Blockchains

Project Report
-M2 Probabilités & Finance-

Pair-trading strategies: From correlation and co-integration to a stochastic control model

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Contents

1 Introduction

In this project, we focused on building trading strategies based on correlations, co-integration as well as causality relationships. As we all know, building a trading strategy is something difficult especially if we want to develop an accurate and efficient one. This accuracy and efficiency comes from including in the strategy different constrains related to the market and to the exchange we are focusing one, as well as to the nature of the traded security (Equity, Commodity, Index,...). This constrains can be explicit ones such as transaction cost, execution cost, currency exchange cost,... or more implicit ones (and very difficult to model and to take into consideration) like the market impact, the slippage, the optimal order placement, optimal execution (both at the order book level),... Not to mention the fact that building a trading strategy should go above and beyond the simple backtesting approach and needs to be based on accurate and very rare and hidden signals.

For the purpose of simplicity, we suggest simple and improvable trading strategies on crypto-assets. We will mainly focus on the spot market. The first part of this report focuses on a Pair-trading strategy, which is a well known and classic trading strategy based on the correlation and the co-integration between two assets. In addition to the main idea of this trading strategy, we will go beyond the notion of correlation to focus on causality between assets's price time series. The main idea is to be able to have a more accurate choice of asset pairs, and hopefully a better performance, taking into account not only the correlation but also the causality.

The second part will focus on a stochastic control approach to pair-trading strategies and the goal is to have the optimal strategy that optimizes a utility/cost function.

2 Pair Trading strategy

Pairs trading is a market-neutral strategy where we use statistical techniques to identify two stocks that are historically highly correlated with each other. When there is a deviation in the price relationship of these stocks, we expect this to be mean reverting and buy the underperforming stock and simultaneously sell the outperforming one. If our mean-reversion assumption is valid then prices should converge to long term average and trade should benefit. However, if the price divergence is not temporary and it is due to structural reasons then there is a high risk of losing the money.
The risk to this type of pairs trade, however, is that sometimes the correlation relationship may be altered by outside forces and this makes the initial trading strategy inefficient.
The overall approach for building the strategy will be:

1. Have a stock Universe and identify Cointegrated Pairs of Stocks.
2. Perform Stationary test for the Selected Pair.
3. Generate Trading Signals using z-score.
4. Portfolio Profit and Loss Calculation.

In this section, we will work with uniformly sampled crypto-assets price data at the scale of one minute. We will also mainly focus on the Open price and work exclusively with crypto pairs with USDT stablecoin as a Quote currency.
We will use data (provided by **Binance**) from $2020-02-18$ to $2022-02-12$. Most of it will be used to backtest the strategy.

2.1 Correlation, co-integration and causality

In what follows we will focus on the pairs "BTC-USDT", "ETH-USDT", "LTC-USDT", "NEO-USDT", "BNB-USDT", "XRP-USDT", "EOS-USDT", "TRX-USDT", "ETC-USDT", "XLM-USDT".
All these pairs will represent our stock universe.
The first step will be to check the correlation between the different assets. We will mainly use Pearson correlation coefficient. The correlations are summurized in the bellow correlation matrix,

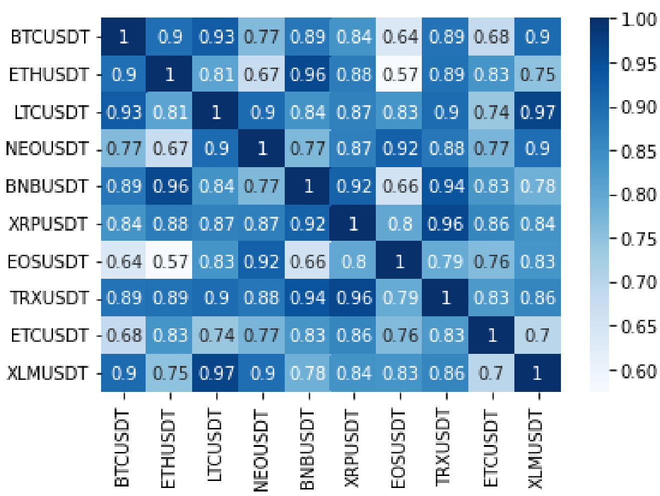


Figure 1: Pearson Correlation Matrix

By looking at the figure (2), we can clearly notice that the different assets are highly correlated, so they represent a good base for the pair co-integration relationship search that will be carried away next. In fact, Co-integration is a statistical property of two or more time-series variables which indicates if a linear combination of the variables is stationary. The statistical test used for the Co-intergration is the Engle-Granger two-step method (non co-integration as a null hypothesis).

