Continuous-Time Fourier Series

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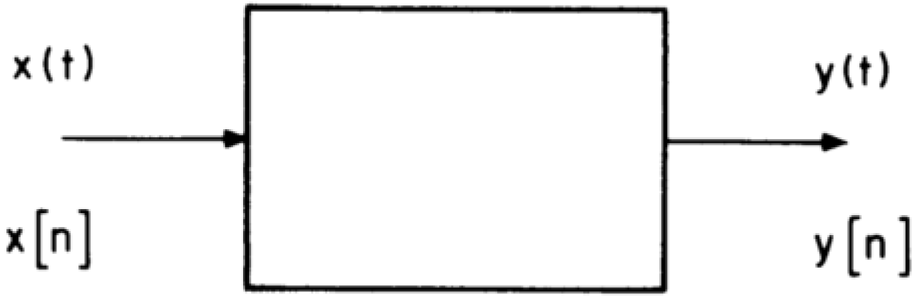


If: $x(t) = a_1\,\phi_1(t) + a_2\,\phi_2(t) + \ldots$

$$\phi_k(t) \longrightarrow \psi_k(t)$$

and system is linear

Then: $y(t) = a_1\,\psi_1(t) + a_2\,\psi_2(t) + \ldots$

Identical for discrete-time

TRANSPARENCY 7.1
The principle of superposition for linear systems.

If: $x = a_1 \phi_1 + a_2 \phi_2 + \dots$

Then: $y = a_1 \psi_1 + a_2 \psi_2 + \dots$

Choose $\phi_k(t)$ or $\phi_k[n]$ so that:

- a broad class of signals can be constructed as a linear combination of ϕ_k's
- response to ϕ_k's easy to compute

TRANSPARENCY 7.2
Criteria for choosing a set of basic signals in terms of which to decompose the input to a linear system.

TRANSPARENCY 7.3
Choice for the basic signals that led to the convolution integral and convolution sum.

LTI SYSTEMS:

- C-T: $\phi_k(t) = \delta(t - k\Delta)$

  $\psi_k(t) = h(t - k\Delta)$

  => Convolution Integral

- D-T: $\phi_k[n] = \delta[n - k]$

  $\psi_k[n] = h[n - k]$

  => Convolution Sum

TRANSPARENCY 7.4
Complex exponentials as a set of basic signals.

$\phi_k(t) = e^{s_k t}$ $\quad$ s_k complex

$\phi_k[n] = z_k^{\ n}$ $\quad$ z_k complex

Fourier Analysis:

- C-T: $s_k = j\omega_k$ $\quad$ $\phi_k(t) = e^{j\omega_k t}$
- D-T: $|z_k| = 1$ $\quad$ $\phi_k[n] = e^{j\Omega_k n}$

s_k complex => Laplace transforms

z_k complex => z-transforms

$$\Phi_k(t) = e^{j\omega_k t}$$

$$e^{j\omega_k t} \longrightarrow H(\omega_k)e^{j\omega_k t}$$

"Proof"

$$e^{j\omega_k t} \rightarrow \int_{-\infty}^{+\infty} h(\tau)\, \underbrace{e^{j\omega_k(t-\tau)}}_{e^{j\omega_k t} e^{-j\omega_k \tau}} d\tau$$

$$e^{j\omega_k t} \rightarrow e^{j\omega_k t} \underbrace{\int_{-\infty}^{+\infty} h(\tau) e^{-j\omega_k \tau} d\tau}_{H(\omega_k)}$$

e-function

e-value.

Periodic Signals

- Fourier Series

Aperiodic Signals

- Fourier Transform

C-T Fourier Series

$$x(t) = x(t+T_0)$$

period

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

$$e^{j\omega_0 t} \qquad T_0 = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t} \qquad \frac{T_0}{k} = \frac{2\pi}{k\omega_0}$$

Harmonically Related

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Fourier series

period:

$$T_0 = \frac{2\pi}{\omega_0}$$

fundamental frequency:

$$\omega_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Complex Exponential Form

$$a_k = A_k e^{j\theta_k}$$

$$= B_k + jC_k$$

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j\sin k\omega_0 t$$

$$x(t) =$$

$$a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$$a_0 + 2\sum_{k=1}^{\infty} \left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right].$$

Trigonometric Form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\int_{T_0} e^{jm\omega_0 t}\,dt = \begin{cases} T_0 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

$$= \underbrace{\int_{T_0} \cos m\omega_0 t\,dt}_{} + j\underbrace{\int_{T_0} \sin m\omega_0 t\,dt}_{}$$

