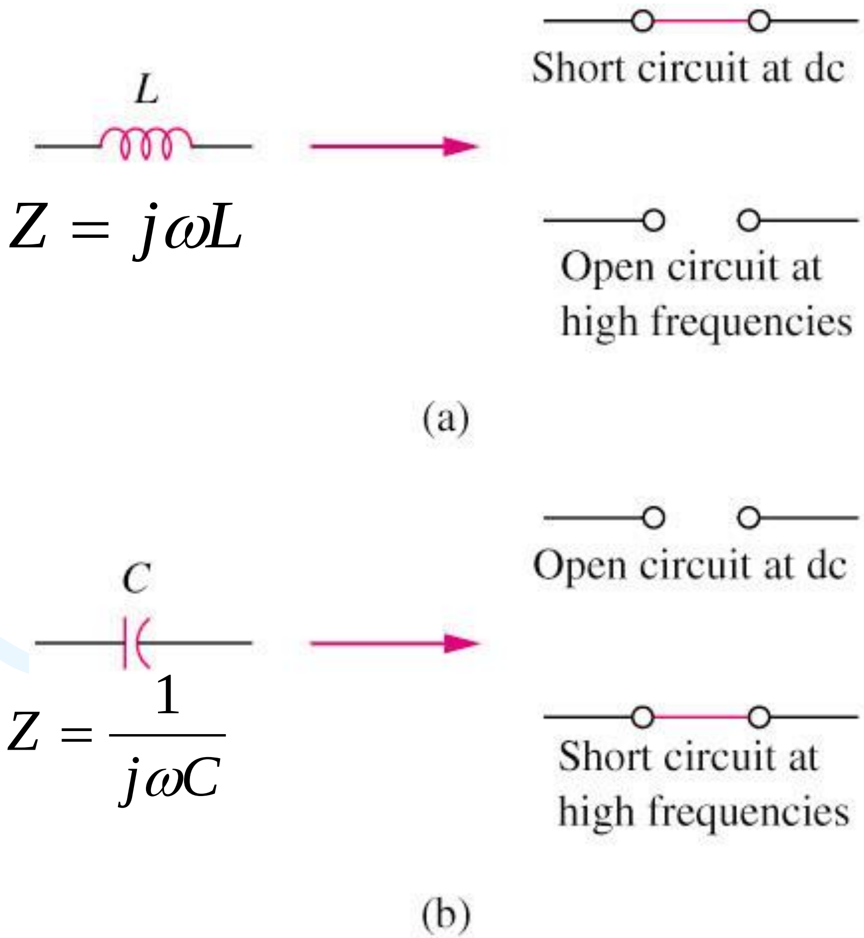
- The admittance Y is the reciprocal of impedance, measured in siemens (S).

$$Y = \frac{1}{Z} = \frac{I}{V}$$

9.5 Impedance and Admittance (2)

Impedances and admittances of passive elements		
Element	**Impedance**	**Admittance**
R	$Z = R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

9.5 Impedance and Admittance (3)



$$\omega = 0; Z = 0$$

$$\omega \to \infty; Z \to \infty$$

$$\omega = 0; Z \to \infty$$

$$\omega \to \infty; Z = 0$$

9.5 Impedance and Admittance (4)

After we know how to convert RLC components from time to phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.

Hence, we can apply the KCL laws and other theorems to directly set up phasor equations involving our target variable(s) for solving.

Impedance and Admittance

Find $v(t)$ and $i(t)$ in the circuit shown in Fig. 9.16.

Example 9.9

Solution:

From the voltage source $10 \cos 4t$, $\omega = 4$,

$$\mathbf{V}_s = 10\angle 0^\circ \text{ V}$$

The impedance is

$$\mathbf{Z} = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

Hence the current

$$\begin{aligned}\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}} &= \frac{10\angle 0^\circ}{5 - j2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2} \\ &= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A}\end{aligned} \tag{9.9.1}$$

The voltage across the capacitor is

$$\begin{aligned}\mathbf{V} = \mathbf{I}\mathbf{Z}_C = \frac{\mathbf{I}}{j\omega C} &= \frac{1.789\angle 26.57^\circ}{j4 \times 0.1} \\ &= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V}\end{aligned} \tag{9.9.2}$$

Converting $\mathbf{I}$ and $\mathbf{V}$ in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

$$i(t) = 1.789 \cos(4t + 26.57^\circ) \text{ A}$$
$$v(t) = 4.47 \cos(4t - 63.43^\circ) \text{ V}$$

Notice that $i(t)$ leads $v(t)$ by 90° as expected.