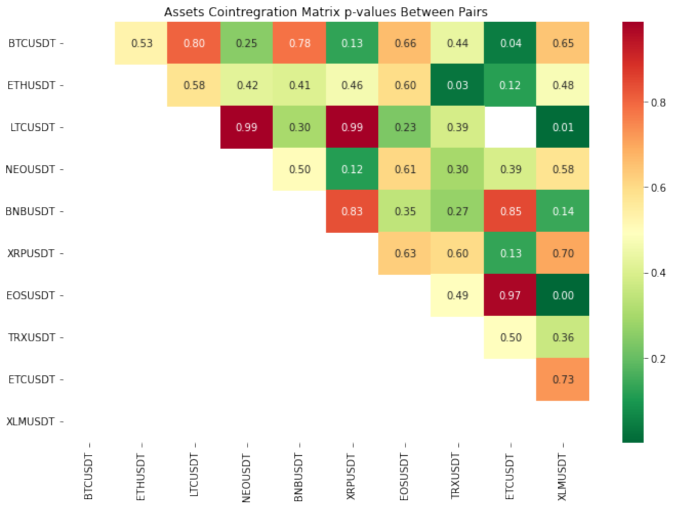


Figure 2: Co-intergration Test Matrix

From the previous figure, we can see that we have very few co-integrated pairs. In fact, if we fix a threshold of 0.05 for the statistical test, we can see that the pairs (BTC-USDT / ETC-USDT), (ETH-USDT / TRX-USDT), (LTC-USDT / XLM-USDT) and (EOS-USDT / XLM-USDT) are co-integrated.
Now that we have selected the pairs that we are going to work with, most pair trading strategies will stop here (concerning the statistical analysis) and go straight to building the trading signals. But we have decided to push a bit further the analysis in order to choose "the optimal trading pair", not only in terms of correlation but also in terms of causality. The goal is to be able to find pairs with valuable relationships and behaviors that are fundamentally linked, and causality between two time series (and thus between the underlying phenomenons) is the best thing that we can try to detect and use.
The notion of causality will require some definitions as well as a study framework to be fixed.
About causality, one can find a lot of articles with different definitions and different approaches to infere and statistically study the causality relationships that can be linear or non-linear. But in our case we will focus on one of the main and most common definition of causality known as the Granger causality.

Definition 1 Let $(X_t)_{t\in\mathbb{Z}}$, $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ be three random processes defined on the same probability space (of course one may consider more than three processes). We define A^t such that $A^t = (X_t; X_{t-1}; \cdots; X_0; Y_t; Y_{t-1}; \cdots; Y_0; Z_t; Z_{t-1}; \cdots; Z_0)$ and let $B^t \subset A^t$ be a subset of A^t. Let $P(Y_t|B^t)$ be the best unbiased predictor of Y_t, based on the information set B^t. We introduce $\epsilon(Y_t|B^t) = Y_t - P(Y_t|B^t)$, the residuals of the prediction. We have $\sigma^2(Y_t|B^t) = \mathbb{E}(\epsilon(Y_t|B^t)^2)$.
We say that $(X_t)_{t\in\mathbb{Z}}$ causes $(Y_t)_{t\in\mathbb{Z}}$ if and only if $\sigma^2(Y_t|B^t) < \sigma^2(Y_t|B^t \setminus X^t)$ for at least one $t \in \mathbb{Z}$.

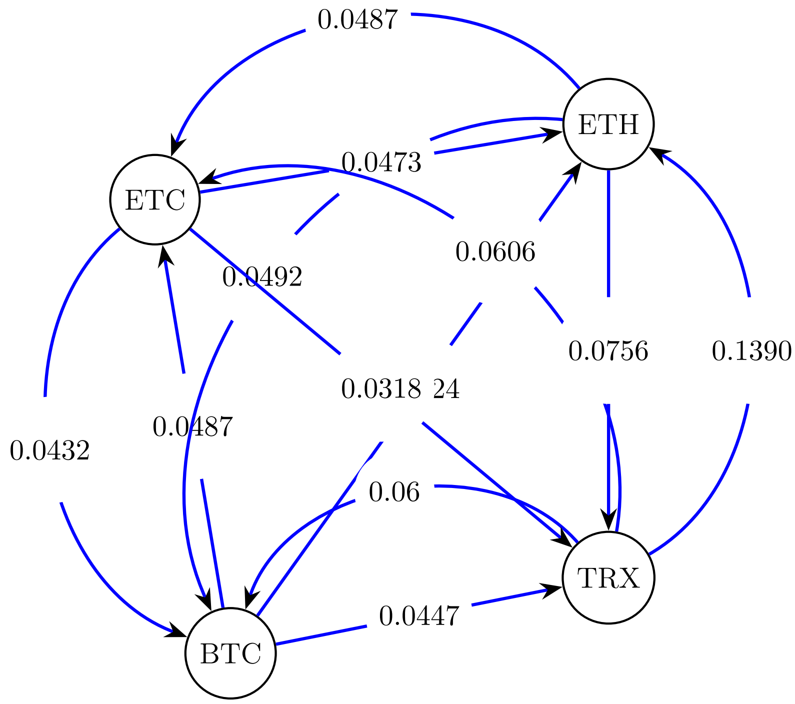
From this definition, we can understand that the granger causality is highly related to the predictive power of the history of a random process (time serie) on another one, and thus the amount of information held by the former.
Characterization of the linear Granger causality:
The linear Granger causality is based on the Vector Autoregressive Models. First, let us consider three random processes $(X_t)_{t\in\mathbb{Z}}$, $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ (one may consider more than three processes). The goal is to identify the causal relationship and its intensity that goes from $(X_t)_{t\in\mathbb{Z}}$ to $(Y_t)_{t\in\mathbb{Z}}$. More precisely, we will build two different regression models of $(Y_t)_{t\in\mathbb{Z}}$.
The first autoregressive model (we will call it the restricted model), will use the time history of $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$. Let $(\epsilon_{1,t})_{t\geq 0}$ be the residual process of this model. The second autoregressive model (we will call it the non-restricted model) will include time history of all the available processes, namely $(X_t)_{t\in\mathbb{Z}}$, $(Y_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ and let $(\epsilon_{2,t})_{t\geq 0}$ be the residual process of this model.
We then define the Granger causality index of the causal relationship from $(X_t)_{t\in\mathbb{Z}}$ to $(Y_t)_{t\in\mathbb{Z}}$ by:

$$F_{(X_t)_{t\in\mathbb{Z}} \Rightarrow (Y_t)_{t\in\mathbb{Z}}} = log(\frac{var((\epsilon_{1,t})_{t\geq 0})}{var((\epsilon_{2,t})_{t\geq 0})})$$

This index allows us to quantify the causal relationships between two time series and the closer the value of this index to 0, the weaker the causal relationship is.

