

Non-asymptotic guarantees for individual eigenvector estimation

Yu-Cheng Hsiao

August 23, 2023

1 Notation

In the following, we denote the eigenspace from eigendecomposition as $\mathbf{U}_K \in \mathbb{R}^{N \times K}$ and the second-largest eigenvector from eigendecomposition as $\mathbf{u}_2 \in \mathbb{R}^N$. Similarly, we denote the eigenspace from centralized orthogonal iteration as $\mathbf{Q}'_K \in \mathbb{R}^{N \times K}$ and the second-largest eigenvector from centralized orthogonal iterations as $\mathbf{q}'_2 \in \mathbb{R}^N$. Moreover, we denote the eigenspace from decentralized orthogonal iteration as $\mathbf{Q}_K \in \mathbb{R}^{N \times K}$ and the second-largest eigenvector from decentralized orthogonal iterations as $\mathbf{q}_2 \in \mathbb{R}^N$. The adjacency matrix is represented by $\mathbf{A} \in \mathbb{R}^{N \times N}$.

2 Proof

Our goal is to show the estimation error of individual eigenvectors are bounded such that:

$$\|\mathbf{u}_i - \mathbf{q}_i\|_2 \leq \epsilon, \forall i \quad (1)$$

where \mathbf{u}_K represents the K_{th} eigenvector, \mathbf{q}'_2 represents the estimated one by the algorithm proposed in [1] and $\epsilon < 1$. From [1], we have

$$\|\mathbf{U}_K \mathbf{U}_K^H - \mathbf{Q}'_K \mathbf{Q}'_K^H\|_2 = \|\mathbf{V}^H \mathbf{Q}'_K\|_2 \leq \mathcal{O}\left(\left|\frac{\lambda_K + 1}{\lambda_K}\right|^T N\right) \leq \epsilon',$$

where $\mathbf{V} \in \mathbb{R}^{N \times (N-K)}$ spans the remaining eigenvectors of \mathbf{A} and \mathbf{Q}'_K denotes the T_{th} output of centralized orthogonal iteration. Consequently,

$$\|\mathbf{V}^H \mathbf{Q}'_K\|_F \leq \sqrt{K} \|\mathbf{V}^H \mathbf{Q}'_K\|_2 \leq \sqrt{K} \epsilon'.$$

By combining this with $\|\mathbf{Q}'_K\|_F^2 = K$ and invoking the orthogonal matrix $\mathbf{Z} = [\mathbf{U}_K, \mathbf{V}] \in \mathbb{R}^{N \times N}$, we have

$$\|\mathbf{Q}'_K\|_F^2 = \|\mathbf{Z}^H \mathbf{Q}'_K\|_F^2 = \|\mathbf{U}_K^H \mathbf{Q}'_K\|_F^2 + \|\mathbf{V}^H \mathbf{Q}'_K\|_F^2.$$

Furthermore, we have

$$\begin{aligned} \|\mathbf{U}_K^H \mathbf{Q}'_K\|_F^2 &= \|\mathbf{Q}'_K\|_F^2 - \|\mathbf{V}^H \mathbf{Q}'_K\|_F^2 \\ &= K - \|\mathbf{V}^H \mathbf{Q}'_K\|_F^2 \geq K - \sqrt{K} \epsilon' \end{aligned}$$

and

$$\begin{aligned}\|\mathbf{U}_K^H \mathbf{Q}'_{K-1}\|_F^2 &\leq \|\mathbf{U}_K^H \mathbf{Q}'_{K-1}\|_F^2 + \|\mathbf{V}^H \mathbf{Q}'_{K-1}\|_F^2 \\ &= \|\mathbf{Q}'_{K-1}\|_F^2 = K - 1.\end{aligned}$$

By the definition of Frobenius norm, we derive

$$\begin{aligned}\|\mathbf{U}_K^H \mathbf{q}'_K\|_2^2 &= \|\mathbf{U}_K^H \mathbf{Q}'_K\|_F^2 - \|\mathbf{U}_K^H \mathbf{Q}'_{K-1}\|_F^2 \\ &\geq (K - \sqrt{K}\epsilon') - (K - 1) = 1 - \sqrt{K}\epsilon'\end{aligned}\tag{2}$$