$m \neq 0$	0	0
$m = 0$	T_0	0

$$\int_{T_0} x(t) e^{-jn\omega_0 t}\,dt =$$

$$\int_{T_0} e^{-jn\omega_0 t} \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\,dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \int_{T_0} e^{j(k-n)\omega_0 t}\,dt$$

$$\int_{T_0} x(t) e^{-jn\omega_0 t}\,dt =$$

$$\sum_{k=-\infty}^{+\infty} a_k \underbrace{\int_{T_0} e^{j(k-n)\omega_0 t}\,dt}_{\begin{array}{l} T_0 \text{ if } k=n \\ 0 \text{ if } k\neq n \end{array}}$$

$$\int_{T_0} x(t) e^{-jn\omega_0 t}\,dt = a_n T_0$$

Synthesis

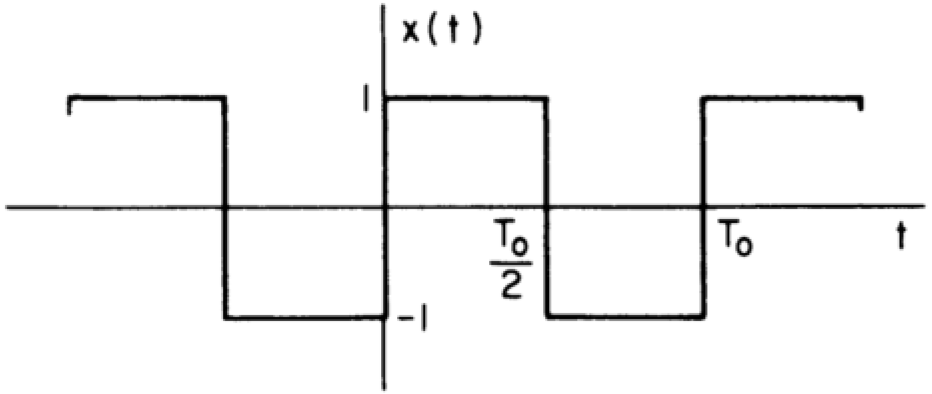
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Analysis

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t}\,dt$$

TRANSPARENCY 7.5
Determination of the Fourier series coefficients for an antisymmetric periodic square wave.

ANTISYMMETRIC PERIODIC SQUARE WAVE



$$a_k = \frac{1}{T_o} \int_{-T_o/2}^{0} (-1)\, e^{-jk\omega_o t}\, dt + \frac{1}{T_o} \int_{0}^{T_o/2} (+1)\, e^{-jk\omega_o t}\, dt$$

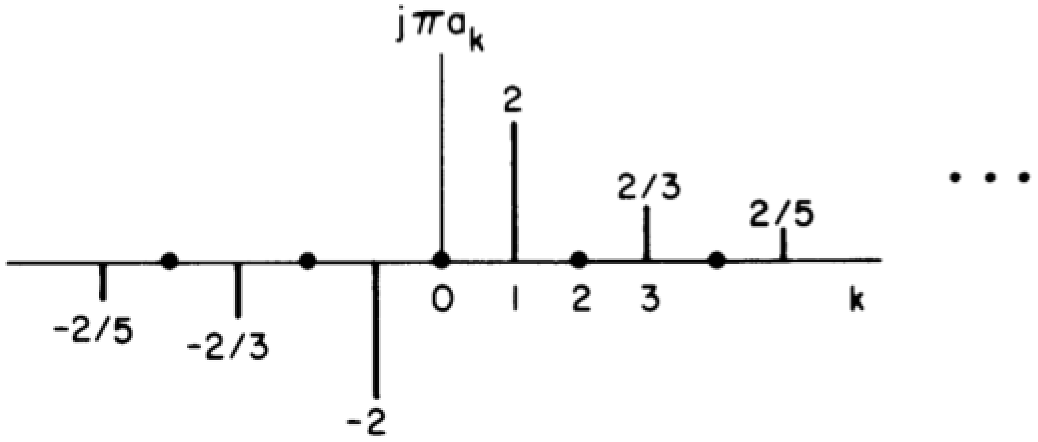
$$\vdots$$

$$= \frac{1}{j\pi k} \left\{ 1 - (-1)^k \right\} \qquad k \neq 0$$

$$a_o = \frac{1}{T_o} \int_{T_o} x(t)\, e^{-jk\omega_o t}\, dt = \frac{1}{T_o} \int_{T_o} x(t)\, dt = 0$$

TRANSPARENCY 7.6
The Fourier series coefficients for an antisymmetric periodic square wave.