Now, we will use this Granger causality to build what we will call causal graphs to have a more accurate idea about the relationship between the different time series (assets). In this report, we will not go into the details of the times series analysis, the VAR models constructions and implementation, but we will directly give the final results.
The following data table summarizes the different Granger causality intensities:

	BTC	ETC	ETH	TRX	LTC	XLM	EOS
BTC	–	0.0487	0.0424	0.0447	0.0218	0.06	0.0512
ETC	0.0432	–	0.0473	0.0318	0.0416	0.0101	0.0478
ETH	0.0756	0.1390	–	0.0487	0.0606	0.0432	0.0441
TRX	0.0492	0.0756	0.0317	–	0.0432	0.05	0.0701
LTC	0.0333	0.05561	0.03318	0.124	–	0.0488	0.0396
XLM	0.0821	0.035	0.0477	0.0416	0.0380	–	0.0310
EOS	0.05616	0.068	0.0332	0.0416	0.05	0.0420	–



Having all the Granger causality intensities between the crypto pairs, we can choose the co-integrated pairs with the highest causality intensity (at least in one direction); in our case the different co-integrated trading pairs have the same level of Granger causality intensity.

We can also notice that some non co-integrated trading pairs have a bigger Granger causality index than the co-integrated ones. This emphasizes the importance of finding a trade-off between the correlation, the co-integration, the causality and the behaviour of the spread (next subsection).

In order to have more accurate causality quantification, one can focus on the non linear causality inference using kernel methods [8] or using information theory [9] that requires a lot more computations.

2.2 Stationary test for the Selected Pair

Now that we have a candidate of a pair for the strategy, selecting the right pair is of the utmost importance as the strategy will not work well if the prices are moving exactly together. They need to be diverging and mean-reverting for our strategy to be profitable. In what follows, we will focus on the co-integrated pairs determined before and further test the stationarity of the spreads using the Augmented Dickey-Fuller test. It is important that the spread is stationary. A time series is considered stationary if parameters such as mean and variance do not change over time and there is no unit root. We will first calculate the hedge ratio between each couple of tickers using OLS regression. Then, using the hedge ratio, we will calculate the spread and run the Augmented Dickey-Fuller test.

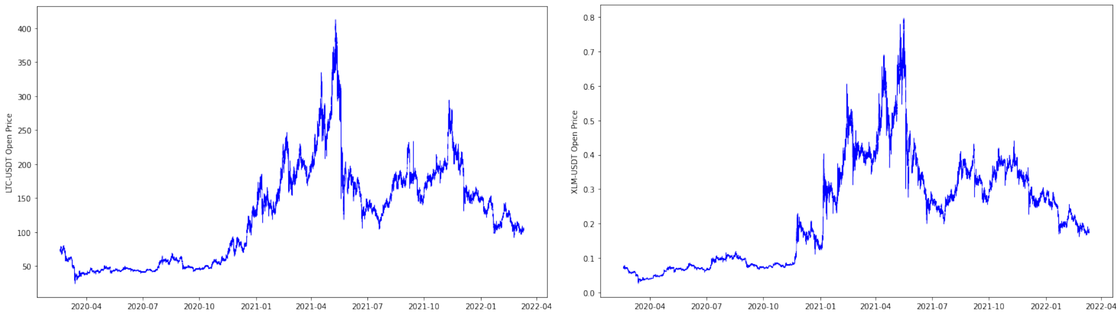


Figure 3

The spread that we will use to generate the trading signals is defined by:

$$Spread_t = X_t - \beta Y_t$$

where β is the hedging ratio. This ratio is computed based on a OLS linear regression of $(X_t)_{t\in\mathbb{Z}}$ on $(Y_t)_{t\in\mathbb{Z}}$ and it is equal to the regression coefficient.
Now, the most important condition is that we need to have a stationary highly mean-reverting spread, which a condition not easy to fulfill. This makes the search for optimal trading pairs even more difficult.
The spread of the co-integrated trading pairs determined previously is shown in figure (4),