On the other side, we have

$$\begin{aligned}\|\mathbf{U}_K^H \mathbf{q}'_K\|_2^2 &= \|\mathbf{U}_{K-1}^H \mathbf{q}'_K\|_2^2 + \|\mathbf{u}_K^H \mathbf{q}'_K\|_2^2 \\ \|\mathbf{U}_{K-1}^H \mathbf{q}'_K\|_2^2 + \|\mathbf{U}_{K-1}^H \mathbf{Q}'_{K-1}\|_2^F &= \|\mathbf{U}_{K-1}^H \mathbf{Q}'_K\|_F^2 \leq \|\mathbf{U}_{K-1}^H\|_F^2 \|\mathbf{Q}'_K\|_2^2 \leq \|\mathbf{U}_{K-1}^H\|_F^2 = K - 1 \\ \|\mathbf{U}_{K-1}^H \mathbf{q}'_K\|_2^2 &\leq K - 1 - \|\mathbf{U}_{K-1}^H \mathbf{Q}'_{K-1}\|_2^F\end{aligned}$$

Then,

$$\begin{aligned}\|\mathbf{U}_K^H \mathbf{q}'_K\|_2^2 &\leq K - 1 - \|\mathbf{U}_{K-1}^H \mathbf{Q}'_{K-1}\|_2^F + \|\mathbf{u}_K^H \mathbf{q}'_K\|_2^2 \\ &\leq K - 1 - (K - 1 - \sqrt{K - 1}\epsilon') + \|\mathbf{u}_K^H \mathbf{q}'_K\|_2^2 \\ &\leq \sqrt{K - 1}\epsilon' + \|\mathbf{u}_K^H \mathbf{q}'_K\|_2^2\end{aligned}$$

Combining the upper bound and the lower bound of $\|\mathbf{U}_K^H \mathbf{q}'_K\|_2^2$,

$$\sqrt{K - 1}\epsilon' + \|\mathbf{u}_K^H \mathbf{q}'_K\|_2^2 \geq \|\mathbf{U}_K^H \mathbf{q}'_K\|_2^2 \geq 1 - \sqrt{K}\epsilon'$$

and we have $\|\mathbf{u}_K^H \mathbf{q}'_K\|_2^2 \geq 1 - (\sqrt{K} + \sqrt{K - 1})\epsilon' \geq 1 - 2\sqrt{K}\epsilon'$.

Finally, assume $\mathbf{u}_K^H \mathbf{q}'_K \geq 0$, we can derive

$$\begin{aligned}\|\mathbf{u}_K - \mathbf{q}'_K\|_2^2 &= \|\mathbf{u}_K\|_2^2 + \|\mathbf{q}'_K\|_2^2 - 2\|\mathbf{u}_K^H \mathbf{q}'_K\|_2 \\ &\leq 2(1 - \|\mathbf{u}_K^H \mathbf{q}'_K\|_2) \leq 2(1 - 1 + 2\sqrt{K}\epsilon') = 4\sqrt{K}\epsilon'\end{aligned}\tag{3}$$

The error caused by decentralized computing has been bounded by Lemma 3.1 in [1], which can written as:

$$\|\mathbf{Q}_K - \mathbf{Q}'_K\|_F \leq \epsilon''.$$

Therefore, we have

$$\|\mathbf{q}_2 - \mathbf{q}'_2\|_2 \leq \epsilon''.$$

Combining with Equation (3), we have

$$\begin{aligned}\|\mathbf{u}_2 - \mathbf{q}'_2\|_2 &\leq \|\mathbf{u}_2 - \mathbf{q}'_2\|_2 + \|\mathbf{q}_2 - \mathbf{q}'_2\|_2 \\ &\leq 2K^{1/4}\sqrt{\epsilon'} + \epsilon''\end{aligned}\tag{4}$$

With appropriate ϵ' and ϵ'' , the proof is completed.

References

- [1] D. Kempe and F. McSherry, “A decentralized algorithm for spectral analysis,” in *Proceedings of the thirty-sixth annual ACM symposium on Theory of computing*, pp. 561–568, 2004.