Non-asymptotic guarantees for individual eigenvector estimation

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1 Notation

In the following, we denote the eigenspace from eigendecomposition as $\boldsymbol{U}_K \in \mathbb{R}^{N\times K}$ and the second-largest eigenvector from eigendecomposition as $\boldsymbol{u}_2 \in \mathbb{R}^N$. Similarly, we denote the eigenspace from centralized orthogonal iteration as $\boldsymbol{Q}'_K \in \mathbb{R}^{N\times K}$ and the second-largest eigenvector from centralized orthogonal iterations as $\boldsymbol{q}'_2 \in \mathbb{R}^N$. Moreover, we denote the eigenspace from decentralized orthogonal iteration as $\boldsymbol{Q}_K \in \mathbb{R}^{N\times K}$ and the second-largest eigenvector from decentralized orthogonal iterations as $\boldsymbol{q}_2 \in \mathbb{R}^N$. The adjacency matrix is represented by $\boldsymbol{A} \in \mathbb{R}^{N\times N}$.

2 Proof

Our goal is to show the estimation error of individual eigenvectors are bounded such that:

$$\|\boldsymbol{u}_i - \boldsymbol{q}_i\|_2 \leq \epsilon, \forall i \tag{1}$$

where $\boldsymbol{u}_K$ represents the K_{th} eigenvector, $\boldsymbol{q}'_2$ represents the estimated one by the algorithm proposed in [1] and $\epsilon < 1$. As described in Theorem 3.1 in [1], we have

$$\|\boldsymbol{U}_K\boldsymbol{U}_K^H - \boldsymbol{Q}'_K\boldsymbol{Q}'^H_K\|_2 = \|\boldsymbol{V}^H\boldsymbol{Q}'_K\|_2 \leq \mathcal{O}(\left|\frac{\lambda_K + 1}{\lambda_K}\right|^T N) \leq \epsilon',$$

where $\boldsymbol{V} \in \mathbb{R}^{N\times(N-K)}$ spans the remaining eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{Q}'_K$ denotes the T_{th} output of centralized orthogonal iteration. Consequently,

$$\|\boldsymbol{V}^H\boldsymbol{Q}'_K\|_F \leq \sqrt{K}\|\boldsymbol{V}^H\boldsymbol{Q}'_K\|_2 \leq \sqrt{K}\epsilon'.$$

By combining this with $\|\boldsymbol{Q}'_K\|_F^2 = K$ and invoking the orthogonal matrix $\boldsymbol{Z} = [\boldsymbol{U}_K, \boldsymbol{V}] \in \mathbb{R}^{N\times N}$, we have

$$\|\boldsymbol{Q}'_K\|_F^2 = \|\boldsymbol{Z}^H\boldsymbol{Q}'_K\|_F^2 = \|\boldsymbol{U}_K^H\boldsymbol{Q}'_K\|_F^2 + \|\boldsymbol{V}^H\boldsymbol{Q}'_K\|_F^2.$$

Furthermore, we have

$$\begin{aligned}\|\boldsymbol{U}_K^H\boldsymbol{Q}'_K\|_F^2 &= \|\boldsymbol{Q}'_K\|_F^2 - \|\boldsymbol{V}^H\boldsymbol{Q}'_K\|_F^2 \\ &= K - \|\boldsymbol{V}^H\boldsymbol{Q}'_K\|_F^2 \geq K - \sqrt{K}\epsilon'\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{U}_K^H\boldsymbol{Q}'_{K-1}\|_F^2 &\leq \|\boldsymbol{U}_K^H\boldsymbol{Q}'_{K-1}\|_F^2 + \|\boldsymbol{V}^H\boldsymbol{Q}'_{K-1}\|_F^2 \\ &= \|\boldsymbol{Q}'_{K-1}\|_F^2 = K-1.\end{aligned}$$