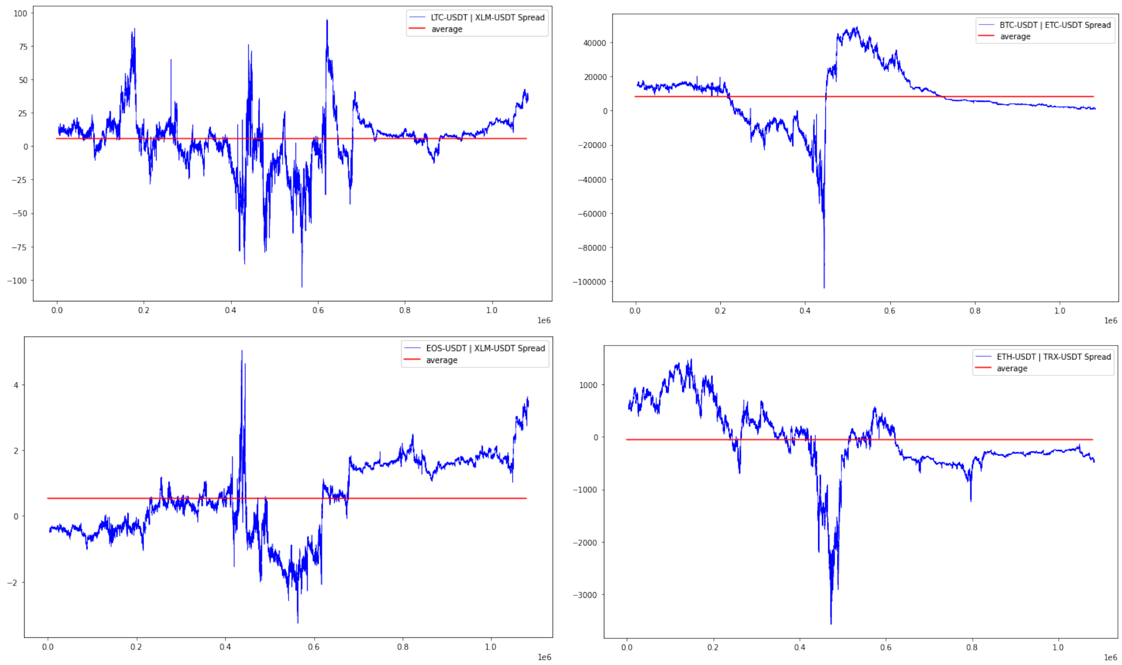


Figure 4

Looking at the previous figure, we can clearly see that the LTC-USDT / XLM-USDT is the only trading pair with mean reverting spread ($\beta = 512.7541$). So it is the only pair that we will be working with.
The next crucial step is to check the stationarity of the spread. Using the ADF test, we end up with a critical value of -7.931 which is less than the value at 1% (-3.430) significance level. Hence, we are able to reject the null hypothesis that spread has a unit root and can conclude that it is stationarity in nature.
Having already established that the spread is mean reverting, we now need to identify the extreme points or threshold levels which when crossed by this signal, we trigger trading orders for pairs trading. To be able to identify these threshold levels, a statistical construct called z-score is widely used in Pairs Trading.

2.3 Generate Trading Signals using z-score

In the previous section, we used the data from $2020-02-18$ to $2022-02-12$ (train set) to do the statistical analysis. from now on, we will focus on the data from $2020-02-19$ to $2022-04-05$ (test set) to ensure trading signal generation.
We will use the z-score of the ratio between the two stock prices to generate trading signals and set the upper and lower thresholds. This will tell us how far a price is from the population mean value. If it is positive and the value is above the upper thresholds then the stock price is higher than the average price value. Therefore, its price is expected to go down hence we want to short (sell) this stock and long (buy) the other one.In our strategy, the upper threshold is $C \times StandardDev(Z-score)$ and the lower one is $-C \times StandardDev(Z-score)$, with $C > 0$ a constant to be fixed.
About the Z-score (or standard score), it is the number of standard deviations by which the value of a raw score (i.e., an observed value or data point) is above or below the mean value of what is being observed or measured. Raw scores above the mean have positive standard scores, while those below the mean have negative standard scores.

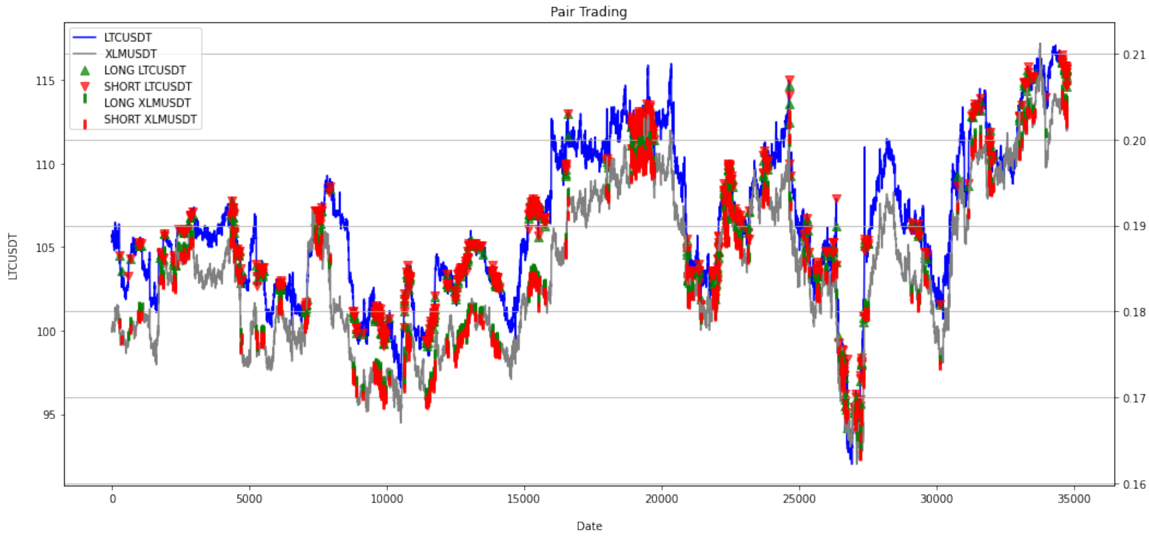


Figure 5

The figure (5) shows the different execution (buy/sell) points of the trading strategy on the historical data for $C = 1$.

2.4 Portfolio Profit and Loss Calculation

If we start with an initial capital of 200000 USDT [1], the evolution of the portfolio value is given in the bellow figure ($C = 1$),

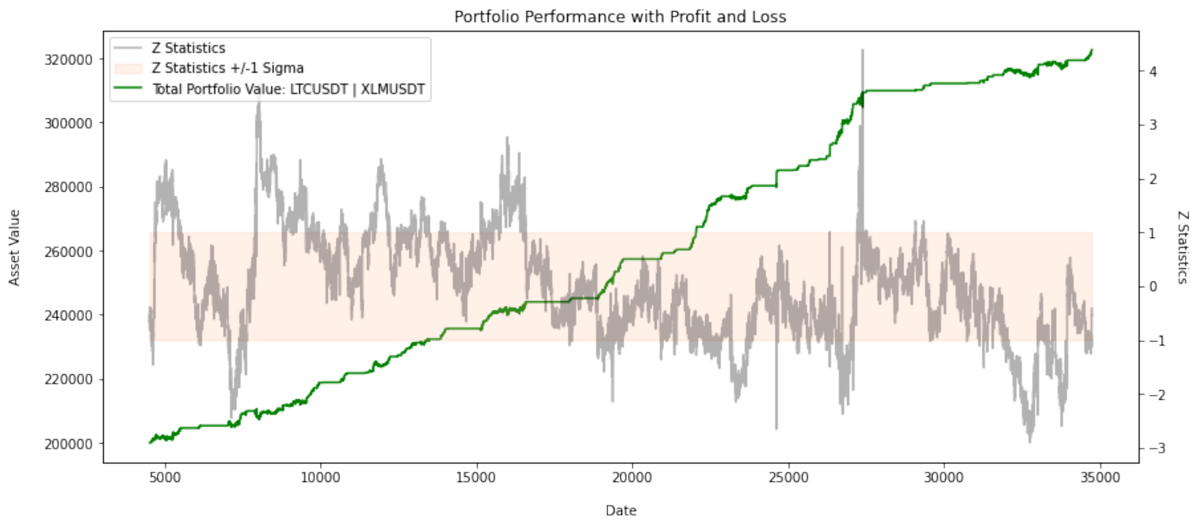


Figure 6

For the sake of comparison, we will also check the performance of the trading strategy on the 3 other co-integrated (but with a non mean reverting spread) trading pairs.