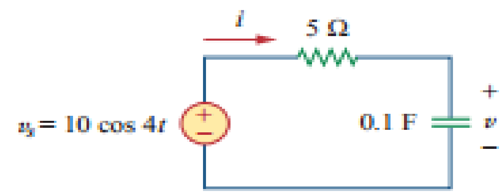
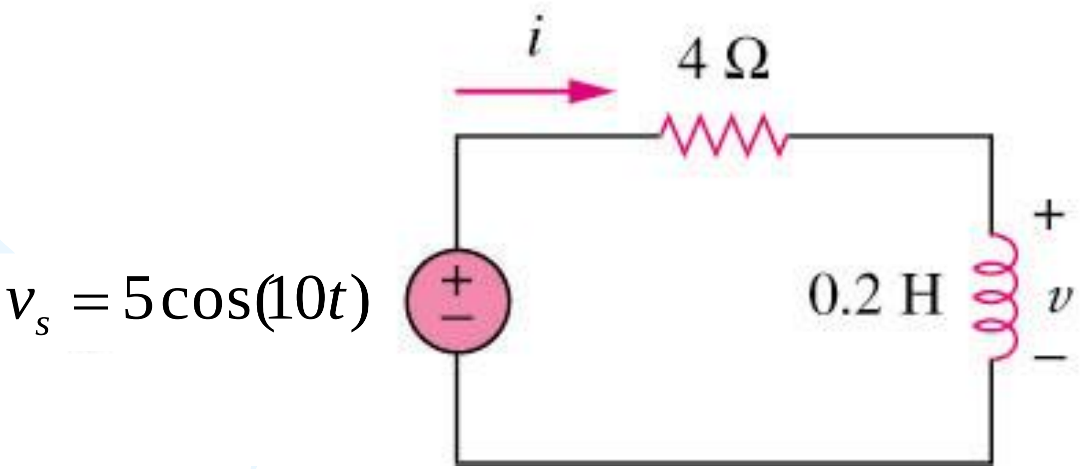


Figure 9.16
For Example 9.9.

9.5 Impedance and Admittance (5)

Example 9.9 and practice problem 9.9

Refer to Figure below, determine *v(t)* and *i(t).*



Answers: *i(t) = 1.118cos(10t – 26.56°) A; v(t) = 2.236cos(10t + 63.43°) V*

9.6 Kirchhoff's Laws in the Frequency Domain (1)

- Both KVL and KCL are hold in the phasor domain or more commonly called frequency domain.

- Moreover, the variables to be handled are phasors, which are complex numbers.

- All the mathematical operations involved are now in complex domain.

9.7 Impedance Combinations (1)

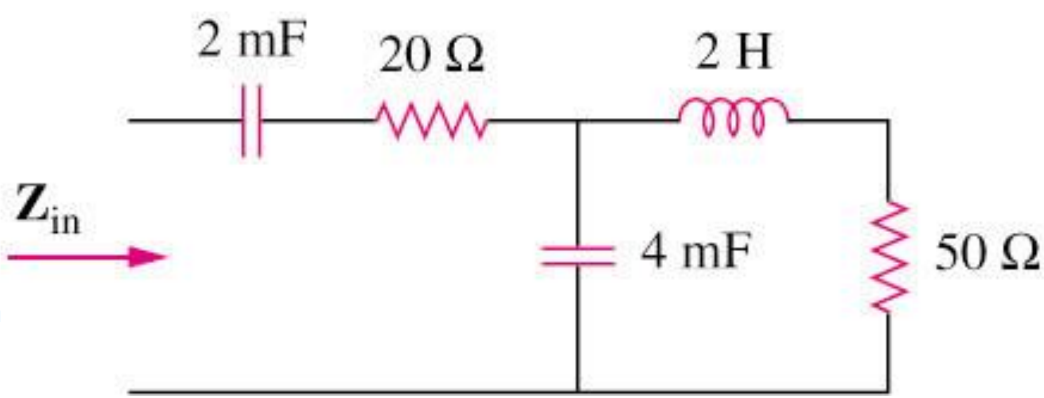
- The following principles used for DC circuit analysis all apply to AC circuit.

- For example:
  a. voltage division
  b. current division
  c. circuit reduction
  d. impedance equivalence
  e. Y-Δ transformation

9.7 Impedance Combinations (2)

Example 9

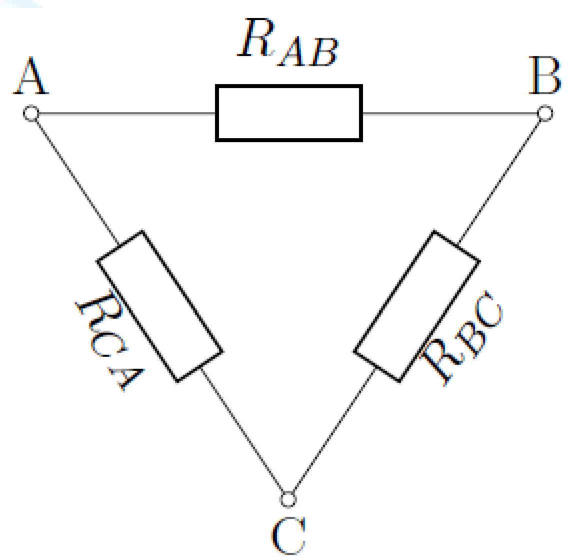
Determine the input impedance of the circuit in figure below at ω =10 rad/s.