ANTISYMMETRIC PERIODIC SQUARE WAVE



$$a_o = 0\,; \quad a_k = \frac{1}{j\pi k}\left\{1-(-1)^k\right\} \qquad k \neq 0$$

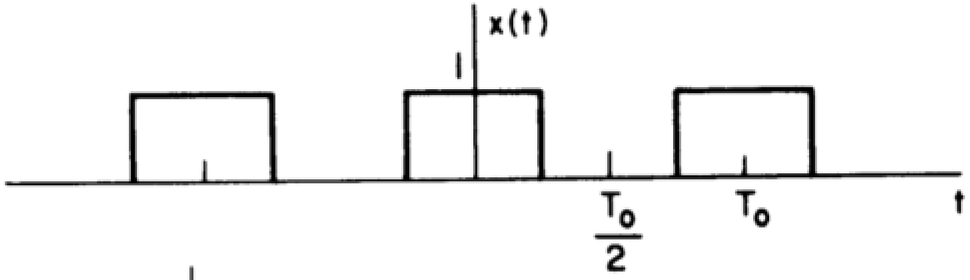
- odd harmonic
- a_k imaginary
- $a_k = -a_{-k}$ (antisymmetric)

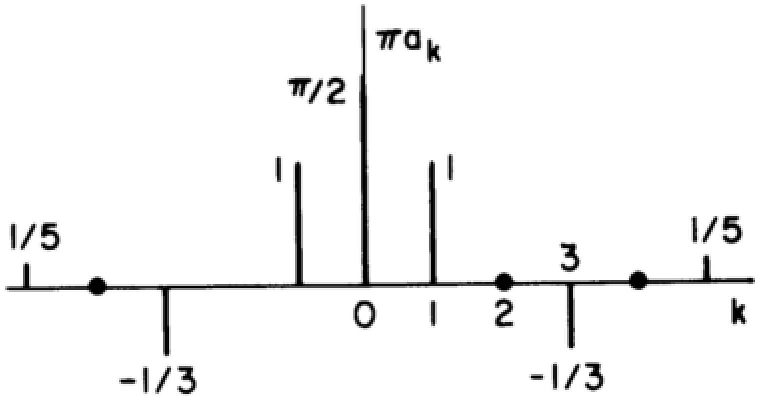
$\Rightarrow$ sine series

$$x(t) = a_o + \sum_{k=1}^{\infty} 2j\, a_k \sin k\omega_o t$$

SYMMETRIC PERIODIC SQUARE WAVE

Example 4.5:





$$a_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{\sin(\pi k/2)}{\pi k} & k \neq 0 \end{cases}$$

- odd harmonic
- a_k real
- $a_k = a_{-k}$ (symmetric)

$\Rightarrow$ cosine series

$$x(t) = a_o + \sum_{k=1}^{\infty} 2a_k \cos k\omega_o t$$

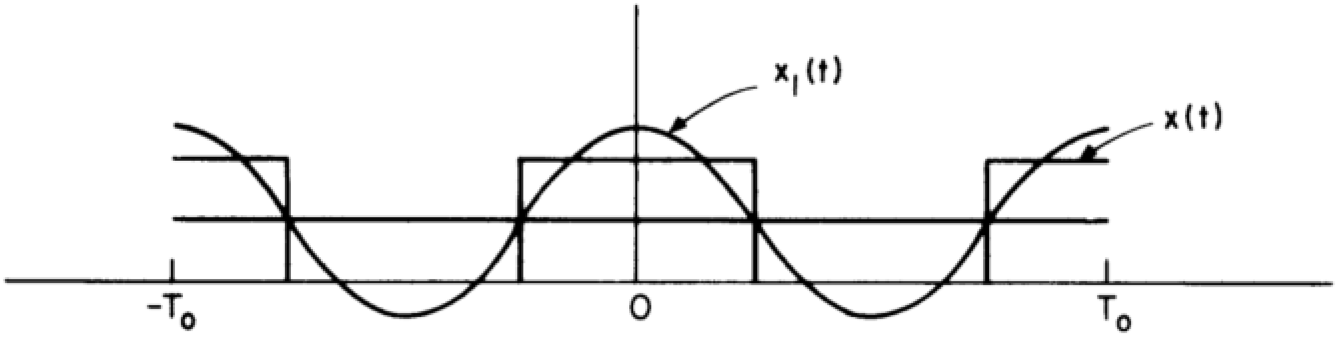
TRANSPARENCY 7.8
Illustration of the superposition of terms in the Fourier series representation for a symmetric periodic square wave. [Example 4.5 from the text.]