[1] This choice of capital is arbitrary. In fact, when we backtest the strategy with different values of the initial capital, we end up having the same behaviour of the portfolio value and its performance.

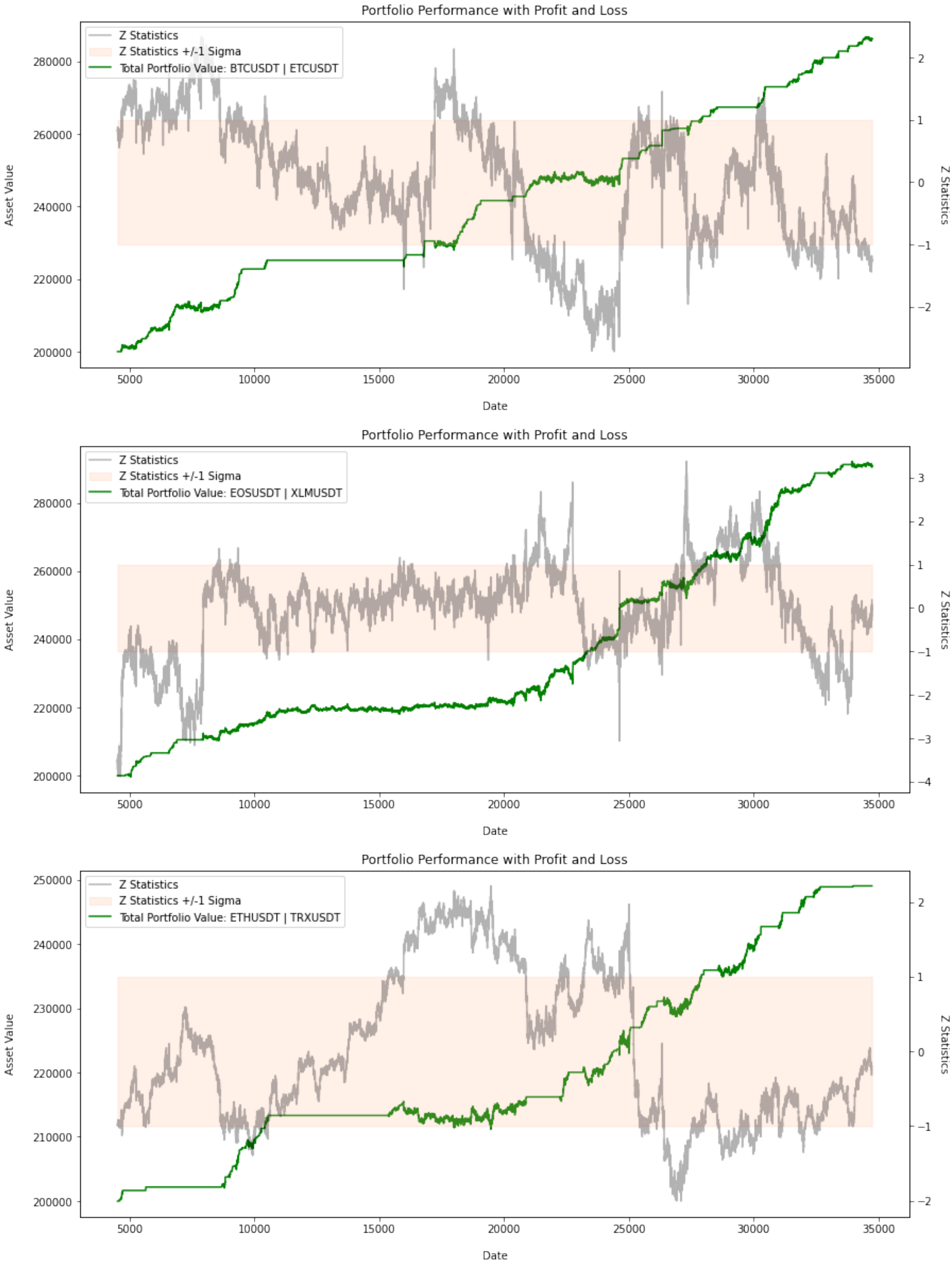

Portfolio Performance with Profit and Loss
Z Statistics
Z Statistics +/-1 Sigma
Total Portfolio Value: BTCUSDT | ETCUSDT
Asset Value
Z Statistics
Date
Portfolio Performance with Profit and Loss
Z Statistics
Z Statistics +/-1 Sigma
Total Portfolio Value: EOSUSDT | XLMUSDT
Asset Value
Z Statistics
Date
Portfolio Performance with Profit and Loss
Z Statistics
Z Statistics +/-1 Sigma
Total Portfolio Value: ETHUSDT | TRXUSDT
Asset Value
Z Statistics
Date

Figure 7

2.5 Comments

Looking at figure (6), we can see that the strategy performs well on the chosen time interval and under the strategy assumptions. At the end of the period, the portfolio value increased by 60% (in USDT). Moreover, if we compare the performance of the trading strategy on $LTC/USDT$ - $XLM/USDT$ (the chosen trading pair) and the one on the other co-integrated trading pairs, we can see that the $LTC/USDT$ - $XLM/USDT$ trading pair performed better than the other ones. But the three other trading pairs still performed well even though their respective spreads were not mean reverting and stationary.
As mentioned in the introduction, we should not rely on these performances because the trading strategy do not include a lot of parameters (trading cost, market impact, order splitting,$\cdots$) that are very important and may change completely the performance of the strategy.

