Take-home 01

General Instructions for take-home exams: Please submit your answers as one pdf document, where you describe the logic and procedure you used to solve the problem and where you plot your results. Attach your python or matlab or R scripts and indicate which code solves which problem. Annotate your codes properly. We will evaluate your answers and codes in terms of correctness and clarity. Compress all your relevant files into one compressed file and submit to tengtianyuan@pku.edu.cn

可以用中文写作业。文件/邮件命名格式请参照“编程语言—本科生/研究生—姓名—学号”。

例：“python—研究生—张三—1901111111”。

1.(4 points) Let $X_1, X_2, \ldots, X_N \sim Possion(\lambda)$. Let $\theta = e^{\lambda}$, whose maximum likelihood estimate (MLE) is $\hat{\theta} = e^{\bar{X}}$. Generate a data set (using $\lambda = 5$) consisting of 20 observations of X.

(1) Write down the derivation for the MLE $\hat{\theta} = e^{\bar{X}}$.

(2) Perform the parametric bootstrap (resampling for 10000 times) on the data set to infer the distribution of $\hat{\theta}$. Plot the histogram of $\hat{\theta}$ and report its 95% confidence interval.

(3) Perform the non-parametric bootstrap (resampling for 10000 times) on the same data set to infer the distribution of $\hat{\theta}$. Plot the histogram of $\hat{\theta}$ and report its 95% confidence interval.

(4) Simulate the true distribution of $\hat{\theta}$ (sampling for 10000 times). Plot the histogram of $\hat{\theta}$ and report its 95% confidence interval.

2. (5 points) Consider the following psychometric experiment (notations same as the in-class example), where a virtual participant is tested on 100 light intensity levels (1, 2, …, 100), one trial for each level.

(1) Generate the correctness of response(C_i, which is either 0 or 1) for the virtual participant whose probability of correct is modeled as $f(I) = \dfrac{1}{1 + e^{-\beta(x-\alpha)}}$. Let $\alpha = 45$ and $\beta = 0.1$. Plot the probability of correct against the stimulus intensity for the generative model. Visualize the simulated data properly.

(2) Suppose α and β are unknown (as in real experiments). Fit the model to the simulated data using MLE to estimate α and β . Report the estimates $\hat{\alpha}$ and $\hat{\beta}$. Plot the estimated psychometric function, $f(I) = \dfrac{1}{1 + e^{-\beta(x-\alpha)}}$.

(3) Use the parametric bootstrap (resampling for 1000 times) to estimate the 95% confidence intervals of $\hat{\alpha}$ and $\hat{\beta}$.

3. (6 points) This is a problem (adapted from Pitt, Myung, & Zhang, 2002, *Psych Rev*) for you to practice the model selection method of BIC and cross-validation. It is also an example showing model selection methods may mis-classify a dataset generated by Model A to Model B.

Two of the most influential models in psychophysics to describe the relationship of psychological magnitudes (e.g. brightness) to the corresponding physical magnitudes (e.g. light intensity) are Stevens' power law and Fechner's logarithmic law.

Stevens' model: $y = \alpha x^{\beta} + \epsilon$

Fechner's model: $y = \alpha \ln(\mathrm{x} + \beta) + \epsilon$

where $\epsilon \sim N(0, \sigma^2)$ is random error, $\alpha > 0, \beta > 0, \sigma > 0$ are free parameters.

① Suppose we run a virtual experiment with 10 trials: $x = 1,2,3,...,10$. One dataset of the experiment is thus defined by 10 data points of (x, y). Please use Stevens' model to generate 50 datasets assuming $\alpha = 2, \beta = 2, \sigma = 0.4$. Use Fechner's model to generate another 50 datasets assuming $\alpha = 2, \beta = 4, \sigma = 0.4$. For the 50 datasets generated from each model, compute the mean of y (denoted $\bar{y}$) for each . Plot $\bar{y}$ against x for the two models in the same figure.

② For each of the 100 datasets generated above, fit Stevens' model and Fechner's model to the dataset using MLE. Compare the BIC of the two models to choose the best model for the dataset. See how many percent of Stevens' datasets or Fechner's datasets are best fit by Stevens' model or by Fechner's model. Summarize your results in the table below.

	Mean –lnL			**Mean BIC**			**Percentage of best fit**	
	Stevens'	Fechner's		Stevens'	Fechner's		Stevens'	Fechner's
Data from Stevens'	?	?		?	?		?	?
Data from Fechner's	?	?		?	?		?	?

③ For each of the 100 datasets generated above, compare Stevens' model and Fechner's model using leave-one-out cross-validation. Define $CV = -\sum_{i} \ln f(y^{i}_{val} \mid \hat{\theta}_{cal})$, where $f(\,.\,)$ denotes likelihood function, $\hat{\theta}_{cal}$ denotes parameters estimated from the calibration set, y_{val} denotes data in the validation set, and the superscript i denotes the choice of the i-th data point as the validation set (the rest as the calibration set). Compare the *CV* of the two models to choose the best model for the dataset. Summarize your results of cross-validation in the table below.

	Mean CV				**Percentage of best fit**	
	Stevens'	Fechner's			Stevens'	Fechner's
Data from Stevens'	?	?			?	?
Data from Fechner's	?	?			?	?

④ Compare the mean **–lnL** in Question ② with the mean **CV** in Question ③. Which has a higher value? Explain why.