By the definition of Frobenius norm, we derive

$$\begin{aligned}\|\boldsymbol{U}_K^H\boldsymbol{q}'_K\|_2^2 &= \|\boldsymbol{U}_K^H\boldsymbol{Q}'_K\|_F^2 - \|\boldsymbol{U}_K^H\boldsymbol{Q}'_{K-1}\|_F^2 \\ &\geq (K-\sqrt{K}\epsilon') - (K-1) = 1-\sqrt{K}\epsilon'\end{aligned} \tag{2}$$

On the other side, we have

$$\begin{aligned}\|\boldsymbol{U}_K^H\boldsymbol{q}'_K\|_2^2 &= \|\boldsymbol{U}_{K-1}^H\boldsymbol{q}'_K\|_2^2 + \|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2 \\ \|\boldsymbol{U}_{K-1}^H\boldsymbol{q}'_K\|_2^2 + \|\boldsymbol{U}_{K-1}^H\boldsymbol{Q}'_{K-1}\|_2^F &= \|\boldsymbol{U}_{K-1}^H\boldsymbol{Q}'_K\|_F^2 \leq \|\boldsymbol{U}_{K-1}^H\|_F^2\|\boldsymbol{Q}'_K\|_2^2 \leq \|\|\boldsymbol{U}_{K-1}^H\|_F^2 = K-1 \\ \|\boldsymbol{U}_{K-1}^H\boldsymbol{q}'_K\|_2^2 &\leq K-1-\|\boldsymbol{U}_{K-1}^H\boldsymbol{Q}'_{K-1}\|_2^F\end{aligned}$$

Then,

$$\begin{aligned}\|\boldsymbol{U}_K^H\boldsymbol{q}'_K\|_2^2 &\leq K-1-\|\boldsymbol{U}_{K-1}^H\boldsymbol{Q}'_{K-1}\|_2^F + \|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2 \\ &\leq K-1-(K-1-\sqrt{K-1}\epsilon') + \|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2 \\ &\leq \sqrt{K-1}\epsilon' + \|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2\end{aligned}$$

Combining the upper bound and the lower bound of $\|\boldsymbol{U}_K^H\boldsymbol{q}'_K\|_2^2$,

$$\sqrt{K-1}\epsilon' + \|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2 \geq \|\boldsymbol{U}_K^H\boldsymbol{q}'_K\|_2^2 \geq 1-\sqrt{K}\epsilon'$$

and we have $\|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2 \geq 1-\left(\sqrt{K}+\sqrt{K-1}\right)\epsilon' \geq 1-2\sqrt{K}\epsilon'$.

Finally, assume $\boldsymbol{u}_K^H\boldsymbol{q}'_K \geq 0$, we can derive

$$\begin{aligned}\|\boldsymbol{u}_K-\boldsymbol{q}'_K\|_2^2 &= \|\boldsymbol{u}_K\|_2^2 + \|\boldsymbol{q}'_K\|_2^2 - 2\|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2 \\ &\leq 2(1-\|\boldsymbol{u}_K^H\boldsymbol{q}'_K\|_2^2) \leq 2(1-1+2\sqrt{K}\epsilon') = 4\sqrt{K}\epsilon'\end{aligned} \tag{3}$$

The error caused by decentralized computing has been bounded by Lemma 3.1 in [1], which can written as:

$$\|\boldsymbol{Q}_K-\boldsymbol{Q}'_K\|_F \leq \epsilon''.$$

Therefore, we have

$$\|\boldsymbol{q}_2-\boldsymbol{q}'_2\|_2 \leq \epsilon''.$$

Combining with Equation (3), we have

$$\begin{aligned}\|\boldsymbol{u}_2-\boldsymbol{q}'_2\|_2 &\leq \|\boldsymbol{u}_2-\boldsymbol{q}'_2\|_2 + \|\boldsymbol{q}_2-\boldsymbol{q}'_2\|_2 \\ &\leq 2K^{1/4}\sqrt{\epsilon'}+\epsilon''\end{aligned} \tag{4}$$

With appropriate ϵ' and ϵ'', the proof is completed.

References

[1] D. Kempe and F. McSherry, "A decentralized algorithm for spectral analysis," in *Proceedings of the thirty-sixth annual ACM symposium on Theory of computing*, pp. 561–568, 2004.