Before going to the next section, we will check the effect of the parameter C on the performance of the strategy on the trading pair LTC/USDT XLM/USDT.

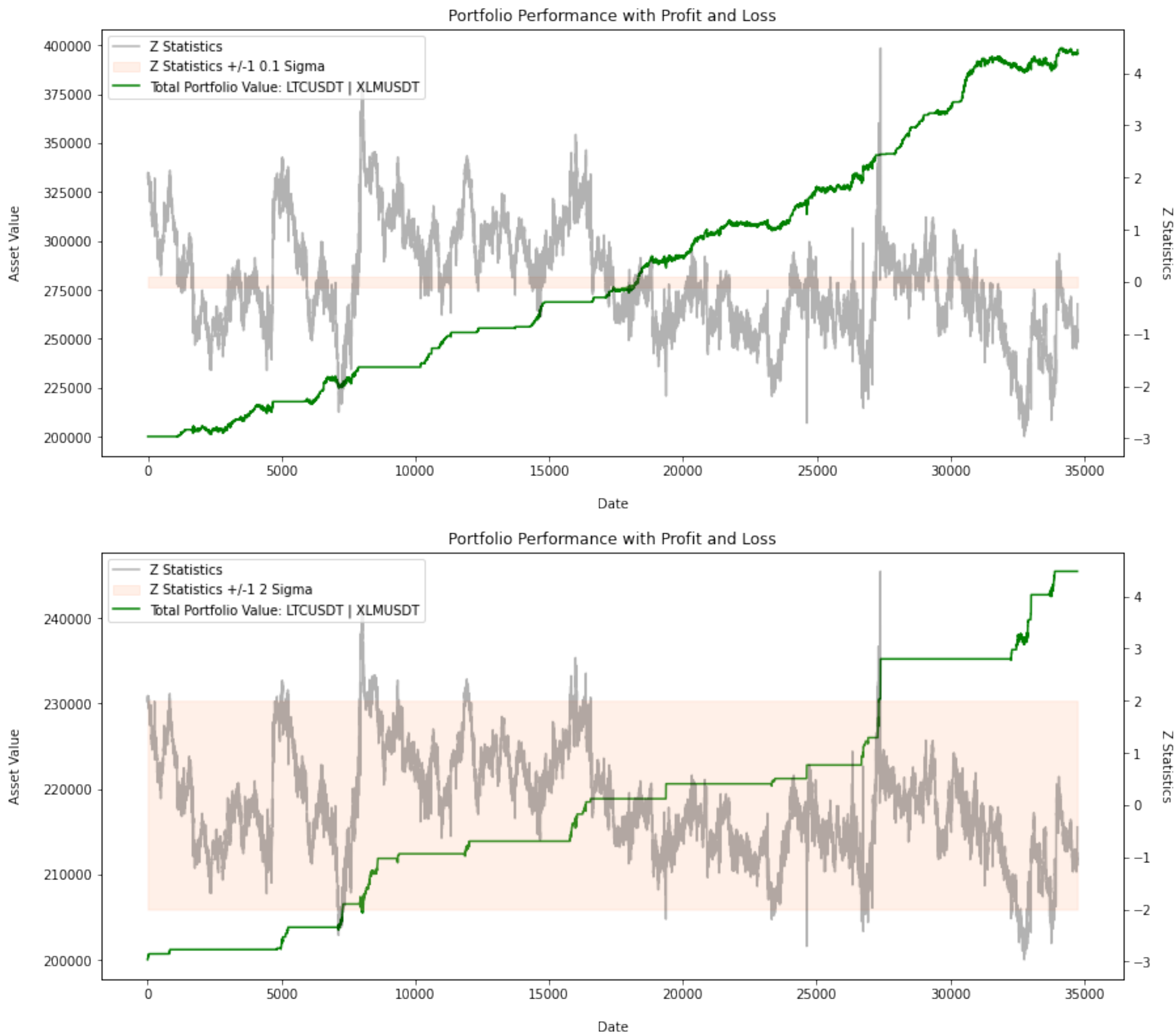


Figure 8

Looking at figure (8), we can see that the strategy performs better for C small ($C = 0.1$). In fact, the smaller C is, the tighter the z-score zone and the more frequent threshold crosses are. This implies more executions (buy/sell). In our case, and on the test historical data, this turned out to be working well.
Now for C big ($C = 2$), the strategy seems to perform well but not as good as in the previous case. This is mainly due to the fact that for such a value of C, the z-score zone is big and thus we have less executions (which explains the constant value of the portfolio on some time intervals) and then less opportunity to generate profit.
We can then conclude that the parameter C is of great importance and needs to be chosen wisely and adapted to the data.

3 A Stochastic Control approach to Pair Trading

In the previous section, we focused on a statistical approach for pair trading (the most traditional one) based on the co-integration relationships as well as the stationarity and mean reversion of the spread between two assets price.
In this section, we will focus on a stochastic control framework in order to derive the optimal trading strategy. the main difference between the two methods is that the latter is not based on a "trading signal", but more on a "utility function optimization".

3.1 Problem Formulation

This section presents the basic asset price models and formulates a pairs trading stochastic control problem. We begin by describing the asset, spread, and wealth dynamics.