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \qquad \omega_o = \frac{2\pi}{T_o}$$

$$x_N(t) \triangleq \sum_{k=-N}^{N} a_k e^{jk\omega_o t}$$

Symmetric square wave: $a_k = a_{-k}$

$$x_N(t) = \frac{1}{2} + \sum_{k=1}^{N} 2a_k \cos k\omega_o t$$



TRANSPARENCY 7.7
Fourier series coefficients for a symmetric periodic square wave.

Matlab simulation!

شرایط دیریکله (دیریشله ـ Dirichlet)

قضیه فوریه

فرض کنید تابع $f(t)$ دارای شرایط زیر موسوم به شرایط دیریشله باشد:

۱. تابع $f(t)$ متناوب با دوره تناوب $2p$ باشد.

۲. تابع $f(t)$ در هر دوره تناوب ، تعریف‌شده باشد.

۳. تعداد نقاط اکستریمم تابع $f(t)$ در هر دوره تناوب محدود باشد.

۴. تعداد نقاط ناپیوستگی تابع $f(t)$ در هر دوره تناوب محدود باشد.

TRANSPARENCY 7.9
Partial sum incorporating $(2N + 1)$ terms in the Fourier series. [The analysis equation should read $a_k = 1/T\int_{T_0} x(t)e^{-jk\omega_0 t}dt$.]

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} \end{aligned}}$$

synthesis

analysis

$$x_N(t) \triangleq \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$e_N(t) \triangleq x(t) - x_N(t)$$

Does $e_N(t)$ decrease as N increases?

TRANSPARENCY 7.10
Conditions for convergence of the Fourier series.

CONVERGENCE OF FOURIER SERIES

- $x(t)$ square integrable:

$$\text{if} \int_{T_0} |x(t)|^2 \, dt < \infty$$

$$\text{then} \int_{T_0} |e_N(t)|^2 \, dt \to 0 \quad \text{as} \quad N \to \infty$$

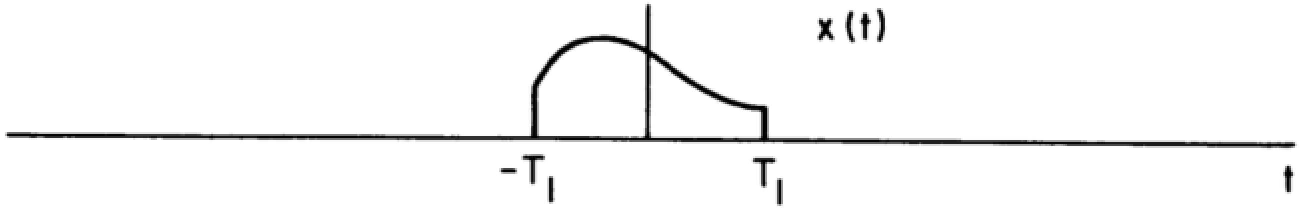
- Dirichlet conditions.

$$\text{if} \int_{T_0} |x(t)| \, dt < \infty \text{ and } x(t) \text{ ``well behaved''}$$

$$\text{then } e_N(t) \to 0 \quad \text{as} \quad N \to \infty$$

except at discontinuities

FOURIER REPRESENTATION OF APERIODIC SIGNALS



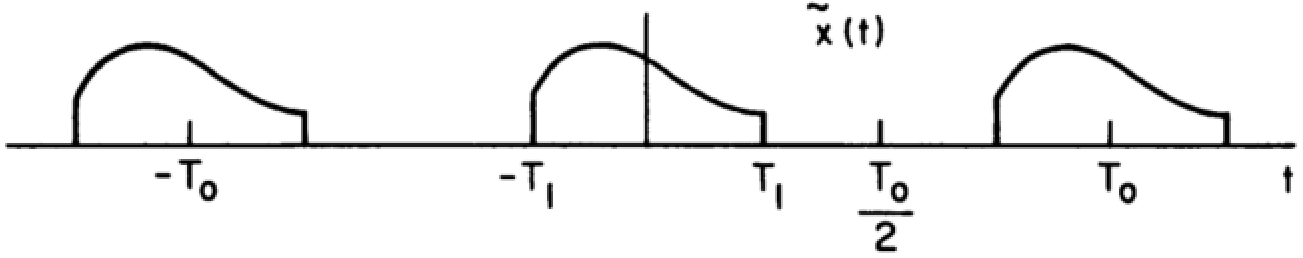
$$\tilde{x}(t) = x(t) \qquad |t| < \frac{T_0}{2}$$

$$\text{As } T_0 \to \infty \qquad \tilde{x}(t) \to x(t)$$

- use Fourier series to represent $\tilde{x}(t)$
- let $T_0 \to \infty$ to represent $x(t)$

TRANSPARENCY 7.11
An aperiodic signal to be represented as a linear combination of complex exponentials.