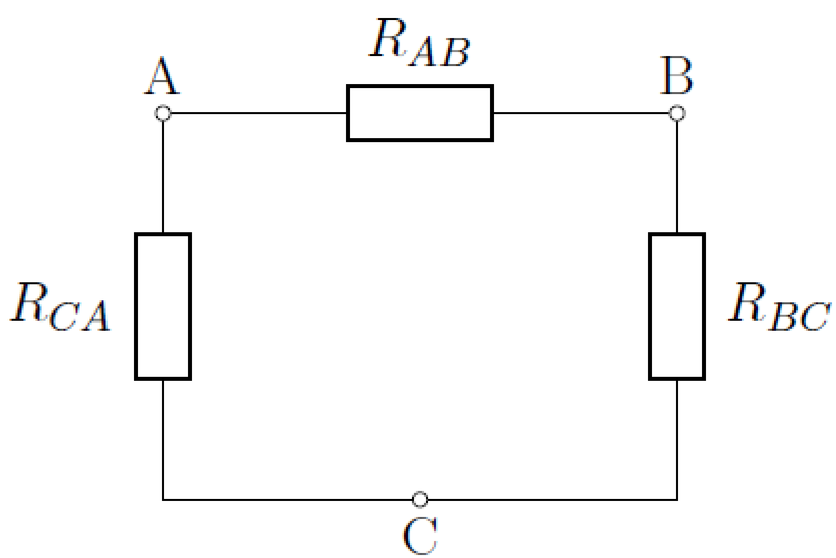


Answer: Z_{in} = **32.38 – j73.76**

Delta-Y conversions

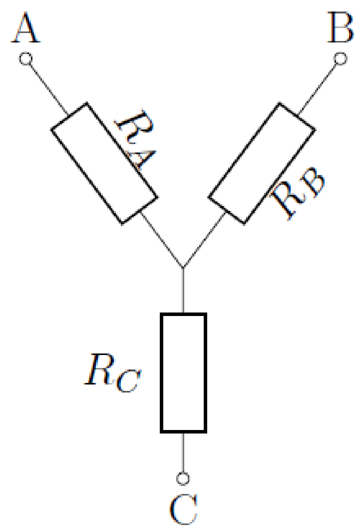
- In many applications circuits or fragments of circuits form a three-terminal network. Components are then connected together in one of two ways: the “Delta,” or *(also* known as the “Pi,” or) configuration, and the “Y” (also known as the “T”) configuration.
- The Delta (left side) and *(right side)*

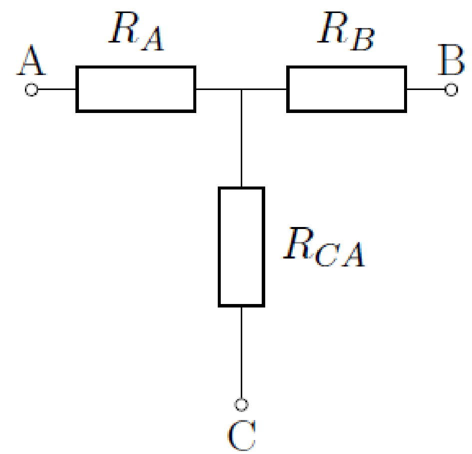




Delta-Y conversions

- The Y (left side) and T (right side) configuration take the form presented below:





Delta-Y conversions

- From Delta (Δ) to Wye (Y).

$$R_A = \frac{R_{AB} * R_{AC}}{R_{AB} + R_{AC} + R_{BC}}$$

$$R_B = \frac{R_{AB} * R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$$

$$R_C = \frac{R_{BC} * R_{AC}}{R_{AB} + R_{AC} + R_{BC}}$$

- From Wye (Y) to Delta (Δ).

$$R_{AB} = \frac{R_A * R_B + R_A * R_C + R_B * R_C}{R_C}$$

$$R_{AC} = \frac{R_A * R_B + R_A * R_C + R_B * R_C}{R_B}$$

$$R_{BC} = \frac{R_A * R_B + R_A * R_C + R_B * R_C}{R_A}$$