3.1.1 Asset, Spread and Wealth Dynamics

Let $(M_t)_{t\geq 0}$ be a risk free rate of r compounded continuously.

$$dM_t = rM_t dt \tag{1}$$

Let $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ denote respectively the prices of the traded pair of stocks A and B at time t. For simplicity, we assume that B follows a geometric Brownian motion

$$dB_t = \mu B_t dt + \sigma B_t dW_t^{(1)} \tag{2}$$

where μ is the drift, σ is the volatility, and $W^{(1)}$ a standard Brownian motion.
Let $(X_t)_{t\geq 0}$ denote the spread of the two stocks at time t; defined as

$$X_t = log(A_t) - log(B_t) \tag{3}$$

We assume that the spread follows an Ornstein-Uhlenbeck process

$$dX_t = k(\theta - X_t)dt + \nu dW_t^{(2)} \tag{4}$$

where $k(\theta - X_t)$ is the drift term that represents the expected instantaneous change in the spread at time t, and θ is the long-term equilibrium level to which the spread reverts. The rate of reversion is represented by the parameter $k \geq 0$ (here again, we see that the mean reversion behaviour is very important, thus the choice of an OU process). ν is the volatility of the spread and $W^{(2)}$ a standard Brownian motion such that $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$, where ρ is the correlation coefficient between the two brownian motions.

A straight forward application of the Itô's lemma, and the use of the dynamics of the asset B and the dynamics of the spread X, the dynamics of the asset A is given by

$$dA_t = (\mu + k(\theta - X_t) + \frac{1}{2}\nu^2 + \rho\sigma\nu)A_t dt + \sigma A_t dW_t^{(1)} + \nu A_t dW_t^{(2)}, \tag{5}$$

Moreover, let $(V_t)_{t\geq 0}$ be the value of a self-financing pairs-trading portfolio and let $(h_t)_{t\geq 0}$ and $(\hat{h}_t)_{t\geq 0}$. denote respectively the portfolio weights for stocks A and B at time t. We will consider a delta-neutral strategy that that always long shares of a stock and short shares of the other stock with equal amount of shares. This delta-neutral strategy was introduced in [2] and [3]. Thus we require that

$$h_t = -\hat{h}_t \tag{6}$$

The wealth dynamics is then

$$dV_t = V_t(h_t \frac{dA_t}{A_t} + \hat{h}_t \frac{dB_t}{B_t} + \frac{dM_t}{M_t}). \tag{7}$$

We can rewrite (7) as

$$dV_t = V_t[(h_t(k(\theta - X_t) + \frac{1}{2}\nu^2 + \rho\sigma\nu) + r)dt + \nu dW_t^{(2)}] \tag{8}$$

3.1.2 Formulation as a stochastic Control Problem

For the optimization problem, we assume that an investor's preference can be represented by the utility function $U(x) = \frac{1}{x}x^\gamma$, with $x \geq 0$ and $\gamma < 1$.Thus the optimization problem is given a maximization of the utility function at a maturity time T,

$$\underset{(h_t)_{t\geq 0}}{sup} \ \mathbb{E}[\frac{1}{\gamma}V_T^\gamma]$$

subject to: $V_0 = v_0$, $X_0 = x_0$.
The supremum is taken over the strategies $(h_t)_{t\geq 0}$ that are adapted to the filtration generated by $W^{(1)}$ and $W^{(2)}$.

3.2 Analytical solution

Let G(t,v,x) (depends on the time parameter, on the wealth and on the portfolio value) denote the value function. Then the Hamilton-Jacobi-Bellman (HJB) equation corresponding to our stochastic control problem is given by

$$\partial_t G + \underset{h}{sup}[\frac{1}{2}(h^2\nu^2 v^2 \partial_{vv}G + \nu^2\partial_{xx}G + 2h\nu^2 v\partial_{vx}G) + (hk(\theta - x) + \frac{1}{2}h\nu^2 + h\rho\nu\sigma + r)v\partial_v G - k(x-\theta)\partial_x G] = 0, \tag{9}$$

with the terminal condition $G(T, v, x) = v^\gamma$.
If we assume that $\partial_{vv}G < 0$,the term in *sup* in (9) reaches its maximum in h^* given by,

$$h^* = -\frac{\nu^2\partial_{vx}G + b\partial_v G}{\nu^2 v\partial_{vv}G} \tag{10}$$

Plugging (10) back into (9) yields

$$\nu^2\partial_t G\partial_{vv}G - \frac{1}{2}\nu^4\partial_{vx}G^2 - \frac{1}{2}b^2\partial_v G^2 - b\nu^2\partial_v G\partial_{vx}G + \frac{1}{2}\nu^4\partial_{vv}G\partial_{xx}G + r\nu^2\partial_v G\partial_{vv}G - k(x-\theta)\nu^2\partial_x G\partial_{vv}G = 0. \tag{11}$$

The optimal strategy is deduced from the solution of the previous equation.

3.2.1 Closed form Solution

In order to solve the HJB equation, we will look for solution G of the form

$$G(t,v,x) = f(t,x)v^{\gamma}, \tag{12}$$

with the condition that

$$f(T,x) = 1, \forall x. \tag{13}$$

Then, using the new expression of G, (11) becomes

$$\begin{aligned}(\gamma-1)\nu^2 f\partial_t f - \frac{1}{2}\gamma\nu^4\partial_x f^2 - \frac{1}{2}\gamma b^2 f^2 - \frac{1}{2}\gamma\nu^4 f\partial_x f - \\ \gamma\rho\sigma\nu^3 f\partial_x f + \frac{1}{2}(\gamma-1)\nu^4 f\partial_{xx} f + \gamma(\gamma-1)r\nu^2 f^2 + k(x-\theta)\nu^2 f\partial_x f = 0.\end{aligned} \tag{14}$$

We will then write the function f like

$$f(t,x) = g(t)e^{x\beta(t)+x^2\alpha(t)}, \tag{15}$$

with $g(T) = 1$, $\beta(T) = 0$, $\alpha(T) = 0$.