FOURIER REPRESENTATION OF APERIODIC SIGNALS



$$\tilde{x}(t) = x(t) \qquad |t| < \frac{T_0}{2}$$

$$\text{As } T_0 \rightarrow \infty \qquad \tilde{x}(t) \rightarrow x(t)$$

- use Fourier series to represent $\tilde{x}(t)$
- let $T_0 \rightarrow \infty$ to represent $x(t)$

TRANSPARENCY 7.12
Representation of an aperiodic signal as the limiting form of a periodic signal with the period increasing to infinity.

خواص سری فوریه پیوسته زمان متناوب

خطی بودن

$$x(t) \xrightarrow{f_s} a_k \quad ; \quad x(t) = x(t+T_0)$$

$$y(t) \xrightarrow{f_s} b_k \quad ; \quad y(t) = y(t+T_0)$$

$$z(t) = Ax(t) + By(t) \xrightarrow{f_s} z(t) = Aa_k + Bb_k \quad ; \quad z(t) = z(t+T_0)$$

انتقال زمانی

$$x(t) \xrightarrow{f_s} a_k \quad ; \quad x(t) = x(t+T_0)$$

$$y(t) = x(t-t_0) \xrightarrow{f_s} b_k = a_k\, e^{-jk(\frac{2\pi}{T_0})t_0} \quad ; \quad y(t) = y(t+T_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{T_0})t} \Rightarrow x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{T_0})(t-t_0)} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T_0})t}\; e^{-jk(\frac{2\pi}{T_0})t_0}$$

وارون سازی زمانی

$$x(t) \xrightarrow{f_s} a_k \quad ; \quad x(t) = x(t+T_0)$$

$$y(t) = x(-t) \xrightarrow{f_s} b_k = a_{-k} \quad ; \quad y(t) = y(t+T_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{T_0})t} \Rightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k\, e^{-jk(\frac{2\pi}{T_0})t} = \sum_{k=-\infty}^{\infty} a_{-k}\, e^{jk(\frac{2\pi}{T_0})t}$$

الف) $x(t) = x(-t) \Rightarrow a_k = a_{-k}$

ب) $x(t) = -x(-t) \Rightarrow a_k = -a_{-k}$

خواص سری فوریه پیوسته زمان متناوب

تغییر مقیاس زمانی

$$x(t) \xrightarrow{f_s} a_k \;; \quad x(t) = x(t+T_0)$$

$$y(t) = x(at) \xrightarrow{f_s} a_k \;; \quad y(t) = y(t+\frac{T_0}{a})$$

اثبات

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{T_0})t} \Rightarrow x(at) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{T_0})at} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{\frac{T_0}{a}})t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jka\omega_0 t}$$

مزدوج گیری

$$x(t) \xrightarrow{f_s} a_k$$

$$x^*(t) \xrightarrow{f_s} a_{-k}^*$$

اثبات

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\frac{2\pi}{T_0})t} \Rightarrow x^*(t) = \sum_{k=-\infty}^{\infty} a_k^*\, e^{-jk(\frac{2\pi}{T_0})at} = \sum_{k=-\infty}^{\infty} a_{-k}^*\, e^{+jk(\frac{2\pi}{T_0})at}$$

خواص سری فوریه پیوسته زمان متناوب

1) اگر سیگنال حقیق باشد چون $x(t) = x^*(t)$ پس $a_k = a_{-k}^*$

2) اگر سیگنال حقیق و زوج باشد $a_k = a_{-k}$ و $a_k = a_{-k}^*$ آن گاه ضرایب سری فوریه مطلقا حقیقی و زوج هستند.

3) اگر سیگنال حقیق و فرد باشد $a_k = a_{-k}^*$ و $a_k = -a_{-k}$ آن گاه ضرایب سری فوریه مطلقا موهومی و فرد هستند.

رابطه پارسوال

$$\frac{1}{T_0}\int_{T_0}|x(t)|^2 dt = \sum_{k=-\infty}^{\infty}|a_k|^2$$

- $|a_k|^2$ نشان دهنده قدرت سیگنال در هارمونی k ام است

$$< x(t), x(t) >= \frac{1}{T_0}\int_{T_0} x(t)\cdot x^*(t)dt$$