Until now, we have made assumptions on the shape of the value function which may seem very restrictive. But we should keep in mind that in order to solve analytically this kind of HJB equation, making assumption/ansatz for the value function is necessary and not relying on these assumptions means that we should adopt a numerical approach (finite difference, finite element) to solve the equation.

The rest of the computation is very classic. In fact, by plugging the final form of the value function into the HJB equation and using the fact that two equal polynoms have the same coefficients, we end up having the following equations for α, β and g,

$$[(\gamma-1)]\nu^2\alpha'^4\alpha^2 + [2k\nu^2]\alpha + [-\frac{1}{2}\gamma k^2] = 0 \tag{16}$$

$$[[(\gamma-1)\nu^2]\beta'^2 - 2\nu^4\alpha]\beta + [-\gamma\nu^4\alpha - 2\gamma\rho\sigma\nu^3\alpha - 2k\theta\nu^2\alpha + \gamma k^2\theta + \frac{1}{2}\gamma k\nu^2 + \gamma k\rho\nu\sigma] = 0 \tag{17}$$

$$\begin{aligned}(\gamma-1)\nu^2 g' + [-\frac{1}{2}\gamma\nu^4\beta - \gamma\rho\sigma\nu^3\beta - \frac{1}{2}\nu^4\beta^2 + (\gamma-1)\nu^4\alpha \\ +\gamma(\gamma-1)r\nu^2 - k\theta\nu^2\beta - \frac{1}{2}\gamma k^2\theta^2 - \frac{1}{2}\gamma k\theta\nu^2 - \gamma k\theta\rho\sigma\nu \\ -\frac{1}{8}\gamma\nu^4 - \frac{1}{2}\gamma\rho\sigma\nu^3 - \frac{1}{2}\gamma\rho^2\sigma^2\nu^2]g = 0\end{aligned} \tag{18}$$

By solving the previous equations, we obtain the solutions in closed form as:

$$\alpha(t) = \frac{k(1-\sqrt{1-\gamma})}{2\nu^2}[1 + \frac{2\sqrt{1-\gamma}}{1-\sqrt{1-\gamma}-(1+\sqrt{1-\gamma})exp(\frac{2k(T-t)}{\sqrt{1-\gamma}})}], \tag{19}$$

$$\begin{aligned}\beta(t) = \frac{1}{2\nu^2[(1-\sqrt{1-\gamma})-(1+\sqrt{1-\gamma})exp(\frac{2k(T-t)}{\sqrt{1-\gamma}})]}. \\ [\gamma\sqrt{1-\gamma}(\nu^2+2\rho\sigma\nu)[1-exp(\frac{2k(T-t)}{\sqrt{1-\gamma}})]^2 - \gamma(\nu^2+2\rho\sigma\nu+2k\theta)[1-exp(\frac{2k(T-t)}{\sqrt{1-\gamma}})]],\end{aligned} \tag{20}$$

and

$$g(t) = exp(\frac{-\int_t^T u(s)ds}{(1-\gamma)\nu^2}) \tag{21}$$

where u is given by,

$$\begin{aligned} u(t) = &-\frac{1}{2}\gamma\nu^4\beta(t) - \gamma\rho\sigma\nu^3\beta(t) - \frac{1}{2}\nu^4\beta^2(t) \\ &+ (\gamma-1)\nu^4\alpha(t) + \gamma(\gamma-1)r\nu^2 - k\theta\nu^2\beta(t) \\ &- \frac{1}{2}\gamma k^2\theta^2 - \frac{1}{2}\gamma k\theta\nu^2 - \gamma k\theta\rho\sigma\nu - \frac{1}{8}\gamma\nu^4 \\ &- \frac{1}{2}\gamma\rho\sigma\nu^3 - \frac{1}{2}\gamma\rho^2\sigma^2\nu^2. \end{aligned} \tag{22}$$

Then, using the previous closed formulas, the optimal strategy (weight) $h^*(t)$ is given by,

$$h^*(t,x) = \frac{1}{1-\gamma}[\beta(t) + 2x\alpha(t) - \frac{k(x-\theta)}{\nu^2} + \frac{\rho\sigma}{\nu} + \frac{1}{2}] \tag{23}$$

We must verify that the smooth candidate solution we derived in the previous section is indeed the value function of the stochastic control problem. This can be achieved by proving a verification result, which connects the HJB equation to the optimal control problem.
First, it is clear that the candidate solution is smooth enough ($C^{1,2}([0:T],(0,+\infty),\mathbb{R})$). Moreover, the optimal control h^* is well defined and is an admissible control. The only tricky assumption to check for the verification theorem is the unique existence of a solution to

$$dV_t = V_t[(h_t^*(k(\theta - X_t) + \frac{1}{2}\nu^2 + \rho\sigma\nu) + r)dt + \nu dW_t^{(2)}] \tag{24}$$

(the existence of a unique solution to (4) is very clear by simple application of Itô's lemma). The goal [2] is to show that the coefficients of the SDE (24) are Lipschitz and use the existence and uniqueness theorem of the solution of the SDE.

Now that we have the optimal control h^*, the only thing left is to estimate the different parameters and calibrate them on the historical data. The different estimators can be found in the appendix of [1].
The application of this stochastic control pair-trading strategy on real data will not be carried away in this report. This is mainly due to the fact that the whole framework of the stochastic control problem assumes a stationarity of the different random processes (Prices and Spread). This ensures a well definition of the the different parameters and their estimators on the trading period. But in our case, the prices' processes are far from being stationary. But the calibration can of course be done after de-trending the price time series and extracting the seasonality component.
Further research was conducted in order to have a more robust stochastic control models for pair-trading strategies that generalize the previous strategy to a non-delta neutral framework and include transaction costs [4], stop-loss exits [5], price jumps [6],... Or Optimal switching for the pairs trading rules [7].

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[2] We will not do the proof in this report and we will assume that the unique solution to (24) exists.

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